VORONOÏ SUMMATION FOR GL_n: COLLUSION BETWEEN LEVEL AND MODULUS

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Abstract. We investigate the Voronoï summation problem for GL_n in the level aspect for $n \geq 2$. Of particular interest are those primes at which the level and modulus are jointly ramified, a common occurrence in analytic number theory when using techniques such as the Petersson trace formula. Building on previous legacies, our formula stands as the most general of its kind; in particular we extend the results of Ichino–Templier [19]. We also give classical refinements of our formula and study the p-adic generalisations of the Bessel transform.

Contents

1. Developments in the Voronoï Summation Problem for GL_n

The Voronoï summation formula for GL_n with $n \geq 2$ is a relative of the Poisson summation formula, which describes the $n = 1$ case. The formula itself expresses a sum of additively twisted Fourier coefficients of automorphic forms on GL_n in terms of a sum over corresponding dual terms. The beauty of this transformation is that the dual side exhibits cancellation in the original summation by redistributing the weight of the summation. This structural shift is intrinsically tied to the functional equation for $\mathrm{GL}_n \times \mathrm{GL}_1$-$L$-functions.

The Voronoï summation formula is an essential tool in the analytic theory of automorphic forms. It features in the study of moments of L-functions and related subconvexity results [1, 26] but also in more esoteric problems, such as reciprocity formulae for spectral averages [5–7] and bounding norms of automorphic forms [4, 8]. Voronoï himself [36] constructed the first such prototype formulae with the

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motivation of giving beyond-trivial bounds for the error terms in both Dirichlet's divisor problem and Gauß' circle problem; an entertaining history is given in [27].

Despite the ubiquity of the Voronoï summation formula, full generality in the level aspect has previously not been addressed. This is the goal of the present work. Our results describe such general summation formulae in the following ways:

(i) We prove a Voronoï summation formula for GL_n in the most general setting, in particular allowing the level and modulus to ramify jointly. This is Theorem 1.1, stated in classical terms. In the adelic language, Theorem 3.4 describes general vectors in an automorphic representation.

(ii) We give two refinements of Theorem 1.1. The first utilises a convenient choice of test vector at which we explicate the Bessel transform; the second restricts the sum to an arithmetic progression. See Corollaries 1.2 & 1.3.

Miller–Schmid [28] prove the level $N = 1$ case of our formula in Theorem 1.1 with the omittance of non-archimedean test functions. They attribute the difficulty of the level aspect to the lack of understanding of Atkin–Lehner theory for GL_n-automorphic forms when $n > 2$. One naturally understands new-form and old-forms, together, as vectors which are respectively fixed by a filtration of compact open subgroups.

Our solution to this problem is to understand the corresponding dual summands locally via a p-adic transform derived from the local functional equation for $\mathrm{GL}_n \times \mathrm{GL}_1$ at primes p dividing the level. Our formulae referred to in (i) describe a *general* choice of vector in such transforms. We go on to give a prototype result for a *particular* choice of vector, as in (ii). This allows one to study the Fourier coefficients away from the level in a more refined and aesthetic way. Whilst our formula explicates, in most general terms, the structural mechanics of such Voronoï summation formula, there are other notable and interesting works exploring similar formulae from varying perspectives; see, for example, [2,7,29,38].

1.1. **A general classical formula.** An automorphic form f on $\mathrm{SL}_n(\mathbb{R})$, or rather on the adele group $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$, naturally generates an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$, which we denote by $\pi_f = \otimes_v \pi_v$. If such a form is a so-called newform then the associated representation is irreducible.

In this work we consider Maaß cusp forms on $\mathrm{SL}_n(\mathbb{R})$ of level $N \geq 1$; these are the eigenvectors in $L^2(\Gamma_1(N)\backslash \mathrm{SL}_n(\mathbb{R}))$ of the generalised Laplacian whose constant terms are zero. If truth be told, our study specifically concerns their Fourier coefficients $A_f \colon \mathbb{Z}^{n-1} \to \mathbb{C}$, as given in Definition 5.1. The normalisation of the A_f coincides with Goldfeld's [17, p. 260, (9.1.2)]; for example, the Ramanujan conjecture predicts that $A_f(m_1, \ldots, m_{n-1}) \ll (m_1 \cdots m_{n-1})^\varepsilon$. Evaluating A_f at integers prime to N one recovers the corresponding Hecke eigenvalues of f, a product of Schur polynomials in the Satake parameters of π_f (see Lemma 5.4). The A_f also occur as the natural L-series coefficients of the Godement–Jacquet

L-function when normalised symmetrically about $s \mapsto 1-s$. As we consider only cuspidal f we have $A_f(m_1, \dots, m_{n-1}) = 0$ for $m_1 \cdots m_{n-1} = 0$.

Let χ be a (Dirichlet) character modulo N. For $f \in L^2(\Gamma_1(N) \backslash \mathrm{SL}_n(\mathbb{R}))$ we say that f has character χ if and only if

$$f(\gamma g) = \chi(\gamma_{n,n}) f(g)$$

for all $\gamma = (\gamma_{i,j}) \in \mathrm{SL}_n(\mathbb{Z})$ such that $\gamma_{n,i} \equiv 0 \pmod N$ for $1 \leq i \leq n-1$. Without loss of generality, we always assume this property of f. Moreover, χ determines the parity of of a Maaß form: f is even, or respectively odd, if and only if χ is.

Theorem 1.1. *Let $N \geq 1$ be an integer and χ a Dirichlet character modulo N. For $n \geq 2$ let $f \in L^2(\Gamma_1(N) \backslash \mathrm{SL}_n(\mathbb{R}))$ be a Maaß cusp form with character χ and assume that f is Hecke eigenform at each prime $p \nmid N$. Let $M \geq 1$ be an integer such that $N \mid M$. For each $p \mid M$ choose some $\phi_p \in \mathcal{C}_c^\infty(\mathbb{Q}_p^\times)$; also pick $\phi_\infty \in \mathcal{C}_c^\infty(\mathbb{R}^\times)$. Let $c := (c_2, \dots, c_{n-1}) \in \mathbb{Z}^{n-2}$ such that $(c_2 \cdots c_{n-1}, M) = 1$. For $a, \ell, q \in \mathbb{Z}$ with $a \neq 0$, $\ell, q \geq 1$, $(a, \ell q) = (q, M) = 1$ and $\ell \mid M^\infty$ we have the following Voronoï summation formula:*

$$\begin{aligned}
&\sum_{m \in \mathbb{Z}_{\neq 0}} e\left(\frac{am}{\ell q}\right) \frac{A_f(m, c_2, \dots, c_{n-1})}{|m|^{\frac{n-1}{2}}} \phi_\infty(m) \prod_{p \mid M} \phi_p(m) \\
&\quad = q^{n-2} \prod_{i=2}^{n-1} |c_i|^{(n-i)(\frac{i}{2}-1)} \times \sum_{\substack{m \in \mathbb{Z}_{\neq 0} \\ (m,M)=1}} \sum_{r \mid M^\infty} \sum_{d_{n-1} \mid qc_2} \sum_{d_{n-2} \mid \frac{qc_2c_3}{d_{n-1}}} \cdots \sum_{d_2 \mid \frac{qc_2 \cdots c_{n-1}}{d_{n-1} \cdots d_3}} \\
&\qquad \mathrm{KL}\left(\overline{a\lambda_\ell} \ell r, m; q, c, d\right) \chi\left(\bar{m} \frac{qc_2 \cdots c_{n-1}}{d_{n-1} \cdots d_2}\right)^{-1} \times \frac{A_f(d_{n-1}, \dots, d_2, m)}{|m|^{\frac{n-1}{2}} \prod_{i=2}^{n-1} d_i^{\frac{i(n-i)}{2}}} \\
&\qquad \mathcal{B}_{\pi_\infty, \phi_\infty}\left(\frac{rm}{\lambda_\ell q^n} \prod_{i=2}^{n-1} \frac{d_i^i}{c_i^{n-i}}\right) \prod_{p \mid M} \mathcal{B}_{\pi_p, \Phi_p^{a/\ell q}}\left(\frac{rm}{\lambda_\ell q^n} \prod_{i=2}^{n-1} \frac{d_i^i}{c_i^{n-i}}\right) \qquad (1)
\end{aligned}$$

where $\lambda_\ell := [\ell, N] \ell^{n-1} L^n$ with L denoting the largest square-free integer dividing M and $[\ell, N] := \mathrm{lcm}(\ell, N)$; we fix inverses $a\bar{a} \equiv \lambda_\ell \bar{\lambda}_\ell \equiv 1 \pmod q$ and $m\bar{m} \equiv 1 \pmod N$; the $(n-1)$-dimensional Kloosterman sum is defined by

$$\begin{aligned}
\mathrm{KL}(x, y; q, c, d) &= \sum_{j=2}^{n-1} \sum_{\alpha_j \in \left(\mathbb{Z}/\left(\frac{qc_2 \cdots c_{n-j+1}}{d_{n-1} \cdots d_i}\right)\mathbb{Z}\right)^\times} e\left((-1)^n \frac{x d_{n-1} \alpha_{n-1}}{q}\right) \\
&\quad \times e\left(\frac{d_{n-2} \alpha_{n-2} \bar{\alpha}_{n-1}}{\frac{qc_2}{d_{n-1}}}\right) \cdots e\left(\frac{d_2 \alpha_2 \bar{\alpha}_3}{\frac{qc_2 \cdots c_{n-2}}{d_{n-1} \cdots d_3}}\right) e\left(\frac{y \bar{\alpha}_2}{\frac{qc_2 \cdots c_{n-1}}{d_{n-1} \cdots d_2}}\right) \qquad (2)
\end{aligned}$$

for $x, y \in \mathbb{Z}$ and $d := (d_2, \dots, d_{n-1})$; and the functions $\mathcal{B}_{\pi_\infty, \phi_\infty}$ and $\mathcal{B}_{\pi_p, \Phi_p^{a/\ell q}}$ are given explicitly in (33) and (34), respectively, whilst $\Phi_p^{a/\ell q}$ is defined in (21).

The proof of Theorem 1.1 is detailed in §5.6. It is a specialisation of our more general summation formula in Theorem 3.4.

1.1.1. *The p-adic Bessel transforms.* The functions $\mathcal{B}_{\pi_p,\Phi_p^{a/\ell q}}$ on $\mathbb{Q}_p^\times$ are crucial as they allow the explicit evaluation of the dual terms in the Voronoï formula at primes $p \mid N$. In this work we explicate these transforms. For particular choices of f, we show they are proportional to sums of twists of GL_n-root numbers when the the level N and modulus ℓ are jointly ramified $(\ell, N) > 1$; see Corollary 1.2.

- The term $\lambda_\ell = [\ell, N]\ell^{n-1}L^n$ is determined by a generic support condition Proposition, 4.6, for $\mathcal{B}_{\pi_p,\Phi_p^{a/\ell q}}$. In special cases, this term and the $r \mid M^\infty$-sum may be greatly improved; see Corollary 1.2.
- Upper bounds are sensitive to the test functions ϕ_p and the local representation π_p attached to π_p. We shall see that the size of the functions $\mathcal{B}_{\pi_p,\Phi_p^{a/\ell q}}$ is related to sums of twisted GL_n-epsilon constants.

1.1.2. *The real Bessel transforms.* If one chooses the support of ϕ_∞ to be contained in $\mathbb{R}_{>0}$ then Kowalski–Ricotta [25, Prop. 3.5] show that for any $A > 0$ we have $\mathcal{B}_{\pi_\infty,\phi_\infty}(y) \ll y^{-A}$. (See Proposition 4.3 for further details.)

1.2. **Refined summation formulae.** Here we consider a certain family of Maaß forms f by placing assumptions upon the local representations of $\pi_f = \otimes_v \pi_v$ at primes $v = p$ for which $p \mid N$.

Assumption 1.1. *For each $p \mid N$, suppose that the twists of the local Euler factor $L(s, \chi\pi_p) = 1$, identically, for each character $\chi\colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$. Moreover suppose that π_p is twist minimal in the sense of Definition 4.2.*

The first part of this assumption is satisfied, for example, by all supercuspidal representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. A representation may be made twist minimal after twisting by a character. For example, a supercuspidal representation whose (log) conductor is indivisible by n is always twist minimal by [12, Prop. 2.2].

1.2.1. *An explicit factor at primes dividing the level.* Here we inserts a clean cut explication of the p-adic Bessel functions. For stylistic reasons we shall assume that ℓ is divisible by the squares of each of its prime divisors, in which case we define the function

$$S_f(m; \frac{a}{\ell q}) = \sum_{\chi \pmod{\ell}}^{*} \chi(-1)^{n-1}\chi(m\bar{a}q)\varepsilon(1/2, f \times \chi)\varepsilon(1/2, \chi^{-1})$$

where we write $a\bar{a} \equiv 1 \pmod{\ell}$ and use the notation $\sum^*$ to indicate that the summation is restricted to just the primitive Dirichlet characters $\pmod{\ell}$. An alternative definition may be made for general $\ell \mid N^\infty$ at th expense of a more complicated expression. This is simply because the p-adic Bessel transforms $v_p(\ell) \leq 1$ depend sensitively to the underlying local representation. Nevertheless, explicit formulas in all cases are given in Proposition 4.7.

Corollary 1.2. *Suppose that, in addition to the hypotheses of Theorem 1.1, Assumption 1.1 holds. Suppose too that ℓ is divisible by the squares of each of its prime divisors. Then we have the following refined Voronoï summation formula:*

$$\sum_{\substack{m\in\mathbb{Z}_{\neq 0}\\(m,N)=1}} e\left(\frac{am}{\ell q}\right)\frac{A_f(m,c_2,\ldots,c_{n-1})}{|m|^{\frac{n-1}{2}}}\phi_\infty(m) = \frac{[N,\ell^n]^{\frac{n-2}{2}}\ell^{-1/2}q^{n-2}}{\prod_{p|N}(1-p^{-1})}S_f(m;\frac{a}{\ell q})$$
$$\times\prod_{i=2}^{n-1}|c_i|^{(n-i)(\frac{i}{2}-1)}\sum_{\substack{m\in\mathbb{Z}_{\neq 0}\\(m,N)=1}}\sum_{d_{n-1}|qc_2}\sum_{d_{n-2}|\frac{qc_2c_3}{d_{n-1}}}\cdots\sum_{d_2|\frac{qc_2\cdots c_{n-1}}{d_{n-1}\cdots d_3}}\mathrm{KL}\left(\overline{a[N,\ell^n]}\ell,m;q,c,d\right)$$
$$\times\ \chi\left(\bar{m}\frac{qc_2\cdots c_{n-1}}{d_{n-1}\cdots d_2}\right)^{-1}\frac{A_f(d_{n-1},\ldots,d_2,m)}{|m|^{\frac{n-1}{2}}\prod_{i=2}^{n-1}d_i^{\frac{i(n-i)}{2}}}\mathcal{B}_{\pi_\infty,\phi_\infty}\left(\frac{m}{[N,\ell^n]q^n}\prod_{i=2}^{n-1}\frac{d_i^i}{c_i^{n-i}}\right). \quad (3)$$

Proof. This formula follows from the explicit computation of the Bessel transforms $\mathcal{B}_{\pi_p,\Phi_p^{a/\ell q}}$ under the refined hypotheses. This is executed in §4.3, specifically in Proposition 4.7. $\square$

The term $S_f(m;\frac{a}{\ell q})$ should be thought of as a 'ramified' Kloosterman sum. It is derived via a transform of a p-adic Bessel function analogue. Unpicking further, the epsilon-factor sum may be shown to equal a Kloosterman sum dependant on the inducing data of the representations π_p for $p\mid\ell$. The explication of these terms is a deep problem in the corresponding p-adic representation theory. At present, it seems only possible to proceed in a case-by-case fashion.

