A Journey Through the Dynamical World of Coupled Laser Oscillators

A thesis submitted to the University of Exeter
by Nicholas Blackbeard

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Image on front cover: Stability diagram for three coupled laser oscillators. Specifically, it is an expanded view from Fig. 4.9(b)–(c) with a different colour scale; yellow-red = periodic intensity fluctuations, and white-black = chaotic intensity fluctuations. This picture received a prize for the Engineering, Mathematics, and Physical Sciences Image Competition.
A Journey Through the Dynamical World of Coupled Laser Oscillators

Submitted by Nicholas Blackbeard, to the University of Exeter as a thesis for the degree of Doctor of Philosophy in Mathematics, January 2012.

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Abstract

The focus of this thesis is the dynamical behaviour of linear arrays of laser oscillators with nearest-neighbour coupling. In particular, we study how laser dynamics are influenced by laser-coupling strength, $\kappa$, the natural frequencies of the uncoupled lasers, $\hat{\Omega}_j$, and the coupling between the magnitude and phase of each lasers electric field, $\alpha$. Equivariant bifurcation analysis, combined with Lyapunov exponent calculations, is used to study different aspects of the laser dynamics. Firstly, codimension-one and -two bifurcations of relative equilibria determine the laser coupling conditions required to achieve stable phase locking. Furthermore, we find that global bifurcations and their associated infinite cascades of local bifurcations are responsible for interesting locking-unlocking transitions. Secondly, for large $\alpha$, vast regions of the parameter space are found to support chaotic dynamics. We explain this phenomenon through simulations of $\alpha$-induced stretching-and-folding of the phase space that is responsible for the creation of horseshoes. A comparison between the results of a simple coupled-laser model and a more accurate composite-cavity mode model reveals a good agreement, which further supports the use of the simpler model to study coupling-induced instabilities in laser arrays. Finally, synchronisation properties of the laser array are studied. Laser coupling conditions are derived that guarantee the existence of synchronised solutions where all the lasers emit light with the same frequency and intensity. Analytical stability conditions are obtained for two special cases of such laser synchronisation: (i) where all the lasers oscillate in-phase with each other and (ii) where each laser oscillates in anti-phase with its direct neighbours. Transitions from complete synchronisation (where all the lasers synchronise) to optical turbulence (where no lasers synchronise and each laser is chaotic in time) are studied and explained through symmetry breaking bifurcations. Lastly, the effect of increasing the number of lasers in the array is discussed in relation to persistent optical turbulence.
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Part I

Introduction and Background
Chapter 1

Motivation and Thesis Outline

The laser is hailed as one of the landmark inventions of the 20th century, and its history is rich, complicated, and controversial (accounts of which are found in Refs. [21, 128, 84, 57, 124]). Theodore Maiman [83] was the first person to produce a working laser, on the 16th May 1960. His design (Fig. 1.1(a)) was relatively simple, in essence consisting of only three components: a ruby rod, a flash lamp and a metal cylinder. At the time there were no preconceived applications for lasers, and so they were dubbed “a solution looking for a problem”. However, over subsequent years they began to be used in a number of fields. For example, in 1961 a ruby laser was used to remove a tumour from a human patient’s retina [70, 44], and in 1969 to accurately measure the distance to the moon. Thus, following its invention, the development of the laser quickly became driven by its potential commercial uses, facilitated by developments in associated technologies.

Today, lasers are produced in many shapes and sizes—from the very small (Fig. 1.1(b))

Figure 1.1: (a) Maiman’s first laser(Credit: HRL Laboratories, LLC). (b) the active medium of a nanocavity laser(Reprinted by permission from Macmillan Publishers Ltd: Nature Photonics [62], copyright 2007). (c) part of a laser at the National Ignition Facility [126](Credit: Lawrence Livermore National Laboratory).
to the very large (Fig. 1.1(c)). Lasers such as that shown in Fig. 1.1(b), which is approximately 300 times smaller than the width of a human hair [62], are central to the development of optical computing. In contrast, Fig. 1.1(c) shows a room containing only part of the complete laser at the National Ignition Facility, California, USA [126]. Composed of 192 smaller lasers, it produces a beam of extremely high-powered light [56] that has potential uses in energy generation.

There are a large variety of modern lasers, but semiconductor lasers are the most widely used. Among other things, being compact, cheap to operate, and relatively efficient makes them ideal for use in consumer electronics. PCs, DVD players, and laser printers, household items in modern-day developed countries, all contain semiconductor lasers. It has been estimated [66], that in 2004 there were 35,890,000 PCs in the United Kingdom alone. The telecommunications market is another area that relies heavily on semiconductor lasers. Anyone who has had a long distance phone call has probably ‘talked’ via a laser. Information is transmitted through fibre cables as optical pulses. In 1988 the worlds first transatlantic cable to use laser technology was installed, connecting the United States to Europe via 6,650km of fibre cable. This cable was comprised of eight single-mode fibres (which included two spares), and to boost the transmission (combating dispersion of the optical pulses), semiconductor lasers were placed on average every 50km. In total the system contained approximately 1,500 semiconductor lasers [133].

As depicted in Fig. 1.2, most semiconductor lasers are based around p-n junctions. On one side of the junction an impurity has been added to the semiconductor medium such that there is a deficiency of electrons on that side, forming “holes”. This side is positively charged and is said to be p-doped. Conversely, on the other side of the junction an impurity is added such that there is an excess of electrons. This side is negatively charged and is said to be n-doped. When a forward-biased electric field is passed through the p-n junction

![Figure 1.2: Simplified schematic of a semiconductor laser.](image-url)
the charged electrons and holes combine, emitting energy as photons. The cleaved ends of the semiconductor material act as mirrors for the photons, which bounce back and forth, amplifying as they collide with other charged electrons. A proportion of these photons escape through the cleaved ends, and provided that there is more amplification than losses, laser light is produced.

Individually, all lasers are nonlinear oscillators. However, when placed in close proximity to each other the dynamics of individual lasers alter due to a phenomenon known as coupling. Thus, laser arrays (arrays of coupled lasers) have come to be studied as individual entities, with their own unique dynamics, and resultant applications [19]. For example, coupled lasers have emerged as compact sources of high-power radiation, which is strongly desirable for applications in medicine, fundamental science, and space communication. On a theoretical level, laser arrays and their dynamical complexities contribute to the field of dynamical systems, with unexplored bifurcation structures and interesting nonlinear phenomena such as excitability [136, 108, 72, 143], various synchronisation types [76, 116, 125, 104, 139], and spatial patterns [1, 89, 106]. Uniquely, coupled laser systems provide an opportunity to exploit these phenomena in real life applications, such as chaos-based secure communication [131, 59, 7], ultra-fast random-number generation [129, 103], and instability-based radars and sensors [78, 30]. This thesis focuses on the analysis of instabilities and synchronisation in semiconductor laser arrays.

