Manna from Heaven or Forty Years in the Desert: Optimal Allocation without Transfer Payments

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Abstract

Often an organization, government or entity must allocate goods without collecting payment in return. This may pose a difficult problem when agents receiving those goods have private information in regards to their values or needs or discriminating among agents is not an option. In this paper, we search for an optimal mechanism to allocate goods when the designer is benevolent. While the designer cannot charge agents, he can receive a costly but wasteful signal from them. We show that for a large class of distributions of valuations, ignoring these costly signals by giving agents equal share (or using lotteries if the goods are indivisible) maximizes the social surplus. In other cases, those that send the highest signal should receive the goods; however, we then show that there exist cases where more complicated mechanisms are superior.

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1 Introduction

And Moses said to them, "It is the bread that the LORD has given you to eat. This is what the LORD has commanded: 'Gather of it.... You shall each take an omer [two quarts], according to the number of the persons that each of you has in his tent.'" Exodus 16:15-16

One of the basic problems in economics is how to allocate scare resources or goods. One of the fundamental difficulties afflicting such allocation is private information: knowing who desires the goods the most. While markets work well with such allocation, the market is not always a feasible or desired mechanism for allocation. In case of kidneys it may be unethical to have a market, while in case of sports or concert tickets it may be undesirable to sell the tickets to the highest bidder.[1] Finally, with the allocation of charitable goods, it is not only undesirable to collect payment in return but those needing it the most are also the least able to pay for it.[2] Hence, we often see markets being replaced with other mechanisms. Psuedo-market systems exist where exogenously given points are substituted for money such as with the allocation of both interviews and courses for MBA students (see Brams and Taylor, 1996; Brams and Kilgour, 2001; Sönmez and Ünver, 2005). However, to work, these require more than one type of objects to be allocated (for an alternative use of points) and may be costly to implement. In addition to market and psuedo-market mechanisms, we see the use of non-market mechanisms to allocate goods.

One method used is instead of goods being allocated to the person who is willing to pay the most; they are allocated to who is willing to work the hardest to get them. Sport and concert tickets are given, often using first-come first-serve mechanism, that is, whoever is willing to wait the longest before the promoters start selling, gets the tickets. Allocation of research funds by agencies like National Science Foundation in USA and Economic and Social Science Research Council in UK to various universities and individuals are

[1] See Roth et al. (2004) for a description of the current method used to allocate kidneys and the ethical difficulties of moving to a market-based system. With tickets, there is sometimes a desire for a wider audience. Indeed, the Metropalitan Opera in New York received a several million dollar grant to widen audience by selling prime orchestra tickets for $20 each (10 percent of their usual price).

[2] Che and Gale (2006) provide further examples of non-market allocation caused by wealth constrained agents.

done based on research proposals (where a well-crafted proposal has a higher chance of being funded). A common feature in these examples is that in order to convey their valuation, individuals must incur a socially wasteful cost. As with waiting overnight in a long line, generally this effort is socially wasted.

Another mechanism that is common with charity, but, surprisingly, also prevalent elsewhere, is to allocate evenly or randomly using a lottery (among those appearing identical when classified according to public information). Often baseball playoff tickets are offered via a lottery.[3] Likewise, NCAA College bowl tickets have a lottery amongst only the season ticket holders. Research funds are often handed evenly amongst certain groups or individuals. Allocating goods equally (ex-ante) has the disadvantage of ignoring any private information, but has the advantage of saving the potential recipients' effort.

In this paper, we analyze the optimal mechanism to allocate homogeneous, not-necessarily-divisible goods when the bids made by the players competing for the goods are socially wasted. In our discussion of the optimal mechanism we seek to maximize the social surplus (ex-ante optimality). For a pure common value it is optimal to allocate the good through a lottery and for a pure private (or interdependent value) allocation problem, the optimal mechanism depends on the distribution of the values of the players. We concentrate on this latter case where values are private. Here we find that if a significant part of the bids are wasted then for a wide class of distributions, allocating the good randomly is optimal. At the same time we show cases where other mechanisms can be optimal, namely giving the objects to those who work the hardest (the all-pay auction) or doing so but randomly allocating the objects amongst any that meet a certain threshold of effort (all-pay auctions with a bid cap). The intuition of our results are that using bids increases the probability the good will be allocated to the person who values it the most; however, this naturally also increases the costs due to bids being wasted. The optimal mechanism depends upon this trade-off determined by the distribution of values.

[3] More precisely, the price is set below the market clearing price. Since the demand exceeds supply, a lottery was used to determine who has the right to buy tickets. Among the teams that have used a lottery system was 2006 NY Mets (baseball). Anyone could register by a certain date for an online lottery. Winners were notified by email and allowed to purchase a limited number of tickets. This was the primary distribution system of tickets available to the general public.

There is a vast literature on mechanisms to allocate goods as well as many papers which analyze all-pay auctions and lotteries in different environments than what we study here. Amongst these Moldovanu and Sela (2001) study the best way to split prize money in a contest, and Gavious, Moldovanu and Sela (2002) analyze contests where depending on the nature of the cost function bid caps may be more profitable or not. While Goeree et al. (2005) rank lotteries and all-pay auctions in fund raising mechanisms and Fullerton and McAfee (1999) model research tournaments and show that it is optimal to limit the number of participants to two.

