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ABSTRACT

Auctions with Private Entry Costs

We study auctions where bidders have private information about their entry costs and the seller does not benefit from these entry costs. We consider a symmetric environment where all bidders have the same value for the object being sold, and also an asymmetric environment where bidders may have different valuations for the object. In these environments, the seller's pay-off as well as the social surplus may either increase or decrease in the number of bidders though not necessarily in the same direction. The auction designs that maximize the social surplus or the seller's pay-off are analysed.

JEL Classification: D44, O31 and O32

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1 Introduction

In auctions with entry costs, each bidder can enter the auction only if he pays an entry cost. The seller, however, does not benefit from the bidders' entry costs. This cost of entering is spent regardless of whether a bidder wins and is independent of his bid. It reflects both an opportunity cost of the time of participating and the cost of the effort needed to learn the rules and prepare a strategy. For large auctions (such as spectrum auctions) it also represents the cost of raising the necessary credit to participate. The usual assumption on entry costs are that they are common knowledge and identical (for example see Samuelson (1985), McAfee and McMillan (1987), Engelbrecht-Wiggans (1993) and Levin and Smith (1994)). We depart from this by assuming that a bidder would have a much better idea about his own opportunity, learning, and fund raising costs than about such costs of his opponent; thus, we assume that the bidders' entry costs are private information.

Another common assumption in the literature on auctions with entry costs is that bidders' decisions on whether or not to enter the auction are made before they learn their private information. This timing assumption and the assumption that bidders are ex-ante symmetric causes the expected profit of each bidder to be zero with all the social surplus going to the seller (as in Levin and Smith (1994)). We depart also from the timing assumption by having a bidder's entry cost known to the bidder before making his entry decision.¹ This complements our first assumption: if the private information about entry costs were discovered after entry decisions were made, then there is no room for this discovery to affect bidding (and obviously entry) since at this time the cost is already sunk. We also depart from the symmetry assumption by investigating when bidders are not necessarily ex-ante symmetric. Together our assumptions allow bidders to earn strictly positive profits and causes the seller's and the social planner's problems to be no longer identical. Instead of the mixed-strategy equilibria found in the literature such as Levin and Smith (1994), our model has cutoff type equilibria, where any bidder with entry cost higher than the equilibrium cutoff will stay out of the auction and any bidder with entry cost lower than the equilibrium cutoff will decide to participate in the auction.

¹Samuelson (1985) considers a model with incomplete information where a player first learns his private value for the object being sold and then decides to enter an English auction. However, in his model all bidders have the same cost of entry which is common knowledge

The questions that we want to address (among others) are: 1) What would happen to the seller's profits (and likewise the social surplus) when the number of bidders considering whether to enter the auction goes up? 2) Does a seller wish to seek a tool to encourage entry into the auction? 3) If one bidder not only values the item more but has lower entry costs than another bidder, would he be more likely to enter.

We find the answer to the first question, by examining three different scenarios. In one the seller's payoff and the social surplus increase in the number of bidders. In another, both decrease in the number of bidders. Surprisingly, in the last scenario, we find the seller's payoff and the social surplus may react in completely opposite ways whereas an increase in the number of bidders yields an increase of the seller's payoff but a decrease of the social surplus.

The second question is answered by our analysis of optimal auctions. Unlike the previous result, we find that given a number of potential bidders, the seller would like to reduce the number of bidders that choose to enter. While reducing the number that enter, the seller collects more from those that decided to enter. Two such methods that do this are to impose a reserve price or alternatively an entry fee. This result is in contrast to the models with common entry costs in which entry fees are useful but reserve price may not be and thus they are not equivalent tools (see, for example, McAfee and McMillan (1987), and Levin and Smith (1994)).

We answer the last question in our section of asymmetric auctions in which bidders may have different valuations for the object. In the type-symmetric equilibrium all the bidders of the same type (valuation) have the same strategy cutoff such that any one of them with a higher cost than the cutoff will stay out of the auction and with a lower cost than the cutoff will enter the auction. The type-asymmetric equilibrium is not necessarily unique, and the auctions are not necessarily efficient: That is, the cutoff of bidders with high valuation may be smaller than the cutoff of bidders with lower valuation. Thus, a bidder with high valuation and low entry cost may choose to stay out of the auction but his opponent with lower valuation and higher entry cost will participate in the auction and may even win the object.

Furthermore, we find results related to the third question: The expected payoff of bidders with low valuations may be larger than the expected payoff of their opponents with higher valuations. In fact, the seller may prefer the inefficient situation, but if the numbers of bidders of each type are identical, the seller prefers efficient auctions.² (The optimal cutoff of bidders

²Gilbert and Klemperer (2000), for example, show that an auctioneer may wish to run an inefficient auction

with the high valuation is always larger than the optimal cutoff of bidders with lower valuation.) Otherwise, if the number of bidders of each type are different, the seller does not necessarily prefer participation of bidders with higher valuation. We show that, independent of the distribution of the bidders' entry costs and the bidders' valuation for the object, the seller always wishes to reduce participation of at least one type of bidders. However, in the asymmetric case usually neither entry fees nor reserve prices are sufficient to implement the optimal cutoffs for the seller, and therefore the seller should find alternative solutions.

We apply the analysis of asymmetric auctions in order to study parallel auctions where each bidder is interested in one object and can participate in one auction only. In equilibrium of parallel auctions, given the choices of entry, no bidder wishes to switch the auction he enters. Usually in parallel auctions with incomplete information an equilibrium is very complex if it exists at all. In our model of parallel auctions where bidders' entry costs are private information, the equilibrium usually exists, and we find it explicitly.

