CONVERGENCE OF TIME AVERAGES NEAR STATISTICAL ATTRACTORS AND RATCHETING OF COUPLED OSCILLATORS

By
Özkan Karabacak

MATHEMATICS RESEARCH INSTITUTE
UNIVERSITY OF EXETER

A thesis submitted to the University of Exeter
for the Degree of

DOCTOR OF PHILOSOPHY
of
UNIVERSITY OF EXETER

September 2010
To Neslihan
Convergence of Time Averages near Statistical Attractors
and
Ratcheting of Coupled Oscillators

Submitted by Özkan Karabacak, to the University of Exeter as a thesis for the degree of Doctor of Philosophy in Mathematics, September 2010.

This thesis is available for Library use on the understanding that it is copyright material and that no quotation from the thesis may be published without proper acknowledgement.

I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

Signature:........................................
ACKNOWLEDGEMENTS

First of all I would like to thank Peter Ashwin for his guidance throughout this project. I have really benefited from his continued input and support. Special thanks to Neslihan Serap Şengör for introducing and teaching dynamical systems to me during my undergraduate and master’s studies. She also encouraged me to start this PhD project and continued to provide her help and support. I am grateful to Mike Field for inviting me to University of Houston for two weeks in October 2009 and for sharing his knowledge and experience. His comments and suggestions on this project have been very useful. I would like to thank Nikita Agarwal, Abul Kalam Al-Azad, Nicholas Blackbeard, Richard Fredlund, Amrita Muallehran, Tolga Esat Özkurt, Ozan Saka and Ahmet Yildiz for discussions related to my work and for their friendly support. Neslihan, Nick and Nikita have also helped in proofreading the thesis. Finally, I thank my family Cevat Karabacak, Sedika Karabacak, Meltem Karabacak, Özlem Ertekin and Nihat Ertekin for their encouragement and support.
Abstract

In this thesis, convergence of time averages near statistical attractors of continuous flows are investigated. A relation between statistical attractor and essential ω-limit set is proved, and using this a general definition for statistical attractor is given. Sufficient conditions are given for an observable to admit a convergent time average along the orbits of the flow. The general results are applied to flows on a torus, and in particular to systems of coupled phase oscillators that admit attracting heteroclinic networks in their phase space.

A particular heteroclinic network that we call heteroclinic ratchet is observed and analysed in detail. Heteroclinic ratchets give rise to a novel phenomenon, unidirectional desynchronization of oscillators (ratcheting). The results obtained about the convergence of time averages near statistical attractors implies that heteroclinic ratchets induce, besides its other interesting consequences, frequency synchronization without phase synchronization. Different coupling structures that can give rise to ratcheting of oscillators are also investigated.
Contents

1 Introduction 11

I Non-ergodicity 18

2 Convergence of time averages 19
   2.1 Essential $\omega$-limit set 21
   2.2 Statistical attractors 23
   2.3 Convergence of time averages 34

3 Heteroclinic networks and ratchets 43
   3.1 Intrinsic definition for heteroclinic networks 44
   3.2 Time averages near embedded heteroclinic networks 47
   3.3 Synchronization for trajectories on a torus 48
   3.4 Heteroclinic networks on a torus 51
   3.5 Heteroclinic ratchets on a torus 54

II Background: Coupled Oscillators 56

4 Coupled systems 57
   4.1 Coupled cell systems 58
   4.2 Balanced partition 63
   4.3 Quotient networks 66
   4.4 Symmetries and fixed point subspaces 67
   4.5 Symmetry-breaking and synchrony-breaking bifurcations 68

5 Coupled oscillators 71
   5.1 From limit cycle oscillators to phase oscillators 72
   5.2 Kuramoto’s model 73
   5.3 Synchronization properties 75
      5.3.1 Phase and frequency synchronization 76
      5.3.2 Sensitivity to detuning and ratcheting 77
III Heteroclinic Ratchets

6 A heteroclinic ratchet for a system of 4 coupled oscillators

6.1 4-cell example
6.1.1 Invariant subspaces
6.1.2 Synchrony-breaking bifurcations
6.2 Robust heteroclinic ratchets
6.2.1 Heteroclinic ratchets for the four coupled oscillators
6.2.2 Stability of the heteroclinic ratchet
6.2.3 Routes to heteroclinic ratchets
6.3 Dynamical consequences of ratchets
6.3.1 Response of the system to detuning
6.3.2 Response of the system to noise and detuning
6.3.3 Frequency synchronization without phase synchronization

7 Further examples of heteroclinic ratchets

7.1 Ratcheting without noise or detuning
7.2 Heteroclinic ratchets in a system without symmetry
7.3 Ratcheting in large networks

IV Conclusion

8 Discussion
List of Figures

6.8 Schematic diagrams demonstrating trajectories switching between saddles of the heteroclinic ratchet in Figure 6.4(b) under small additive noise. ........................................... 114
6.9 Winding frequency $\Omega$ plotted against $\log(\Delta)$ for (a) $\Delta < 0$, (b) $\Delta > 0$ and additive noise of amplitude $\varepsilon$. ................................. 115

7.1 A heteroclinic attractor ($\{p_1\} \cup q_1 \cup \{p_2\} \cup q_2$) for the system (7.1.2). 120
7.2 A non-symmetric coupled cell network that allows heteroclinic ratchets. ................................................. 122
7.3 A solution of the system (7.2.1) with additive white noise (amplitude $= 10^{-6}$) for $\alpha_1 = 0.01$ and $\alpha_2 = 0.02$. ................................. 123
7.4 A coupled cell network that consists of 6 identical cells and allows heteroclinic ratchets. ................................. 124
7.5 A solution of the coupled oscillator network in Figure 7.4 under small additive white noise (amplitude = $10^{-4}$) with zero initial states. 125
7.6 $2N$-cell coupled cell networks that may support heteroclinic ratchets in higher dimension. ................................. 127
7.7 $(n_1 + n_2 + 2)$-cell coupled cell networks that may support heteroclinic ratchets in higher dimension $(n := n_1 + n_2 + 1)$. ................................. 128
7.8 A solution of (7.3.1) with $n_1 = 4$ and $n_2 = 3$ under small additive white noise (amplitude = $10^{-4}$). ................................. 129
List of Tables

6.1 Invariant subspaces forced by the coupling structure in Figure 6.1 for the system (6.1.1) ............................................. 88
6.2 Quotient networks for three dimensional invariant subspaces $V_3^3$, $V_3^1$, and $V_3^2$ of the 4-cell system (6.1.1) .................. 89
6.3 Adjacency matrix of the network in Figure 6.1 with eigenvalues and eigenvectors ......................................................... 93
“We shall definitely go to the moon.  
And maybe further away.  
To the places even  
Telescopes cannot show us.  
But shall we reach to the age.  
where no one suffers from hunger.  
no one threatens people.  
and no one gives commands?”  

Nazım Hikmet

An efficient approach to modelling complex physical systems is the network approach, namely assuming a system of interacting identical (or similar) entities [65]. Chemical reactions, animal populations, and the internet are examples of physical systems for which the network approach is useful (see [24, 65] and the references therein). Although individual entities can always be modelled more realistically as being non-identical, depending on the considered aspect different entities may behave almost identically. This approach articulates the effect of connectivity and the effect of the nature of individual entities on the considered aspect of the overall physical system. Examples of such aspects which can be effectively analysed using network approach are synchronization and clustering [43].

Systems considered in this thesis are networks of dynamical systems (units). Such networks can be thought as digraphs for which vertices correspond to dynam-
Chapter 1. Introduction

... units and directed edges correspond to the coupling between these dynamical units. The overall system is then called a coupled cell system and proved to be useful in modeling physical systems in different areas [43]. It is important to note that one can always use general techniques from the theory of dynamical systems for analyzing the overall system behavior; however, modeling networks of dynamical units using coupled cell systems has various advantages.

Firstly, the above-mentioned articulation of the effect of the connection structure and individual dynamics is possible. Secondly, some degenerate behavior of dynamical systems can occur robustly in coupled cell systems. One example of this is the heteroclinic behavior of dynamical systems which arises due to the existence of a heteroclinic cycle in the phase space. Although heteroclinic cycles are not structurally stable, they can exist robustly in coupled cell systems. In other words, for some coupled systems as long as the coupling structure is preserved, a heteroclinic cycle may persist under small perturbation in parameters of the system. Therefore, one might expect to observe heteroclinic behavior in a physical system with an underlying network structure (e.g., neural systems [71]).

Another interesting behavior of dynamical systems is the so-called “historic behavior”, where time averages of orbits do not converge. In this case, the system “keeps having new ideas about what it wants to do” [72]. Such irregular, messy [73] behavior is generally believed to be an anomaly. “There is however no justification for this belief” [81]. More precisely, the open question is “whether there are persistent classes of smooth dynamical systems such that the set of initial states
Chapter 1. Introduction

which give rise to orbits with historic behavior has positive Lebesgue measure” [81]. The Birkhoff Ergodic Theorem implies that the time averages of continuous observables along the orbits converge for almost all initial states and therefore exhibit non-historic behavior, where the term *almost all* is used with respect to an invariant measure [67, 75]. However, the Lebesgue measure may not be an invariant measure for the considered system and therefore the non-historic orbits predicted by the Birkhoff Ergodic Theorem may have Lebesgue measure zero.

One approach to the open problem mentioned above is to consider attractors of the system, i.e., the limit sets in the phase space to which *considerable* amount of trajectories *converge*. For a general class of dynamical systems, namely for dissipative systems, attractors always exist [28, Chapter 1]. However, what is understood by the terms “*considerable*” and “*converge*” can give rise to different definitions for attractors, such as global attractors [28], measure attractors [64], or statistical attractors [10]. Once the type of attractor is chosen, one can consider the *basin* of the attractor $B(A)$, namely the set of initial points for which the orbits converge to the attractor $A$. If for all continuous functions the time averages are equal to the space averages with respect to an invariant measure $\mu$ on $A$ for all initial states in a subset $V \subset B(A)$ of positive Lebesgue measure, the attractor $A$ is then called *ergodic* or *$\mu$-ergodic* [69]. Otherwise, it is called *non-ergodic*. The Bowen-Sinai-Ruelle Theorem gives sufficient conditions for an attractor to be ergodic [32, 75]. If Lebesgue almost all points in the basin of an attractor give rise to historic behavior, the attractor is then non-ergodic according to the
Chapter 1. Introduction

an aforementioned definition.

An example of a non-ergodic attractor is an attracting heteroclinic cycle. For a system with a globally attracting heteroclinic cycle, invariant measures are supported on the equilibria in the heteroclinic cycle, therefore the consequence of Birkhoff Ergodic Theorem is trivial. In fact, Gaunersdorfer [38] has proved that for such systems Lebesgue almost all points give rise to historic orbits. Yet, this example does not give a positive answer to the above-mentioned question, since heteroclinic cycles are not structurally stable.

Here, we ask a different question: If an attractor is non-ergodic for which observables does it exhibit historic behavior for Lebesgue almost all points in the basin? We show that the answer of this question is related to the notion of statistical attractor. Our conclusion is that if an observable has convergent time averages for all orbits on the statistical attractor and if the limit time averages are equal for these, then time averages converge for all points in the basin of attractor.

In the first part of this thesis (Part I), continuous flows on compact manifolds are considered, and it is proved that convergence of time averages for a set of initial states with positive Lebesgue measure depends only on the statistical attractors of the system. In order to prove this, we give a general definition for statistical attractor that reveals the relation between statistical attractors and essential $\omega$-limit sets [11]. The definition we introduce is more general than the original definition in [10] as it extends the original definition to the idea of non-maximal statistical attractors. Moreover, we introduce a new concept, namely the basin of statistical
attrac tors and prove some elementary results on the relations between measure attractors, statistical attractors, and their basins. The results are applied to systems with an attracting heteroclinic network, in particular, to such systems defined on a torus. In Chapters 6 and 7, we apply these general result to networks of coupled oscillators that admit a particular heteroclinic network. All material in Part I is original unless declared otherwise. New results in Chapter 2 are also published in a research article by myself and Peter Ashwin [55] for which I was lead author.

Many physical processes that are periodic in time can be modelled as nonlinear oscillators. As a type of coupled cell system, networks of coupled oscillators have been used as models of interacting units each of which produce rhythmic behavior. Due to interaction, synchronization of distinct units is possible. Recently, it was shown that the existence of heteroclinic cycles may give rise to interesting phenomena related to the synchronization of oscillators, such as slow switching [47] and extreme sensitivity to detuning [14, 13]. In this thesis, we describe a novel phenomenon, namely ratcheting of coupled oscillators which manifests itself as unidirectional effect of noise and/or detuning on frequency synchronization. Usually, if natural frequencies of the oscillators are close enough, coupling of oscillators gives rise to frequency synchronization. Normally, one expects that synchronization happens irrespective of which oscillator has a smaller natural frequency. However, we observe that a particular type of heteroclinic network, which we call heteroclinic ratchet, causes synchronization to take place only if a certain oscillator has a slightly smaller frequency than the other. Similarly, noise gives rise to desynchro-
Chapter 1. Introduction

Figure 1.1: Mechanical ratchet. A device that allows rotary motion on applying a torque in one direction but not in the opposite direction. Dynamics of coupled oscillators with an attracting heteroclinic ratchet resembles to the dynamics of a mechanical ratchet in that the oscillators desynchronize only when a positive (or negative) detuning of natural frequencies is applied.

oscillators in a certain way such that a certain oscillator ends up having a larger observed frequency after the loss of synchrony. We call this phenomenon ratcheting and the underlying heteroclinic network heteroclinic ratchet, since the observed consequences are similar to that of a mechanical ratchet, a device that allows rotary motion on applying a torque in one direction but not in the opposite direction (see Figure 1.1). Even without noise or detuning of natural frequencies, ratcheting can give rise to perpetual phase slips, and therefore phase differences of oscillators grow unboundedly. Nevertheless, using results from Part I, we show that an important time average, namely the average difference in observed frequencies, converges to zero resulting in frequency synchronization of oscillators, even though a typical time average does not converge.
Chapter 1. Introduction

In the second part of the thesis (Part II), an overview of coupled system theory and networks of coupled oscillators is given, as well as some recent results of the heteroclinic behavior in such systems. All material in Part II except Section 5.3.2 is a review of previous works.

In the last part of the thesis (Part III), ratcheting phenomenon is introduced. Existence of robust heteroclinic ratchets in networks of coupled oscillators gives rise to interesting effects related to synchronization. All material presented in Part III is original research. Some of the results in Part III are also published in [54].

During this PhD project we first observed the ratcheting phenomenon for coupled oscillator systems, and then studied the convergence of time averages to explain the frequency synchronization in the ratcheting phenomenon. However, this order is reversed in the thesis since the theoretical results on the convergence of time averages (Part I) are applied to the ratcheting phenomenon (Part III). Moreover, before explaining the ratcheting phenomenon with an example in Part III, we present a general definition for heteroclinic ratches in Chapter 3 and applied the results in Chapter 2 to this general notion of a heteroclinic ratchet.
Part I

Non-ergodicity
Convergence of time averages

Attractors are subsets of the phase space to which a considerable amount of trajectories converge. What is understood by 'considerable' and 'converge' gives rise to different concepts of attractor; see [64, 28] for different definitions of attractors. We will be interested in two of these, namely measure (or Milnor) and statistical (or Ilyashenko) attractors. We will describe an analogy between these as follows: measure attractors are defined considering the $\omega$-limit sets of trajectories [64]; in the same way, statistical attractor can be characterized using the essential $\omega$-limit sets [11] of trajectories (see Theorem 2.2.2). This unifying approach leads to a more general definition for a statistical attractor (see Definition 2.2.5).

Each attractor is accompanied with a set of physical measures that are related to the limits of the time averages of typical trajectories in the basin of attractor. If this set consists of one element (namely the pointwise SRB measure in the
Chapter 2. Convergence of time averages

terminology of [23]) then the attractor is ergodic; i.e., a typical trajectory in the basin has convergent time averages over continuous functions. For non-ergodic attractors, time averages may not exist along a typical trajectory. For instance, it is known that time averages of trajectories asymptotic to a heteroclinic cycle do not converge in general [38, 75], namely they show historic behavior (in the terminology of Ruelle [72]) (for more about non-convergence of time averages see [81, 80, 66, 51]). For a particular system and observable it may be non-trivial to check whether an average converges or not. We show that, in order to prove the convergence of the average or otherwise, one only needs to consider the values of the observable on statistical attractors for the flow (see Corollary 2.3.7).

Although not all the notions described in this chapter are original, new equivalent definitions for the essential $\omega$-limit set, statistical limit set and statistical attractor are introduced in Definitions 2.1.1, 2.2.3 and 2.2.5, respectively. All lemmas, theorems and corollaries are original if not indicated otherwise. These results are also published in [55].

**Notation:** We will use $\ell$ to denote Lebesgue (resp. Riemannian) measure on $\mathbb{R}^n$ (resp. on a compact manifold $X$). All subsets will be assumed to be measurable (Borel) unless otherwise stated. For two subsets $A$ and $B$, we will write as in [21] $A =_o B$ to mean $\ell((A \setminus B) \cup (B \setminus A)) = 0$, and $A \supset_o B$ to mean $\ell(B \setminus A) = 0$. 
2.1 Essential $\omega$-limit set

We consider throughout a continuous flow (or semiflow) $\gamma_t$ on a compact manifold $X$. For a trajectory passing through the point $x \in X$, the $\omega$-limit set of $x$ [56] is defined as follows:

$$\omega(x) = \bigcap_{T=0}^{\infty} \left( \bigcup_{t>T} \gamma_t x \right).$$

It follows that, for any $x \in X$, $\omega(x)$ is closed, connected, non-empty, flow-invariant. It consists of the accumulation points of sequences $\{\gamma_{T_k} x\}$ where $T_k \to \infty$. Hence, for a point $y \in \omega(x)$ a small neighborhood of $y$ is visited perpetually by the trajectory $\gamma_t x$. However, for some points in $\omega(x)$, the frequency of these visits may decrease with time such that as time proceeds the possibility of finding the trajectory near $x$ tends to zero. Another type of limit set for a trajectory that excludes such rarely-visited points is the essential $\omega$-limit set defined in [11] for maps and in [15] for flows. The essential $\omega$-limit set of a trajectory can be thought of as the set of points in the $\omega$-limit set whose arbitrary small neighborhoods are visited with a non-zero frequency. For an open set $U \subset X$ and a finite trajectory $\{\gamma_t x\}_{0 \leq t \leq T}$, the frequency of the trajectory being in $U$ is defined to be the function

$$\rho(x,U,T) = \frac{\ell(\{t: 0 \leq t \leq T, \gamma_t x \in U\})}{T}. \quad (2.1.1)$$

We give a slightly different definition of $\omega_{\text{ess}}$ to that in [15] and show in Theorem 2.3.3 that these definitions are equivalent.
Figure 2.1: Bowen’s example. There is one attracting heteroclinic cycle that consists of two hyperbolic equilibria $p_1$, $p_2$ and two heteroclinic trajectories $q_1$ and $q_2$. A typical trajectory in the region bounded by the heteroclinic cycle converges to the heteroclinic cycle.

**Definition 2.1.1 (essential $\omega$-limit set)** Let $\gamma_t$ be a continuous flow on a compact manifold $X$. For $z \in X$, let $U_z$ be the set of open neighborhoods of $z$. The essential $\omega$-limit set is defined as

$$\omega_{\text{ess}}(x) = \{ z \in X : \limsup_{t \to \infty} \rho(x, U, t) > 0, \forall U \in U_z \}. \quad (2.1.2)$$

If $z \in \omega_{\text{ess}}(x)$ then, for all $U \in U_z$, there exist arbitrary large values of $T$ for which $\gamma_T x \in U$. Hence,

$$\omega_{\text{ess}}(x) \subset \omega(x). \quad (2.1.3)$$

For example, consider the so-called Bowen’s Example [80] given in Figure 2.1, namely a heteroclinic cycle which attracts almost all trajectories in the interior of
the cycle. A typical trajectory with initial point $x(0) = x \neq s$ converges to the heteroclinic cycle, therefore $\omega(x) = \{p_1\} \cup q_1 \cup \{p_2\} \cup q_2$. However, the trajectory stays near the equilibria $p_1$ and $p_2$ longer at each turn, whereas the time it spends between equilibria, namely along the heteroclinic connections $q_1$ and $q_2$ remains constant. Hence $\omega_{\text{ess}}(x) = \{p_1, p_2\}$.

### 2.2 Statistical attractors

Milnor defines the likely limit set $\Lambda_{\text{likely}}$ (also called measure or Milnor attractor in the literature) as the smallest closed subset that contains the $\omega$-limit sets of almost all trajectories [64]. Similarly, Ilyashenko defines the statistical limit set $\Lambda_{\text{stat}}$ (also called statistical or Ilyashenko attractor in the literature) as the smallest closed subset for which almost all trajectories spend almost all time near $\Lambda_{\text{stat}}$ [10]. In the following, we show that statistical limit sets can also be defined using the essential $\omega$-limit set. In addition, we give a more general definition for a statistical attractor that covers non-maximal statistical attractors. Finally, some basic relations between measure and statistical attractors are proved.

**Definition 2.2.1 (statistical limit set [10, 50])** Let $\rho$ be defined as in (2.1.1). The statistical limit set $\Lambda_{\text{stat}}$ is the smallest closed subset of $X$ for which any open neighborhood $U$ of $\Lambda_{\text{stat}}$ satisfies $\lim_{t \to \infty} \rho(x, U, t) = 1$ for almost all $x \in X$, where $\rho$ is as in 2.1.1.
Using the concept of essential $\omega$-limit set, one can characterize the statistical limit set as follows: (proof given later)

**Theorem 2.2.2** The statistical limit set is the smallest closed subset that contains the essential $\omega$-limit sets of almost all trajectories.

This implies that one can define statistical attractors in analogy to measure attractors [64] by replacing the $\omega$-limit set with the essential $\omega$-limit set (see Section 2.2). For example, consider Bowen's example in Figure 2.1. The phase space $X$ is assumed to be the union of the heteroclinic cycle and the region bounded by the heteroclinic cycle. For almost all points in $X$, namely all points in $X$ except the points on the heteroclinic cycle and the unstable equilibrium $s$, the $\omega$-limit set is the whole cycle, whereas the essential $\omega$-limit set is the union of two equilibria $p_1$ and $p_2$. Therefore, the likely limit set of the system is the heteroclinic cycle while the statistical limit set is $\{p_1, p_2\}$.

