

**Stochastic Finite Element Modelling of Flow and  
Solute Transport in Dual Domain System**

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AND SOLUTE TRANSPORT IN DUAL DOMAIN SYSTEM**

Submitted by

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to the University of Exeter as a thesis for the degree of  
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M. Mousavi Nezhad

## **ABSTRACT**

Hydrological processes are greatly influenced by the characteristics of the domain through which the process occurs. It is generally accepted that earth materials have extreme variations from point to point in space. Consequently this heterogeneity results in high variation in hydraulic properties of soil. In order to develop a reliable predictive model for transport processes in soil, the effects of this variability must be considered. Soil heterogeneity due to presence of macropores (micro-) and to spatial variability in hydraulic properties (macro-heterogeneity) coexists in the real field conditions. The challenge is to incorporate the effects of both types of soil heterogeneity in simulation models.

This thesis presents development and application of a 2D/3D numerical model for simulation of advection and diffusion-dispersion contaminant transport considering both types of soil heterogeneity. Stochastic finite element approach is used to incorporate the effects of the spatial variability of soil hydraulic properties on contaminant fate. The soil micro heterogeneity effects are modelled with a dual domain concept in which a first order kinetic expression is used to describe the transfer of the solute between the two domains. Also, the capability of the model in 3D simulation of field problems improves the accuracy of the results, since it is possible to avoid the generally applied assumption in 2D simulations.

From comparison of the model results with experimental and analytical results, it is concluded that the model performs well in predicting contaminant fate and the incorporation of the both types of micro- and macro- heterogeneity in the simulation models improves the accuracy of the prediction. Also, capability of the model in evaluation of the concentration variation coefficient as an index of reliability of the model outputs makes it possible to estimate a probable interval (mean concentration minus and plus standard deviation) for the range of oscillations of possible realizations of solute distribution. Moreover, comparison of the results of the proposed method with the results obtained using the Monte Carlo approach yields a pronounced reduction in the computation cost while resulting in virtually the same response variability as the Monte Carlo technique.

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