SELF-ORGANIZED BRAIDING AND THE STRUCTURE OF CORONAL LOOPS

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ABSTRACT

The Parker model for heating of the solar corona involves reconnection of braided magnetic flux elements. Much of this braiding is thought to occur at as yet unresolved scales, for example, braiding of threads within an extreme-ultraviolet or X-ray loop. However, some braiding may be still visible at scales accessible to *TRACE* or *Hinode*. We suggest that attempts to estimate the amount of braiding at these scales must take into account the degree of coherence of the braid structure. In this paper, we examine the effect of reconnection on the structure of a braided magnetic field. We demonstrate that simple models of braided magnetic fields which balance the input of topological structure with reconnection evolve to a self-organized critical state. An initially random braid can become highly ordered, with coherence lengths obeying power-law distributions. The energy released during reconnection also obeys a power law. Our model gives more frequent (but smaller) energy releases nearer to the ends of a coronal loop.

Key words: MHD – Sun: corona – Sun: flares – Sun: magnetic fields

Online-only material: color figures

1. INTRODUCTION

Observations of coronal loops by *TRACE* and *Hinode* reveal a beautiful and highly regular structure (Figure 1). In many regions, the loops seem to be of similar diameters and almost parallel. This coherence in and between loops is surprising, considering the fragmented nature of magnetic field generation and release in the solar atmosphere. How can the random processes thought to exist in the corona lead to such well-organized structures?

This puzzle may relate to two others: how are the coronal loops heated (Klimchuk 2006)? And why is the distribution of flare energies a power law (Charbonneau et al. 2001)? The structure of a loop must be strongly affected by magnetic reconnection within the loop, as well as larger scale reconnection with other loops. In the Parker theory of coronal heating, a large number of small flares (*nanoflares*) exist as reconnection of small flux elements within loops (Parker 1988; Sturrock et al. 1990; Cargill 1994). The net effect of these tiny unresolvable events provides an in situ heating of the coronal plasma. On larger scales, *microflares*, of energies of about 10^{26} erg or more, may involve reconnection between loops. These nanoflares and microflares reorganize the magnetic field within the loops. This paper examines how a highly braided field responds to this reorganization using simple mathematical models. As such we hope to supplement numerical simulations of weakly braided line-tied fields (e.g., Rappazzo et al. 2008).

The classic picture of solar activity set out by Parker (1979, 2004), Sturrock & Uchida (1981), and others describes a highly conducting plasma evolving due to the random motions of the photospheric footpoints. Random rotations of the footpoints twist the field lines above, while a random walk of the footpoints braid the overlying loops about each other. The corona field finds an equilibrium state (i.e., the state of minimum energy) given its topology, but must gradually adjust this equilibrium as the topology changes. In Parker's theory, smooth equilibria do not exist for sufficiently complex

topologies, so electric current sheets form. There is some controversy as to whether (in ideal MHD) true current sheets of zero thickness form, rather than thin current layers (Ng & Bhattacharjee 1998; Wilmot-Smith et al. 2009). The physical consequences for a finite conductivity plasma will be the same, however; resistive effects will change the field topology and release magnetic energy. Numerical (Galsgaard & Nordlund 1996) and theoretical evidence (Démoulin et al. 1996) strongly suggest that current layers of exponentially decreasing thickness will soon be produced when the magnetic field topology reaches even a small level of complexity. These current layers may either burn slowly as, say, a tearing mode, or burn quickly in a rapid reconnection event. Numerical simulations (Dahlburg et al. 2005) suggest that rapid reconnection does not occur until sufficient stresses have built up in the magnetic field.

Delaying rapid reconnection allows the buildup of substantial energy reserves in the magnetic field. As the magnetic field becomes more and more braided and twisted, its energy will generally increase quadratically in time (Parker 1983; Berger 1991, 1993). If reconnection occurs too early, there will not be enough energy stored to power flares or coronal heating.

This classic picture has recently run into some difficulties. First, as will be discussed in the following section, present observations of loops do not clearly display a braided structure. Even if most of the braiding is at smaller scales, some braid structure might be expected to be visible at the subarcsecond resolutions of *TRACE* and *Hinode*. Furthermore, random braiding of field lines within loops may expand the diameters of the loops beyond what we observe (Galloway et al. 2006).