We define statistical attractors in analogy to measure attractors [64]; namely, we say a closed set is a statistical attractor if it is the minimal closed subset that contains the essential $\omega$-limit set of almost all points in a given positive measure subset of $X$. If one replaces the term “essential $\omega$-limit set” with “$\omega$-limit set”, one obtains the definition for measure attractors. Both of these attractors can be defined using the following set valued set functions:

**Definition 2.2.3** For a given subset $Y$ of $X$, the likely limit set of $Y$ and the
Chapter 2. Convergence of time averages

statistical limit set of \( Y \) are defined, respectively, as follows:

\[
\Lambda_M(Y) := \bigcap_{V=\circ Y} \left( \bigcup_{x \in V} \omega(x) \right)
\]

(2.2.1)

\[
\Lambda_S(Y) := \bigcap_{V=\circ Y} \left( \bigcup_{x \in V} \omega_{\text{ess}}(x) \right).
\]

(2.2.2)

**Lemma 2.2.4** Let \( \zeta \) be any map from \( X \) to the set of closed subsets of \( X \). Let \( \{U_i\} \) be a countable base for \( X \). Consider two subsets \( A, Y \subset X \); then the following statements are equivalent:

(a) \( A^c = \bigcup \{U_i: U_i \cap \zeta(x) = \emptyset \text{ for } \ell\text{-a.e. } x \in Y\} \);

(b) \( A = \bigcap_{V=\circ Y} \left( \bigcup_{x \in V} \zeta(x) \right) \);

(c) \( A \) is the smallest closed subset of \( X \) that contains \( \zeta(x) \) for almost every point in \( Y \). In other words, there exists a full measure subset \( W \) of \( Y \) such that 
\[
A = \bigcup_{x \in W} \zeta(x), \text{ and for any other } W' \subset Y \text{ with } W' =_0 Y, \ A \subset \bigcup_{x \in W'} \zeta(x);
\]

where \( A^c \) denotes the complement of \( A \) in \( X \).

**Proof.** (a) \( \Leftrightarrow \) (b): We need to show that

\[
U^c := \bigcup \{U_i: U_i \cap \zeta(x) = \emptyset \text{ for almost all } x \in Y\}^c = \bigcap_{V=\circ Y} \left( \bigcup_{x \in V} \zeta(x) \right).
\]

We show the contrapositives for both inclusions:

“\( \supset \)” Assume that \( x \notin \bigcap_{V=\circ Y} \left( \bigcup_{x \in V} \zeta(x) \right) \), then there exists \( V =_o Y \) such that
Chapter 2. Convergence of time averages

$x \notin \bigcup_{x \in V} \zeta(x)$. Then, there exists an open neighborhood $U \in \{U_i\}$ of $x$ such that $U \cap \bigcup_{x \in V} \zeta(x) = \emptyset$. Hence, $U \cap \zeta(x) = \emptyset$ for almost all $x$ in $Y$, namely $x \notin \mathcal{U}^c$.

“$\supset$”. Assume $x \notin \mathcal{U}^c$, that is, there exists an open neighborhood $U \in \{U_i\}$ of $x$ such that $U \cap \zeta(x) = \emptyset$ for almost all $x$ in $Y$. Hence, $U \cap \bigcup_{x \in V} \zeta(x) = \emptyset$ for some $V =_o Y$. Since $U$ is open, we have $U \cap \bigcup_{x \in V} \zeta(x) = \emptyset$. Thus, $x \notin \bigcap_{V =_o Y} \left( \bigcup_{x \in V} \zeta(x) \right)$. (b) \(\Rightarrow\) (c): (b) implies that $A$ is closed and contained in any closed subset that contains $\zeta(x)$ for almost every $x$ in $Y$. We only need to show that $A \supset \bigcup_{x \in V} \zeta(x)$ for some $V \subset Y$ with $V =_o Y$. From (a), for each $U_i \in \{U_i\}$ with $U_i \subset \mathcal{U}$, there exists $V_i \subset Y$ with $V_i =_o Y$ and $U_i \cap \bigcup_{x \in V_i} \zeta(x) = \emptyset$. Let $W$ be the intersection of such (at most countably many) $V_i$’s. Hence, $W \subset Y$, $W =_o Y$ and $\bigcup_{x \in W} \zeta(x) \subset \mathcal{U}^c = A$. (c) \(\Rightarrow\) (b): This follows from the statement of (c).

By Lemma 2.2.4, $\Lambda_M(Y)$ (resp. $\Lambda_S(Y)$) is the smallest closed subset of $X$ that contains the $\omega$-limit set (resp. essential $\omega$-limit set) of almost every point in $Y$. Now, we can define a statistical attractor in analogy with the definition of a measure attractor as follows:

**Definition 2.2.5 (Measure and statistical attractor)** Let $\Lambda_M$ and $\Lambda_S$ be defined as in Definition 2.2.3. A subset $A$ of $X$ is called a measure attractor if there exists a subset $V \subset X$, such that $\ell(V) > 0$ and $A = \Lambda_M(V)$. $A$ is called statistical attractor if there exists a subset $V \subset X$, such that $\ell(V) > 0$ and $A = \Lambda_S(V)$.

The notions of measure (or Milnor) and statistical (or Ilyashenko) attractor are not new. The novelty of the Definition 2.2.5 is that the statistical attractor is
Chapter 2. Convergence of time averages

defined in terms of essential $\omega$-limit set and that non-maximal statistical attractors
are considered (when $\ell(V) < \ell(X)$). Note that the maximal measure attractor
$\Lambda_M(X)$, that is, the smallest closed subset that contains the $\omega$-limit sets of almost
all points in $X$ is the likely limit set $\Lambda_{\text{likely}}$. Similarly, by Theorem 2.2.2, the
maximal statistical attractor $\Lambda_S(X)$ is equal to the statistical limit set $\Lambda_{\text{stat}}$. In
other words, Definition 2.2.5 covers the previous definition of a statistical limit set,
introduced by Ilyashenko [10] as a special case (the latter also called the “statistical
attractor”; see [49, 57]), but is more general.

**Proof of Theorem 2.2.2.** By definition, $z \notin \omega_{\text{ess}}(x)$ if and only if there exists
an open neighborhood of $U$ of $z$ such that $\lim_t \rho(x, U, t) = 0$. In other words, for any
open $U$, $\omega_{\text{ess}}(x) \cap U = \emptyset$ if and only if $\lim_t \rho(x, U, t) = 0$. In addition, from Definition
2.2.1, $\Lambda_{\text{stat}}^c = \bigcup \{ U \subset X : U \text{ is open and } \lim_t \rho(x, U, t) = 0 \text{ for } \ell\text{-a.e. } x \in X \}$. 
Therefore, $\Lambda_{\text{stat}}^c = \bigcup \{ U \subset X : U \text{ is open and } \omega_{\text{ess}}(x) \cap U = \emptyset \text{ for } \ell\text{-a.e. } x \in X \}$. 
Thus, the statement follows from Lemma 2.2.4.

We say a measure (statistical) attractor is *minimal* if it does not strictly contain
any other measure (statistical) attractor. Note that both measure and statistical
attractors are closed and invariant under the flow. We can define the measure basin $B_M(A)$ and the statistical basin $B_S(A)$ for a subset $A$ as follows:

$$B_M(A) = \{ x \in X \mid \omega(x) \subset A \} \quad (2.2.3)$$

$$B_S(A) = \{ x \in X \mid \omega_{\text{ess}}(x) \subset A \} \quad (2.2.4)$$

27
Lemma 2.2.6 Let $\Lambda_M$, $\Lambda_S$, $\mathcal{B}_M$ and $\mathcal{B}_S$ be defined as in (2.2.1), (2.2.2), (2.2.3) and (2.2.4), respectively. Then the following statements hold, where $A, Y_i, Y$ are (Borel) subsets of $X$:

(i) $\mathcal{B}_M(A) \subset \mathcal{B}_S(A)$ for any subset $A$ of $X$.

(ii) $\Lambda_S(Y) \subset \Lambda_M(Y)$ for any subset $Y$ of $X$.

(iii) If $Y_1 \supset Y_2$, then $\Lambda_M(Y_2) \subset \Lambda_M(Y_1)$ and $\Lambda_S(Y_2) \subset \Lambda_S(Y_1)$.

(iv) For any subset $Y$, $\mathcal{B}_M(\Lambda_M(Y)) \supset Y$ and $\mathcal{B}_S(\Lambda_S(Y)) \supset Y$.

(v) For any subset $A$, $\Lambda_M(\mathcal{B}_M(A)) \subset \overline{A}$ and $\Lambda_S(\mathcal{B}_S(A)) \subset \overline{A}$.

(vi) If $A$ is a measure attractor, then $\Lambda_M(\mathcal{B}_M(A)) = A$. If $A$ is a statistical attractor, then $\Lambda_S(\mathcal{B}_S(A)) = A$.

Proof. (i) and (ii) follow from (2.1.3). (iii) From Lemma 2.2.4, there exists $W_1 \subset Y_1$ with $\ell(W_1) = \ell(Y_1)$ such that $\Lambda_M(Y_1) = \bigcup_{x \in W_1} \omega(x)$. Let $W_2 := W_1 \cap Y_2$, then $W_2 \subset Y_2$ and $\ell(W_2) = \ell(Y_2)$. Hence, using Lemma 2.2.4 again, $\Lambda_M(Y_2) \subset \bigcup_{x \in W_2} \omega(x) \subset \bigcup_{x \in W_1} \omega(x) = \Lambda_M(Y_1)$. Same argument holds if $\Lambda_M$ and $\omega$ are replaced with $\Lambda_S$ and $\omega_{\text{stat}}$, respectively. (iv) From Lemma 2.2.4, there exists $W \subset Y$ with $W = \omega_\infty Y$ such that $\Lambda_M(Y) = \bigcup_{x \in W} \omega(x)$. Therefore, $\mathcal{B}_M(\Lambda_M(Y)) \supset W$. This implies $\ell(Y \setminus \mathcal{B}_M(\Lambda_M(Y))) \leq \ell(Y \setminus W) = 0$. Hence $\mathcal{B}_M(\Lambda_M(Y)) \supset Y$. Same argument holds if $\Lambda_M$, $\omega$ and $\mathcal{B}_M$ are replaced with $\Lambda_S$, $\omega_{\text{stat}}$ and $\mathcal{B}_S$, respectively. (v) Consider $x \in \Lambda_M(\mathcal{B}_M(A))$. If $\mathcal{B}_M(A)$ has zero measure then $\Lambda_M(\mathcal{B}_M(A)) = \emptyset$. Hence, we assume $\mathcal{B}_M(A)$ has positive measure. Choose a subset $V \subset \mathcal{B}_M(A)$
with $V = _o B_M(A)$. Then $x \in \bigcup_{y \in V} \omega(y)$. Since for all $y \in B_M(A)$, $\omega(y) \subset A$, we have $\bigcup_{y \in V} \omega(y) \subset \overline{A}$. Hence, $x \in \overline{A}$. Same argument holds if $\Lambda_M$, $\omega$ and $B_M$ are replaced with $\Lambda_S$, $\omega_{\text{ess}}$ and $B_S$, respectively. (vi) Assume $A$ is a measure attractor. Then $A$ is closed and therefore (v) implies $\Lambda_M(B_M(A)) \subset A$. We will show that $\Lambda_M(B_M(A)) \supset A$. Since $A$ is a measure attractor, there exists a positive measure subset $V$ such that $A = \Lambda_M(V)$. By Lemma 2.2.4, there exists $W \subset V$ such that $\Lambda_M(V) = \bigcup_{x \in W} \omega(x)$. Hence, for all $x \in W$, $\omega(x) \subset A$, therefore, $W \subset B_M(A)$. From (iii), this implies $A = \Lambda_M(V) = \Lambda_M(W) \subset \Lambda_M(B_M(A))$. Similarly, assume that $A$ is a statistical attractor. Then $A$ is closed and therefore (v) implies $\Lambda_S(B_S(A)) \subset A$. We need to show that $\Lambda_S(B_S(A)) \supset A$. Since $A$ is a statistical attractor, there exists a positive measure subset $V$ such that $A = \Lambda_S(V)$. By Lemma 2.2.4, there exists $W \subset V$ such that $\Lambda_S(V) = \bigcup_{x \in W} \omega_{\text{ess}}(x)$. Hence, for all $x \in W$, $\omega_{\text{ess}}(x) \subset A$, therefore, $W \subset B_S(A)$. From (iii), this implies $A = \Lambda_S(V) = \Lambda_S(W) \subset \Lambda_S(B_S(A))$. 

Measure attractors and statistical attractors can be related to each other as shown in Lemma 2.2.6. However, they are not in one-to-one correspondence as we will see in Example 2.2.9 below. Nevertheless, we show that for each measure attractor there is a smaller statistical attractor, and for each statistical attractor there is a larger measure attractor:

**Theorem 2.2.7** Suppose that $\gamma_t$ is a continuous flow on a compact manifold $X$.

(a) If $A$ is a measure attractor for the flow, then $A_S := \Lambda_S(B_M(A))$ is a statistical
attraction contained in $A$ with $\mathcal{B}_S(A_S) \supset \mathcal{B}_M(A)$.

(b) If $A$ is a statistical attractor for the flow, then $A_M := \Lambda_M(\mathcal{B}_S(A))$ is a measure attractor that contains $A$ with $\mathcal{B}_M(A_M) \supset \mathcal{B}_S(A)$.

Proof. (a) If $A$ is a measure attractor, then $\mathcal{B}_M(A)$ has positive measure and therefore $A_S := \Lambda_S(\mathcal{B}_M(A))$ is a statistical attractor. From Lemma 2.2.6(iv), it follows that $\mathcal{B}_S(A_S) = \mathcal{B}_S(\Lambda_S(\mathcal{B}_M(A))) \supset \mathcal{B}_M(A)$. It remains to show that $A_S \subset A$. From Lemma 2.2.6(ii) and (vi), $A_S = \Lambda_S(\mathcal{B}_M(A)) \subset \Lambda_M(\Lambda_S(\mathcal{B}_M(A))) = A$. (b) If $A$ is a statistical attractor, then $\mathcal{B}_S(A)$ has positive measure and therefore $A_M := \Lambda_M(\mathcal{B}_S(A))$ is a measure attractor. From Lemma 2.2.6(iv), it follows that $\mathcal{B}_M(A_M) = \mathcal{B}_M(\Lambda_M(\mathcal{B}_S(A))) \supset \mathcal{B}_S(A)$. It remains to show that $A_M \supset A$. From Lemma 2.2.6(ii) and (vi), $A_M = \Lambda_M(\mathcal{B}_S(A)) \supset \Lambda_S(\mathcal{B}_S(A)) = A$. 

The simplest examples where statistical attractors are different than measure attractors are systems such as Bowen's example mentioned in Section 2.1. We discuss two other illustrative examples for the remainder of this section.

Example 2.2.8 The heteroclinic cycle illustrated in Figure 2.2 arises as a minimal measure attractor of the flow on $\mathbb{R}^3$ given by

\[
\begin{align*}
\dot{x} &= kx + (ax^2 + by^2 + cz^2)x \\
\dot{y} &= ky + (ay^2 + bz^2 + cx^2)y \\
\dot{z} &= kz + (az^2 + bx^2 + cy^2)z
\end{align*}
\]
Chapter 2. Convergence of time averages

where the real parameters are chosen such that \( k > 0, a < 0 \) and \( b < -c < 0 \) \cite{26, 46}. The system admits the rotation symmetries \((x, y, z) \rightarrow (z, x, y)\) and the reflection symmetries \((x, y, z) \rightarrow (\pm x, \pm y, \pm z)\). The reflection symmetries imply that each octant is invariant and there exist symmetric copies of this cycle that attract almost all initial conditions in each octant. These are clearly minimal measure attractors. On the other hand, the statistical attractor in the positive octant consist of the equilibria \( p_1, p_2, \) and \( p_3 \) only and in each region there is a different minimal statistical attractor that consist of three equilibria each of which is a symmetric copy of \( p_1, p_2, \) and \( p_3 \).

In Example 2.2.8, there is a one-to-one map between measure attractors and statistical attractors. The following example shows that this is not always the case.

**Example 2.2.9** Figure 2.3 shows the phase portrait of a flow with a measure attractor that contains six smaller measure attractors. Three of them are minimal measure attractors: one heteroclinic cycle \( \{p_1, q_2, p_2, q_3\} \) and two homoclinic cycles \( \{p_1, q_1\} \) and \( \{p_2, q_1\} \). The statistical limit set consists of two points \( p_1 \) and \( p_2 \) each of which is a minimal statistical attractor. There are two minimal statistical attractors but three minimal measure attractors. As a result, unlike Example 2.2.8, there is no one-to-one map between measure and statistical attractors in this case.
Chapter 2. Convergence of time averages

Figure 2.2: A trajectory for Example 2.2.8 that converges to an attracting heteroclinic cycle on the boundary of the positive octant of $\mathbb{R}^3$. The $\omega$-limit set of each point except $p_4$ in the interior of the octant $\{x, y, z > 0\}$ is the whole cycle. Due to the symmetry, there exist seven symmetric copies of this cycle in the other regions that are attracting almost all points in their interiors and therefore are measure attractors. Similarly, there are eight statistical attractors each of which consists of three fixed points contained in a heteroclinic cycle. There is a one-to-one map between measure attractors (cycles) and statistical attractors (triples of equilibria on cycles) in this example.
Chapter 2. Convergence of time averages

Figure 2.3: An invariant set \( \{p_1, p_2, q_1, q_2, q_3, q_4\} \) containing seven measure attractors and three statistical attractors. The measure attractors are two homoclinic cycles \( \{p_1, q_1\} \) and \( \{p_2, q_4\} \), the heteroclinic cycle \( \{p_1, q_2, p_2, q_3\} \) and various combinations of these, namely \( \{p_1, p_2, q_1, q_2, q_3\} \), \( \{p_1, p_2, q_2, q_3, q_4\} \), \( \{p_1, p_2, q_1, q_4\} \) and \( \{p_1, p_2, q_1, q_2, q_3, q_4\} \). The statistical attractors are \( \{p_1\} \), \( \{p_2\} \), and \( \{p_1, p_2\} \). Hence, in this example there is no one-to-one map between measure and statistical attractors.
2.3 Convergence of time averages

In this section, we attempt to precisely characterize those observables whose time averages converge for a given statistical attractor. Our main theoretical result will be Theorem 2.3.2 and Theorem 2.3.6, In terms of applications, we give two corollaries that have more easily checkable assumptions.

Time averages of continuous functions along trajectories can be thought as functionals on the space of continuous functions. Hence the convergence of time averages for all continuous functions is related to weak* limits of such functionals. It is well-known that these limit functionals correspond to certain invariant measures as we explain in the next paragraph. Using a relation between such measures and the essential \( \omega \)-limit set (Theorem 2.3.3), we explain the relation between the essential \( \omega \)-limit set and convergence of time averages (Theorem 2.3.6), and therefore the relation between statistical attractor and convergence of time averages (Corollary 2.3.7).

Consider the measures \( \mu_{x,T} = \frac{1}{T} \int_0^T \delta_{\gamma_t x} \, dt \) where \( T > 0 \) and \( \delta_x \) is the Dirac measure. Note that

\[
\mu_{x,T}(U) = \rho(x, U, T),
\]

where \( \rho \) is defined in (2.1.1). Let \( \mathcal{C}(X) \) denote the set of continuous functionals on \( X \). For each \( f \in \mathcal{C}(X) \) we have \( \int_X f \, d\mu_{x,T} = \frac{1}{T} \int_0^T f(\gamma_t x) \, dt \). Define a functional
Chapter 2. Convergence of time averages

on $\mathcal{C}(X)^*$ by $\varphi_{x,T} : f \to \int_X f d\mu_{x,T}$. Then,

$$|\varphi_{x,T}| = \sup_{\|f\|=1} \frac{1}{T} \int_0^T |\gamma(t)x| dt \leq \sup_{\|f\|=1} \|f\| = 1.$$ 

Since the unit $(\cdot, \cdot)$-ball in $\mathcal{C}(X)^*$ is weak* compact (Alaoglu’s theorem), the set of accumulation points of $\varphi_{x,T}$ as $T \to \infty$ in the weak* topology of $\mathcal{C}(X)^*$, namely,

$$\Theta(x) = \bigcap_{T > 0} \left\{ \varphi_{x,T} : \tilde{T} > T \right\}$$  (2.3.2)

is non-empty and bounded, where the closure is in the weak* topology. The Riesz Representation Theorem implies that for each $\tilde{\varphi} \in \Theta(x)$ there exists a unique Borel probability measure $\mu(\tilde{\varphi})$ such that

$$\tilde{\varphi}(f) = \int_X f d\mu(\tilde{\varphi}).$$  (2.3.3)

The set of such measures $\{\mu(\tilde{\varphi}) : \tilde{\varphi} \in \Theta(x)\}$ can also be written as

$$\Omega(x) = \bigcap_{T > 0} \left\{ \mu_{x,T} : \tilde{T} > T \right\},$$  (2.3.4)

where the closure is under the weak topology of measures. We say a sequence of measures $\{\mu_k\}$ converges weakly to the measure $\mu$ ($\mu_k \rightharpoonup \mu$) if and only if

$$\lim_k \int f d\mu_k = \int f d\mu$$

for every continuous function $f$ [22]. Since $\Theta(x)$ is non-empty, $\Omega(x)$ is also non-empty. We can use $\Theta(x)$ and $\Omega(x)$ to classify the behavior of time averages of continuous observables as follows.
Chapter 2. Convergence of time averages

Lemma 2.3.1 For every \( f \in C(X) \),

\[
\bigcap_{T > 0} \left\{ \frac{1}{T} \int_{0}^{T} f(\gamma t x) \, dt : \tilde{T} > T \right\} = \{ \varphi(f) : \varphi \in \Theta(x) \},
\]

where \( \Theta(x) \) is defined in (2.3.2).

Proof. We prove equality in two stages. “\( \subseteq \)” Let \( \tilde{f} \in \mathbb{R} \) be a limit point of \( \frac{1}{T} \int_{0}^{T} f(\gamma t x) \, dt \) as \( T \to \infty \). Then, there exists a sequence \( \{t_k\} \to \infty \) such that \( \frac{1}{t_k} \int_{0}^{t_k} f(\gamma t x) \, dt = \int_{X} f \, d\mu_{x,t_k} = \varphi_{x,t_k}(f) \to \tilde{f} \). By Alaoglu’s theorem, there exists a subsequence \( \{t_{n_k}\} \) such that \( \{\varphi_{x,t_{n_k}}\} \) converges to some functional, say \( \tilde{\varphi} \), in weak* topology. That is, for each \( f \in C(X) \), \( \varphi_{x,t_{n_k}}(f) \to \tilde{\varphi}(f) \). Therefore, \( \tilde{f} = \tilde{\varphi}(f) \). It is clear that \( \tilde{\varphi} \in \Theta(x) \). “\( \supseteq \)” Let \( \tilde{f} = \tilde{\varphi}(f) \) for some \( \tilde{\varphi} \in \Theta(x) \). By the definition of \( \Theta(x) \), there exists a sequence \( \{t_k\} \to \infty \) such that \( \varphi_{x,t_k}(f) \to \tilde{\varphi}(f) \) for all \( f \in C(X) \). Therefore, \( \frac{1}{t_k} \int_{0}^{t_k} f(\gamma t x) \, dt \to \tilde{f} \).

Using (2.3.3), we can rewrite (2.3.5) as

\[
\bigcap_{T > 0} \left\{ \frac{1}{T} \int_{0}^{T} f(\gamma t x) \, dt : \tilde{T} > T \right\} = \left\{ \int_{X} f \, d\mu : \mu \in \Omega(x) \right\}.
\]

We can now state the main result of this section:

Theorem 2.3.2 Let \( \gamma_t \) be a continuous flow on a compact metric space \( X \) and \( x(t) \subset X \) be a trajectory with \( x_0 = x(0) \). Let \( f : X \to \mathbb{R} \) be a continuous function.
Then the following limit
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\gamma_t(x_0)) \, dt = \bar{f} \]
exists, if and only if,
\[ \int_X f \, d\mu = \bar{f} \text{ for all } \mu \in \Omega(x_0). \]

\textbf{Proof.} This follows as a special case of Lemma 2.3.1 where the sets on both sides of (2.3.6) reduce to single points.