While we talk about entry costs in auctions, our results can be applied to the Bertrand price competition with entry costs. Our results add to this literature, particularly in regards to the first question (number of potential entrants). Lang and Rosenthal (1991) show that if the number of entrants is unknown at the time of bidding but there is symmetry and complete information about values and costs, then the total welfare (which equals the seller's surplus) decreases with the number of potential entrants. Elberfeld and Wolfstetter (1999) show that when there is complete information about the symmetric entry costs and about the number of entrants, but incomplete information about the values, then total welfare decreases in the number of potential entrants.³ Thomas (2002), on the other hand, shows that with complete information about entry costs and about the number of entrants, but with asymmetric entry costs, the total welfare can increase in the number of potential entrants. Our results in this paper show that all the situations described above are possible in one model.⁴

The Bertrand setting can also make use of our optimal design analysis. Here the optimal design for the seller is the same as an optimal design for a regulatory agencies with the con-to attract weaker bidders to enter the auction.

³Elberfeld and Wolfstetter also show that with a small amount of incomplete information about entry costs, the result continues to hold.

⁴Samuelson (1985) finds similar results in a different model with complete information about entry costs, but incomplete information about values.

sumer interest as the objective. This may shed light on which policies may work best in price competitions like Bertrand competition.

The paper is organized as follows: In Section 2, we describe the general environment. Our analysis of the symmetric environment is carried out in Section 3, while the analysis of the asymmetric environment is carried out in Section 4. Finally, we discuss future extensions in Section 5.

2 The Model

Consider an auction with n bidders competing for an indivisible item. Bidder i 's valuation for the item, $v_i \geq 0$, is common knowledge. Participating in the auction generates a fixed cost c_i for bidder i which is private information and is drawn independently from the interval $[0, 1]$ according to the distribution function F .⁵ We assume that F is continuously twice differentiable with $F(0) = 0$ and is common knowledge.⁶ The bidders' entry costs are wasted in the sense that the seller does not benefit from these costs. We assume that each bidder knows his entry cost and his value before he makes his decision. This decision made by bidders can be split into two parts: whether to enter or stay out and what to bid if entering. Denote by $d_i(c_i, v_i)$ the entry decision (the probability of entering) if one has cost c_i and value $v_i > 0$, and $b_i(v_i)$ be the bid if one indeed enters and has value v_i .

We will study both a symmetric and an asymmetric environment. In the symmetric environment, the Revenue Equivalence Theorem (see Myerson (1981) and Riley & Samuelson (1981)) holds whether or not bidders observe how many others have decided to enter before bidding in the auction (and across this condition).⁷ Therefore, for simplicity of analysis, we study second-price auctions (see Vickrey (1961)) with private entry costs: The bidder with the highest bid wins the item and pays the second-highest bid with ties broken randomly. If there is no second-highest bid, then the price of the item is zero.

⁵The choice of the interval $[0, 1]$ is a normalization.

⁶To avoid a trivial solution assume that $F(v) > 0$ (there is a chance that a player has a cost lower than v).

⁷In the Discussion section, we discuss this in more detail about when this would hold. Also when bidders are risk-averse the revenue equivalence theorem does not hold in auctions with or without entry costs. Levin and Smith (1996) show that for risk averse bidders, usually, but not always, the seller prefers the first-price auction to the second price auction when there are entry costs.

3 Symmetric Auctions

We consider first the symmetric case where all the n bidders have the same valuation v for the item being sold. For simplicity, we write $d_i(c_i, v)$ as $d_i(c_i)$.

Proposition 1 *The symmetric equilibrium is given by $b_i(v) = v$ and*

$$d_i(c) = \begin{cases} 1 & \text{if } c_i \leq c^* \\ 0 & \text{if } c_i > c^* \end{cases}$$

where the equilibrium cutoff $c^* > 0$ is the solution of

$$c^* = v(1 - F(c^*))^{n-1} \quad (1)$$

Proof. See Appendix.

The symmetric equilibrium described by Proposition 1 is such that any bidder with an entry cost higher than the equilibrium cutoff c^* will stay out of the auction and any bidder with an entry cost lower than the equilibrium cutoff c^* will participate in the auction.

In the symmetric equilibrium a bidder has a positive payoff only if he is the only entrant. Thus, the payoff of a bidder with entry cost $c \leq c^*$ is $v(1 - F(c^*))^{n-1} - c$ or simply $c^* - c$. Thus, a bidder's expected profits is

$$\int_0^{c^*} (c^* - c)dF(c) \quad (2)$$

It can be verified that the equilibrium cutoff decreases in the number of bidders, and therefore the expected payoff of each bidder decreases in the number of bidders as well.

The seller's expected payoff is actually equal to the total surplus minus the bidders' surplus. The total surplus for the seller and the bidders together must equal the chance that at least one bidder enters times v , minus the expected cost of entry. The chance that at least one enters is $[1 - (1 - F(c^*))^n]$ where c^* is the equilibrium cutoff. The expected entry cost of a bidder is $\int_0^{c^*} cdF$. Hence, we can write the seller's expected payoff as:

$$\begin{aligned} \pi_s &= [1 - (1 - F(c^*))^n]v - n \int_0^{c^*} cdF - n \int_0^{c^*} (c^* - c)dF \\ &= [1 - (1 - F(c^*))^n]v - nc^*F(c^*) \end{aligned} \quad (3)$$

Unlike in Levin and Smith (1994), the seller's payoff is not equivalent to the social surplus. This social surplus π_{ss} is the chance that someone enters and gets the object minus the expected

costs of all the entrants. Given that all bidders use a symmetric cutoff strategy c^* , the social surplus is given by:

$$\pi_{ss} = [1 - (1 - F(c^*))^n]v - n \int_0^{c^*} c dF \quad (4)$$

By looking at the first order condition, we see that a maximum is obtained when $v(1 - F(c^o))^{n-1} = c^o$ which is the same as (1), the condition for a symmetric equilibrium. The intuition behind this is that an individual bidder gains only when no one else has entered which is also the social gains. The amount gained by the bidder is the same as the social amount gained, while the amount that the bidder spends entering is also equal to the social cost.