Close in spirit to our paper, Che and Gale (2006) have also motivated non-market mechanisms for allocating goods and services. They find that when agents have wealth constraints in a pure market those that value goods the most do not necessarily receive them, and sometimes a random allocation can be superior. In our paper, the cost of allocating goods competitively is in the wasted effort of signalling one's value. Hence, while those that value the goods the most receive them, sometimes the cost of an efficient ex-post allocation is too high.

In the next section we discuss the allocation problem and convert it into a mechanism design problem in Section 3. In Section 4, we present the results of our analysis. Finally, in Section 5, we make our final remarks and present our conclusions.

2 Allocation Problem

The designer's problem is to allocate M homogeneous, not-necessarily-divisible goods among N agents (bidders) where $M < N$. The designer is benevolent and wishes to maximize the social surplus. Each agent i has a privately known type (signal) $\theta_i \in \mathbb{R}_+$ that is drawn independently from cumulative distribution F. Agent i has value $v(\theta_i) \geq 0$ for at most one object, such that $v'(\theta_i) \geq 0$. (If goods are divisible, the value to agent i is $\min\{q_i, 1\} \cdot v(\theta_i)$ where $q_i \geq 0$ is the fraction of good agent i receives.) Each agent i is able to send a range of costly messages $x_i \in \mathbb{R}_+$ to the designer. (The agents are also able to send costless messages.) The cost to the agent of sending message x_i depends upon his type and equals $c(x_i) \cdot g(\theta_i) \geq 0$. The function $g(\theta_i)$

captures how the agent's type affects the cost of bidding. So if for instance $g(\theta_i) = \theta_i^{-1}$, then the higher the type of the player, the less costly it is for him to make a high bid. Likewise, if $g(\theta_i) = \theta_i$, then the higher the type of the player, the more costly it is for him to make a high bid. When $g(\theta_i)$ depends upon θ_i, the designer is able to see the message x_i, but does not know the agent's cost of sending the message. For instance, if the message is standing in line x_i hours, the designer is able to see how long the agent stands in line, but is unable to determine the (opportunity) cost to the agent. Finally, we assume that $v'(\theta_i)/v(\theta_i) > g'(\theta_i)/g(\theta_i)$ for $\theta_i > 0$.

The designer then receives these costly signals $(x_1, \ldots, x_N)$ and uses them to allocate the M goods by rule $a : \mathbb{R}_+^N \rightarrow [0,1]^N$ where $\sum_i a_i(x_1, \ldots, x_N) \leq M$ guarantees feasibility. (Note that a_i indicates the probability that agent i receives the good when the goods are indivisible and the fraction of the good received.) Denote A as the set of feasible allocation rules. Given allocation rule a, the agents form a Bayes-Nash equilibrium by choosing a strategy $x_i(\theta_i, a)$ to maximize their expected surplus given the strategies of other agents. The designer's problem is to choose rule a to maximize the equilibrium social surplus of the agents given the future Bayes-Nash equilibria of the agents, that is, the designer solves

$$\max_{a \in A} \sum_i E[v(\theta_i) \cdot a_i(x_1(\theta_1, a), \ldots, x_N(\theta_N, a)) - c(x_i(\theta_i, a)) \cdot g(\theta_i)]$$

3 Mechanism Design Problem

For simplicity of analysis we will invoke the revelation principle and look at direct mechanisms where each agent i sends a costless (but not necessarily truthful) message $\widetilde{\theta}_i$. Given the messages $\widetilde{\theta}_i$, the mechanism gives an object to agent i with probability $Pwin_i(\widetilde{\theta}_1, \ldots, \widetilde{\theta}_i, \ldots, \widetilde{\theta}_N)$. (Under divisibility, this will represent the fraction good that agent i receives.) Likewise, the mechanism charges agent i an amount $e_i(\widetilde{\theta}_i)$. Note that this charge depends only on $\widetilde{\theta}_i$. Feasibility requires $\sum_i Pwin_i(\widetilde{\theta}_1, \ldots, \widetilde{\theta}_N) \leq M$ and $Pwin_i(\widetilde{\theta}_1, \ldots, \widetilde{\theta}_N) \geq 0$. Although the agent incurs a cost $e_i(\widetilde{\theta}_i) \cdot g(\theta_i)$, the designer just receives the signal $\widetilde{\theta}_i$, and the cost actually incurred by the agent is wasted. The above formulation allows for a more general formulation of the mechanism. A lottery charges each agent 0 and allocates objects with probability $Pwin_i(\widetilde{\theta}_1, \ldots, \widetilde{\theta}_N) = M/N$. The mechanism is truthful if it is incentive compatible (IC) to report truthfully and individually rational. Once we solve for the optimal direct mechanism, then we can implement the solution by choosing an appropriate allocation rule, that is by setting $a_i(x) = Pwin_i(e_1^{-1}(c(x_1)), \ldots, e_N^{-1}(c(x_N)))$, we have $c(x_i(\theta_i, a)) = e_i(\theta_i)$. Notice that this can be implemented since $e_i(\theta_i)$ does not depend upon θ_j $(j \neq i)$.

By restricting ourselves to symmetric mechanisms, we can denote $Pwin(\widetilde{\theta}_i)$ as the probability of agent i receiving an object with message $\widetilde{\theta}_i$ when everyone else reports truthfully and $e(\widetilde{\theta}_i)$ as the expected cost given that everyone else reports truthfully.[4] For simplicity of notation, we drop the i subscript. Both $Pwin(\widetilde{\theta})$ and $e(\widetilde{\theta})$ are assumed to be increasing in $\widetilde{\theta}$. Now an agent of type θ reporting $\widetilde{\theta}$ (with all others reporting truthfully) has payoff $\pi(\theta, \widetilde{\theta}) \equiv Pwin(\widetilde{\theta})v(\theta) - e(\widetilde{\theta}) \cdot g(\theta)$. The agent solves $\pi(\theta) \equiv \max_{\widetilde{\theta}} \pi(\theta, \widetilde{\theta})$ which in a truthful mechanism should equal $\pi(\theta, \theta)$.

