

# Classification of robust heteroclinic cycles for vector fields in $\mathbb{R}^3$ with symmetry

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## Abstract

We consider a classification of robust heteroclinic cycles in the positive octant of  $\mathbb{R}^3$  under the action of the symmetry group  $\mathbb{Z}_2^3$ . We introduce a coding system to represent different classes up to a topological equivalence, and produce a characterization of all types of robust heteroclinic cycle that can arise in this situation. These cycles may or may not contain the origin within the cycle. We proceed to find a connection between our problem and meandric numbers. We find a direct correlation between the number of classes of robust heteroclinic cycle that do not include the origin and the ‘Mercedes-Benz’ sequence of integers characterizing meanders through a ‘Y-shaped’ configuration. We investigate upper and lower bounds for the number of classes possible for robust cycles between  $n$  equilibria, one of which may be the origin.

## 1 Introduction

Heteroclinic cycles have been noted to appear robustly as attracting dynamics in a wide range of physical problems, notably in population dynamics with models of three or more competing species [6, 13]. Hofbauer and Sigmund [6] provide numerous references of further examples in population dynamics where heteroclinic cycles appear in cases where the system is non-permanent (i.e. where there is an attractor on which one or more of the populations get arbitrarily close to extinction). Guckenheimer and Holmes analyse an example where attracting robust cycles can bifurcate from equilibrium dynamics [4]. They also appear in certain models of rotating convection [2]; see also the review by Krupa [10].

These cycles appear in the attracting dynamics as a slow switching between a sequence of equilibrium states. This manifests itself as intermittency of trajectories that spend an increasing amount of time at the equilibria each time they pass near one. Interspersed

between these are rapid transitions that shadow the connecting trajectories. The period of recurrence increases as the cycle is approached, giving asymptotically a constant geometric rate of ‘slowing down’.

In all cases where robust heteroclinic cycles arise, a critical ingredient is the presence of invariant subspaces (forced by symmetries or on other physical grounds) such that connections are robust within these subspaces. Such robust heteroclinic cycles arise easily for vector fields in  $\mathbb{R}^n$  ( $n \geq 3$ ) with quite simple symmetry groups, for example  $\mathbf{O}(2)$ ,  $\mathbb{D}_4$  or  $\mathbb{Z}_2^3$ , all of which occur in models of physical problems, for example in [1, 9, 11, 12, 14, 15, 16].

This paper focuses on the classification of vector fields with the symmetry group  $\mathbb{Z}_2^3$  in  $\mathbb{R}^3$  (we discuss some simpler cases in Appendix A). After some basic definitions we introduce a coding that characterises cycles up to an appropriate topological equivalence. This is used to classify robust cycles not including the origin (RHCs) into equivalence classes given by considering the order of visiting the equilibria on the cycle. We perform a similar classification of robust cycles including the origin (RHCOs), motivated by a recent investigation into such cycles [5]. Using this coding we investigate connections between the numbers of classes and sequences of integers known as ‘meandric numbers’ (see for example Di Francesco *et al* [3]), and the ‘Mercedes-Benz’ integer sequence. We provide conjectures for upper and lower bounds for the number of RHCs and RHCOs given certain configurations of axes equilibria.

## 1.1 Robust Heteroclinic Cycles

Consider the set  $X_G$  of  $C^1$  vector fields on  $\mathbb{R}^n$  that respect some group of symmetries  $G$  acting linearly on  $\mathbb{R}^n$  (such that  $f \in X_G$  means that  $gf(x) = f(gx)$  for all  $x \in \mathbb{R}^n$  and  $g \in G$ ). We give  $X_G$  the topology of  $C^1$  convergence on compact sets. For  $f \in X_G$  we define an ODE  $\dot{x} = f(x)$  and introduce some definitions that will be used throughout the rest of the paper. The definition we use for an heteroclinic cycle is slightly more restrictive than that given in [12]. However, any cycle according to [12] will trivially contain at least one cycle according to our definition.

**Definition 1** *A heteroclinic cycle  $\{x_i, c_i\}$  for an ODE on  $\mathbb{R}^n$  consists of  $(k + 1)$  equilibria  $\{x_i\}$ ,  $i = 0..k$ ,  $k \geq 1$  and  $(k + 1)$  trajectories  $c_i(t)$ , where  $c_i \rightarrow x_i$  as  $t \rightarrow -\infty$  and  $c_i \rightarrow x_{i+1 \pmod{k+1}}$  as  $t \rightarrow +\infty$ , such that the union of these equilibria and trajectories is homeomorphic to a circle.*

In this paper we are only interested in heteroclinic cycles that are robust, namely those that appear for open sets of vector fields in  $X_G$ . More precisely, we define a *robust heteroclinic cycle* (RHC) in the following way.

**Definition 2** *Given some  $f \in X_G$ , any heteroclinic cycle  $\{x_i, c_i\}$ ,  $i = 0..k$ , is an RHC if there are  $\Sigma_i$  isotropy subgroups of  $G$  such that:*

- *The  $x_i$  are hyperbolic saddles with  $\dim(W^u(x_i)) = 1$ .*
- *For all  $i$ ,  $\{c_i\} \subset \text{Fix}(\Sigma_i)$ .*

- The  $x_{i+1 \pmod{k+1}}$  are sinks in the fixed-point subspaces  $\text{Fix}(\Sigma_i)$ .

Furthermore,  $\{x_i, c_i\}$  is an RHCO if we can choose  $x_0$  to be the origin.

This definition implies that for any  $f \in X_G$  with an RHC (RHCO)  $\{x_i, c_i\}$ , there is an open set of perturbed vector fields possessing an RHC (RHCO)  $\{y_i, d_i\}$  with equilibria  $y_i$  near  $x_i$  that are connected in the same ordering. This is because the connections  $c_i$  are from saddle to sink within some invariant subspace, and hence they are robust to perturbation. In other words, there will be an open set of  $g \in X_G$  near  $f$  that give rise to an equivalent cycle in the following sense.

**Definition 3** We say that two RHCs (RHCOs)  $\{x_i, c_i\}$  and  $\{y_i, d_i\}$  for vector fields  $f, g \in X_G$  are equivalent if there is a homeomorphism  $\Pi$  of  $\mathbb{R}^3$  that preserves all symmetry subspaces (i.e. for all  $\Sigma$ ,  $\Pi(\text{Fix}(\Sigma)) = \text{Fix}(\Sigma)$ ), and such that  $\Pi(x_i) = y_i$  and  $\Pi(\{c_i\}) = \{d_i\}$  for all  $i = 0, \dots, k$ .

Definition 3 implies that the homeomorphism preserves the ordering of the equilibria, and so in particular the time-direction of any heteroclinic cycle will also be preserved under equivalence.

## 1.2 Meandric Numbers

We recall the definitions of some integer sequences referred to as meandric numbers and their generalisations. These arise in the classification of curves in the plane, analogous to the number of ways a road (with no junctions) can cross a meandering river using a fixed number of bridges [3, 7, 8, 18]. Consider an infinite oriented line  $L$  in the plane and fix  $n$  a positive integer. We define a *closed meander of order  $n$*  to be any non-selfintersecting closed curve in the plane that transversally intersects  $L$  at  $2n$  points. We similarly say an *open meander of order  $n$*  is a segment of a non-selfintersecting curve in the plane starting on one side of  $L$  that transversally intersects  $L$  at  $n$  points. Two meanders are said to be *equivalent* if one can be mapped onto the other via an isotopy of the plane, such that the line  $L$  is fixed. Otherwise they are said to be *distinct*. A few examples of closed and open meanders are shown in Figure 1. Meandric numbers arise in counting the equivalence classes of meanders as follows.