Theorem 2.3.2 implies that the time average of a given observable exists if and only if the observable has a constant integral with respect to all measures in \( \Omega(x) \). We now show the relation between \( \Omega(x) \) and the essential \( \omega \)-limit set \( \omega_{\text{ess}}(x) \).

\textbf{Theorem 2.3.3} \(^1\) Let \( \gamma_t \) be a continuous flow on a compact manifold \( X \) and \( \Omega(x) \) be defined as in (2.3.4). Then, for all \( x \in X \),
\[ \omega_{\text{ess}}(x) = \bigcup_{\mu \in \Omega(x)} \text{supp} \, \mu. \]  
(2.3.7)

Note that Theorem 2.3.3 implies that our definition of essential \( \omega \)-limit set (Definition 2.1.1) is equivalent to the original definition in [15]. Definition 2.1.1 is in some sense simpler in that it does not depend on measure theoretical notions such as weak convergence. It also makes the relation between the concepts \textit{essential} \( \omega \)-limit set and \( \omega_{\text{ess}}(x) \) explicit.

\(^1\) (2.3.7) was given in [11] for maps without proof.
Chapter 2. Convergence of time averages

$\omega$-limit set and statistical attractor clearer.

To prove Theorem 2.3.3 we use the following lemmas:

**Lemma 2.3.4** Let $\Omega$ be any set of measures on $X$. Then $z \in \bigcup_{\mu \in \Omega} \text{supp } \mu$ if and only if for every open neighborhood $U$ of $z$, there exists a $\mu \in \Omega$ such that $\mu(U) > 0$.

**Proof.** ("only if") Assume that there exists a sequence $\{z_k\} \to z$ such that for all $k$, $z_k \in \text{supp } \mu_k$ where $\mu_k \in \Omega$. Then, for each open neighborhood $U$ of $z$ there exists $K > 0$ such that $z_K \in U$. Choose an open neighborhood $V$ of $z_K$ such that $V \subset U$. Since $\mu_K(V) > 0$, then $\mu_K(U) > 0$. ("if") Assume that for every open neighborhood $U$ of $z$ there exists a $\mu \in \Omega$ such that $\mu(U) > 0$. Hence, $\text{supp } \mu \cap U \neq \emptyset$, that is, there exists a $\bar{z} \in U$ such that $\bar{z} \in \text{supp } \mu$. This implies that $z \in \bigcup_{\mu \in \Omega} \text{supp } \mu$. \hfill \blacksquare

We quote the following result from [22, Theorem 2.1] without proof.

**Lemma 2.3.5** ([22]) Let $\mu$ and $\mu_k$, $k = 1, 2, \ldots$ be Borel probability measures. The following statements are equivalent:

- $\mu_k \rightharpoonup \mu$, (i.e. $\mu_k$ converges weakly to $\mu$).
- $\liminf_{k} \mu_k(U) \geq \mu(U)$ for every open set $U$.
- $\limsup_{k} \mu_k(F) \leq \mu(F)$ for every closed set $F$.

**Proof of Theorem 2.3.3.** ("$\supset$") Assume $z \in \bigcup_{\mu \in \Omega(x)} \text{supp } \mu$. From Lemma 2.3.4, for each open neighborhood $U$ of $z$, there exists a $\mu \in \Omega(x)$ such that
Chapter 2. Convergence of time averages

\( \mu(U) > 0 \). Note that \( \mu \in \Omega(x) \) implies there exists a sequence \( \{T_k\} \to \infty \) such that \( \mu_{x,T_k} \rightharpoonup \mu \). Hence, from Lemma 2.3.5 and Equation (2.3.1), \( \liminf \rho(x, U, T_k) = \mu_{x,T_k}(U) \geq \mu(U) > 0 \). Therefore, \( \limsup_{t \to \infty} \rho(x, U, t) > 0 \), and from Definition 2.1.1, \( z \in \omega_{\text{ess}}(x) \). (\( \subset \)) Assume that \( z \in \omega_{\text{ess}}(x) \) and \( U \) is an arbitrary open neighborhood of \( z \). Let \( U' \) and \( F \) be open and closed neighborhoods of \( z \), respectively, that satisfy \( U' \subset F \subset U \). From Definition 2.1.1, \( \limsup_{t \to \infty} \rho(x, U', t) > 0 \) since \( z \in U' \) and \( U' \) is open. Then, \( \limsup_{t \to \infty} \rho(x, F, t) \geq \limsup_{t \to \infty} \rho(x, U', t) > 0 \). Therefore, there exists a sequence \( \{t_k\} \to \infty \) such that \( \lim_k \rho(x, F, t_k) = \lim_k \mu_{x,t_k}(F) > 0 \). By compactness, there exists a subsequence \( \{t_{k_m}\} \to \infty \) such that \( \mu_{x,t_{k_m}} \) converges weakly to a measure \( \mu \in \Omega(x) \). From Lemma 2.3.5, \( \limsup \mu_{x,t_{k_m}}(F) \leq \mu(F) \). Hence,

\[
\mu(U) \geq \mu(F) \geq \limsup \mu_{x,t_{k_m}}(F) = \lim_k \mu_{x,t_k}(F) > 0.
\]

Finally, Lemma 2.3.4 implies \( z \in \bigcup_{\mu \in \Omega} \text{supp } \mu \).

Using the Ergodic Decomposition Theorem [56, Theorem 4.1.12] one can restrict the condition on measures in Theorem 2.3.2 to ergodic measures supported on the essential \( \omega \)-limit set. Let \( \mathcal{E}(X) \) denote the set of ergodic \( \gamma_l \)-invariant probability measures supported on \( X \). Namely, \( \mathcal{E}(X) = \{\mu \in \mathcal{M}(X) : \text{supp}(\mu) \subset X\} \), where \( \mathcal{M}(X) \) is the set of invariant ergodic measures of the flow \( (X, \gamma_l) \). Since the supports of the measures in \( \Omega(x) \) are contained in \( \omega_{\text{ess}}(x) \), the Ergodic Decompo-
Chapter 2. Convergence of time averages

Theorem 2.3.6 Suppose that $\gamma_t$ is a continuous flow on a compact manifold $X$. Assume that, for a given continuous function $f : X \to \mathbb{R}$, there exists a constant $\bar{f} \in \mathbb{R}$ such that, for all $y \in \omega_{\text{ess}}(x_0)$, $\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\gamma_t y) \, dt = \bar{f}$. Then the limit $\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\gamma_t x_0) \, dt$ exists and equals $\bar{f}$.

**Proof.** Let $\mu$ be an ergodic measure supported on $\omega_{\text{ess}}(x_0)$, namely $\mu \in \mathcal{E}(\omega_{\text{ess}}(x_0))$. From a corollary of Birkhoff Ergodic Theorem [75, p. 223], there exists a point $y \in \omega_{\text{ess}}(x_0)$ such that $\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\gamma_t y) \, dt = \int f \, d\mu$. Then, by assumption, $\int f \, d\mu = \bar{f}$. Let $\mathcal{F}$ denote the set of accumulation points of $\frac{1}{T} \int_0^T f(\gamma_t x_0) \, dt$ as $T \to \infty$. From (2.3.6) $\mathcal{F} = \{ \int_X f \, d\mu : \mu \in \Omega(x_0) \}$. Using (2.3.8), we conclude

$$\mathcal{F} \subset \text{conv} \left( \left\{ \int_X f \, d\mu : \mu \in \mathcal{E}(\omega_{\text{ess}}(x_0)) \right\} \right) = \text{conv}(\{ \bar{f} \}) = \{ \bar{f} \}.$$

Theorem 2.3.6 gives a sufficient condition for the existence of time averages of continuous functions along a trajectory. In order to see that this condition is not
Chapter 2. Convergence of time averages

necessary, one can consider an Axiom-A attractor (see [32] for definition) for which there is an SRB measure \( \mu \) supported on the attractor \( A \). The essential \( \omega \)-limit set of a typical point in the basin is then the whole attractor \( A \), and one can construct a continuous function that has different time averages on the periodic orbits in \( A \) and therefore does not satisfy the condition in Theorem 2.3.6.

The results given above are related to statistical attractors in the following way: If an observable \( f \) has a constant time average along all trajectories in a statistical attractor, then there is a positive measure subset of initial states for which time averages of \( f \) exist. Both the following corollaries follow directly from Theorem 2.3.6.

**Corollary 2.3.7** If \( A \) is a statistical attractor and \( \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(\gamma t x) dt = \bar{f} \) for all \( x \in A \). Then, for all \( x \in B_S(A) \), \( \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(\gamma t x) dt \) exists and is equal to \( \bar{f} \).

**Proof.** If \( x \in B_S(A) \), then \( \omega_{\text{stat}}(x) \subset A \). Therefore the assumption of Theorem 2.3.6 is satisfied.

**Corollary 2.3.8** If time averages of an observable \( f \) have the same limit for all trajectories in the statistical limit set, then time averages of \( f \) exist for almost all trajectories.

**Proof.** Since the statistical limit set is a statistical attractor whose basin is equal to the whole phase space up to a zero measure set, Corollary 2.3.7 implies
Chapter 2. Convergence of time averages

the result.
Heteroclinic networks and ratchets

Heteroclinic networks are invariant sets in phase space that contain heteroclinic cycles, a sequence of trajectories forming a cycle. This phenomenon is highly degenerate, but it has important effects on the dynamics. Heteroclinic networks exist, in general, as non-ergodic attractors, hence giving rise to non-convergent, namely historic behaviour in the terminology of [72]. They also have interesting consequences in coupled oscillator systems (see Section 5.4 and Part III).

In Section 3.1, we give a formal definition of heteroclinic networks from [15]. The results of Chapter 2 have direct implications to heteroclinic networks; these are given in Theorem 3.2.2, Theorem 3.4.4 and in Theorem 3.5.3. We consider heteroclinic networks on a torus and therefore define some properties of invariant sets on a torus in Sections 3.3 and 3.4. Finally, a particular type of heteroclinic network on a torus is defined in Section 3.5, whose effects on the dynamics of
coupled oscillators will be analysed in the next chapters.

The material in Sections 3.1, 3.2 and 3.3 are introductory and not original except Theorem 3.2.2, which is a consequence of the results in Chapter 2. All material stated in Section 3.4 and Section 3.5 is original and unpublished.

3.1 Intrinsic definition for heteroclinic networks

In this section, we summarize the definitions of heteroclinic networks given in [15]. For this we need to define various notions related to the recurrence properties of a flow, such as pseudo-trajectory, recurrent set, chain-recurrent set, etc.

Let \((X, \| \cdot \|)\) be a compact connected metric space. Consider a continuous flow on \(X\):

\[ \gamma_t : X \to X , \quad t \in \mathbb{R} . \tag{3.1.1} \]

We refer to (3.1.1) as \((X, \gamma)\), or in short as \(X\) if the flow \(\gamma\) is clear from the context. A trajectory of \((X, \gamma)\) is a function \(x : \mathbb{R} \to X\) that satisfies \(x(t) = \gamma_tx_0, \quad t \in \mathbb{R}\) for some \(x_0 \in X\). A finite trajectory is a function \(x : [t_0, T] \to X\), where \(t_0, T \in \mathbb{R}\) and \(t_0 < T\) such that \(x(t) = \gamma_{t-t_0}(x(t_0)), \quad t \in [t_0, T]\). We sometimes denote a finite trajectory as \(x([t_0, T])\) if the time domain needs to be stated. A finite trajectory \(x([t_0, T])\) is called periodic to \(y\) if \(x(t_0) = x(T) = y\).

**Definition 3.1.1 \(((\epsilon, \tau)\text{-pseudo-trajectory})\)** Let \(\epsilon\) and \(T\) be positive real num-
Chapter 3. Heteroclinic networks and ratchets

 bers. A function $x_{\epsilon, \tau} : [t_0, T] \to X$, $t_0, T \in \mathbb{R}$ defined as

$$
x_{\epsilon, \tau}(t) =
\begin{cases}
  x_0(t), & t_0 \leq t < t_1 \\
  \vdots & \\
  x_n(t), & t_n \leq t \leq t_{n+1} = T
\end{cases}
$$

is called an $(\epsilon, \tau)$-pseudo-trajectory of (3.1.1) (with $n$ discontinuities) if $t_{k+1} - t_k > \tau$, $\epsilon > \|x_{k+1}(t_{k+1}) - x_k(t_{k+1})\| > 0$ and $x_k(t)$ is a trajectory of (3.1.1) for all $k = 0, \ldots, n$.

A pseudo-trajectory $x_{\epsilon, \tau}([t_0, T])$ is said to be periodic to $y$ if $x_{\epsilon, \tau}(t_0) = x_{\epsilon, \tau}(T) = y$.

We sometimes write a pseudo-trajectory with $n$ discontinuities as

$$x_{\epsilon, \tau} = \{x_0, \ldots, x_n\}.$$

**Remark 3.1.2** A finite trajectory of (3.1.1) is an $(\epsilon, \tau)$-pseudo trajectory with zero discontinuities for some $T > 0$ and for any $\epsilon > 0$.

The set of all limit points of a trajectory passing through $x$ as $t \to \infty$ ($t \to -\infty$) is called the $\omega$-limit set ($\alpha$-limit set) of $x$ and denoted by $\omega(x)$ ($\alpha(x)$). A point $x \in X$ is called a recurrent point if $x \in \omega(x) \cap \alpha(x)$. The set of recurrent points of $X$ is denoted by $\mathcal{R}(X)$. The chain-recurrent set $\mathcal{R}_{ch}(X)$ is the set of all points $x \in X$ such that for all $\epsilon, \tau > 0$ there is a periodic $(\epsilon, \tau)$-pseudo-trajectory to $x$. In $\mathcal{R}(X)$, each point returns back to its arbitrary small neighborhood in forward
and backward time. The same is true for a point in $\mathcal{R}_{ch}(X)$ up to arbitrary small error in the estimation of trajectories. Note that, by definition, $\mathcal{R}(X) \subset \mathcal{R}_{ch}(X)$. Let $\mathcal{V}$ be a compact subset of $X$. Define
\[
\lambda^1(\mathcal{V}) = \bigcup_{x \in \mathcal{V}} \omega(x) \cup \alpha(x)
\]
and for $n \geq 2$ define $\lambda^n(\mathcal{V}) = \lambda^1(\lambda^{n-1}(\mathcal{V}))$, inductively. The sequence $\lambda^0(X) = X$, $\lambda^1(X), \lambda^2(X), \ldots$ is called the asymptotic filtration of the flow $(X, \gamma)$ \cite{5}. Note that $\lambda^n(X)$ contains $\mathcal{R}(X)$ for $n \geq 0$. We say that $X$ has depth $N$ if $\lambda^N(X) = \mathcal{R}(X)$ and $\lambda^n(X) \supseteq \mathcal{R}(X)$ for all non-negative integers $n < N$.

**Definition 3.1.3 (Heteroclinic Network, \cite{15})** Let $\Sigma$ be a compact connected metric space and $\gamma$ a continuous flow on $\Sigma$. We say the flow $(\Sigma, \gamma)$ is a heteroclinic network if

a) $\mathcal{R}_{ch}(\Sigma) = \Sigma$

b) $\mathcal{R}(\Sigma)$ consists of the finite union of $M$ disjoint, compact, connected flow-invariant sets.

c) $\Sigma$ has finite depth.

By definition, if $\Sigma$ is a heteroclinic network then $\mathcal{R}(\Sigma)$ is closed. The components of $\mathcal{R}(\Sigma)$ are called the nodes of the heteroclinic network. The set of nodes is called the nodal set of $\Sigma$ and denoted by $\mathcal{N}$. The set $\mathcal{C}(\Sigma) = \Sigma \setminus \mathcal{R}(\Sigma)$ is called the set of connections.
Chapter 3. Heteroclinic networks and ratchets

The (measure) attractors in Example 2.2.8 and 2.2.9 in Chapter 2 are heteroclinic networks where the equilibria form the nodal set and the trajectories connecting the equilibria form the set of connections. Note that more complex heteroclinic networks can exist where the nodes consist of chaotic sets and connections may be higher dimensional [16].

3.2 Time averages near embedded heteroclinic networks

In Chapter 2 we give some results on the convergence of time averages along the trajectories of a flow. The application of these results to heteroclinic networks is straightforward. Consider a heteroclinic network $\Sigma$ embedded in a flow $(X, \gamma)$, namely $\Sigma \subset X$ is a flow-invariant set. We assume that there exists a trajectory approaching $\Sigma$.

**Lemma 3.2.1 ([15])** Let $\Sigma \subset X$ be a heteroclinic network and $x \in X$ be such that $\omega(x) \subset \Sigma$. Then

$$\omega_{\text{ess}}(x) \subset \mathcal{R}(\Sigma).$$  \hspace{1cm} (3.2.1)

**Theorem 3.2.2** Let $\Sigma \subset X$ be a heteroclinic network embedded in a compact metric space $X$ and $x(t)$ be a trajectory with $\omega(x(0)) \subset \Sigma$. Let $f : X \rightarrow \mathbb{R}$ be a continuous function and assume that there exists $\bar{f} \in \mathbb{R}$ such that $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\gamma y_0) \, dt = \bar{f}$.
Chapter 3. Heteroclinic networks and ratchets

\[ \tilde{f} \text{ for all } y_0 \in \mathcal{R}(\Sigma). Then } \]

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t)) \, dt = \tilde{f}. \quad (3.2.2)$$

**Proof.** By Lemma 3.2.1, \( \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t)) \, dt = \tilde{f} \in \mathbb{R} \) for any \( x_0 \in \omega_{\text{ess}}(y(0)) \). Hence, Theorem 2.3.6 implies the result. \( \blacksquare \)

We will mostly consider a heteroclinic network \( \Sigma \) that is a measure attractor. Then, it contains the statistical attractor \( \Sigma_S = \Lambda_S(B_M(\Sigma)) \) (see Section 2.2). Clearly, \( \Sigma_S \) is contained in \( \mathcal{R}(\Sigma) \).

### 3.3 Synchronization for trajectories on a torus

In this section, we define various notions of synchronization for trajectories of a flow on a torus \( X = \mathbb{T}^N = [0,1)^N \). We consider a continuous flow

$$\gamma_t : \mathbb{T}^N \to \mathbb{T}^N, t \in \mathbb{R}. \quad (3.3.1)$$

We give definitions of synchronization in terms of a function, called the *winding vector*, of trajectories. In the next section, we will relate the winding vector to a time average of a particular continuous function along trajectories.

We define the winding vector similar to the definition of winding number in [39]. Firstly, we need to consider the lifted flow \( \gamma_t^L \) of (3.3.1), that is, the continuous
flow

\[ \gamma^L_t : \mathbb{R}^N \to \mathbb{R}^N, \quad t \in \mathbb{R} \quad (3.3.2) \]

satisfying \( \gamma_t^L(x) \mod 1 = \gamma_t(x \mod 1) \) for all \( x \in \mathbb{R}^N, \ t \in \mathbb{R} \). For a finite trajectory \( x([t_0, T]) \) of (3.3.1), the finite trajectory \( x^L([t_0, T]) \) of (3.3.2) that has the same initial state \( x(t_0) = x^L(t_0) \) is called the lift of \( x \). Similarly, one can define the lift of a pseudo-trajectory: the lift of an \( (\epsilon, \tau) \)-pseudo-trajectory \( x \) of (3.3.1) with \( n \) discontinuities is the \( (\epsilon, \tau) \)-pseudo-trajectory \( x^L \) of (3.3.2) with \( n \) discontinuities whose continuous parts \( x^L_i \) satisfy \( x^L_i(t_i) = x_i(t_i) \mod 1 \). This is well-defined if \( \epsilon \) is small enough.

We assume that a basis on \( \mathbb{T}^N \) is fixed and an element \( x \in \mathbb{T}^N \) is shown as \( x = (x_1, \ldots, x_n) \) with respect to this basis. Later, for a system of coupled oscillators in Chapter 5, the basis will be chosen as the set of states of the individual oscillators.

**Definition 3.3.1 (Winding vector)** The winding vector of a (pseudo-) trajectory \( x([t_0, T]) \) of (3.3.1) is defined as \( \rho(x) = x^L(T) - x^L(t_0) \).

Note that although a winding vector of an arbitrary (pseudo-)trajectory does not necessarily consist of integer numbers, the winding vector \( \rho = (\rho_1, \ldots, \rho_N) \) of a periodic (pseudo-)trajectory \( p \) is a vector of integers \( (\rho(p) \in \mathbb{Z}^N) \).

**Lemma 3.3.2** Let \( x_{\epsilon, \tau} = \{x_0, \ldots, x_n\} \) be a pseudo-trajectory, then

\[ \sum_{i=0}^n |\rho(x_i)| \leq n\epsilon + |\rho(x_{\epsilon, \tau})|. \]
Chapter 3. Heteroclinic networks and ratchets

**Proof.** This is a straightforward result of the triangle inequality. ■

Define the functions $\rho_{i,j}$ on a trajectory $x([t_0,T])$ as $\rho_{i,j}(x) = \rho_i(x) - \rho_j(x)$.

**Definition 3.3.3 (Phase Synchronization)** A trajectory $x(t)$ of (3.3.1) is said to be $(i,j)$-phase-synchronized if $\|\rho_{i,j}(x([0,T]))\|$ is bounded for all $T > 0$.

**Definition 3.3.4 (Frequency Synchronization)** A trajectory $x(t)$ of (3.3.1) is said to be $(i,j)$-frequency-synchronized if $\|\rho_{i,j}(x([0,T]))\|/T \to 0$ as $T \to \infty$.

Note that Definition 3.3.3 and 3.3.4 are consistent with the corresponding definitions in the coupled oscillator systems literature. Assume $\theta(t)$ is the vector of the states of oscillators. If $\theta(t)$ is $(i,j)$-phase-synchronized then $|\theta_i(t) - \theta_j(t)|$ remains bounded for all $t > 0$. Similarly, $(i,j)$-frequency-synchronization of $\theta(t)$ implies that the average frequencies $\Omega_i$ and $\Omega_j$ are equal (if they exist), where the average frequency for an oscillator $k$ is defined as

$$\Omega_k := \lim_{t \to \infty} \frac{\theta_k^L(t)}{t}.$$ 

Note that a trajectory $\theta(t)$ can be $(i,j)$-frequency-synchronized, although the average frequencies $\Omega_i$ and $\Omega_j$ do not exist. In this case, the rate of phase growth $\theta_{i,j}(t)/t$ of oscillators do not converge, yet tend to be equal. It is clear that phase-synchronization implies frequency-synchronization, although the converse is not true.
3.4 Heteroclinic networks on a torus

Let $\Sigma \subset \mathbb{T}^N$ be a heteroclinic network embedded in an $N$-torus. We assume that there is a trajectory $x(t)$ converging to $\Sigma$. We will show that the frequency synchronization of $x(t)$ is strongly related to the time average of a particular continuous function along $x(t)$.

Since we will be interested in frequency synchronization of two oscillators, say $i$ and $j$, the case where $\rho_i \neq \rho_j$ in the winding vector of a periodic (pseudo-)trajectory is important.

