Decelerating defects and non-ergodic critical behaviour in a unidirectionally coupled map lattice

Peter Ashwin*and Rob Sturman†
School of Mathematical Sciences,
University of Exeter, Exeter EX4 4QE, UK

February 4, 2003

Abstract

We examine a coupled map lattice (CML) consisting of an infinite chain of logistic maps coupled in one direction by inhibitory coupling. We find that for sufficiently strong coupling strength there are dynamical states with 'decelerating defects', where defects between stable patterns (with chaotic temporal evolution and average spatial period two) slow down but never stop. These defects annihilate each other when they meet. We show for certain states that this leads to a lack of convergence (non-ergodicity) of averages taken from observables in the system and conjecture that this is typical for these states.

1 Introduction

Coupled map lattices (CMLs) are simple dynamical systems consisting of arrays of maps that are coupled together in some way. They have been proposed since the mid-1980s as a paradigm in which one can understand many properties of high-dimensional spatially extended nonlinear systems while keeping within a framework that allows for very easy simulation and analysis, and enables progress in understanding the phenomenology of spatially extended systems. They have been particularly successful in illuminating the transition from turbulent isotropic space-time chaos to chaos that has spatial structure [11, 14, 15], in understanding the persistence of space-time chaos under the influence of weak coupling [3, 6, 10], and in examining critical coupling effects [12].

Many studies of coupled map lattices have concentrated on asymptotic behaviour that is characterized by natural (SRB) ergodic invariant measures. However, in the dynamics of systems with invariant subspaces it has been recognised for several years that attractors of heteroclinic type can appear robustly, see for example [9] for a review of robust heteroclinics in symmetric systems, [7] for cycles in flows on simplices. These cycles may also be robust between chaotic invariant sets [1, 4]. The dynamics of such maps $F : M \to M$ possessing such cycles are non-ergodic in the sense that averages of a smooth observable $\phi : M \to \mathbb{R}$

$$\frac{1}{n}\sum_{i=0}^{n-1}\phi(F^i(X))$$

*P.Ashwin@ex.ac.uk
†rsturman@amsta.leeds.ac.uk, Also: Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK.

do not converge as $n \to \infty$ but continue to oscillate progressively at a slower rate *ad infinitum* for an open set of initial conditions; see [5, 13]. A consequence of this lack of convergence is that the trajectory is not typical for any ergodic invariant measure of the system and so, loosely speaking, we term the trajectory as being non-ergodic.

This means that averaged observed quantities such as Lyapunov exponents are not well defined but will similarly oscillate. This paper examines a CML that shows non-ergodic behaviour in a robust way. Although the results we prove in detail are for a specific system, we discuss in Section 4 some ingredients for this behaviour to remain typical.

Consider an infinite chain of coupled maps

$$X_{n+1}^{k} = f(X_n^k)e^{-\gamma X_n^{k-1}} \tag{1}$$

where $f(x) = rx(1-x)$ is the logistic map and $\mathbf{X} = \{X_n^k\}$ is the state of the system indexed at spatial location $k \in \mathbb{Z}$ and time $n \in \mathbb{N}$. We have two parameters; r determining the local behaviour of the maps, and γ determining the coupling strength. We are only interested in the case that $r > 2$, meaning that there can be non-trivial dynamics of (1).

The CML (1) has coupling in one direction only (from left to right) allowing information to flow only one way along the chain. However as we will discuss in Section 4, the assumption of one-way (or local) coupling is not necessary for the behaviour we discuss. Although the model is not intended as a model of any specific spatially extended system, it represents a chain of coupled chaotic units that have a parametric inhibiting effect on their neighbours and so is analogous to a neural circuit where neighbouring connections are inhibitory.

In Section 2 we examine the spatio-temporal dynamics of the CML (1). For small γ the dynamics of the CML are qualitatively close to that for $\gamma = 0$, namely spatio-temporal chaos for large enough r. For this regime we observe well-defined ergodic behaviour. There is a $\gamma_c > 0$ such that for $\gamma > \gamma_c > 0$ we see the appearance of stable patterns of spatially periodic stripes where for fixed $k \in \mathbb{Z}$, X_n^{2k} are chaotic and X_n^{2k+1} tend to zero (cf [15]). For an infinite chain, the stable patterns will form locally with defects between them. We hence numerically observe and conjecture the existence of a state of *decelerating defects* for typical initial conditions and any $\gamma > \gamma_c$. The defects propagate in the direction of coupling and slow down, but never stop unless they are pinned by one of the sites being exactly zero. The defects annihilate on collision leading to a gradual reduction in the density of defects in the system.

In Section 3 we attempt a rigorous description of the periodic patterns and the decelerating defect state. We obtain estimates for how these defects move along the chain; they move such that the times between propagation of the defect by one site increase geometrically. For the special case where the initial condition is spatially periodic with period-3 we prove that the behaviour is non-ergodic for large enough γ and conjecture that this is true for a full measure set of initial conditions on the infinite lattice and $\gamma > \gamma_c$.

Finally, Section 4 discusses several aspects including the genericity of the model and its results.