Second, the distribution of energy deposition along a loop is not easy to determine and remains controversial (Patsourakos & Klimchuk 2005; Buchlin et al. 2007; Aschwanden 2008). Some observations suggest that the deposition is largely at the base of the loops (Aschwanden 2001), while others suggest a uniform deposition along the loop (Priest et al. 2000). If most of the deposition is at the base of the loop, then the

Figure 1. Magnetic loops observed by *Hinode*. JAXA/NASA.
(A color version of this figure is available in the online journal.)

predominant heating mechanism occurs at low heights, for example, reconnection with field lines in the magnetic network or carpet (Priest et al. 2002; Aschwanden et al. 2007). Note that coronal loops will only be observed if they contain a sufficient density of plasma, as emission scales with the square of the density. This plasma can be supplied in upflows from the chromosphere. In chromospheric heating models, the upflows are caused by heat deposition in the chromosphere or transition region. In nanoflare heating models, energy is first released in the corona; this energy quickly spreads along the field lines down to the chromosphere, which then heats up and sends plasma upwards.

There are at least two ways of characterizing braiding inside a loop, depending on whether the magnetic field is continuous or possesses tangential discontinuities. For magnetic fields which are not continuous, but instead consist of distinct threads separated by current sheets, we can measure the braiding between threads. The most convenient measure of braiding is *crossing number*: when we look at a braid in projection, we see a certain number of crossings between the individual threads (for example, the braid in Figure 2 has six crossings).

For continuous fields where distinct threads are not well defined, we can instead choose some number N of individual field lines within the loop, and quantify braiding between these lines. We can then ask what the average amount of braiding is over many sets of N field lines, or what the maximum is. For example, Wilmot-Smith et al. (2009) made an important breakthrough in deriving a semianalytic force-free model of a braided magnetic field. For their model based on a pigtail braid, there are sets of $N = 3$ field lines that cross each other six times. But as the field is continuous, the braid pattern varies between sets of field lines, and many sets will have less than six.

For the purposes of our analysis, we will assume that the field divides into discrete threads or flux elements. Partly, this greatly simplifies the modeling while still capturing the large-scale geometry of a tangled magnetic field. But also there is some scientific justification for dealing with discontinuous fields: the magnetic flux is highly localized at the photosphere, so the effect of random photospheric motions will be to braid whole bundles of coronal magnetic flux about each other. Also, even without localization at the photosphere, coronal fields driven by photospheric motions are likely to develop strong thin current

Figure 2. (a) Braid with six crossings. (b) A cartoon of three coronal loops with the same braid structure. The loops are drawn (using PovRay) as optically thin tubes.

(A color version of this figure is available in the online journal.)

layers (Démoulin et al. 1996; Galsgaard & Nordlund 1996; Rappazzo et al. 2008).

Observations show that the distribution of solar flare hard X-ray bursts is a power law in peak photon flux with logarithmic slope near 1.8 (Datlowe et al. 1974; Dennis 1985; Lin et al. 1984; Einaudi & Velli 1999; Charbonneau et al. 2001). The power-law index of the distribution is independent of solar cycle. This indicates that the underlying mechanism giving rise to the power-law distribution is insensitive to the level of coronal activity. Lu & Hamilton (1991) suggest that there is a connection between the distribution of solar flares and a phenomenon known as self-organized criticality (SOC).

In order to explain the prevalence of power-law correlations extending over many decades in complex dynamical systems, Bak et al. (1987, 1988) proposed the concept of SOC. Their models suggest that extended systems with many metastable states can naturally evolve into a critical state with no intrinsic length. Experiments (Babcock & Westervelt 1990), as well as numerical simulations (Carlson & Langer 1989; Kadanoff et al. 1989), demonstrate the existence of such critical states. In the avalanche model, sand settles on a sand pile or snowflakes fall on a mountainside. As the sand or snow accumulates, the slope of the pile or mountainside increases, until it reaches a critical state, where avalanches balance the addition of new material. In the SOC models, the critical state is found to be insensitive to initial conditions. Furthermore, it does not require any fine tuning of parameters. As soon as the perturbations cause disturbances that are able to propagate the length of the system, the system becomes stationary. When the system is in a critical state, it has a distribution of minimally stable regions of all sizes. Therefore, small perturbations give rise to avalanches of all sizes from the smallest possible avalanche such as a single sand grain up to the size of the system. A featureless power-law distribution of avalanche sizes results since there are no characteristic length

scales in between the very small scale of the sand grain and the very large scale of the sand pile itself.