### Definition 4

- The closed meandric number  $M_n$  is the number of distinct closed meanders of order  $n$ .
- The open meandric number  $m_n$  is the number of distinct open meanders of order  $n$ .

The first few open meandric numbers are shown in Table 1. We first generalise meandric numbers to consider crossings of more general sets  $L$ , see Di Francesco *et al* [3]. A *configuration  $L$*  is a union of line segments embedded in the plane (these may join at one or more

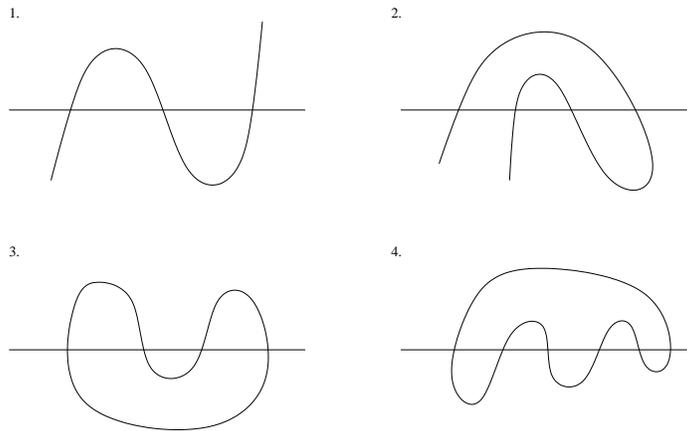


Figure 1: Examples of meanders crossing a fixed oriented line. 1 and 2 show open meanders of order 3 and 4 respectively. 3 and 4 show closed meanders of orders 2 and 3.

$n$	0	1	2	3	4	5	6	7	8	9	10
$m_n$	1	1	1	2	3	8	14	42	81	262	538

Table 1: The first eleven entries in the open meander number integer sequence. Note the sequence is strictly increasing for  $n \geq 2$ .

points). A closed meander through  $L$  is defined analogously to that for a line and the closed meandric number  $M_n(L)$  for the configuration  $L$  is defined similarly.

Meandric numbers are known to have a number of interesting properties (for example one can see that  $M_n = m_{2n-1}$ ). However, there is no known formula for calculating meandric numbers directly for any configuration. The first few terms can be found computationally, for example using a transfer matrix method as Jensen and Guttmann [8]. Jensen [7] gives an algorithm for calculating  $M_n$  that is of computational complexity of order  $2.5^n$ , a great improvement on the method of direct enumeration [18] that is of order  $12.26^n$ .

## 2 Codes of cycles for symmetry group $\mathbb{Z}_2^3$

We consider smooth ODEs  $\dot{\mathbf{x}} = f(\mathbf{x})$  with  $f \in X_G$ ,  $(x, y, z) = \mathbf{x} \in \mathbb{R}^3$  that respect the group of symmetries  $G = \mathbb{Z}_2^3$  generated by the reflections  $\kappa_i$ ,  $i = 1, 2, 3$  in each coordinate plane. These act linearly on  $\mathbb{R}^3$  as follows

$$\begin{aligned}\kappa_1(x, y, z) &= (-x, y, z) \\ \kappa_2(x, y, z) &= (x, -y, z) \\ \kappa_3(x, y, z) &= (x, y, -z).\end{aligned}$$

In what follows we restrict our attention to looking at just the positive octant  $\mathcal{O} = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$ . This is flow invariant as it is bounded by invariant

Typical point	Invariant subspaces	Isotropy subgroup	Generators	Conjugates
$(0, 0, 0)$	origin	$\mathbb{Z}_2^3$	$\langle \kappa_1, \kappa_2, \kappa_3 \rangle$	1
$(x, 0, 0)$	coordinate axes	$\mathbb{Z}_2^2$	$\langle \kappa_2, \kappa_3 \rangle$	3
$(x, y, 0)$	coordinate planes	$\mathbb{Z}_2$	$\langle \kappa_3 \rangle$	3
$(x, y, z)$	general point	$\{e\}$	$\{\}$	8

Table 2: The invariant subspaces corresponding to different isotropy subgroups for the action of  $\mathbb{Z}_2^3$  on  $\mathbb{R}^3$ .

subspaces (see Table 2), and thus any RHC or RHCO contained in it will be restricted to the boundary, namely the axes and coordinate planes. Note that  $G\mathcal{O} = \mathbb{R}^3$ .

Observe that any  $f \in X_G$  can be written in ‘Lotka-Volterra’ form by noting that

$$\begin{aligned} \dot{x} &= x f_1(x^2, y^2, z^2) \\ \dot{y} &= y f_2(x^2, y^2, z^2) \\ \dot{z} &= z f_3(x^2, y^2, z^2) \end{aligned} \tag{1}$$

for some  $f_i : \mathbb{R}^3 \rightarrow \mathbb{R}$ , and changing to coordinates  $X = x^2$ ,  $Y = y^2$ ,  $Z = z^2$  so that

$$\begin{aligned} \dot{X} &= X f_1(X, Y, Z) \\ \dot{Y} &= Y f_2(X, Y, Z) \\ \dot{Z} &= Z f_3(X, Y, Z). \end{aligned} \tag{2}$$

We note that in such a setting, Schreiber [17] gives some very weak general conditions that imply the existence of robust heteroclinic cycles.

## 2.1 A coding system for RHCs and RHCOs in $\mathcal{O}$

We describe a coding for different equivalence classes of robust cycles. For RHCOs we assume without loss of generality that the unstable manifold of the origin is within the  $x$ -axis. This means that all connections will occur within one of the following invariant subspaces:

$$C_0 = (x, 0, 0), \quad C_1 = (x, y, 0), \quad C_2 = (x, 0, z), \quad C_3 = (0, y, z).$$

We number the equilibria included in the cycle along the axes, working out from the origin and numbering the  $x$ ,  $y$  and  $z$  axes one after the other. Suppose that there are  $n_x$  equilibria on the  $x$ -axis,  $n_y$  on the  $y$ -axis and  $n_z$  on the  $z$ -axis (we write  $n = n_x + n_y + n_z$  for RHCs or  $n = n_x + n_y + n_z + 1$  for RHCOs). This gives a unique ordering of the equilibria which for  $n_x > 0$  and  $n_y > 0$  is

$$\begin{aligned} P_0 &= \text{the origin} \\ P_1 \dots P_{n_x} &= \text{points on } x \text{ axis} \\ P_{n_x+1} \dots P_{n_x+n_y} &= \text{points on } y \text{ axis} \\ P_{n_x+n_y+1} \dots P_n &= \text{points on } z \text{ axis.} \end{aligned} \tag{3}$$