The following example consists of three cases and shows that an increase in the number of potential bidders has an ambiguous effect both on the seller's expected payoff and on the social surplus.

Example 1 Consider an auction where $v = 1$.

By (1), (3) and (4) the equilibrium cutoff, the seller's payoff and the social surplus respectively are given for three different cases as follows:

Case 1: The bidders' entry costs are distributed according to a uniform distribution on $[0, 1]$.

number of bidders	equilibrium cutoff	seller's payoff	social surplus
2	0.5	0.25	0.5
3	0.381966	0.326238	0.545085
4	0.317672	0.379581	0.581412
5	0.275508	0.420873	0.610635
6	0.245122	0.454453	0.634708
7	0.22191	0.482624	0.654979
8	0.203456	0.506785	0.672362
9	0.188348	0.527854	0.68749
10	0.175699	0.546468	0.700819
1000	0.00524	0.9673	0.9810

In this case, an increase in the number of potential bidders yields an increase of both the seller's payoff and the social surplus.

Case 2: The bidders' entry costs are distributed according to a uniform distribution on $[0.5, 0.75]$.

number of bidders	equilibrium cutoff	seller's payoff	social surplus
2	0.6	0.16	0.2
3	0.5625	0.15625	0.17969
4	0.5457	0.15501	0.17171
5	0.53608	0.15443	0.16745
6	0.52983	0.15412	0.1648
7	0.52543	0.15394	0.16299
8	0.52216	0.15382	0.16168
9	0.51964	0.15374	0.16068
10	0.51764	0.15368	0.1599
1000	0.50017	0.15335	0.15340

In this case, an increase in the number of potential bidders yields a decrease of both the seller's payoff and social surplus.

Case 3: The bidders' entry costs are distributed according to a uniform distribution on $[0.5, 1]$.

number of bidders	equilibrium cutoff	seller's payoff	social surplus
2	0.66667	0.11111	0.16667
3	0.60961	0.12311	0.15915
4	0.58244	0.12947	0.15665
5	0.56626	0.13355	0.1555
6	0.55547	0.13642	0.15488
7	0.54773	0.1385	0.1545
8	0.5419	0.1402	0.15425
9	0.53735	0.14153	0.15408
10	0.5337	0.1426	0.15396
1000	0.5003	0.1533	0.1534

In this case, an increase in the number of bidders yields an increase of the seller's payoff but a decrease of the social surplus. In all three cases, as the number of potential entrants increase,

the seller captures more of the social surplus. \square

Consider now that the seller could influence the equilibrium cutoff. We show that the seller always wishes to decrease the equilibrium cutoff, namely, he wishes to reduce the participation of bidders in the auction.⁸

Proposition 2 *The optimal cutoff c^{op} for the seller is strictly positive and always smaller than the equilibrium cutoff c^* .*

Proof. See Appendix.

In example 1 we showed that the seller’s payoff in equilibrium may either increase or decrease in the number of bidders. The following example shows that the seller’s payoff with the optimal cutoff (seller’s optimal payoff) may also decrease in the number of bidders. (While we will not show it here, the seller’s optimal payoff may also increase in the number of bidders).

Example 2 *Consider an auction where $v = 1$ and F is a uniform distribution function on $[0.5, 0.75]$.*

By (2) and (17) the seller’s optimal cutoff and the seller’s optimal payoff are as follows:

number of bidders	seller’s optimal cutoff	seller’s optimal payoff
2	0.58333	0.16667
3	0.55481	0.15915
4	0.54122	0.15665
5	0.53313	0.15550
6	0.52773	0.15488
7	0.52387	0.15450
8	0.52095	0.15425
9	0.51868	0.15408
10	0.51685	0.15396
1000	0.500173	0.1534

⁸The optimal cutoff is optimal given the limitation of symmetry (all bidders should have identical strategies). Any mechanism that induces behavior according to this optimal cutoff is an optimal mechanism (as shown in the appendix).

Note that the seller's optimal cutoff as well as the seller's optimal payoff decrease in the number of bidders. Thus, the optimal number of bidders for the seller in this example is 2. \square

In the symmetric case, the optimal critical entry cost can be obtained by imposing an entry fee or, alternatively, a reserve price.

When the seller imposes an entry fee e , the symmetric equilibrium is given by $b_i(v) = v$ and

$$d_i(c) = \begin{cases} 1 & \text{if } c_i \leq c^e \\ 0 & \text{if } c_i > c^e \end{cases}$$

where the equilibrium cutoff c^e is the solution of

$$c^e + e = v(1 - F(c^e))^{n-1} \quad (5)$$

This can easily be shown in the same manner as the proof of Proposition 1 (see the appendix).

Now, if we set $e = \frac{F(c^{op})}{F'(c^{op})}$, then the solution of (5) yields the optimal cutoff.⁹

When the seller imposes a reserve price r , the symmetric equilibrium is given by $b_i(v) = v$ and

$$d_i(c) = \begin{cases} 1 & \text{if } c_i \leq c^r \\ 0 & \text{if } c_i > c^r \end{cases}$$

where the equilibrium cutoff c^r is the solution of

$$c^r = (v - r)(1 - F^{n-1}(c^r))^{n-1}$$

Notice that the reserve price r only affects the profit when exactly one bidder enters the auction. Instead of receiving the object for a price of 0, he must pay r . Note that $c^r = c^e$ if $r = e/(1 - F(c^e))^{n-1}$. Therefore, we can obtain the optimal cutoff by setting $r = F(c^{op})/(F'(c^{op})(1 - F(c^{op}))^{n-1})$.

