Blowout bifurcation in a system of coupled chaotic class B lasers

Peter Ashwin and John R. Terry *
Department of Mathematics and Statistics, University of Surrey
Guildford GU2 5XH, UK

K. Scott Thornburg, Jr. and Rajarshi Roy
School of Physics, Georgia Institute of Technology,
Atlanta, GA 30332, USA
(February 10, 1998)

We show that loss of synchronization of two identical coupled chaotic class B lasers can occur via a *blowout bifurcation*. This is when a transverse Lyapunov exponent governing the stability of the synchronized subspace passes through zero. A system of two laterally-coupled lasers with modulated parameters is investigated numerically in a region of chaotic behaviour and evidence of the blowout is found in Lyapunov exponents and in the presence of on-off intermittency for small enough coupling strengths. At all parameter values investigated, the phase of the electric fields are shown to be precisely synchronized (coherent) even though the intensities may fluctuate chaotically and independently.

Understanding the synchronization and desynchronization of signals from two or more nearly identical lasers is a matter that has important technological applications. Notably, it is important for designing high power coherent laser sources from arrays of low power lasers or for high-speed communication using synchronized optical systems. There have been several studies of the problem of chaos and synchronization in lasers over the last few years [9,6,10,13,12,2].

We consider coupling a pair of identical class B lasers in a way that preserves the interchange symmetry. In addition to the obvious interchange symmetry, they have phase-shift symmetries of the electric fields. An elementary but surprising consequence of these symmetries is the existence of states that are exactly phase synchronized (coherent) but not amplitude synchronized. The fact that these (anti)coherent states are always attracting for any non-zero coupling means that the issue of synchronization of amplitudes has been overlooked for this system.

We present what we believe is a new route to loss of synchronization in laser systems. It is a symmetry breaking that is purely dynamical, i.e. caused by loss of stability of a synchronized attractor through a 'blowout' bifurcation, where a coherent synchronized state loses stability to fluctuations that preserve coherence while breaking synchrony. Such bifurcations have been seen in numerically in maps [1,15] and experimentally in electronic and other systems [3,16,18] but this is, to our knowledge, the first observation of a blowout bifurcation in a laser system of any sort.

For real systems, symmetry is only an idealization that is broken by imperfections in the system e.g. [2] where a perfectly synchronized state no longer exists or by noise within the system that moves the trajectory away from any synchronized state [5]. However, the mechanism we discuss leads to desynchronization without the need to appeal to either of these effects. Moreover it is a form of desynchronization that only occurs when chaotic dynamics is present in the system.

The system we consider is a pair of coupled lasers where the coupling is purely via overlap of the electric field. The lasers under consideration are Class B [17], where only the field and gain variables need be considered. The lasers are subjected to identical periodic modulations of the loss and may become chaotic in certain parameter regimes.

The coupled laser system we consider is motivated by the two coupled single-mode Class B lasers studied by Roy and Thornburg [5,6] and Roy and Colet [10]. We include a periodic forcing of the loss or pump so that the two lasers are modulated at a rate close to the natural relaxation oscillation frequency. This frequency represents the timescale with which the laser intensity fluctuates [8,7].

The lasers are assumed to be identical in all their parameters and they are subject to the same modulated loss. We assume that the two lasers are not detuned from a common cavity mode and therefore are governed by the following equations defining an evolution in the five dimensional phase space.

$$\begin{aligned}\frac{dX_1}{dT} &= [F_1 - \alpha_0 (1 + \alpha_M \cos \omega t)] X_1 - \beta X_2 \cos \Phi \\ \frac{dF_1}{dT} &= \gamma \left[A_0(1 + A_M \cos \omega t) - F_1 - F_1 X_1^2\right] \\ \frac{dX_2}{dT} &= [F_2 - \alpha_0 (1 + \alpha_M \cos \omega t)] X_2 - \beta X_1 \cos \Phi \\ \frac{dF_2}{dT} &= \gamma \left[A_0(1 + A_M \cos \omega t) - F_2 - F_2 X_2^2\right]\end{aligned} \quad (1)$$

*PA and JT would like to thank the support of the EPSRC via grant GR/K77365.

$$\frac{d\Phi}{dT} = \beta\ (X_2 X_1^{-1} + X_1 X_2^{-1}) \sin\Phi$$

X_i represents the electric field amplitude and F_i the gain of laser $i = 1, 2$; Φ the difference in phases between the electric fields of the two lasers $\phi_2 - \phi_1$. The equations are nondimensionalised with time being expressed in units of the round-trip time of light around the cavity, τ_c. The parameter γ represents the ratio of the timescales of the electric field τ_c and the upper level spontaneous emission lifetime of the laser material τ_f. For the Nd:YAG lasers considered by Roy and Thornburg, and Thornburg *et al.* [6,5] $\gamma \sim 10^{-6}$, meaning that the equations are very stiff. Other class B laser media, and different resonator configurations may have more moderate values of γ. This is an important fact, since for the purposes of the numerics we have had to consider moderate values of γ.