1.2.2. *Voronoi summation in arithmetic progressions.* An amusing variant of the Theorem 1.1 is to restrict the summation to an arithmetic progression. The formula we present here is suited to applications such as in [25]. For a set S let us define $\mathrm{Char}_S(x)=1$ is $x\in S$ and $\mathrm{Char}_S(x)=0$ otherwise.

Corollary 1.3. *With the hypotheses of Theorem 1.1 and Assumption 1.1, fix an integer $M\geq 1$ and for each $p\mid M$ and define $\phi_p=\mathrm{Char}_{1+M\mathbb{Z}_p}\in\mathcal{C}_c^\infty(\mathbb{Q}_p^\times)$. Once again assume that ℓ is divisible by the squares of each of its prime divisors. Then the left-hand side of the equation in Theorem 1.1 becomes*

$$\sum_{\substack{m\in\mathbb{Z}_{\neq 0}\\(m,N)=1\\m\equiv 1\pmod{M}}} e\left(\frac{am}{\ell q}\right)\frac{A_f(m,c_2,\ldots,c_{n-1})}{|m|^{\frac{n-1}{2}}}\phi_\infty(m)$$

whilst, on the right-hand side the r-sum, indexing the variable r/λ_ℓ is simply replaced by the variable $1/r$ for $r\mid[M,\ell]$. In this case, the transforms $\mathcal{B}_{\pi_p,\Phi_p^{a/\ell q}}$ are given in Proposition 4.10, taking $k=v_p(M)$.

2. Background Representation Theory and Notation

Here we give a bite-sized recap of the theory of automorphic representations for GL_n. The main purpose of this section is to fix notation for §3. It may be referred back to should confusion present itself. The content here may thus be sidestepped without causing injury to our discourse on Voronoï summation.

2.1. **Notable matrix groups.** For a commutative ring R, with unit $1 \in R$, let us introduce notation for certain subgroups and elements of $\mathrm{GL}_n(R)$. Let $B_n(R) = T_n(R) \ltimes U_n(R)$ denote the standard Borel subgroup, consisting of upper triangular matrices in $\mathrm{GL}_n(R)$, given by the semi-direct product of the maximal torus of diagonal matrices $T_n(R)$ and the subgroup $U_n(R)$ of upper triangular matrices whose n eigenvalues are all equal to 1. Denote the centre of $\mathrm{GL}_n(R)$ by $Z(R) \cong R^\times$, which acts on $\mathrm{GL}_n(R)$ by scalar multiplication.

Let 1_n denote the $n \times n$ identity matrix. Let $w = w_n$ be the longest Weyl element of $\mathrm{GL}_n(R)$, defined recursively by $w_n = \left(\begin{smallmatrix} & w_{n-1} \\ 1 & \end{smallmatrix}\right)$ and $w_1 = 1$. Define a second Weyl element by $w' = \left(\begin{smallmatrix} 1 & \\ & w_{n-1} \end{smallmatrix}\right)$. We assign specialist notation to the matrices

$$a(y) := \begin{pmatrix} y & \\ & 1_{n-1} \end{pmatrix} \in T_n(R) \quad \text{and} \quad n(x) := \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1_{n-2} \end{pmatrix} \in U_n(R). \tag{4}$$

For $g \in \mathrm{GL}_n(R)$ we consider the involution $g^\iota := {}^t g^{-1}$, noting in particular that $n(x)^\iota = {}^t n(-x)$ and $a(y)^\iota = a(y^{-1})$. For a function $f\colon \mathrm{GL}_n(R) \to \mathbb{C}$, denote the right regular action of $g \in \mathrm{GL}_n(R)$ by $\rho(g)f(x) = f(xg)$ for $x \in \mathrm{GL}_n(R)$.

2.2. **Local and global fields.** Let F be a number field. Let $\mathbb{A}_F$ denote the ring of F-adeles and $\mathfrak{o}_F$ ring of algebraic integers contained in F. At each place v of F let F_v denote the completion of F at v. Let S_∞ denote the set of archimedean places of F. Suppose $v \notin S_\infty$. Then denote by $\mathfrak{o}_v$ the ring of integers of F_v; $\mathfrak{p}_v$ the maximal (prime) ideal of $\mathfrak{o}_v$; ϖ_v a choice of uniformising parameter, that is a generator of $\mathfrak{p}_v$; and let $q_v = \#(\mathfrak{o}_v/\mathfrak{p}_v)$. Let $|\cdot|_v$ denote the absolute value on F_v, normalised so that $|\varpi_v|_v = q_v^{-1}$. The place v itself denotes the discrete valuation on F_v, satisfying $|x|_v = q_v^{-v(x)}$ for $x \in F$. At real places $v \in S_\infty$ set $|x|_v = \mathrm{sgn}(x)x$ for $x \in \mathbb{R}$ and at complex places $v \in S_\infty$ set $|z|_v = z\bar{z}$ for $z \in \mathbb{C}$.

2.3. **Additive characters.** We say that an (additive) character $\psi = \otimes_v \psi_v$ of $\mathbb{A}_F/F$ is *unramified* if $\mathfrak{o}_v \subset \ker_{F_v}(\psi_v)$ for each $v \notin S_\infty$ and *generic* if $\psi_v \neq 1$ for each v. We henceforth fix such a choice of ψ throughout. Note that the dual group of $\mathbb{A}_F/F$ is identified by the set $\{x \mapsto \psi(ax) : a \in F\}$. We shall abuse notation by letting ψ denote the following character of each subgroup $H \leq U_n(\mathbb{A}_F)$: if $h \in H$ is given by $h = (h_{i,j})$ then define

$$\psi(h) = \psi(h_{1,2} + h_{2,3} + \cdots + h_{n-1,n}). \tag{5}$$

By the F-invariance of ψ we also have $(H \cap U_n(F)) \subset \ker_H(\psi)$.

2.4. **Multiplicative characters.** Let $\hat{F}_v^\times$ denote the group of unitary characters of $F_v^\times$. Suppose v is non-archimedean. For $\mu \in \hat{F}_v^\times$ we define the *conductor* of μ to be the non-negative integer

$$a(\chi) = \min\{r \geq 0 : \ker(\chi) \subset \mathfrak{o}_v^\times \cap (1 + \varpi_v^r \mathfrak{o}_v)\}.$$

Moreover define

$$\mathfrak{X}_v = \{\, \chi \colon F_v^\times \to \mathbb{C}^\times \mid \chi(\varpi_v) = 1 \,\} \subset \hat{F}^\times.$$

Then $\mathfrak{X}_v$ is a discrete group, isomorphic to the unitary dual of $\mathfrak{o}_v^\times$. Considering the 'polar coordinates' $y = u\varpi_v^{v(y)}$ for $y \in F_v^\times$, one identifies

$$\hat{F}_v^\times = \mathfrak{X}_v \times \hat{\mathbb{Z}} \cong \mathfrak{X}_v \times \mathbb{R}/\mathbb{Z}.$$

2.5. **The Whittaker model.** let $\pi = \otimes_v \pi_v$ be an irreducible, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, realised in a space of automorphic forms $\mathcal{V}_\pi$. There is a $\mathrm{GL}_n(\mathbb{A}_F)$-intertwining map from $\mathcal{V}_\pi$ into the space of functions $W\colon \mathrm{GL}_n(\mathbb{A}_F) \to \mathbb{C}$ that satisfy $W(ug) = \psi(u)W(g)$ for each $u \in U_n(\mathbb{A}_F)$. This carries the right regular representation ρ of $\mathrm{GL}_n(\mathbb{A}_F)$. Explicitly, one maps $\varphi \in \mathcal{V}_\pi$ to the element

$$W_\varphi\colon g \longmapsto \int_{U_n(F)\backslash U_n(\mathbb{A})} \varphi(ug)\overline{\psi(u)}du, \tag{6}$$

where we choose du to be the invariant probability measure on $U_n(F)\backslash U_n(\mathbb{A})$. Let $\mathcal{W}(\pi,\psi)$ denote the image of $\mathcal{V}_\pi$ under (6), another irreducible $\mathrm{GL}_n(\mathbb{A}_F)$-module. One calls $\mathcal{W}(\pi,\psi)$ the ψ-*Whittaker model* of π. This generalises the classical realisation of Fourier coefficients.

Similarly, at each place v of F there is the notion of a ψ_v-Whittaker model; this is the space $\mathcal{W}(\pi_v,\psi_v)$ of functions $W\colon \mathrm{GL}_n(F_v) \to \mathbb{C}$ satisfying $W_v(ug) = \psi_v(u)W_v(g)$ for each $u \in U_n(F_v)$. It is again a $\mathrm{GL}_n(F_v)$-module under ρ and one has a non-zero module homomorphism $\pi_v \cong \mathcal{W}(\pi_v,\psi_v)$. The uniqueness of Whittaker models implies that $\mathcal{W}(\pi,\psi)$ is given by the restricted tensor product $\mathcal{W}(\pi,\psi) = \otimes_v' \mathcal{W}(\pi_v,\psi_v)$. This equality is significant as it identifies $\mathcal{W}(\pi,\psi)$ as a space of *factorisable* functions on $\mathrm{GL}_n(\mathbb{A}_F)$.

Remark 2.1. Global multiplicity one for GL_n was proved by Shalika [32, Theorem 5.5]. The local multiplicity one theorem was also completed by Shalika in [32, Theorem 3.1], who showed that the non-archimedean result of Gelfand–Každan [15] was true for archimedean places too.

It shall be of use to speculate on the support of non-archimedean Whittaker functions. The following lemma gives a first observation. In the unramified setting, we can say much more; see §2.7.

Lemma 2.2. *Suppose $v \notin S_\infty$ and let $W_v \in \mathcal{W}(\pi_v,\psi_v)$ such that W_v is $U_n(\mathfrak{o}_v)$-fixed. Then, $W_v(a(y)) = 0$ for all $y \in F_v^\times$ with $|y|_v > 1$.*

Proof. Let $y \in F_v^\times$. Then, for all $x \in \mathfrak{o}_v$ we have

$$W_v(a(y)) = W(a(y)n(x)) = W(n(xy)a(y)) = \psi_v(xy)W(a(y)).$$

If $|y|_v > 1$, we can always find $x \in \mathfrak{o}_v$ such that $\psi_v(xy) \neq 1$. □

Remark 2.3. In particular, the hypothesis of Lemma 2.2 is satisfied by all new-forms and old-forms.

2.6. **The Kirillov model.** The classical results of both Gelfand–Každan [15] and Jacquet–Shalika [22, Prop. 3.8] imply that an irreducible representation of $\mathrm{GL}_n(F_v)$ isomorphic to π_v may be realised in the restriction of the functions $W_v \in \mathcal{W}(\pi_v, \psi_v)$ to the 'mirabolic subgroup' $P_n(F_v)$ of $\mathrm{GL}_n(F_v)$, the stabiliser of $(0, \ldots, 0, 1)$ on the right. This space of vectors is known as the *Kirillov model*.

Proposition 2.4. *Let v be a place of F. Then the space of functions on $P_n(F_v)$ given by $W_v|_{P_n(F_v)}$, for some $W_v \in \mathcal{W}(\pi_v, \psi_v)$, contains the entire space of compactly supported Bruhat–Schwartz functions Φ on $P_n(F_v)$ such that $\Phi\left(\left(\begin{smallmatrix} n & \\ & 1\end{smallmatrix}\right)p\right) = \psi_v(n)\Phi(p)$ for each $n \in U_{n-1}(F_v)$ and $p \in P_n(F_v)$.*

In the present article, it shall suffice to consider just those Bruhat–Schwartz functions on $P_n(F_v)$ with support on the matrices $a(y)$ for $y \in F_v^\times$.

Remark 2.5. If v is non-archimedean and π_v is supercuspidal, then the space of functions on $P_n(F_v)$ given by $W_v|_{P_n(F_v)}$ is *precisely* the space of locally constant, compactly supported functions Φ described in Proposition 2.4. In general, the co-dimension for the containment in Proposition 2.4 is at most n.

2.7. **Spherical Whittaker functionals.** In [33], Shintani evaluated the (spherical) Whittaker functionals on $\mathrm{GL}_n(F_v)$ in terms of certain polynomials of the inducing data. Let v be a non-archimedean place of v and suppose π_v is unramified; that is,

$$\pi_v = \mathrm{Ind}_{B_n(F_v)}^{\mathrm{GL}_n(F_v)}(\mu_1 \otimes \cdots \otimes \mu_n) \tag{7}$$

for unramified characters $\mu_i \colon F_v^\times \to \mathbb{C}^\times$ for $i = 1, \ldots, n$. This criterion includes all but finitely many places of F. Such characters are determined by their value on ϖ_v and, in turn, π_v is completely determined by the n complex numbers $\mu_i(\varpi_v)$, its 'Satake parameters', for $i = 1, \ldots, n$.