Hence, setting the optimal reserve price or, alternatively, setting the optimal entry fee are equivalent operations in the symmetric setup, both of which yields the optimal entry in the auction. This result is in contrast to the models with common entry costs in which a reserve

⁹>From the proof of Proposition 2 (see the appendix), the optimal entry cost is such that $F(c^{op}) > 0$. In addition $F'(c^{op}) > 0$ since if $F'(c^{op}) = 0$ then $\frac{d\pi_s}{dc^o}(c^{op}) < 0$ which would be a contradiction given our continuity assumptions that implies $\frac{d\pi_s}{dc^o}(c^{op}) = 0$.

price and an entry fee are not equivalent tools. For example, Levin and Smith (1994) showed that in a common value auctions the seller should discourage entry by charging a positive entry fee, but no reserve price.

In the next section we will see that the asymmetric case is more complex and particularly such tools as entry fees and reserve prices are not sufficient to obtain the optimal entry in asymmetric auctions.

4 Asymmetric Auctions

Consider a second-price auction where n_1 bidders have a valuation of v_1 and n_2 bidders have a valuation of v_2 for the item being sold, where $v_1 > v_2$. We say that an equilibrium is type-symmetric if all bidders of the same type (same v) follow the same strategy.

Proposition 3 *A type-symmetric equilibrium exists and satisfies $b_i(v_i) = v_i$ and*

$$d_i(c) = \begin{cases} 1 & \text{if } c_i \leq c_i^* \\ 0 & \text{if } c_i > c_i^* \end{cases}$$

where the equilibrium cutoffs c_i^* , $i = 1, 2$ are given by¹⁰

$$c_1^* = (v_1 - v_2)(1 - F(c_1^*))^{n_1 - 1} + v_2(1 - F(c_2^*))^{n_2}(1 - F(c_1^*))^{n_1 - 1} \quad (6)$$

$$c_2^* = v_2(1 - F(c_1^*))^{n_1}(1 - F(c_2^*))^{n_2 - 1} \quad (7)$$

Proof. See Appendix.

Proposition 3 implies¹¹ that if there are more than one bidder of each type, then in the type-symmetric equilibrium all bidders have some chance of participating in the auction and some chance of staying out. The type-symmetric equilibrium is not necessarily unique as we can see in the following example.

Example 3 *Consider an auction where $n_1 = 2$, $n_2 = 1$, $v_1 = 2.25$, $v_2 = 2$ and F is a uniform distribution on $[0, 1]$.*

¹⁰If $n_1 = 1$ or $n_2 = 1$ the type-symmetric equilibrium can be non interior with $c_1^* \geq 1$, $c_2^* = 0$ or $c_2^* \geq 1$, $0 < c_1^* < 1$. (A cutoff bigger than 1 implies that everyone would enter.)

¹¹If $F'(0) > 0$.

By (6) and (7) the equilibrium interior cutoffs are given by:

$$\begin{aligned} c_1^* &= (2.25 - 2)(1 - c_1^*) + 2(1 - c_2^*)(1 - c_1^*) \\ c_2^* &= 2(1 - c_1^*)^2 \end{aligned}$$

There are two solutions to this system of equations:

1. $c_1^* = 0.34255$ and $c_2^* = 0.8644$
2. $c_1^* = 0.62993$ and $c_2^* = 0.2739$

Note that in the first solution the bidders exhibit paradoxical behavior in the following sense. The equilibrium cutoff of the bidder with the low valuation v_2 is higher than the equilibrium cutoff of the bidders with the higher valuation v_1 . That is, a bidder with a relatively high valuation and low entry cost may decide to stay out of the auction whereas a bidder with a relatively low valuation and high entry cost may decide to participate in the auction.¹² In particular, the expected payoff of the bidder with the low valuation v_2 is larger than the expected payoff of his opponents with the higher valuations v_1 . \square

Due to similar reasons as in the symmetric case, the payoff of a bidder with value v_i and entry cost $c \leq c_i^*$ is $c_i^* - c$. Thus, the expected payoff of a bidder with value v_i is

$$\int_0^{c_i^*} (c_i^* - c) dF(c) \quad (8)$$

Likewise, the seller's expected surplus is the total surplus minus the bidders' surplus. Hence, the seller's expected payoff in the asymmetric case is given by:

$$\pi_s(c_1^*, c_2^*) = (1 - (1 - F(c_1^*))^{n_1})v_1 + (1 - F(c_1^*))^{n_1}(1 - (1 - (F(c_2^*))^{n_2})v_2 - n_2 c_2^* F(c_2^*) - n_1 c_1^* F(c_1^*))$$

We can now use this expression to show that the seller always wish to eliminate participation of at least one type of bidders.

Proposition 4 *For at least one type v_i , the optimal cutoff c_i^{op} is smaller than the equilibrium cutoff c_i^* , that is, either $c_1^{op} < c_1^*$ or $c_2^{op} < c_2^*$.*

Proof. See Appendix.

¹²When $v_1 > 2v_2$, such a paradoxical equilibrium cannot exist. In a similar vein, there always exists an equilibrium where the cutoff c_1 is larger than the cutoff c_2 , but as the example shows it is not always unique.

In example 3 where $n_1 = 2, n_2 = 1$, $v_1 = 2.25, v_2 = 2$ and F is a uniform distribution on $[0, 1]$, it can be shown that the optimal cutoffs are $c_1^{op} = .452, c_2^{op} = .3$. Recall that there were two equilibria: $c_1^* = 0.343$ and $c_2^* = 0.864$; $c_1^* = 0.62993$ and $c_2^* = 0.2739$. This shows that the optimal cutoff c_i^{op} can be either smaller or larger than the equilibrium cutoff c_i^* .

A consequence of Proposition 4 is that the seller may wish to either decrease or increase the equilibrium cutoff of either type of bidders. However, if the number of bidders from each type is identical, the seller will always prefer participation of bidders with the higher type.