There have been numerous theoretical and experimental studies concerning the nonlinear dynamics of coupled lasers. These studies have considered arrays of various sizes [20, 127, 106, 42] and geometries [76, 116, 106], as well as different coupling types, including nearest-neighbour [147, 95, 76, 20, 127, 106, 42], global [107, 76, 71], and time-delayed coupling [58, 71, 149]. Many theoretical studies focused on simple ordinary differential equation (ODE) models [120, 95, 109, 76, 147, 20, 127, 106, 42], bifurcations in two-laser systems [120, 27, 109, 127, 64, 138, 141], and stability of synchronous solutions in larger arrays [20, 106] with circular geometry (periodic boundary conditions) [95, 116, 76, 77]. Some effort has been devoted to the analysis of more accurate composite-cavity mode models [112, 141] and partial differential equation (PDE) models [1, 61, 115, 87, 137, 32, 67] including the Maxwell-Bloch equations [92]. Despite extensive and important previous work on the subject there are still many unexplored problems concerning nonlinear behaviour in laser arrays.

This thesis considers three such problems. The First problem is to understand the dynamical behaviour of nearest-neighbour coupled lasers with the linear array geometry found in commercially available laser arrays. Specifically, we explore how laser dynamics are influenced by the coupling strength between lasers, $\kappa$, the natural frequencies of the uncoupled lasers, $\tilde{\Omega}_j$, and the coupling between the magnitude and phase of each laser’s
electric field, $\alpha$. On the one hand, by varying $\kappa$ and $\tilde{\Omega}_j$ we study laser-coupling conditions required to achieve phase locking, coupling-induced instabilities, and chaos. On the other hand, by varying $\alpha$, we uncover drastic differences in the ability of different laser types (as characterised by $\alpha$) to phase-lock or produce chaos.

The second problem concerns modelling approaches for laser arrays. The Maxwell–Bloch PDEs give a very accurate physical description of strong optical nonlinearities in coupled laser systems [109, 61, 1]. However, they are not well suited for stability and bifurcation analysis, which can provide insight into the mechanisms underlying synchronisation, multistability, or coupling-induced instabilities. Alternatively, a simple coupled-laser model (comprised of ODEs) neglects certain spatial effects in optical coupling between lasers, but is well suited for stability and bifurcation analysis, even for very large arrays [105, 106]. The coupled-laser model has been shown to work well for just two coupled lasers [41], but it is not clear whether it accurately captures all the essential nonlinearities of larger arrays. To further verify its validity, and understand its limitations, we compare results of the coupled-laser model with the more accurate composite-cavity mode model (ODEs with equations for spatial mode profiles).

The third problem concerns synchronisation properties of laser arrays. Since lasers are examples of self-sustained nonlinear oscillators they have the remarkable ability to synchronise [99]. However, there are many different types of synchronisation, so we first have to decide on a suitable definition for laser synchronisation. This needs to be general enough to capture relevant synchronisation types, but it also needs to be tractable enough to allow for a mathematical analysis. Once a suitable definition of laser synchronisation is fixed, we address the following questions: Is it possible to choose laser parameters such that different subsets of lasers synchronise? And, if so, what are the corresponding synchronised dynamics? The answers to these questions take us a step further towards understanding the complicated, self-organised patterns that emerge, in coupled lasers in particular, and coupled self-sustained nonlinear oscillators in general.

### 1.1 Outline of Thesis

This thesis is organised as follows: For the remainder of Part I we review the dynamics of a solitary laser and show that for typical parameter values it can be considered as a nonlinear oscillator. We then introduce physical setups that lead to different laser array configurations, and discuss the two modelling approaches that we use: the simpler coupled-laser approach and the more accurate composite-cavity mode approach. Part I is concluded with a brief overview of the tools and techniques from dynamical systems theory that are used throughout the thesis. In Part II, Lyapunov exponent calculations
are combined with (equivariant) bifurcation theory, to provide a detailed stability analysis of an array of three coupled laser oscillators, which stresses the effect of amplitude-phase coupling, $\alpha$, on the dynamics. Large regions of chaotic dynamics are found in the coupling strength versus frequency detuning parameter plane for non-zero $\alpha$. While many studies have also shown that positive $\alpha$ is conducive to the creation of chaos, the underlying mechanisms have not been clearly identified. Here we give an intuitive explanation of this effect in terms of $\alpha$-induced stretching and folding of the phase space. Furthermore, a comparison between the more accurate composite-cavity mode model and the simpler coupled-laser model demonstrates excellent agreement between the two, thus the bulk of the analysis has been carried out using the coupled-laser model. In Part III we study synchronisation properties of laser arrays. The coupled-laser model is easily generalised to an array of $M$ lasers, and is used to provide conditions, in terms of the natural frequencies of the lasers, that guarantee the existence of synchronised solutions. We then focus on two special cases of synchronisation and provide analytical conditions for their stability. Focusing again on an array of three laser oscillators, the transitions between different degrees of synchronisation are studied. In particular, large regions of optical turbulence (chaotic oscillations where no lasers are synchronised) are uncovered. By studying the underlying chaotic attractors we investigate properties of the optical turbulence, and how it is affected by the number of lasers, $M$, and the laser-coupling strength, $\kappa$. In particular, we use the Lyapunov spectrum of the underlying chaotic attractors to determine the intensity of the chaos [100] (given by the largest Lyapunov exponent), the number of unstable directions (given by the number of positive Lyapunov exponents), and the Lyapunov dimension [47]. Finally, our concluding comments are given in Part IV along with some open questions.
Chapter 2

A Dynamical Systems Approach for Laser Systems

The dynamics of laser systems are well described by low dimensional rate equation models in the form of ODEs [123, 141]. This desirable property makes them amenable to a dynamical systems approach, where the ultimate goal is to find and classify all invariant objects of the system’s phase space along with their stability and parameter dependencies [75]. In this chapter such an approach is applied to a solitary laser revealing that, for appropriately chosen parameter values, it can be considered as a limit cycle oscillator. We then discuss two different modelling approaches for an array of coupled lasers and highlight their strengths and weaknesses. While it is possible to provide a complete picture of the dynamics associated to a solitary laser, it becomes difficult (if not impossible) to achieve the equivalent for an array of coupled lasers. We therefore end this chapter with a brief overview of the theory and techniques that have been used in this thesis to provide as complete a picture as currently possible of the dynamics of an array of coupled lasers.

2.1 A Solitary Laser

Three main ingredients are required to create a laser: (i) a pump source that provides energy to the laser; (ii) an active medium, also known as the gain medium, which consists of excited atoms, molecules, or electron-hole pairs that can give rise to Light Amplification by Stimulated Emission of Radiation; and (iii) a ‘leaky’ optical resonator that is usually formed by two mirrors, one of which is partially transmitting.