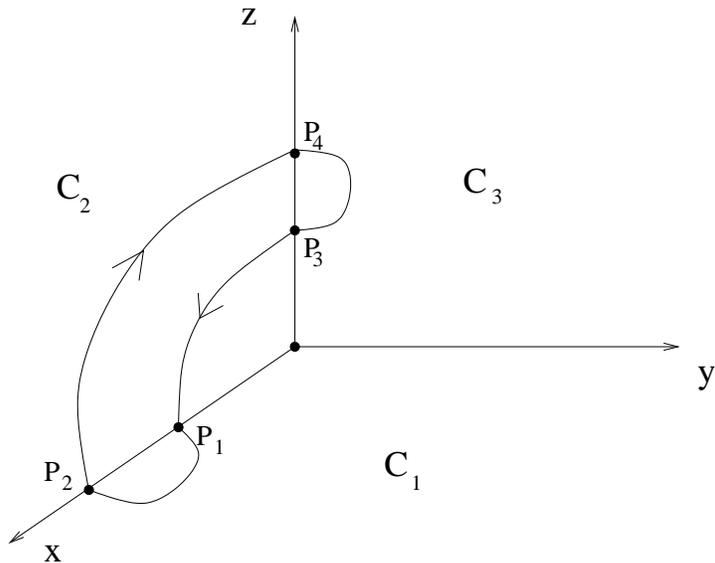


Figure 2: An example of a member of a class of RHCs in  $\mathbb{R}^3$  with symmetry group  $\mathbb{Z}_2^3$  defined by the coding  $P_1C_1P_2C_2P_4C_3P_3C_2$ . Note the coding also determines the direction; the same cycle traced in reverse has a different coding and thus falls within a different class of RHCs.

In the case where  $n_x = 0$ , the point  $P_1$  is taken to be the first point on the  $y$ -axis, and when both  $n_x = 0$  and  $n_y = 0$  it is taken as the first point on the  $z$ -axis. Our assumption on RHCs means that that  $P_0$  has an unstable manifold on the  $x$ -axis and thus  $P_1$  will always appear on the  $x$ -axis, and  $n_x > 0$ . Other than this first connection there can be no further robust connections along coordinate axis once we leave  $P_1$ .

By Definition 1, we have that each point  $P_i$  appears at most once in the cycle, so w.l.o.g. we can start coding the cycle at point  $P_1$  for RHCs and the point  $P_0$  for RHCs. We build up the coding for the cycle simply by listing the equilibria and invariant subspaces in the order they appear on the cycle; for example, a cycle with coding  $P_1C_1P_2C_2P_4C_3P_3C_2$  is shown in Figure 2.

**Theorem 1** *Two RHCs are equivalent if and only if they have the same coding.*

**Proof:** We start by showing that the equivalent cycles must have the same coding. To this end, suppose we have two equivalent robust cycles with codings

$$C_1 = P_1C_{j_1}P_{i_1}C_{j_2}\dots C_{j_n}, \quad C_2 = P_1C_{l_1}P_{k_1}C_{l_2}\dots C_{k_n}.$$

Since the cycles are equivalent there is an equivariant homeomorphism mapping one cycle to the other, and in particular preserving all the coordinate planes. Hence we have  $C_{j_\alpha} = C_{l_\alpha}$  for all  $1 \leq \alpha \leq n$ . The choices of the planes being fixed forces the equilibria at each stage to be on the same axes. Since the homeomorphism preserves orientation, the ordering of the equilibria will be preserved and hence the codings identical.

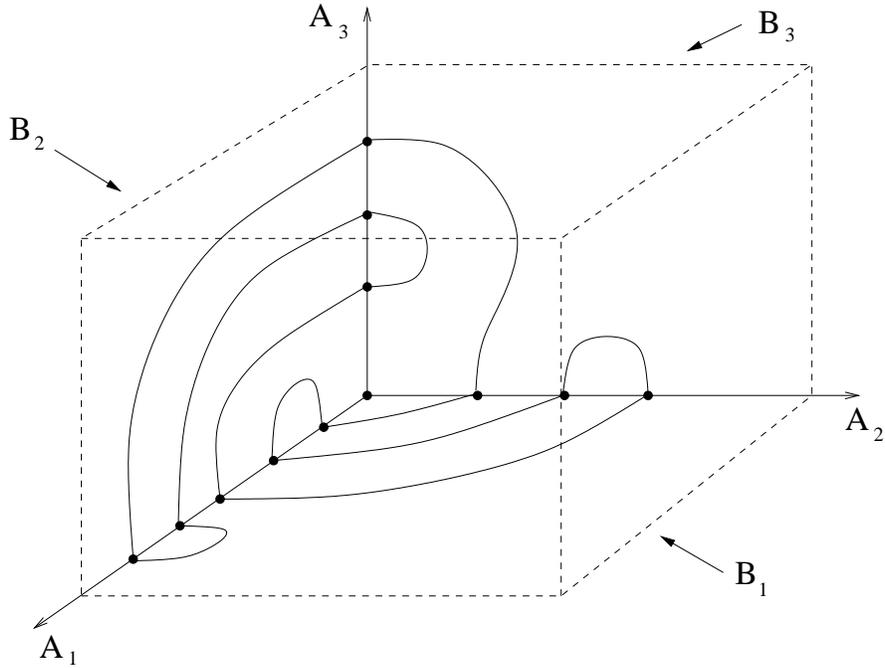


Figure 3: An example of an RHC  $\Sigma_i$  in  $Q = [0, L]^3 \subset \mathbb{R}^3$ . The axes and planes are labelled as in the proof of Theorem 1. Note all trajectories start and finish on the boundary of the  $B_j$  so there are no closed loops (holes) in the planes. Thus  $B_j \setminus \Sigma_i$  consists of a union of 2D regions that are homeomorphic to discs.

To show the converse, suppose we have two RHCs  $\Sigma_1$  and  $\Sigma_2$  with the same coding  $\mathcal{C} = P_1 C_{j_1} P_{i_1} C_{j_2} \dots C_{j_n}$ . We wish to find a homeomorphism between the two that extends to the whole space  $\mathbb{R}^3$ . This we do by a series of extensions. First, we choose  $L \in \mathbb{R}$  with  $L > 0$  such that

$$\Sigma_1, \Sigma_2 \subseteq [0, L]^3 \equiv Q.$$

Let  $A_1, A_2, A_3$  be the intersection of  $Q$  with the coordinate axes  $x, y, z$  respectively, and let  $B_1, B_2, B_3$  be the intersection of  $Q$  with the coordinate planes  $xy, xz, yz$ . Then

$$\partial Q = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$$

where the other  $B_i$  are the remaining faces of the cube  $\partial Q$ .

Consider the equilibria of  $\Sigma_1$  on the axes  $A_i$ . Note that these can be mapped bijectively to the equilibria of  $\Sigma_2$  on  $A_i$ . This can then be extended, for example through linear interpolation, to some homeomorphism

$$\Pi_1 : \bigcup_{i=1}^3 A_i \longrightarrow \bigcup_{i=1}^3 A_i$$

on the axes.