**Definition 3.4.1** $(i, j)$-winding. An invariant set $X$ is $(i, j)$-winding if for every $\epsilon, \tau > 0$ there exists a periodic $(\epsilon, \tau)$-pseudo-trajectory on $X$ with winding number $\rho$ satisfying $\rho_i > \rho_j$. 

**Lemma 3.4.2** If for two invariant sets $V_1$ and $V_2$, $V_1 \subset V_2$ and $V_2$ is not $(i, j)$-winding then $V_1$ is not $(i, j)$-winding.

Let $\mathcal{L}_\gamma f : X \rightarrow \mathbb{R}$ be the Lie derivative of the function $f$ along the flow $\gamma$, that is $\mathcal{L}_\gamma f(x) = \frac{d}{dt}f(\gamma_t(x))|_{t=0}$. Consider the function

$$\pi_{i,j}(x) = x_i - x_j. \quad (3.4.1)$$

Then

$$\rho_{i,j}(x[0,T]) = \int_0^T \mathcal{L}_\gamma \pi_{i,j}(\gamma_t x(0)) \, dt. \quad (3.4.2)$$
**Lemma 3.4.3** Assume that a recurrent set \( V \subset X \) is neither \((i, j)\)-winding nor \((j, i)\)-winding. Then

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T L_{\gamma} \pi_{i,j}(\gamma t x_0) \, dt = 0 \text{ for all } x_0 \in V.
\]  

(3.4.3)

**Proof.**

Let \( x(t) \subset V \) be a trajectory with \( x(0) = x_0 \). Since \( V \) is neither \((i, j)\)-winding nor \((j, i)\)-winding, there exist \( \varepsilon, \tilde{\varepsilon} > 0 \) such that \( \varepsilon \leq \tilde{\varepsilon} \) and \( \tau \geq \tilde{\tau} \) implies a periodic \((\varepsilon, \tau)\)-pseudo-trajectory \( p_{\varepsilon, \tau} \subset V \) has \( \rho_{i,j}(p_{\varepsilon, \tau}) = 0 \).

Assume that (3.4.3) is not true. Then, there exists a sequence \( \{T_k\} \to \infty \) such that \( \frac{1}{T_k} \int_0^{T_k} L_{\gamma} \pi_{i,j}(\gamma t x_0) \, dt \to c \neq 0 \). Since \( V \) is closed and \( X \) is compact, \( V \) is compact, and therefore \( \{x(T_k)\} \) has a convergent subsequence, say \( \{x(T_{k_m})\} \to x^* \in V \). Then, there exists a sequence \( \{m_n\} \) such that \( \|x(T_{k_m}) - x^*\| < \tilde{\varepsilon} \cdot 2^{-n-1} \) for \( n = 0, 1, \ldots \). Note that the sequence \( \{m_n\} \) can be chosen such that \( |T_{k_m} - T_{k_{m+1}}| > 2 \cdot \tilde{\tau} \) for \( n = 0, 1, \ldots \). Let us define the periodic \((\tilde{\varepsilon} \cdot 2^{-n}, \tilde{\tau})\)-pseudo-trajectories \( p_n = \{x([T_n^*, T_{k_{m+1}}]), x([T_{k_m}, T_n^*])\} \) for \( n = 0, 1, \ldots \), where \( T_n^* = (T_{k_m} + T_{k_{m+1}})/2 \). Since \( p_n \subset V \), \( \rho_{i,j}(p_n) = 0 \). Therefore, from Lemma 3.3.2,

\[
|\rho_{i,j}(x([T_{k_m}, T_{k_{m+1}}]))| = |\rho_{i,j}(x([T_{k_m}, T_n^*])) + \rho_{i,j}(x([T_n^*, T_{k_{m+1}}]))| \\
\leq |\rho_{i,j}(x([T_n^*, T_{k_{m+1}}]))| + |\rho_{i,j}(x([T_{k_m}, T_n^*]))| \\
\leq \tilde{\varepsilon} \cdot 2^{-n+1} + \rho_{i,j}(p_n) \\
= \tilde{\varepsilon} \cdot 2^{-n+1}.
\]
This proves that

\[ |\rho_{i,j}(x([0, T_{kmn}]))| \leq |\rho_{i,j}(x([0, T_{km0}]))| + 2 \cdot \bar{\epsilon}, \]

namely \( \rho_{i,j}(x([0, T_{kmn}])) = \int_{0}^{T_{kmn}} \mathcal{L}_{\gamma} \pi_{i,j}(\gamma_t x_0) \, dt \) is bounded. Therefore,

\[ \frac{1}{T_{kmn}} \int_{0}^{T_{kmn}} \mathcal{L}_{\gamma} \pi_{i,j}(\gamma_t x_0) \, dt \to 0, \]

which is a contradiction. \[ \blacksquare \]

**Theorem 3.4.4** Let \( \Sigma \subset \mathbb{T}^N \) be a heteroclinic network and \( \mathcal{R}(\Sigma) \) is neither \((i, j)\)-winding nor \((j, i)\)-winding. Let \( x \in \mathbb{T}^N \) such that \( \omega(x) \subset \Sigma \), then

\[ \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \mathcal{L}_{\gamma} \pi_{i,j}(\gamma_t x) \, dt = 0. \]  \hspace{1cm} (3.4.4)

In other words, trajectories approaching \( \Sigma \) are \((i, j)\)-frequency-synchronized.

**Proof.** Since \( \mathcal{R}(\Sigma) \) is closed, by Lemma 3.4.3, we have that the limit exists and is equal to zero whenever \( x \in \mathcal{R}(\Sigma) \). Therefore, by Theorem 3.2.2, the limit exists and is equal to zero for any \( x \) that satisfies \( \omega(x) \subset \Sigma \). \[ \blacksquare \]
3.5 Heteroclinic ratchets on a torus

We consider a particular type of heteroclinic network (see Definition 3.1.3) embedded in $T^N$ that has an interesting effect on synchronization of trajectories approaching the heteroclinic network. Roughly speaking, a heteroclinic ratchet is a heteroclinic network on which all trajectories are $(i, j)$-phase-synchronized, and thus $(i, j)$-frequency-synchronized, although there are $(\epsilon, \tau)$-pseudo-trajectories that destroy frequency synchronization in one-way even for arbitrary small $\epsilon$ and arbitrary large $\tau$. By the term “one-way” we mean that after the synchrony loss, a specific oscillator always has a larger rate of phase growth. This justifies the use of the term “ratchet”. (See [54] for the first example of heteroclinic ratchets in networks of coupled oscillators). We give the definition of the heteroclinic ratchet in terms of the winding property (see Definition 3.4.1) of certain subsets.

**Definition 3.5.1 (heteroclinic ratchet)** A heteroclinic network $\Sigma$ embedded in $T^N$ is a heteroclinic $(i, j)$-ratchet (or $(i, j)$-ratcheting) if

a) $\Sigma$ is $(i, j)$-winding,

b) $\Sigma$ is not $(j, i)$-winding,

c) and $R(\Sigma)$ is not $(i, j)$-winding.

**Remark 3.5.2** If $\Sigma$ is a heteroclinic ratchet, then

a) $\Sigma$ has depth at least one, since otherwise $\Sigma = R(\Sigma)$ and the conditions (a) and (c) are in contradiction.
Chapter 3. Heteroclinic networks and ratchets

b) There exists no periodic trajectories in $\Sigma$ with $\rho_i > \rho_j$.

c) There exists an $(i, j)$-ratcheting heteroclinic cycle in $\Sigma$.

d) There exists no $(j, i)$-winding heteroclinic cycle in $\Sigma$.

Note that for a heteroclinic network being $(i, j)$-ratcheting is a stronger condition than being $(i, j)$-winding. Ratcheting implies that the winding is only due to the connections of the network. Namely, in heteroclinic $(i, j)$-ratchets there are no $(i, j)$-winding (pseudo-)trajectories contained in the nodes of the heteroclinic ratchet, but there is at least one $(i, j)$-winding pseudo-trajectory that contains heteroclinic connections.

**Theorem 3.5.3** Let $\Sigma \subset T^N$ be a heteroclinic $(i, j)$-ratchet. Consider a point $x$ for which $\omega(x) \subset \Sigma$. Then, $\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathcal{L}_{i,j}(\gamma_t x) \, dt = 0$. In other words, trajectories approaching a heteroclinic $(i, j)$-ratchet are $(i, j)$-frequency-synchronized.

**Proof.** By Lemma 3.4.2, $\mathcal{R}(\Sigma)$ is not $(j, i)$-winding. Therefore, Theorem 3.4.4 implies the result.

■
Part II

Background: Coupled Oscillators
“Science would not give you anything, unless you devote yourself to it. Once you are devoted, it is still not for sure whether it will give you anything.”

el Nazzam

4

Coupled systems

When a group of dynamical systems (cells) are coupled together with a certain coupling structure, they clearly form a new higher dimensional dynamical system which we call a coupled system. Although a coupled system can be analysed using general techniques from the theory of dynamical systems, new questions arise such as how the dynamics of cells affect the overall dynamics of the coupled system or which properties of the overall dynamics only depend on the coupling structure? For instance, it is well known that symmetries in the coupling structure induce symmetries in the overall dynamics. These symmetries give rise to certain invariant subspaces in the phase space and impose special types of bifurcations, namely symmetry-breaking bifurcations [42, 44, 37, 19, 30, 31, 45, 7, 8, 9, 33, 40]. On the other hand, even non-symmetric coupling structures may have similar effects on dynamics [41, 45, 43, 36, 2, 41, 29, 4, 52, 53]. Authors of [77] show that (not
necessarily symmetric) network structures impose invariant subspaces in dynamics. They also give a definition of a coupled cell network, a digraph with certain equivalence relations on the set of vertices (or cells) and on the set of edges (or arrows). This definition is improved in [45] so as to represent coupling structures for a larger class of coupled systems.

In this chapter, we introduce a definition of a coupled cell system similar to the one given in [45]. In Section 4.2 we describe the so-called balanced partitions of the set of cells, which give rise to invariant subspaces. In Section 4.3 we briefly explain the notion of a quotient network. This governs the reduced dynamics in an invariant subspace of a coupled cell system. In Section 4.4 we summarize the effect of symmetry using the theory given in [42]. This theory can be applied to coupled cell systems with symmetries. Finally, in the last section, we mention symmetry-breaking and synchrony-breaking bifurcations.

This chapter is a review of the material needed for the remaining chapters and is not original work with the exception that Definition 4.1.2 has been modified from the usual definition in [43].

4.1 Coupled cell systems

The coupling structure of a coupled system can be represented by a digraph where each vertex (cell) corresponds to individual dynamical units and directed edges (arrows) represent the coupling between these dynamical units. If we take into account that some dynamical units and/or couplings may be identical, a different
concept, called coupled cell network, arises that describes a digraph structure with certain equivalence classes on vertices and edges (see Definition 4.1.2). Then, we can describe the coupled cell system as a dynamical system of coupled dynamical units where the coupling respects a certain coupled cell network.

First of all, we need some definitions related to equivalence relations on sets. Let $\sim_1$ and $\sim_2$ be equivalence relations on a set $X$. We say $\sim_1$ is finer than $\sim_2$ and write $\sim_1 \preceq \sim_2$ if and only if $x \sim_1 y$ implies $x \sim_2 y$ for all $x, y \in X$. We say $\sim_1$ is coarser than $\sim_2$ and write $\sim_1 \succeq \sim_2$ if and only if $\sim_2$ is finer than $\sim_1$. By $\sim_1$, we denote the trivial equivalence, that is, $x \sim_1 y \iff x = y$.

**Definition 4.1.1** (Subset Equivalence) Given a set $X$ and an equivalence relation $\sim$ on $X$ we define subset equivalence $\sim^P$ on the set $P(X)$, that is, on the set of subsets of $X$ (or, more generally, on the set of all multisets whose elements belong to $X$), as

\[
V \sim^P W \iff \exists \text{ a bijection from } V \text{ to } W \text{ that preserves } \sim \text{-equivalence}.
\]

In other words, there exist some number of elements in the sets $V$ and $W$ from any given $\sim$-equivalence class.

In the following, we use the terms “cells” and “arrows” as usual in coupled systems literature for vertices and edges in graph theory literature.

**Definition 4.1.2** (Coupled Cell Network, adapted from Definition 5.1 in [43]) A coupled cell network $\mathcal{G} = \{C, E, \sim_C, \sim_E\}$ consists of
(a) two finite sets: a set $\mathcal{C} = \{1, \ldots, N\}$ of cells and a multiset $\mathcal{E} \subseteq \mathcal{C} \times \mathcal{C}$ of arrows. (The cells indicate individual dynamical systems and the arrows represent coupling between these dynamical units.)

(b) two equivalence relations: ‘cell equivalence’ $\sim_{\mathcal{C}}$ and ‘arrow equivalence’ $\sim_{\mathcal{E}}$ defined on $\mathcal{C}$ and $\mathcal{E}$, respectively. (The corresponding equivalence classes represent the type of cells and arrows in order to take into account the possible identity of dynamical units and the identity of couplings between them.)

(c) two maps $\mathcal{H}: \mathcal{E} \to \mathcal{C}$ and $\mathcal{T}: \mathcal{E} \to \mathcal{C}$. For an arrow $e \in \mathcal{E}$, $\mathcal{H}(e)$ is the head of $e$ and $\mathcal{T}(e)$ is the tail of $e$.

that satisfy two consistency conditions:

- Equivalent arrows have equivalent tails and heads. That is, for any $e_1, e_2 \in \mathcal{E}$, $e_1 \sim_{\mathcal{E}} e_2$ implies

  $$\mathcal{H}(e_1) \sim_{\mathcal{C}} \mathcal{H}(e_2), \quad \mathcal{T}(e_1) \sim_{\mathcal{C}} \mathcal{T}(e_2).$$

- Equivalent cells have equivalent input structure. That is, $c_1 \sim_{\mathcal{C}} c_2$ implies $\mathcal{H}^{-1}(c_1) \sim_{\mathcal{E}} \mathcal{H}^{-1}(c_2)$. ($\mathcal{H}^{-1}(c)$ is the set of arrows whose heads are $c$, which is called the input set of $c$)

The second consistency condition is not included in the original definitions of coupled cell systems in [77, 45]. Instead, the so-called ‘input equivalence’ is introduced as an equivalence relation on $\mathcal{C}$ finer than $\sim_{\mathcal{C}}$. Since by means of
Figure 4.1: A coupled cell network with different cell and arrow types. Cell types are depicted by the shapes of cells (square or circle), and arrow types are shown by different line types (solid, dashed or dot-dashed).

Synchrony and bifurcations there is no need to consider cells having the same cell type but different input structures, we assume that the input equivalence is identical to the cell equivalence here, as done in the combinatorial definition of coupled cell systems in [36].

Figure 4.1 shows a network of cells with different cell and arrow types. Note that cells of the same type receive the same number of arrows of a given arrow type (solid, dashed or dot-dashed). Moreover, arrows of the same type have the same type of cells on their tails and heads. Therefore, this network is a coupled cell network.

One can consider a coupled system with any cell dynamics, such as continuous-time, discrete-time or hybrid systems. Here, we restrict ourselves to the continuous-
Chapter 4. Coupled systems

time case, in particular, coupled ODEs:

\[ \dot{x} = F(x) \] (4.1.1)

where \( F : \mathcal{P} \rightarrow \mathcal{P} \) is a vector field on a smooth manifold \( \mathcal{P} \). We assume that, corresponding to each cell \( c \), there is a phase space \( \mathcal{P}_c \), which is a smooth manifold, such that \( c_1 \sim_c c_2 \implies \mathcal{P}_{c_1} = \mathcal{P}_{c_2} \) and that the phase space \( \mathcal{P} \) is the direct product of the spaces \( \mathcal{P}_c, c \in \mathcal{C} \). The condition that the system (4.1.1) has a given coupling structure (i.e. respects a coupled cell network \( G \) as defined in Definition 4.1.2) implies certain conditions on the vector field \( F \). In fact, a definition for the so-called \( G \)-admissibility of vector fields given in [43] is required. Here, instead of reproducing the formal definition of \( G \)-admissibility, we explain this with an example. Consider the coupled cell network \( G \) in Figure 4.1 and let the state for the cell \( i \) be \( x_i \). Then, we can write an admissible vector field, namely, the \textit{coupled cell system} (of ODEs) that respects \( G \) as follows:

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1; x_1, x_3) \\
\dot{x}_2 &= f_1(x_2; x_1, x_3) \\
\dot{x}_3 &= f_2(x_3; x_1, x_3, x_4) \\
\dot{x}_4 &= f_2(x_4; x_2, x_3, x_3)
\end{align*}
\] (4.1.2)

For each cell there is a Lipschitz continuous map \( f_i \) that governs the dynamics
Chapter 4. Coupled systems

of the cell. Note that cells of same type have identical dynamics represented by $f_1$ and $f_2$. Each cell has internal dynamics, hence the state of each cell appears in the first argument. The arguments that are interchangeable are specified by an overline. In other words, the order of the arguments under an overline does not affect the value of the function. For example, cell 1 receives two different input types: first argument after the semicolon represents solid edge coming from cell 1 and the next one represents the dashed edge coming from cell 3. On the other hand, cell 3 receives three input, two of same type and one of different type: The first argument after the semicolon represents the input from cell 1 whereas the last two arguments show inputs from cell 3 and cell 4. These two inputs marked by an overline are of same type, hence their effect are the same. This means that

$$f_2(x_3; x_1, \overline{x_3}, x_4) = f_2(x_3; x_1, x_4, x_3).$$

4.2 Balanced partition

The coupling structure of a coupled cell system may impose certain properties on the dynamics. One of these is the invariance of certain subspaces in the phase space. For example, for the system (4.1.2), the subspace given by $x_1 = x_2$ is invariant, which can be seen directly from the equations (Setting $x_1 = x_2 = y$ one gets the same dynamics for $y$ from the first and second equations in (4.1.2)). This is due to the coupling structure given in Figure 4.1. In other words, for any coupled cell system that respects the network given in Figure 4.1 the subspace
Chapter 4. Coupled systems

given \( x_1 = x_2 \) is invariant. This is related to the existence of the so-called balanced partitions (also called balanced coloring) of the set of cells \( C \) in the network.

Let \( \Sigma = \{ \Pi_1, \Pi_2, \ldots, \Pi_K \} \) be a partition of the set \( C \), i.e., \( C = \bigcup_{i=1}^{K} \Pi_i \) and \( \Pi_i \cap \Pi_j = \emptyset \) whenever \( i \neq j \). We call each \( \Pi_i \) a cluster of the partition \( \Sigma \). Then there exists an equivalence relation \( \sim_\Sigma \) induced by the partition \( \Sigma \). This can be defined as follows:

\[
x \sim_\Sigma y \text{ if and only if } x, y \in \Pi_i \text{ for some } i.
\]

Note that, the above mentioned partial ordering of equivalence relations (see the second paragraph of Section 4.1) induces a partial ordering of partitions:

\[
\Sigma_1 \preceq \Sigma_2 \text{ if and only if } \sim_{\Sigma_1} \preceq \sim_{\Sigma_2}
\]

If \( c \in C \) and \([e]\) is a \( \sim_{\mathcal{E}} \)-equivalence class, we denote the \([e]\)-type input set of \( c \) by \( I_{[e]}(c) = T(\{ e \in \mathcal{E} \mid e \in [e], \mathcal{H}(e) = c \}) \). Namely, \( I_{[e]}(c) \) is the multiset of cells in \( C \) which provide an \([e]\)-type input to the cell \( c \).

**Definition 4.2.1 (Balanced Partition, [77])** Let \( N = \{ C, \mathcal{E}, \sim_{C}, \sim_{\mathcal{E}} \} \) be a coupled cell network and \( \sim_{\Sigma} \preceq \sim_{C} \) be an equivalence relation corresponding to a partition \( \Sigma = \{ \Pi_1, \ldots, \Pi_N \} \) of \( C \). We say \( \Sigma \) is balanced if for any edge class \([e]\)

\[
x \sim_{\Sigma} y \Rightarrow I_{[e]}(x) \sim_{\Sigma}^{P} I_{[e]}(y).
\]

64
Chapter 4. Coupled systems

We can rewrite the above definition as follows: A partition of cells is balanced if and only if, for each $\sim_{\xi}$-equivalence class $[e]$, any pair of cells in a cluster receive equal number of type-$[e]$ inputs from cells in any other cluster and each cluster consists of cells of same type. It follows that the minimal partition where each cluster consists of one cell is always balanced. On the other hand, the cell-type partition induced by $\sim_{\mathcal{C}}$ is also balanced and this is the maximal balanced partition. In fact, the set of balanced partitions of a coupled cell network form a complete lattice with respect to the ordering given by $\preceq$ even for the coupled cell networks with infinitely many cells [76].

For the network given in Figure 4.1 the minimal and maximal balanced partitions are $\Sigma_1 := \{\{1\}, \{2\}, \{3\}, \{4\}\}$ and $\Sigma_3 = \{\{1, 2\}, \{3, 4\}\}$. One can check that $\Sigma_2 := \{\{1, 2\}, \{3\}, \{4\}\}$ is also a balanced partition, and there is no other balanced partition for this network.

Each balanced partition imposes an invariant subspace in the phase space of the coupled cell system, namely in $\mathcal{P}$, and vice versa. Let $\Sigma$ be a partition of $\mathcal{C}$, then the polydiagonal subspace $V_{\Sigma}$ corresponding to the partition $\Sigma$ is defined as follows:

$$V_{\Sigma} = \{x \in \mathcal{P} \mid c_1 \sim_{\Sigma} c_2 \implies x_{c_1} = x_{c_2} \text{ for all } c_1, c_2 \in \mathcal{C}\}$$

**Proposition 4.2.2** ([77, Theorem 6.5]) A partition $\Sigma$ of cells of a coupled cell network $G$ is balanced if and only if the polydiagonal subspace $V_{\Sigma}$ is invariant for all coupled cell system that respects $G$.  

65
4.3 Quotient networks

Recall the example in the previous section: the coupled cell system with dynamics given in (4.1.2) and the coupling structure depicted in Figure 4.1. Since \( \{\{1, 2\}, \{3\}, \{4\}\} \) is a balanced partition, Proposition 4.2.2 implies that the polydiagonal subspace \( \{x \in \mathcal{P} \mid x_1 = x_2\} \) is invariant. The dynamics on this subspace can be obtained by substituting \( x_1 = x_2 = y \) in (4.1.2):

\[
\begin{align*}
\dot{y} &= f_1(y; y, x_3) \\
\dot{x}_3 &= f_2(x_3; x_1, x_3, x_4) \\
\dot{x}_4 &= f_2(x_4; x_2, x_3, x_3)
\end{align*}
\]  

(4.3.1)

The system of ODEs in (4.3.1) also describes a coupled cell system whose connection structure is as in Figure 4.2. This 3-cell network can also be obtained from the 4-cell network in Figure 4.1 as a quotient by the partition \( \{\{1, 2\}, \{3\}, \{4\}\} \). In general, if a network \( G \) admits a balanced partition \( \Sigma = \{\Pi_1, \ldots, \Pi_K\} \), then the quotient network \( G/\Sigma \) consists of \( K \) cells \( c_1, \ldots, c_K \), where cell \( c_i \) represents the cluster \( \Pi_i \). The quotient has \( n [e] \)-type arrows from \( c_i \) to \( c_j \) if and only if for the original network \( G \) a cell in cluster \( \Pi_j \) receives \( n [e] \)-type arrows from the cells in cluster \( \Pi_i \).
Chapter 4. Coupled systems

Figure 4.2: The quotient network of the network in Figure 4.1 corresponding to the balanced partition \(\{1, 2\}, \{3\}, \{4\}\).