The pump source excites the active medium resulting in a population inversion where more members of the active medium are in an excited energy state. The optical resonator imposes resonant light oscillations, such as standing waves, that are called optical modes.
An optical mode can exhibit sustained oscillation if its frequency falls within the amplification band of the active medium and its amplification exceeds the losses. Otherwise, the mode decays exponentially to zero. Here, we assume that the laser is a single-mode laser, meaning that only one of these modes is amplified.

The three ingredients, (i)-(iii), interact in a specific way to amplify a single optical mode according to a set of simple ordinary differential equations.

2.1.1 Rate Equations For a Solitary Laser

The dynamics of a single-mode laser can be described by the normalised slowly-varying electric field, \( E(t) \in \mathbb{C} \), and the normalised population inversion, \( N(t) \in \mathbb{R} \), whose time evolution is determined by (see Appendix A and [41])

\[
\frac{dE}{dt} = \beta \gamma (1 - i\alpha) N(t) E(t) - i(\Omega - \nu) E(t), \tag{2.1}
\]

\[
\frac{dN}{dt} = \Lambda - N(t) - (1 + \beta N(t)) |E(t)|^2. \tag{2.2}
\]

Equations (2.1)–(2.2) form a three-dimensional dynamical system that is rotationally symmetric under the transformation \( (E, N) \rightarrow (e^{i\theta} E, N) \) for \( \theta \in [0, 2\pi) \). The laser variables are normalised\(^1\) so that \( |E| = 0 \) corresponds to zero electric field, and that \( N = -1 \) corresponds to zero population inversion. The linewidth enhancement factor, \( \alpha \), quantifies the coupling between the magnitude, \( |E| \), and phase, \( \text{arg}(E) \), of the slowly-varying electric field \( E \). It plays a crucial role for the analysis in this thesis and is discussed in more detail in Ch. 2.1.3. The normalised natural frequency of the laser is given by \( \Omega \), and \( \nu \) is a conveniently chosen reference frequency. The normalised pump strength, \( \Lambda \), corresponds to zero pump strength when \( \Lambda = -1 \). The remaining parameters, \( \beta \) and \( \gamma \), are the normalised gain coefficient, and the ratio of field and population inversion decay rates. The parameter values used in this thesis are representative of typical (semiconductor) lasers and are given in Table A.1 of the Appendix.

\(^1\)See Appendix A for the normalisations.
2.1.2 Dynamics of a Solitary Laser

It is a simple task to find and classify the attractors of a solitary laser analytically. There is one equilibrium at \((E, N) = (0, \Lambda)\) which represents the ‘off state’ of the laser. This equilibrium is exponentially stable if \(\Lambda < 0\) and unstable if \(\Lambda > 0\). A limit cycle (if \(\nu \neq \Omega\)) given by

\[
E(t) = \sqrt{\Lambda} e^{-i(\Omega - \nu)t}, \quad N(t) = 0,
\]

(2.3)

bifurcates out of the equilibrium at \(\Lambda = 0\). It is generally difficult to calculate the stability of a limit cycle analytically. However, in this case we can choose the reference frequency, \(\nu = \Omega\), and freeze the limit cycle into a ring of infinitely many nonhyperbolic equilibria. The stability of the limit cycle is determined by the eigenvalues of the Jacobian of (2.1)–(2.2) evaluated at any of these non-hyperbolic equilibria. If

\[
0 < \Lambda < \frac{4\gamma \left(1 - \sqrt{1 - 1/(2\gamma)}\right) - 1}{\beta} \approx 2 \times 10^{-4},
\]

(2.4)

then the eigenvalues are given by

\[
\mu_1 = 0, \quad \mu_{2,3} = -a \pm b,
\]

where

\[
a = \frac{1 + \beta \Lambda}{2} > 0 \quad \text{and} \quad b = |\sqrt{a^2 - 2\beta \gamma \Lambda}| > 0.
\]

(2.5)

For \(\Lambda\) satisfying (2.4) the limit cycle (2.3) is over-damped. If

\[
\frac{4\gamma \left(1 - \sqrt{1 - 1/(2\gamma)}\right) - 1}{\beta} < \Lambda < \frac{4\gamma \left(1 + \sqrt{1 - 1/(2\gamma)}\right) - 1}{\beta} \approx 154,
\]

(2.6)

then the eigenvalues are given by

\[
\mu_1 = 0, \quad \mu_{2,3} = -a \pm ib,
\]

where \(a\) and \(b\) are defined in (2.5). For \(\Lambda\) satisfying (2.6) the limit cycle (2.3) is under-damped. In the laser literature the oscillations of transient dynamics about the under-damped limit cycle (2.3) are known as relaxation oscillations [143]—an example of these oscillations can be seen in Fig. 2.1(c). Note that laser relaxation oscillations are not to
be confused with relaxation oscillations from the nonlinear dynamics literature [130, 49].

From the above analysis, it is clear that the pump strength, $\Lambda$, is vital for lasing to take place. Figure 2.1 contains a one-parameter bifurcation diagram in $\Lambda$ that summarises the previous results. If $\Lambda < 0$ the ‘off state’ (equilibrium) is stable (Fig. 2.1(b)), however, if $\Lambda > 0$ the ‘on state’ (limit cycle) is stable (Fig. 2.1(c)). The value at which the stable ‘on state’ is created ($\Lambda = 0$) is known as the laser threshold. Mathematically, it is a supercritical Hopf bifurcation (if $\nu \neq \Omega$). For the rest of this thesis we set $\Lambda = 2$ so that the laser is operating in the ‘on state’, and can hence be considered as a limit cycle oscillator.

### 2.1.3 Shear and the $\alpha$-parameter

Amplitude-phase coupling is a universal property of nonlinear oscillators and appears in various scientific disciplines under different names. In dynamical systems and biology, one speaks of shear, twist or nonisochronicity [9, 79], in physics, of nonlinear dispersion [136], and in engineering, of self-phase modulation or chirp [60]. For a solitary laser oscillator the amplitude-phase coupling is quantified by the linewidth enhancement factor, $\alpha$, henceforth called the $\alpha$-parameter. Specifically, $\alpha$ couples the magnitude, $|E|$, and phase, $\arg(E)$, of the complex-valued electric field, $E$. Its physical origin is the dependence of the refractive index of the active medium—and hence the laser resonant frequency—on the population inversion [60]. The $\alpha$-parameter takes values between 0 and 1 for gas and solid state lasers, and between 1 and 10 for typical semiconductor lasers.

While $\alpha$ does not influence the stability of a single laser, it introduces a special
property in the laser’s phase space. For $\alpha = 0$, trajectories with different $|E|$ rotate with the same frequency about the line defined by $|E| = 0$. However, if $\alpha > 0$, trajectories with larger $|E|$ rotate faster, giving rise to $\alpha$-dependent phase space stretching along the angular direction in the complex $E$-plane.