Let $W_v^\circ \in \mathcal{W}(\pi_v, \psi_v)$ denote the unique $\mathrm{GL}_n(\mathfrak{o}_v)$-fixed vector with $W_v^\circ(1) = 1$. Then, by the Iwasawa decomposition, W_v° is completely determined by its values on $T_n(F_v)/T_n(\mathfrak{o}_v)$. These values are given in terms of a *Schur polynomial*, or rather a (trace) character value of an irreducible representation of $\mathrm{GL}_n(\mathbb{C})$; see [14, Chapter 6] for instance. To this end we define the function

$$s_\lambda(t_1, \ldots, t_n) := \prod_{1 \le i < j \le n} (t_i - t_j)^{-1} \det \begin{pmatrix} t_1^{\lambda_1+n-1} & t_2^{\lambda_1+n-1} & \cdots & t_n^{\lambda_1+n-1} \\ t_1^{\lambda_2+n-2} & t_2^{\lambda_2+n-2} & \cdots & t_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\lambda_n} & t_2^{\lambda_n} & \cdots & t_n^{\lambda_n} \end{pmatrix} \tag{8}$$

evaluated on a toral element $t = \mathrm{diag}(t_1, \ldots, t_n) \in \mathrm{GL}_n(\mathbb{C})$ and indexed by a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ satisfying

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \tag{9}$$

Proposition 2.6 (Shintani's formula [33])**.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ *and consider the element* $\varpi_v^\lambda := \mathrm{diag}(\varpi_v^{\lambda_1}, \ldots, \varpi_v^{\lambda_n}) \in T_n(F_v)$*. Then* $W_v^\circ(\varpi_v^\lambda) = 0$ *unless* λ *satisfies* (9)*, in which case*

$$W_v^\circ(\varpi_v^\lambda) = q_v^{\sum_{i=1}^n \lambda_i\left(\frac{n+1}{2}-i\right)} s_\lambda(\mu_1(\varpi_v), \ldots, \mu_n(\varpi_v)).$$

These values are equal to a certain Hecke eigenvalue of W_v°. We refer to [10, Lecture 7] for wider exposition and context of Shintani's result. Note that the power of q_v in Shintani's formula is precisely (the square-root of) the modular character of $B_n(F_v)$. It comes directly from our *unitary* normalisation of the unramified principal series π_v.

2.8. **The contragredient representation.** The involution ι on $\mathrm{GL}_n(\mathbb{A}_F)$, given by $g^\iota = {}^t g^{-1}$, determines an injection of π into its contragredient representation $\tilde{\pi}$. Explicitly by defining the functions $\tilde{\varphi} = \varphi \circ \iota$ for $\varphi \in \mathcal{V}_\pi$. By (6), we obtain the $\bar{\psi}$-Whittaker function $W_{\tilde{\varphi}}$ which satisfies $W_{\tilde{\varphi}}(g) = W_\varphi(wg^\iota)$ and

$$W_{(\rho(h)\varphi)}(g^\iota) = \rho(h^\iota) W_{\tilde{\varphi}}(g) \tag{10}$$

for $g, h \in \mathrm{GL}_n(\mathbb{A})$. The Whittaker model $\mathcal{W}(\tilde{\pi}, \bar{\psi})$ (resp. $\mathcal{W}(\tilde{\pi}_v, \bar{\psi}_v)$) is then given by the space of functions $\tilde{W}$, defined by $\tilde{W}(g) = W(wg^\iota)$, for each $W \in \mathcal{W}(\pi, \psi)$ (resp. $\mathcal{W}(\tilde{\pi}_v, \bar{\psi}_v)$).

2.9. **Euler factors and epsilon constants.** Let v be a place of F. For any character χ of $F_v^\times$ we can define the twist $\chi\pi_v = (\chi \circ \det) \otimes \pi_v$. We follow Godement–Jacquet [16] in defining the local Euler factors $L(s, \chi\pi_v)$, epsilon constants $\varepsilon(s, \chi\pi_v, \psi_v)$, and gamma factors

$$\gamma(s, \chi\pi_v, \psi_v) = \varepsilon(s, \chi\pi_v, \psi_v) \frac{L(1-s, \chi^{-1}\tilde{\pi}_v)}{L(s, \chi\pi_v)}. \tag{11}$$

For $v \notin S_\infty$ we have the formula

$$\varepsilon(s, \pi_v, \psi_v) = \varepsilon(1/2, \pi_v, \psi_v)\, q_v^{a(\pi_v)(\frac{1}{2}-s)} \tag{12}$$

in which the *conductor* $a(\pi_v)$ of π_v is implicitly defined (see [16, Theorem 3.3 (4) & (3.3.5)]). The *root number* $\varepsilon(1/2, \pi_v, \psi_v)$ takes its value on the unit circle. Of course, these factors can also be reinterpreted via the local Langlands correspondence for GL_n (see [31, 37] for such a description); this viewpoint is useful should one want to compute them explicitly for a given π_v. One has $a(\pi_v) = 0$ if and only if π_v is unramified. Thus one makes sense of the following global definition.

Definition 2.1. We call a positive integer $N(\pi)$ the *level* (or *arithmetic conductor*) of an irreducible automorphic representation $\pi = \otimes_v \pi_v$ if $N(\pi) = \prod_{v \notin S_\infty} q_v^{a(\pi_v)}$.

Then the global L-function $L(s,\pi) = \prod_v L(s,\pi_v)$, initially defined for $\mathrm{Re}(s)$ sufficiently large, has analytic continuation to all $s \in \mathbb{C}$, is bounded in vertical strips, and satisfies the functional equation

$$L(s,\pi) = \varepsilon(1/2,\pi)N(\pi)^{(1/2-s)}L(1-s,\tilde{\pi})$$

where $\varepsilon(1/2,\pi) = \prod_v \varepsilon(1/2,\pi_v,\psi_v)$ is independent of ψ (see [16, Theorem 13.8]).

3. Voronoï Summation via the Whittaker Model

Ichino–Templier's recasting of Voronoï summation formulae in an adelic framework is a tremendously important step in understanding the mechanisms governing such identities. Here we prove a strengthening of their results as given in [19].

Throughout this section, fix a number field F and an unramified, generic character $\psi = \otimes_v \psi_v$ of $\mathbb{A}_F/F$ (see §2.3). Also fix $n \geq 2$ and let $\pi = \otimes_v \pi_v$ denote an irreducible, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. We now pose a Voronoï summation formula for the 'Fourier coefficients' in the Whittaker model $\mathcal{W}(\pi,\psi)$.

3.1. **The generalised Bessel tranformation.** We now describe the key tool in handling local ramification in the Voronoï summation formula: the generalised Bessel tranformation. Presently we give a criterion for the existence of such a transform. But a key feature of our work is the explication of these transforms, for which we give full details in §4 in the general case.

Let v be a place of F. If $v \in S_\infty$ (resp. $v \notin S_\infty$) let $\mathcal{C}^\infty(F_v^\times)$ denote the space of smooth (resp. locally constant) functions on $F_v^\times$. In either case, let $\mathcal{C}_c^\infty(F_v^\times) \subset \mathcal{C}^\infty(F_v^\times)$ denote the subspace of functions whose support is compact.

Proposition 3.1. *Let $\Phi\colon F_v^\times \to \mathbb{C}$ be a function defined by $\Phi(y) = W(\left(\begin{smallmatrix} y & \\ & 1_n\end{smallmatrix}\right))$ for some $W \in \mathcal{W}(\pi_v,\psi_v)$; by Proposition 2.4, this includes the set $\mathcal{C}_c^\infty(F_v^\times)$. Then there exists a unique function $\mathcal{B}_{\pi_v,\Phi}\colon F_v^\times \to \mathbb{C}$, the Bessel transform of Φ, such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large and for all characters χ of $F_v^\times$ we have*

$$\int_{F_v^\times} \mathcal{B}_{\pi_v,\Phi}(y)\chi(y)^{-1}|y|_v^{s-\frac{n-1}{2}}\,d^\times y = \chi(-1)^{n-1}\gamma(1-s,\chi\pi_v,\psi_v)\int_{F_v^\times}\Phi(y)\chi(y)|y|_v^{1-s-\frac{n-1}{2}}\,d^\times y \quad (13)$$

where $d^\times y$ denotes a Haar measure on $F_v^\times$.

Proof. This result follows directly from the local functional for $\mathrm{GL}_n \times \mathrm{GL}_1$ as proved by Jacquet–Piatetski-Shapiro–Shalika [21, 23]; see also Cogdel's notes [10, Theorems 6.2 & 8.2]. This result is a generalisation of [19, Lemma 5.2]. We remark that $\mathrm{Re}(s)$ is chosen sufficiently large to ensure the convergence of the left-hand side and furthermore to be greater than the poles of $\gamma(1-s,\chi\pi_v,\psi_v)$. For such s, the integral on the right-hand side of (13) converges due to the compact support of Φ, thus (13) holds as an identity without invoking analytic continuation. $\square$

Remark 3.2. For $\Phi \in \mathcal{C}_c^\infty(F_v^\times)$, the transform $\mathcal{B}_{\pi_v,\Phi}$ satisfies the following properties: sub-polynomial decay at infinity (for each $\delta > 0$ we have $\tilde{\Phi}(y) \ll |y|_v^{-\delta}$ as $|y|_v \to \infty$) and polynomial growth at zero (there exists a $\delta > 0$ such that $\tilde{\Phi}(y) \ll |y|_v^{\delta}$ as $y \to 0$).

3.2. **The hyper-Kloosterman sum.** At a place $v \notin S_\infty$, denote by $T_v^1 \subset T_{n-2}(F_v)$ the set of diagonal matrices of the form $t = \mathrm{diag}(t_2, \ldots, t_{n-1})$ with $|t_i|_v \geq 1$ for each $2 \leq i \leq n-1$. Then $\mathfrak{o}_v$ acts (additively) on t_i if $|t_i|_v > 1$, whence we define the quotients

$$\Lambda_{t_i} = \begin{cases} t_i\mathfrak{o}_v^\times/\mathfrak{o}_v & \text{if } |t_i|_v > 1 \\ \{1\} & \text{if } |t_i|_v = 1. \end{cases} \tag{14}$$

We shall consider a summation over the set

$$\Lambda_t := \{0\} \times \Lambda_{t_2} \times \cdots \times \Lambda_{t_{n-1}}$$

containing elements of the form

$$x = (0, x_2, \ldots, x_{n-1}) \in \Lambda_t.$$

The component $\{0\}$ is to ensure that the sum is non-empty in the case $n = 2$, in which case the only summand is equal to 1. Fix a 'modulus' $\zeta = (\zeta_v) \in \mathbb{A}_F$ and a 'shift' $\xi = (\xi_v) \in T_n(\mathbb{A}_F)$. We write $\xi_v = \mathrm{diag}(\xi_1, \cdots, \xi_n)$. Let R be a set of places $v \notin S_\infty$ such that π_v is unramified and either $|\zeta_v|_v > 1$ or $\xi_v \notin T_n(\mathfrak{o}_v)$. For $v \in R$ let $t \in T_v^1$. If $|\zeta_v\xi_1^{-1}\xi_2|_v \geq 1$ then for $y \in F_v^\times$ define the $(n-1)$-dimensional hyper-Kloosterman sum by

$$\begin{aligned} \mathcal{K}\ell_v(y, t; \zeta, \xi) &= |\xi_2\zeta_v|_v^{n-2}|\xi_3\cdots\xi_n|_v^{-1}\psi_v(-\xi_2\xi_3^{-1}) \\ &\times \sum_{x \in \Lambda_t} \psi_v((-1)^n y\zeta_v^{-1}\xi_2^{-1}\xi_n x_{n-1}^{-1}\cdots x_2^{-1}) \prod_{j=2}^{n-1} \psi_v(\xi_{n-j+1}\xi_{n-j+2}^{-1}x_j). \end{aligned} \tag{15}$$

If $|\zeta_v\xi_1^{-1}\xi_2|_v \leq 1$ then define $\mathcal{K}\ell_v(y, t; \zeta, \xi) = \mathcal{K}\ell_v(y, t; \xi_1\xi_2^{-1}, \xi)$ so that

$$\begin{aligned} \mathcal{K}\ell_v(y, t; \zeta, \xi) &= |\xi_1|_v^{n-2}|\xi_3\cdots\xi_n|_v^{-1}\psi_v(-\xi_2\xi_3^{-1}) \\ &\times \sum_{x \in \Lambda_t} \psi_v((-1)^n y\xi_1^{-1}\xi_n x_{n-1}^{-1}\cdots x_2^{-1}) \prod_{j=2}^{n-1} \psi_v(\xi_{n-j+1}\xi_{n-j+2}^{-1}x_j). \end{aligned} \tag{16}$$

More generally, define the product $T_R^1 = \prod_{v\in R} T_v^1$, which we view embedded in $T_{n-2}(\mathbb{A}_F)$ by extending trivially at places $v \notin R$. Moreover, for $y \in \bigcap_{v\in R} F_v$ and $t = (t_v) \in T_R^1$ define

$$\mathcal{K}\ell_R(y, t; \zeta, \xi) = \begin{cases} \prod_{v\in R} \mathcal{K}\ell_v(y, t_v; \zeta, \xi) & \text{if } R \neq \emptyset \\ 1 & \text{if } R = \emptyset. \end{cases} \tag{17}$$

Finally, for $t = (t_v) \in T^1_R$ with $t_v = \mathrm{diag}(t_2, \ldots, t_{n-1})$, define the matrix

$$\delta_R(t; \zeta, \xi) = (\delta_v) \in T_n(\mathbb{A}_F) \tag{18}$$

whose local factors are $\delta_v = 1$ for $v \notin R$;

$$\delta_v = \begin{pmatrix} \zeta_v^{-1}(\det t)^{-1}\xi_2^{-1} & & & & \\ & t_2\xi_n^{-1} & & & \\ & & \ddots & & \\ & & & t_{n-1}\xi_3^{-1} & \\ & & & & \zeta_v\xi_1^{-1} \end{pmatrix}$$

for $v \in R$ such that $|\zeta_v\xi_1^{-1}\xi_2| > 1$; and

$$\delta_v = \begin{pmatrix} \xi_1^{-1}(\det t)^{-1} & & & & \\ & t_2\xi_n^{-1} & & & \\ & & \ddots & & \\ & & & t_{n-1}\xi_3^{-1} & \\ & & & & \xi_2^{-1} \end{pmatrix}$$

for $v \in R$ such that $|\zeta_v\xi_1^{-1}\xi_2|_v \leq 1$.

Remark 3.3. In practice, we only consider sums of finitely many terms in T^1_R, as determined by the support of unramified Whittaker new-vectors on $W_v(a(y)\delta_v)$ for $v \in R$ (See Proposition 2.6 and Theorem 3.4). In particular we consider

$$|y\zeta^{-1}(\det t_v)^{-1}\xi_2^{-1}|_v \leq |t_2\xi_n^{-1}|_v \leq \cdots \leq |t_{n-1}\xi_3^{-1}|_v \leq |\zeta_v\xi_1^{-1}|_v \tag{19}$$

for $\xi_v = \mathrm{diag}(\xi_1, \ldots, \xi_n)$ and $|\zeta_v\xi_1^{-1}\xi_2|_v \geq 1$. This is an artefact of the more general notion of a *Kloosterman integral* as studied by Stevens [34, Def. 2.6], wherefrom (15) and (16) are derived in the proof of Theorem 3.4.