Proposition 5 *If $n_1 = n_2$, then the optimal cutoff of the bidders with the high valuation c_1^{op} is always larger than the optimal cutoff c_2^{op} of the bidders with the lower value.*

Proof. See Appendix.

If the number of bidders of each type is not identical, that is, $n_1 \neq n_2$, then the seller does not necessarily prefer participation of bidders with the higher type as we can see in the following example.

Example 4 *Consider an auction where $n_1 = 2, n_2 = 1, v_1 = 1.5, v_2 = 1.4$ and F is a uniform distribution on $[.8, 1]$.*

By (18) and (19) the optimal cutoffs are $c_1^{op} = 0$ and $c_2^{op} = 1$. That is, the seller would prefer to get the bidder with the lower valuation v_2 to always enter while leaving the bidders with the higher valuation v_1 to always stay out. \square

In contrast to symmetric auctions, in the asymmetric auctions the seller's aim to obtain the optimal entry costs (c_1^{op}, c_2^{op}) is not simple. In these auctions using tools such as entry fees or reserve prices are not always sufficient for the seller to reach this goal. Moreover, these tools are not equivalent as they are in the symmetric auctions.

Example 5 *Consider an auction where $n_1 = n_2 = 1, v_1 = 1, v_2 = 0.5$ and F is a uniform distribution on $[0, 1]$.*

The unique equilibrium is $c_1^* = 1, c_2^* = 0$. The optimal cutoffs obtained by the solution of equations (18) and (19) are: $c_1^{op} = \frac{7}{15}, c_2^{op} = \frac{2}{15}$.

Therefore the aim of the seller is decreasing of the equilibrium cutoff of the bidder with the high valuation and increasing of the equilibrium cutoff of the bidder with the low valuation.

If the seller imposes an entry fee e , the equilibrium cutoffs c_i^e $i = 1, 2$ are given by

$$c_1^e + e = (v_1 - v_2) + v_2(1 - F(c_2^e)) = 0.5 + 0.5(1 - F(c_2^e))$$

$$c_2^e + e = v_2(1 - F(c_1^e)) = 0.5(1 - F(c_1^e))$$

The solution of these equations yields $c_1 = 1 - 2/3e$ and $c_2 = -2/3e$. If the seller imposes a different entry fee to each type of bidder, then the solution is

$$\begin{aligned} c_1^* &= 1 - \frac{4}{3}e_1 + \frac{2}{3}e_2 \\ c_2^* &= -\frac{4}{3}e_2 + \frac{2}{3}e_1 \end{aligned}$$

The seller can reach the optimal solution by setting $e_1 = \frac{F(c_1^{op})}{F'(c_1^{op})} = 7/15$ and $e_2 = \frac{F(c_2^{op})}{F'(c_2^{op})} = 2/15$.

Naturally, one can't reach the optimal cutoffs by imposing entry fees were restricted to be the same. Then, the above equations become

$$\begin{aligned} c_1^* &= 1 - \frac{2}{3}e \\ c_2^* &= -\frac{2}{3}e \end{aligned}$$

This means that the seller cannot induce an interior solution with a uniform e . Any positive e will result in $c_2 = 0$ and $c_1 = 1 - e$. Any negative e will result in $c_1 = 1$ and $c_2 = -e$. The negative e would not be profitable since it will only cause additional not useful entry. With a positive e , the seller's profit will be $e(1 - e)$ which reaches its maximum at $e = 1/2$ with a profit of $1/4$.

On the other hand, if the seller imposes a reserve price r (less than v_2), then the equilibrium cutoffs c_i^r $i = 1, 2$ are given by

$$c_1^r = (v_1 - v_2) + (v_2 - r)(1 - F(c_2^r)) = 0.5 + (0.5 - r)(1 - c_2^r)$$

$$c_2^r = (v_2 - r)(1 - F(c_1^r)) = (0.5 - r)(1 - c_1^r)$$

for every $0 < r < 0.5$, by using a reserve price r , the equilibrium cutoff of the bidder with the high valuation decreases and the equilibrium cutoff of the bidder with the low valuation increases.

The solution to the above equations yields

$$\begin{aligned} c_1 &= \frac{3 - 4r^2}{3 + 4r - 4r^2} \\ c_2 &= \frac{2r - 4r^2}{3 + 4r - 4r^2} \end{aligned}$$

and the seller's profit is

$$\pi_s(r) = \frac{4r(3 - 2r^2 - 4r^3)}{(3 - 4(r - 1)r)^2}$$

The maximum profit by using a reservation price is .252366 which although close is still different than the optimal profit .26666. However, it is better than one can do with uniform entry fees which yields a profit of .25.

The social optimal cutoffs can be derived in a similar method to the symmetric case. As with the symmetric case, the first-order conditions are identical to the equilibrium conditions. Naturally, when there are multiple equilibria, the social optimum would be the one that obtains the highest social surplus. In any case, the social optimal cutoffs are different from the seller's optimal cutoffs.

We now turn to some interesting properties of the comparative statics on the values in an asymmetric environment.

Example 6 Consider an auction where $n_1 = n_2 = 1$ and F is a uniform distribution on $[1/2, 1]$.

Take $v_1 = b$ and $v_2 = a$ where $a < b$ and a, b are in $(1/2, 1)$. The only potential interior equilibrium is given by

$$c_1^* = \frac{4a^2 - (a + b)}{4a^2 - 1} \text{ and } c_2^* = \frac{2a(a + b - 1)}{4a^2 - 1}$$

It is indeed interior when $b < (4a^2 - 1)/2a + 1 - a$ which is possible when $a > .78$. It is clear that c_1^* decreases with b and c_2^* increases with b . Furthermore, one can show that c_1^* increases with a and c_2^* decreases with a .