This is best illustrated using sets called isochrons [53]. Each point, $p(0) \in \Gamma$, on an arbitrary hyperbolic limit cycle, $\Gamma$, has an isochron defined by

$$\{x \in \mathbb{R}^n : x(t) \to p(t) \text{ as } t \to \infty\}.$$  

Isochrons are invariant under the time-$T$ map (where $T$ is the period of $\Gamma$), codimension-one manifolds that intersect the limit cycle, $\Gamma$, transversally. The isochrons of the limit

Figure 2.2: Three isochrons (2.7) of a solitary laser oscillator with (a) $\alpha = 0$ and (b) $\alpha = 2$.

Figure 2.3: A sketch illustrating how the amount of shear in a solitary laser is related to the angle, $\theta$, between a vector normal to the limit cycle at $p(0)$ (black) and a vector tangent to the isochron at $p(0)$ (green). If $\alpha = 0$ then $\theta = 0$ and there is no shear. If $\alpha > 0$ then $\theta > 0$ and there is shear.
cycle (2.3) of a solitary laser are logarithmic spirals ([99, Ch. 7],[140]):

\[
\arg(E) + \alpha \ln(|E|) = C \quad \text{for} \quad C \in [0, 2\pi).
\] (2.7)

Three isochrons (blue surfaces) are shown in Fig. 2.2 for a solitary laser oscillator (red limit cycle) with two different values of \(\alpha\). For \(\alpha = 0\) (Fig. 2.2(a)) the isochrons are planes orthogonal to the limit cycle (2.3). However, for \(\alpha = 2\) (Fig. 2.2(b)) the isochrons are two-dimensional surfaces that are not orthogonal to the limit cycle (2.3). The amount of shear can be approximated by the inclination of the isochrons to a plane orthogonal to the limit cycle (see Fig. 2.3). With this in mind, Eqn. (2.7) and Fig. 2.2, indicate that there is no shear for \(\alpha = 0\) and that there is shear for \(\alpha > 0\). For this reason the \(\alpha\)-parameter quantifies the amount of shear in the laser phase space. The resulting stretch-and-fold action that the shear introduces is important for the discussion of coupling-induced chaos in Ch. 4.8.

### 2.2 Coupled Lasers

As we saw in Ch. 2.1, a solitary laser is fairly mundane from a dynamical systems perspective. However, when two or more lasers are coupled together, the resulting systems are rich in interesting nonlinear phenomena. In this thesis we consider a linear array of lasers, denoted with a subscript \(j = 1, \ldots, M\), that are coupled via their optical fields to their nearest neighbour(s) (Fig. 2.4). As in the solitary laser case (Ch. 2.1) we assume that each laser is a single-mode laser, meaning that it operates with a single optical mode of natural frequency \(\tilde{\Omega}_j\). Furthermore, all the lasers are assumed to be identical apart from a possible natural-frequency detuning. With the exception of Ch. 9, this frequency detuning is between the (resonant) middle lasers, \(\tilde{\Omega}_j = \tilde{\Omega}_m\) for \(j = 2, \ldots, M - 1\), and the (resonant) outer two lasers, \(\tilde{\Omega}_1 = \tilde{\Omega}_M = \tilde{\Omega}_{\text{out}}\).

Two different physical realisations of optical coupling in a linear laser array are sketched in Fig. 2.4. Small arrows indicate the direction of coupling, and large arrows indicate the direction in which the laser beam propagates. For side-to-side coupled lasers (Fig. 2.4(a)), the coupling is due to the evanescent electric field transverse to the direction of the laser beam. Such coupling realisation has been studied for example in Refs. [69, 27, 147, 95, 1, 45, 76, 109, 116, 77, 20, 89, 127, 125, 104, 101, 106, 41, 81, 42]. For face-to-face coupled lasers (Fig. 2.4(b)), the coupling is due to the electric field in the propagation direction of the laser beam. Such coupling realisation has been studied for example in Refs. [120, 46, 26, 86, 150, 141, 43].

The laser arrays sketched in Fig. 2.4 have been studied in different mathematical
frameworks, all of which are approximations of the Maxwell–Bloch equations [55, 85]. We introduce the Maxwell–Bloch equations in this section along with an outline of the two different modelling approaches taken in this thesis.

2.2.1 Different Modelling Approaches

In semiclassical laser theory light is treated classically and is therefore governed by Maxwell’s Equations:

\[
\nabla \times \mathcal{E}(\mathbf{r}, \hat{t}) = -\frac{\partial}{\partial \hat{t}} \mathcal{B}(\mathbf{r}, \hat{t}),
\]

\[
\nabla \times \mathcal{H}(\mathbf{r}, \hat{t}) = j(\mathbf{r}, \hat{t}) + \frac{\partial}{\partial \hat{t}} \mathcal{D}(\mathbf{r}, \hat{t}),
\]

\[
\nabla \cdot \mathcal{D}(\mathbf{r}, \hat{t}) = \rho(\mathbf{r}, \hat{t}),
\]

\[
\nabla \cdot \mathcal{B}(\mathbf{r}, \hat{t}) = 0,
\]

where \(\mathcal{E}(\mathbf{r}, \hat{t})\) is the electric field, \(\mathcal{H}(\mathbf{r}, \hat{t})\) is the magnetic field, \(\mathcal{D}(\mathbf{r}, \hat{t})\) is the electric field displacement, \(\mathcal{B}(\mathbf{r}, \hat{t})\) is the magnetic induction, \(j(\mathbf{r}, \hat{t})\) is the current density, and \(\rho(\mathbf{r}, \hat{t})\) is the density of electric charges. When an external electromagnetic field is applied to matter, atomic or molecular electric/magnetic dipoles are generated [82]. The dipole moment per unit volume is called the electric/magnetic polarisation and we denote them by \(\mathcal{P}(\mathbf{r}, \hat{t})\) and \(\mathcal{J}(\mathbf{r}, \hat{t})\) respectively. The constitutive equations are:

\[
\mathcal{D}(\mathbf{r}, \hat{t}) = \varepsilon_0 \mathcal{E}(\mathbf{r}, \hat{t}) + \mathcal{P}(\mathbf{r}, \hat{t}),
\]

\[
\mathcal{B}(\mathbf{r}, \hat{t}) = \mu_0 \mathcal{H}(\mathbf{r}, \hat{t}) + \mathcal{J}(\mathbf{r}, \hat{t}),
\]

where \(\varepsilon_0 = 8.8542 \times 10^{-12}\) is the electric constant and \(\mu_0 = 4\pi \times 10^{-7}\) is the magnetic constant.