Lu & Hamilton (1991) suggest that the coronal magnetic field behaves in a similar manner. In their model, the random stressing of magnetic fields by photospheric convective motions provides the input of energy, and swarms of small flares provide the output. First, nanoflares are excited by local increases in magnetic energy. These can then trigger more nanoflares nearby, resulting in a larger flare.

Note that the observed flare energy distribution does not integrate to give sufficient coronal heating. It is possible that the slope steepens at nanoflare energies (Vlahos et al. 1995; Georgoulis & Vlahos 1996); a steeper slope means more small events and hence more heating.

Several subsequent authors have presented more refined SOC models (Zirker & Cleveland 1993b; Vlahos et al. 1995; Macpherson & MacKinnon 1997; Charbonneau et al. 2001). Most of these models involve a grid (two-dimensional or three-dimensional) with magnetic field vectors at the nodes. At each time step, small perturbations are added to the field, in analogy with the input of the structure to the coronal field at the photosphere. The models assume that nodes become unstable if some criterion is satisfied, for example, high field strength or high field gradients. In this case a small “nanoflare” occurs, and the node shares some of its excess magnetic energy with neighboring nodes. This spread of energy may make neighboring nodes unstable, leading to more nanoflares. Thus, a single small event can avalanche to a much larger set of events. A grid set up in an arbitrary initial state will evolve due to both the input perturbations and the output events until a statistical steady state is reached.

One great challenge of solar SOC theory is making a close correspondence with the specific geometry of the coronal magnetic field. The grid models do not explicitly possess a photospheric boundary. Also, one might wish to see coronal loops, or at least distinct flux regions bounded by separatrices (Longcope 2005).

Zirker & Cleveland (1993a) created an SOC model where the two-dimensional grid represents a cross section through a set of coronal flux tubes. They tuned the energy inputs and outputs according to what would be expected for a set of flux tubes undergoing twisting and braiding (Berger 1991). Longcope & Noonan (2000) relate the build up of stresses in a two-dimensional grid to current flows along separatrices. In an interesting departure from grid models, Hughes et al. (2003) considered a set of magnetic loops randomly rising through the photosphere. When two loops meet, they reconnect and release energy. The energy releases obeyed a power law with a slope of -3 .

In Section 2, we estimate the extent of braiding inside a coronal loop, assuming that most of the free energy stored in the loop results from braiding of flux elements. Section 3 briefly discusses whether braiding can be detected with present instruments. Section 4 reviews several mechanisms for generating braid structure within or between loops.

Section 5 presents a simple analytic model for the evolution of a loop where braiding is added by endpoint motions and removed by reconnection. Because the reconnection selectively occurs at critical points along the braid, the loop evolves to a self-organized state with a power-law distribution of coherence lengths. The energy release distribution is also power law. These analytic results are then verified in Monte Carlo simulations in Section 6. Our model resembles forest fire models more than avalanche models: in avalanche models a small reconnection

triggers more reconnection; but this triggering has not been included here.

Conclusions are given in Section 7.

2. HOW MUCH BRAIDING EXISTS INSIDE A LOOP?

Suppose a braid has N strings of diameter D and exhibits C crossings. If N is small, say $N = 3$ or $N = 4$, then each crossing will require two neighboring strings to rotate around each other. If the strings are magnetic flux tubes, this means that the flux tubes must have a transverse magnetic field $B_{\perp}$, hence extra magnetic energy. Thus, the magnetic energy increases with the number of crossings. Inequalities relating magnetic energy to crossing number C are derived in (Berger 1993).

Let us make a simple estimate of how the transverse field grows with C . Suppose the braided tubes stretch vertically between planes $z = 0$ and $z = L$. We ignore the contribution to the transverse field from the twisting of individual flux elements, i.e., we assume that the transverse field arises predominantly from braiding.

Each crossing involves two strings out of the N strings. Thus, on average, each string takes part in $2C/N$ crossings. Let us assume that these $2C/N$ crossings are uniformly distributed along the height L . Then a single crossing takes place in a vertical distance of about

$$\delta z = \frac{NL}{2C}. \quad (1)$$

Consider the center lines of two tubes that rotate about each other between heights z and $z + \delta z$. Assuming the tubes have a circular horizontal cross section, the transverse distance each tube travels is $\delta \ell = \pi D/2$. Then the typical ratio of transverse field strength to axial field strength will be

$$\frac{B_{\perp}}{B_{\parallel}} = \frac{\delta \ell}{\delta z} = \frac{\pi D}{2\delta z} \quad (2)$$

