

**Orbit Space Reduction for
Symmetric Dynamical Systems
with an Application
to Laser Dynamics**

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Victoria Crockett

Abstract

This work considers the effect of symmetries on analysing bifurcations in dynamical systems. We consider an example of a laser with strong optical feedback which is modelled using coupled non-linear differential equations. A stationary point can be found in space, which can then be continued in parameter space using software such as AUTO. This software will then detect and continue bifurcations which indicate change in dynamics as parameters are varied. Due to symmetries in the equations, using AUTO may require the system of equations to be reduced in order to study periodic orbits of the original system as (relative) equilibria of the reduced system. Reasons for this are explored as well as considering how the equations can be changed or reduced to remove the symmetry. Invariant and Equivariant theory provide the tools for reducing the system of equations to the orbit space, allowing further analysis of the lasers dynamics.

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