Notation We write $\mathbf{X} = \{X^k\} \in S := [0,1]^{\mathbb{Z}}$ as the initial vector and assume that $2 < r < 4$ and $0 < \gamma$ are chosen so that the forward orbit is well-defined within S, i.e. we write (1) as a map

$$\Phi : S \to S.$$

Note that the maximum that any site can reach under iteration by (1) is $f(1/2) = r/4$ and so we define

$$\tilde{S} = \{\mathbf{X} \in S \ : \ X^k \leq r/4 \text{ for all } k \in Z\}$$

and note that $\Phi(S) \subset \tilde{S}$. Hence by ignoring the first iterate we can assume that $\mathbf{X} \in \tilde{S}$. We define as usual the ℓ^∞-metric on S by $\|\mathbf{X}\|_\infty = \sup_{k\in\mathbb{Z}} |X^k|$ for all n, k and note that one can define a probability measure μ on S given by product of Lebesgue measure in each factor. The system possesses an uncountably infinite number of invariant subspaces; given any subset $I \subset \mathbb{Z}$ the set of $\{\mathbf{X} \;:\; X^k = 0 \;\text{ if } k \in I\}$ is an invariant subspace. In particular if we set any spatial site to zero it will remain zero thereafter.

2 Stable patterns

In this paper we consider values of r such that the logistic map f has chaotic dynamics and examine the stable dynamics of the CML (1).

2.1 Turbulent dynamics

The CML (1) shows a variety of stable dynamics; the simplest dynamics is for values of γ close to zero, where an uncoupled attractor is apparently only deformed. If there is a natural measure for f that is weak-mixing, note that this means that the infinite product measure is ergodic for the uncoupled system and hence there is a natural measure in this case. For example, Figure 1(a) illustrates the turbulent state for $r = 3.8$ and $\gamma = 0$ and apparent decay of correlations in both space and time. We find that this behaviour is qualitatively preserved for small values of γ, though not for any interval. There are circumstances where one can prove persistence of the fully turbulent state (for example [3, 6, 8]) but these rely on having coupled particular expanding maps. For our system the fragility of chaotic attractors of the logistic map [2] means that we expect a positive measure (but not open) set of γ that have essentially the same dynamics as $\gamma = 0$.

2.2 Period-2 spatial dynamics

For larger values of γ we obtain attractors that are chaotic patterns such that for any continuous observable $\phi : [0, 1] \to \mathbb{R}$ the average at each site

$$\overline{\phi}^k := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(X_j^k)$$

has period two in k; in such a case we say the state has *spatial period two on average*.

In between this and the turbulent dynamics there are a number of transitions that we have not examined in detail but they show right propagating fluctuations within the patterns, for example Figure 1(b). The period-2 patterns are stable when sites are suppressed on average by their neighbours. We refer to the suppressed sites where $X_n^k \to 0$ as *quiescent*. We can estimate that these are stable when the geometric average of the growth rate $re^{-\gamma x_n}$ of a quiescent state near 0 is less than unity, i.e. when $\ln r - \gamma\langle x_n\rangle < 0$, giving that this state is stable when

$$\gamma > \gamma_c = \frac{\ln r}{A_\infty}, \tag{2}$$

where $A_\infty = \langle x_n\rangle$ is the average of x for the natural measure of the logistic map f. For $r = 3.8$ we numerically estimate this as $\gamma_c = 2.079$.

2.3 Decelerating defects

Consider a spatially periodic state with period N for $\gamma > \gamma_c$. Clearly if N is odd, the pattern with average spatial period 2 cannot occur and there must be at least one defect; numerical experiments suggest that almost all initial conditions lead to a state where there is exactly one defect and an average spatial period 2 pattern of length $N-1$; (see Section 4 for a further discussion). We estimate how long it takes for a defect to move by seeing how close a quiescent state comes to zero. In the time it takes for a defect to move forwards by one step, a quiescent state will decay at average rate $re^{-\gamma A_\infty}$ for $N-2$ time periods and then grow at a rate r. In the case that

$$(re^{-\gamma A_\infty})^{N-2} r < 1 \tag{3}$$

then the smallest quiescent state will be even smaller after the defect has moved, meaning that the defect will take longer to move next time. The inequality (3) implies that slowing down will occur for

$$\gamma > \frac{(N-1)\ln r}{(N-2)A_\infty}$$

For $N = 3$ this confirms the case examined in [1] while for $N \to \infty$ this converges to γ_c. Hence for any $\gamma > \gamma_c$ in the infinite system we expect to observe decelerating defects when the domains separating them become large enough.

3 Rigorous characterisation of the dynamics

We now prove the existence and nature of some of the dynamics discussed above. The following results are not optimal with regard to estimates for γ in particular, but serve to illustrate the dynamics for large γ.