$$= \frac{\pi C D}{NL}. \quad (3)$$

The free energy is proportional to the square of the crossing number (Berger 1993). Let

$$E_{\text{free}} = \frac{1}{2\mu} \int B_{\perp}^2 d^3x. \quad (4)$$

The volume of the N tubes is approximately $N\pi D^2 L$. Then by Equation (3)

$$E_{\text{free}} \approx \left(\frac{N\pi D^2 L}{2\mu} \right) \left(\frac{\pi C D B_{\parallel}}{NL} \right)^2 = \left(\frac{\pi^3 D^4 B_{\parallel}^2}{2\mu NL} \right) C^2. \quad (5)$$

How large is C ? Now, $\tan^{-1} B_{\perp}/B_{\parallel}$ gives the typical angle of the flux tubes with respect to the vertical. Two crossing flux tubes will be misaligned by as much as twice this angle. When the crossing number becomes large enough, neighboring tubes will be sufficiently misaligned to trigger reconnection (Linton et al. 2001). If a misalignment of about $\pi/6$ triggers reconnection, then $B_{\perp}/B_{\parallel} \approx \tan \pi/12 \approx 0.27$, and the crossing number will be

$$C_{\text{critical}} = \frac{NL}{\pi D} \tan \pi/12 \approx 0.085 \frac{NL}{D}. \quad (6)$$

For flux elements with aspect ratio $L/D = 100$ and $N = 3$, we find $C_{\text{critical}} \approx 25$.

3. CAN WE OBSERVE BRAIDED CORONAL LOOPS?

Observations of coronal loops, like the *Hinode* image (Figure 1), display a well-combed set of almost parallel loops. This observed structure seems far away from mathematical diagrams of braids, such as Figure 2. The mathematical diagrams, however, are designed to most clearly display the geometrical structure. If we elongate, compress, and arch the strings in the braid diagram, then the braid structure becomes much more difficult to see. Figure 2 demonstrates this. One can see that a braid with, say, six crossings might be difficult to distinguish from a collection of slightly irregular but parallel loops. Detailed forward modeling (beyond the scope of this paper) would be likely to further obscure the braid structure. For example, Figure 2 shows tubes with circular cross sections. Three circular tubes bundled together, as in the diagram, have an empty space in between which makes them easier to see. However, more physically realistic magnetic flux tubes should not leave any empty spaces (consider a cross section of three neighboring loops. If the loops are modeled with circular cross sections, then we obtain three close-packed circles in a plane; they can only touch each other at three points. But in cross section three more realistic flux elements might resemble one big circle divided into three pie slices).

The possibility of directly observing braiding also depends on details of the braid structure. A braid in mathematics is a set of curves stretched between two planes. The simplest non-trivial braid consists of two curves twisted about each other. Suppose curves 1 and 2 wind about each other through an angle θ_{12} . In what follows, it will be convenient to let $w_{12} = \theta/\pi$ be the number of half-turns between curves 1 and 2, as the number of half-turns (ignoring sign) equals the number of crossings seen in projection, $C_{12} = |w_{12}|$.

In general, two magnetic flux tubes braided in this fashion will also possess internal twists. Of course, if we are looking at two neighboring coronal loops or two threads in an erupting prominence then the large-scale structure represented by w_{12} will be easier to see than the smaller scale twists. Suppose, however, that the two tubes are tightly wound about each other. In this case, they might appear as a single tube. In the particular case where the internal twists match the winding between the tubes (so field lines within tube 1 twist about the axis of tube 1 through an angle of πw_{12} , and similarly for tube 2), the set of two flux tubes will be topologically identical to a single tube with uniform twist. If the internal and external twists differ from each other, then current sheets will develop between the two tubes. This will lead to reconnection (Linton et al. 2001). If that reconnection is not too fast, then the two tubes may persist as topologically distinct entities. However, both tubes will light up due to emission caused by the reconnection, so in practical terms they may still be indistinguishable from one single tube.