4.4 Symmetries and fixed point subspaces

Invariant subspaces are sometimes related to symmetries in the coupling structure. Assume that a coupled cell network \(G\) is invariant under a permutation of cells and edges. To be more precise, let \(\sigma_C\) be a cell-type preserving permutation on \(C\) and \(\sigma_E\) be an edge-type preserving permutation on \(E\). If \((\sigma_C, \sigma_E)(G) = (\sigma_C(C), \sigma_E(E), \sim_C, \sim_E) = G\), then we say \((\sigma_C, \sigma_E)\) is a symmetry of the network \(G\). Obviously, all symmetries of a network \(G\) form a group, which we denote by \(\Gamma(G)\). Note that the permutation \(\sigma_C\) can be seen as acting on the phase space \(\mathcal{P}\) by permutations of the corresponding axes. Hence, we can define the so-called fixed point subspace
Chapter 4. Coupled systems

of a subgroup $T \subset \Gamma(G)$ as follows:

$$\text{Fix}(T) = \{ x \in \mathcal{P} \mid \sigma_c(x) = x \text{ for all } (\sigma_c, \sigma_e) \in \Gamma(G) \}$$

One can see that for a coupled cell system $\dot{x} = F(x)$ that respects the coupled cell network $G$, $\sigma_c \circ F = F \circ \sigma_c$ for all $\sigma_c \in \Gamma(G)$. Namely, $F$ is $\Gamma(G)$-equivariant (For a general introduction to equivariant dynamical systems see [42]). This implies that, for any subgroup $T \subset \Gamma(G)$, $\text{Fix}(T)$ is an invariant subspace [42, Theorem 1.17]. In fact, these invariant subspaces are polydiagonals, hence by Proposition 4.2.2, one can find balanced partitions related to these. These partitions would simply be obtained by the group orbit of $T$ on $\mathcal{C}$. In conclusion, some (but not all) balanced partitions, and therefore some invariant subspaces arise as a result of the symmetry in the coupling structure.

4.5 Symmetry-breaking and synchrony-breaking bifurcations

Symmetries give rise to invariant subspaces that are fixed point subspaces of the subgroups of the symmetry group. A solution in such an invariant subspace would be fixed by the transformations in the corresponding subgroup, hence would have a certain amount of symmetry. As the subgroup gets larger, a solution fixed by this subgroup would have more symmetries. Therefore, in symmetric systems, the
Chapter 4. Coupled systems

phase space is decomposed into regions of solutions with different symmetries. In order to explain the effect of symmetries on the phase portrait and on the bifurcations of the system, one needs to define more notions. Here, instead of reproducing the theory we refer to the book [42] and mention only the existence of certain type of bifurcations: symmetry-breaking and synchrony-breaking bifurcations.

Assume that a $\Gamma$-equivariant system of ODEs varies continuously by a parameter, say $\alpha$. Namely, we consider the systems

$$\dot{x} = F(x, \alpha).$$

If this system has an equilibrium $x^*$, we call the group $\{\sigma \in \Gamma \mid \sigma x^* = x^*\}$ the symmetry group of $x^*$. When the parameter $\alpha$ varies, the equilibrium $x^*$ may undergo a bifurcation and new equilibria $y_1^*, \ldots, y_n^*$ may emanate from $x^*$. Now we can ask whether the new equilibria have the same symmetry group or a smaller symmetry group. The *Equivariant Branching Lemma* [42, Lemma 1.31] answers this question by stating that under certain conditions one can expect symmetry-breaking bifurcations, namely the new equilibria have smaller symmetry group.

On the other hand, the existence of invariant subspaces induced by the non-symmetric coupling structure, those which can found by the balanced coloring method explained above, suggests that some bifurcations may gives rise to less synchronized solution. Namely, one might expect that new equilibria stay in some smaller synchrony subspaces. Such bifurcations can be named as *synchrony-breaking* bifurcations. Although there is no well-established theory on this type
Chapter 4. Coupled systems

of bifurcation, there are some results along this line in [6, 29, 4, 53]. Note that some bifurcations may be both synchrony-breaking and symmetry-breaking. In the next chapter we will explain the emergence of a heteroclinic ratchet from such a synchrony- and symmetry-breaking bifurcation (Theorem 6.1.1).
“It is contrary to the mode of thinking in science to conceive of a thing which acts itself, but which cannot be acted upon.”

Albert Einstein

Coupled oscillators

Many physical processes that are time-periodic in nature can be modelled by limit cycle oscillators, by which we mean ODEs with hyperbolic, attracting limit cycles. A state of a limit cycle oscillator can be represented by its amplitude and phase. When several oscillators are coupled, various phenomena related to the synchronization of oscillators can arise [68], such as complete (phase and amplitude) synchronization, phase synchronization and frequency synchronization. Recently, interesting phenomena such as extreme sensitivity to detuning [14, 13] and ratcheting [54] of coupled oscillators are observed. These last two phenomena are related to the presence of attracting robust heteroclinic networks in the phase space of coupled oscillator system.

In the following, in Section 5.1 and 5.2 we first give some background on coupled phase oscillators; a simplification of limit cycle oscillators in case of weak
Chapter 5. Coupled oscillators

coupling [48, 20]. Then, in Section 5.3, we define the above-mentioned phenomena in coupled oscillator systems. Finally, in Section 5.4, some recent works on heteroclinic networks in coupled oscillator systems are summarized.

The material in this chapter is a review of the literature except Section 5.3.2 where new concepts, tolerance to positive and negative detuning and ratcheting of oscillators, are introduced.

5.1 From limit cycle oscillators to phase oscillators

By a limit cycle oscillator, we mean a dynamical system \( \dot{x} = f(x) \) on a manifold \( M \) that has an attracting hyperbolic periodic solution \( \gamma(t) \). Coupled limit cycle oscillator systems are dynamical systems of the form

\[
\dot{x} = F(x, \kappa), \quad x \in M^N
\]  

(5.1.1)

that reduce to \( N \) uncoupled limit cycle oscillators when the coupling strength is zero. In this uncoupled case \( (\kappa = 0) \), the \( N \)-torus

\[
\tau^N = \{x_i = \gamma_i(t + \theta_i): (\theta_1, \ldots, \theta_N) \in T^N \},
\]

defined as the direct product of the limit cycles of each oscillator, is an attracting, normally hyperbolic, invariant manifold. Therefore, one can predict that this attracting \( N \)-torus persists in the weak coupling case \( \kappa \ll 1 \). As a result, for the weak
Chapter 5. Coupled oscillators

coupling case, the asymptotic dynamics of (5.1.1) can be reduced to dynamics on this $N$-torus. Note that, a point on this torus represents phase variables $t + \theta_i \in \mathbb{T}$ of oscillators. Using an averaging technique [20], one can obtain a coupled phase oscillator system of the form

$$\dot{\theta} = \bar{F}(\theta, \kappa), \quad \theta \in \mathbb{T}^N,$$

(5.1.2)

where $\theta_i \in \mathbb{T}$ represents the phase of the oscillator $i$ and $F$ is invariant under the action of $\mathbb{S}^1$ given by $\theta \mapsto \theta + \epsilon(1, \ldots, 1)$, $\epsilon \in [0, 2\pi)$ (see e.g. [20] for details). This phase-shift symmetry gives rise to a further reduction of the system on $N$-torus to a system on the quotient space $\mathbb{T}^N / \mathbb{S}^1$, which is an $(N - 1)$-torus,

$$\dot{\phi} = \tilde{F}(\phi, \kappa), \quad \phi \in \mathbb{T}^{N-1},$$

(5.1.3)

where $\phi_i$’s can be chosen as independent phase difference variables $\theta_{m_i} - \theta_{n_i}$. In the remaining part, we refer to the space of phase differences $\mathbb{T}^{N-1}$ as phase difference space of the coupled oscillator system (5.1.2).

5.2 Kuramoto’s model

The idea of reducing coupled phase oscillator systems to limit cycle oscillators was first proposed by Winfree in 1967. However, coupled phase oscillator systems began to be studied widely after Kuramoto’s works in 1984 ([79, 1] and the references...
Chapter 5. Coupled oscillators

\[ \dot{\theta}_i = \omega_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_i - \theta_j), \quad (5.2.1) \]

where \( \theta_i \in \mathbb{T} = [0, 2\pi) \) is the phase and \( \omega_i \) is the natural frequency of the oscillator \( i \).

Kuramoto’s model consists of \( N \) phase oscillators that are coupled globally with a sinusoidal coupling function. That is, the governing equation for each oscillator is

Considering an arbitrary coupling structure and a more general coupling function, the coupled phase oscillator dynamics can be written as follows:

\[ \dot{\theta}_i = \omega_i + \frac{\kappa}{N} \sum_{j=1}^{N} c_{ij}g(\theta_i - \theta_j). \quad (5.2.2) \]
Chapter 5. Coupled oscillators

Here, the connection matrix \( \{c_{ij}\} \) represents the coupling between oscillators. \( c_{ij} = 1 \) if the oscillator \( i \) receives an input from the oscillator \( j \) and \( c_{ij} = 0 \) otherwise. The coupling function \( g \) is a \( 2\pi \)-periodic function. Therefore, it is natural to consider a Fourier series expansion of \( g \)

\[
g(x) = \sum_{k=1}^{\infty} r_k \sin(kx + \alpha_k)
\] (5.2.3)

Note that, by scaling the time, we can set \( \kappa = N \) and \( r_1 = 1 \). In this case, the coupling is modulated by the parameters \( \alpha_1, \alpha_2, \ldots \) and \( r_2, r_3, \ldots \).

Several truncated cases of the general case (5.2.3) were considered in the literature. Considering the first Fourier term only (as in the Kuramoto model (5.2.1)), frequency synchronization and clustering phenomena have been analyzed [62, 74]. Hansel et al. used the first two Fourier terms and observed a new phenomenon, called slow switching, as a result of the presence of an asymptotically stable robust heteroclinic cycle [47, 58]. Recently, using the first three harmonics an attracting heteroclinic ratchet was found for a non-symmetric connection structure [54].

5.3 Synchronization properties

Although we have given definitions of synchronization for general systems on the torus in Section 3.3, here we give definitions for synchronization and similar concepts in the more special context of coupled oscillators. These are consistent with the general definitions in Section 3.3.
Chapter 5. Coupled oscillators

In the literature, there are various definitions for phase/frequency synchronization of oscillators. Moreover, one can define other concepts related to the synchronization, such as sensitivity to detuning [14]. For an ordered pair of oscillators, we call all such properties synchronization properties of the oscillator pair, including phase locking, phase synchronization, frequency synchronization, sensitivity to detuning and ratcheting. The former three are discussed for example in [68] while the latter two are discussed in [14, 54].

5.3.1 Phase and frequency synchronization

For a solution \( \theta(t) = (\theta_1(t), \ldots, \theta_N(t)) \) of (5.2.2), let \( \theta^L(t) = (\theta^L_1(t), \ldots, \theta^L_n(t)) \) denote the lifted phase variables. We say the oscillator pair \((i, j)\) is phase synchronized on the solution \( \theta(t) \) if \( \theta^L_i(t) - \theta^L_j(t) \) is bounded for all \( t \) and phase locked if \( \lim_{t \to \infty} (\theta^L_i(t) - \theta^L_j(t)) \) exists. We say oscillators are frequency synchronized if \( \lim_{t \to \infty} \frac{\theta_i(t) - \theta_j(t)}{t} = 0 \). Note that phase locking implies phase synchronization and phase synchronization implies frequency synchronization. However, the converses are not true in general. For example, on a typical solution approaching a heteroclinic network, particular pairs of oscillators are never phase locked, but they can be phase synchronized if the heteroclinic network is contractible to the diagonal in \( \mathbb{T}^n \). More interesting is the effect of heteroclinic ratchets on the synchronization properties of oscillators. On a solution approaching a heteroclinic ratchet, some oscillator pairs can be frequency synchronized but not phase synchronized, as we will see in the next chapter.
5.3.2 Sensitivity to detuning and ratcheting

It is well-known that when oscillators are synchronized, a mismatch (detuning) in natural frequencies may cause loss of synchronization depending on how large the detuning is. Let $\omega_{ij} = \omega_i - \omega_j$ denote the detuning and $\Omega_{ij} = \Omega_i - \Omega_j$ the difference in observed average frequencies. Here $\Omega_i = \lim_{t\to\infty} \frac{\theta_i}{t}$. The typical $(\omega_{ij}, |\Omega_{ij}|)$ characteristic of coupled oscillators is as in Fig. 5.2a.

For an ordered oscillator pair $(i, j)$, we generalize notions in [14] to define the tolerance to positive detuning and tolerance to negative detuning as

$$
\Delta^+_ij := \sup\{\Delta : 0 \leq \omega_{ij} < \Delta \implies (i,j) \text{ is phase synchronized.} \},
$$

$$
\Delta^-ij := \sup\{\Delta : -\Delta < \omega_{ij} \leq 0 \implies (i,j) \text{ is phase synchronized.} \},
$$


Figure 5.2: Different $(\omega_{ij}, |\Omega_{ij}|)$-characteristics of coupled oscillators. (a) Usual case: Frequency synchronization of the oscillators persist in a certain tolerance range of detuning. (b) Extreme sensitivity to detuning: Although there is a dynamically stable frequency synchronized behaviour at $\omega_{ij} = 0$, synchronization is broken by arbitrarily small detuning. This can happen if there is an attracting heteroclinic cycle in state space (see Section 5.4). (c) Unidirectional extreme sensitivity to detuning (or ratcheting): Under small detuning synchronization is broken only if the detuning is of one sign.
Chapter 5. Coupled oscillators

respectively. We call \( \Delta_{ij} := \min(\Delta_{ij}^-, \Delta_{ij}^+) \) the tolerance to detuning of \((i, j)\). If \( \Delta_{ij} = 0 \) then the oscillator pair \((i, j)\) is said to have extreme sensitivity to detuning.

If \( \Delta_{ij}^+ = 0 \) but \( \Delta_{ij}^- > 0 \), we say that the oscillator pair \((i, j)\) is ratcheting (see Fig. 5.2) (for details see Chapter 6). Note that ratcheting is an asymmetric relation on the set of oscillators, that is, if \((i, j)\) is ratcheting then \((j, i)\) is not ratcheting. In the following, we will show that heteroclinic networks may result in extreme sensitivity to detuning (Section 5.4) and heteroclinic ratchets give rise to ratcheting of some oscillator pairs (see the next chapter).

5.4 Heteroclinic phenomena in coupled oscillators

An attracting heteroclinic cycle in the phase difference space of a coupled oscillator system has a strong effect on the synchronization properties of oscillators. For instance, a solution approaching a heteroclinic cycle implies the absence of phase locking of certain oscillator pairs. Moreover, heteroclinic cycles are related to extreme sensitivity phenomena [14].

Heteroclinic cycles induce an intermittent behaviour called slow switching where the dynamics stays long time near one cluster and then passes to another cluster. Slow switching behaviour of coupled oscillator systems was first studied by Hansel et al. in [47]. They found heteroclinic cycles for four globally coupled phase oscillator system with a coupling function up to second order Fourier terms \((\alpha_1 = 1.25, r_2 = 0.5)\). Heteroclinic cycles associated with slow switching were also studied for different oscillator types, such as delayed pulse-coupled integrate-and-fire oscilla-
Chapter 5. Coupled oscillators

tors [59, 25] and limit cycle oscillators [58]. In the following, we will describe the heteroclinic behaviour observed in coupled phase oscillators [13, 14, 12, 18], and explain its effect on synchronization properties. This effect has been investigated for fully symmetric (all-to-all coupled) systems but not in many other configurations.

All-to-all coupling of identical units gives rise to an $S_N$-permutation symmetry for the system. This imposes many dynamically invariant subspaces arising as fixed point subspaces of subgroups of $S_N$. Therefore, the dynamics is trapped in invariant regions bounded by these fixed point subspaces. Let us choose the phase difference variables as $\phi_i = \theta_i - \theta_i + 1$, $i = 1, \ldots, N - 1$. Then, the invariant regions are $\{ \phi \in \mathbb{T}^{N-1}: \phi_{\sigma(1)} \leq \phi_{\sigma(2)} \leq \cdots \leq \phi_{\sigma(N-1)} \}$ where $\sigma$ is a permutation of oscillators. When $\sigma$ is identity, this region is called canonical invariant region [20]. Since all these regions are symmetric images of each other, it suffices to study the dynamics on the canonical invariant region. Note that, since the dynamics is trapped in these invariant regions in the phase difference space, oscillators are always phase synchronized and therefore frequency synchronized (the subspace $\theta_i = \theta_j$ being invariant implies phase synchronization of oscillators $i$ and $j$ [39]). We will be more interested in the extreme sensitivity properties of oscillators for which the existence of heteroclinic networks are crucial.

For $N$ coupled phase oscillators, heteroclinic behaviour can arise if $N \geq 3$. The case $N = 3$ and $N = 4$ is analyzed in detail by Ashwin et al. in [13]. Using a second order Fourier truncation of the coupling function, they show that for $N = 3$ a heteroclinic cycle appears as a codimension one phenomenon in phase difference
space (see Figure 5.3). This heteroclinic cycle connects the saddles labelled by \( P \) and \( Q \) on the invariant lines, which have \( S_2 \times S_1 \) isotropy [13]. Note that, the heteroclinic network on \( \mathbb{T}^{N-1} \) formed by these heteroclinic cycles contains winding heteroclinic cycles in each \( \theta_i - \theta_j \) direction. Therefore any detuning \( \Delta_{ij} \) gives rise to a periodic orbit that breaks the synchronization of the oscillators \( i \) and \( j \) (see [14] for details). As a result, this heteroclinic network leads to extreme sensitivity to detuning (see [14]). However, this phenomenon is not robust for \( N = 3 \) as it occurs at a bifurcation point.

For the case \( N = 4 \), one can observe robust heteroclinic cycles (see Fig. 5.4). In this case the canonical invariant region is a tetrahedron whose lines have either \( S_2 \times S_2 \) or \( S_3 \times S_1 \) isotropy. The heteroclinic cycle shown in Figure 5.4 exists robustly for an open set in the parameter space (see [13] for details.) This time
the heteroclinic network formed by these heteroclinic cycles in different invariant regions does not contain any winding heteroclinic cycle, except for the critical case when the heteroclinic cycles first appear and lie on the invariant lines. As a result, although the heteroclinic behaviour is robust when $N = 4$, the extreme sensitivity phenomenon is again not robust.

Robust extreme sensitivity behaviour arises when one considers an all-to-all coupled oscillator system with $N \geq 5$. It is numerically shown in [14] that for $N = 5$, the extreme sensitivity is robust. In [12], a heteroclinic network for the 5-oscillator all-to-all coupled system is shown to exist on the phase difference space $\mathbb{T}^4$. In this case, the heteroclinic network contains winding heteroclinic cycles in any direction breaking the frequency synchronization of oscillators, and this happens robustly under small parameter changes. This robust extreme sensitivity behaviour is bidirectional due to the presence of full permutation symmetry.
Chapter 5. Coupled oscillators

Figure 5.4: A robust heteroclinic cycle for the all-to-all coupled 4-oscillator system. The heteroclinic cycle consists of two saddle equilibria $P_1$ and $P_2$ with $S_2 \times S_2$ isotropy and two connections $\Gamma_1$ and $\Gamma_2$ on the two dimensional invariant subspaces. The invariant subspaces are embedded in a cube that represents a unit cell for the torus of phase difference space- in this representation all vertices of the cube represent in-phase solutions where all oscillators are synchronized. (Adapted from [13]).
Part III

Heteroclinic Ratchets
A heteroclinic ratchet for a system of 4 coupled oscillators

We have already mentioned ratcheting phenomenon in Section 5.3.2 and defined a heteroclinic ratchet as a heteroclinic network with certain properties in Definition 3.5.1. In this chapter, we explain the first example that exhibits ratcheting phenomena. This is a coupled oscillator system that consists of four identical phase oscillators. The coupling structure gives rise to certain invariant subspaces and a particular type of synchrony-breaking bifurcation (see Section 6.1). The invariant subspaces persist as far as the coupling structure is conserved (see Chapter 4), therefore a robust heteroclinic network can exist on these subspaces. For some parameter values, we identify a heteroclinic ratchet that seems to appear after the above-mentioned synchrony-breaking bifurcation of the zero solution (see Section
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

6.2. Heteroclinic ratchets have interesting effects on the synchronization properties of coupled oscillators (see Chapter 5), especially when small noise or detuning of the natural frequencies of the oscillators are considered (see Section 6.3).

All material in this chapter is original and published in [54].

6.1 4-cell example

In this section, we consider four oscillators coupled by a connection structure shown in Figure 6.1. This can be seen as a coupled cell system (see Section 4.1) where each cell is a phase oscillator as in (5.1.2). More specifically, the system we consider is

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1 + f(\theta_1; \theta_2, \theta_3) \\
\dot{\theta}_2 &= \omega_2 + f(\theta_2; \theta_1, \theta_4) \\
\dot{\theta}_3 &= \omega_3 + f(\theta_3; \theta_1, \theta_2) \\
\dot{\theta}_4 &= \omega_4 + f(\theta_4; \theta_1, \theta_2),
\end{align*}
\]

(6.1.1)

where \( \theta_i \in \mathbb{T} = [0, 2\pi) \) is the phase of the \( i \)th oscillator and \( f \) is a \( 2\pi \) periodic continuous function that represents the coupling.

We first assume identical oscillators, that is

\[
\omega = \omega_1 = \cdots = \omega_4.
\]

(6.1.2)
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

![Diagram of a 4-cell coupled cell network](image)

Figure 6.1: A 4-cell coupled cell network that gives coupled cell systems of the form (6.1.1). The network has a single symmetry given by the permutation (12)(34).

Oscillators with different natural frequencies will be considered in Section 6.3. The overlines in the function $f$ indicates that the inputs to each cell are indistinguishable, i.e.

$$f(x; \overline{y}, \overline{z}) = f(x; \overline{z}, \overline{y}) \text{ for all } x, y, z \in \mathbb{T}.$$  \hspace{1cm} (6.1.3)

We will also assume the phase-shift symmetry

$$f(x + \epsilon; \overline{y + \epsilon}, \overline{z + \epsilon}) = f(x; \overline{y}, \overline{z}) \text{ for all } x, y, z, \epsilon \in \mathbb{T}.$$  \hspace{1cm} (6.1.4)

This $S^1$-symmetry arises for example in weakly coupled limit cycle oscillators via averaging [20]. Note that, for the present section, the form of coupling we assume will be more general than (5.2.2).