We can extend this to the faces of  $Q$  by considering one of the faces  $B_j \setminus \Sigma_1$ . This is a union of topological discs since all trajectories in  $\Sigma_1$  must start and finish somewhere on the boundaries of  $B_j$  (see Figure 3). In the same way  $B_j \setminus \Sigma_2$  will also consist of topological discs. Hence we can extend  $\Pi_1$  to

$$\Pi_2 : \bigcup_{j=1}^6 B_j \longrightarrow \bigcup_{j=1}^6 B_j$$

such that  $\Pi_2|_{\bigcup_{j=1}^3 A_j} = \Pi_1$  and  $\Pi_2|_{\bigcup_{j=4}^6 B_j} = id$ .

To extend the homeomorphism from  $\partial Q$  to  $Q$ , choose an arbitrary point  $x$  in the interior of  $Q$ . We can interpolate along the line from  $x$  to  $y \in \partial Q$  to construct an homeomorphism

$$\Pi_3 : Q \longrightarrow Q$$

such that  $\Pi_3|_{\partial Q} = \Pi_2$ .

Finally then we can define an equivariant homeomorphism  $\Pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$  on the whole space, extending  $\Pi_3$  by

$$\Pi(x) \equiv \begin{cases} \kappa^{-1} \Pi_3(\kappa x) & \text{if } \kappa \in G \text{ such that } \kappa x \in Q \\ x & \text{otherwise.} \end{cases}$$

Note then in particular  $\Pi$  is continuous with continuous inverse for all  $x \in \partial Q$  since  $\Pi(B_j) = B_j$ . Hence by Definition 3 the two cycles are equivalent. **QED**

**Corollary 1** *Two RHCs are equivalent if and only if they have the same coding.*

**Proof:** The proof follows in a similar way to Theorem 1.

**QED**

The following Theorem precisely characterises the set of possible codings of RHCs and RHCOs:

**Theorem 2** Consider a set of distinct points  $P_1, \dots, P_n$  on the axes bounding  $\mathcal{O}$ , numbered sequentially as in (3). The sequence

$$\mathcal{C} = P_1 C_{j_1} P_{i_1} \cdots P_{i_{n-1}} C_{j_n}$$

is a coding for an RHC with equilibria  $P_i$  if and only if  $C_{j_1} \neq C_{j_n}$  and the following are satisfied.

- (i) The set  $\{i_1, \dots, i_{n-1}\}$  is a permutation of  $\{2, \dots, n\}$ .
- (ii) For every  $l = 1, \dots, n-1$  we have  $C_{j_l} \neq C_{j_{l+1}}$ .
- (iii) For every  $l = 1, \dots, n-1$  we have  $P_{j_l} \subset C_{i_l} \cap C_{i_{l+1}}$ .
- (iv) The crossing conditions in Appendix B are satisfied.

Similarly, the word

$$\mathcal{C} = P_0 C_{j_1} P_{i_1} \cdots P_{i_{n-1}} C_{j_n}$$

is a coding for an RHCO between equilibria  $P_i$  and the origin (where we assume unstable manifold of origin is in  $x$ -direction) if and only if

$$C_{j_1} = C_0, \quad P_{i_1} = P_1, \quad C_{j_n} = C_3$$

and the conditions (i-iv) above are satisfied.

**Proof:** We first show necessity of the conditions for RHCs. Point (i) follows because each equilibrium has only one part of its unstable manifold contained within  $\mathcal{O}$ . Moreover, equilibria must be contained in the nontrivial intersection of two invariant subspaces, implying (ii) is necessary. Similarly, (iii) is clearly necessary. Finally, two different connections cannot intersect, which implies that the crossing conditions (iv) are necessary.

To show sufficiency, embed a number of curves in the planes  $C_{i_{l+1}}$  that connect  $P_{j_k}$  to  $P_{j_{k+1}}$ . (iii) and (iv) imply one can do this such that none of the curves intersect. By (i) these curves form a continuous non-selfintersecting loop  $\ell$ . By (ii) we can find vector fields with hyperbolic saddles at each  $P_{i_l}$ , such that  $W^u(P_{i_l})$  is one dimensional and contained within  $C_{j_{l+1}}$  and such that  $W^s(P_{i_l}) \subset C_{j_l}$ . Taking a tubular neighbourhood of  $\ell$  we can find a vector field as above with connections from  $P_{i_l}$  to  $P_{i_{l+1}}$  lying within  $C_{j_{l+1}}$ . Such a vector field will have an RHC with the required coding.

For the analogous result for RHCOs, one can argue in a similar way.

**QED**

$n$	0	1	2	3	4	5	6	7	8	9	10
$M_n(MB)$	3	0	3	1	9	6	45	42	279	320	1977

Table 3: The entries in the Mercedes-Benz integer sequence up to  $n = 10$ .

### 3 Classification of RHCs and RHCOs in $\mathcal{O}$

#### 3.1 RHCs in $\mathcal{O}$ and the ‘Mercedes-Benz’ problem

In this section we apply the coding of Theorems 1 and 2 to classify RHCs in  $\mathcal{O}$ . We define  $N(n)$  to be the number of distinct classes of RHCs between  $n$  equilibria. The first nontrivial case starts with  $n = 2$ . One can see immediately that there can be only six distinct heteroclinic cycles of this type, namely loops around each axis, one travelling in each direction. Their codings are:

$$\begin{aligned}
 &P_1C_1P_2C_2 \text{ and } P_1C_2P_2C_1 \\
 &P_1C_1P_2C_3 \text{ and } P_1C_3P_2C_1 \\
 &P_1C_2P_2C_3 \text{ and } P_1C_3P_2C_2
 \end{aligned}$$

For  $n = 3$  we get only two distinct classes of RHC, which contain in particular the example studied by Guckenheimer and Holmes [4] but without the additional cyclic symmetry property. The cycles simply pass through one point on each axis. The codings are thus:

$$P_1C_1P_2C_3P_3C_2 \text{ and } P_1C_2P_3C_3P_2C_1.$$

As  $n$  is increased beyond 3, the number of loops  $N(n)$  crossing the axes  $n$  times increases rapidly. The problem of working out the numbers of classes has a strong connection with meandric numbers (Section 1.1 for more details) and in particular is identical (up to a factor of two) to the sequence generated by looking at the ‘Mercedes Benz problem’.

The Mercedes Benz problem examines the number of ways an undirected loop in the plane can cross three roads meeting in a ‘Y’ configuration  $n$  times. The name seems to be due to Wilde and Sloane [18]. This corresponds in our earlier notation to meanders through the configuration  $L = MB$  of three lines meeting at a single point, in the style of the Mercedes-Benz symbol (see Figure 4). The first few terms of the sequence  $M_n(MB)$  are listed in Table 3. For more observations on this sequence see Appendix C.

**Lemma 1** *The number of classes of RHCs with  $n$  equilibria is related to the Mercedes-Benz number  $M_n(MB)$  by*

$$N(n) = 2M_n(MB).$$

**Proof:** If we map our system onto  $\mathbb{R}^2$  by taking the projection along the line  $x = y = z$  then we obtain a picture identical to that corresponding to the Mercedes-Benz problem, i.e. we can find a one-to-one mapping between the projection and the Mercedes-Benz diagram. Thus the number of undirected closed loops crossing the Mercedes-Benz symbol will be identical to the number of undirected RHCs. The factor of two corresponds to the fact that the heteroclinic cycles are directed and hence we can trace the closed loops in either direction.