With three or more curves, there are many more possible braid patterns. Rather than merely recording the twist between each pair of curves, we must record the actual sequence of windings. Not all braided fields will look the same, even if the number of curves N and the number of crossings C are the same. Figure 3 shows two braid structures which both have four strings and 13 crossings. The first structure is randomly generated. The second structure (Figure 3(b)), however, is much more coherent. Apart from the crossing in the middle, the braid consists of two distinct pairs of twisting tubes. Such a braid pattern would look well combed in comparison to the randomly tangled pattern of Figure 3(a). Some descriptive terminology will be useful here. For three-braids (Figure 4(a)), the sections of the braid where

Figure 3. Figure on the left shows a randomly generated four string braid with 13 crossings. Contrast this with the figure on the right, showing a highly coherent four string braid with 13 crossings. The bottom six crossings involve two pairs of curves which wind about each other, and similarly for the top six crossings. An *interchange* crossing in the middle swaps two of the strings between the two pairs. Without the interchange crossing, the braid would appear as two distinct twisted pairs.

(A color version of this figure is available in the online journal.)

the two strings on the left twist about each other will be called *coherent sequences*. The single crossings which swap the middle string on the right will be called *interchanges*.

In this paper, we suggest that self-organization will add coherence to the braid pattern. This additional coherence has some practical consequences for any attempt to observe coronal braiding. First, the interchanges may be the most visible signature of braiding. If two tubes twisting about each other appear like one larger tube, then at an interchange this larger tube will appear to branch and/or merge with a neighboring tube. Second, a coherent braid pattern may not spread out as much in radius as a random pattern. This may affect attempts to infer the amount of braiding of fine threads below resolution. Galloway et al. (2006) suggest that limits can be placed on braiding of threads within an X-ray loop because randomly braided field lines will random walk perpendicular to the axis of the loop. As a consequence, the loop spreads out laterally as it rises from the photosphere. However, a more well-combed braid pattern within a loop will not spread out as much.

4. GENERATION OF BRAID STRUCTURE

There are several mechanisms which will braid coronal magnetic flux. First, random motions of photospheric magnetic footpoints will braid the flux elements above. Second, a coronal flux tube may be fragmented at its footpoints. Footpoints have a finite lifetime; individual fragments of magnetic flux within the footpoint of a large coronal loop can eventually disperse, presumably as a consequence of turbulent motions below the photosphere. Suppose two or more neighboring footpoints break up. The individual fragments will move around with the granular

Figure 4. (a) Three-braid with two coherent sequences separated by an interchange. (b) Reconnection removes the interchange. (c) The curves can now relax, canceling out their crossings.

(A color version of this figure is available in the online journal.)

motion and eventually be swept into the edges of the granules or supergranules. By this mechanism or some other mechanism they eventually coalesce into new flux concentrations. However, the resulting new footpoints will generally have a different mixture of flux fragments than the old footpoints. This process can increase the topological complexity of the field and may contribute to coronal heating (Berger 1994).

Third, the fragmentation of the footpoints of large coronal loops can be interpreted in terms of the coronal flux interacting with small-scale loops. These small loops can reconnect with the large loops, leading to an effective motion of the endpoints of the coronal field lines (Schrijver et al. 1998; Priest et al. 2002). Evidence for fragmentation and reconnection near footpoints of coronal loops can be seen in Shibata et al. (2007).

Fourth, reconnection is not always clean and simple! For example, if two tubes with five units and seven units of flux reconnect, then there will be two units of flux left over in a topologically distinct third tube. Moreover, if there are multiple reconnection sites, then the end product may be several topologically distinct tubes.

Finally, internal braiding may already be present in newly emerging flux.

5. A SIMPLE ANALYTIC BRAID MODEL EXHIBITING SELF-ORGANIZATION

5.1. Description of the Model

Here we present a simple model which gives a power-law distribution of energy releases. We consider a braid on three strings (Figure 4). At the boundaries, braid structure is generated in the strings below by a succession of motions. We suppose at the boundaries the endpoints line up on the x -axis.

In this simple model, two motions alternate with each other. The first motion rotates the leftmost and middle endpoints through some net twist angle θ , where θ is a multiple of π . This generates a winding w in the strings between the boundaries. (For upper boundary motions $w = \theta/\pi$; for lower boundary motions $w = -\theta/\pi$.) These windings correspond to $C = |w|$ crossings. The sign of w tells us whether the crossing is right-handed or left-handed.

The second motion gives a single half-twist between the middle- and right-hand end point, and generates an interchange. For example, the braid of Figure 4(a) can be generated as follows: start with three straight curves. First, let the leftmost endpoints rotate at the top boundary through three anti-clockwise half-turns ($w = +3$), giving the strings below three positive (right-handed) half-twists. Next, the second motion exchanges the third string with the middle string. Finally, a twist of three clockwise turns gives the two strings on the left (no longer the same two strings!) three negative half-twists ($w = -3$). Note that the twists cannot cancel because of the interchange. This constraint can only be removed by reconnection. The middle figure shows the result immediately after reconnecting the curves involved in the interchange. Now the positive and negative twists connect to each other. They can now cancel out through purely ideal motions.

