Energy-Crossing Number Relations for Braided Magnetic Fields

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This Letter derives lower bounds on the energy of braided magnetic fields, based on crossing number techniques pioneered by Freedman and He. Mean crossing numbers are defined for magnetic fields inside a cylinder; for configurations with a uniform axial component the square of the crossing number provides a lower bound for the free magnetic energy. An energy-crossing number relation is also derived for a random field with Gaussian amplitude distribution, in the limit of small correlation lengths. These results provide information on the structure of solar coronal loops and on the problem of coronal heating.

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The lines of force within a magnetic field can be twisted, knotted, or entangled. Recently, Freedman and He [1] employed knot theory techniques to find lower bounds on the energy of topologically complex magnetic fields. These lower bounds can be used to constrain the equilibrium states accessible to a magnetized fluid with a given field topology.

Magnetic helicity conservation has often been invoked as a constraint on the final state of a resistive plasma undergoing relaxation [2]. The helicity integral [3, 4] measures the total pairwise linking of all the field lines in the volume of integration, but does not take into account the structure of individual lines. Under a wide variety of boundary conditions the minimum energy states for a given helicity are linear ($\nabla \times \mathbf{B} = \lambda \mathbf{B}$, with λ constant)[5–7].

Suppose we wish to include more or different information on the topology of the field than total magnetic helicity. If we take into account the topology of individual field lines, then the magnetohydrodynamics equilibrium equations become nonlinear and often extremely difficult to solve [2, 8]. To make matters worse, for many topologies solutions must necessarily contain singular sheets where the electric current density is divergent [9–11]. Numerical modeling of braided magnetic fields extending between two parallel plates shows current densities increasing exponentially with topological complexity [12]. Here the Freedman and He techniques become important, as these do not involve solving nonlinear partial differential equations. Instead, they define a quantity called the "asymptotic crossing number" which provides a precise measure of field-line entanglement. This quantity is not a topological invariant, but like the magnetic energy it has a positive minimum value for a given magnetic topology. The minimum asymptotic crossing number, times a constant coefficient, provides a lower bound for the equilibrium energy.

This Letter presents crossing number-energy relations for *braided* magnetic fields. Such fields are strongly aligned in one direction but possess significant transverse structure. The derivation for braided fields is simpler than for knotted fields, and minimum crossing numbers are easier to calculate [13, 14].

Fluid dynamical techniques can assist in the study of knots and links: For example, the minimum energy of a knotted magnetic flux tube provides a potentially powerful knot invariant [15]. Conversely, topological techniques have led to new and interesting questions about the behavior of fluids [16]. The criticism is sometimes raised, however, that traditional fluid problems do not depend on topological insight for their solution. One traditional problem in astrophysical fluid dynamics concerns the source of heat for solar and stellar coronas. Parker [17, 18] has suggested that numerous small flaring events convert magnetic energy into heat. The "microflares" are triggered when randomly generated magnetic fields in a corona become unstable to reconnection. For this problem we need to understand the structure, or at least the energy content, of highly tangled magnetic fields. As discussed at the end of this Letter, energy-crossing number relations can make an essential contribution to this understanding.

First consider two field lines stretching between parallel plates at $z = 0$ and $z = L$. Let ϕ be the polar angle in the x-y plane. Now observe the curves from the viewing angle $\phi = \pi/2$ (equivalently, project them onto the x-z plane). The two curves will exhibit a certain number of crossovers, $c(\pi/2)$. Different viewing angles ϕ yield different crossing numbers $c(\phi)$ (see Fig. 1). However,

$$\bar{c} = \frac{1}{\pi}\int_0^{\pi} c(\phi)d\phi \tag{1}$$

is independent of viewing angle.

The crossing number can be computed directly from the form of the curves. Let the two field lines follow the curves $\mathbf{x}_1(z)$ and $\mathbf{x}_2(z)$, where $\mathbf{x}_1(z) = (x_1, y_1)$. The displacement vector $\mathbf{r}_{12}(z) = \mathbf{x}_2(z) - \mathbf{x}_1(z)$ makes an angle $\theta_{12}(z)$ with respect to the x axis. Now, an observer viewing the curves from the angle ϕ will see crossovers wherever $\theta_{12}(z) = \phi$ or $\phi + \pi$. If $\mathbf{r}_{12}(z)$ sweeps out a net angle $\Delta\theta_{12} = \int |d\theta_{12}/dz|dz$, then a proportion $\Delta\theta_{12}/\pi$ of observers will see a crossover. Thus