3.1 Stability of the period 2 spatial dynamics

The space-periodic dynamics can be characterized as the state where the individual cells alternate between being *active* (with average value positive) or quiescent. We define

$$P_{\mu,\nu} = \{\mathbf{X} \ : \ X^{2k} < \mu \text{ and } X^{2k+1} > \nu\}$$

and

$$P_{0,\nu} = \{\mathbf{X} \ : \ X^{2k} = 0 \text{ and } X^{2k+1} > \nu\} = \bigcap_{\mu > 0} P_{\mu,\nu}.$$

We assume that $\mathbf{X} \in \tilde{S}$.

Theorem 1. *For any $2 < r < 4$ and small enough ν, if $\gamma > \frac{\ln r}{\nu}$ then $P_{0,\nu}$ attracts all nearby trajectories in the ℓ^∞ norm.*

Note that the estimate $\frac{\ln r}{\nu}$ is considerably greater than γ_c defined in (2); this is because we prove a stronger form of stability (asymptotic stability). Figure 1(c) and (d) illustrate these period two stripes appearing within more complex patterns; they only become unstable through invasion by other patterns. Again, if r is such that $x \mapsto rx(1-x)$ has a natural invariant measure μ that is weakly mixing then we can infer that there is a natural invariant measure for the system in Theorem 1. This is the product of μ at all active sites and a delta function at 0 at all quiescent sites. We now prove Theorem 1 via the following lemmas. Assume that $\mathbf{X} \in \tilde{S}$.

Lemma 1. *Suppose that $X_n^k < \mu$, $X_n^{k-1} > \nu$, $\gamma > \gamma_1 = \frac{\ln r}{\nu}$ and let $\rho = re^{-\gamma\nu}$. Then we have $X_{n+1}^k < \rho\mu$ (and $\rho < 1$ as $\gamma > \ln r/\nu$).*

Proof. Observe that $X_{n+1}^k = f(X_n^k)e^{-\gamma X_n^{k-1}} < rX_n^k e^{-\gamma X_n^{k-1}} < re^{-\gamma\nu}X_n^k < \rho\mu$. □

Lemma 2. *If $X_n^k < \mu$ and $X_n^{k+1} > \nu$ where $\mu < \mu_1 = -\ln\left(\frac{4\nu}{r^2(1-r/4)}\right)/\gamma$ and $\nu < \nu_1 = f^2(1/2) = \frac{r^2}{4}(1-r/4)$, then we have $X_{n+1}^{k+1} > \nu$.*

Proof. Observe that for the logistic map $f(x)$ with parameter r, the maximum value of an iterate of the map is given by $f(1/2) = r/4$, and that the map sends the interval $[\nu, f(1/2)]$ into itself for any $\nu < \nu_1$ (recall that $\mathbf{X} \in \tilde{S}$). Including the effect of the coupling, this means that

$$
\begin{aligned}
X_{n+1}^{k+1} &= f(X_n^{k+1})e^{-\gamma X_n^k} > \frac{r^2}{4}(1-r/4)e^{-\gamma X_n^k} \\
&> \frac{r^2}{4}(1-r/4)e^{-\gamma\mu} > \nu_1 > \nu.
\end{aligned}
$$

□

Proof. (of Theorem 1.) For $2 < r < 4$, choose $\nu = \frac{1}{2}\frac{r^2}{4}(1-r/4)$ and assume that $\gamma > \gamma_1 = (\ln r)/\nu$ and $\mu < \min(\mu_1, \frac{\ln 2}{\nu})$ (note that $\frac{1}{2} < e^{\gamma\mu} < 1$ and so $\nu < \nu_1$ where μ_1 and ν_1 are as in Lemma —reflemtwo). Then the set $P_{\mu,\nu}$ is forward invariant in that

$$
\begin{aligned}
\Phi(P_{\mu,\nu}) &= \left\{\Phi(\mathbf{X}) \ : \ X^{2k} < \mu \text{ and } X^{2k+1} > \nu\right\} \\
&\subset \left\{\mathbf{X} \ : \ X^{2k} < \rho\mu \text{ and } X^{2k+1} > \nu\right\} \\
&\quad \text{(by lemmas 1 and 2)} \\
&= P_{\rho\mu,\nu},
\end{aligned}
$$

where $\rho = re^{-\gamma\nu}$. Since $\rho < 1$, we have $\Phi^N(P_{\mu,\nu}) \subset P_{\rho^N\mu,\nu} \to P_{0,\nu}$ as $N \to \infty$. □

3.2 Propagation of a single decelerating defect

We now turn our attention to defects that appear at the boundaries of domains of period two spatial patterns discussed in the previous section, as illustrated in Figure 1(c), (d). A defect creates a shift of one cell in the phase of the spatial periodicity. We say a pattern $\mathbf{X}$ has a single defect at location m if $\mathbf{X} \in D_{\mu,\nu,m,\theta}$ where

$$
D_{\mu,\nu,m,\theta} = \left\{ \mathbf{X} \ : \ \begin{array}{l} X^{m+2k+1} > \nu, \\ X^{m-2k-2} > \nu, \\ X^{m+2k+2} < \mu, \\ X^{m-2k-1} < \mu, \\ X^m = \theta \end{array} \ \text{for all } k \geq 0 \right\}
$$

A schematic illustration of such a defect is shown in Figure 2. More generally, we say $\mathbf{X}$ has a defect at location m for threshold $\mu > 0$ if $X^{m-1} < \mu$ and $X^{m+2} < \mu$. Note that the threshold μ needs to be chosen small enough for this to be meaningful (see the statements of the theorems that follow).