The designer chooses $Pwin(\widetilde{\theta})$ and $e(\widetilde{\theta})$ to maximize $N \cdot E[\pi(\theta)] = N \cdot E[Pwin(\theta) \cdot v(\theta) - e(\theta) \cdot g(\theta)]$ subject to $Pwin(\theta)$ being consistent with feasibility, IC ($\pi(\theta) \geq \pi(\theta, \widetilde{\theta})$) and IR ($\pi(\theta) \geq 0$).

Before getting to our results, we wish to simplify the designer's problem only as a selection of the $Pwin(\theta)$ function subject to feasibility but without incentive constraints. First, satisfying the first-order condition $\pi_{\widetilde{\theta}}(\theta, \theta) = 0$, having $\pi(\underline{\theta}) \geq 0$ and limiting $Pwin(\widetilde{\theta})$ & $e(\widetilde{\theta})$ to be increasing is sufficient to satisfy incentive compatibility and individual rationality, since the single-crossing property is satisfied by our assumption of $v'(\theta)/v(\theta) > g'(\theta)/g(\theta)$. Second, we can also take advantage of the first-order condition and use the envelope theorem to find the agents' surpluses. By doing so, we have

$$\pi'(\theta) = \pi_\theta(\theta, \theta) + \pi_{\widetilde{\theta}}(\theta, \theta) = \pi_\theta(\theta, \theta) = Pwin(\theta)v'(\theta) - e(\theta) \cdot g'(\theta)$$

Therefore

$$\pi(\theta) = \int_{\underline{\theta}}^{\theta} \pi'(\hat{\theta})d\hat{\theta} + \pi(\underline{\theta}) = \int_{\underline{\theta}}^{\theta} \left[Pwin(\hat{\theta})v'(\hat{\theta}) - e(\hat{\theta})g'(\hat{\theta})\right] d\hat{\theta} + \pi(\underline{\theta})$$

As mentioned before, the designer cares only about the total expected utility of the agents subject to

[4] This symmetric assumption is not crucial for our results. In addition it is not optimal to restrict the number of participants as opposed to Fullerton and McAfee (1999)

feasibility (all collected $e(\theta)$ is wasted) and has payoff:

$$
\begin{aligned}
N \cdot E[\pi(\theta)] &= N\int_{\underline{\theta}}^{\overline{\theta}} \pi(\theta)dF + N\pi(\underline{\theta}) = N\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\theta}\left[Pwin(\hat{\theta})v'(\hat{\theta}) - e(\hat{\theta})g'(\hat{\theta})\right]d\hat{\theta}dF + N\pi(\underline{\theta}) = \\
&\quad N\int_{\underline{\theta}}^{\overline{\theta}}\left[\frac{1-F(\theta)}{f(\theta)}\left(Pwin(\theta)v'(\theta) - e(\theta)g'(\theta)\right)\right]d\theta + N\pi(\underline{\theta})
\end{aligned} \tag{1}
$$

(the last part by integration by parts.)

Finally, we can further simplify (1) since $e(\theta)$ is dictated in the first-order condition $\pi_{\widetilde{\theta}}(\theta, \theta) = 0$ by $Pwin(\theta)$:

$$e'(\theta) = Pwin'(\theta)\frac{v(\theta)}{g(\theta)}$$

Hence,

$$e(\theta) = \int_{\underline{\theta}}^{\theta} Pwin'(\hat{\theta})\frac{v(\hat{\theta})}{g(\hat{\theta})}d\hat{\theta} + e(\underline{\theta}) \tag{2}$$

(Note the designer would always want to set $e(\underline{\theta}) = 0$.) The designer's payoff now becomes

$$N\int_{\underline{\theta}}^{\overline{\theta}}\left[\frac{1-F(\theta)}{f(\theta)}\left(Pwin(\theta)v'(\theta) - g'(\theta)\int_{\underline{\theta}}^{\theta} Pwin'(\hat{\theta})\frac{v(\hat{\theta})}{g(\hat{\theta})}d\hat{\theta}\right)\right]d\theta + N\pi(\underline{\theta})$$

Integrating by parts of the second expression yields:

$$
\begin{aligned}
&N\int_{\underline{\theta}}^{\overline{\theta}}\left[\frac{1-F(\theta)}{f(\theta)}Pwin(\theta)v'(\theta)\right]d\theta - N\int_{\underline{\theta}}^{\overline{\theta}}\frac{1-F(\theta)}{f(\theta)}g'(\theta)d\theta\int_{\underline{\theta}}^{\overline{\theta}} Pwin'(\hat{\theta})\frac{v(\hat{\theta})}{g(\hat{\theta})}d\hat{\theta} + \\
&N\int_{\underline{\theta}}^{\overline{\theta}}\left[Pwin'(\hat{\theta})\frac{v(\hat{\theta})}{g(\hat{\theta})}\int_{\underline{\theta}}^{\theta}\frac{1-F(\hat{\theta})}{f(\hat{\theta})}g'(\hat{\theta})d\hat{\theta}\right]d\theta + N\pi(\underline{\theta})
\end{aligned} \tag{3}
$$