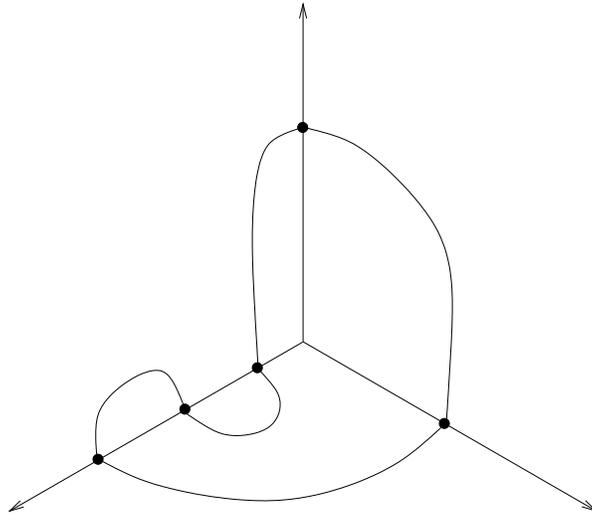


Figure 4: A closed meander through a ‘Y-shaped’ configuration in the plane; the Mercedes-Benz problem enumerates such meanders up to equivalence. Shown here is an example of a closed meander of order 5.

**QED**

We also consider the number of possible classes of RHC given a particular configuration of equilibria, i.e. where the number of equilibria on each axis  $n_x, n_y, n_z$  is known. In order to do this we introduce some new notation.

**Definition 5** *Let  $Y$  be the subset of  $X_G$  containing  $n_x, n_y, n_z$  equilibria that are hyperbolic sinks within the non-negative  $x, y$  and  $z$  axes respectively. Let  $T(n_x, n_y, n_z)$  be the total number of RHC classes in  $Y$  that include a subset of those equilibria  $n_x, n_y, n_z$ . Let  $K(n_x, n_y, n_z)$  be the number of classes of RHCs including precisely those equilibria.*

**Lemma 2** *The quantities above can be related as follows:*

$$T(n_x, n_y, n_z) = \sum_{p_i \leq n_i \forall i \in \{x, y, z\}} \binom{n_x}{p_x} \binom{n_y}{p_y} \binom{n_z}{p_z} K(p_x, p_y, p_z).$$

**Proof:** The quantity  $T(n_x, n_y, n_z)$  is simply a sum of the classes contained by choice of  $p_i$  out of  $n_i$  of the equilibria on each of the axes, times the number of classes visiting these equilibria. Each class is clearly distinct, giving the sum as required.

**QED**

We can also note that the total number of classes obtainable using precisely  $n$  equilibria (see Lemma 1) is related to the quantity  $K$  above by

$$N(n) = \sum_{n_x + n_y + n_z = n} K(n_x, n_y, n_z).$$

We were not able to find any formulae or recursion relations to help in the calculation of either of the quantities  $T$  or  $K$ . However we did find that certain choices of  $n_x, n_y, n_z$  appeared to consistently give the minimum and maximum possible number of classes of RHC. Hence we make the following conjecture.

**Conjecture 1** *The total number of classes of RHCs possible with  $n$  axes equilibria is bounded below by the number of classes of RHCs possible when  $n_x = \frac{n}{2}, n_y = \frac{n}{2}, n_z = 0$  when  $n$  is even or by  $n_x = \frac{n+1}{2}, n_y = \frac{n-1}{2}, n_z = 0$  when  $n$  is odd. It is bounded above by the number of classes obtained by setting  $n_x = n, n_y = 0, n_z = 0$ , where  $n = n_x + n_y + n_z$ . More succinctly,*

$$T\left(\frac{n}{2}, \frac{n}{2}, 0\right) \leq T(n_x, n_y, n_z) \leq T(n, 0, 0)$$

for  $n$  even, or

$$T\left(\frac{n+1}{2}, \frac{n-1}{2}, 0\right) \leq T(n_x, n_y, n_z) \leq T(n, 0, 0)$$

otherwise.

We have verified this computationally up to  $n = 7$ . Furthermore it appears that as  $n$  is increased, the bounds separate farther from any other possible value of  $T$ . We can also rewrite the conjecture above in terms of coefficients  $K$  using our definition of  $T$  above, by writing:

$$\begin{aligned} T(n, 0, 0) &= \sum_{p \leq n} \binom{n}{p} K(p, 0, 0) \\ &= \sum_{p \leq n} \binom{n}{p} 2M_{\frac{p}{2}} \end{aligned}$$

where  $M_{\frac{p}{2}}$  is the closed meandric number of order  $\frac{p}{2}$  and  $p$  is even. Also for  $n$  even we have

$$T\left(\frac{n}{2}, \frac{n}{2}, 0\right) = \sum_{p_i \leq \frac{n}{2} \forall i \in \{x, y\}} \binom{\frac{n}{2}}{p_x} \binom{\frac{n}{2}}{p_y} K(p_x, p_y, 0)$$

for all combinations  $K$  such that both of  $p_x, p_y$  are even. Finally we have the result for  $n$  odd where

$$T\left(\frac{n+1}{2}, \frac{n-1}{2}, 0\right) = \sum_{p_i \leq \frac{n}{2} \forall i \in \{x, y\}} \binom{\frac{n+1}{2}}{p_x} \binom{\frac{n-1}{2}}{p_y} K(p_x, p_y, 0)$$

where again we must choose  $p_x, p_y$  to be both even. This follows by firstly noticing from the definition of the  $K$  that  $K(p, 0, 0)$  is precisely the number of equivalence classes of directed closed loops passing through  $p$  points (i.e.  $2M_{\frac{p}{2}}$ ). It also makes use of the following observation.

**Lemma 3** *All  $K$  are zero except for when we have an even number of points on all axes or when we have an odd number of points on all axes. The former configuration gives rise to a closed loop with the origin on the outside, the latter to a closed loop surrounding the origin.*

**Proof:** Consider  $K(p_x, p_y, p_z)$  with (w.l.o.g.)  $p_x$  even. If  $p_y = p_z = 0$  then we can obtain a cycle equivalent to the closed meander of order  $\frac{p_x}{2}$  around the  $x$ -axis. If one of the other two axes is unused, say  $p_z = 0$  and  $p_y \neq 0$ , then  $p_y$  must be even. Since suppose  $p_y$  were odd. Then there are an odd number of trajectories passing through these points, some subset of which may interconnect. However this subset will contain an even number of trajectories, hence the remainder that must connect to the  $x$  axis are odd in number. But there will be an even number of trajectories leaving the  $x$ -axis that must connect to these (since  $p_x$  is even). This gives a contradiction, so  $p_y$  is even. Lastly if all the  $p_i$  are non-zero then we can apply a similar argument again to see that  $p_y$  and  $p_z$  must both be even to obtain an RHC. It follows then that if there are an odd number of points on one axis, then there must be an odd number on all axes. **QED**

Because of this result it is helpful to think of the ‘Mercedes-Benz’ sequence as being two different sequences depending on whether  $n$  (and thus the  $n_x, n_y, n_z$ ) is even or odd. We note also that there may well be more than one RHC in existence for a particular choice of equilibria (for example if we choose  $n_x = n_y = n_z = 2$  then we can obtain two RHCs of 3 points, say  $P_1C_1P_3C_3P_5C_2P_1$  and  $P_2C_1P_4C_3P_6C_2P_2$  which do not intersect and can thus theoretically coexist). However if the origin is a part of the cycle we can only ever obtain one class of RHCO for a given ordering of equilibria in the cycle.