3.3. **The general Voronoï summation formula.** We now state the most general Voronoï formula for $\mathrm{GL}_n(\mathbb{A}_F)$, extending the results of Ichino–Templier. In particular, what follows in Theorem 3.4 subsumes their main results, [19, Theorems 1, 3, & 4], as well as generalising them to the case of joint level–modulus ramification. We further refine this formula by later explicating the generalised Bessel transforms; see §4.

Theorem 3.4. *Let $n \geq 2$ and let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let S denote a finite set of places of F which at least contains S_∞ and each place v at which π_v ramifies; that is $a(\pi_v) > 0$. Without loss of generality (with respect to the present hypotheses) let $\psi = \otimes_v \psi_v$ denote a non-trivial, unramified additive character of $\mathbb{A}_F/F$. For each $v \in S$ pick some $\phi_v \in \mathcal{C}_c^\infty(F_v^\times)$. Fix a 'modulus' $\zeta = (\zeta_v) \in \mathbb{A}_F$ and a 'shift' $\xi = (\xi_v) \in T_n(\mathbb{A}_F)$ such that $\xi_v = 1$ for $v \in S$. Denote by R the set of places $v \notin S$ such that either*

$|\zeta_v|_v > 1$ or $\xi_v \notin T_n(\mathfrak{o}_v)$. Finally, let $W = \otimes_v W_v \in \mathcal{W}(\pi,\psi)$ such that W_v is right-$\mathrm{GL}_n(\mathfrak{o}_v)$-invariant for almost all $v \notin S_\infty$. Then

$$\sum_{\gamma\in F^\times} \psi(\gamma\zeta)\, W\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\xi\right)\prod_{v\in S}\phi_v(\gamma) = \\ \sum_{\gamma\in F^\times}\sum_{t\in T_R^1} \mathcal{K}\ell_R(\gamma,t;\zeta,\xi)\,\tilde{W}_S\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\delta_R(t;\zeta,\xi)\right)\prod_{v\in S}\mathcal{B}_{\pi_v,\Phi_v^{\zeta_v}}(\gamma) \tag{20}$$

where the hyper-Kloosterman sum $\mathcal{K}\ell_R(\gamma,t;\zeta,\xi)$ is defined in (17); $\delta_R(t,\zeta,\xi)$ in (18); we define $\tilde{W}_S(g) := \prod_{v\notin S}\tilde{W}_v(g_v)$ for $g=(g_v)\in\mathrm{GL}_n(\mathbb{A}_F)$ where the dual vector is given by $\tilde{W}_v(g) = W_v(w_n g^\iota)$, as in §2.8; and lastly, for $v\in S$, one defines the function Φ_v on $y\in F_v^\times$ by

$$\Phi_v(y) := \phi_v(y) W_v\left(\begin{pmatrix} y & \\ & 1_{n-1}\end{pmatrix}\right) \tag{21}$$

and its twists by ζ_v via

$$\Phi_v^{\zeta_v}(y) := \psi_v(y\zeta_v)\Phi_v(y). \tag{22}$$

Remark 3.5 (On the hypotheses). The details with which we formulate Theorem 3.4 impose no real restrictions on the generality of the result. The additional generality in comparison to [19] may be pinpointed as follows:

- For those $v\notin S_\infty$ with $|\zeta_v|_v > 1$, we allow $v\in S$. In other words, the denominator and level may jointly ramify.
- We make more general allowances for $W\in\mathcal{W}(\pi,\psi)$: we allow old forms.
- We allow a shift by a diagonal element ξ, corresponding to choosing more general Fourier coefficients $A_f(m,c_2,\dots,c_{n-1})$ with respect to the c_i.

Proof. Our proof of Theorem 3.4 closely follows [19, §2], to which we shall refer for the sake of concision. Here we outline the core of this argument and describe in detail the modifications we make; in particular to [19, §2.6 & §2.7].

One starts with the following fundamental identity:

$$\sum_{\gamma\in F^\times} W_\varphi\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\right) = \sum_{\gamma\in F^\times}\int_{\mathbb{A}^{n-2}} \widetilde{W}_\varphi\left(\begin{pmatrix}\gamma & & \\ x & 1_{n-2} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & \\ & w_{n-1}\end{pmatrix}\right)dx \tag{23}$$

for any $\varphi\in\mathcal{V}_\pi$, letting $\mathcal{V}_\pi$ denote the space of automorphic forms carrying the representation π. See §2.5 for discussion on Whittaker functions. The identity (23) follows from [19, Prop. 1.1 & Lem. 2.1]. It also features crucially in the construction of the global functional equation for $\mathrm{GL}_n\times\mathrm{GL}_1$. (Although, as noted by the authors, a proof does not appear in the literature until that in [19, §4].)

The subsequent goal is to evaluate (23) for an appropriate choice of $\varphi\in\mathcal{V}_\pi$. Typically, one synthesises the left-hand side as desired and picks up the pieces on the right.

To reconstruct the left-hand side of (20), we first pick a vector $\varphi' \in \mathcal{V}_\pi$ and, without loss of generality, suppose $W_{\varphi'} = \otimes_v W'_v$. Firstly, note that the right translate $\rho(n(\zeta)\xi)\varphi'$ satisfies

$$W_{\rho(n(\zeta)\xi)\varphi'}(a(\gamma)) = W_{\varphi'}(a(\gamma)n(\zeta)\xi) = \psi(\gamma\zeta)W_{\varphi'}(a(\gamma)\xi). \tag{24}$$

(See (4) for definitions of the matrices $a(y)$ and $n(x)$.) Next, recall that the element $W = \otimes_v W_v \in \mathcal{W}(\pi,\psi)$ has been selected in the hypotheses of Theorem 3.4. We impose our choice upon each W'_v as follows:

- If $v \in S$, consider the function $\Phi_v(y) = \phi_v(y)W_v(a(y))$. By Proposition 2.4, there exists $W'_v \in \mathcal{W}(\pi_v,\psi_v)$ such that $W'_v(a(y)) = \Phi_v(y)$ for all $y \in F_v^\times$.
- If $v \notin S$ then directly choose $W'_v = W_v$.

We thus choose the test vector $\varphi := \rho(n(\zeta)\xi)\varphi'$. By construction, the left-hand side of (23) is equal to the left-hand side of (20).

It remains to compute the right-hand side of (23) after applying $\varphi = \rho(n(\zeta)\xi)\varphi'$. As is common practice in any trace formula, one observes that the geometric integrals factorise into local components: for $y \in F_v^\times$ define

$$\mathfrak{H}_v(y;\zeta_v,\xi_v) = \int_{F_v^{n-2}} \tilde{W}_v\left(\begin{pmatrix} y & & \\ x & 1_{n-2} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & \\ & w_{n-1}\end{pmatrix}{}^t n(-\zeta_v)\xi_v^{-1}\right)dx.$$

Then the right-hand side of (23) is equal to $\sum_{\gamma\in F^\times}\prod_v \mathfrak{H}_v(\gamma;\zeta_v,\xi_v)$. We divide the argument into two genres dependent on whether or not they are contained in S.

Suppose $v \notin S$. We first show that $\mathfrak{H}_v(y;\zeta_v,\xi_v)$ is equal to a certain hyper-Kloosterman integral, and then refine this integral in terms of Kloosterman sums; see also [19, Theorem 3]. Let us always denote the factors of $\xi = (\xi_v)$ by

$$\xi_v = \mathrm{diag}(\xi_1,\xi_2,\ldots,\xi_n).$$

Note that if $|\zeta_v\xi_1^{-1}\xi_2|_v \le 1$ then $\mathfrak{H}_v(y;\zeta_v,\xi_v) = \mathfrak{H}_v(y;\xi_1\xi_2^{-1},\xi_v)$, since

$$\xi_v{}^t n(-\zeta_v)\xi_v^{-1} = {}^t n(-\zeta_v\xi_1^{-1}\xi_2).$$

In such unramified cases the integral may be computed directly; see [19, §2.5]. However, in the spirit of austerity, we proceed by executing our computations with the assumption $|\zeta_v\xi_1^{-1}\xi_2|_v \ge 1$. For example, for almost all places v we have $\xi_v \in T(\mathfrak{o}_v)$ and $\zeta_v \in \mathfrak{o}_v$, in which case $\mathfrak{H}_v(y;\zeta_v,\xi_v) = \mathfrak{H}_v(y;1,1)$.

For $x \in F_v^{n-2}$, rewrite

$$\begin{pmatrix}1 & & \\ x & 1_{n-2} & \\ & & 1\end{pmatrix} = \sigma\begin{pmatrix}1_{n-2} & x & \\ & 1 & \\ & & 1\end{pmatrix}\sigma^{-1} \quad\text{where}\quad \sigma := \begin{pmatrix} & 1 & \\ 1_{n-2} & & \\ & & 1\end{pmatrix}.$$

Changing variables from x to $y^{-1}x$ we obtain

$$\mathfrak{H}_v(y;\zeta_v,\xi_v) = \\ |y|_v^{n-2}\int_{F_v^{n-2}} \tilde{W}_v\left(\sigma\begin{pmatrix}1_{n-2} & x & \\ & 1 & \\ & & 1\end{pmatrix}\sigma^{-1}a(y)\begin{pmatrix}1 & \\ & w_{n-1}\end{pmatrix}\xi_v^{-1}\,{}^t n(-\zeta_v\xi_1^{-1}\xi_2)\right)dx.$$

Consider the commutation relations

$$\sigma^{-1}a(y)\sigma\sigma^{-1}\begin{pmatrix}1 & \\ & w_{n-1}\end{pmatrix}\xi^{-1}\begin{pmatrix}1 & \\ & w_{n-1}\end{pmatrix}\sigma = \begin{pmatrix}\xi_n^{-1} & & & & \\ & \ddots & & & \\ & & \xi_3^{-1} & & \\ & & & y\xi_1^{-1} & \\ & & & & \xi_2^{-1}\end{pmatrix}$$

and, after applying the Bruhat decomposition to ${}^t n(-\zeta_v\xi_1^{-1}\xi_2)$,

$$\sigma^{-1}\begin{pmatrix}1 & \\ & w_{n-1}\end{pmatrix}n(-\zeta_v\xi_1^{-1}\xi_2)\begin{pmatrix}1 & \\ & w_{n-1}\end{pmatrix}\sigma = \begin{pmatrix}1_{n-2} & & \\ & 1 & -\zeta_v^{-1}\xi_1\xi_2^{-1} \\ & & 1\end{pmatrix}$$
$$\times\begin{pmatrix}1_{n-2} & & \\ & -\zeta_v^{-1}\xi_1\xi_2^{-1} & \\ & & -\zeta_v\xi_1^{-1}\xi_2\end{pmatrix}\begin{pmatrix}1_{n-2} & & \\ & & -1 \\ & 1 & -\zeta_v^{-1}\xi_1\xi_2^{-1}\end{pmatrix},$$

where the final factor above and $\sigma^{-1}\begin{pmatrix}1 & \\ & w_{n-1}\end{pmatrix}$ are both elements of $\mathrm{GL}_n(\mathfrak{o}_v)$. By the right-$\mathrm{GL}_n(\mathfrak{o}_v)$-invariance of W_v we obtain

$$\mathfrak{H}_v(y;\zeta_v,\xi_v) = |y|_v^{n-2}\int_{F_v^{n-2}}\tilde{W}_v\left(\sigma A(x)\right)dx. \tag{25}$$

where

$$A(x) := \begin{pmatrix}1_{n-2} & x & -y\zeta_v^{-1}x \\ & 1 & -y\zeta_v^{-1} \\ & & 1\end{pmatrix}\begin{pmatrix}\xi_n^{-1} & & & & \\ & \ddots & & & \\ & & \xi_3^{-1} & & \\ & & & -y\zeta_v^{-1}\xi_2^{-1} & \\ & & & & -\zeta_v\xi_1^{-1}\end{pmatrix}.$$

Then, if $n=2$ we deduce that

$$\mathfrak{H}_v(y;\zeta_v,\xi_v) = \psi_v(y\zeta_v^{-1})\tilde{W}_v\left(\begin{pmatrix}-y\zeta_v^{-1}\xi_2^{-1} & \\ & -\zeta_v\xi_1^{-1}\end{pmatrix}\right), \tag{26}$$

noting that $\tilde{W}_v$ is invariant by the character $\bar{\psi}_v$. Now suppose that $n \geq 3$. We have the identity

$$\sigma \begin{pmatrix} 1_{n-2} & x & -y\zeta_v^{-1}x \\ & 1 & -y\zeta_v^{-1} \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & -y\zeta_v^{-1} \\ & 1_{n-2} & -y\zeta_v^{-1}x \\ & & 1 \end{pmatrix} \sigma \begin{pmatrix} 1_{n-2} & x & \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{27}$$

Write $x = {}^t(x_1, \ldots, x_{n-2})$ for $x \in F_v^{n-2}$. Consider the integral in (25) and apply (27), commuting the last matrix of $A(x)$ with the last in (27). Then making the substitution from $x_i \mapsto -\xi_{n+1-i}^{-1} x_i y^{-1} \zeta_x \xi_2$ for each $i = 1, \ldots, n-2$ in the integral over $x \in F_v^{n-2}$ and defining

$$\tau := \begin{pmatrix} & & 1 \\ 1_{n-2} & & \\ & & 1 \end{pmatrix} \begin{pmatrix} \xi_n^{-1} & & & & \\ & \ddots & & & \\ & & \xi_3^{-1} & & \\ & & & -y\zeta_v^{-1}\xi_2^{-1} & \\ & & & & -\zeta_v \xi_1^{-1} \end{pmatrix} \tag{28}$$

we derive the formula

$$\begin{aligned} \mathfrak{H}_v(y; \zeta_v, \xi_v) &= |\xi_2 \zeta_v|_v^{n-2} |\xi_3 \cdots \xi_n|_v^{-1} \\ &\quad \times \int_{F_v^{n-2}} \psi_v(-\xi_2 \xi_3^{-1} x_{n-2}) \tilde{W}_v \left(\tau \begin{pmatrix} 1_{n-2} & x & \\ & 1 & \\ & & 1 \end{pmatrix} \right) dx. \end{aligned} \tag{29}$$

In this form, $\mathfrak{H}_v(y; \zeta_v, \xi_v)$ is known as a 'hyper-Kloosterman integral' [34, Def. 2.6]. It remains to express it as a sum of $(n-1)$-dimensional hyper-Kloosterman sums. We remark that the following calculation and subsequent result is not contained in [19]. We now refer to and amend their lemmata directly.