We will consider a random process which evolves the braid by both adding new structure and removing or simplifying old structure. We start at $t = 0$ with some arbitrary initial braid with m sequences separated by $m - 1$ interchanges. At each time step, we add one new coherent sequence and one new interchange. At the same time, a reconnection randomly removes one of the pre-existing interchanges, allowing the sequences on either side to merge or cancel. After many time steps, the braid should forget its initial structure, and reach a statistical steady state. This process would then be an example of self-organization. We wish to find the structure of the self-organized braid.

The number of sequences with crossing number w is $n(w)$. Thus,

$$m = \sum_{w=-\infty}^{\infty} n(w). \quad (7)$$

Also, the total number of crossings (ignoring sign) is

$$C_{\text{tot}} = \sum_{w=-\infty}^{\infty} |w|n(w). \quad (8)$$

We will normalize by letting $f(w) = n(w)/m$ be the probability distribution function for the sequence lengths w .

5.2. A Simple Self-organized Braid: Analytic Model

We first give a simplified analytic treatment, then provide the results of a Monte Carlo simulation. The main simplification involves letting the random input of new sequences and interchanges occur anywhere in the braid, rather than the boundaries. With the Monte Carlo simulation we can easily remove this assumption, and thus allow a more realistic model (where the boundaries correspond to the photosphere).

In our analytical treatment, we also simplify the calculation by assuming w is a continuous rather than discrete variable. In this case, the number of crossings between w and $w + \delta w$ is (considered as an ensemble average) $n(w)\delta w$.

At each time step, one new sequence and one new interchange is added, while reconnection removes one of the interior interchanges. Let the probability distribution of new sequence lengths be $p(w)$. At each time step, the probability distribution function $f(w)$ changes by $\delta f(w)$: first, there is a probability of $p(w)$ that the new sequence will add to $f(w)$. Next, the sequence to the left of the reconnecting interchange may have winding w . This sequence disappears, so there is a probability $f(w)$ that the number $n(w) = mf(w)$ of sequences with crossing number w will decrease by one. Similarly, the sequence to the right of the obstruction has probability of $f(w)$ of removing a w -sequence. Finally, if the left sequence has w_1 windings and the right sequence has $w_2 = w - w_1$ windings, a new w -sequence will be created. Thus,

$$m\delta f(w) = p(w) - 2f(w) + \int_{-\infty}^{\infty} f(w_1)dw_1 \times \int_{-\infty}^{\infty} f(w_2)d(w_2)\delta(w - (w_2 + w_1)), \quad (9)$$

$$= p(w) - 2f(w) + \int_{-\infty}^{\infty} f(w_1)f(w - w_1)dw_1. \quad (10)$$

In a steady state, the left-hand side vanishes. So

$$p(w) - 2f(w) + (f * f)(w) = 0, \quad (11)$$

where $f * g$ is the Fourier convolution. To solve this, we take the Fourier transform,

$$\tilde{p}(k) - 2\tilde{f}(k) + \tilde{f}^2(k) = 0. \quad (12)$$

This has solution

$$\tilde{f}(k) = (1 \pm \sqrt{1 - \tilde{p}(k)}). \quad (13)$$

Note that we must choose the negative square root in order to ensure that $\tilde{f}(k) \rightarrow 0$ as $k \rightarrow \infty$.

Say the input is a Poisson process, so that for some λ ,

$$p(w) = \frac{\lambda}{2} e^{-\lambda|w|}. \quad (14)$$

Then

$$\tilde{p}(k) = \frac{\lambda^2}{\lambda^2 + k^2}, \quad (15)$$

$$\tilde{f}(k) = \left(1 - \frac{|k|}{\sqrt{\lambda^2 + k^2}}\right). \quad (16)$$

We can solve the inverse transform using standard integrals

$$f(w) = \frac{\lambda}{2} (I_1(\lambda w) - L_{-1}(\lambda w)), \quad (17)$$

where I_0 is a Bessel- I function and L_0 is a Struve- L function (see Figure 5). The function $f(w)$ falls asymptotically as w^{-2} .

Figure 5. Distribution $f(w)$ of braid sequence lengths given by Equation (17), shown with a log-log plot. The asymptotic slope is -2 . (A color version of this figure is available in the online journal.)