We continue in the next section with a detailed look at RHCOs and we provide a similar classification to that for RHCs.

### 3.2 RHCOs in $\mathcal{O}$

We now shift our attention to the robust cycles including the origin as a point on the cycle. We present a classification as with the RHCs, however this time, since we have chosen the starting point as the origin, we assume the direction of the unstable manifold as passing along the  $x$ -axis, and are confined to  $\mathcal{O}$ , the direction of travel is determined. We define  $N_0(n)$  to be the number of distinct classes of RHCOs between  $n$  hyperbolic equilibria (this time including the origin). Once again there is a strong connection with meandric numbers, but unfortunately this time we are not afforded the luxury of association with a previously studied sequence.

We can count classes of RHCOs of vector fields in  $X_G$  for  $n$  a small number of equilibria in the cycle, via observation or computation.

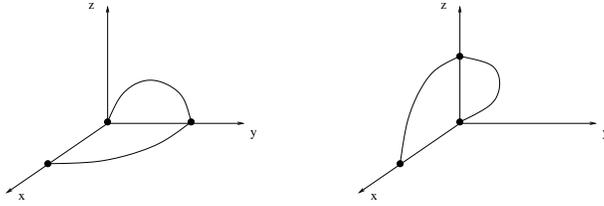
**Theorem 3** *The number of possible inequivalent RHCOs in  $X_G$  is given by  $N_0(n) = 0$  for  $n \leq 2$ ,  $N_0(3) = 2$ ,  $N_0(4) = 2$ ,  $N_0(5) = 8$ ,  $N_0(6) = 12$ . Examples of these cycles up to  $n = 5$  can be seen in Figures 5.*

**Proof:** This is computed by using Theorem 2 to enumerate all possible allowable codings. For larger  $n$  the results are found computationally. `Maple` was used to generate many of the results listed in Table 4.

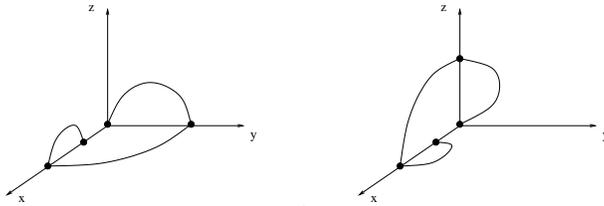
**QED**

As before we went on to look at particular configurations of axes equilibria, and introduce here definitions in a similar fashion to the RHCs.

n=3



n=4



n=5

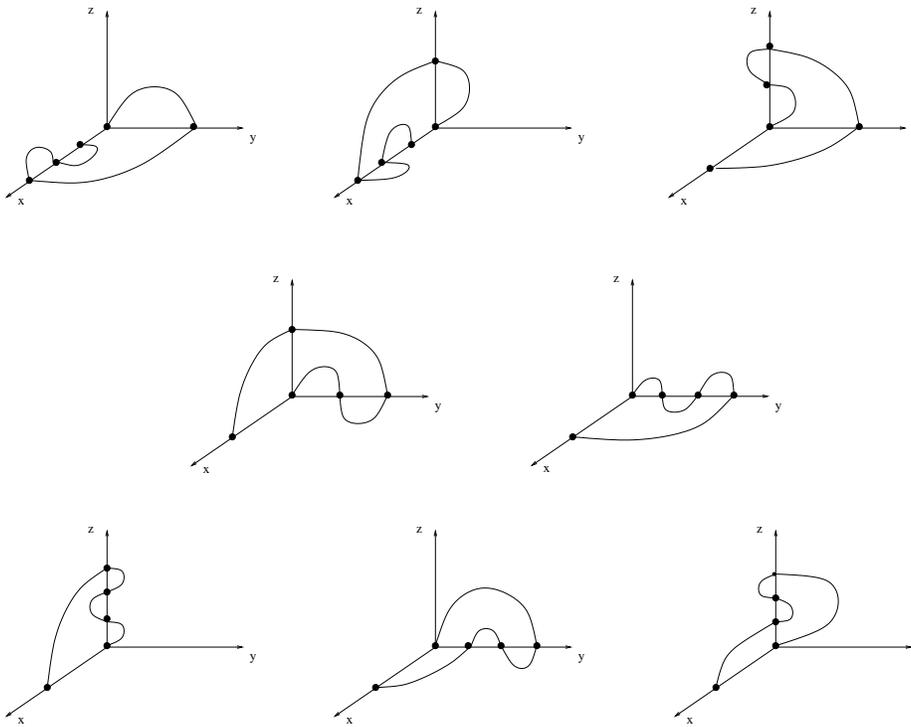


Figure 5: Possible classes of RHCO for  $n = 3, 4$  and  $5$ . We see here that  $N_0(3) = 2$ ,  $N_0(4) = 2$  and  $N_0(5) = 8$ .

**Definition 6** Let  $Y$  be the subset of  $X_G$  containing  $n_x, n_y, n_z$  equilibria that are hyperbolic sinks within the non-negative  $x, y$  and  $z$  axes respectively. Let  $T_0(n_x, n_y, n_z)$  be the total number of RHC classes in  $Y$  that include a subset of those equilibria  $n_x, n_y, n_z$ . Let  $K_0(n_x, n_y, n_z)$  be the number of classes of RHCs including precisely those equilibria.

Similarly to Lemma 2 one can show that

$$T_0(n_x, n_y, n_z) = \sum_{p_i \leq n_i \forall i \in \{x, y, z\}} \binom{n_x - 1}{p_x - 1} \binom{n_y}{p_y} \binom{n_z}{p_z} K_0(p_x, p_y, p_z)$$

where  $p_x > 0$  (or if  $p_x = 0$  then we set  $T_0(n_x, n_y, n_z) = 0$ ). One can get an expression for  $N_0(n)$  in terms of the  $K_0$  exactly as for  $N(n)$ . We obtain the following conjecture upon calculation of the  $T$  for small values of  $n$ .

**Conjecture 2** The total number of classes of RHCs possible with  $n$  equilibria (including the origin) is bounded below by the number of classes of RHCs possible when  $n_x = n - 3, n_y = 2, n_z = 0$  and above by the number when  $n_x = 1, n_y = n - 2, n_z = 0$ , where  $n = n_x + n_y + n_z + 1$ .

$$T_0(n - 3, 2, 0) \leq T_0(n_x, n_y, n_z) \leq T_0(1, n - 2, 0)$$

Both of these bounds can be written as before in terms of open meandric numbers by observing that  $K_0(i, 1, 0) = m_{i-1}$ , and  $K_0(1, j, 0) = m_j$  for  $j$  odd or zero for  $j$  even. If the conjecture is correct, this would imply for  $n$  odd that

$$2 \binom{n-4}{0} m_0 + \dots + 2 \binom{n-4}{n-4} m_{n-4} \leq T_0(n_x, n_y, n_z),$$

$$T_0(n_x, n_y, n_z) \leq \binom{n-2}{1} m_1 + 0m_2 + \dots + 0m_{n-3} + \binom{n-2}{n-2} m_{n-2},$$

or if  $n$  is even then

$$2 \binom{n-4}{0} m_0 + \dots + 2 \binom{n-4}{n-4} m_{n-4} \leq T_0(n_x, n_y, n_z),$$

$$T_0(n_x, n_y, n_z) \leq \binom{n-2}{1} m_1 + 0m_2 + \dots + 0m_{n-4} + \binom{n-2}{n-3} m_{n-3} + 0m_{n-2}.$$

We have verified analytically that these hold at least up to  $n = 7$ .