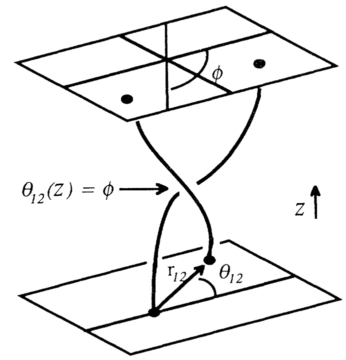

FIG. 1. From the viewing (projection) angle ϕ, two curves will be seen to cross wherever the displacement vector $\mathbf{r}_{12}(z) = \phi$ or $\phi + \pi$.

$$\bar{c} = \frac{1}{\pi}\int_0^L \left|\frac{d\theta_{12}}{dz}\right| dz. \tag{2}$$

Consider a magnetic field confined inside the cylinder $x^2 + y^2 \le R^2$. We sum $\bar{c}$ over all pairs of lines (counting each pair once and weighting by flux) to obtain the crossing number for the magnetic field:

$$C \equiv \frac{1}{2\pi}\int_0^L \int\int B_{z1}B_{z2}\left|\frac{d\theta_{12}}{dz}\right| d^2x_1\, d^2x_2\, dz. \tag{3}$$

Let

$$\mathbf{b}_1 \equiv d\mathbf{x}_1(z)/dz = B_\perp(\mathbf{x}_1)/B_z(\mathbf{x}_1). \tag{4}$$

Then

$$\frac{d\theta_{12}}{dz} = \frac{1}{r_{12}^2}(\mathbf{b}_2 - \mathbf{b}_1)\cdot\hat{\boldsymbol{\theta}}_{12}, \tag{5}$$

where $\hat{\boldsymbol{\theta}}_{12} = \hat{\mathbf{z}} \times \hat{\mathbf{r}}_{12}$. Equation (3) for C can be written as the integral of dC/dz, with

$$\frac{dC}{dz} = \int\int \frac{B_{z1}B_{z2}}{2\pi r_{12}}|(\mathbf{b}_2 - \mathbf{b}_1)\cdot\hat{\boldsymbol{\theta}}_{12}|d^2x_1\, d^2x_2. \tag{6}$$

Without the absolute value signs, Eq. (6) would give the z derivative of open (relative) helicity [19]. Freedman and He [1] obtained analogous expressions for crossing numbers averaged over all possible projection angles in three dimensions (i.e., averaged over S_2 rather than S_1).

This Letter concerns fields with a strong axial component B_z. For such fields the topology manifests itself in the structure of the transverse field rather than in the relatively small variations in B_z [20]. We will let $B_z = \text{const}$ so that

$$\mathbf{B} = B_z(b_x, b_y, 1) = B_z(\mathbf{b} + \hat{\mathbf{z}}). \tag{7}$$

We wish to look for lower bounds on the free energy

$$E_f \equiv \frac{B_z^2}{8\pi}\int b^2\, d^3x. \tag{8}$$

Apply the triangle inequality to the integrand in Eq. (6):

$$\frac{dC}{dz} \le \frac{B_z^2}{2\pi}\int\int r_{12}^{-1}(|\mathbf{b}_1\cdot\hat{\boldsymbol{\theta}}_{12}| + |\mathbf{b}_2\cdot\hat{\boldsymbol{\theta}}_{12}|)d^2x_1\, d^2x_2. \tag{9}$$

After relabeling the second term, and writing $\mathbf{b}_1 = b_1\hat{\mathbf{b}}_1$, the inequality becomes

$$\frac{dC}{dz} \le \frac{B_z^2}{\pi}\int b_1 \int \frac{|\hat{\mathbf{b}}_1\cdot\hat{\boldsymbol{\theta}}_{12}|}{r_{12}}d^2x_2\, d^2x_1. \tag{10}$$

Now estimate the x_2 integral: First find the vector field $\hat{\mathbf{n}}(\mathbf{x}_1)$ which maximizes $I(\mathbf{x}_1) = \int(|\hat{\mathbf{n}}\cdot\hat{\boldsymbol{\theta}}_{12}|/r_{12})d^2x_2$. This turns out to be $\hat{\mathbf{n}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}_1$. The maximum is $I_{\max}(\mathbf{x}_1) = Rf(a)$, where $a = |\mathbf{x}_1|/R$ and

$$f(a) = 2\left(1 + \ln(1 + a^2) + \frac{1 - a^2}{2a}\tan^{-1}\frac{2a}{1 - a^2}\right). \tag{11}$$