For large enough γ, this pattern is preserved while θ remains small but travels two spatial units to the right when it becomes active. Recall from Theorem 1 that the spatially 2-periodic state is stable if $\gamma > \frac{\ln r}{\nu}$.

Theorem 2. *Suppose that $2 < r < 4$ and let $\nu = \frac{1}{2}\frac{r^2}{4}(1-\frac{r}{4})$, pick $\gamma > \frac{2\ln r}{\nu}$ and $\mu < \min(\frac{\ln 2}{r}, \mu_1)$ (with μ_1 as in Lemma 2). Suppose that $\mathbf{X}_{-1} \in D_{\mu,\nu,m-2,\theta_{-1}}$ and $\mathbf{X}_0 = \Phi(\mathbf{X}_{-1}) \in D_{\mu,\nu,m,\theta_0}$. Then there are M, N with $0 < M < N$ such that such that for $n < M$ and some θ_n*

$$\Phi^n(\mathbf{X}_0) \in D_{\mu,\nu,m,\theta_n}$$

while for some some θ_N.

$$\Phi^N(\mathbf{X}) \in D_{\mu,\nu,m+2,\theta_N}.$$

Moreover, one can estimate

$$A - B\ln\theta_0 < N < A' - B'\ln\theta_0$$

and

$$\chi C\theta_0^D < \theta_N < C'\theta_0^{D'}$$

where $X_0^{m+2} = \chi$, for constants A, B, C, C', D and D' that depend only on r, ν, μ and γ, such that $D > D' > 1$. The value of D' can be made arbitrarily large by increasing γ.

Note that $D_{\mu,\nu,m,\theta}$ is open in ℓ^∞ if we allow θ to lie in an open interval; hence the theorem holds for an open set of defects. We say the state $\mathbf{X}$ *switches* at the first N such that $\phi^N(\mathbf{X})$ has a defect that is translated (cf [1]) and the theorem above provides an estimate of the time it takes between switches. In the following we write $\theta = \theta_0$.

Lemma 3. *Suppose that the parameters are as in Theorem 2 and $\mathbf{X} \in D_{\mu,\nu,m,\theta}$. Then*

$$\Phi^n(\mathbf{X}) \in D_{\mu,\nu,m,\theta_n}$$

for $n < \frac{\ln(\ln r) - \ln\gamma\theta}{\ln r - \mu\gamma}$ and some θ_n.

Proof. This is because after n iterates we have

$$\theta_n \le (re^{-\mu\gamma})^n\theta$$

until $\theta_n > \frac{\ln r}{\gamma}$ at which point the active site at $m+1$ will start to decay. □

To estimate bounds for N we observe that the state containing the defect, X^m must first grow in i iterates from θ to $X_i^m > \ln r/\gamma$, at which value it begins to have a suppressing effect on X^{m+1} variable. Then the state at X^{m+1} must decay for j iterates until $X_{i+j}^{m+1} < \mu$ and hence we have

$$\Phi^{i+j}(\mathbf{X}) \in D_{\mu,\nu,m+2,\theta'}$$

for some $\theta' = \theta_N$ and $N = i + j$. Observe that for any state $X_n^k \le \frac{1}{2}$, the next iterate is bounded by $\frac{r}{2}X_n^k e^{-\gamma\max\{X_n^{k-1}\}} < X_{n+1}^k < rX_n^k e^{-\gamma\min\{X_n^{k-1}\}}$. We now obtain upper and lower estimates

$$i_{\min} \le i \le i_{\max} \quad \text{and} \quad j_{\min} \le j \le j_{\max}.$$

Lemma 4. *For*

$$i_{\min} = \left\lfloor \frac{\ln(\frac{\ln r}{\gamma}) - \ln\theta}{\ln r} \right\rfloor,$$
$$i_{\max} = \left\lceil \frac{\ln(\frac{\ln r}{\gamma}) - \ln\theta}{\ln(r/2) - \gamma\mu} \right\rceil$$

we have $X_i^m < \frac{\ln r}{\gamma}$ for $i < i_{\min}$ and $X_i^m > \frac{\ln r}{\gamma}$ for some $i > i_{\max}$.

Proof. Whilst X^m is growing it is forced by X_n^{m-1} which is bounded by $0 < X_n^{m-1} < \mu$ for all $n < i$. Hence we have bounds

$$\frac{r}{2} X_n^m e^{-\gamma\mu} < \quad X_{n+1}^m \quad < r X_n^m$$

so that

$$\theta (re^{-\gamma\mu}/2)^i < \quad X_i^m \quad < r^i \theta$$

and hence we obtain the expressions for $i_{\min}$ and $i_{\max}$. □

Lemma 5. *For*

$$j_{\min} = \left\lfloor \frac{\ln\nu - \ln\mu}{\gamma r/4 - \ln(r/2)} \right\rfloor$$
$$j_{\max} = \left\lceil \frac{\ln(r/4) - \ln\mu}{\gamma\nu - \ln r} \right\rceil$$

we have $X_{i+j}^{m+1} > \mu$ *for* $j < j_{\min}$ *and* $X_{i+j}^{m+1} < \mu$ *for some* $j > j_{\max}$.