The lasers can either be modulated in their losses ($A_M = 0$, $\alpha_M \neq 0$) or in their pumps ($A_M \neq 0$, $\alpha_M = 0$) relative to their mean losses α_0 and mean pump parameters A_0. In the absence of modulation both lasers are stable and exhibit damped oscillations to their fixed-point values. Note that both lasers experience identical modulations and are assumed to have identical parameters. The coupling via β can be caused by overlap of the laser electric fields in a laser crystal. If the beams have $1/e^2$ radii w_0 [14,5] and the coupling is assumed to be proportional to the area of overlap between the two lasers then $\beta \sim e^{-d^2/w_0^2}$ [14].

Because we are interested in the problem of synchronization, we introduce the sum and difference variables, $X_+ = \frac{1}{2}(X_1 + X_2), X_- = \frac{1}{2}(X_1 - X_2), F_+ = \frac{1}{2}(F_1 + F_2), F_- = \frac{1}{2}(F_1 - F_2)$, to facilitate the stability analysis of the synchronized state.

The transformed system is equivariant under the action of the symmetry

$$\kappa(X_+, F_+, X_-, F_-, \Phi) = (X_+, F_+, -X_-, -F_-, -\Phi)$$

corresponding to interchanging the two lasers. There is another not so obvious symmetry of the system, namely

$$\mu(X_+, F_+, X_-, F_-, \Phi) = (X_+, F_+, X_-, F_-, -\Phi)$$

as the only coupling is via $\cos(\Phi)$ terms. This corresponds to interchanging the phases of the beams without interchanging their amplitudes.

There is also a symmetry involving the parameter β; this adds π onto Φ while reversing the sign of the parameter β. We use the parameter symmetry to simplify the numerics; however this is not physically relevant as $\beta \geq 0$ in practise.

Because Φ is a periodic variable μ will fix the subspaces where $\Phi = 0$ or π and so there are in total five distinct dynamically invariant subspaces that are forced to exist purely by virtue of their symmetry. These are listed in Table I. Of particular interest are the existence of states we call simply *coherent* that are not synchronized and *anticoherent* that are not antisynchronized.

Possibly even more surprising is the existence of states we denote as *incoherent synchronized* where the amplitudes are identical but the phases are not! Such states are however never observed to be attracting for this system.

For $0 < \gamma \ll 1$ the system undergoes a period doubling cascade to chaos as the strength of modulation is increased (*c.f.* [8]). Any attractor is contained either in the coherent or anticoherent subspaces. This is because for any X_1, X_2 bounded away from zero we have $\frac{d\Phi}{dt} = \beta F \sin\Phi$ with F positive and bounded below. Therefore $\Phi \to \pi$ as $t \to \infty$ for almost any initial condition, and any attractor must be contained within the anticoherent subspace. Note that if $\beta < 0$ then the above holds but with the coherent and anticoherent subspaces exchanged.

The simulations were performed using Bulirsch-Stoer and Runge-Kutta integrators. We consider here only the case of modulated loss but note that we have found similar results for modulated pumping [7]. For a typical value of $\gamma = 0.01$ and $A_0 = 1.2$, $\alpha_0 = 0.9$ and $\alpha_M = 2/9$ we see the following behaviour on varying the coupling strength β we find that there is a critical value of β, $\beta_c \sim 0.002234$ such that a randomly chosen initial condition evolves as follows. For $0 < \beta < \beta_c$ the trajectory is attracted onto the anticoherent subspace to a chaotic attractor that intersects but is not contained within the antisynchronized subspace. For $\beta > \beta_c$ there is an attractor within the antisynchronized subspace. As explained above, the phase difference Φ evolves to anticoherence π in all cases.