The designer wants to maximize this expression by choosing $Pwin(\theta)$ (and implicitly choosing $e(\theta)$ by equation (**?**))) that is increasing and consistent with feasibility. By choosing such a $Pwin(\theta)$, he will also satisfy the IC and IR constraints of the agents.[5]

We recognize that any mechanism that is feasible with increasing $Pwin(\theta)$ can be decomposed into a mechanism $Pwin_i(\widetilde{\theta}_1, \ldots, \widetilde{\theta}_N)$ that is symmetric and satisfies monotonicity: if $\widetilde{\theta}_i > \widetilde{\theta}_i'$ then $Pwin_i(\widetilde{\theta}_1, \ldots, \widetilde{\theta}_i, \ldots, \widetilde{\theta}_N) \geq Pwin_i(\widetilde{\theta}_1, \ldots, \widetilde{\theta}_i', \ldots, \widetilde{\theta}_N)$. Likewise, any mechanism that satisfies monotonicity and symmetry has an increasing $Pwin(\theta)$. Henceforth, we look at mechanisms that satisfy this monotonicity condition.

[5] We assume that the designer is able to commit to the mechanism.

4 Results

We now use the simplified designer's problem to derive our first results about the optimal mechanism.

Proposition 1 *If* $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$ *is decreasing in* θ *and* $g'(\theta) \geq 0$*, then a lottery is socially optimal. If* $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$ *is increasing in* θ *and* $g'(\theta) = 0$*, then an all-pay auction is socially optimal.*

Proof. Let us first prove the first part. Note that if $g'(\theta) \geq 0$, then $\int_{\underline{\theta}}^{\theta} \frac{1-F(\hat{\theta})}{f(\hat{\theta})} g'(\hat{\theta}) d\hat{\theta}$ is increasing w.r.t. θ. This then implies that

$$N \int_{\underline{\theta}}^{\overline{\theta}} \left[Pwin'(\theta) \frac{v(\theta)}{g(\theta)} \int_{\underline{\theta}}^{\theta} \frac{1-F(\hat{\theta})}{f(\hat{\theta})} g'(\hat{\theta}) d\hat{\theta} \right] d\theta \leq N \int_{\underline{\theta}}^{\overline{\theta}} \frac{1-F(\theta)}{f(\theta)} g'(\theta) d\theta \int_{\underline{\theta}}^{\overline{\theta}} Pwin'(\hat{\theta}) \frac{v(\hat{\theta})}{g(\hat{\theta})} d\hat{\theta}$$

Examining the social surplus in (3), the second + third expression is negative and hence maximized (at zero) when $Pwin'(\theta) = 0$. Now notice that the first expression of (3) is maximized when $Pwin'(\theta) = 0$, since $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$ is decreasing and it is best to allocate as many objects as possible to the lower valuations. Finally, notice that again the last expression of (3), $N\pi(\underline{\theta})$, is maximized when $Pwin'(\theta) = 0$ (a lottery gives the highest possible surplus to the lowest type.) The second part of the proposition is easily shown since the second and third expressions of (3) vanish when $g'(\theta) = 0$, the remaining is maximized by, whenever possible giving the object to the highest-value player. ■

Note that the conditions given by the above proposition, are sufficiency conditions for lotteries to be an optimal mechanism to allocate goods.[6] All-pay auctions or other combined mechanisms can be a better mechanism to allocate goods when one or both of these conditions doesn't hold. However, when these do not hold, other mechanisms can also be optimal. We provide two stylized examples here for exposition of the above proposition:

Example 1 *F is uniform on* $[0, 1]$*,* $N = 2$*,* $M = 1, v(\theta) = \theta$*, and* $g(\theta) = 1$.

[6] Without monotonicity, a lottery is not a unique optimal mechanism. For instance, when $N = 2$, $M = 1$ and F is uniform on $[0, 1]$, one can ask players to send a costly signal s_i of their types. If $s_i > s_j$ then only if $s_i - s_j < 0.5$, the mechanism would allocate it to i.If players truth tell, the expected probability of winning will be the same. Monotonicity is violated since if $s_2 = 0.8$, player 1 wins with a signal of $s_1 = 0.1$, but loses with a signal of $s_1 = 0.4$.

In this case $\frac{1-F(\theta)}{f(\theta)}v'(\theta) = 1-\theta$ decreases on $[0,1]$. Here the social planner does better by running a lottery $\int_0^1 (1-\theta)\, d\theta = \frac{1}{2}$ than the all-pay auction $2\int_0^1 (1-\theta)\theta d\theta = \frac{1}{3}$ (or any other mechanism). □

Example 2 *F is uniform on* $[0,1]$, $N=2$, $M=1$, $v'(\theta) = \frac{1}{(1-\theta)^{1.5}}$, *and* $g(\theta) = 1$.

Notice that $\frac{1-F(\theta)}{f(\theta)}v'(\theta) = \frac{1}{(1-\theta)^{0.5}}$ is increasing on $[0,1]$. Surplus from a pure all-pay auction is $2\int_0^1 \frac{1}{(1-\theta)^{.5}}\theta d\theta = 2\frac{2}{3}$ and is more than that from a lottery $\int_0^1 \frac{1}{(1-\theta)^{.5}} d\theta = 2$. Here, the all-pay auction does best. □

Note that in the examples for simplicity $g(\theta)$ is taken as constant, i.e., the cost of bidding is just the bid $e(\theta)$. As in the proposition, if $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$ is increasing, then the social surplus may increase if the good gets allocated to the player who values the good more (as the case when $g'(\theta) = 0$). In reverse, if $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$, falls and $g'(\theta) \geq 0$ then surplus decreases as the good is allocated to the higher valued player. In this case the bids made by the higher valued players are too costly for the society to waste and therefore it is better to run a lottery or allocate the good randomly. If the above conditions are not met then we get the following result.