The first few values for the upper and lower bounds here and for the RHC problem are shown in Table 4. Note that if the conjecture is true these will be optimum bounds since we always have the possibility that the set-up is identical to that needed for one of the bounds. To tie in with the rest of the paper we include the open meandric numbers, and values of  $N_0(n)$  and  $N(n)$ .

It can be seen in Table 4 that the quantity  $N_0(n)$  is very close to the corresponding values of the open meandric number  $m_n$ . To demonstrate this we illustrate the two in Figure 6. Only the values for  $n \leq 10$  are shown. The last four entries for  $N_0(n)$  have been estimated by hand but not yet confirmed computationally.

$n$	$T_0(1, n-2, 0)$	$T_0(n-3, 2, 0)$	$T(n, 0, 0)$	$T(\frac{n}{2}, \frac{n}{2}, 0)$ or $T(\frac{n+1}{2}, \frac{n-1}{2}, 0)$	$N_0(n)$	$N(n)$	$m_n$
2	0	0	2	0	0	6	1
3	1	0	6	2	2	2	2
4	2	2	16	6	2	18	3
5	5	4	40	14	8	12	8
6	12	8	106	30	12	90	14
7	33	18	294	76	46	84	42
8	94	44	868	198	84?	558	81
9	287	118	2676	526	304?	640	262
10	904	338	8594	1430	566?	3954	538
11	2959	1024	28446	4266	1982?	5220	1828

Table 4: Upper and Lower bounds on classes of RHCs for small values of  $n$ , together with similar fields for RHCs. Also shown are the values of  $N(n)$  and  $N_0(n)$  and the open meandric numbers for comparison.

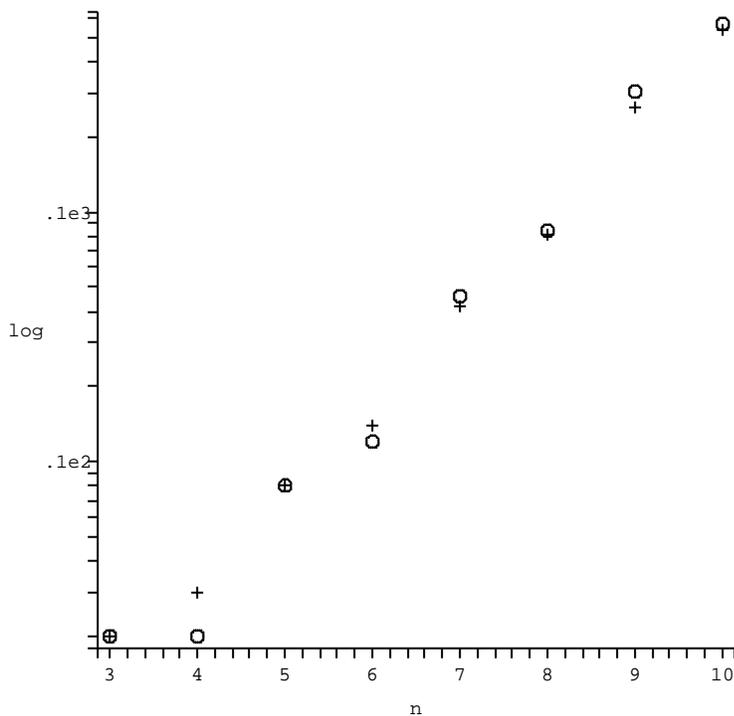


Figure 6: A plot of  $\log(m_n)$  and  $\log(N_0(n))$  against  $n$ , the former given by the crosses and the other by the circles. Notice the apparently strong correlation between the two.

## 4 Discussion

We have introduced a new coding invariant that allows one to completely classify RHCs and RHCOs in  $\mathcal{O}$  with the symmetry group  $\mathbb{Z}_2^3$  using a coding of the equilibria and planes involved. Each coding represents an unique class of robust cycle and thus by finding all possible codings we can classify all types of possible cycles up to equivalence. This classification should allow for example, a better understanding of the ‘successionally stable vector fields’ in the three-species population models investigated by Schreiber [17] in that it classifies precisely the possible orderings of visits to one-species equilibria.

As a new feature, we have demonstrated connections to the study of the combinatorics of meanders and meandric numbers, and more precisely an equivalence of the Mercedes Benz sequence to classes of RHCs. For RHCs the problem is closely related to that of meanders, but we could find no such sequence for  $N_0(n)$  in the literature, though were able to calculate the first few terms both analytically and computationally. The latter sequence is similar to the open meandric number sequence but it is unclear whether this is significant. We have conjectured upper and lower bounds based on meanders for specific configurations of equilibria.

We do not discuss the stability of the RHCs and RHCs but note that robust stability is possible; see for example Krupa and Melbourne [11, 12] for conditions on the asymptotic stability of RHCs. Note that RHCs do not fall into the class of ‘simple’ robust cycles they consider, with the consequence that other eigenvalues become important for stability of RHCs; see [5].

One might suspect that the results here (being driven by consideration of the topology of plane meanders) do not apply to higher dimensional systems. However this is not the case, as long as some invariant subspaces of dimension two contain some of the connections. For example, one extension of this work would be to consider RHCs for vector fields in  $\mathbb{R}^4$  with respect to a symmetry group that has no fixed point subspaces of dimension 3 (type *A* in [12]). The RHCs we study in this paper are all of type *A* since we are working in  $\mathbb{R}^3$ .

Finally, robust cycles with higher dimensional manifolds of connections can also appear in more general symmetric systems; by analogy with the meanders here there should be a rich topological structure in any case where the connecting manifolds are of codimension one within some invariant subspace.

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## A The symmetry group $\mathbb{Z}_2^2$

In order for an RHC or an RHCO to arise, the group symmetry must preserve more than one distinct invariant subspace; hence no robust cycle can be obtained if we have only the symmetries produced by  $\mathbb{Z}_2$ .

For  $\mathbb{Z}_2^2$  we can obtain RHCs (there is no distinction with RHCOs as the origin is not a special point). These will all be of the form where all the equilibria in the cycle are on the intersection of the two symmetry-invariant planes (i.e. on an axis). The trajectories will be contained in these planes, and so we will have four copies of any cycle: one in each quadrant of  $\mathbb{R}^3$ . Furthermore we can completely classify the different equivalence classes of cycle possible. This is done quite simply by noticing that given  $n$  hyperbolic saddles on the intersection (we need at least one equilibria between these to produce the required stabilities), the number of classes is given precisely by the closed meandric number of order  $\frac{n}{2}$ . This again comes straight from the definitions in the previous section, upon noticing that we can effectively make each cycle 2 dimensional by looking at it from a point on the line  $x = z, y = 0$  (i.e we can form a projection of the cycle onto the plane). This will then have precisely the form of a closed meander. Note this method of projection is used implicitly throughout the paper to show the relationship between RHCs and various types of meander. See Figure 7 for an example.