Figure 1 shows this change in synchronization on varying β. The transition at β_c is strongly suggestive of a blowout bifurcation [1] at this value of β. Fig. 1(a) shows totally the unsynchronized behavior of uncoupled ($\beta = 0$) lasers. Fig. 1(b) demonstrates intensity dynamics that are almost synchronized with occasional very large fluctuations away from the antisynchronized subspace and on-off intermittent behaviour. Fig. 1(c) shows completely antisynchronized intensity dynamics and corresponds to $\beta > \beta_c$.

To investigate the loss of synchronization at $\beta = \beta_c$ and confirm the blowout scenario we numerically compute the Lyapunov exponents of the attractors we have found; recall that a blowout bifurcation occurs when the largest transverse Lyapunov exponent of the attractor for the system within an invariant subspace passes through zero. This Lyapunov exponent governs the exponential rate of growth of almost all perturbations away from the invariant subspace and in particular if it is negative then the attractor within the subspace is the attractor for the full system.

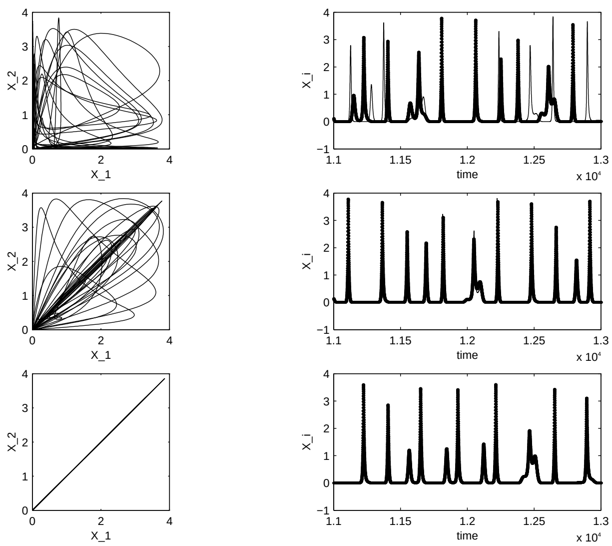

FIG. 1. Numerically calculated electric field amplitudes in a loss-modulated laser, computed by integrating Eqs. 1 with randomly chosen initial conditions. On the left is shown (X_1, X_2), while on the right (X_i, t), after transients have been allowed to decay away. In all cases the phase difference evolves quickly to anticoherence $\Phi = \pi$. On the right-hand figures, the dotted line indicates the evolution of X_1 and the solid, that of X_2. Throughout, $A_0 = 1.2$, $\alpha_0 = 0.9$, $\alpha_M = 2/9$, $\omega = 0.045$, $\gamma = 0.01$ and the coupling strength varies from (top) $\beta = 0$, (middle) $\beta = 0.001$ and (bottom) $\beta = 0.0025$. Observe how the random but independent fluctuations of the uncoupled systems (top) give way to an (anti)synchronized attractor (bottom). The transition occurs at approximately $\beta = 0.00225$. Time is measured in arbitrary units.

Suppose we have a trajectory $(x_+(t), f_+(t), 0, 0, \pi)$ for an initial condition chosen randomly for the system (1) in the subspace $\mathbf{Z}_2(\kappa)^-$ (the antisynchronized subspace) and consider the behaviour of a point $(x_+(t) + \delta x_+, f_+(t) + \delta f_+, \delta x_-, \delta f_-, \pi + \delta\phi)$ linearized about the δ variables. The δ terms represent small perturbations away from the trajectory. Perturbations with $\delta x_- = \delta f_- = \delta\phi = 0$ correspond to perturbations within the antisynchronized subspace and these grow at a rate $e^{\lambda t}$ where λ is some *tangential* Lyapunov exponent Λ_1 or Λ_2. Any other perturbation will grow at a rate $e^{\lambda t}$ where λ is a *transverse* Lyapunov exponent, λ_1, λ_2 or λ_3. If any of these is positive, the antisynchronized subspace is unstable. Since the antisynchronized subspace is codimension three, there are three transverse Lyapunov exponents. We can divide these up into a pair, λ_1 and λ_2, corresponding to perturbations within the anticoherent subspace and one λ_3 that breaks anticoherence. It is easy to compute from the linearisation of the last equation of (1) that $\lambda_3 = -2\beta$.