Proof. Whilst X^{m+1} is decaying it is forced by X^m which is bounded by $\nu < X_l^m < r/4$ for all $i < l < i+j$. Also, the active state X^{m+1} is bounded by $\nu < X_l^{m+1} < r/4$. Hence we have bounds

$$\frac{r}{2} X_l^{m+1} e^{-\gamma r/4} < \quad X_{l+1}^{m+1} \quad < r X_l^{m+1} e^{-\gamma\nu}$$

for l in this range, and so

$$\nu (re^{-\gamma r/4}/2)^j < \quad X_{i+j}^{m+1} \quad < r(re^{-\gamma\nu})^j/4.$$

Hence we get the estimate for $j_{\min}$ and $j_{\max}$ by examining the first j such that $X_{i+j}^{m+1} < \mu$. □

Proof. (of Theorem 2.) Note that the first part is proven by Lemma 3. Combining the estimates from Lemmas 4 and 5 means that we can obtain

$$i_{\min} + j_{\min} \leq N \leq i_{\max} + j_{max}$$

and so

$$\frac{\ln(\frac{\ln r}{\gamma}) - \ln\theta}{\ln r} + \frac{\ln\nu - \ln\mu}{\gamma r/4 - \ln(r/2)} < N < \frac{\ln(\frac{\ln r}{\gamma}) - \ln\theta}{\ln(r/2) - \gamma\mu} + \frac{\ln(r/4) - \ln\mu}{\gamma\nu - \ln r}$$

which gives the required estimate for N. As before, let $\rho = re^{-\gamma\nu}$ and note that $\rho < 1$. For $l < i_{\min}$, $X_l^{m+2} < \mu$, and $X_l^{m+1} > \nu$, and so we can apply lemma 1 to give

$$\theta_N < X_{i_{\min}}^{m+2} < \rho^{i_{\min}} \mu$$

and hence $\theta_N < C'\theta^{D'}$, where $C' = \rho^{\frac{\ln(\ln r/\gamma)}{\ln r}} \mu$ and $D' = \frac{-\ln\rho}{\ln r} = \frac{\gamma\nu}{\ln r} - 1$. Since we have assumed that $\gamma > \frac{2\ln r}{\nu}$ this implies that $D' > 1$. Similarly, we can obtain a lower bound for θ_N by observing that the maximum forcing X_i^{m+2} can receive is bounded by $X_l^{m+1} < r/4$ for $l < i_{\max}$, and so, writing $X_0^{m+2} = \chi$ we have

$$\chi (re^{-\gamma r/4})^{i_{\max}} < \theta_N$$

and hence $\theta_N > \chi C \theta^D$, where $C = (\frac{\ln r}{\gamma})^{(\frac{\ln r - \gamma r/4}{\ln(r/2) - \gamma\mu})}$. Observe that

$$D = \frac{\gamma r/4 - \ln r}{\ln(r/2) - \gamma\mu} > \frac{\gamma r/4 - \ln r}{\ln r} > \frac{\gamma\nu}{\ln r} - 1 = D'$$

and so we can choose D' arbitrarily large by appropriately choice of large γ. □

3.3 Annihilation of defects.

Given an initial state with two defects at say 0 and $K > 0$, there are a number of possible behaviours depending on the path of each defect. If the paths cross at some point in the future then the two defects annihilate each other to create a stable spatial period-2 domain.

We make the following conjecture for the behaviour of the system for large γ based on numerical experiments. Suppose that we can divide a spatial pattern of length l into K domains of stable spatial period 2 patterns and defects between them, then we say the pattern has *defect density* K/l. Similarly one can define such a quantity for an infinite domain assuming that defects are distributed in a regular fashion.

Conjecture 1. *Given $2 < r < 4$ and γ sufficiently large there is a full measure set of initial conditions $\mathbf{X} \in S$ such that $\Phi^n(\mathbf{X})$ is a state with defect density $\rho(n)$, and*

$$0 < \rho(n) \to 0$$

as the time $n \to \infty$.

There are a number of obstacles to proving such a result. Firstly, during the initial transient there will be states that are not easily classified into being defects. Secondly, one needs to have a good understanding of the time it takes for an annihilation to occur to overcome problems of more than two defects meeting at some point in space-time.

3.4 Failure of ergodicity

For large values of γ we expect the convergence of long-time averages to fail under some fairly weak assumptions on the initial condition. We prove this explicitly for the case of a single defect propagating in a spatial period-3 lattice, and then explain why we believe the extension to arbitrary numbers of defects in other periodic lattices should follow. For infinite lattices we expect some (carefully chosen) initial conditions will violate this result even for large γ. Let

$$S' = \{\mathbf{X} \in S \ : \ X^{k+3} = X^k \quad \text{for all } k \in \mathbb{Z}\}$$

be the set of initial conditions that are periodic with spatial period 3, and note that $\Phi(S') \subset S'$. Most initial conditions in S' will give a defect at every third site, but for example initial states with X^k all equal will not.