In the following we describe the invariant subspaces of (6.1.1) and give a result about the solution branches on invariant subspaces that emanate at bifurcation from a fully synchronized solution.
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

6.1.1 Invariant subspaces

In order to identify invariant subspaces forced by the coupling structure in Figure 6.1 we use the theory of coupled cell systems summarized in Chapter 4. The network in Figure 6.1 has a symmetry that we characterize as follows. Let \( \Gamma \) be an \( S_2 \)-action on \( \mathbb{T}^4 \) generated by

\[
\sigma: (\theta_1, \theta_2, \theta_3, \theta_4) \to (\theta_2, \theta_1, \theta_4, \theta_3).
\]

The symmetry of the network implies that the system (6.1.1) is \( \Gamma \)-equivariant and the fixed point subspace of \( \Gamma \), that is,

\[
\text{Fix}(\Gamma) = \{ x \in \mathbb{T}^4 | \sigma x = x \text{ for all } \sigma \in \Gamma \}
\]

is invariant under the dynamics of (6.1.1) (see Section 4.4). On the other hand, there are many other invariant subspaces of which not all appear because of the symmetries of the network but because of the groupoid structure of the input sets of cells (see [43] for groupoid formalism). These invariant subspaces corresponds to the balanced partitions (see Section 4.2) of cells in Figure 6.1. Recall that a partition of cells into a number of clusters is called balanced if, for any arrow type \([e]\), two cells in a cluster receive same number of type-\([e]\) arrows from any given cluster. Each balanced partition of cells gives rise to an invariant subspace where the states of cells in clusters are equal (4.2.2). Moreover, each balanced
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>Invariant Subspaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$V_1 = T^4$</td>
</tr>
<tr>
<td>3</td>
<td>$V_3^s = { \theta \in T^4 \mid \theta_3 = \theta_4 }$</td>
</tr>
<tr>
<td>3</td>
<td>$V_3^1 = { \theta \in T^4 \mid \theta_2 = \theta_4 }$</td>
</tr>
<tr>
<td>3</td>
<td>$V_3^2 = { \theta \in T^4 \mid \theta_1 = \theta_3 }$</td>
</tr>
<tr>
<td>2</td>
<td>$V_2 = { \theta \in T^4 \mid \theta_1 = \theta_3, \theta_2 = \theta_4 }$</td>
</tr>
<tr>
<td>2</td>
<td>$V_2^{s1} = { \theta \in T^4 \mid \theta_2 = \theta_3 = \theta_4 }$</td>
</tr>
<tr>
<td>2</td>
<td>$V_2^{s2} = { \theta \in T^4 \mid \theta_1 = \theta_3 = \theta_4 }$</td>
</tr>
<tr>
<td>2</td>
<td>$V_2^{s3} = { \theta \in T^4 \mid \theta_1 = \theta_2, \theta_3 = \theta_4 }$</td>
</tr>
<tr>
<td>1</td>
<td>$V_1 = { \theta \in T^4 \mid \theta_1 = \theta_2 = \theta_3 = \theta_4 }$</td>
</tr>
</tbody>
</table>

Table 6.1: Invariant subspaces forced by the coupling structure in Figure 6.1 for the system (6.1.1)

Partition corresponds to a quotient network which gives the dynamics reduced to the corresponding invariant subspace (see Section 4.3).

For the system (6.1.1) the invariant subspaces that correspond to the balanced partitions of cells are listed in Table 6.1. The subscripts indicate the dimensions of the invariant subspaces and the superscript $s$ labels the subspaces related to the $S_3$ symmetry of the quotient network for $\theta_3 = \theta_4$ (see Table 6.2). There exists a partial ordering for the set of these subspaces given by containment, that is, $V_x \prec V_y \iff V_x \subset V_y$. This ordering of invariant subspaces is illustrated in Figure 6.2.

Consider the balanced partition $\{\{1\}, \{2\}, \{3, 4\}\}$, where only cell 3 and cell 4 are clustered. The corresponding invariant subspace is $V_2^{s3}$ and the quotient network is the $S_3$-symmetric all-to-all coupled 3-cell network (see Table 6.2). Necessarily all the fixed point subspaces of this 3-cell quotient lift to some invariant subspaces of the 4-cell system and these are labelled by the superscript $s$. Note that $V_2^{s3}$ is
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

<table>
<thead>
<tr>
<th>Balanced partitions</th>
<th>Invariant subspaces</th>
<th>Quotient networks</th>
</tr>
</thead>
<tbody>
<tr>
<td>{{1}, {2}, {3, 4}}</td>
<td>(V_3^s)</td>
<td>(N_1):</td>
</tr>
<tr>
<td>{{1}, {3}, {2, 4}}</td>
<td>(V_3^1)</td>
<td>(N_2):</td>
</tr>
<tr>
<td>{{2}, {4}, {1, 3}}</td>
<td>(V_3^2)</td>
<td>(N_3):</td>
</tr>
</tbody>
</table>

Table 6.2: Quotient networks for three dimensional invariant subspaces \(V_3^s\), \(V_3^1\), and \(V_3^2\) of the 4-cell system (6.1.1)

the only one of these that arises from the symmetry of the system (6.1.1) \((V_2^{s3} = \text{Fix}(\Gamma))\), but there are some pairs of subspaces for which one subspace is related to the other by the symmetry of the system, namely \(\sigma(V_2^{s2}) = V_2^{s1}\) and \(\sigma(V_2^2) = V_3^1\). As a result, the quotient networks corresponding the subspaces \(V_3^1\) and \(V_3^2\) are also symmetrically related (see Table 6.2).

Exploiting the phase-shift symmetry (6.1.4), the 4-dimensional system (6.1.1)
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

Figure 6.2: Containment of the invariant subspaces given in Table 6.1. \( V_x \rightarrow V_y \) means \( V_x \subset V_y \). The subscripts indicate the dimensions of the invariant subspaces and the superscript \( s \) labels the fixed point subspaces related to the \( S_3 \) symmetry of the quotient network for \( \theta_3 = \theta_4 \).

and (6.1.2) can be reduced to a 3-dimensional one by defining new variables

\[
(\phi_1, \phi_2, \phi_3) := (\theta_1 - \theta_3, \theta_2 - \theta_4, \theta_3 - \theta_4)
\]

so that

\[
\begin{align*}
\dot{\phi}_1 &= f(\phi_1; \phi_2 - \phi_3, 0) - f(0; \phi_1, \phi_2 - \phi_3) \\
\dot{\phi}_2 &= f(\phi_2; \phi_1 + \phi_3, 0) - f(0; \phi_1 + \phi_3, \phi_2) \quad (6.1.5) \\
\dot{\phi}_3 &= f(\phi_3; \phi_1 + \phi_3, \phi_2) - f(0; \phi_1 + \phi_3, \phi_2).
\end{align*}
\]

The symmetry of the system (6.1.1) has implications for this system. Let \( \tilde{\Gamma} \) be an
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

![Diagram](image)

Figure 6.3: Invariant subspaces given in Table 1 projected onto $T^3$ (represented by a $2\pi$-cube in $\mathbb{R}^3$) and the synchrony-broken branches given in Theorem 6.1.1. Subscripts indicate the subspace dimensions on $T^4$. The bifurcating branches of equilibria given in Theorem 6.1.1 are represented by disks filled by black and gray colors for pitchfork and transcritical branches, respectively.

$S_2$-action on $T^3$ generated by

$$
\rho: (\phi_1, \phi_2, \phi_3) \to (\phi_2, \phi_1, 2\pi - \phi_3). \quad (6.1.6)
$$

Then the system (6.1.5) is $\tilde{\Gamma}$-equivariant. In this case the fixed point subspace of $\tilde{\Gamma}$ is the union of the lines \( \{ \phi \in T^3 \mid \phi_1 = \phi_2, \phi_3 = 0 \} \) and \( \{ \phi \in T^3 \mid \phi_1 = \phi_2, \phi_3 = \pi \} \). Other invariant subspaces can be obtained by projecting the previously found invariant subspaces onto $T^3$. These are illustrated in Figure 6.3, where the previous notation for subspaces is used. That is, subscripts indicate dimensions of the subspaces in $T^4$. 

91
6.1.2 Synchrony-breaking bifurcations

For this section, we assume that $f$ depends on a parameter $\alpha$. Hence, we can rewrite (6.1.5) as

\[
\begin{align*}
\dot{\phi}_1 &= f(\phi_1; \phi_2 - \phi_3, 0; \alpha) - f(0; \phi_1, \phi_2 - \phi_3; \alpha) \\
\dot{\phi}_2 &= f(\phi_2; \phi_1 + \phi_3, 0; \alpha) - f(0; \phi_1 + \phi_3, \phi_2; \alpha) \\
\dot{\phi}_3 &= f(\phi_3; \phi_1 + \phi_3, \phi_2; \alpha) - f(0; \phi_1 + \phi_3, \phi_2; \alpha).
\end{align*}
\]

In [3], it is shown that any coupled cell system that has a connection structure as in Figure 6.1 admits an $S_3$-transcritical bifurcation on $V_2^s$ at the origin. More specifically, there exist three transcritical branches of unstable solutions on $V_2^{s1}$, $V_2^{s2}$, and $V_2^{s3}$ simultaneously emanating from the origin if $f_x(0, 0) - f_y(0, 0) = 0$ and some transversality inequalities are satisfied (The zero vector is denoted by $0$, and $f(0, \alpha) = f(0; 0, 0; \alpha)$). However, for the coupled phase oscillators of type (6.1.1), apart from the connection structure, dynamical properties affect the bifurcation scheme. Now we will show in Theorem 6.1.1 how the $S_2$-symmetry of the 4-cell network gives rise to a pitchfork bifurcation on $V_2$ that takes place simultaneously with the transcritical bifurcations mentioned above. The simultaneous occurrence of branches on invariant lines is not only a consequence of the Equivariant Branching Lemma [42] but also a result of the connection structure and the property of the individual dynamics, that is the $S^1$-symmetry of $f$.  

92
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

<table>
<thead>
<tr>
<th>Adjacency matrix</th>
<th>eigenvalues and eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = \begin{pmatrix} 0 &amp; 1 &amp; 1 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 1 \ 1 &amp; 1 &amp; 0 &amp; 0 \ 1 &amp; 1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\mu_1 = -1, \quad \nu_1 = (1, -1, 0, 0)^T$</td>
</tr>
<tr>
<td></td>
<td>$\mu_2 = -1, \quad \nu_2 = (0, -1, 1, 1)^T$</td>
</tr>
<tr>
<td></td>
<td>$\mu_3 = 0, \quad \nu_3 = (1, -1, 1, -1)^T$</td>
</tr>
<tr>
<td></td>
<td>$\mu_4 = 2, \quad \nu_4 = (1, 1, 1, 1)^T$</td>
</tr>
</tbody>
</table>

Table 6.3: Adjacency matrix of the network in Figure 6.1 with eigenvalues and eigenvectors

**Theorem 6.1.1** Assume that $f$ satisfies $f_x(0, \alpha^*) = 0$, $f_{x\alpha}(0, \alpha^*) \neq 0$, $f_{xx}(0, \alpha^*) \neq f_{yy}(0, \alpha^*)$, and $f_{yy}(0, \alpha^*) \neq 0$. Then there exists a pitchfork bifurcation of the origin of (6.1.7) on $V_2$ at $\alpha = \alpha^*$ appearing simultaneously with the transcritical bifurcations on $V^{*2}_2$, $V^{*2}_2$ and $V^{*3}_2$.

**Remark 6.1.2** A direct consequence of the Theorem 6.1.1 is that a generic bifurcation of the fully synchronized periodic solution $(x, x, x, x)$ of (6.1.1) will give rise to three branches of periodic solutions of the form

$(x, y, x, x)$

$(y, x, x, x)$

$(x, x, y, y)$

and two other branches of the form

$(x, y, x, y)$,

where the first three appear by transcritical bifurcations and the final two via a pitchfork bifurcation.
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

**Proof.** Consider the adjacency matrix $A$ of the network (see Table 6.3). The eigenvalues of $A$ and partial derivatives of $f$ ($f_x$, $f_y$ and $f_z$) at the origin determine the stability of the origin (see Proposition 2 in [3]). The eigenvalues of (6.1.7) at the origin are

$$\lambda_i = f_x(0, \alpha) + \mu_i f_y(0, \alpha)$$  \hspace{1cm} (6.1.8)

where $\mu_i$ is an eigenvalue of $A$ and $i = 1, 2, 3$. The eigenvectors of (6.1.7) are the same as the eigenvectors of $A$ that correspond to its nonzero eigenvalues. It is important to note that the $S^1$-phase-shift symmetry of (6.1.1) induce a relation between partial derivatives:

$$f_x(u, v, w, \alpha) + f_y(u, v, w, \alpha) + f_z(u, v, w, \alpha) = 0, \ \forall u, v, w, \alpha \in \mathbb{R}.$$ \hspace{1cm} (6.1.9)

This can be obtained by taking the derivative of (6.1.4) with respect to $\epsilon$, and (6.1.3) implies

$$f_y(u, v, w, \alpha) = f_z(u, w, v, \alpha), \ \forall u, v, w, \alpha \in \mathbb{R}.$$  \hspace{1cm} (6.1.10)

Thus, from (6.1.9) and (6.1.10), there exists a linear relationship between the partial derivatives:

$$f_x(0, \alpha) = -2 f_y(0, \alpha) = -2 f_z(0, \alpha), \ \forall \alpha \in \mathbb{R}.$$  \hspace{1cm} (6.1.11)
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

Similarly, the derivatives of (6.1.9) and (6.1.10) with respect to \( \alpha \) give

\[
f_{x\alpha}(0, \alpha) = -2f_{y\alpha}(0, \alpha) = -2f_{z\alpha}(0, \alpha), \quad \forall \alpha \in \mathbb{R}, \tag{6.1.12}
\]

and the derivative of (6.1.10) with respect to \( v \) gives

\[
f_{yy}(0, \alpha) = f_{zz}(0, \alpha), \quad \forall \alpha \in \mathbb{R}. \tag{6.1.13}
\]

Finally, deriving (6.1.10) twice with respect to \( v \) gives

\[
f_{zzz}(0, \alpha) = f_{yyy}(0, \alpha), \quad \forall \alpha \in \mathbb{R}. \tag{6.1.14}
\]

Equation (6.1.8) and Equation (6.1.11) imply that all eigenvalues \( \lambda_i \) become zero simultaneously when \( f_x(0, \alpha) = 0 \). To see that there exists a pitchfork branch on \( V_2 \) we consider the solutions of type \((x, x + u, x, x + u)\). Substituting this into (6.1.7) and using Equation (6.1.14), one gets \( \dot{u} = F(u) := f(0; 0, -u, \alpha) - f(0; u, 0, \alpha) \). Thus the assumptions \( f_x(0, \alpha^*) = 0, f_{xx}(0, \alpha^*) \neq 0, f_{yy}(0, \alpha^*) \neq 0 \) and Equations (6.1.11)-(6.1.14) imply the pitchfork bifurcation conditions \((\partial F/\partial u)(0, \alpha^*) = 0\), \((\partial^2 F/\partial u^2)(0, \alpha^*) = 0\), \((\partial^2 F/\partial \alpha \partial u)(0, \alpha^*) \neq 0\) and \((\partial^3 F/\partial u^3)(0, \alpha^*) \neq 0\). Since these and the condition \( f_{xx}(0, \alpha^*) \neq f_{yy}(0, \alpha^*) \) also imply the assumptions of Theorem 1 in [3], there exist simultaneous transcritical bifurcations on \( V_2^{s1}, V_2^{s2} \) and \( V_2^{s3} \). \( \blacksquare \)
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

**Remark 6.1.3** The existence of pitchfork branches can also be explained by considering the $S_2$ interior symmetry of the set of cells $\{3, 4\}$ in Figure 6.1 (see [41] for the concept of interior symmetry and the interior symmetry branching lemma). However, this does not imply the simultaneous occurrence of transcritical and pitchfork branches for the system (6.1.7).

### 6.2 Robust heteroclinic ratchets

For a vector field $F : \mathbb{R}^N \to \mathbb{R}^N$ (or $\mathbb{T}^N \to \mathbb{T}^N$), a *heteroclinic cycle* consists of a set of saddle equilibria $\xi_0, \ldots, \xi_{m-1}$ and trajectories (connections) $x_0(t), \ldots, x_{m-1}(t)$ such that $\lim_{t \to -\infty} x_i(t) = \xi_i$ and $\lim_{t \to \infty} x_i(t) = \xi_{i+1 \pmod{m}}$ for $i = 0, \ldots, m - 1$. We call a connected invariant set a *heteroclinic network* if it is a union of heteroclinic cycles.

In the previous section, it is shown that the connection structure of the system (6.1.1) forces a number of invariant subspaces to exist. These subspaces persist under the perturbations that preserve the connection structure. For this reason, as in symmetric systems, one can find robust heteroclinic networks lying on the invariant subspaces of the system (6.1.1). By “robust” we mean the persistence under small perturbations that preserve the coupling structure. We will see that for the phase-difference system (6.1.5) some unusual heteroclinic networks exist, which are not seen for symmetric systems. We distinguish one type of these heteroclinic networks, which we call a heteroclinic ratchet because it includes connections that wind around the torus in one direction only. For a more general definition of
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

heteroclinic ratchets see Definition 3.5.1.

**Definition 6.2.1** For a system on $\mathbb{T}^N$, a heteroclinic network is a heteroclinic ratchet if it includes a heteroclinic cycle with nontrivial winding in one direction but no heteroclinic cycles winding in the opposite direction. More precisely, we say a heteroclinic cycle $C \subset \mathbb{T}^N$ parametrized by $x(s) \ (s: [0, 1) \to \mathbb{T}^N)$ has nontrivial winding in some direction if there is a projection map $P: \mathbb{R}^N \to \mathbb{R}$ such that the parametrization $\tilde{x}(s) \ (\tilde{x}: [0, 1) \to \mathbb{R}^N)$ of the lifted heteroclinic cycle $\tilde{C} \subset \mathbb{R}^N$ satisfies $\lim_{s \to 1} P(\tilde{x}(s)) - P(\tilde{x}(0)) = 2k\pi$ for some positive integer $k$. A heteroclinic cycle winding in the opposite direction would satisfy the same conditions for a negative integer $k$.

In this section, we will first explain how a heteroclinic ratchet emerges for the system (6.1.5) after a synchrony-breaking bifurcation. Then, we will discuss the stability of the heteroclinic ratchet and exhibit a coupling function $g$ for which the heteroclinic ratchet is an attractor. Finally, different routes that lead to heteroclinic cycles will be discussed.

**6.2.1 Heteroclinic ratchets for the four coupled oscillators**

We consider a particular case of (6.1.1), where the coupling has the same form as in (5.2.2):

$$f(x; y, z) = g(x - y) + g(x - z). \quad (6.2.1)$$

Using (6.2.1), we can write the phase-difference system with identical natural
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

frequencies given in Equation (6.1.5) in the form

\[
\begin{align*}
\dot{\phi}_1 &= g(\phi_1 + \phi_3 - \phi_2) + g(\phi_1) - g(-\phi_1) - g(\phi_3 - \phi_2) \\
\dot{\phi}_2 &= g(\phi_2 - \phi_3 - \phi_1) + g(\phi_2) - g(-\phi_3 - \phi_1) - g(-\phi_2) \quad (6.2.2) \\
\dot{\phi}_3 &= g(-\phi_1) + g(\phi_3 - \phi_2) - g(-\phi_3 - \phi_1) - g(-\phi_2).
\end{align*}
\]

We consider the coupling function \( g \) with up to three harmonics:

\[
g(x) = -r_1 \sin(x + \alpha_1) + r_2 \sin(2x + \alpha_2) + r_3 \sin(3x + \alpha_3), \quad (6.2.3)
\]

where by scaling the time variable, \( r_1 \) can be set to 1, and \( \alpha_2, \alpha_3 \) are assumed to be zero for simplicity. For this coupling function, there may exist different types of robust heteroclinic networks for different parameter values of \( \alpha_1, r_2 \) and \( r_3 \). We first demonstrate a heteroclinic ratchet that exists for an open set of parameters.

Heteroclinic networks are usually exceptional phenomena, but they can be robust if the associated heteroclinic connections are contained within invariant subspaces [60]. For (6.2.2) and (6.2.3) there are invariant subspaces that are found in the previous section for a more general system (6.1.5) (see Figure 6.3). For the parameter set

\[
(\alpha_1, r_2, r_3) = (1.4, 0.3, -0.1), \quad (6.2.4)
\]

we identify robust heteroclinic connections between two equilibria on the invariant subspaces \( V_3^1 \) and \( V_3^2 \), using the simulation tool XPPAUT [34]. Note that the
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

symmetry (6.1.6) of (6.1.5) acts on $V_2 = V_3^1 \cap V_3^2$ ($\phi_3$ axis) as $(0, 0, x) \rightarrow (0, 0, 2\pi - x)$. Therefore, an equilibrium $p = (0, 0, p_3)$ on $V_2$ has its symmetric counterpart on $V_2$ as $q = \sigma(p) = (0, 0, 2\pi - p_3)$. These equilibria $p$ and $q$ with the connections between them on the invariant planes $V_3^1$ and $V_3^2$ form the heteroclinic network in Figure 6.4.

Recall that the subspaces $V_3^1$ and $V_3^2$ are mapped to each other by the symmetry (6.1.6). Thus, the presence of a connection from $p$ to $q$ on $V_3^1$ implies the presence of another connection on $V_3^2$ that connects $q$ to $p$. Therefore, in order to verify the existence of a heteroclinic network in $\mathbb{T}^3$, it suffices to identify connections from $p$ to $q$ on $V_3^1$, as done in Figure 6.4(a) for the parameter set (6.2.4). Note that this is a heteroclinic ratchet since it includes phase slips in the directions $+\phi_1$ and $+\phi_2$ only (see the winding trajectories in Figure 6.4(b)).

The winding connections of the heteroclinic ratchet are contained in symmetrically related subspaces $V_3^1$ and $V_3^2$ (Figure 6.3), where the dynamics are governed by the quotient networks $N_2$ and $N_3$ illustrated in Table 6.2. However, neither $N_2$ nor $N_3$ has a network symmetry and this can be related to the existence of the heteroclinic ratchet, since a symmetry in these networks may leave out the possibility for a winding orbit or may result in symmetric connections winding in opposite directions.
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

Figure 6.4: Heteroclinic ratchet for the system (6.2.2,6.2.3) with the parameter set (6.2.4). Sources, saddles and sinks are indicated by small disks filled with white, gray or black color, respectively. (a) Phase portrait on $V_3^1$ (projected onto $\mathbb{T}^2$). (b) The heteroclinic ratchet on $\mathbb{T}^3$ (represented by a $2\pi$-cube in $\mathbb{R}^3$).