Thus

$$\frac{dC}{dz} \le \frac{RB_z^2}{\pi}\int b_1 f\, d^2x_1. \tag{12}$$

Define

$$\overline{f^2} \equiv (\pi R^2)^{-1}\int f^2\, d^2x_1 \approx 13.6. \tag{13}$$

Then by the Schwarz inequality

$$\left(\frac{dC}{dz}\right)^2 \le \pi^{-1}R^4B_z^4\overline{f^2}\left(\int b_1^2\, d^2x_1\right). \tag{14}$$

Noting that

$$C^2 \le L\int_0^L (dC/dz)^2\, dz, \tag{15}$$

one finds

$$C^2 \le \pi^{-1}LR^4B_z^4\overline{f^2}\int b^2\, d^3x. \tag{16}$$

Turn this around to obtain a lower bound for the free energy [Eq. (8)]:

$$E_f \ge \left(8\overline{f^2}LR^4B_z^2\right)^{-1} C^2 \tag{17}$$

$$= 9.18 \times 10^{-3}\left(LR^4B_z^2\right)^{-1} C^2. \tag{18}$$

How far off is the lower bound in Eq. (17)? We can test this for a particular configuration: Assume that $\mathbf{b}(\mathbf{x}, z)$ is a random vector field with correlation length much smaller than R. Also assume that the distribution of field strengths is Gaussian. Consider the average of dC/dz [Eq. (6)] over many realizations. First, $\langle|(\mathbf{b}_2 - \mathbf{b}_1)\cdot\hat{\boldsymbol{\theta}}_{12}|\rangle$ should only depend on the separation r_{12}. In the limit of zero correlation length this quantity will be a constant.

Thus

$$\langle dC/dz\rangle = \frac{B_z^2}{2\pi}\langle|(\mathbf{b}_2-\mathbf{b}_1)\cdot\hat{\boldsymbol{\theta}}_{12}|\rangle \int\int \frac{1}{r_{12}} d^2x_1\, d^2x_2. \tag{19}$$

For a Gaussian distribution the bracketed term equals $\sqrt{2/\pi}b_{\rm rms}$ where $b_{\rm rms}^2 = \langle b^2\rangle^{1/2}$. Second, the double integral of r_{12}^{-1} yields $16\pi R^3/3$. Thus

$$\langle dC/dz\rangle = (8/3)\sqrt{2/\pi}B_z^2R^3b_{\rm rms}, \tag{20}$$

which leads to

$$E_f = 9\pi 2^{-10}\langle C\rangle^2 B_z^{-2}L^{-1}R^{-4} \tag{21}$$

$$= 2.76\times 10^{-2}\langle C\rangle^2 B_z^{-2}L^{-1}R^{-4}. \tag{22}$$

This is 3.01 times the lower limit.

Energy-crossing number relations can only be useful if there is a method for calculating C. Suppose the field can be divided into N mutually entangled flux tubes. Then we can approximate C by counting the crossovers of the N axis curves of the flux tubes. These N curves form a braid between the planes $z=0$ and $z=L$. Let the dimensionless crossing number for the N axis curves be C_N with minimum $\mathcal{C}_{\rm min}$, and assume for simplicity that all N tubes have equal flux $\Phi = \pi R^2 B_z/N$. Note that C counts more crossings than C_N; in particular, C counts crossings between two field lines within the same tube whereas C_N only counts crossings between different tubes. Thus $C \geq \Phi^2 C_N \geq \Phi^2\mathcal{C}_{\rm min}$ so the lower bound [Eq. (17)] becomes

$$E_f \geq (\pi^2/8\overline{f^2})\mathcal{C}_{\rm min}^2\Phi^2/N^2L \tag{23}$$

$$= 9.06\times 10^{-2}\mathcal{C}_{\rm min}^2\Phi^2/N^2L. \tag{24}$$

The quadratic dependence on $\mathcal{C}_{\rm min}$ deserves a remark. The braided fields with constant B_z treated here are similar to a system with two spatial dimensions plus time (let $z\to t$). In contrast, Freedman and He [1] considered fully three-dimensional fields; they found the minimum energy growing linearly with crossing number.