Starting with [19, Def. 6.2], define $\mathcal{K}\ell^{\mathrm{IT}}(\psi_t, \psi', \tau)$ as it is there (we have appended an additional superscript). For the following new variables: τ, as in (28); $\psi_t(u) := \psi_v(tut^{-1})$ for $u \in U_n(F_v)$, $t \in T_n(F_v)$; and $\psi'(a) := \psi_v(-\xi_2 \xi_3^{-1} a)$ for $a \in F_v$. (See [34, Def. 2.10] for the original definition.) Then, [34, Theorem 2.12] implies

$$\mathfrak{H}_v(y; \zeta_v, \xi_v) = |\xi_2 \zeta_v|_v^{n-2} |\xi_3 \cdots \xi_n|_v^{-1} \sum_{t \in T_n(F_v)/T_n(\mathfrak{o}_v)} \tilde{W}_v(t) \mathcal{K}\ell^{\mathrm{IT}}(\psi_t, \psi', t^{-1}\tau).$$

By [34, Cor. 3.11], the Kloosterman sum $\mathcal{K}\ell^{\mathrm{IT}}(\psi_t, \psi', t^{-1}\tau)$ factorises into an $(n-1)$-dimensional and a 1-dimensional term by the decomposition of

$$t^{-1}\tau = \begin{pmatrix} & & & -y\zeta_v^{-1}\xi_2^{-1}t_1^{-1} & \\ \xi_n^{-1}t_2^{-1} & & & & \\ & \ddots & & & \\ & & \xi_3^{-1}t_{n-1}^{-1} & & \\ & & & & -\zeta_v \xi_1^{-1} t_n^{-1} \end{pmatrix},$$

for $t = \operatorname{diag}(t_1, \ldots, t_n) \in T_n(F_v)/T_n(\mathfrak{o}_v)$, into $\mathrm{GL}_{n-1}(F_v) \times \mathrm{GL}_1(F_v)$-parabolic factors. The 1-dimensional factor (in the bottom right corner) vanishes unless $|t_n|_v = |\zeta_v \xi_1^{-1}|_v$, in which case it equals the constant $\psi_v(-\xi_2\xi_3^{-1})$.

We pick this moment to reorder summation by exchanging the variables t_i with $t_i\xi_{n-i+2}$, for each $2 \le i \le n-1$, and t_1 with $\xi_2 t_1$. For the $(n-1)$-dimensional factor to be non-zero, by [34, Th. 3.12], we require the determinant to be a unit, $|t_1 \cdots t_{n-1}|_v = |y\zeta_v^{-1}|_v$, and each exposed sub-determinant to be integral. Checking the definition of an exposed sub-determinant [34, Def. 3.3]; one finds that this condition is equivalent to that $|t_i|_v \ge 1$ for $2 \le i \le n-1$. Collecting these observations we obtain the refined expression

$$\begin{aligned}\mathfrak{H}_v(y;\zeta_v,\xi_v) &= |\xi_2\zeta_v|_v^{n-2}|\xi_3\cdots\xi_n|_v^{-1}\psi_v(-\xi_2\xi_3^{-1})\\ &\times \sum_{t\in T_v^1} \tilde{W}_v\left(\begin{pmatrix} y\zeta_v^{-1}(\det t)^{-1} & & \\ & t & \\ & & \zeta_v\end{pmatrix}\begin{pmatrix} & 1 & \\ & & w_{n-2} \\ 1 & & \end{pmatrix}\xi_v^{-1}\right)\mathcal{K}\ell^{\mathrm{IT}}(\psi_{t'},\psi',\tau').\end{aligned}$$

where T_v^1 was introduced in §3.2 and we define the variables

$$t' = \begin{pmatrix} y\zeta_v^{-1}(\det t)^{-1} & \\ & t\end{pmatrix}\begin{pmatrix}\xi_2^{-1} & & & \\ & \xi_n^{-1} & & \\ & & \ddots & \\ & & & \xi_3^{-1}\end{pmatrix}; \quad \tau' = \begin{pmatrix} & -\det t \\ t^{-1} & \end{pmatrix}.$$

The final step is to compute the terms $\mathcal{K}\ell^{\mathrm{IT}}(\psi_{t'},\psi',\tau')$. This is executed recursively via [19, Prop. 6.4]. Applying [19, Cor. 6.5 & Lem. 6.6] with the parameters τ' and t' we obtain

$$\mathcal{K}\ell^{\mathrm{IT}}(\psi_{t'},\psi',\tau') = \sum_{x_{n-1}\in\Lambda_{t_{n-1}}} \psi_v(-\xi_2\xi_3^{-1}x_{n-1})\mathcal{K}\ell^{\mathrm{IT}}(\psi_{t''},\psi'',\tau'')$$

where Λ_{t_i} was introduced in §3.2, $\psi''(a) := \psi_v(-\xi_3\xi_4^{-1}a)$ for $a \in F_v$, and we define the new variables

$$t'' = \begin{pmatrix} y\zeta_v^{-1}(\det t)^{-1}\xi_2^{-1} & & & \\ & t_2\xi_n^{-1} & & \\ & & \ddots & \\ & & & t_{n-2}\xi_4^{-1}\end{pmatrix};$$

$$\tau'' = \begin{pmatrix} & & & x_{n-1}^{-1}(\det t) \\ t_2^{-1} & & & \\ & \ddots & & \\ & & t_{n-2}^{-1} & \end{pmatrix}.$$

Note that we have multiplied the top right-hand corner of τ' by $-x_{n-1}^{-1}$ to obtain the top right-hand corner of τ''. Continuing recursively, terminating the evaluation with [19, Lem. 6.6], we deduce that

$$\mathcal{K}\ell^{\mathrm{IT}}(\psi_{t'\xi^{-1}}, \psi', \tau') = \sum_{x_{n-1}\in\Lambda_{t_{n-1}}} \cdots \sum_{x_2\in\Lambda_{t_2}} \psi_v(-\xi_2\xi_3^{-1}x_{n-1}) \prod_{j=2}^{n-2} \psi_v(\xi_{n-j+1}\xi_{n-j+2}^{-1}x_j) \\ \times \psi_v\left((-1)^{n-1}y\zeta_v^{-1}\xi_2^{-1}\xi_n x_{n-1}^{-1}\cdots x_2^{-1}\right),$$

which is equal to $(|\xi_2\zeta_v|_v^{n-2}|\xi_3\cdots\xi_n|_v^{-1}\psi_v(-\xi_2\xi_3^{-1}))^{-1}\mathcal{K}\ell_v(y,t;\zeta,\xi)$, as defined in §3.2, on the nose. (To be compared with [19, Cor. 6.7].) Note that, unlike [19, Prop. 6.4], we subsume the cases $|t_i|_v > 1$ and $|t_i|_v = 1$ into one via our definition of the sets Λ_{t_i} in (14). In particular, the above equality holds for $|t_2|_v = 1$ since then we have

$$|y\zeta_v^{-1}\xi_2^{-1}\xi_n x_{n-1}^{-1}\cdots x_2^{-1}|_v = |y\zeta_v^{-1}\xi_2^{-1}\xi_n(\det t)^{-1}|_v \leq 1$$

from the support of $\tilde{W}_v$ (see (19)); indeed, ψ_v is trivial on $\mathfrak{o}_v$. As a last remark we simply note that for all places $v \notin R \cup S$ we have

$$\mathfrak{H}_v(y;\zeta_v,\xi_v) = \mathfrak{H}_v(y;1,1) = \tilde{W}_v(a(y)).$$

Now consider the remaining places $v \in S$. Theorem 3.4 shall follow duly from the observation that $\mathfrak{H}_v(y;\zeta_v,\xi_v) = \mathcal{B}_{\pi_v,\Phi_v^{\zeta_v}}(y)$. Or argument follows [19, §2.7], except that we identify a different element of the Whittaker model upon shifting by the matrix $n(\zeta_v)$. By assumption, the function $\Phi_v^{\zeta_v}$ satisfies

$$\Phi_v^{\zeta_v}(y) = \psi_v(y\zeta_v)\Phi_v(y) = W_v'(n(\zeta_v y)a(y)) = W_v'(a(y)n(\zeta_v)).$$

We denote this right-translate by $W_v^{\zeta_v} := \rho(n(\zeta_v))W_v'$, thus defining a new element $W_v^{\zeta_v} \in \mathcal{W}(\pi_v,\psi_v)$ satisfying $\Phi_v^{\zeta_v}(y) = W_v^{\zeta_v}(a(y))$ for all $y \in F_v^\times$. Now [19, Lemma 2.3] applies to $\Phi_v^{\zeta_v}$, thus determining its unique transform that satisfies

$$\mathcal{B}_{\pi_v,\Phi_v^{\zeta_v}}(y) = \mathfrak{H}_v(y;\zeta_v,\xi_v)$$

for all $y \in F_v^\times$. We remark that the hypotheses of [19, Lemma 2.3] demand that $\Phi_v^{\zeta_v} \in \mathcal{C}_c^\infty(F_v^\times)$; in the case $v \in S$ we have $\Phi_v^{\zeta_v}(y) = \psi_v(y\zeta_v)\phi_v(y)W_v(a(y))$ is again smooth (resp. locally constant) of compact support. However, the argument there applies to all such functions $y \mapsto W_v(a(y))$ for any $W_v \in \mathcal{W}(\pi_v,\psi_v)$.

□

4. Explicit Bessel Transforms

Here we give an explicit description of the generalised Bessel transforms $\mathcal{B}_{\pi_v,\Phi}$, as introduced in §3.1. We consider their analytic behaviour at all places and, for non-archimedean places v, we detail the 'joint ramification' case with $\Phi = \Phi_v^{\zeta_v}$ for $|\zeta_v|_v > 1$. In this section let us retain the notation of §3.

4.1. **The Mellin inversion formula for local fields.** Let v be any place of F and consider a Bruhat–Schwartz function $\Phi\colon F_v^\times \to \mathbb{C}$. The fundamental principle of harmonic analysis on locally compact abelian groups [30] is to study the frequencies of Φ contained in the unitary dual group $\hat{F}^\times$; this is the set of continuous unitary characters on $F_v^\times$.

Definition 4.1. Let $\mu \in \hat{F}_v^\times$. The *Mellin transform* of Φ is given by

$$\mathcal{M}(\Phi,\mu) = \int_{F_v^\times} \Phi(y)\mu(y)d^\times y$$

for a Haar measure $d^\times y$ on $F_v^\times$. Similarly, for a Schwartz function $\tilde{\Phi}\colon \hat{F}_v^\times \to \mathbb{C}$, the *inverse Mellin transform* of $\tilde{\Phi}$ evaluated at $y \in F_v^\times$ is given by

$$\mathcal{M}^{-1}(\Phi,\mu) = \int_{\hat{F}_v^\times} \tilde{\Phi}(\mu)\mu(y)^{-1}d\mu$$

for a Haar measure $d\mu$ on $\hat{F}_v^\times$.

Proposition 4.1 (Mellin inversion)**.** *There exist 'self-dual' normalisations of the pair of Haar measures $(d^\times y, d\mu)$ such that*

$$\mathcal{M}^{-1}\circ\mathcal{M} = \mathcal{M}\circ\mathcal{M}^{-1} = \mathrm{Id}\,.$$

Proof. See [30, §3] for instance. □

4.1.1. *Archimedean Mellin inversion.* Suppose that v is real so that $F_v = \mathbb{R}$. Any unitary character on $\mathbb{R}^\times$ is of the form $\mu = \mathrm{sgn}^r|\cdot|_v^{it} \in \hat{\mathbb{R}}^\times$ for $r \in \{0,1\}$ and $t \in \mathbb{R}$. Let us normalise

$$\mathcal{M}(\Phi,\mu) = \int_{\mathbb{R}^\times} \Phi(y)\,\mathrm{sgn}(y)^r|y|_v^{it}\,d^\times y$$

and

$$\mathcal{M}^{-1}(\tilde{\Phi},y) = \frac{1}{4\pi i}\sum_{r\in\{0,1\}}\int_{\mathbb{R}} \tilde{\Phi}(\mathrm{sgn}^r|\cdot|_v^{it})\,\mathrm{sgn}(y)^r|y|_v^{-it}\,dt$$

where $d^\times y = \mathrm{sgn}(y)y^{-1}dy$ and dy, dt both denote the Lebesgue measure on $\mathbb{R}$. With this choice of Haar measures, Proposition 4.1 holds.

Remark 4.2 (The Mellin transform for $\mathbb{R}_{>0}$)*.* For $s \in \mathbb{C}$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_{>0})$ let

$$\mathfrak{m}(\phi,s) := \int_0^\infty \phi(y)y^{s-1}\,dy.$$

Defining $\Phi(y) = \phi(y)y^\sigma$ for $y > 0$, with $\sigma = \mathrm{Re}(s)$ sufficiently large, and $\Phi(y) = 0$ for $y \leq 0$ we have $\mathcal{M}(\Phi, \mathrm{sgn}^r|\cdot|_v^{\mathrm{Im}(s)}) = \mathfrak{m}(\phi,s)$, constant on $r \in \{0,1\}$. Proposition (4.1) implies the usual Mellin inversion formula

$$\phi(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} \mathfrak{m}(\phi,s)\,y^{-s}\,ds. \tag{30}$$

Suppose that v is complex so that $F_v = \mathbb{C}$. Expressing a complex number $z \in \mathbb{C}^\times$ in polar coordinates, $z = |z|_v^{1/2}e^{i\arg(z)}$, the unitary dual of $\mathbb{C}^\times$ may be identified as $\hat{\mathbb{C}}^\times = \hat{\mathbb{R}}_{>0}\times\widehat{\mathbb{R}/\mathbb{Z}} \cong \mathbb{R}\times\mathbb{Z}$. We omit the details of the complex case in this article.

4.1.2. *Non-archimedean Mellin inversion.* Let v be non-archimedean and recall the notation of §2.2. Recall that, as described in §2.4, any unitary character $\mu \in \hat{F}_v^\times$ is of the form $\mu = \chi|\cdot|_v^{it}$ for some $\chi \in \mathfrak{X}_v$ and $t \in \mathbb{R}$ satisfying $-\pi/\log q_v < t \leq \pi/\log q_v$. We normalise measures by choosing $d^\times y$ to be the Haar measure on $F_v^\times$ such that $\mathrm{Vol}(\mathfrak{o}_v^\times, d^\times y) = 1$ and suppose that dt is the usual Lebesgue measure on $\mathbb{R}$. Then

$$\mathcal{M}(\Phi, \mu) = \sum_{k \in \mathbb{Z}} q_v^{ikt} \int_{\mathfrak{o}_v^\times} \Phi(y \varpi_v^{-k}) \chi(y)\, d^\times y$$

and

$$\mathcal{M}^{-1}(\tilde{\Phi}, y) = \frac{\log q_v}{2\pi} \sum_{\chi \in \mathfrak{X}_v} \chi(y)^{-1} \int_{-\pi/\log q_v}^{\pi/\log q_v} \tilde{\Phi}(\chi|\cdot|_v^{it}) |y|_v^{-it}\, dt.$$

One may refer to Taibleson's book [35, §II.4] for background material.