Assuming a charge-free, \(\rho(\mathbf{r}, \hat{t}) = 0\), and non-magnetic medium, \(\mathcal{J}(\mathbf{r}, \hat{t}) = 0\), a single inhomogeneous wave equation describing the propagation of the real-valued electric field,
\( \mathbf{E}(\mathbf{r}, \tilde{t}) \), can be derived as follows:

\[
\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, \tilde{t}) = -\frac{\partial}{\partial \tilde{t}} \left( \mathbf{B}(\mathbf{r}, \tilde{t}) \right) \quad \text{(the curl of Eqn. (2.8))},
\]

\[
= -\mu_0 \frac{\partial}{\partial \tilde{t}} \nabla \times \mathbf{H}(\mathbf{r}, \tilde{t}) \quad \text{(subst. Eqn. (2.13) for } \mathbf{B}(\mathbf{r}, \tilde{t})),
\]

\[
= -\mu_0 \frac{\partial}{\partial \tilde{t}} \left( \mathbf{J}(\mathbf{r}, \tilde{t}) + \frac{\partial}{\partial \tilde{t}} \mathbf{D}(\mathbf{r}, \tilde{t}) \right) \quad \text{(subst. Eqn. (2.9) for } \nabla \times \mathbf{H}(\mathbf{r}, \tilde{t})),
\]

\[
= -\mu_0 \frac{\partial}{\partial \tilde{t}} \left( \sigma \mathbf{E}(\mathbf{r}, \tilde{t}) + \frac{\partial}{\partial \tilde{t}} \mathbf{D}(\mathbf{r}, \tilde{t}) \right) \quad \text{(using Ohm’s law: } j(\mathbf{r}, \tilde{t}) = \sigma \mathbf{E}(\mathbf{r}, \tilde{t})),
\]

where \( \sigma \) is the electric conductivity of the medium. From this point we use Eqn. (2.12), the identity

\[
\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, \tilde{t}) = \nabla (\nabla \cdot \mathbf{E}(\mathbf{r}, \tilde{t})) - \nabla^2 \mathbf{E}(\mathbf{r}, \tilde{t}),
\]

and the fact that light vectors vary little along the directions in which they point, \( \nabla (\nabla \cdot \mathbf{E}(\mathbf{r}, \tilde{t})) \approx 0 \), to give

\[
-\nabla^2 \mathbf{E}(\mathbf{r}, \tilde{t}) + \mu_0 \sigma \frac{\partial}{\partial \tilde{t}} \mathbf{E}(\mathbf{r}, \tilde{t}) + \frac{1}{c^2} \frac{\partial^2}{\partial \tilde{t}^2} \mathbf{E}(\mathbf{r}, \tilde{t}) = -\mu_0 \frac{\partial^2}{\partial \tilde{t}^2} \mathbf{P}(\mathbf{r}, \tilde{t}),
\]

(2.14)

where \( c = \sqrt{1/(\mu_0 \varepsilon_0)} \) is the speed of light in a vacuum. Light, in a laser array is confined by changes in the refractive index of the medium. To take this into account we need to replace \( c \) in Eqn. (2.14) with \( c/n(\mathbf{r}) \), where \( n(\mathbf{r}) \) is the space-dependent refractive index of the medium. Making this substitution leads to the inhomogeneous electromagnetic wave equation [111]

\[
-\nabla^2 \mathbf{E}(\mathbf{r}, \tilde{t}) + \mu_0 \sigma \frac{\partial}{\partial \tilde{t}} \mathbf{E}(\mathbf{r}, \tilde{t}) + \frac{n^2(\mathbf{r})}{c^2} \frac{\partial^2}{\partial \tilde{t}^2} \mathbf{E}(\mathbf{r}, \tilde{t}) = -\mu_0 \frac{\partial^2}{\partial \tilde{t}^2} \mathbf{P}(\mathbf{r}, \tilde{t}),
\]

(2.15)

The left-hand side of this PDE describes the electric field propagation and losses within the laser array, whose physical structure is specified by the space-dependent refractive index, \( n(\mathbf{r}) \), and appropriate boundary conditions [41]. The inhomogeneous term on the right-hand side involves the active-medium polarisation, \( \mathbf{P}(\mathbf{r}, \tilde{t}) \), and represents the source of the propagating electric field. To calculate polarisation and population inversion within each laser, Eqn. (2.15) has to be combined with a suitable quantum-mechanical description of the active medium [111]. The resulting PDE model is known as the Maxwell–Bloch equations [55, 85].

In semiclassical laser theory [111], the inhomogeneous wave equation (2.15) is solved by expanding the electric field

\[
\mathbf{E}(\mathbf{r}, \tilde{t}) = \frac{1}{2} \sum_j U_j(\mathbf{r}) A_j(\tilde{t}) + \overline{U_j(\mathbf{r})} \overline{A_j(\tilde{t})},
\]

(2.16)
(where the bar denotes complex conjugation) and the active-medium polarisation

\[ P(r, t) = \frac{1}{2} \sum_j U_j(r) B_j(t) + \bar{U}_j(r) \bar{B}_j(t), \]  

(2.17)

in terms of real-valued optical modes, \( U_j(r) \), that are solutions to the homogeneous wave equation

\[-\nabla^2 E(r, t) + \frac{n^2(r)}{c^2} \frac{\partial^2 E(r, t)}{\partial t^2} = 0.\]  

(2.18)

Equation (2.18) is obtained by setting \( P(r, t) = 0 \) and \( \sigma = 0 \) in Eqn. (2.15). Here, we assume passive optical modes \( U_j(r) \) that do not depend on the instantaneous population inversion. This assumption is justified when population-induced contributions to the refractive index, or their variations, remain negligible. However, when those contributions vary enough to change the mode profiles, one needs to consider active optical modes that do depend on the instantaneous population inversion [137, 115]. Substituting a field of a passive eigenmode with a frequency \( \tilde{\nu}_j \),

\[ E(r, t) = U_j(r) e^{-i \tilde{\nu}_j t}, \]  

into Eqn. (2.18) gives the Helmholtz equation for \( U_j(r) \):

\[ \nabla^2 U_j(r) + \frac{n^2(r)}{c^2} \tilde{\nu}_j^2 U_j(r) = 0. \]  

(2.19)

For suitably chosen boundary conditions (see for example Ch. 5), passive optical modes are orthogonal with a weight function \( n^2(r) \), meaning that they satisfy the orthogonality relation

\[ \int_{\mathbb{R}^3} n^2(r) U_j(r) U_{j'}(r) \, dr = \mathcal{N} \delta_{jj'}, \]  

(2.20)

with an arbitrary normalisation constant, \( \mathcal{N} \), and the Kronecker delta, \( \delta_{jj'} = 1 \) if \( j = j' \) and zero otherwise. The time-dependent and complex-valued coefficients of the field expansion (2.16), \( A_j(\tilde{t}) \), are the corresponding complex-valued electric fields. Given the passive optical modes \( U_j(r) \), equations for \( A_j(\tilde{t}) \) are obtained by substituting expansions (2.16)–(2.17) into (2.15) and projecting onto \( U_j(r) \) [111]. It is common to separate in \( A_j(\tilde{t}) \) a term oscillating at a fast optical frequency, \( \tilde{\nu} \), by writing

\[ A_j(\tilde{t}) = \tilde{E}_j(\tilde{t}) e^{-i \tilde{\nu} \tilde{t}}, \]  

(2.21)

and study the slowly-varying, complex-valued electric field, \( \tilde{E}_j(\tilde{t}) \). This is accomplished
through a number of approximations described in [111, 28], including the *rotating wave approximation* that removes the complex-conjugated terms in Eqs. (2.16)–(2.17) and introduces the rotational symmetry discussed in Ch. 4.1 and 8.4.