6.2.2 Stability of the heteroclinic ratchet

A necessary and sufficient condition for stability of a heteroclinic cycle in $\mathbb{R}^3$ whose connections are included in 2-dimensional invariant regions is given in terms of the eigenvalues of equilibria by Melbourne [63] (for more results on stability of heteroclinic cycles see [35, 61]). Melbourne proves that the eigenvalues $\lambda^0(\xi_i) < 0$, corresponding to the eigenvectors tangent to the intersection of the invariant regions, are irrelevant for the stability of heteroclinic cycles and only the saddle quantities $\sigma_i = |\lambda^+(\xi_i)/\lambda^-(\xi_i)|$ determine the stability, where $\xi_i$ is a saddle in the heteroclinic cycle and $\lambda^-(\xi_i) < 0$ ($\lambda^+(\xi_i) > 0$) is the eigenvalue at $\xi_i$ corresponding to the eigenvector on the stable (unstable) manifold of $\xi_i$ that is not contained
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

in the intersection of the invariant regions. Note that the eigenvalues $\lambda^0(\xi_i)$ that correspond to the eigenvectors in the intersection of the invariant regions are necessarily negative for robustness (saddle-to-sink connections on invariant regions). Under some generic assumptions, a heteroclinic cycle in $\mathbb{R}^3$ whose connections are contained in two-dimensional invariant regions is asymptotically stable if $\prod_i \sigma_i < 1$ and is unstable if $\prod_i \sigma_i > 1$.

Now, we will show that the heteroclinic ratchet depicted in Figure 6.4 is asymptotically stable. Note that the equilibria $p = (0, 0, p_3)$ and $q = (0, 0, 2\pi - p_3)$ in $V_2$ are given by

$$p_3 = \cos^{-1} \left( \frac{\cos \alpha_1}{2r_2 \cos \alpha_2} \right).$$

This can be obtained from (6.2.2) by setting $\dot{\phi}_1 = \phi_2 = \dot{\phi}_3 = 0$. Let us calculate the eigenvalues at $p$. Linearizing (6.2.2) at $p$ gives

$$\lambda^+(p) = g'(\mp p) + 2g'(0), \quad \lambda^0 = g'(p) + g'(-p),$$

where $\lambda^+(p)$ and $\lambda^-(p)$ correspond to the eigenvectors in $V_3^1 \setminus V_2$ and $V_3^2 \setminus V_2$, respectively, and $\lambda^0(p)$ is the eigenvalue corresponding to the eigenvector in $V_2 = V_3^1 \cap V_3^2$. A heteroclinic cycle in [63] is defined as a set of saddle equilibria and their one-dimensional unstable manifolds and it is assumed that each of these unstable manifolds is contained in a stable manifold of some equilibrium inside the heteroclinic cycle. Therefore, the heteroclinic ratchet in Figure 6.4 satisfies this definition. Since in our example the equilibria $p$ and $q$ are symmetrically related,
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

it follows from [63] that the heteroclinic ratchet in Figure 6.4 is asymptotically stable if $|\lambda^+(p)/\lambda^-(p)| < 1$ and unstable if $|\lambda^+(p)/\lambda^-(p)| > 1$. For the parameter set (6.2.4), the equilibrium is at $p = 1.4432$. Then, $\lambda^+(p)$ and $\lambda^-(p)$ are 0.74 and $-1.2$ by linearizing (6.2.2) at $p$. This implies the asymptotic stability of the heteroclinic ratchet.

Since the condition for the asymptotic stability is open and the heteroclinic connections are robust, one can find an open set in parameter space $\{(\alpha_1, r_2, r_3) \mid 0 \leq r_2, r_3 \text{ and } 0 \leq \alpha_1 < 2\pi\}$, for which the system (6.2.2) admits an asymptotically stable robust heteroclinic ratchet. On the other hand, for the system (6.2.2), the robust heteroclinic ratchet connecting a pair of saddles $p$ and $q$ on $V_3^1$ cannot be asymptotically stable if $r_3 = 0$ (see Appendix). Therefore, the heteroclinic ratchets for the system (6.2.2) cannot be asymptotically stable unless the third or higher harmonics of the coupling function $g$ are taken into account.

Finally, we show that a robust heteroclinic ratchet that connects the symmetrically related equilibria $p$ and $q$ (as the one in Figure 6.4) cannot be asymptotically stable if only the first two harmonics of the coupling function are considered, namely, if

$$g(x) = -\sin(x + \alpha_1) + r_2 \sin(2x + \alpha_2).$$ (6.2.7)

Without loss of generality, we assume that the heteroclinic network connects the equilibrium $p$ to its symmetric image $q = \rho(p)$ on $V_3^1$ and $q$ to $p$ on $V_3^2$. Note that for the robustness of the heteroclinic ratchet, it is necessary that the heteroclinic connections are contained in invariant subspace of codimension at least
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

one, therefore the connections are necessarily contained in $V_3^1$ and $V_3^2$.

For the asymptotic stability of the heteroclinic network, the following conditions are necessary:

Existence of saddles $p$ and $q$ (from (6.2.5)): 
\[ \frac{\cos \alpha_1}{2r_2 \cos \alpha_2} < 1. \]  
(6.2.8)

Existence of connections on $V_3^1$:
\[ \lambda^0(p) = g'(p) + g'(-p) < 0, \]
\[ \lambda^+(p) > 0, \lambda^-(p) < 0. \]  
(6.2.9, 6.2.10)

Asymptotic stability condition [63]: 
\[ \left| \frac{\lambda^+(p)}{\lambda^-(p)} \right| < 1. \]  
(6.2.11)

Conditions (6.2.10) and (6.2.11) imply
\[ \lambda^+(p) + \lambda^-(p) = g'(p) + g'(-p) + 4g'(0) < 0. \]  
(6.2.12)

We first assume $r_2 \cos \alpha_2 < 0$. From (6.2.9) we have

\[ -2 \cos p \cos \alpha_1 + 4r_2 \cos 2p \cos \alpha_2 < 0 \]  
(6.2.13)

\[ -2 \cos p \cos \alpha_1 + 8r_2 \cos^2 p \cos \alpha_2 - 4r_2 \cos \alpha_2 < 0. \]  
(6.2.14)
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

Substituting (6.2.5) we get

\[
\frac{\cos^2 \alpha_1}{r_2 \cos \alpha_2} - 4r_2 \cos \alpha_2 < 0.
\]

Our assumption then follows

\[
\frac{\cos^2 \alpha_1}{4r_2^2 \cos^2 \alpha_2} > 1,
\]

which contradicts (6.2.8). On the other hand, if we assume \( r_2 \cos \alpha_2 > 0 \), the condition (6.2.12) cannot be satisfied since

\[
\lambda^+ (p) + \lambda^- (p) = g'(p) + g'(-p) + 4g'(0)
\]

\[
= -2 \cos p \cos \alpha_1 + 8r_2 \cos^2 p \cos \alpha_2 - 4r_2 \cos \alpha_2 - 4 \cos \alpha_1
\]

\[
+ 8r_2 \cos \alpha_2
\]

and substituting (6.2.5) one gets

\[
\lambda^+ (p) + \lambda^- (p) = \frac{-\cos^2 \alpha_1}{r_2 \cos \alpha_2} + \frac{2 \cos^2 \alpha_1}{r_2 \cos \alpha_2} + 4r_2 \cos \alpha_2 - 4 \cos \alpha_1
\]

\[
= \left( \frac{\cos \alpha_1}{\sqrt{r_2 \cos \alpha_2}} - 2 \sqrt{r_2 \cos \alpha_2} \right)^2 \geq 0.
\]

Thus, (6.2.12) can not be satisfied for two harmonics coupling.
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

6.2.3 Routes to heteroclinic ratchets

The equilibria \( p \) and \( q \) in \( V_2 = V_3^1 \cap V_3^2 \) bifurcate from the origin via a pitchfork bifurcation simultaneously with other transcritical branches of solutions on \( V_3^{s1} \), \( V_3^{s2} \) and \( V_3^{s3} \). This synchrony-breaking bifurcation is discussed in Theorem 6.1.1. Although we cannot rule out the possibility of the presence of more complex behavior near this bifurcation, we numerically find the heteroclinic ratchet for the parameter values close to the bifurcation point. This suggests that the bifurcation given in Theorem 6.1.1 may be associated with a global bifurcation to a heteroclinic ratchet.

Although the subspace \( V_3^a \) does not include any part of the heteroclinic network, the dynamics restricted to this subspace, that is, the dynamics of the network \( N_1 \) (see Table 6.2) gives rise to another bifurcation to a heteroclinic ratchet as shown in Figure 6.5. The detailed bifurcation analysis of the 3-cell all-to-all coupled oscillators with a coupling function having the first two harmonics is given in [13]. There, it is stated that apart from the transcritical bifurcation of the origin there exists a saddle-node bifurcation on invariant lines. This bifurcation should also exist for nonzero \( r_3 \) values. In Figure 6.5(a)-(c), phase portraits on \( V_3^1 \) are illustrated for \( \alpha_1 = 1.2, \alpha_1 \approx 1.327 \) and \( \alpha_1 = 1.4 \), respectively, while \( r_2 = 0.3 \) and \( r_3 = -0.1 \) are fixed. As \( \alpha_1 \) increases, a sink and a saddle equilibrium on \( V_2^{s1} \) (see Figure 6.5(a)) collide (Figure 6.5(b)) and disappear by a reverse saddle-node bifurcation giving rise to a winding connection from \( p \) to \( q \) (see Figure 6.5(c)). With this disappearance of the sink on \( V_2^{s1} \) (and on \( V_2^{s2} \) by symmetry), a heteroclinic
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

Figure 6.5: Phase portraits for the system 6.2.2, 6.2.3 on $V_1^1$ demonstrating a bifurcation from a heteroclinic cycle (d) to a heteroclinic ratchet (e). Parameters are chosen as $r_2 = 0.3$, $r_3 = -0.1$ and different values for $\alpha$ are considered: (a) $\alpha_1 = 1.2$, (b) $\alpha_1 \simeq 1.327$, and (c) $\alpha_1 = 1.4$. As $\alpha_1$ increases, the reverse saddle-node bifurcation indicated in (b) takes place resulting in disappearance of the sink $s$ and therefore changes the structure of the unstable manifold of $p$. This gives rise in a global bifurcation from a heteroclinic cycle (d) to a heteroclinic ratchet (e). (For each graph sources, saddles and sinks are indicated by small disks filled with white, gray or black color, respectively. The unstable manifolds of $p$ are shown by thick lines.)
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

cycle (see Figure 6.5(d)) that exists for the parameter set

$$(\alpha_1, r_2, r_3) = (1.2, 0.3, -0.1),$$  \hspace{1cm} (6.2.15)

bifurcates to the heteroclinic ratchet which is observed in the previous section for the parameter set (6.2.4) (see Figure 6.5(e)). Therefore, this bifurcation describes another route to heteroclinic ratchets where $S^1$-symmetry is not necessary (see Section 7.2 for a heteroclinic ratchet in a system without symmetry).

Although the heteroclinic cycle seen for the parameter set (6.2.15) satisfies $|\lambda^+(p)/\lambda^-(p)| = |0.68/ -0.7| < 1$, it is not stable because $p$ has an unstable manifold which approaches a sink $s$ outside the heteroclinic ratchet (see Figure 6.5(a) and Figure 6.5(d)). This type of heteroclinic cycle is also unusual for symmetric systems. It attracts nearby trajectories with initial states $\phi(0)$ close to $p$ and with $\phi_1(0), \phi_2(0)$ on the left of $0 \in T^1$, whereas other nearby trajectories with initial states $\phi(0)$ close to $p$ and with $\phi_1(0)$ or $\phi_2(0)$ on the right of $0 \in T^1$ converge to the sink $s$ because of the connection from $p$ to $s$. (see Figure 6.5(d)). Therefore, this heteroclinic cycle has a basin with positive measure, so it is a measure attractor (see Definition 2.2.5), though not asymptotically stable.

6.3 Dynamical consequences of ratchets

We have so far demonstrated that the system of four coupled oscillators in Figure 6.1 with identical natural frequencies $\omega_i$ can support a robust heteroclinic
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

ratchet. In this section, we consider the response of such an attractor to imperfections in the system. In particular, we consider the effect of setting the detunings

$$\Delta_{ij} = \omega_i - \omega_j$$

to be nonzero, and the effect of adding noise to the system. The frequency locking response to detuning and/or noise is an indicator of the heteroclinic ratchet.

For typical trajectories, in terms of the original phases $\theta_i(t) \in \mathbb{R}$, one can define the average frequency of the $i$th oscillator $\Omega_i = \lim_{t \to \infty} \frac{\theta_i(t)}{t}$ and the frequency difference

$$\Omega_{ij} = \lim_{t \to \infty} \frac{\theta_i(t) - \theta_j(t)}{t}.$$

**Definition 6.3.1** We say the $i$th and $j$th oscillators are frequency synchronized on an attractor of the system if all trajectories approaching the attractor satisfy $\Omega_{ij} = 0$.

Note that a stronger notion is phase synchronization; we say the $i$th and $j$th oscillators are *phase synchronized* if all trajectories approaching the attractor have $\theta_i(t) - \theta_j(t)$ bounded in $t$. Phase synchronization is a sufficient condition for frequency synchronization, but the converse is not always true as we see below.

### 6.3.1 Response of the system to detuning

Note that in the case of identical natural frequencies, the oscillators of the original system are frequency synchronized for all trajectories; this follows because
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

trajectories of the reduced phase-difference system are trapped inside a bounded invariant region, namely the boundary of the $2\pi$-cube in Figure 6.3, and so they are phase synchronized. As soon as $\Delta_{ij} \neq 0$ for some $i, j$, this may no longer be the case. Here, we choose three independent detuning variables as $\Delta_{13}, \Delta_{24}, \text{and } \Delta_{34}$ so that the natural frequencies can be written as

\[
\begin{align*}
\omega_1 &= \omega + \Delta_{13} + \Delta_{34} \\
\omega_2 &= \omega + \Delta_{24} \\
\omega_3 &= \omega + \Delta_{34} \\
\omega_4 &= \omega.
\end{align*}
\]

(6.3.1)

Using (6.3.1) instead of (6.1.2), the phase-difference system (6.1.5) can be rewritten as

\[
\begin{align*}
\dot{\phi}_1 &= \Delta_{13} + f(\phi_1; \phi_2 - \phi_3, 0) - f(0; \phi_1, \phi_2 - \phi_3) \\
\dot{\phi}_2 &= \Delta_{24} + f(\phi_2; \phi_1 + \phi_3, 0) - f(0; \phi_1 + \phi_3, \phi_2) \tag{6.3.2} \\
\dot{\phi}_3 &= \Delta_{34} + f(\phi_3; \phi_1 + \phi_3, \phi_2) - f(0; \phi_1 + \phi_3, \phi_2).
\end{align*}
\]

An interesting property of heteroclinic ratchets (such as that illustrated in Figure 6.4) is that the qualitative response to detuning depends on the sign of the detuning. An example showing $\Omega_{13}$, the difference between the observed average frequencies of the oscillators 1 and 3, as a function of $\Delta_{13}$ is given in Figure 6.6. Considering (6.3.2), one can observe that since the heteroclinic ratchet includes

109
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

Figure 6.6: The main graph shows the frequency difference $\Omega_{13}$ for (6.1.1) with parameters (6.2.4) as a function of detuning $\Delta_{13}$. The other independent detuning variables are chosen as $\Delta_{24} = \Delta_{34} = 0$. Note that oscillators remain frequency synchronized for $\Delta_{13} \leq 0$ but quickly break synchrony for $\Delta_{13} > 0$; this is the evidence of the attractor being a heteroclinic ratchet. The insets show time evolution of the phase differences $\phi_i$ for a positive and a negative value of $\Delta_{13}$; observe that oscillators 1 and 3 are phase and frequency synchronized for $\Delta_{13} < 0$ but neither phase nor frequency synchronized for $\Delta_{13} > 0$. 
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

winding connections in the $+\phi_1$ direction but no connections winding in the $-\phi_1$ direction, the oscillator system responds to $\Delta_{13} > 0$ by breaking the frequency synchronization of the oscillator pair (1, 3), whereas $\Delta_{13} \leq 0$ leaves the frequency synchronization unchanged, $\Omega_{13} = 0$. There is a similar response for the difference between oscillators 2 and 4 as can be seen by the symmetry of the original system (6.1.1). Small positive and/or negative detunings $\Delta_{34}$ do not have any qualitative effect on dynamics of (6.3.2) near the heteroclinic ratchet considered, since the heteroclinic ratchet does not include winding connections in the $+\phi_3$ or $-\phi_3$ directions.

6.3.2 Response of the system to noise and detuning

Here, we consider the effect of additive white noise with amplitude $\varepsilon$ for the system (6.3.2) with $\Delta_{34} = 0$ and $\Delta_{13} = \Delta_{24} = \Delta$. The noise terms are added to all equations in (6.2.2) independently and have the same Gaussian distribution with mean 0, amplitude $10^{-6}$ and variance 1. Recall that the heteroclinic cycle shown in Figure 6.4(b) contains two non-winding and two winding trajectories, and a solution converging to the heteroclinic ratchet oscillates near the non-winding trajectories (in the absence of detuning or noise). However, addition of noise to the system (without detuning) will cause phase slips in $+\phi_1$ and $+\phi_2$ directions such that winding will be present even for arbitrary low amplitude $\varepsilon$ (see Figure 6.7).

We define a winding frequency of the system (6.1.1) as $\Omega = (\Omega_{13} + \Omega_{24})/(2\pi)$ and the corresponding winding period as $T = \Omega^{-1}$. For a given noise amplitude $\varepsilon$,
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

Figure 6.7: A solution of the system (6.2.2) with no detuning and additive white noise for the parameter set (6.2.4). The noise terms are added to all equations in (6.2.2) independently and have the same Gaussian distribution with mean 0, amplitude $10^{-6}$ and variance 1. The noise causes the system to have repeated phase slips in the $+\phi_1$ and $+\phi_2$ directions.
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

and detuning $\Delta$, the winding frequency $\Omega(\varepsilon, \Delta)$ can be obtained numerically as in Figure 6.9. Even in the presence of negative detuning $\Delta < 0$, arbitrarily low amplitude noise will eventually cause fluctuations such that the winding trajectories in the ratchet are visited. This can be seen from Figure 6.9(a), where $\Omega$ is plotted as a function of $\Delta < 0$ for different noise amplitudes $\varepsilon$.

The effect of noise on the dynamics near the heteroclinic ratchet is different when $\Delta > 0$ is considered. In this case noise can cause fluctuations such that non-winding trajectories are visited more frequently than in the case of positive detuning without noise. This happens only when $0 < \Delta \ll \varepsilon$, and diminishes the observed winding frequency $\Omega$.

Note that the winding period $T$, in the absence of noise, varies linearly with $\log(\Delta)$ for $0 < \Delta \ll 1$ (see Figure 6.9(c)). This is because $T$ can be expressed in terms of $\Delta$ as

$$T(0, \Delta) = \Omega(0, \Delta) \approx -\frac{1}{\lambda} \ln(\Delta) = -\frac{\ln(10)}{\lambda} \log(\Delta),$$

as expected from the residence time near an equilibrium of a perturbed homoclinic cycle [78], where $\lambda$ is the most positive eigenvalue at the saddle and $\log = \log_{10}$.

In our case, $\lambda = 0.74$ as found in Section 6.2 and the corresponding slope of line representing the relation between $T$ and $\log(\Delta)$ is $-\ln(10) / \lambda = -3.11$, consistent with simulations (see Figure 6.9(c)).

$$\Omega(\varepsilon, \Delta) \approx \Omega(0, \varepsilon) / 2 \text{ for } 0 < \Delta \ll \varepsilon.$$
Chapter 6. A heteroclinic ratchet for a system of 4 coupled oscillators

Figure 6.8: Schematic diagrams demonstrating trajectories switching between saddles of the heteroclinic ratchet in Figure 6.4(b) under small additive noise. (a) A trajectory switching randomly between saddles $p$ and $q$ is shown on the lift of $\mathbb{T}^3$ to $\mathbb{R}^3$. (b) All possible switchings between saddles $p$ and $q$ with probabilities under homogeneous noise are plotted as projected onto $\phi_1 - \phi_2$ plane.
Figure 6.9: Winding frequency $\Omega$ plotted against $\log(\Delta)$ for (a) $\Delta < 0$, (b) $\Delta > 0$ and additive noise of amplitude $\varepsilon$. The corresponding winding period $T = \Omega^{-1}$ is plotted in (c) for $\Delta > 0$. Note that for $|\Delta| \ll \varepsilon$, noise dominates causing a $\Delta$-independent winding, while $\Delta > \varepsilon$ implies winding and $\Delta < -\varepsilon$ gives no winding. The winding period $T$ varies linearly with $\log(\Delta)$ until noise effects dominate.
6.3.3 Frequency synchronization without phase synchronization

Adding homogeneous (same distribution for all equations) noise without detuning can lead to frequency synchronization \textit{without} phase synchronization; one can have a situation where $\phi_1$ and $\phi_2$ are frequency synchronized but $\phi_1 - \phi_2$ is unbounded. This occurs because the presence of unbiased noise means that the average frequency of the phase slips in the $+\phi_1$ and $+\phi_2$ directions should be equal, that is $\lim_{t \to \infty} \frac{\phi_1 - \phi_2}{t} = 0$.

Using the usual phase variables we can write this as

$$\lim_{t \to \infty} \frac{\theta_1 - \theta_3 - \theta_2 + \theta_4}{t} = 0. \quad (6.3.3)$$

Due to the symmetry of the system when the detunings are zero, we have $\Omega_{34} = \lim_{t \to \infty} \frac{\theta_3}{t} = \lim_{t \to \infty} \frac{\theta_1 - \theta_3}{t} = 0$. Thus, (6.3.3) implies $\Omega_{12} = \lim_{t \to \infty} \frac{\theta_1 - \theta_2}{t} = 0$. As a result, the oscillator pairs (1, 2) and (3, 4) are frequency synchronized.

On the other hand, arbitrary small homogeneous noise will cause all oscillator pairs to lose phase synchronization. Moreover, the oscillator pairs (1, 3) and (2, 4) lose their frequency synchronization since noise results in repeated forward phase slips of the oscillators 1 and 2 due to the winding connections of the ratchet, whereas the pairs (1, 2) and (3, 4) maintain their frequency synchronization without phase synchronization.
“Intuition is science’s most powerful and yet most untrust-worthy engine.”

Lewis Carroll Epstein

Further examples of heteroclinic ratchets

In this chapter, we mention some interesting examples of heteroclinic ratchets found in various systems of coupled oscillators. Firstly, we give an example where phase slips between oscillators occur even without noise or detuning. In this case a trajectory approaching the heteroclinic ratchet perpetually undergoes phase slips, therefore, phase differences are not bounded and some oscillators are not phase synchronized. However, an application of Theorem 3.5.3 shows that oscillators maintain frequency synchronization. In Section 7.2, we give an example where reduction to phase difference space is not possible, but still one can observe a heteroclinic ratchet connecting limit cycles. This system has no symmetry unlike the main example in the previous chapter. Therefore, it shows that ratcheting does not rely on symmetry in the system. Finally, in Section 7.3, we give two examples of heteroclinic ratchets in high dimensional systems. We consider two
Chapter 7. Further examples of heteroclinic ratchets

generalizations of the coupling structure given in Figure 6.1 and show ratcheting phenomena for high dimensional coupled systems with these coupling structures.

This chapter contains original material. The first three examples are published in [54] and [55], whereas the last two examples are unpublished.