4.2. **Explicating the Bessel transform via Mellin inversion.** We now give a general and explicit description of the Bessel transforms introduced in §3.1. This expression is obtained by applying the Mellin inversion formula to the identity (13) between Φ and its dual $\mathcal{B}_{\pi_v,\Phi}$. Let $s = \sigma + it \in \mathbb{C}$ where, as in §3.1, we assume $1-\sigma$ is sufficiently large. The left-hand side of (13) is precisely the Mellin transform of the function $|\cdot|_v^{\sigma - \frac{n-1}{2}} \cdot \mathcal{B}_{\pi_v,\Phi}$ evaluated at $\chi^{-1}|\cdot|_v^{it} \in \hat{F}_v^\times$. Explicitly,

$$\mathcal{M}(|\cdot|_v^{\sigma-\frac{n-1}{2}} \cdot \mathcal{B}_{\pi_v,\Phi}, \chi^{-1}|\cdot|_v^{it}) = \chi(-1)^{n-1}\gamma(1-s, \chi\pi_v, \psi_v) \int_{F_v^\times} \Phi(y)\chi(y)|y|_v^{1-s-\frac{n-1}{2}}\, d^\times y. \quad (31)$$

Bifurcating according to the place v, we proceed by using Proposition 4.1 to invert this expression: we substitute (31) into the identity

$$\mathcal{B}_{\pi_v,\Phi}(y) = |y|_v^{\frac{n-1}{2}-\sigma} \mathcal{M}^{-1}(\mu \mapsto \mathcal{M}(|\cdot|_v^{\sigma-\frac{n-1}{2}} \cdot \mathcal{B}_{\pi_v,\Phi}, \mu), y). \quad (32)$$

4.2.1. *Archimedean Bessel transforms and estimates.* Let v be a real-archimedean place of F so that $F_v = \mathbb{R}$. Solving (32) with (31), for all $y \in \mathbb{R}^\times$ we have

$$\mathcal{B}_{\pi_v,\Phi}(y) = \frac{1}{4\pi i} \sum_{r \in \{0,1\}} (-1)^{r(n-1)} \operatorname{sgn}(y)^r \int_{\mathrm{Re}(s)=\sigma} \gamma(1-s, \operatorname{sgn}^r \pi_v, \psi_v)|y|_v^{\frac{n-1}{2}-s} \times \int_{\mathbb{R}^\times} \Phi(x) \operatorname{sgn}(x)^r |x|_v^{1-s-\frac{n-1}{2}}\, d^\times x\, ds. \quad (33)$$

To quote Kowalski–Ricotta [25, §3] on the analytic behaviour of $\mathcal{B}_{\pi_v,\Phi}$, we assume that Φ is compactly supported in $\mathbb{R}_{>0}$. The Bessel transform may then be expressed in terms of the classical Mellin transform;

$$\mathcal{B}_{\pi_v,\Phi}(y) = \frac{1}{2} \sum_{r\in\{0,1\}} (-1)^{r(n-1)} \operatorname{sgn}(y)^r \times \int_{\mathrm{Re}(s)=\sigma} \mathfrak{m}\left(\Phi, 1-s-\tfrac{n-1}{2}\right) \gamma(1-s, \operatorname{sgn}^r \pi_v, \psi_v) |y|_v^{\frac{n-1}{2}-s} ds.$$

Decomposing the operator $\Phi \mapsto \mathcal{B}_{\pi_v,\Phi}$ into its r-summands, we find that they are unitary with respect to the L^2-norm on $\mathbb{R}_{>0}$ computed with respect to the Lebesgue measure [25, Prop. 3.3]; this depends on the parity of n. In [25, Cor. 3.6], estimates are given for the sum of $\mathcal{B}_{\pi_v,\Phi}$ juxtaposed with the Fourier coefficients of an automorphic form in an interval. Moreover, we record the following asymptotic estimates of [25, Prop. 3.5].

Proposition 4.3 (Kowalski–Ricotta [25])**.** *Suppose that the support of Φ is a compact subset of $\mathbb{R}_{>0}$. Then if $0 < y \leq 1$ we have*

$$\mathcal{B}_{\pi_v,\Phi}(y) \ll y^{-\left(\frac{1}{2}+\frac{1}{n^2+1}\right)}.$$

And for all $y, A \in \mathbb{R}_{>0}$ we have

$$\mathcal{B}_{\pi_v,\Phi}(y) \ll_{A,\pi_v,\Phi} y^{-A}.$$

Remark 4.4. In the notation of [25, §3], our transform $\mathcal{B}_{\pi_v,\Phi}$ coincides with their function "$\mathcal{B}_{\alpha_\infty(f)}[w]$" by assigning $w = \Phi$.

4.2.2. *Non-archimedean Bessel transforms.* Let v be a non-archimedean place of F. The simultaneous solution of (31) and (32) implies that for all $y \in F_v^\times$ we have

$$\mathcal{B}_{\pi_v,\Phi}(y) = \frac{\log q_v}{2\pi} \sum_{\chi\in\mathfrak{X}_v} \chi(-1)^{n-1}\chi(y) \int_{\sigma-i\pi/\log q_v}^{\sigma+i\pi/\log q_v} \gamma(1-s,\chi\pi_v,\psi_v)\,|y|_v^{\frac{n-1}{2}-s} \times \int_{F_v^\times} \Phi(x)\chi(x)|x|_v^{1-s-\frac{n-1}{2}}\, d^\times x\, ds. \tag{34}$$

This is the most general description of the Bessel transform.

4.3. **Non-archimedean Bessel transforms in detail.** We now consider the Bessel transforms $\mathcal{B}_{\pi_v,\Phi_v^{\zeta_v}}$, at a non-archimedean place v. In the general case, we give a bound for the support. However, in practice, we shall not always require the full generality of Theorem 3.4. Making an assumption on the local factor π_v, we give a refined formula for the Bessel transform. We then show how to choose the test functions ϕ_v to determine a Voronoï formulae on arithmetic progressions.

A natural factor occurring at places for which $|\zeta_v|_v > 1$ is the *Gauß* sum

$$\mathfrak{G}_v(y,\chi) := \int_{\mathfrak{o}_v^\times} \psi_v(yu)\,\chi(u)\,d^\times u \tag{35}$$

for $y \in F_v^\times$ and $\chi \in \mathfrak{X}_v$. Recall that $a(\chi)$ denotes the conductor of χ (see §2.4).

Lemma 4.5. *Let $y \in F_v^\times$ and $\chi \in \mathfrak{X}_v$. If $a(\chi) = 0$, or equivalently $\chi = 1$, then*

$$\mathfrak{G}_v(y,1) = \begin{cases} 1 & \text{if } |y|_v \le 1 \\ \frac{1}{1-q_v} & \text{if } |y|_v = q_v \\ 0 & \text{if } |y|_v > q_v. \end{cases}$$

If $a(\chi) > 0$ then $\mathfrak{G}_v(y,\chi) = 0$ unless $|y|_v = q_v^{a(\chi)}$, in which case

$$\mathfrak{G}(y,\chi) = \zeta(1)|y|^{-1/2}\chi(y)^{-1}\varepsilon(1/2,\chi^{-1}).$$

Proof. This result is well-known. For instance, a proof is given by the author in [13, Lemma 2.3] using [30, (7.6) & Lem. 7-4]. □

4.3.1. *Support of the Bessel transforms.* Let $v \in S$ and let ψ_v, W_v, Φ_v, and $\Phi_v^{\zeta_v}$ be as in Theorem 3.4. There are no additional assumptions in the following, in particular not on π_v.

Proposition 4.6. *Let $y \in F_v^\times$. If $|\zeta_v|_v \le 1$ then $\mathcal{B}_{\pi_v,\Phi_v^{\zeta_v}}(y) = 0$ whenever $|y|_v > q_v^{n+a(\pi_v)}$. If $|\zeta_v|_v > 1$ then $\mathcal{B}_{\pi_v,\Phi_v^{\zeta_v}}(y) = 0$ whenever $|y|_v > |\zeta_v|_v^{n-1}q_v^{n+\max\{a(\pi_v),-v(\zeta_v)\}}$.*

Proof. By Lemma 2.2 on the support of $y \mapsto W_v(a(y))$, the inner integral becomes

$$\int_{F_v^\times} \Phi(x)\chi(x)|x|_v^{1-s-\frac{n-1}{2}}\, d^\times x$$
$$= \sum_{r\ge 0} W_v(a(\varpi_v^r)) \int_{\mathfrak{o}_v^\times} \psi_v(\zeta_v\varpi_v^r u)\chi(u)q^{-r(1-s-\frac{n-1}{2})}\phi_v(\varpi_v^r u)d^\times u.$$

The r-sum converges by the compact support of ϕ_v. Also note that in the special case $\phi_v|_{\mathfrak{o}_v^\times} = 1$ one may directly evaluate the Gauß sum $\mathfrak{G}_v(\zeta_v\varpi_v^r,\chi)$ using Lemma 4.5. To solve the s-integral we use the formulae (11) and (12) to evaluate the terms $\gamma(1-s,\chi\pi_v,\psi_v)$. We also write down a generic geometric series for the quotient of L-factors. This is possible by our choice of $\mathrm{Re}(s)$ being larger than the real part of any poles of $L(s,\chi^{-1}\tilde{\pi}_v)$. This series has no lower powers of q_v^{-s} than $(q_v^{-s})^{-n}$, thus we incur this factor in the support bound. The estimate then follows from bounding the conductor of $a(\chi\pi_v)$ using [12, Theorem 2.7]. □

4.3.2. *An explicit formula for minimal supercuspidal representations.* Let us now enforce the following.

Assumption 4.1. *Let π_v satisfy $L(s,\chi\pi_v) = 1$, identically, for all $\chi \in \mathfrak{X}_v$ with $a(\chi) \le \max\{-v(\zeta_v), 0\}$.*

This assumption is satisfied, for example, by all supercuspidal representations of $\mathrm{GL}_n(F_v)$.

Definition 4.2. We say π_v is *twist minimal* if $a(\pi_v) = \min\{a(\chi\pi_v) : \chi \in \mathfrak{X}_v\}$.

Tautologically, the property of being twist minimal may always be obtained via twisting by some $\chi \in \mathfrak{X}_v$. For example, any supercuspidal representation π_v such that $n \nmid a(\pi_v)$ is twist minimal by [12, Prop. 2.2]. Without loss of generality, we further impose that, for $v \in S$, $W_v(a(y)) = \mathrm{Char}_{\mathrm{supp}(\phi_v)}(y)$ so that $\Phi_v = \phi_v$. This may be chosen by Proposition 2.4.

Proposition 4.7. *Assume that π_v is twist minimal (Definition 4.2) and satisfies Assumption 4.1. Let $\phi_v = \mathrm{Char}_{\mathfrak{o}^\times}$. Then $\mathcal{B}_{\pi_v,\phi_v^{\zeta_v}}(y) = 0$ if $|y|_v \neq q_v^{\max\{a(\pi_v),-nv(\zeta_v)\}}$. Otherwise, suppose $|y|_v = q_v^{\max\{a(\pi_v),-nv(\zeta_v)\}}$. If $|\zeta_v|_v \leq 1$ then*

$$\mathcal{B}_{\pi_v,\phi_v^{\zeta_v}}(y) = \varepsilon(1/2,\pi_v,\psi_v) q_v^{a(\pi_v)\frac{n-2}{2}}.$$

If $|\zeta_v|_v = q_v$ then

$$\mathcal{B}_{\pi_v,\phi_v^{\zeta_v}}(y) = \varepsilon(1/2,\pi_v,\psi_v)\frac{q_v^{a(\pi_v)\frac{n-2}{2}}}{1-q_v} + \frac{q_v^{\max\{a(\pi_v),n\}\frac{n-2}{2}+\frac{1}{2}}}{1-q_v^{-1}} \sum_{\substack{\chi\in\mathfrak{X}_v\\ a(\chi)=1}} \chi(-1)^{n-1}\chi(\zeta_v^{-1}y)\varepsilon(1/2,\chi\pi_v,\psi_v)\varepsilon(1/2,\chi^{-1},\psi_v).$$

If $|\zeta_v|_v > q_v$ then

$$\mathcal{B}_{\pi_v,\phi_v^{\zeta_v}}(y) = \frac{1}{1-q_v^{-1}} q_v^{\max\{a(\pi_v),-nv(\zeta_v)\}\frac{n-2}{2}+\frac{v(\zeta_v)}{2}} \times \sum_{\substack{\chi\in\mathfrak{X}_v\\ a(\chi)=-v(\zeta_v)}} \chi(-1)^{n-1}\chi(\zeta_v^{-1}y)\varepsilon(1/2,\chi\pi_v,\psi_v)\varepsilon(1/2,\chi^{-1},\psi_v).$$

Proof. We compute the expression (34) under Assumption 4.1 so that

$$\mathcal{B}_{\pi_v,\phi_v^{\zeta_v}}(y) = \sum_{\chi\in\mathfrak{X}} \mathfrak{G}_v(\zeta_v,\chi)\varepsilon(1/2,\chi\pi_v,\psi_v)\chi(-1)^{n-1}\chi(y)q_v^{a(\chi\pi_v)\frac{n-2}{2}}\delta(v(y),-a(\chi\pi_v)).$$

We evaluate this expression by the explicit formula for the Gauß sum in Lemma 4.5. In particular, the assumption that π_v is minimal allows us to use the formula

$$a(\chi\pi_v) = \max\{a(\pi_v), na(\chi)\}$$

given in [12, Prop. 2.2]. □

Estimating trivially we obtain the following upper bound.