In this thesis we consider two different approaches to modelling laser arrays within the framework of semiclassical laser theory: the *coupled-laser approach* and the *composite-cavity mode approach*. These approaches arise from different ways of calculating the passive optical modes $U_j(r)$ or, in other words, from different eigenbases used in expansions (2.16)–(2.17).

### 2.2.2 Coupled-Laser vs. Composite-Cavity Mode Approach

In the coupled-laser approach the spatial modes used in expansions (2.16)–(2.17) are modes of the individual lasers (Fig. 2.5(a)), meaning that laser-coupling is completely neglected in the spatial part of the problem. The spatial modes are obtained for each individual laser by solving the Helmholtz equation (2.19) for a constant refractive index, $n(r) = n$, which reflects the spatial structure of each individual laser. The coupling between lasers is then included as extra source terms in Eqn. (2.15) that satisfy appropriate boundary conditions, i.e. the total electric field and its first derivative must be continuous at the laser boundaries which are determined by discontinuities in the refractive index (for details see [120, 41]). The final system of coupled ordinary differential equations can be used to study both spatial configurations in Fig. 2.4. Such a model is well suited to be analysed in a dynamical systems framework but has the caveat that it is limited to weak coupling strengths between the lasers.

In the composite-cavity mode approach the spatial modes used in expansions (2.16)–(2.17) are modes of the entire coupled laser structure (Fig. 2.5(b)) and are known as *composite-cavity modes*. The composite-cavity modes are obtained by solving the Helmholtz equation (2.19) for the refractive index $n(r)$ which describes the entire coupled laser structure. The coupling between the lasers is fully taken into account in the spatial part of the problem and hence the model is not limited to weak coupling. The resulting model is a system of coupled ordinary differential equations with algebraic constraints, that has to be reformulated for the different geometries shown in Fig. 2.4. Such a model can be studied in a dynamical systems framework but this requires additional analysis that is explained in Ch. 5. The composite-cavity mode model has the advantage of being valid for arbitrary coupling strengths, and as such, serves as a benchmark with which to compare other simpler models such as the coupled-laser model.
2.3 Tools and Techniques from Dynamical Systems Theory

In both the coupled-laser approach and the composite-cavity mode approach the governing equations define a dynamical system comprised of first-order ODEs of the form

\[ \frac{dx}{dt} = F(x(t), p), \tag{2.22} \]

where \( x(t) \in \mathbb{R}^n \) is a vector of state variables, \( p \in \mathbb{R}^m \) is a vector of parameters, and \( F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is a vector valued function. In general, a system of nonlinear ODEs is very difficult (if not impossible) to solve analytically. There is, however, well developed theory [51, 75, 73] that allows one to reach a good understanding of the qualitative behaviour of solutions to system (2.22). In this section we provide an overview of the relevant tools and techniques that are used throughout this thesis.

2.3.1 Bifurcation Theory

The setting for bifurcation theory is the phase space—defined as the set of all possible initial conditions of the system. A solution, \( x(t) \), of system (2.22) passing through a point in the system’s phase space is known as a trajectory. The set of all trajectories for a given parameter vector is called the phase portrait, it provides a global qualitative picture of the system’s dynamics. As one varies the parameters of (2.22) the phase portrait may deform slightly but not change its qualitative features, alternatively, it could be modified significantly and produce a qualitative change in the system dynamics. Bifurcation theory is concerned with these qualitative changes in the phase portrait, i.e. changes in stability.
and the disappearance or creation of invariant sets such as equilibria, limit cycles, and strange attractors. The notion of ‘qualitative change’ is rigorously defined in terms of topological equivalence [75, Ch. 2]—two systems are said to be topologically equivalent if there is a homeomorphism between their phase portraits that preserves the direction of time.

As a simple example of a system undergoing a bifurcation consider the following one-dimensional dynamical system:

\[
\frac{dx}{dt} = p + x^2, \tag{2.23}
\]

where \( p \in \mathbb{R} \) is a parameter. For \( p < 0 \) system (2.23) has one stable equilibrium at \( x = -\sqrt{-p} \), and one unstable equilibrium at \( x = +\sqrt{-p} \) (Fig. 2.6(a)). As \( p \) is increased the two equilibria move toward each other and then collide forming a single equilibrium when \( p = 0 \) (Fig. 2.6(b)). This equilibrium is said to be neutral. For \( p > 0 \) system (2.23) has no equilibria and trajectories drift toward increasing \( x \). The bifurcation point \( p = 0 \) is known as a saddle-node (or fold) bifurcation.

The saddle-node bifurcation is an example of a local bifurcation. Local bifurcations can be classified and analysed through a Taylor series (hence the term local) of a vector field or map at a single point [54, Ch. 3]. Other examples of local bifurcations include pitchfork of equilibria, Hopf, saddle-node of limit cycles, period doubling, and torus bifurcations. There are also bifurcations that cannot be determined from local knowledge of the flow alone. These are known as global bifurcations [54, Ch. 6]. To illustrate the difference between local and global bifurcations consider the following two-dimensional dynamical

Figure 2.6: In each panel the black line with arrows (indicating the direction of flow) is a phase portrait for system (2.23) (a) before, (b) at, and (c) after the saddle-node bifurcation. Equilibria are represented by circles and shading indicates their stability: full=stable, empty=unstable, and half-full=neutral.
Figure 2.7: Phase portraits (a) before, (b) at, and (c) after the saddle-node homoclinic bifurcation in system (2.24). Arrows indicate the direction of flow. Equilibria are represented by circles and shading indicates their stability: full=stable, empty=unstable, and half-full=neutral.

system [75, Ch. 8.4.2]:
\[
\begin{align*}
\frac{dx_1}{dt} &= x_1(1 - x_1^2 - x_2^2) - x_2(1 + p + x_1), \\
\frac{dx_2}{dt} &= x_1(1 + p + x_1) + x_2(1 - x_1^2 - x_2^2),
\end{align*}
\]
(2.24)

where \( p \in \mathbb{R} \) is a parameter. In polar coordinates \((r, \theta)\), system (2.24) becomes
\[
\begin{align*}
\frac{dr}{dt} &= r(1 - r^2), \\
\frac{d\theta}{dt} &= 1 + p + r \cos(\theta).
\end{align*}
\]

From the polar coordinate representation one can verify that system (2.24) has two invariant sets for any value of \( p \): an unstable equilibrium at the origin, and the unit circle. For \( p < 0 \) there are two additional equilibria that lie on the unit circle (Fig. 2.7(a)). Increasing \( p \) causes these equilibria to move toward each other and collide when \( p = 0 \) (Fig. 2.7(b)). For \( p > 0 \) the two equilibria on the unit circle have been destroyed leaving behind a limit cycle (red curve in Fig. 2.7(c)). The important point here is that if we had only focused on a small neighbourhood of the neutral equilibrium for \( p = 0 \) (Fig. 2.7(b)), we would have missed the creation of the stable limit cycle (Fig. 2.7(c)). The bifurcation point \( p = 0 \) is known as a saddle-node homoclinic bifurcation.