Proposition 2 *If* $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$ *is non decreasing in* θ *or* $g'(\theta)$ *is strictly decreasing, then another more complicated mechanism such as an all-pay auction with a bid cap can be optimal.*

We illustrate the variety of possible mechanisms by means of examples. In example 3, we show that an all-pay auction with bid caps can be optimal, while in example 4 we indicate that a lottery, when all values are low, followed by an all-pay auction for higher values is optimal. Finally, in example 5, we show that an all-pay auction followed by a lottery and then again followed by an all-pay auction is the optimal mechanism. Note that in all the examples the main intuition for the use of a particular mechanism, is if the virtual surplus increases or falls. If the virtual surplus increases then it means that efficiency increases with a non-cooperative bidding mechanism while if the surplus falls a random allocation is better.

Example 3 *F is uniform on* $[0,1]$, $N=2$, $M=1, v(\theta) = \frac{\theta^2}{2}$ *and* $g(\theta) = 1$.

Notice here that $\frac{1-F(\theta)}{f(\theta)}v'(\theta) = (1-\theta)\,\theta$ increases and then decreases in θ. Consider the following allocation where θ_1 and θ_2 denote the types of the players. If $\theta_1, \theta_2 \geq \theta^*$, then the good is allocated randomly, otherwise whichever θ is higher gets the good. Such an allocation results from running an all-pay auction with an appropriate bid cap. Under such an allocation the social surplus is $N\int_0^{\theta^*}\left[\frac{1-F(\theta)}{f(\theta)}\theta Pwin(\theta)\right]d\theta + N\int_{\theta^*}^1\left[\frac{1-F(\theta)}{f(\theta)}\theta\overline{P}\right]d\theta$. A bid cap allows us to implement the above allocation by choosing θ^*, the probability of winning with $\theta \geq \theta^*$ is $(1+\theta^*)/2$. Surplus is then $2\int_0^{\theta^*}(1-\theta)2\theta^2 d\theta + 2\int_{\theta^*}^1(1-\theta)(1+\theta^*)\theta d\theta = \frac{1}{3}\theta^* - \theta^{*2} + \theta^{*3} - \frac{1}{3}\theta^{*4} + \frac{1}{3}$ which achieves its maximum of 0.36849 at $\theta^* = 1/4$.[7] This is an optimal mechanism in order to maximize the designer's surplus under the limitation that any incentive-compatible mechanism must have an increasing probability of receiving the object.

Examine the thin line in Figure 1. This represents the virtual surplus of giving the object to an agent of type θ. For all θ, it is also worthwhile to give the object than not to give the object. Notice that for points to the right of the graph, such as $\theta = 0.9$ and $\theta = 0.8$, one would prefer to give the object to the player with lower θ. However, with the restriction in probability, the designer can at most keep the probability of receiving the object the same (holding a lottery). While the surplus reaches the peak at $\theta = 0.5$, we would still want to hold a lottery between someone with someone with $\theta = 0.5$ and someone with $\theta = 0.4$, since under the increasing probability restriction (monotonicity), we must choose between either holding a lottery amongst someone with $\theta = 0.4$ and all those with $\theta \geq 0.5$ or always giving the object to all those with $\theta \geq 0.5$ over someone with $\theta = 0.4$. This is necessary to be consistent with monotonicity. For instance, if we choose someone with $\theta = 0.6$ over someone with $\theta = 0.4$ while choosing someone with $\theta = 0.4$ over someone with $\theta = 0.7$, then $Pwin_1(0.6, 0.4) > Pwin_1(0.7, 0.4)$. This leads us to the thick line in the graph. This line represents the average virtual surplus of all $\theta' \geq \theta$. The optimal mechanism will weigh this average against the virtual surplus of θ. When the surplus of θ is higher, then θ will be added to the lottery. When the average above θ is higher, then higher θ will be preferred as in an all-pay auction. □

[7] The equivalent bid cap of $\overline{x}$ sets $c(\overline{x}) = 0.00716$.

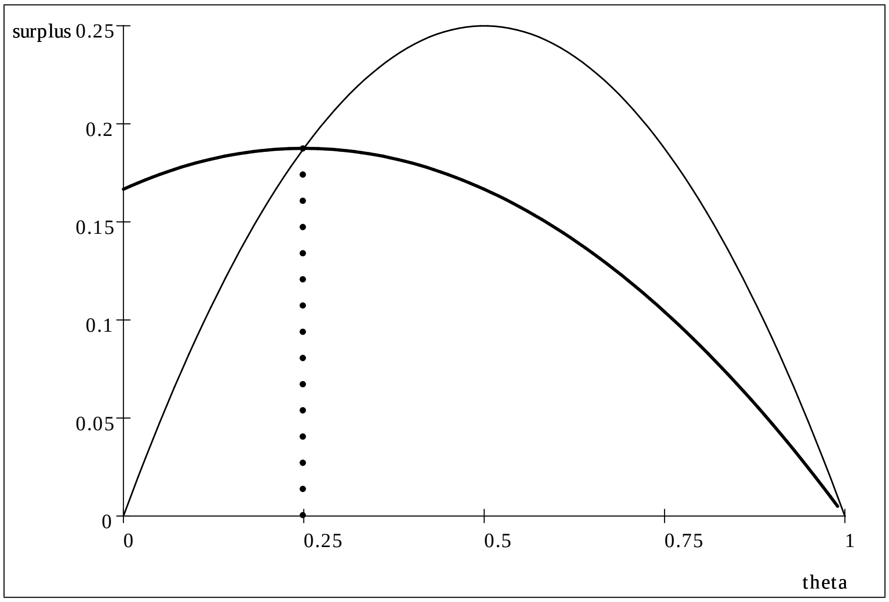

Figure 1: In example 3, the winner is by highest signal and after a threshold $\theta^* = 1/4$, by lottery. Thin line is virtual surplus and thick line is average virtual surplus above θ