This shows that an increase in a bidder's value may reduce his entry and hurt his surplus while doing the opposite for the other bidder. This may have policy implications in areas such as international trade. If there are two countries each with a local firm competing internationally through Bertrand competition. Each firm has the possibility of entering a market with some sunk cost. Afterwards, each firm has constant marginal cost that for instance is related to the

wage. This shows that if the local country tries to subsidize wages or output on a per-unit basis, it can hurt entry and expected profits for that firm. Ironically, a firm may campaign not to get subsidized.

4.1 Parallel Auctions

Consider now that two identical items are sold through two parallel second-price auctions. Assume that n_1 bidders have a valuation of v_1 and n_2 bidders have a valuation of v_2 for each item being sold, where $v_1 > v_2$. Each bidder is interested in one item and can participate in one auction only.

In equilibrium of parallel auctions, given the choices of entry, no bidder wishes to switch the auction he enters. This is just whether a bidder's expected value of entry is higher in his own auction than if he switches to the other auction. In this context, a type-symmetric equilibrium is where each bidder of the same type (same v) **and** in the same auction follows the same strategy. Denote by n_i^K the number of bidders with valuations $v_i, i = 1, 2$ in auction $K (= A, B)$. Note that $n_1 = n_1^A + n_1^B$ and $n_2 = n_2^A + n_2^B$.

Proposition 6 *Suppose that $n_i^K \geq 1, i = 1, 2, K = A, B$. Then, the type-symmetric equilibrium in auction K is given by $b_i^k(v_i) = v_i$ and*

$$d_i^K(c) = \begin{cases} 1 & \text{if } c_i \leq c_i^k \\ 0 & \text{if } c_i > c_i^k \end{cases}$$

where the equilibrium cutoffs $c_i^k, i = 1, 2, K = A, B$, are given by¹³

$$c_1^A = (v_1 - v_2)(1 - F(c_1^A))^{n_1^A - 1} + v_2(1 - F(c_2^A))^{n_2^A} (1 - F(c_1^A))^{n_1^A - 1} \quad (9)$$

$$c_2^A = v_2(1 - F(c_1^A))^{n_1^A} (1 - F(c_2^A))^{n_2^A - 1} \quad (10)$$

$$c_1^B = (v_1 - v_2)(1 - F(c_1^B))^{n_1^B - 1} + v_2(1 - F(c_2^B))^{n_2^B} (1 - F(c_1^B))^{n_1^B - 1} \quad (11)$$

$$c_2^B = v_2(1 - F(c_1^B))^{n_1^B} (1 - F(c_2^B))^{n_2^B - 1} \quad (12)$$

In addition, there are four entry-incentive constraints.

¹³If $n_1^K = 1$ and/or $n_2^K = 1$ the type-symmetric equilibrium can also be $c_1^K = 1, c_2^K = 0$ or $c_2^K = 1, 0 < c_1^K < 1$.

Bidder with value v_2 : (not switching from B to A, and A to B).

$$v_2(1 - F(c_1^A))^{n_1^A} (1 - F(c_2^A))^{n_2^A} < v_2(1 - F(c_1^B))^{n_1^B} (1 - F(c_2^B))^{n_2^B - 1} = c_2^B \quad (13)$$

$$c_2^A = v_2(1 - F(c_1^A))^{n_1^A} (1 - F(c_2^A))^{n_2^A - 1} > v_2(1 - F(c_1^B))^{n_1^B} (1 - F(c_2^B))^{n_2^B} \quad (14)$$

bidder with value v_1 : (not switching from B to A, and A to B).

$$\begin{aligned} & (v_1 - v_2)(1 - F(c_1^A))^{n_1^A} + v_2(1 - F(c_2^A))^{n_2^A} (1 - F(c_1^A))^{n_1^A} \\ < & (v_1 - v_2)(1 - F(c_1^B))^{n_1^B - 1} + v_2(1 - F(c_2^B))^{n_2^B} (1 - F(c_1^B))^{n_1^B - 1} = c_1^B \end{aligned} \quad (15)$$

$$\begin{aligned} c_1^A &= (v_1 - v_2)(1 - F(c_1^A))^{n_1^A - 1} + v_2(1 - F(c_2^A))^{n_2^A} (1 - F(c_1^A))^{n_1^A - 1} \\ > & (v_1 - v_2)(1 - F(c_1^B))^{n_1^B} + v_2(1 - F(c_2^B))^{n_2^B} (1 - F(c_1^B))^{n_1^B} \end{aligned} \quad (16)$$

Example 7 Consider an auction where $n_1 = n_2 = 2$, $v_1 = 0.5$, $v_2 = 0.25$ and F is a uniform distribution on $[0, 1]$.

The bidders are symmetrically distributed among the auctions such that $n_i^K = 1, i = 1, 2$, $K = A, B$. The type-symmetric equilibrium in auction K is given by $b_i^K(v_i) = v_i$ and

$$d_i^K(c) = \begin{cases} 1 & \text{if } c_i \leq c_i^K \\ 0 & \text{if } c_i > c_i^K \end{cases}$$

where, the equilibrium cutoffs c_i^K $i = 1, 2$, $K = A, B$, are given by (9,10,11,12) as follows

$$\begin{aligned} c_1^A &= c_1^B = 0.25 + 0.25(1 - c_2^A) \\ c_2^A &= c_2^B = 0.25(1 - c_1^A) \end{aligned}$$

The solution to this system of equations is: $c_1^A = c_1^B = \frac{7}{15}$, $c_2^A = c_2^B = \frac{2}{15}$.

Because of the symmetry of both auctions, we actually have only two entry-incentive constraints which are trivially satisfied since a bidder switching from A to B not only faces a potential entrant of the opposite type but another entrant of his own type.