In the remainder of this Letter we consider magnetic fields (or braids) generated by random motions. When investigating how photospheric motions induce structure in the coronal field, it is natural to employ a model with discrete filaments: Magnetic flux is highly localized at the photosphere. The flux from one photospheric flux element may bifurcate and connect to several photospheric elements of the opposite polarity. This possibility may be accommodated by increasing N and allowing for variable flux.

The procedure involves letting N points in a plane correspond to the N foot points of the filaments inside a coronal loop. These N points move about each other randomly according to some law (e.g., random walk, diffusion, motion in a stochastic velocity field). The filaments above become braided as a result. One must then find the minimum value $\mathcal{C}_{\rm min}$ for the braid structure at time t. This gives $\mathcal{C}_{\rm min}(t)$; the relations between $\mathcal{C}_{\rm min}$ and the magnetic energy given above yield $E_f(t)$ and can be used in a coronal heating calculation.

In a recent simulation [21] three photospheric flux elements were allowed to random walk about each other. They were confined to a disk of radius 1000 km; when they reached the edge of the disk they bounced back inside. They moved with an average velocity of $V=1$ km/s. For step sizes of $\lambda = 1000$ km it was found that $C(t)\approx \frac{1}{4}Vt/\lambda$, i.e., C increased by about one unit each hour.

How does crossing number $\mathcal{C}_{\rm min}$ vary with N when the braid is generated by random motions? Fix the flux Φ and the typical diameter D of a flux tube, i.e., $\Phi = \pi D^2 B_z/4$. Then $R = N^{1/2}D/2$. The rms field strength $b_{\rm rms}$ is an intensive quantity: it should depend on the "amount of tangling per unit area" but not on N. From Eq. (22)

$$b_{\rm rms} \geq 0.27\mathcal{C}_{\rm min}DL^{-1}N^{-3/2}. \tag{25}$$

Thus $\mathcal{C}_{\rm min}$ increases as $N^{3/2}$. Intuitively, think of the N foot points in a square array, with $N^{1/2}$ points in each row, and a distance D between points. If one point moves by D or so, then in projection up to $N^{1/2}$ additional crossings will be made. The $N^{1/2}$ dependence remains after averaging over projection angle; it also remains if we drop the assumption of a square array. Thus if all N points move the increase in $\mathcal{C}_{\rm min}$ should go as $N^{3/2}$.

Coronal heating by random boundary motions has been suggested by Sturrock and Uchida [22], Parker [17], van Ballegooijen [23], and Berger [24]. For a coronal active region the power requirement is approximately 10^7 ergs cm^2 s^{-1} [25]. Again consider N foot points in random motion about each other. If the typical photospheric distance between foot points is d, then we can write

$$\mathcal{C}_{\rm min}(t) = \beta\frac{Vt}{d}N^{3/2}. \tag{26}$$

Here β is a dimensionless efficiency parameter. The photospheric distance d may be less than the diameter of coronal tubes D because of the clumping of flux at supergranule boundaries.

We wish to calculate the power per unit area going into a loop consisting of N flux tubes. The power input per unit area [using the lower bound of Eq. (24)] is then

$$P = (N\pi D^2/4)^{-1}\, dE_f/dt \tag{27}$$

$$= (\pi^3/16\overline{f^2})\,\beta\mathcal{C}_{\rm min}N^{-3/2}\,(B_z^2D^2V/Ld). \tag{28}$$

We now suppose that the energy input saturates when the mean value of b reaches some critical value μ. This happens when reconnection liberates energy at the same rate as energy is pumped in at the photosphere. In this case, $E_f = LN\pi(D^2/4)B_z^2\mu^2$. From Eq. (24) (again for the lower bound)

$$C_{\min} = (2\overline{f^2}^{1/2} N^{3/2} L\mu/\pi D). \tag{29}$$

The power at saturation is then

$$P = (\pi^2/8\overline{f^2}^{1/2})\beta\mu(VB_z^2 D/d). \tag{30}$$

Here $\pi^2/8\overline{f^2}^{1/2} = 0.334$. Let $B_z = 100$ G, $\beta = 0.06$ (as suggested by the random walk simulation [18]), $V = 1$ km/s, and $\mu = 0.25$. Then $P = 5 \times 10^6 D/d$ $\mathrm{ergs\,cm^{-2}\,s^{-1}}$. This gives sufficient power for an active region, provided $D/d \gtrsim 2$ (as d measures typical distances between foot points at the photosphere, clumping of foot points at the boundaries of supergranules should reduce d relative to the coronal flux tube diameter D). Using the zero-correlation estimate increases the heating rate by a factor of 1.7.

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