Theorem 3. *Suppose that $2 < r < 4$, γ is sufficiently large, $\mathbf{X} \in S'$ is any initial condition that has $X^k \neq 0$ for all k and $\Phi^n(\mathbf{X})$ has a defect for some n. Then the ergodic sum*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} X^0_m$$

does not converge.

The system on S' is formally equivalent to the 'free-running' dynamics of the system in [1] and so X^0_n will alternate between epochs of length T_n where every third epoch it is active and in between it is quiescent.

Lemma 6. *Suppose that $0 < \nu < X^0_n < 1$ for the duration of every third epoch, $0 < X^0_n < \mu$ for the other two and the duration of the epochs satisfies*

$$\lambda < \frac{T_{n+1}}{T_n} < \Lambda$$

where μ can be chosen arbitrarily small and λ arbitrarily large. Then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} X^0_j \neq \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} X^0_j.$$

Proof. Observe that bounds for the extrema of the sum will occur when the first $n-1$ epochs are as long as possible and the nth epoch is as short as possible. Hence taking $T_i = \Lambda^{i-1}$ for $i < n$, $T_n = \lambda\Lambda^{n-1}$ and defining

$$S_n = \sum_{i=0}^{n-1} T_i$$

one can compute that

$$\frac{T_n}{S_n} \to 1 - \frac{\Lambda}{\lambda(\Lambda-1)+\Lambda} = 1-\epsilon$$

for $n \to \infty$, where $\epsilon \sim \frac{1}{\lambda}$. Using this and the assumptions on the values of X^0_n during each epoch we get

$$\liminf_{n\to\infty} \frac{1}{S_n}\sum_{j=0}^{S_n-1} X^0_j < \nu\epsilon + (1-\epsilon)\mu$$

and

$$\limsup_{n\to\infty} \frac{1}{S_n}\sum_{j=0}^{S_n-1} X^0_j > (1-\epsilon)\nu$$

where $\epsilon \to 0$ as $\lambda \to \infty$. Hence, by choosing μ small enough we can ensure that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} X^0_j > \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} X^0_j.$$

□

Proof. **(of Theorem 3)** By the assumption that no $X^j = 0$ we know that a defect will arrive and pass through X^0 infinitely often. Suppose that the nth defect remains at X^0 for a time T_n. By Theorem 2 we can associate each time a defect passes with θ_n such that

$$\tilde{C}\theta_n^D < \theta_{n+1} < C'\theta_n^{D'}$$

for some $D > D' > 1$, and we seek an expression for T_{n+1}/T_n (the assumption that the spatial period is three means that the new location of the defect has just decayed from being active, and so we can take $\chi = \mu$ and define $\tilde{C} = \mu C$). Taking logarithms and dropping the tilde, we obtain

$$\ln C + D\ln\theta_n < \theta_{n+1} < \ln C' + D'\ln\theta_n$$

and, also from Theorem 2,

$$A - B\ln\theta_n < T_n < A' - B'\ln\theta_n.$$

Combining these two inequalities we produce:

$$\begin{aligned}
A - B\ln\theta_{n+1} &< T_{n+1} < A' - B'\ln\theta_{n+1} \\
A - B(\ln C' + D'\ln\theta_n) &< T_{n+1} < A' - B'(\ln C + D\ln\theta_n) \\
E - BD'\ln\theta_n &< T_{n+1} < E' - B'D\ln\theta_n \\
E + \frac{BD'}{B'}(T_n - A') &< T_{n+1} < E' + \frac{B'D}{B}(T_n - A) \\
F + \frac{BD'}{B}T_n &< T_{n+1} < F' + \frac{B'D}{B}T_n \\
\lambda = \frac{BD'}{B'} &< \frac{T_{n+1}}{T_n} < \frac{B'D}{B} = \Lambda \text{ for sufficiently large } n
\end{aligned}$$

where $B = 1/\ln r$ and $B' = 1/(\ln(r/2) - \gamma\mu)$, for suitable constants E and F depending only on r, ν, γ and μ. By noting that given r we can choose $\nu > 0$ and then we can attain an arbitrarily large value of $\lambda > 0$ by choice of γ such that, on choosing μ sufficiently small we have non-ergodicity of X_n^0 on applying Lemma 6. □

4 Discussion

We briefly discuss some aspects of this work including some exceptional initial conditions and the genericity of the behaviour. Up to now we have concentrated on the case where r is large enough that the logistic map is chaotic; but the above applies equally for periodic cases as long as $r > 2$, subject to some obvious simplifications.