Example 4 *F is uniform on $[0,1]$, $N = 2$, $M = 1$, $v(\theta) = \frac{1}{2\sqrt{(1-\theta)}} + \theta$ and $g(\theta) = 1$.*

Here and $\frac{1-F(\theta)}{f(\theta)} v'(\theta) = (1-\theta)\left(\frac{0.25}{(1-\theta)^{1.5}} + 1\right)$ first decreases till $\theta = 0.75$, and then increases till $\theta = 1$. Consider the following allocation where θ_1 and θ_2 denote the types of the players. If $\theta_1, \theta_2 \leq \theta^*$, then the good is allocated randomly, otherwise whichever θ is higher gets the good. Such an allocation results from running an all-pay auction with a minimum bid and allocating the good randomly if no one meets the minimum bid. From this, the social surplus is $\int_0^{\theta^*} \left[(1-\theta)\left(\frac{0.25}{(1-\theta)^{1.5}} + 1\right)\theta^*\right] d\theta + N \int_{\theta^*}^{1} \left[(1-\theta)\left(\frac{0.25}{(1-\theta)^{1.5}} + 1\right)\theta\right] d\theta$. We will now see that this is the optimal mechanism under the probability limitation.

In Figure 2, as before, the thin line in the graph represents the virtual surplus of giving the object to an agent of type θ. Again, for all θ, it is also worthwhile to give the object than not to give the object. Notice that now for points to the left of the graph, such as $\theta = 0.2$ and $\theta = 0.1$, a designer prefers to give the object to the player with lower θ. Hence, under the probability restriction, the designer would choose a lottery for those points. While the surplus reaches the minimum at $\theta = 0.75$, we would still want to hold a lottery beyond this point, for example between someone with $\theta = 0.75$ and $\theta = 0.8$. This is for similar reasons to those in example 3. Namely, since under the monotonicity restriction, we need to make the choice between holding a lottery amongst someone with $\theta = 0.8$ and all those with $\theta \leq 0.75$ or always giving the object to someone with $\theta = 0.8$ over all those with $\theta \leq 0.75$. Otherwise, monotonocity is broken. For instance, if we choose someone with $\theta = 0.2$ over someone with $\theta = 0.8$ while choosing someone with $\theta = 0.8$ over someone with $\theta = 0.75$, then $Pwin_1(0.2, 0.8) > Pwin_1(0.75, 0.8)$.

This leads us to the thick line in the graph that represents the average virtual surplus of all $\theta' \leq \theta$. The optimal mechanism will weigh this average against the virtual surplus of θ. When the surplus of θ is higher, then θ will be preferred. When the average below θ is higher, then θ will be added to the lottery. □

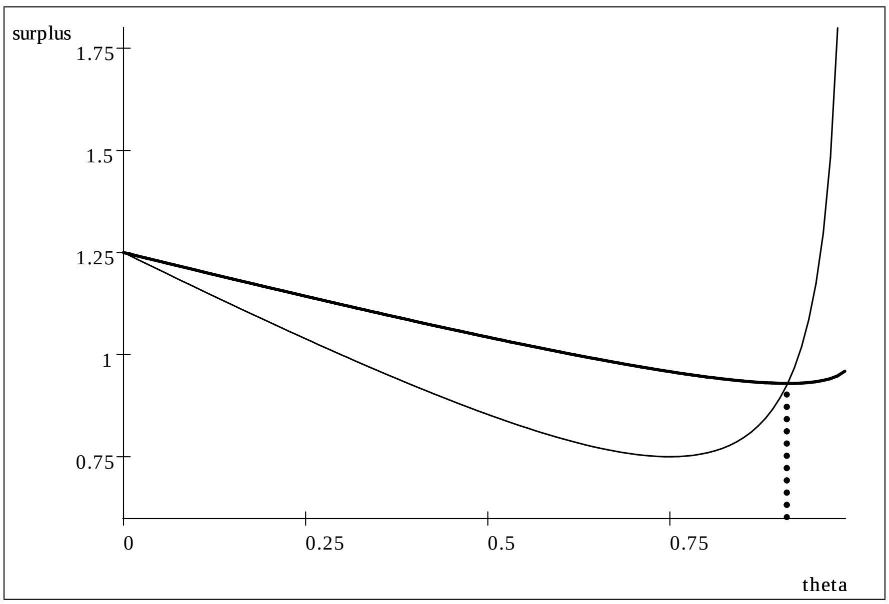

Figure 2: In Example 4, the winner is by lottery, and then by highest signal. Thin line is virtual surplus and thick line is average virtual surplus above θ.