Corollary 4.8. *For $y \in F_v^\times$ we have*

$$\mathcal{B}_{\pi_v,\phi_v^{\zeta_v}}(y) \ll q_v^{\max\{a(\pi_v),-nv(\zeta_v)\}\frac{n-2}{2}+\frac{3v(\zeta_v)}{2}}.$$

4.3.3. *Summation in arithmetic progressions.* Let us maintain Assumption 4.1 for simplicity and fix the test function

$$\phi_v = \mathrm{Char}_{1+\varpi_v^k \mathfrak{o}_v}$$

for some $k \geq 1$. Consider the following variant of the Gauß sum:

$$\mathfrak{G}_v^k(a, \chi) := \int_{1+\varpi_v^k \mathfrak{o}_v} \psi_v(ay)\chi(y) d^\times y. \tag{36}$$

For now, to determine the support of $\mathfrak{G}_v^k(a, \chi)$ we only consider its size.

Lemma 4.9. *Suppose $|a|_v > 1$. Then $\mathfrak{G}_v^k(a, \chi) = 0$ unless $a(\chi) \leq \max\{k, -v(a)\}$, in which case*

$$|\mathfrak{G}_v^k(a, \chi)|^2 = \mathrm{Vol}(1 + \varpi_v^k \mathfrak{o}_v, d^\times y) \, \mathrm{Vol}(1 + \varpi_v^{\max\{k, -v(a)\}} \mathfrak{o}_v, d^\times y) \ll q_v^{\max\{2k, k-v(a)\}}.$$

Proof. Expanding the integral, orthogonality of additive characters implies that

$$|\mathfrak{G}_v^k(a, \chi)|^2 = \mathrm{Vol}(1 + \varpi_v^k \mathfrak{o}_v, d^\times y) \int_{(1+\varpi_v^k \mathfrak{o}_v) \cap (1+\varpi_v^{-v(a)} \mathfrak{o}_v)} \chi(y) d^\times y.$$

Orthogonality of multiplicative characters now verifies the lemma. □

Proposition 4.10. *Under Assumption 4.1, assume $\phi_v = \mathrm{Char}_{1+\varpi_v^k \mathfrak{o}_v}$. Then*

$$\mathcal{B}_{\pi_v, \phi_v^{\zeta_v}}(y) = \sum_{\chi \in \mathfrak{X}} \mathfrak{G}_v^k(\zeta_v, \chi) \varepsilon(1/2, \chi\pi_v, \psi_v) \chi(-1)^{n-1} \chi(y) q_v^{a(\chi\pi_v)\frac{n-2}{2}} \delta(v(y), -a(\chi\pi_v)).$$

Proof. Unfolding definitions as before, we recover the the expression after noting that $\mathfrak{G}_v^k(\zeta_v, \chi)$ is supported on $a(\chi) \leq \max\{k, -v(\zeta_v)\}$ by Lemma 4.9. □

Corollary 4.11. *For minimal representations, $\mathcal{B}_{\pi_v, \phi_v^{\zeta_v}}(y)$ is supported on the compact set defined by $v(y) = \max\{a(\pi_v), nr\}$ for $0 \leq r \leq \max\{k, -v(\zeta_v)\}$.*

5. A Classical Formulation

In this final section, we translate our results into a more classical parlance; that of Maaß forms on $\mathrm{SL}_n(\mathbb{R})$. We apply our representation theoretic results to such forms by considering the special case $F = \mathbb{Q}$.

5.1. **Specialist notation.**

5.1.1. *Valuations.* Recall that for $F = \mathbb{Q}$ there is a single archimedean place and it is denoted by ∞. Here we have $\mathbb{Q}_\infty = \mathbb{R}$ and the absolute value is the usual one: $|y|_\infty = |y| := \mathrm{sgn}(y)y$. All other places are non-archimedean and indexed by a rational prime p, denoting this property by $p < \infty$. For integers $a, b \geq 1$, we make the convention that $a \mid b^\infty$ if and only if $a \mid b^k$ for some $k \geq 0$. For all $a, b \in \mathbb{Z}$ define $[a, b] := \mathrm{lcm}(|a|, |b|)$ and $(a, b) := \gcd(|a|, |b|)$ as usual.

5.1.2. *The standard additive character.* We use the notation $e(z) := e^{2\pi i z}$ for $z \in \mathbb{C}$. Fix the character $\psi \colon \mathbb{A}_\mathbb{Q}/\mathbb{Q} \to \mathbb{C}$ given by $\psi = \otimes_v \psi_v$ with $\psi_\infty(x) = e(-x)$ for $x \in \mathbb{R}$ and, for $x_p \in \mathbb{Q}_p$, $\psi_p(x_p) = 1$ if and only if $x_p \in \mathbb{Z}_p$. Explicitly, for $p < \infty$, we have $\psi_p(x) = e(\mathrm{fr}_p(x))$ where $\mathrm{fr}_p(x) \in \mathbb{Q}$ is the 'fractional part' of x satisfying $0 \leq \mathrm{fr}_p(x) < 1$ and $\mathrm{fr}_p(x) - x \in \mathbb{Z}_p$.

Lemma 5.1. *For $x \in \mathbb{Q}$ we have $\sum_{p<\infty} \mathrm{fr}_p(x) \equiv x \mod \mathbb{Z}$.*

Proof. Note that for almost all p $\mathrm{fr}_p(x) = 0$ as $x \in \mathbb{Z}_p$ for all p not dividing the denominator of x. We need to show $x - \sum_p \mathrm{fr}_p(x) \in \mathbb{Z}$. Fix primes $q \neq p$. Then $\mathrm{fr}_p(x) \in \mathbb{Z}_q$ and (by definition) $x - \mathrm{fr}_q(x) \in \mathbb{Z}_q$. Hence

$$x - \sum_p \mathrm{fr}_p(x) = x - \mathrm{fr}_q(x) - \sum_{p \neq q} \mathrm{fr}_p(x) \in \mathbb{Z}_q.$$

Since this is true for each prime q we must in fact have $x - \sum_p \mathrm{fr}_p(x) \in \mathbb{Z}$. □

Corollary 5.2. *The character $\psi = \otimes_v \psi_v$ is trivial on $\mathbb{Q}$.*

One may further show that all characters of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ are of the form $x \mapsto \psi(ax)$ for some $a \in \mathbb{Q}$.

5.2. **Level structure.** There is a natural right-action of the group $\mathrm{SL}_n(\mathbb{Z})$ on $(\mathbb{Z}/N\mathbb{Z})^n$, whence we introduce the congruence subgroups

$$\Gamma_1(N) := \mathrm{Stab}_{\mathrm{SL}_n(\mathbb{Z})}((0, \ldots, 0, 1)) \subset \mathrm{SL}_n(\mathbb{Z})$$

for each integer $N \geq 1$, thus defining a filtration of $\mathrm{SL}_n(\mathbb{Z})$-subgroups with respect to successive multiples of N. We also introduce the following p-adic analogues: by the right-action of $\mathrm{GL}_n(\mathbb{Z}_p)$ on $(\mathbb{Z}/N\mathbb{Z})^n$ we define

$$K_1(N)_p := \mathrm{Stab}_{\mathrm{GL}_n(\mathbb{Z}_p)}((0, \ldots, 0, 1)) \subset \mathrm{GL}_n(\mathbb{Z}_p)$$

for each integer $N \geq 1$. Also define $K_1(N) = \{1\} \times \prod_{p<\infty} K_1(N)_p \subset \mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. We thus realise $\Gamma_1(N)$ embedded into $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ by

$$\Gamma_1(N) = \mathrm{GL}_n(\mathbb{Q}) \cap (\mathrm{GL}_n(\mathbb{R})^+ \times K_1(N)). \tag{37}$$

These filtrations are a good choice on which to study the level structure of $\mathrm{SL}_n(\mathbb{R})$-Maaß forms since there is a robust theory of newforms.

Remark 5.3. For $n = 2$, newform theory was proposed by Atkin–Lehner [3] and later developed by Casselman [9] in the context of representation theory. For $n \geq 2$ the theory has been constructed by Gelfand—Každan [15]. Critically, in the general case, Jacquet–Piatetski-Shapiro–Shalika [20] prove that the conductor associated to a newform in the sense of a level structure is equal to that which occurs in the ε-factor of the local functional equation [16, Theorem 3.3].

5.3. **Dirichlet and Hecke characters.** Central characters of automorphic representations are given by Hecke characters of $\mathbb{A}_{\mathbb{Q}}^\times$. Briefly recall here the correspondence between Dirichlet characters $\chi \pmod N$ and finite order Hecke characters ω of conductor at most N. (See [24, §12.1] for a detailed reference.) Explicitly, given χ, we define a character $\omega\colon \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ using the strong approximation theorem $\mathbb{A}_{\mathbb{Q}}^\times = \mathbb{Q}^\times \cdot (\mathbb{R}_{>0} \times \prod_{p<\infty} \mathbb{Z}_p^\times)$ by

$$\omega(y) = \prod_{p\mid N} \chi(y_p) \tag{38}$$

where $y = (y_p') \in \mathbb{R}_{>0} \times \prod_{p<\infty} \mathbb{Z}_p^\times$ and $y_p \in (\mathbb{Z}/p^{v_p(N)}\mathbb{Z})^\times$ is the image of y_p' for $p \mid N$ obtained via the isomorphism $\mathbb{Z}_p^\times/(1+N\mathbb{Z}_p) \cong (\mathbb{Z}_p/N\mathbb{Z}_p)^\times$. As for any continuous Hecke character, we have the factorisation $\omega = \otimes_p \omega_p$. For each $p \nmid N$ and $y \in \mathbb{Q}_p^\times$ the local factors satisfy

$$\omega_p(y) = \chi(p)^{-v_p(y)}. \tag{39}$$

Moreover, for each integer $d \geq 1$ with $(d,N) = 1$ we have $\prod_{p\mid d} \omega_p(d) = \chi(d)^{-1}$.

5.4. **Lifting Maaß forms to adele groups.** The dictionary between classical Maaß forms and automorphic forms on $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ hinges on the following strong approximation theorem:

$$\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}) \cong \mathrm{GL}_n(\mathbb{Q}) \cdot (\mathrm{GL}_n(\mathbb{R}) \times K_1(N)) \tag{40}$$

for each $N \geq 1$ (cf. [18, Prop. 13.3.3]). Explicitly, given $f \in L^2(\Gamma_1(N)\backslash \mathrm{SL}_n(\mathbb{R}))$, we define a function $\varphi_f \in L^2(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})/K_1(N))$ by

$$\varphi_f(\gamma g_\infty k) = f(g_\infty) \tag{41}$$

for $\gamma \in \mathrm{GL}_n(\mathbb{Q})$, $g_\infty \in \mathrm{GL}_n(\mathbb{R})$ and $k \in K_1(N)$. Note that this definition is well defined by (37). Then φ_f generates the automorphic representation π_f of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ with the central character $\omega := \pi_f|_{Z(\mathbb{A})}$. As in Theorem 1.1, without loss of generality we assume that ω corresponds to a Dirichlet character $\chi \pmod N$.

Moreover, we now have a notion of (normalised) ψ_p-Whittaker function W_{φ_f} associated to f, as in §2.5. Under the assumption that f is a Hecke eigenform with respect to the operators T_p (as defined in [17, (9.3.5)] for instance) for each $p \nmid N$ we have that

$$W_{\varphi_f} = W_\infty \otimes \bigotimes_{p<\infty} W_p. \tag{42}$$

We fix the ongoing assumption that $W_p(1) = 1$ for all primes $p < \infty$ so that (42) imposes a constraint on the normalisation of $W_\infty \in \mathcal{W}(\pi_\infty, \psi_\infty)$, the so-called 'Jacquet Whittaker function'. This assumption only concerns finitely many primes p since, by definition, $W_p = W_p^\circ$ is the unique $\mathrm{GL}_n(\mathbb{Z}_p)$-fixed vector satisfying $W_p(1) = 1$ for almost all $p \nmid N$; see Proposition 2.6.

5.5. **Fourier coefficients and Whittaker functions.** Henceforth, let us consider a cuspidal Maaß form $f \in L^2(\Gamma_1(N)\backslash \mathrm{SL}_n(\mathbb{R}))$ which is a Hecke eigenform with respect to the operators T_p for each $p \nmid N$. Without loss of generality, suppose $W_{\varphi_f} = \otimes_v W_v$ such that $\prod_{p<\infty} W_p(1) = 1$, as before.

Definition 5.1. For $(m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$ with $\prod_i m_i \neq 0$ let

$$A_f(m_1, \dots, m_{n-1}) = \prod_{i=1}^{n} |m_i|^{\frac{i(n-i)}{2}} \prod_{p<\infty} W_p\left(\begin{pmatrix} m_1 \cdots m_{n-1} & & & \\ & \ddots & & \\ & & m_{n-1} & \\ & & & 1 \end{pmatrix}\right).$$

When $\prod_i m_i = 0$, we extend the definition by requiring $A_f(m_1, \dots, m_{n-1}) = 0$.

At least when $(m_1 \cdots m_{n-1}, N) = 1$, the following lemma implies that the coefficients $A_f(m_1, \dots, m_{n-1})$ are the Hecke eigenvalues of f (cf. [11, Lecture 7]).

Lemma 5.4 (Shintani's formula)**.** *Fix a prime $p \nmid N$; here, as in §2.7, the local component π_p of $\pi_f = \otimes_p \pi_p$ is an unramified principal series representation with Satake parameters $\mu_1(p), \dots, \mu_n(p)$. Then for integers $k_i \geq 0$, $i = 1, \dots, n-1$ we have*

$$A_f(p^{k_1}, \dots, p^{k_{n-1}}) = s_\lambda(\mu_1(p), \dots, \mu_n(p))$$

for the partition $\lambda = (\lambda_1, \dots, \lambda_{n-1}, 0)$ with $\lambda_i = k_i + \cdots + k_{n-1}$ for $1 \leq i \leq n-1$.

Proof. The (square root of the) modular character determines the constant

$$p^{\sum_{i=1}^{n} \lambda_i \left(\frac{n+1}{2} - i\right)} = \prod_{i=1}^{n} p^{k_i \frac{i(n-i)}{2}}.$$

Recalling the reciprocity of absolute values, $|\gamma|_\infty = \prod_{p<\infty} |\gamma|_p^{-1}$ for $\gamma \in \mathbb{Q}^\times$, the claim now follows immediately from Proposition 2.6. ☐

Remark 5.5 (Dual Maaß forms). Consider the isomorphism between π_f and its contragredient given by mapping φ to the function $\varphi^\iota(g) := \varphi({}^t g^{-1})$. We define the *dual Maaß form* $f^\iota \in L^2(\Gamma_1(N)^\iota \backslash \mathrm{SL}_2(\mathbb{R}))$. At least for $(m_1 \cdots m_{n-1}, N) = 1$, we have

$$A_{f^\iota}(m_1, \dots, m_{n-1}) = \chi(m_1 \cdots m_{n-1}) A_f(m_{n-1}, \dots, m_1). \tag{43}$$

See [17, §9.2] for corresponding discussion in the case $N = 1$.