All the bifurcations relevant to this thesis are given in Table 2.1. Notice that in Table 2.1 some bifurcations are represented by coloured lines while others are represented by black dots. The reason for this is related to the codimension of the bifurcations. Following [54, Ch. 3.1] we define the codimension of a bifurcation as the lowest dimension of a
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Key</th>
<th>Bifurcation/Solution</th>
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<tr>
<td>S</td>
<td></td>
<td>Saddle-node bifurcation</td>
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<tr>
<td>Shom</td>
<td></td>
<td>Saddle-node homoclinic bifurcation</td>
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<td>P</td>
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<td>Pitchfork bifurcation</td>
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<td>H</td>
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<td>Hopf bifurcation</td>
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<td>SL</td>
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<td>PL</td>
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<td>Pitchfork of limit cycle bifurcation</td>
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<td>PD</td>
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<td>Torus (Neimark–Sacker) bifurcation</td>
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<td>Homoclinic bifurcation</td>
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<td>SH</td>
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<td>Saddle-node-Hopf bifurcation</td>
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<td>BT</td>
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<td>Bogdanov–Takens bifurcation</td>
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<td>PSL</td>
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<td>Pitchfork-saddle-node of limit cycle bifurcation</td>
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<td>PT</td>
<td>•</td>
<td>Pitchfork-torus bifurcation</td>
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<td>1:2</td>
<td>•</td>
<td>1:2 resonance</td>
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<tr>
<td>δ_{-1}</td>
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<td>homoclinic Belyakov bifurcation where ( \Re(\lambda_s)/\lambda_u = -1 )</td>
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<tr>
<td>δ_{-0.5}</td>
<td>•</td>
<td>homoclinic Belyakov bifurcation where ( \Re(\lambda_s)/\lambda_u = -0.5 )</td>
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<tr>
<td>ShH</td>
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<td>Relative Shilnikov–Hopf bifurcation</td>
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<tr>
<td>NC</td>
<td>•</td>
<td>Non-central saddle-node homoclinic bifurcation</td>
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<td>*</td>
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<td>Double bifurcations of conjugate equilibria and conjugate limit cycles that are not ( R_{\mathbb{Z}_2} )-invariant</td>
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<td>Stable fixed equilibria/phase-locked solutions (4.20)</td>
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<tr>
<td></td>
<td></td>
<td>Stable conjugate equilibria/phase-locked solutions (4.21)</td>
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Table 2.1: The labelling, colour coding, and shading of the bifurcation diagrams. \( \lambda_s \) and \( \lambda_u \) are the stable and unstable eigenvalues of a saddle within the homoclinic/heteroclinic centre manifold.

The parameter space that contains the bifurcation in a persistent way. Codimension-one bifurcations are typically points in a one-dimensional parameter space (see Fig. 4.10), curves in a two-dimensional parameter space (see Fig. 4.2), surfaces in a three-dimensional parameter space, and so on. Similarly, codimension-two bifurcations are typically points in two-dimensional parameter space (see Fig. 4.2), curves in a three-dimensional parameter space, and surfaces in a four-dimensional parameter space. In this thesis we concentrate on a three-dimensional parameter space and study its two-dimensional cross sections. Bifurcation diagrams for these cross sections typically comprise codimension-one bifurcation curves and codimension-two bifurcation points.

Locating codimension-two bifurcation points is thus an important step towards gaining an understanding of the dynamical behaviour of system (2.22). To further demonstrate
why, we consider the following two-dimensional system which contains a codimension-two bifurcation known as the Bogdanov–Takens bifurcation:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2, \\
\frac{dx_2}{dt} &= p_1 + p_2 x_1 + x_1^2 - x_1 x_2,
\end{align*}
\] (2.25)

where \( p_1, p_2 \in \mathbb{R} \) are parameters. Equilibria of system (2.25) lie on the line \( x_2 = 0 \) and satisfy

\[
x_1^2 + p_2 x_1 + p_1 = 0.
\] (2.26)

The discriminant parabola of (2.26),

\[ S = \{(p_1, p_2) : p_2^2 - 4p_1 = 0\}, \]

corresponds to a curve of saddle-node bifurcations (blue curve in Fig. 2.25) which delineates the \( (p_1, p_2) \) parameter plane into two regions. To the right of \( S \) there are no equilibria, and to the left of \( S \) there are two equilibria given by:

\[
x_e^\pm = \left( \frac{p_2 \mp \sqrt{p_2^2 - 4p_1}}{2}, 0 \right)^T.
\]

Varying \( (p_1, p_2) \) so system (2.26) moves from region (1) to (2) results in the creation of a stable node \( x_e^- \) and a saddle \( x_e^+ \). Whereas, going from region (1) to (4) results in
the creation of an unstable node $x_e^-$ and a saddle $x_e^+$. Therefore there must be further bifurcations that involve $x_e^-$. There is in fact a curve a supercritical Hopf bifurcation (red line in Fig. 2.8) occurring along the line

$$H = \{(p_1, p_2): p_1 = 0, p_2 < 0\}.$$ 

The Hopf bifurcations along $H$ are supercritical, meaning that a stable limit cycle bifurcates out of $x_e^-$ when $(p_1, p_2)$ is varied from region (2) to (3). There are no further local bifurcations in system (2.25).

Since there are no equilibria to the left of $S$ in Fig. 2.8 no limit cycles can exist in region (1) [121, Ch. 6.8]. Therefore, there has to be other global bifurcations that destroy the stable limit cycle created along $H$. It can be shown that there is a unique curve of homoclinic bifurcations emanating from BT [75, Ch. 8.4.2]. For small $p_2 \ll 1$ an approximation to this curve is given by

$$\text{hom} = \{(p_1, p_2): p_1 = -\frac{6}{25} p_2^2, p_2 < 0\}.$$ 

Along this curve (whose approximation is plotted in black in Fig. 2.25) the stable limit cycle from the Hopf bifurcation collides with the stable and unstable manifolds of the saddle equilibrium $x_e^+$ to form a homoclinic orbit. With the addition of $\text{hom}$ it is possible to make a trip in parameter space around BT accounting for any invariant objects and changes in their stability.