Example 5 *F is uniform on $[0, 1]$, $N = 2$, $M = 1$, $v(\theta) = \frac{1}{2(1-\theta)^{0.5}} + \frac{3}{2}\theta^2$ and $g(\theta) = 1$.*

Here $\frac{1-F(\theta)}{f(\theta)} v'(\theta) = (1 - \theta) \left(\frac{0.25}{(1-\theta)^{1.5}} + 3\theta \right)$. This first increases, then decreases, and then again increases till $\theta = 1$. In this case, the following mechanism is optimal, where the social planner first runs an all-pay auction then a lottery and then runs an all-pay auction for the high value players. This yields the following allocation: θ_1 and θ_2 denote the types of the players. If θ_1, θ_2 are in $[0.45, 0.91]$, then the good is allocated randomly. Otherwise, it is allocated to the one with the highest θ. Note that from $\theta = 0.45$ to $\theta = 0.91$ the social planner will run a lottery and in the rest of the range an all pay auction will be used. Therefore, the social surplus is

$$2\int_0^{0.45}\left[(1-\theta)\left(\frac{0.25}{(1-\theta)^{1.5}}+3\theta\right)\theta\right]d\theta+\int_{0.45}^{0.91}\left[(1-\theta)\left(\frac{0.25}{(1-\theta)^{1.5}}+3\theta\right)(0.91-0.45)\right]d\theta$$
$$+2\int_{0.91}^{1}\left[(1-\theta)\left(\frac{0.25}{(1-\theta)^{1.5}}+3\theta\right)\theta\right]d\theta.$$

This is a combination of our two previous examples with the lottery range in the middle. Denote the lottery range from θ_a to θ_b. We must compare the average virtual surplus of those in the range to those out of the range. We would prefer a θ in $[\theta_a,\theta_b]$ to those below θ_a if the average surplus is higher than the surplus of all those below and prefer those above θ_b if the average surplus is lower than the surplus of all these values. Since the virtual surplus is increasing (and continuous) outside of this range, this can only happen if the average virtual surplus is equal to the virtual surplus on both ends: $\int_{\theta_a}^{\theta_b} s(\theta)d\theta = s(\theta_a) = s(\theta_b)$. We see this in Figure 3. Again, the thin line is the virtual surplus. Here, the thick line helps demonstrate the range of types where a lottery should be used. With this line, both endpoints have the same virtual surplus. This virtual surplus should also equal the average virtual surplus in the range of the thick line. In order for this to happen, the area above it and below the thin line and below it and above the thin line should cancel (be equal).

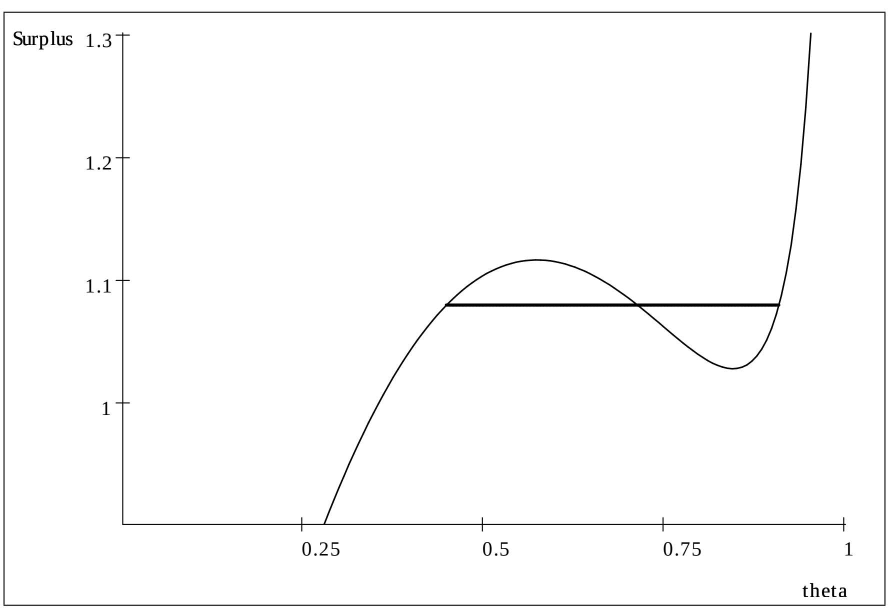

Figure 3: In Example 5, the winner is by highest signal except for the interval $\theta \in [0.45, 0.91]$ Thin line is virtual surplus and thick line is the interval $[0.45, 0.91]$. The area below the thin line and above the thick

line is equal to the area above the thin line and below the thick line.

Note that with all the above examples $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$ is non-monotonic and since $g(\theta)$ is constant, we can design the allocation mechanism with the help of the function $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$. We know that we can rank all-pay auctions and lotteries according to their social surplus by analyzing the slope of $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$. In case of an upward sloping $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$, the gains from allocating the good to higher valued players increase and therefore all-pay auctions are optimal for those regions of the distribution of θ, and if $\frac{1-F(\theta)}{f(\theta)}v'(\theta)$ falls then waste of the bids are more than the gains from allocating it to the high value players, therefore lottery is optimal for those regions of θ.□