By (13), bidder with value v_2 : (not switching from B to A and similarly from A to B):

$$0.5(1 - \frac{7}{15})(1 - \frac{2}{15}) < 0.5(1 - \frac{7}{15})$$

By (15), bidder with value v_1 : (not switching from B to A and similarly from A to B).

$$\begin{aligned} & 0.25\left(1 - \frac{7}{15}\right) + 0.25\left(1 - \frac{2}{15}\right)\left(1 - \frac{7}{15}\right) \\ & < 0.25 + 0.25\left(1 - \frac{2}{15}\right) \end{aligned}$$

Consequently, no bidder wishes to switch to the other auction and we have an equilibrium in the parallel auctions. Similarly, we can show that no other combination of entry can form an equilibrium.

5 Discussion

In our symmetric model with private entry costs, the Revenue Equivalence Theorem holds whether or not bidders observe how many others have decided to enter before bidding in the auction (see Kaplan and Zamir (2002)). This implies that our results from Section 2 will hold for instance if the auction was first price and bidders were uninformed about who entered the auction before bidding. However, in our asymmetric model there may be a difference among the auctions, in particular, the first-price auction when the bidders are uninformed about who enters may generate lower revenue than the other cases. For example, consider an auction with two bidders with values: $v_1 = 1, v_2 = 2/3$. Costs are either 0 or 2 with equal probability. This simplifies the entry decision to enter only if your costs are 0. In a second-price auction, everyone bids his value irrespective of who else is in the auction and the seller's expected payoff is $1/6$. In a first-price auction, when it is known who enters, the price is 0 unless both enter. In that case, the high bidder wins at a price of $2/3$ and the seller's expected payoff is the same as in the second-price auction. In a first-price auction when it is unknown who enters, both would bid a mixed strategy on $[0, 1/3]$.¹⁴ This equilibrium yields an expected payoff of .0856. This shows us that there is room to study other auction forms in the asymmetric, uninformed case.

Surprisingly, our environments have the advantage that the introduction of asymmetry does not obstruct solvability. While we looked at different values, future work could analyze asymmetric environments where the distributions of costs are asymmetric. It would also be of interest to investigate an environment with two areas of private information: costs and values. This will

¹⁴The high-value bidder will bid with cumulative distribution function $G_h(x) = 2/(2 - 3x) - 1$ and the low-value bidder will bid with cumulative distribution function $G_l(x) = 4/(3 - x) - 1$.

bridge the gap between our work and that of Samuelson (1985) and Elberfeld and Wolfstetter (1999). If values were made public before entry decisions are made, then the subgames would resemble our asymmetric environment.

Intuition may run counter to some of our results. We feel future work is needed to address equilibrium selection in our model and in particular whether and under what conditions some of the equilibria found are stable. To our knowledge this has not thus far been addressed in the auction with entry costs literature.

6 Appendix

6.1 Proof of Proposition 1

The standard result that bidding your value in a second-price auction is the weakly dominant strategy (see Vickrey (1961)) holds in our environment. It can be easily verified that in the symmetric equilibrium where all bidders have a positive probability of entering, this is the unique symmetric bidding strategy.

From the equilibrium bidding strategy, a bidder with cost c that enters earns $v - c$ when he is the only entrant and $-c$ if there is more than one entrant. Thus, the profit of a bidder that decide to enter is $\alpha \cdot v - c$ where α is the probability that this bidder is the only entrant. Given that in the symmetric equilibrium, everyone should be using the same cutoff c^* , the chance of being the only one entering must be the probability that everyone else has a value above c^* , that is, $\alpha = (1 - F(c^*))^{n-1}$. Since the bidder with the cost c^* is indifferent between entering or not, we obtain that the cutoff c^* is determined by (1). Note that for all $c_1 < c^*$, we have $\alpha \cdot v - c_1 > 0$ so $d(c_1) = 1$, and for all $c_2 > c^*$, we have $\alpha \cdot v - c_2 < 0$ so $d(c_2) = 0$. Hence, no bidder has an incentive to deviate from his entry decision.

The solution of c^* exists and unique, since the LHS of (1) is strictly increasing in c^* and trivially equal to zero at $c^* = 0$, and on the other hand, the RHS of (1) is decreasing (and continuous) in c^* and is strictly positive at $c^* = 0$ and equal to zero at $c^* = 1$ (from our assumptions on F). \square

6.2 Proof of Proposition 2

We will start by showing that an auction with an equilibrium with an optimal cutoff is the optimal mechanism for the seller. Since our participation costs are wasted, any bidder that agrees to participate in the mechanism must incur his costs. This eliminates the possibility of a mechanism that queries the bidders about their costs before they are incurred. We also restrict mechanisms to being symmetric in regards to the individual bidders and assume that the equilibrium of such mechanisms are also symmetric. (If this restriction were lifted it may be indeed possible for the optimal mechanism to be asymmetric.)

If in equilibrium of the mechanism m , a bidder with cost c_1 enters and receives expected payoff $m(c_1)$, then a bidder with cost $c_2 < c_1$ would also have to enter in equilibrium since he can always imitate a bidder with cost c_1 . This, as before, leads to cutoff strategies. Since at the cutoff anyone above the cutoff can not pretend to have cost c^o , the expected profits of the cutoff must be zero, $m(c^o) - c^o = 0$. Also, since any bidder with cost below c^o can always receive payoff $m(c^o)$ and the bidder with cutoff c^o can always pretend to have lower costs, the payoff to entering must be $m(c^o)$. Thus, the expected profits of a bidder entering with costs c must be $m(c^o) - c = c^o - c$. Therefore, the expected profits of each bidder is the same as shown for the auction and likewise, for the seller's profits.