By choosing initial states that are spatially periodic with period N we have observed asymptotic states that have a finite density of defects even in the case as time goes to infinity for N odd. If $N = 3$ this system is identical to the 'free-running' system investigated in [1] which has robust attracting cycling chaos. In this case our numerical studies suggest that for typical initial conditions with spatial period N, the defect density $\rho(n) \to 0$ as $n \to \infty$ for N even and $\rho(n) \to 1/N$ for N odd. Clearly there are initial conditions (e.g. homogeneous) that give a higher asymptotic density of defects or converge to states that are not classifiable as having defects but we believe that these states are exceptional.

Similar behaviour is expected for generalizations of (1) as long as the local map $f(x)$ has a fixed point whose location is unaffected by the coupling (although its stability can be affected). This preserves the dynamically invariant subspaces that are important for the robust appearance of heteroclinic attractors and non-ergodic behaviour.

The one-way coupling is not necessary; we obtain very similar behaviour e.g. for

$$x_{n+1}^k = f(x_n^k)e^{-\gamma x_n^{k-1} - \gamma' x_n^{k+1}}$$

as long as $0 < \gamma' \ll \gamma$. Similarly, one could choose coupling that is not nearest neighbour. The translation invariance of the CML is also not necessary; it should make no difference if there are anisotropies in the chain, for example if $f(x)$ is replaced by $f_k(x) = r_k x(1-x)$ with r_k chosen randomly within a certain range, or even choosing different maps f_k at each site.

Acknowledgements

We thank A. Politi for drawing our attention to the references [14, 15] and A. Rucklidge for discussions related to this work.

References

[1] P. Ashwin and A. M. Rucklidge and R. Sturman, Infinities of periodic orbits near robust cycling. *Phys. Rev.E* **66**:035201(R) (2002).

[2] E. Barreto and B. Hunt and C. Grebogi and J. Yorke, From high dimensional chaos to stable periodic orbits: the structure of parameter space. *Phys. Rev. Lett.* **78**:4561–4564 (1997).

[3] L.A. Bunimovich, Coupled map lattices: some topological and ergodic properties. *Physica D* **103**:1–17 (1997).

[4] M. Dellnitz and M. Field and M. Golubitsky and A. Hohmann and J. Ma, I.E.E.E. Transactions on Circuits and Systems: I. Fundamental Theory and Applications, **42**:821–823 (1995).

[5] A. Gaunersdorfer, Time averages for heteroclinic attractors, *SIAM J. Appl. Math.* **52** 1476–1489 (1992).

[6] V.M. Gundlach and D. Rand, Spatio-temporal chaos. I. Hyperbolicity, structural stability, spatio-temporal shadowing and symbolic dynamics and II. Unique Gibbs states for higher-dimensional symbolic systems, *Nonlinearity* **6**:165–200 (1993) and *ibid.*, 201–213 (1993), Corrigenda: *ibid.* **9**:605–606 (1996).

[7] J. Hofbauer and K. Sigmund, *The Theory of Evolution and Dynamical Systems*, Cambridge University Press, Cambridge, (1988).

[8] G. Keller, M. Künzle and T. Nowicki, Some phase transitions in coupled map lattices, *Physica D* **59**:39–51 (1992).

[9] M. Krupa, Robust heteroclinic cycles, *Journal of Nonlinear Science* **7**:129–176 (1997).

[10] R. Livi, Models of spatially extended systems, In; *Dynamics of complex interacting systems (Santiago 1994)*, pp1–33, Kluwer, Dordrecht (1996).

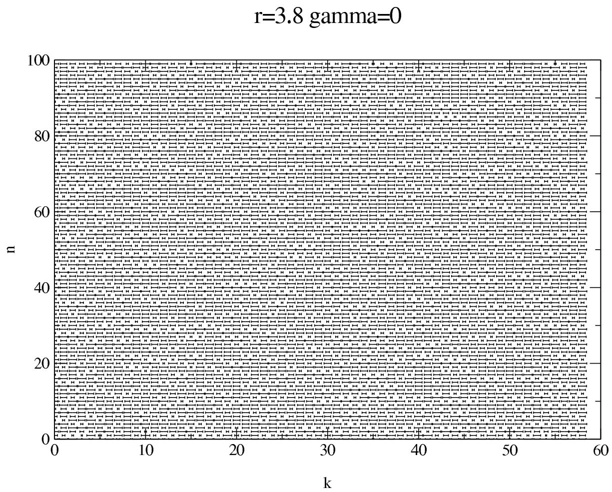
[11] J. Losson and M.C. Mackey, Statistical cycling in coupled map lattices. *Phys. Rev. E* **50**:843–856 (1994).

[12] F. Schmüser and W. Just, Non-equilibrium behaviour in unidirectionally coupled map lattices. *J. Stat. Phys.* **105**:525–559 (2001).