5.3. The Energy Distribution

From Section 2, the free energy of a set of braided magnetic flux tubes is $E_{\text{free}} = a C^2$, where the constant a depends on the length and diameter of the tubes. Suppose that a reconnection reduces the number of crossings from C_{initial} to C_{final} . We ignore the constant a and define the “flare energy” to be

$$E = \Delta E_{\text{free}}/a = C_{\text{initial}}^2 - C_{\text{final}}^2. \quad (18)$$

Given the distribution $f(w)$ of coherent braid sequences, what is the corresponding distribution of flare energies $f(E)$?

We calculate the energy distribution assuming that reconnection occurs when the total crossing number reaches some critical value C_{crit} . Here two coherent sequences of length w_1 and w_2 merge. Suppose that $|w_2| > |w_1|$ and that they are of opposite sign. They merge to form a single sequence of length $|w_2| - |w_1|$. As we had two sequences of total length $|w_2| + |w_1|$ beforehand, the loss of crossings is $2|w_1|$. The change in crossing number squared is

$$E = C_{\text{crit}}^2 - (C_{\text{crit}} - 2|w_1|)^2 = 4C_{\text{crit}}|w_1| - 4w_1^2. \quad (19)$$

We assume $w_1 \ll w_{\text{crit}}$ and neglect the $4w_1^2$ term. For convenience, we choose w_1 to be positive and w_2 negative. There is a constraint here: the second sequence has a larger crossing number, i.e., $|w_2| > |w_1|$. Thus, the probability of obtaining E is

$$f(E) = 2 \int_0^{\infty} \int_{-\infty}^{-w_1} f(w_1) f(w_2) \delta(E - 4w_1 C_{\text{crit}}) dw_2 dw_1. \quad (20)$$

For a power-law distribution of coherence lengths, $f(w_1) = b|w_1|^{-\beta}$ with $\beta > 1$, we have

$$f(E) = 2b^2 \int_0^{\infty} w_1^{-\beta} \frac{w_1^{1-\beta}}{1-\beta} \delta(E - 4w_1 C_{\text{crit}}) dw_1 \quad (21)$$

$$= \left(\frac{2b^2}{\beta - 1} \left(\frac{1}{4C_{\text{crit}}} \right)^{2-2\beta} \right) E^{1-2\beta}. \quad (22)$$

Adopting the convention of using the Greek letter α for the energy slope, we find

$$f(E) \propto E^{-\alpha} \Rightarrow \alpha = 2\beta - 1. \quad (23)$$

Figure 6. Results from a simulation of a periodic braid with $N = 16,000$ steps, averaged over an ensemble of 1000 runs. For $|w| = 5$ up to $|w| = 110$, $f(w)$ obeys a power law with best-fit exponent -2.02 . The energy release variable E is normalized so that the smallest energy release is assigned the value 1. The energy distribution from $E = 5$ to $E = 50$ has best-fit exponent -3.09 .

(A color version of this figure is available in the online journal.)

Figure 7. Results from a simulation of a bounded braid with $N = 16,000$ steps, averaged over an ensemble of 2000 runs. For $|w| = 5$ up to $|w| = 37$, $f(w)$ obeys a power law with best-fit exponent -2.84 . The energy distribution from $E = 15$ to $E = 85$ has best-fit exponent -3.00 .

(A color version of this figure is available in the online journal.)

For example, if $\beta = 2$ as in the model of the previous section, then the distribution of flares with energy E decreases as the third power of E , i.e., $f(E) \propto E^{-3}$.

The distribution will be modified for smaller E where the influence of new sequences with exponentially decreasing $|w|$ is important. Furthermore, at low energies, the energy of the interchange cannot be neglected. The unit of crossing corresponding to the interchange gives an effective length scale to the problem, which modifies the power-law behavior at low E .

6. MONTE CARLO SIMULATIONS

These simple analytic results can be verified and extended by numerical simulation. We first set up a braid with a random exponential distribution of coherent sequences (with a probability distribution given by Equation (14) with $\lambda = 1$). We then run the simulation for N steps; at each step, we randomly choose one interchange to be removed; we also balance this removal by inserting one or more new coherent sequences (together with new interchanges) to maintain a constant braid length $C_{\text{crit}} = 1000$. Each interchange removal we call a flare event. After each removal of an interchange, there is either a merger of the two neighboring sequences (if they are of the same sign), or a partial cancellation (if they are of opposite signs). For the last quarter of the run, we record the energies of the flares. (In the first part of the run, the braid will still remember its initial configuration so the distribution of flare energies will not have settled to its final state.)