The derivative of the seller's profit with respect to c^o yields:

$$\frac{d\pi_s}{dc^o}(c^{op}) = n[v(1 - F(c^{op}))^{n-1} - c^{op}]F'(c^{op}) - nF(c^{op}) \quad (17)$$

Substituting the equilibrium entry cost c^* (1) in (17) yields that

$$\frac{d\pi_s}{dc^o}(c^*) = -nF(c^*) < 0$$

Furthermore, $v(1 - F(c^{op}))^{n-1} - c^{op}$ is decreasing in c^{op} . Also, $-nF(c^{op})$ is decreasing. Therefore, for any $c > c^*$ the $\frac{d\pi_s}{dc^o}(c) < 0$ as well. Thus, the optimal critical entry cost c^{op} is always smaller than the equilibrium critical entry cost c^* .

The optimal critical entry cost is strictly positive (with strictly positive entry $F(c^{op}) > 0$). We can see this since the profits for no entry is zero. Thus, we need to only show that there is a possibility for the seller to make a profit. Our assumption that $F(v) > 0$ and continuity of F imply that there exists a c' such that $v(1 - F(c'))^{n-1} - c' > 0$ and $F(c') > 0$. If the seller set an

additional entry fee $e = v(1 - F(c'))^{n-1} - c'$, all bidders with $c < c'$ will enter. Hence, the seller would make profit of at least $F(c') \cdot e > 0$. \square

6.3 Proof of Proposition 3

The optimal bid function of bidding one's valuation and the existence of a cutoff entry strategy can be shown in the same manner as Proposition 1. Thus, we only need to find the indifference condition to determine the equilibrium cutoffs. Given the equilibrium bid function, a bidder with a low valuation v_2 will profit only when he is in the auction alone. The probability of this is $(1 - F(c_1^*))^{n_1}(1 - F(c_2^*))^{n_2-1}$ which implies equation (7). On the other hand, a bidder with a high valuation v_1 will profit v_2 when he is in the auction alone and will profit the difference $v_1 - v_2$ when he is in the auction with only bidders with valuations of v_2 . These happen with probability $(1 - F(c_2^*))^{n_2}(1 - F(c_1^*))^{n_1-1}$ and $(1 - F(c_1^*))^{n_1-1}$ which implies equation (6). The existence of the equilibrium is derived by Brower's Fixed Point Theorem. The RHS of equations (6) and (7) form a bounded function from $[0, v_1] \times [0, v_2]$ to $[0, v_1] \times [0, v_2]$ that is continuous since F is continuous. Therefore, a fixed point must exist. (Note that if one cutoff c_i^* of the fixed point is above 1, then it would imply that everyone with value v_i enters.)

In the following we show that if $n_1 \geq 2$ and $n_2 \geq 2$, the fixed point belongs to $[0, 1] \times [0, 1]$ and therefore it is our equilibrium.¹⁵ The RHS of equations (6) and (7) are decreasing in c_1^* and c_2^* . If $F(c_1^*) = 0$, then the RHS of (6) is strictly positive for any value of c_2^* – a contradiction. If $F(c_1^*) = 1$, then the RHS of (6) is zero – also a contradiction. Hence, $0 < F(c_1^*) < 1$. A similar argument shows that $0 < F(c_2^*) < 1$ as well. \square

6.4 Proof of Proposition 4

The derivative of the seller's profit with respect to the cutoff of bidders with value v_1 is:

$$\frac{d\pi_s}{dc_1^o} = n_1[(v_1 - v_2)(1 - F(c_1^o))^{n_1-1} + v_2(1 - F(c_1^o))^{n_1-1}(1 - (F(c_2^o))^n - c_1^o)F'(c_1^o) - n_1F(c_1^o)] \quad (18)$$

Substituting the equilibrium cutoff (6) in (18) yields

$$\frac{d\pi_s}{dc_1^o}(c_1^*, c_2^*) = -n_1F(c_1^*) < 0 \quad (19)$$

¹⁵We assume that $F'(0) > 0$.

Since the RHS of (18) is decreasing in both c_1^o and c_2^o , we obtain that for any $c_1 \geq c_1^*$ and $c_2 \geq c_2^*$, the expression $\frac{d\pi_s}{dc_1^o}(c_1, c_2) < 0$.

Likewise,

$$\frac{d\pi_s}{dc_2^o} = n_2(v_2(1 - F(c_1^o))^{n_1}(1 - (F(c_2^o))^{n_2-1} - c_2^o)F'(c_2^o) - n_2F(c_2^o))$$

Substituting the equilibrium entry cost (7) in (19) yields

$$\frac{d\pi_s}{dc_2^o}(c_1^*, c_2^*) = -n_2F(c_2^*) < 0$$

Since the RHS of (19) is decreasing in both c_1^o and c_2^o , then we obtain that for any $c_1 \geq c_1^*$ and $c_2 \geq c_2^*$, the expression $\frac{d\pi_s}{dc_2^o}(c_1, c_2) < 0$. Thus, $\pi_s(c_1, c_2) < \pi_s(c_1^*, c_2^*)$ for all $c_1 \geq c_1^*, c_2 \geq c_2^*$ with at least one strict inequality. This implies that it is better off to decrease at least one of the equilibrium cutoffs c_1^*, c_2^* , and particularly, these equilibrium cutoffs are not optimal. \square

6.5 Proof of Proposition 5

Assume that $c_1^{op} < c_2^{op}$. Let us compare the sellers profits using the optimal cutoffs to the sellers profits reversing the optimal cutoffs, i.e. using c_2^{op} for the cutoff for a bidder with value v_1 and vice versa. The advantage of the optimal cutoffs to this new set of cutoffs is $\pi_s(c_1^{op}, c_2^{op}) - \pi_s(c_2^{op}, c_1^{op})$. Since $n_1 = n_2$, we have $\pi_s(c_1^{op}, c_2^{op}) - \pi_s(c_2^{op}, c_1^{op}) = -(v_1 - v_2)((1 - F(c_1^{op}))^{n_1} - (1 - F(c_2^{op}))^{n_1}) < 0$. This is a contradiction to the optimality of the cutoffs and therefore the optimal cutoffs must satisfy $c_1^{op} > c_2^{op}$. \square

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