Our definition of the terms $A_f(m_1, \dots, m_{n-1})$ coincides with that given by Goldfeld [17, p. 260, (9.1.2)] and Kowalski–Ricotta [25, §2], whose coefficients are denoted by "A" and "a_f", respectively; there the assumption $N = 1$ is enforced. In particular, by [17, Lem. 9.1.3] we have the trivial bound

$$A_f(m_1, \dots, m_{n-1}) \ll \prod_{k=1}^{n-1} |m_i|^{i(n-i)/2}.$$

We make the following additional observations.

- The Fourier coefficients are multiplicative:
  $$A_f(m_1m_1', \dots, m_{n-1}m_{n-1}') = A_f(m_1, \dots, m_{n-1})A_f(m_1', \dots, m_{n-1}')$$
  whenever $(m_1 \cdots m_{n-1},\ m_1' \cdots m_{n-1}') = 1$.
- The Fourier coefficients "arithmetically" normalised: $A_f(1, \dots, 1) = 1$.
- The coefficients $A_f(m_1, \dots, m_{n-1})$ are precisely those occurring in the various L-series [16] attached to f, or rather π_f. (See [25, §2] for a concise explanation of this remark.)

5.6. **Derivation of the classical summation formula.** Here we give a proof of Theorem 1.1 obtained by specialising our choices in Theorem 3.4.

5.6.1. *The landscape.* The integer at which the Voronoï summation problem ramifies is denoted by $M \geq 1$ with $N \mid M$ where N is the level of f and M determines the set of primes $p \mid M$ at which we choose local test functions $\phi_p \in \mathcal{C}_c^\infty(\mathbb{Q}_p^\times)$, alongside $\phi_\infty \in \mathcal{C}_c^\infty(\mathbb{R}^\times)$. Define the modulus $\zeta = (\zeta_v) \in \mathbb{A}_\mathbb{Q}$ by first choosing $a, \ell, q \in \mathbb{Z}$ with $a \neq 0$, $\ell, q \geq 1$, $(a, \ell q) = (q, M) = 1$ and $\ell \mid M^\infty$. Then we take $\zeta_\infty = 0$ and $\zeta_p = a/\ell q$ for each $p < \infty$. We evaluate the additive character ψ by Corollary 5.2 so that for each $\gamma \in \mathbb{Q}^\times$ we have
$$\psi(\zeta\gamma) = e(a\gamma/\ell q). \tag{44}$$
Following Theorem 3.4, define the set
$$S = \{\infty\} \cup \{\, p \text{ prime} \,:\, p \mid M \,\}\,.$$
For $y \in \mathbb{Q}_p$, recall the definition $\Phi_p(y) = \phi_p(y)W_p(a(y))$ for each $p \mid M$. At ∞ we use Proposition 2.4 to make the assumption that $W_\infty(a(y)) = \mathrm{Char}_{\mathrm{supp}(\phi_\infty)}(y)$, simply so that $\Phi_\infty = \phi_\infty$. Define the shift $\xi \in T_n(\mathbb{A}_\mathbb{Q})$ as follows: for $i = 2, \dots, n-1$ make a choice of integers $c_i \geq 0$ such that $(c_i, M) = 1$. Then, without loss of generality, for $p \nmid M$ let the component of ξ at p equal $\mathrm{diag}(\xi_1, \dots, \xi_n)$ such that $\xi_i = c_i \cdots c_{n-1}$ for $2 \leq i \leq n-1$, $\xi_1 = \xi_2$, and $\xi_n = 1$. Otherwise, let the components of ξ at ∞ and $p \mid M$ equal 1.

5.6.2. *The left-hand side.* With our specialist assumptions, the left-hand side of (20) is equal to
$$\sum_{\gamma \in \mathbb{Q}^\times} e\left(\frac{am}{\ell q}\right) \prod_{p<\infty} W_p\left(\begin{pmatrix} \gamma c_2 \cdots c_{n-1} & & & & \\ & c_2 \cdots c_{n-1} & & & \\ & & \ddots & & \\ & & & c_{n-1} & \\ & & & & 1 \end{pmatrix}\right) \prod_{v \in S} \phi_v(\gamma). \tag{45}$$
Here we see the choice of ξ was to be contained in the support of $\prod_{p<\infty} W_p$. Moreover, the summands of (45) are non-zero only for those $\gamma \in \mathbb{Q}^\times$ such that

$|\gamma c_2 \cdots c_{n-1}|_p \le |c_2 \cdots c_{n-1}|_p$. Thus, letting $\gamma = m \in \mathbb{Z} \cap \mathbb{Q}^\times$ and applying Definition 5.1 we obtain that (45) is equal to

$$\sum_{m \in \mathbb{Z}_{\neq 0}} e\left(\frac{am}{\ell q}\right) \frac{A_f(m, c_2, \ldots, c_{n-1})}{|m|^{\frac{n-1}{2}} \prod_{i=2}^{n-1} |c_i|^{\frac{i(n-i)}{2}}} \phi_\infty(m) \prod_{p | M} \phi_p(m). \tag{46}$$

5.6.3. *The right-hand side.* Firstly, reconsider the set R in Theorem 3.4:

$$R = \{ p \text{ prime} : p \mid q c_2 \cdots c_{n-1} \}.$$

Recalling that $\xi_1 = \xi_2$ by assumption, the problem bifurcates according to whether $p \in R$ satisfies $p \mid q$ or not. On the right-hand side of (20), one incurs the set $T_R^1 = \otimes_{p \in R} T_p^1$; the Kloosterman sum $\mathcal{K}\ell_R(\gamma, t; \zeta, \xi)$; the Whittaker functions $\tilde{W}_p$ at primes $p \nmid M$; and the local Bessel transforms at $p \mid M$. The support of the T_p^1-sum is determined by the Whittaker function, according to Proposition 2.6. In particular, the support contains only those $\mathrm{diag}(t_2, \ldots, t_{n-1}) \in T_p^1$ whence we make the following inductive change of variables

$$t_i^{-1} = q \frac{c_{n-i+1} \cdots c_2}{d_i \cdots d_{n-1}} \text{ for some } d_i \mid q \frac{c_{n-i+1} \cdots c_2}{d_{i+1} \cdots d_{n-1}}$$

with respect to $i = n-1, \ldots, 2$. In this coordinate system, the Whittaker function on the right-hand side of (20) reads

$$\prod_{p \nmid M} \tilde{W}_p \left(\frac{1}{q c_2 \cdots c_{n-1}} \begin{pmatrix} \gamma \det t^{-1} q^2 & & & & \\ & q c_2 \cdots c_{n-1} t_2 & & & \\ & & \ddots & & \\ & & & q c_2 t_{n-1} & \\ & & & & 1 \end{pmatrix} \right)$$
$$= \chi(q c_2 \cdots c_{n-1})^{-1} \frac{A_{f^\iota}(m, d_2, \ldots, d_{n-1})}{|m|^{\frac{n-1}{2}} \prod_{i=2}^{n-1} d_i^{\frac{i(n-i)}{2}}} \tag{47}$$

where $m \in \mathbb{Z}$ with $(m, M) = 1$ and $\gamma_0 \in \mathbb{Q}^\times$ such that $|\gamma_0|_p = 1$ for each $p \nmid M$, together determining the change of variables

$$\gamma = \gamma_0 \frac{m d_2 \cdots d_{n-1}}{q^n \prod_{i=2}^{n-1} \left(\frac{c_{n-i+1}}{d_i}\right)^{i-1}}.$$

Noting our observation (43) on the coefficients of the dual Maaß form f^ι, (47) is equal to

$$\chi\left(\frac{q c_2 \cdots c_{n-1}}{m d_{n-1} \cdots d_2}\right)^{-1} \frac{A_f(d_{n-1}, \ldots, d_2, m)}{|m|^{\frac{n-1}{2}} \prod_{i=2}^{n-1} d_i^{\frac{i(n-i)}{2}}}. \tag{48}$$

To unwrap the hyper-Kloosterman sums (of §3.2), first note that $\xi_1 \xi_2^{-2} = 1$. We have the following two cases $p \nmid q$, so that $|\zeta_p|_p = |a|_p \le 1$, and $p \mid q$, so that

$|\zeta_p|_p = |q|_p^{-1} > 1$. In either case, noting $\psi_p(-\xi_2\xi_3^{-1}) = e(-c_2) = 1$, we incur the following constant

$$\prod_{p|qc_2\cdots c_{n-1}} |\xi_2 q^{-1}|_p^{n-2}|\xi_3\cdots\xi_n|_p^{-1}\psi_p(-\xi_2\xi_3^{-1}) = \frac{q^{n-2}}{\prod_{i=2}^{n-1}|c_i|^{n-i}}.$$

Applying the Chinese remainder theorem to the product $\mathcal{K}\ell_R(\gamma,t;\zeta,\xi)$, let us make the following inductive change of variables in the Λ_{t_i}-sum: $x_{n-1} = \left(\frac{qc_2}{r_{n-1}}\right)^{-1}\alpha_{n-1}$ with $\alpha_{n-1} \in \left(\mathbb{Z}/\left(\frac{qc_2}{r_{n-1}}\right)\mathbb{Z}\right)^\times$ and

$$x_j = \left(\frac{qc_2\cdots c_{n-j+1}}{r_j\cdots r_{n-1}}\right)^{-1}\alpha_j\bar{\alpha}_{j+1} \text{ with } \alpha_j \in \left(\mathbb{Z}/\left(\frac{qc_2\cdots c_{n-j+1}}{r_j\cdots r_{n-1}}\right)\mathbb{Z}\right)^\times$$

for each $n-2 \geq j \geq 2$, noting the congruence $\alpha_{j+1}\bar{\alpha}_{j+1} \equiv 1 \pmod{\frac{qc_2\cdots c_{n-j+1}}{r_j\cdots r_{n-1}}}$ is well defined. The resulting sum $\mathcal{K}\ell_R(\gamma,t;\zeta,\xi)$ contains the summands

$$\prod_{p|qc_2\cdots c_{n-1}} \psi_p(\xi_{n-j+1}\xi_{n-j+2}^{-1}x_j) = e\left(\frac{c_{n-j+1}\alpha_j\bar{\alpha}_{j+1}}{\frac{qc_2\cdots c_{n-j+1}}{d_{n-1}\cdots d_j}}\right) = e\left(\frac{d_j\alpha_j\bar{\alpha}_{j+1}}{\frac{qc_2\cdots c_{n-j}}{d_{n-1}\cdots d_{j+1}}}\right),$$

for $n-2 \geq j \geq 2$; the term $e\left(\frac{d_{n-1}\alpha_{n-1}}{q}\right)$; and

$$\prod_{p|qc_2\cdots c_{n-1}} \psi_p((-1)^n\gamma\zeta_v^{-1}\xi_2^{-1}\xi_n x_{n-1}^{-1}\cdots x_2^{-1})$$
$$= e\left((-1)^n\frac{\gamma_0 m d_2\cdots d_{n-1}}{q^n\prod_{i=2}^{n-1}\left(\frac{c_{n-i+1}}{r_i}\right)^{i-1}}\frac{\frac{\ell q}{a}}{c_2\cdots c_{n-1}}q^{n-2}\prod_{i=2}^{n-1}\left(\frac{c_{n-i+1}}{r_i}\right)^{i-1}\bar{\alpha}_2\right)$$
$$= e\left((-1)^n\frac{m\gamma_0\frac{\ell}{a}\bar{\alpha}_2}{\frac{qc_2\cdots c_{n-1}}{d_{n-1}\cdots d_2}}\right).$$

Shifting each α_j-variable by $\ell\gamma_0/a$, we remove it from the above exponential and recover it in the term $e\left(\frac{\bar{a}\ell d_{n-1}\alpha_{n-1}}{q}\right)$. Altogether we obtain that

$$\mathcal{K}\ell_R(\gamma,t;\zeta,\xi) = \mathrm{KL}_{n-1}(\bar{a}\ell\gamma_0, m; q, c, d)$$

where $a\bar{a} \equiv 1 \pmod{q}$, $d := (d_2,\ldots,d_{n-1})$, $c := (c_2,\ldots,c_{n-1})$, and the classical Kloosterman sum was defined in (2).

We now consider the M-part of the summands $\gamma \in \mathbb{Q}^\times$, indexed by γ_0. Let L denote the largest square free integer such that $L \mid M$. Suppose that the product $\prod_{p|M}\mathcal{B}_{\pi_p,\Phi_p^{a/\ell q}}(\gamma)$ is non-zero. Then Proposition 4.6 implies that there exists $r \mid M^\infty$ such that

$$\gamma_0 = \frac{r}{[\ell,N]\ell^{n-1}L^n}.$$

Altogether, the right-hand side of (20) is equal to

$$\frac{q^{n-2}}{\prod_{i=2}^{n-1}|c_i|^{n-i}} \sum_{\substack{m\in\mathbb{Z}_{\neq 0}\\(m,M)=1}} \sum_{r|M^\infty} \sum_{d_{n-1}|qc_2} \sum_{d_{n-2}|\frac{qc_2c_3}{d_{n-1}}} \cdots \sum_{d_2|\frac{qc_2\cdots c_{n-1}}{d_{n-1}\cdots d_3}} \mathrm{KL}(\overline{a\lambda_\ell}\ell, m; q, c, d)$$
$$\times\ \chi\left(\bar{m}\frac{qc_2\cdots c_{n-1}}{md_{n-1}\cdots d_2}\right)^{-1} \frac{A_f(d_{n-1},\ldots,d_2,m)}{|m|^{\frac{n-1}{2}}\prod_{i=2}^{n-1} d_i^{\frac{i(n-i)}{2}}} \mathcal{B}_{\pi_\infty,\phi_\infty}\left(\frac{rmd_2\cdots d_{n-1}}{\lambda_\ell q^n\prod_{i=2}^{n-1}\left(\frac{c_{n-i+1}}{d_i}\right)^{i-1}}\right)$$
$$\times \prod_{p|M} \mathcal{B}_{\pi_p,\Phi_p^{a/\ell q}}\left(\frac{rmd_2\cdots d_{n-1}}{\lambda_\ell q^n\prod_{i=2}^{n-1}\left(\frac{c_{n-i+1}}{d_i}\right)^{i-1}}\right)$$

where $\lambda_\ell := [\ell, N]\ell^{n-1}L^n$, $a\bar{a} \equiv \lambda_\ell\bar{\lambda}_\ell \equiv 1 \pmod{q}$, and $m\bar{m} \equiv 1 \pmod{N}$. Thus we conclude the proof of Theorem 1.1.

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