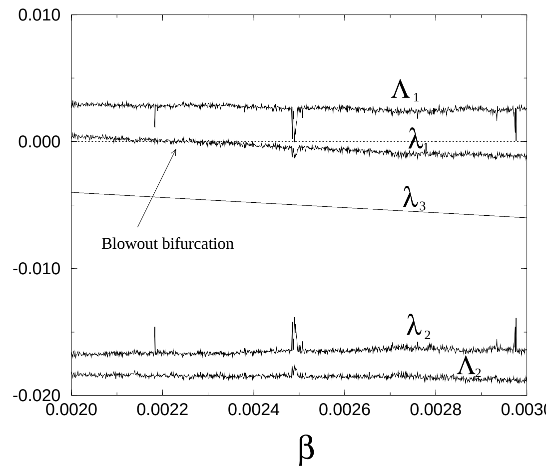

FIG. 2. Scan through parameter space showing the variation of the Lyapunov exponents with β for an initial condition started within the antisynchronized subspace, parameter values as in Fig. 1. The Λ_i are tangential Lyapunov exponents whereas the λ_i are transverse Lyapunov exponents. NB (a) The most positive λ_1 passes through 0 at $\beta \sim 0.00223$ indicating a blowout bifurcation occurs here. (b) Λ_1 is positive, indicating the presence of chaotic behaviour. (c) There are occasional dips in Λ_1 corresponding to windows of stabilisation of periodic attractors in the antisynchronized subspace. (d) $\lambda_3 = -2\beta$ exactly.

To see precisely when the antisynchronized state is attracting, we have numerically approximated $\lambda_i < 0$ for $i = 1, 2$. Figure 2 shows a scan through a range of β showing the tangential and transverse Lyapunov exponents. This was computed using timeseries with length $150,000$ and orthonormalisation. By making a linear fit the obtained values of λ_1 we compute that the blowout occurs at approximately $\beta_c = 0.0022336$. For $\beta < \beta_c$ there is an attracting on-off intermittent state that persists up to the point of the blowout bifurcation. We have also investigated cases for $\gamma = 10^{-3}$ and 5×10^{-4} and observed similar behaviour, with the additional observation that the blowout bifurcations occur for progressively smaller values of β as the stiffness parameter γ is reduced. Due to the increased stiffness of the equations, the numerics become much harder to evaluate accurately in this limit, and machine precision becomes an important issue when computing the normal Lyapunov exponents.

In conclusion, we report the existence of a blowout bifurcation in a system of two coupled lasers with periodically modulated parameters, for certain parameter values. This is a dynamical symmetry breaking within the system that leads to a desynchronization of the intensities of the two chaotic lasers. The desynchronized attractors are nonetheless exactly (anti)coherent due to the symmetries of the system. To perform the numerical investigations accurately we were forced to look at more moderate values of the stiffness parameter γ than in the experiment reported in [6] but we suspect the mechanism here plays an important role in the latter system.

One physical consequence of this investigation is that we expect the coupling strength for synchronizing such linearly coupled chaotic lasers will be intimately related to the magnitude of the positive Lyapunov exponent of the synchronized chaos, as discussed by Schuster *et al.* [20]. In particular, if the (anti)synchronized state is non-chaotic, e.g. attracting periodic then it will be stable and no blowout will be in evidence. As is evident in the 'coarseness' of the graph of Lyapunov exponents against β in Fig. 2, the fact that β is not a normal parameter [4] (i.e. β varies the dynamics within the synchronized subspace as well as that transverse to it) means that we do not expect these exponents to vary smoothly or even continuously; see [11].

One should not finish without mentioning the effects of noise and imperfections on the discussed dynamics. Firstly, noise will have the effect of causing dynamics similar to on-off intermittency (called *bubbling* in [3]) to appear even when the Lyapunov exponents of the noise-free system give stable synchronization. If there is a mismatch in the parameters of the lasers, for example if there is detuning then this will destroy the invariant subspace and cause the phase dynamics to become more nontrivial, as discussed for example in [5].

[1] E. Ott and J.C. Sommerer, Blowout bifurcations; the occurrence of riddled basins and on-off intermittency. *Physics Letters A* **188**: 39–47 (1994)

[2] Yudong Liu and J.R. Roi Leite, Coupling of two chaotic lasers. *Physics Letters A* **191**:134–138 (1994).

[3] P. Ashwin, J. Buescu and I. Stewart, Bubbling of attractors and synchronisation of oscillators. *Phys. Lett. A* **193** 126 (1994).