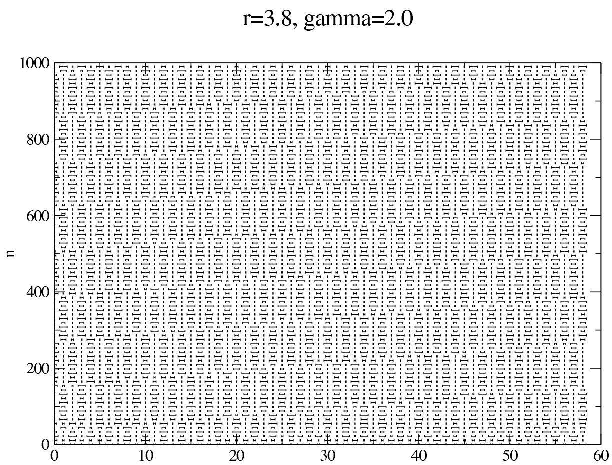
[13] K. Sigmund, Time averages for unpredictable orbits of deterministic systems, *Annals of Operations Research* **37**:217–228 (1992).

[14] F.H. Willeboorde and Kunihiko Kaneko, Bifurcations in spatial chaos in an open flow model. *Phys. Rev. Lett.* **73**:533–539 (1994).

[15] F.H. Willeboorde and Kunihiko Kaneko, Periodic lattices of chaotic defects. *Phys. Rev. E* **52**:1516–1519 (1995).



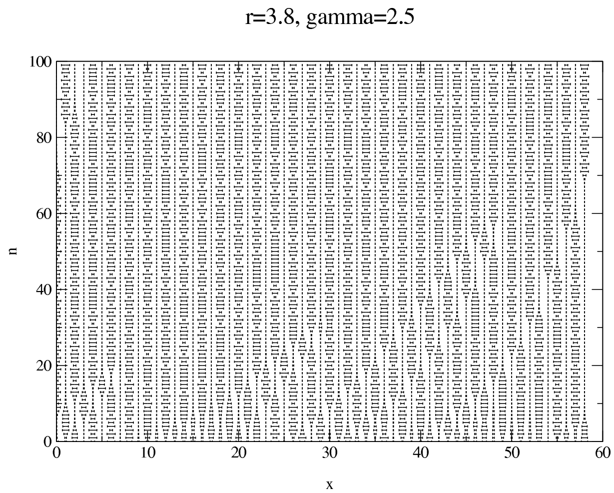
(a)



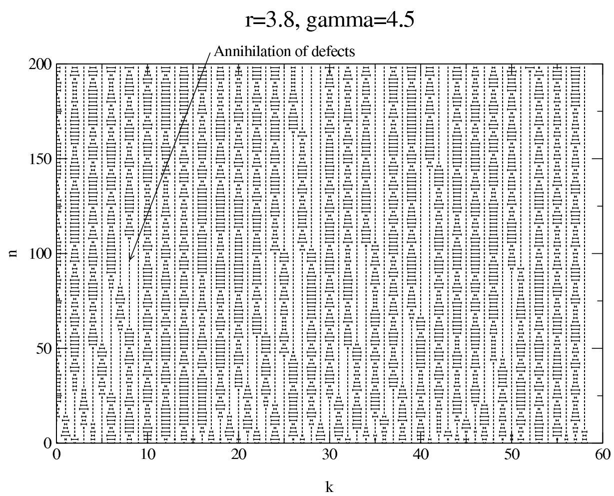
(b)

Figure 1:

Space-time plots showing the behaviour of 1 where the length of each horizontal line indicates the value of X_n^k at time n and spatial location k. In all cases a random i.i.d initial condition was chosen at $n = 0$ and spatially periodic boundary conditions on a domain of size 59 are chosen, for $r = 3.8$. (a) $\gamma = 0$ showing the evolution of the uncoupled system. For (b) $\gamma = 2$ and X_n^k (plotted every 11 timesteps) settles into a state where a single defect propagates at a finite velocity. Moving to the regime shown in (c) at $\gamma = 2.5$ we can see the formation of domains of pattern with spatial period-2 separated by right moving decelerating defects. Finally, in case (d) $\gamma = 4.5$ (plotted every 2 timesteps) we can see fast approach to a set of domains separated by decelerating right-moving defects. These annihilate, for example, at the location indicated. This behaviour is typical for larger values of γ. Note that in cases (c,d) the active sites are oscillating chaotically and independently of each other, and that after a long enough period all defects annihilate except for one.



(c)



(d)

Figure 1 continued.

time				$m-2$	$m-1$	m	$m+1$	$m+2$				
N	$-$	$*$	$-$	$*$	$-$	$*$	$-$	θ_N	$*$	$-$	$*$	$-$
$N-1$	$-$	$*$	$-$	$*$	$-$	θ_{N-1}	X	$-$	$*$	$-$	$*$	$-$
0	$-$	$*$	$-$	$*$	$-$	θ_0	$*$	$-$	$*$	$-$	$*$	$-$
-1	$-$	$*$	$-$	θ_{-1}	X	$-$	$*$	$-$	$*$	$-$	$*$	$-$

Figure 2:

The bottom line shows the spatial pattern for a defect at location m at time n that has just moved. This propagates forward to arrive at location $m+2$ after a further N iterates. Here, '' denotes an active site with value $> \nu$, '-' a quiescent site with value $< \mu$ and X denotes a site that has value $> \mu$. The value θ_i increases for the defect to move along.*