## B The crossing conditions

Suppose we have a coding  $\mathcal{C}_1 = P_1 C_{j_1} P_{i_1} C_{j_2} \dots C_{j_n}$  for an RHC (or the similar coding for an RHCO), and suppose  $C_{j_m} = C_{j_n}$  for some  $m \neq n$ . Then let the four points surrounding these be relabelled as:

$$P_{i_{m-1}} = Q_0, P_{i_m} = Q_1, P_{i_{n-1}} = Q_2, P_{i_n} = Q_3.$$

Then we must consider the possibilities below. By ‘crossing trajectories’ here we mean that for any two trajectories contained in a plane  $C_j$ , there can be no point at which they intersect. In what follows, if  $Q_i$  and  $Q_j$  are on the same axis,  $Q_i < Q_j \Leftrightarrow \|Q_i\| < \|Q_j\|$

- **If all 4 points are on the same axis.** If  $(Q_0 < Q_2 < Q_1 < Q_3)$  or  $(Q_0 < Q_3 < Q_1 < Q_2)$  or  $(Q_1 < Q_2 < Q_0 < Q_3)$  or  $(Q_1 < Q_3 < Q_0 < Q_2)$  or  $(Q_2 < Q_0 < Q_3 < Q_1)$  or  $(Q_2 < Q_1 < Q_3 < Q_0)$  or  $(Q_3 < Q_0 < Q_2 < Q_1)$  or  $(Q_3 < Q_1 < Q_2 < Q_0)$  then we have crossing trajectories.

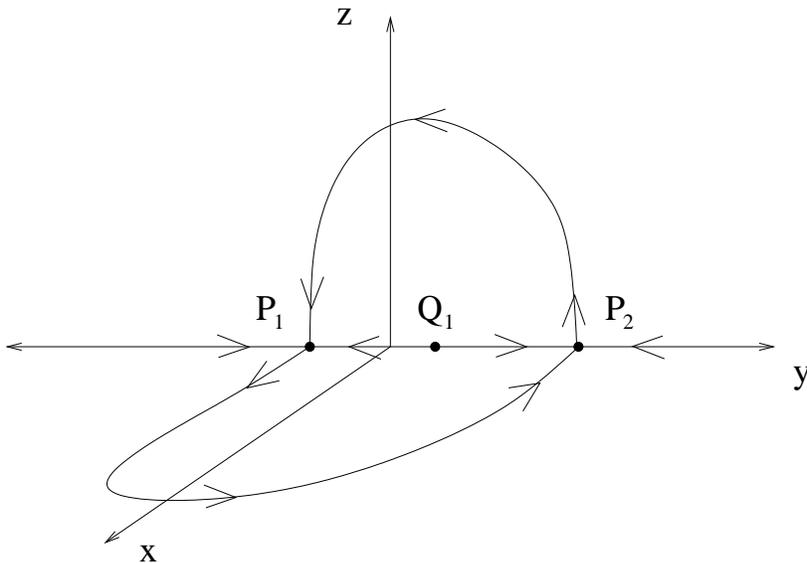


Figure 7: An example of an RHC in  $\mathbb{R}^3$  with the symmetry group  $\mathbb{Z}_2^2$ . Both equilibria  $P_1, P_2$  are hyperbolic sinks: the additional point  $Q_1$  between them produces the correct stabilities. All connections are of saddle-sink type and thus the cycle is robust. This example is equivalent to the closed meander of order one.

- **If 3 points are on the same axis.** If for example  $Q_0$  is on a different axis, then if  $(Q_2 < Q_1 < Q_3)$  or  $(Q_3 < Q_1 < Q_2)$  then we have crossing trajectories. Similar conditions apply when the other three points are chosen.
- **If 2 points are on each axis.** Then only possibilities of crossings are when  $Q_0$  and  $Q_1$  are on different axes. So for example if  $Q_0$  and  $Q_2$  are on the same axis, then if we have  $(Q_0 < Q_2)$  and  $(Q_3 < Q_1)$  or  $(Q_2 < Q_0)$  and  $(Q_1 < Q_3)$  we have crossing trajectories. Similarly, we get two more inequality sets if  $Q_0$  and  $Q_3$  are on the same axis.

If *none* of these inequalities hold for a particular coding, then we say that the *crossing conditions are satisfied*.

## C Mercedes-Benz numbers; additional information

Note that for most values of  $n$ , the Mercedes-Benz numbers are divisible by three. This is since most patterns of equilibria and trajectories can be rotated by  $\pm \frac{2\pi}{3}$  radians to create a different class of cycles with the same overall pattern. Thus in general there will be three classes arising from the rotations of any given cycle. However if the cycle possesses rotational symmetry of order three, then we have extra classes that can appear. In such a case we must have the same (odd) number of points on each axis so that  $n_x = n_y = n_z$ , and further more

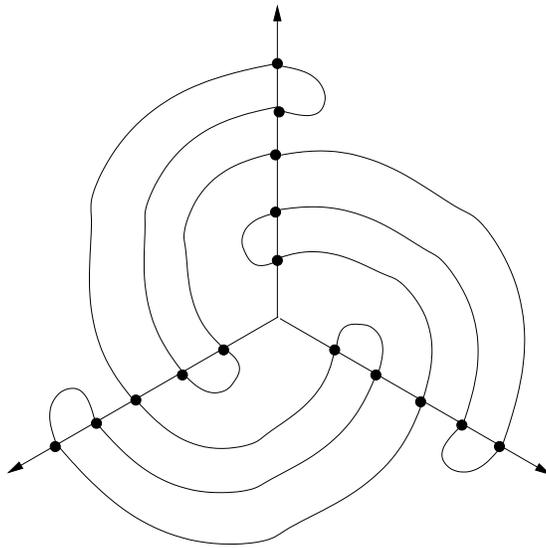


Figure 8: The additional pattern possible for  $n = 15$  with third-order rotational symmetry as referred to in the text of Appendix C.

the pattern of trajectories about each axis must be identical. Thus all the numbers will be divisible by three except for when  $n$  is three multiplied by an odd number. We can say a little more by looking at the number of different patterns we can choose around each axis that retain the rotational symmetry. For  $n = 3$  this will be  $m_1 = 1 = 1 \pmod{3}$  and for  $n = 9$ ,  $m_3 = 2 = 2 \pmod{3}$ . For  $n = 15$ , we have an additional pattern other than those arising from  $m_5$ , namely that seen in Figure 8 and the same pattern flipped over, i.e. we have two extra cycles, so the total number of classes will be  $m_5 + 2 = 10 = 1 \pmod{3}$ . As  $n$  is increased we find more and more of these additional symmetric cycles, e.g. for  $n = 21$ , we can see that the total number of classes should be  $m_7 + 24 = 66 = 0 \pmod{3}$  as we find twelve unique extra symmetric patterns, which along with their reverses give the value of 24. According to the sequence these patterns hold for all the numbers calculated (given up to  $n = 19$  in [18]).