We observe in numerous instances where goods are allocated by one of the more complicated methods of examples 3 to 5, that is, a method beyond a straight lottery or contest. For instance, the way example 3 could work in practice, is to allocate tickets for an event by having a lottery for anyone that waits x hours for tickets and if there are tickets left after that allocate the tickets through lottery. Another illustration of this is the ticket distribution by All England Tennis and Croquet Club for the Wimbledon tennis tournament. The club first holds a lottery to allocate the tickets almost six months before the Wimbledon tournament and then gives them away in first come serve or person willing to stand longest in the queue. (We presume that buying tickets six-months prior takes more effort.) We see a system like example 4 with the distribution of entries in the New York marathon.[8] Those that put in greater effort can qualify automatically (by completing a number of sanctioned races or making a qualifying time), remaining entries are distributed by a lottery system. Finally, example 5, we see where it is optimal to run an all-pay auction first, and then allocate the good randomly and for the higher values again run an all-pay auction. One possible example of this is admissions to top US universities among students with high test scores. Writing an essay is part of the application. As most lecturers know, most essays are indistinguishable in level. A few good ones stand out as well as a few bad ones. It is possible that an admissions officer would first admit the good

[8] Some may be surprised to discover that the right to run in the major marathons needs to be rationed. There are logistic limits to supply and demand for 26 miles of punishment is high. We also note that transfer of numbered bibs is currently prohibited on both medical and fairness grounds (see Blecher, 2006 for details).

essays and then randomly select among the middle pile. If there are still slots left, the officer may start to offer admissions to the top of the lower group. Similarly, there are more students that apply to Oxford or Cambridge Universities with the highest score on the admissions tests (three As in the A-level exams) than places. To select students, interviews are held. We can interpret the interviews (where preparation can help) as the socially wasteful but necessary to signal the type of the students.

5 Concluding remarks

This paper makes a contribution in the allocation of goods when signalling one's desire for the good is a wasteful activity. Under such conditions, there is a trade-off between efficient allocation and wasted resources. A mechanism such as an all-pay auction, which allocates by the highest signal, will allocate goods to the people who value them the most but the act of signalling will be wasteful. Allocating an equal share to everyone (or a random allocation by a lottery) saves any waste from signalling, but leads to an ex-post inefficient allocation. Here, we have analyzed when the waste of signalling will exceed the benefit of efficiency in an independent private value environment. In different environments such as common values or where a significant part of the valuation is common, the optimal mechanism will further favour lotteries. Changing the environment by relaxing our key assumption of wasteful signalling such as when there isn't complete waste of the signals, will favour the all-pay auctions. However, as long as some of the valuation is private and some of the bids are wasted, there is a possibility that an even allocation or lottery will be optimal. We can also partially relax the assumption on payments to the designer. The necessary element for our analysis to apply is that there is a maximum price that the designer can charge and at this price, there is an excess demand (as the case with playoff tickets). The timing of the signals in our mechanism can also be changed while keeping the same nature of our results. For instance, a war of attrition can be used to allocate goods in place of an all-pay (first-price) auction. A war of attrition with a time limit can be used in lieu of an all-pay auction with a bid cap.

This analysis has many applications. We have already referred to allocation of concert and sport tickets, and distribution of research grants. Standing in line or filling in paper work for grant application can be socially wasteful. Contests are also used to grant the Olympic games, where the individual cities submit bids, and part of the bids are often socially wasted. In the UK, governmental research funds are distributed through two main channels: research councils or quality-related (QR) funding. Research councils allocate funds to institutions by gathering private signals through research grant applications, which are costly to make. QR funding allocates funds through publicly available information such as publications, which presumably is less costly to gather (this is done through the Research Assessment Exercise, RAE). Our analysis can help design an appropriate mechanism. If the cost incurred by the institutions to make the case for grants is too high, the government should favour QR funding. Policy research into which system is best is an important area in which our paper contributes.

Our results also has implications for bidding rings (cartels) in auctions. In this literature, McAfee and McMillan (1992) find an optimal mechanism for collusion that agrees with our results. Namely, if the hazard rate is decreasing, bidders should participate in the auction non-cooperatively; however, if the hazard rate is increasing, then bidders should bid the reserve price whenever they value the object more than the reserve. In this application, the cartel's objective is congruent to that of our designer while the bids are analogous to our wasteful signal. Hence, our results indicate that the McAfee and McMillan results apply more generally.[9] Moreover, a connection would show that in may cases the optimal collusive policy would be something more complicated such as an increasing bid function that reaches a peak or bidding the reserve price for low values and then jump to bidding higher values.

While in this paper, we examined only the case where each agent has one of two possible allocations: with an object or without an object, we can apply this in any case with two possible outcomes. Think of the case where a course is offered twice and students have to decide which time they want to be scheduled for (with all students being assigned to one of the two slots). If there is an excess demand for one of the time slots, then one can use our analysis to determine how to allocate the slots to the students demanding the popular time slot. (All students demanding the less popular slot will get it.)

[9] There is still the discrepancy of the all-pay nature of our signals vs. the first-price payments in an auction

A natural extension for our paper would be to consider several types of goods. Doing so would make it possible to explore a link to papers on pseudo-markets (markets using points rather than money), except we will optimize the method for obtaining points as a function of effort (better grades yield more points in course markets). An exogenous allocation of points is similar to a lottery while points solely as a function of effort is like an all-pay auction.

As a concluding remark, we will further explain the title of the paper. When God sends manna (food) from heaven, it floats to earth and is evenly distributed (two quarts per day per person). This refers to the lottery mechanism of our paper, where the expected amount per agent is also even. The second mechanism from the bible can be interpreted as God wanting only those worthy entering the holyland. To determine who is worthy, God has the Jews wander the desert for forty years. This was a weeding out process. Only those families that where willing to put forth the costly signal of wandering the desert were permitted to enter the Holy Land. This refers to the all-pay auction mechanism of our paper. We will leave it to the theologists to why omniscient God didn't make use of the agents types and just decide. For this, we presume that God wished to use a non-discriminatory mechanism.

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