For a braided solar flux rope, new braid structure enters at the ends (e.g., at the photosphere or via reconnection with low

lying loops, as discussed in Section 4). This implies there may be some inhomogeneity in the overall braid structure: near the ends there will be fresh small coherent sequences, while in the middle of the braid larger sequences will have had more time to develop. Here we first verify the analytic results of the previous section and then examine the influence of the photospheric ends.

The analytic results did not take into account end effects. Thus, in our first simulation new sequences can appear anywhere in the braid. Also we make the braid periodic; i.e., one end of the braid connects to the other as in a ring. The results are given in Figure 6.

To improve the statistics, we plot $f(|w|)$ versus $|w|$, ignoring the signs of w (both signs are equally likely in these runs). At very low values of w , the distribution is influenced by the new sequences being added at each step. At very high values of w , the distribution drops off because of the finite size of the braids ($= C_{\text{crit}}$), and the time needed for self-organization to produce very large sequences. However, in between these extremes, intermediate scales do indeed display power-law behavior. Figure 6 shows excellent agreement with the prediction that the sequence distribution $f(w)$ obeys a power law with slope $\beta = 2$, and the energy release distribution $f(E)$ obeys a power law with slope $\alpha = 3$.

In a second experiment (Figure 7), we input new sequences at the boundaries. Here the sequence distribution $f(w)$ steepens to a slope of $\beta = 2.84$. However, the flare energy distribution changes little, staying near $\alpha = 3$. The relation $\alpha = 2\beta - 1$ no longer works.

Figure 8 provides some clues for this behavior. The braided loop has been divided into two parts: one part combines the two

Figure 8. Energy release distribution from the previous figure (Figure 7) has been divided into separate plots. The left plot is for events occurring near the ends of the loop (first or last quarter of the length); the right plots events in the middle of the loop (between 0.25 and 0.75 of the braid's length). For the ends, there are more deviations from a straight power-law fit; also the slope is much steeper ($E = 15$ to $E = 58$, exponent -3.91). For the middle, the energy distribution from $E = 9$ to $E = 85$ has best-fit exponent -2.90 .

(A color version of this figure is available in the online journal.)

ends (or legs), comprising the first and last 25% of the length of the braid pattern (as measured by crossings C), while the second part consists of the middle 50%. The flares in the middle part have a well-developed power-law distribution with $\alpha = 2.90$. However, the flare energy distribution in the end parts drops off much more quickly. Large coherent sequences migrate to the center of the braid as they increase in length due to mergers; meanwhile the ends of the braided loop fill with fresh shorter sequences (corresponding to input from the base of the corona).

We can also compare the total number of flares and their average energies. If the flares were distributed homogeneously, then middle and end parts would both have 50% of the flares and 50% of the flare energy release. In fact, only 19% of the flares were located in the middle part, but the average middle flare was 1.7 times more powerful than the average flare near the ends. Overall, flares in the middle of the braid loop accounted for 29% of the total energy release.

7. CONCLUSIONS

This paper has presented a statistical model of braiding in coronal loops. The model demonstrates that braid patterns can organize themselves so that coherence lengths and flare energies obey power-law energy distributions. The model assumes that the random input of a new braid structure is balanced by the loss of braid structure caused by reconnection. The reconnection is assumed to occur selectively at points of maximum stress in the braid. This selective reconnection organizes the braid pattern into smoother structures. Analytic and numerical modeling both demonstrate the development of a self-organized state with power-law distributions.

These models add to the literature on SOC in the solar atmosphere, by explicitly including the braid structure, as well as the influence of placing the input at the two ends of a loop. The results have implications for coronal observations: first, attempts to directly observe large-scale braiding between resolvable loops must take into account the level of coherence of the braid pattern. Second, large coherence lengths will affect the amount of spreading of fine structure within a loop, and hence how much the diameter of a loop varies between its base and its top. Third, our model gives many small flares near the base of the loop and fewer but larger flares near the top. This has implications for observations of the spatial distribution of energy deposition in loops. Directions for future development

include allowing flare events to trigger other flare events (as in avalanche models), considering the influence of a net magnetic helicity inside a loop, and looking for evidence of coherent braiding in MHD simulations of braided magnetic fields.

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