

Relating forced climate change to natural variability and emergent dynamics of the climate-economy system

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Owen Kellie-Smith

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Owen Kellie-Smith

Abstract

This thesis is in two parts. The first part considers a theoretical relationship between the natural variability of a stochastic model and its response to a small change in forcing. Over a large enough scale, both the real climate and a climate model are characterised as stochastic dynamical systems. The dynamics of the systems are encoded in the probabilities that the systems move from one state into another. When the systems' states are discretised and listed, then transition matrices of all these transition probabilities may be formed. The responses of the systems to a small change in forcing are expanded in terms of the eigenfunctions and eigenvalues of the Fokker-Planck equations governing the systems' transition densities, which may be estimated from the eigenvalues and eigenvectors of the transition matrices. Smoothing the data with a Gaussian kernel improves the estimate of the eigenfunctions, but not the eigenvalues. The significance of differences in two systems' eigenvalues and eigenfunctions is considered. Three time series from HadCM3 are compared with corresponding series from ERA-40 and the eigenvalues derived from the three pairs of series differ significantly.

The second part analyses a model of the coupled climate-economic system, which suggests that the pace of economic growth needs to be reduced and the resilience to climate change needs to be increased in order to avoid a collapse of the human economy. The model condenses the climate-economic system into just three variables: a measure of human wealth, the associated accumulation of greenhouse gases, and the consequent level of global warming. Global warming is assumed to dictate the pace of economic growth. Depending on the sensitivity of economic growth to global warming, the model climate-economy system either reaches an equilibrium or oscillates in century-scale booms and busts.

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