7.1 Ratcheting without noise or detuning

The heteroclinic ratchet described in Chapter 6 exists robustly for the 4-cell coupled system (6.1.1). This is due to the persistence of invariant subspaces in which the heteroclinic connections are saddle-to-sink type, and therefore robust. However, as seen in the previous chapter, these invariant subspaces can trap the dynamics such that phase slips (winding of the trajectory around the phase difference torus) cannot occur. The example shown in this section implies that this is not always the case. Therefore, one directional phase slips occur repeatedly even in the absence of noise or detuning. This gives rise to the question as to whether oscillators stay frequency synchronized in this case. We provide an answer to this question using Theorem 3.5.3.
Consider the following system of ODEs:

\[
\begin{align*}
\dot{\theta}_1 &= \omega + f(\theta_1, \theta_2, \theta_3) + 2\sin(\theta_1 - \theta_3) \\
\dot{\theta}_2 &= \omega + f(\theta_2, \theta_1, \theta_4) \\
\dot{\theta}_3 &= \omega + f(\theta_3, \theta_1, \theta_2) + 2\sin(\theta_1 - \theta_3) \\
\dot{\theta}_4 &= \omega + f(\theta_4, \theta_1, \theta_2),
\end{align*}
\]

(7.1.1)

where \( f \) is given in (6.2.1) and (6.2.3). Choosing phase difference variables \( \phi_1 := \theta_1 - \theta_3, \phi_2 := \theta_2 - \theta_4 \) and \( \phi_3 := \theta_3 - \theta_4 \), we can write the reduced system as follows:

\[
\begin{align*}
\dot{\phi}_1 &= f(\phi_1, \phi_2 - \phi_3, 0) - f(0, \phi_1, \phi_2 - \phi_3) \\
\dot{\phi}_2 &= f(\phi_2, \phi_1 + \phi_3, 0) - f(0, \phi_1 + \phi_3, \phi_2) \\
\dot{\phi}_3 &= f(\phi_3, \phi_1 + \phi_3, \phi_2) - f(0, \phi_1 + \phi_3, \phi_2) + 2\sin \phi_1.
\end{align*}
\]

(7.1.2)

Note that the surfaces \( \phi_1 = 0 \) and \( \phi_2 = 0 \) are invariant subspaces for any system of the form (7.1.2), but \( \phi_3 = 0 \) may not be invariant. Setting \( \alpha_1 = 1.4, r_2 = 0.3 \) and \( r_3 = -0.1 \), one can verify, using numerical simulation and examination of the flows in the invariant subspaces, that there is an attracting heteroclinic ratchet contained within the invariant subspaces (see Figure 7.1a). This heteroclinic network consists of a heteroclinic cycle between the equilibria \( p_1 \) and \( p_2 \) that winds in \(-\phi_3\) direction. Therefore, when lifted to \( \mathbb{R}^3 \), a typical trajectory converging to the attractor has \( \phi_3^L \to -\infty \), where \( \phi^L \) denotes the lifted trajectory (see Figure 7.1b). Namely, the phase difference \( \theta_4 - \theta_3 \) increases unboundedly. Therefore, oscillators 3 and 4 are not
Chapter 7. Further examples of heteroclinic ratchets

Figure 7.1: A heteroclinic attractor \((\{p_1\} \cup q_1 \cup \{p_2\} \cup q_2)\) for the system (7.1.2). (a) Schematic illustration of the attractor in \(\mathbb{T}^3 = \{\phi_1, \phi_2, \phi_3\}\) lifted to \(\mathbb{R}^3\). (b) The \(\phi_3\) component of a trajectory converging to the attractor; note that a lift of this component will be unbounded, but it grows so slowly that its average converges to zero.

Phase synchronized. However, one can show that they are frequency synchronized. This follows directly from Theorem 3.5.3 as follows: Observe that the heteroclinic ratchet shown in Figure 7.1 is a \((3, 4)\)-ratchet according to Definition 3.5.1. The recurrent set of the heteroclinic ratchet consists of two constant solutions, namely \(p_1\) and \(p_2\). For these, the condition \(\lim_{T \to \infty} \frac{1}{T} \mathcal{L}_\gamma \pi_{i,j}(x(t)) \, dt = 0\) is satisfied because the Lie derivative of a function vanishes along constant solutions.
7.2 Heteroclinic ratchets in a system without symmetry

Although the system (6.1.1) has $S_2$ permutation and $S^1$ phase-shift symmetries, we show in this section that these symmetries are not necessary for the existence of a heteroclinic ratchet. In fact, for the system (6.1.1), the $S_2$ symmetry merely simplifies the existence and stability discussions in Section 6.2.1 and 6.2.2 and gives rise to a clear explanation for the emergence of the heteroclinic ratchet via the synchrony-breaking bifurcation in Theorem 6.1.1. On the other hand, $S^1$ symmetry makes it possible to describe heteroclinic connections between periodic orbits of (6.1.1) by heteroclinic connections between saddle equilibria of (6.1.5) with the help of the phase-difference reduction. In order to see that $S_2$ and $S^1$ symmetries are not necessary for the existence of heteroclinic ratchets, we consider a perturbed system of (6.1.1) on $\mathbb{T}^4$:

\[
\begin{align*}
\dot{\theta}_1 &= \omega + f(\theta_1; \theta_2, \theta_3) + \alpha_1 \cos(\theta_1) \\
\dot{\theta}_2 &= \omega + f(\theta_2; \theta_1, \theta_4) + \alpha_2 \cos(\theta_2) \\
\dot{\theta}_3 &= \omega + f(\theta_3; \theta_1, \theta_2) + \alpha_1 \cos(\theta_3) \\
\dot{\theta}_4 &= \omega + f(\theta_4; \theta_1, \theta_2) + \alpha_2 \cos(\theta_4).
\end{align*}
\]  

(7.2.1)

Note that the above system has the same coupling structure as in Figure 6.1, but with two different cell types, namely the cells 1 and 3 are of one type and the
Figure 7.2: A non-symmetric coupled cell network that allows heteroclinic ratchets. The coupling structure is the same as the network in Figure 6.1 but there are two different cell types.

cells 2 and 4 are of another type as illustrated in Figure 7.2. This is due to the $\alpha_j \cos(\theta_k)$ terms in (7.2.1). The balanced coloring method from Section 6.1.1 only gives three non-trivial invariant subspaces $V_2$, $V_3^1$ and $V_3^2$, since in the case of different cell types, only cells of the same type can have the same color. Note that these invariant subspaces are the ones that contain the saddles and the connections of the heteroclinic ratchet for the symmetric system (6.1.1). Therefore, we expect robustness of the heteroclinic ratchet for (7.2.1) that exists when $\alpha_1 = \alpha_2 = 0$. Here, by robustness we mean persistence under small enough perturbations of the system that preserve the connection structure and cell types, specifically perturbations of the parameters $\alpha_1$ and $\alpha_2$. We denote by $\bar{\bar{p}}(0)$ and $\bar{\bar{q}}(0)$ the saddle periodic orbits of (7.2.1) in $V_2$ for $\alpha_1 = \alpha_2 = 0$ corresponding to the saddle equilibria $p$ and $q$ in Figure 6.4 for the phase-difference system (6.1.5). In $T^4$, the heteroclinic ratchet is between the saddle periodic orbits $\bar{\bar{p}}(0)$ and $\bar{\bar{q}}(0)$, whereas the connections between these are the two dimensional unstable manifolds of $\bar{\bar{p}}(0)$.
Chapter 7. Further examples of heteroclinic ratchets

Figure 7.3: A solution of the system (7.2.1) with additive white noise (amplitude=10^{-6}) for $\alpha_1 = 0.01$ and $\alpha_2 = 0.02$. Repeated phase slips in specific directions indicate the presence of an attracting heteroclinic ratchet. The inset shows the oscillations in detail when the first oscillator undergoes a forward phase slip relative to the others.

and $\bar{q}(0)$, which are contained in $V_3^1$ and $V_3^2$ respectively. Similar to the case in the phase-difference system, $\bar{p}(0)$ ($\bar{q}(0)$) is a sink in $V_3^2$ ($V_3^1$) and a saddle in $V_3^1$ ($V_3^2$). By robustness, we expect the heteroclinic ratchet in $T^4$ between the perturbed periodic orbits $\bar{p}(\alpha)$ and $\bar{q}(\alpha)$ to persist for small enough $\alpha_1$ and $\alpha_2$. In Figure 7.3, a solution of the perturbed system with additive white noise is shown for $\alpha_1 = 0.01$ and $\alpha_2 = 0.02$. Repeated forward phase slips of the oscillators 1 and 2 are indicators of the presence of a heteroclinic ratchet. The inset in this figure shows clearly the transition from one periodic orbit to another as discussed above, accompanied by a forward phase slip of oscillator 1 relative to the other oscillators.
Chapter 7. Further examples of heteroclinic ratchets

Figure 7.4: A coupled cell network that consists of 6 identical cells and allows heteroclinic ratchets.

7.3 Ratcheting in large networks

One may expect to observe heteroclinic ratchets in larger oscillator networks. However, due to the growth in phase space dimension, analysis of heteroclinic ratchets can be quite complex as these may include unstable manifolds of saddles with dimension greater than one. Here, we give two examples of large coupled cell network structures, which can support heteroclinic ratchets.

Let us consider the 6-cell network illustrated in Figure 7.4 and simulate the same coupled oscillator dynamics in (5.2.2) where \( N = 6 \), \( c_{ij} \)'s are determined by the given network structure and the coupling function \( g \) is the same as in (6.2.3). Similar to the example in Section 6.2, ratcheting solutions are found when small additive noise is applied. The phase differences are illustrated in Figure 7.5, which suggests the existence of an attracting heteroclinic ratchet in \( \mathbb{T}^6 \).
Figure 7.5: A solution of the coupled oscillator network in Figure 7.4 under small additive white noise (amplitude = $10^{-4}$) with zero initial states. The equations for the dynamics are as in (5.2.2) and (6.2.3) and the parameters are $\alpha_1 = 1.15$, $r_2 = 0.3$ and $r_3 = -0.1$. The phase differences between certain pairs of oscillators increase monotonically. This suggests the existence of an attracting heteroclinic ratchet including connections winding in $\theta_1 - \theta_4$, $\theta_2 - \theta_5$ and $\theta_3 - \theta_6$ directions.
Chapter 7. Further examples of heteroclinic ratchets

The network structure given in Figure 7.4 can be generalized to \(2N\)-cell networks to give heteroclinic ratchets in higher dimensional tori (see Figure 7.6). Another generalization of the 4-cell network in Figure 6.1 is given in Figure 7.7. This network consists of \(n_1 + n_2 + 2\) cells. A coupled cell system that has this network structure can be written as

\[
\begin{align*}
\dot{\theta}_1 &= f(\theta_1, \theta_2, \ldots, \theta_{n_1+n_2}, \theta_{n_1+n_2+1}) \\
\dot{\theta}_2 &= f(\theta_2, \theta_1, \theta_3, \ldots, \theta_{n_1+n_2}, \theta_{n_1+n_2+1}) \\
&\vdots \\
\dot{\theta}_{n_1} &= f(\theta_{n_1}, \theta_1, \ldots, \theta_{n_1-1}, \theta_{n_1+1} \ldots \theta_{n_1+n_2}, \theta_{n_1+n_2+1}) \\
\dot{\theta}_{n_1+1} &= f(\theta_{n_1+1}, \theta_1, \ldots, \theta_{n_1}, \theta_{n_1+2} \ldots \theta_{n_1+n_2}, \theta_{n_1+n_2+2}) \\
&\vdots \\
\dot{\theta}_{n_1+n_2} &= f(\theta_{n_1+n_2}, \theta_1, \ldots, \theta_{n_1+n_2-1}, \theta_{n_1+n_2+2}) \\
\dot{\theta}_{n_1+n_2+1} &= f(\theta_{n_1+n_2+1}, \theta_1, \ldots, \theta_{n_1+n_2}) \\
\dot{\theta}_{n_1+n_2+2} &= f(\theta_{n_1+n_2+2}, \theta_1, \ldots, \theta_{n_1+n_2}).
\end{align*}
\]

Defining new variables \(\phi_i = \theta_i - \theta_{n_1+n_2+1}\) for \(i = 1, \ldots, n_1\), \(\phi_i = \theta_i - \theta_{n_1+n_2+2}\) for \(i = n_1 + 1, \ldots, n_1 + n_2\) and \(\phi_{n_1+n_2+1} = \theta_{n_1+n_2+1} - \theta_{n_1+n_2+2}\), one can write the reduced dynamics on the phase difference space \(\mathbb{T}^{n_1+n_2+1}\) as follows:
Chapter 7. Further examples of heteroclinic ratchets

Figure 7.6: $2N$-cell coupled cell networks that may support heteroclinic ratchets in higher dimension. Note that for $N = 2$ and $N = 3$ one gets the networks given in Figure 6.1 and 7.4, respectively.

For $i = 1, \ldots, n_1$,

$$
\dot{\phi}_i = f(\phi_i, \phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_{n_1}, \phi_{n_1+1} - \phi_{n_1+n_2+1}, \ldots, \phi_{n_1+n_2} - \phi_{n_1+n_2+1}, 0)
- f(0, \phi_1, \ldots, \phi_{n_1}, \phi_{n_1+1} - \phi_{n_1+n_2+1}, \ldots, \phi_{n_1+n_2} - \phi_{n_1+n_2+1}),
$$

for $i = n_1 + 1, \ldots, n_1 + n_2$,

$$
\dot{\phi}_i = f(\phi_i, \phi_1 + \phi_{n_1+n_2+1}, \ldots, \phi_{n_1} + \phi_{n_1+n_2+1}, \phi_{n_1+1}, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_{n_1+n_2}, 0)
- f(0, \phi_1 + \phi_{n_1+n_2+1}, \ldots, \phi_{n_1} + \phi_{n_1+n_2+1}, \phi_{n_1+1}, \ldots, \phi_{n_1+n_2}),
$$

and for the last variable

$$
\dot{\phi}_{n_1+n_2+1} = f(\phi_{n_1+n_2+1}, \phi_1 + \phi_{n_1+n_2+1}, \ldots, \phi_{n_1} + \phi_{n_1+n_2+1}, \phi_{n_1+1}, \ldots, \phi_{n_1+n_2})
- f(0, \phi_1 + \phi_{n_1+n_2+1}, \ldots, \phi_{n_1} + \phi_{n_1+n_2+1}, \phi_{n_1+1}, \ldots, \phi_{n_1+n_2}).
$$

(7.3.1)
Chapter 7. Further examples of heteroclinic ratchets

Figure 7.7: $n_1 + n_2 + 2$-cell coupled cell networks that may support heteroclinic ratchets in higher dimension ($n := n_1 + n_2 + 1$). Note that for $n_1 = n_2 = 1$ this network is identical to the network in Figure 6.1.

Here, we again assume the phase shift symmetry of $f$, namely $f(x) = f(x + \varepsilon(1, \ldots, 1))$ for all $x \in \mathbb{T}^{n_1 + n_2 + 1}$ and for all $\varepsilon \in \mathbb{T}$. Note that permutations within the first group of $n_1$ cells or within the second group of $n_2$ cells would not change the network. Therefore, the coupled cell system has $S_{n_1} \times S_{n_2}$-symmetry, which means that ratcheting cannot occur between the cells in the first group or between the cells in the second group, but ratcheting of a cell in one of these groups can occur with respect to the $(n_1 + n_2 + 1)$th or $(n_1 + n_2 + 2)$th cell. Note that the subspaces

- $\{\phi_i = 0\}$ for $i = 1 \ldots, n_1 + n_2 + 1$
- $\{\phi_i = \phi_j\}$ for $i, j \in \{1, \ldots, n_1\}$
- $\{\phi_i = \phi_j\}$ for $i, j \in \{n_1 + 1, \ldots, n_1 + n_2\}$
Figure 7.8: A solution of (7.3.1) with $n_1 = 4$ and $n_2 = 3$ under small additive white noise (amplitude = $10^{-4}$). The parameters for the coupling function are chosen as $\alpha_1 = 1.4$, $r_2 = 0.3$ and $r_3 = -0.1$. The transient dynamics shows a switching from one ratcheting behaviour to another.

- $\{\phi_{n_1+n_2+1} = 0, \phi_i = \phi_j\}$ for $i,j \in \{n_1 + n_2\}$

are invariant. All the other invariant subspaces of the system (7.3.1) arise as intersections of these subspaces.

For an example, we choose $n_1 = 4$ and $n_2 = 3$ and employ the same coupling as in (5.2.2) and (6.2.3) with parameters $\alpha_1 = 1.4$, $r_2 = 0.3$ and $r_3 = -0.1$. For these parameters, a solution of the system (7.3.1) is shown in Figure 7.8. The transient dynamics switches between different ratcheting behaviour. From $t = 0$ to $t \approx 500,$
Chapter 7. Further examples of heteroclinic ratchets

ratcheting occurs for the first four oscillators with respect to the 8th oscillator and then ratcheting occurs for the second group of oscillators (5-7) with respect to the 9th oscillator. After long run the dynamics settles downs at a stable equilibrium. Nevertheless, the effect of the coupling structure manifests itself in the transient dynamics.

Analyzing the structure of high dimensional robust heteroclinic ratchets and finding conditions for networks that allow heteroclinic ratchets are interesting topics motivated by the examples given here. For the latter, the balanced partition method (see Section 4.2) can be used even for larger networks to see whether there exist invariant subspaces on which robust heteroclinic ratchets can be found.
Part IV

Conclusion
“I wish he, the old savant who knows all mysteries, would not hide anything from me. Last night, he came quietly and told me: ‘Do not ask! You must feel, in order to learn the things impossible to tell.’”

Rumi

Discussion

In this thesis, heteroclinic networks have been investigated in terms of their effects on time averages and, in particular, on the synchronization properties of coupled oscillators. A new phenomenon that we call ratcheting has been analyzed in detail. In order to explain the frequency synchronization in ratcheting, the convergence of time averages of continuous functions has been studied. Some results on different notions of limit sets, attractors, and on the convergence of time averages have been obtained in Chapter 2 for general continuous flows. In particular, for continuous flows on a torus that admit attracting heteroclinic networks, convergence of time averages and synchronization properties have been investigated in Chapter 3. These two chapters form the first part of the thesis (Part I) where new abstract results are stated. In Part II, some background for coupled cell systems and networks of coupled oscillators have been given. In particular, recent results
on the effect of heteroclinic cycles on synchronization properties of coupled oscillators have been reviewed in Section 5.4. Finally, in Part III, a new interesting example of a heteroclinic network, called a heteroclinic ratchet, has been discussed in terms of its effect on an important time average, namely, the average frequency. The results obtained in this thesis give rise to new problems, on the one hand, about the general theory for convergence of time averages and attractors, and on the other hand, about ratcheting phenomenon for coupled oscillators.

A relation between statistical attractors and convergence of time averages has been described in Corollary 2.3.7. Naturally, the question arises as to whether the conditions in this corollary can be weakened. Another interesting question is the following: Lemma 2.2.4 provides a unifying approach for different definitions of attractors and implies that the statistical attractor can be defined in terms of the essential $\omega$-limit set (Theorem 2.2.2). Is it possible to define other new and useful notions of attractors using this approach or can this approach help us in understanding the existing ones better? For example, as pointed out by an anonymous referee of [55], a relation between minimal attractor\(^1\) and the limit set $\omega_{\min}(x) := \{z \in X : \liminf_{t \to \infty} \rho(x, U, t) > 0, \forall U \in \mathcal{U} \}$ may be possible. Consider the modified Bowen's example studied by Kleptsyn [57], namely the heteroclinic cycle shown in Figure 2.1 where $p_1$ is non-hyperbolic with exponential contraction and $p_2$ is hyperbolic. For this example, as shown below, $\omega_{\text{min}}(x) = \{p_1, p_2\}$ and $\omega_{\text{min}}(x) = \{p_1\}$ for a typical initial point $x \in X$, which suggests that $\omega_{\text{min}}$

\(^{1}\) In [57], the minimal attractor is defined as the complement of the union of open sets $U$ that satisfy $\frac{1}{T} \int_0^T \ell(\gamma_{-t}(U)) \, dt \to 0$ as $T \to \infty$. 

133
may be related to the minimal attractor introduced by Il'yashenko. (the point \( p_1 \) for the modified Bowen’s example). However, there is no reason to assume that \( \omega_{\min}(x) \) is non-empty in general, and in addition, the minimal attractor is a set-wise definition. This suggests that the relationship between minimal attractor and \( \omega_{\min} \) may be non-trivial. In order to see that \( \omega_{\text{ess}}(x) = \{p_1, p_2\} \) and \( \omega_{\min}(x) = \{p_1\} \), let \( \tau_{n,1}(x) \) and \( \tau_{n,2}(x) \) denote the period of time spent by the orbit \( \gamma_{t}x \) in some neighborhoods of \( p_1 \) and \( p_2 \), respectively, on the \( n \)th turn (the first turn starting from the first entrance of the trajectory to the neighborhood of \( p_1 \)). By [57, Proposition 1] a typical trajectory asymptotically spends comparable periods of time near \( p_1 \) and \( p_2 \), namely \( \tau_{n,2}(x)/\tau_{n,1}(x) \to c \neq 0 \) as \( n \to \infty \).

Therefore, \( \omega_{\text{ess}}(x) = \{p_1, p_2\} \). However, [57, Equation 1 and Equation 2] imply that \( \tau_{n+1,1}(x)/\tau_{n,1}(x) \to \infty \) as \( n \to \infty \). Hence, for a sufficiently small open neighborhood \( U_2 \) of \( p_2 \), \( \liminf_{t \to \infty} \rho(x, U_2, t) = 0 \), but for all open neighborhoods \( U_1 \) of \( p_1 \), \( \liminf_{t \to \infty} \rho(x, U_1, t) > 0 \). Namely, \( \omega_{\min}(x) = \{p_1\} \).

The ratcheting phenomenon introduced in Part III is novel for coupled oscillators literature. It has strong effect on the synchronization properties of oscillators. Although heteroclinic networks, and therefore also heteroclinic ratchets, are not structurally stable for generic dynamical systems, they can exist robustly for families of coupled dynamical systems with certain coupling structures. Some examples of such connection structures have been given in Section 7.3. Identifying such connection structures that may support heteroclinic ratchets needs further work which may require new results in coupled systems theory summarized in Part II. For ex-
ample the idea of inflation, used for generating larger networks from smaller ones [2], might be useful if it can be done in such a way that certain properties of the network are preserved. One can further discuss that the connection structure required for robustness are also highly degenerate. Here, we should point out that small perturbation in the coupling structure, for instance varying the coupling strengths of some arrows can yield to bifurcations to periodic orbits winding in certain directions (for bifurcations of nonsingular heteroclinic cycles see [82, 27]). Such systems close to a ratcheting system may be structurally stable. Moreover, the winding vectors of these periodic orbits would depend on the properties of the heteroclinic ratchet as well as on the perturbation term, and give rise to certain frequency-locked oscillations. In other words, a system with a heteroclinic ratchet can play a role of an organizing center. Another question motivated by the work presented in Part III is about the possibility of finding ratcheting phenomenon in physical systems. In this thesis, we have shown the existence of ratcheting using computer simulation. However, whether the ratcheting phenomenon takes place in physical systems and whether it plays an important role is not known. It may be possible to observe ratcheting phenomenon in neural systems for which coupled phase oscillator models have been proved to be useful [71] and heteroclinic networks might play an important role in dynamics [70, 17].
Bibliography


Bibliography


Bibliography


Bibliography


Bibliography


Bibliography


Bibliography


Bibliography