[4] P. Ashwin, J. Buescu and I.N. Stewart. From attractor to chaotic saddle: a tale of transverse instability. *Nonlinearity* **9** 703 (1996).

[5] K. S. Thornburg, Jr. and M. Möller, R. Roy, T. W. Carr, R.-D. Li, and T. Erneux, Chaos and Coherence in Coupled Lasers. *Phys. Rev. E* **55**,3865 (1997).

[6] R. Roy and K. S. Thornburg, Jr., Experimental Synchronization of Chaotic Lasers. *Phys. Rev. Lett.* **72**, 2009 (1994)

[7] J. R. Tredicce, N. B. Abraham, G. P. Puccioni, and F. T. Arecchi, On chaos in lasers with modulated parameters: A comparative analysis. *Optics Comm.* **55**, 131 (1985)

[8] D. Dangoisse, P. Glorieux, and D. Hennequin, Chaos in a CO_2 laser with modulated parameters: Experiments and numerical simulations. *Phys. Rev. A* **36**, 4775 (1987)

[9] Herbert G. Winful and Lutfur Rahman, Synchronized chaos and spatiotemporal chaos in arrays of coupled lasers. *Phys Rev. Lett.* **65**,1575 (1990)

[10] Pere Colet and Rajarshi Roy, Digital communication with synchronized chaotic lasers. *Opt. Lett.* **19**,2056 (1994)

[11] P. Ashwin, E. Covas and R. Tavakol, *Transverse instability for non-normal parameters* Technical Report 98/3, Dept of Maths and Stats, University of Surrey (1998).

[12] Yudong Lie and P. C. de Oliveira and M. B. Danailov and J. R. Rios Leite, Chaotic and periodic passive Q switching in coupled CO_2 lasers with a saturable absorber. *Phys. Rev. A* **50**,3464 (1994)

[13] T. Sugawara, M. Tachikawa, T. Tsukamoto, and T. Shimizu, Observation of synchronization in laser chaos. *Phys. Rev. Lett.* **72**,3502 (1994)

[14] Larry Fabiny, Pere Colet, Raj Roy, and Daan Lenstra, Coherence and phase dynamics of spatially coupled solid-state lasers. *Phys. Rev. A*, **47**,4287 (1993)

[15] Hirokazu Fujisaka and Tomoji Yamada, Stability Theory of Synchronized Motion in Coupled-Oscillator Systems. *Prog. of Theor. Phys.*, **69**,32 (1983)

[16] Louis M. Pecora, Thomas L. Carroll, and James F. Heagy, Riddled Basins and Other Practical Problems in Coupled, Synchronized Chaotic Circuits. *SPIE* **2612**, 25 (1995)

[17] F. T. Arecchi, G. L. Lippi, G. P. Puccioni, and J. R. Tredicce, Deterministic chaos in laser with injected signal. *Opt. Commun.* **51**, 308 (1984)

[18] M. Wada, Y. Nishio, A. Ushida, Blowout Bifurcation and Riddled Basin in simple coupled chaotic circuits *Proceedings of ECCTD'97*, 1269 (1997)

[19] L. M. Pecora, T. L. Carroll, *Phys. Rev. A* **64**, 821 (1990)

[20] H.G. Schuster, S. Martin and W. Martienssen, *Phys Rev A* 33, 3547 (1986).

Symmetry	Representative point	Dim.	Name
$\mathbf{Z}_2(\kappa)^+$	$(X_+, F_+, 0, 0, 0)$	2	Synchronized
$\mathbf{Z}_2(\kappa)^-$	$(X_+, F_+, 0, 0, \pi)$	2	Antisynchronized
$\mathbf{Z}_2(\mu)^+$	$(X_+, F_+, X_-, F_-, 0)$	4	Coherent
$\mathbf{Z}_2(\mu)^-$	$(X_+, F_+, X_-, F_-, \pi)$	4	Anticoherent
$\mathbf{Z}_2(\kappa\mu)$	$(X_+, F_+, 0, 0, \Psi)$	3	Incoherent synchronized

TABLE I. The symmetry-forced invariant subspaces of the equations for two coupled lasers. The first column gives a symbol for the subgroup of symmetries that fix a typical point in this invariant subspace with coordinates given by the second column (X_+, F_+, X_-, F_-, Ψ are arbitrary values for these coordinates). The third column gives the dimension of this invariant subspace within the 5 dimensional phase space.