In practice, a bifurcation analysis approach starts with the simplest stable object of a system’s phase space, usually an equilibrium. Codimension-one bifurcations curves then make up this objects stability boundary. The stability boundary changes type at codimension-two bifurcations. In system (2.25) the stability boundary of the equilibrium, $x_e^-$, consists of curves of saddle-node and Hopf bifurcations that are connected via the Bogdanov–Takens bifurcation point. Once codimension-two bifurcations have been located one expects to find certain other bifurcation curves emanating from them. Furthermore, there are only so many possibilities for each bifurcation type. For example, a Bogdanov–Takens bifurcation can be identified without any global knowledge of the flow, and once found, we know that there is a curve of homoclinic bifurcations emanating from it. For this reason, codimension-two bifurcations are also known as ‘organising centres’.

Another important aspect of this approach is that one can identify areas of parameter space where the dynamical system is likely to contain chaotic dynamics. A classic example is close to a chaotic Shilnikov bifurcation [75, Ch. 6.3] (see Fig. 2.9). This global bifurcation involves an equilibrium whose Jacobian matrix has one real eigenvalue, $\lambda_u > 0$, and a pair of complex conjugate eigenvalues, $\lambda_s \pm i\omega$, with $\lambda_s < 0$. At the bifurcation
Figure 2.9: The formation of a homoclinic orbit to a saddle-focus equilibrium. (a) before, (b) at, and (c) after a homoclinic Shilnikov bifurcation.

point $p = p_{\text{hom}}$ the system has a homoclinic orbit to this equilibrium (Fig. 2.9(b)). If the quantity

$$\delta \equiv \frac{\lambda_s}{\lambda_u} < -1$$

is satisfied at the bifurcation point, then there are a countable infinity of saddle limit cycles in a neighbourhood of the homoclinic orbit. In addition, there are an infinite number of horseshoes, each of which is considered a hallmark of chaos in its own right (see for example Ref. [75, Ch. 6.3] and [50]). Chaos generated close to a chaotic Shilnikov bifurcation is known as homoclinic chaos, and typically has a distinct spiral structure, see for example Fig. 4.14 and Refs. [8, 6].

One of the main aims of bifurcation theory is to categorise and study all possible types of bifurcations. There is however, a whole wealth of different bifurcations—take a glance at Table 2.1, for example, and it is by no means exhaustive! How then, does one decide which are the most relevant? It seems logical to focus on bifurcations that one would expect to see when varying a parameter of a given system. Such bifurcations are called \textit{generic}—for a mathematical description of this term see [110, Ch. 8.7]. The focus of traditional bifurcation theory is on the generic bifurcations of system (2.22), whose defining vector field, $\mathbf{F}$, does not have any special properties. For example, the only generic local (codimension-one) bifurcations of equilibria in such systems are saddle-node and Hopf bifurcations. However, if $\mathbf{F}$ in (2.22) has special properties—such as those imposed by symmetry—then the set of generic bifurcations can be different.
2.3.2 Symmetries and Equivariance

An important concept for the analysis in this thesis is the notion of a symmetry of system (2.22). A real $n \times n$ matrix, $R \in O(n)$, is said to be a symmetry of (2.22) if for every solution $x_1(t)$ of (2.22), $x_2(t) = Rx_1(t)$ is also a solution. Physical aspects of a system can lead to such symmetries, and in turn, the symmetries mean that the vector field of (2.22) has a special property. To see this, consider a solution $x_1(t)$ to (2.22). If $R$ is a symmetry of $F$ then by definition

$$x_2(t) = Rx_1(t)$$

is also a solution of (2.22). Differentiating (2.27) gives

$$\frac{dx_2}{dt} = R\frac{dx_1}{dt} = RF(x_1).$$

(2.28)

Since $x_2(t)$ is a solution of (2.22),

$$\frac{dx_2}{dt} = F(x_2) = F(Rx_1).$$

(2.29)

Equations (2.28) and (2.29) lead to the condition,

$$F(Rx_1) = RF(x_1),$$

(2.30)

which is satisfied if $R$ is a symmetry of (2.22). A vector field $F$ that satisfies this condition is called $R$-equivariant. The collection of symmetries of system (2.22), along with the $n \times n$ identity matrix, $I_n$, form a group $\Gamma$. $F$ is then said to be $\Gamma$-equivariant.

The study of bifurcations in system (2.22) with a $\Gamma$-equivariant vector field is an active area of research known as equivariant bifurcation theory [51]. $\Gamma$-equivariance has important consequences on the systems dynamics. Of particular relevance to this thesis is the existence of invariant manifolds and changes to the generic bifurcations. As an example, consider systems with $\mathbb{Z}_2$ symmetry where $\Gamma$ consists of the identity matrix, $I_n$, and $R$ such that $R^2 = I_n$; the only generic local (codimension-one) bifurcations of equilibria are saddle-node, Hopf, and pitchfork bifurcations.

2.3.3 Numerical Continuation

Bifurcation analysis is often difficult to perform analytically and needs to be done numerically. To carry out numerical bifurcation analysis we used a powerful software package

\footnote{where $O(n)$ is the orthogonal group consisting of invertible $n \times n$ matrices, $R$, such that $R^T = R^{-1}$.}
called AUTO [36]. AUTO implements numerical continuation methods which are particularly good because their errors converge fast. As such, the corresponding bifurcation analysis can in principle be made as accurate as desired. See [73] for a good overview of numerical continuation and its practical importance for the analysis of nonlinear systems of ODEs. Below we provide an illustration of how it works [88].

Consider system (2.22) with one state variable, $x \in \mathbb{R}$, and one parameter, $p \in \mathbb{R}$. Equilibria of this system are solutions to

$$F(x(p), p) = 0,$$

(2.31)

where $F$ is assumed to be continuously differentiable in $x$ and $p$, and parameter dependence has been explicitly denoted. Differentiating (2.31) with respect to $p$ gives

$$\frac{\partial F}{\partial x} \frac{dx}{dp} + \frac{\partial F}{\partial p} = 0,$$

which can be rearranged (providing $\partial F/\partial x \neq 0$) for $dx/dp$:

$$\frac{dx}{dp} = -\frac{\partial F/\partial p}{\partial F/\partial x}.$$  

(2.32)

Equation (2.32) is an ODE that can be solved as an initial value problem once an equilibrium for a particular parameter value is determined. Using numerical techniques, a particular solution $x(p)$ of (2.32) can be traced out (or continued), thus providing an understanding of the parameter dependence of equilibria of the system.

Furthermore, from bifurcation theory it is known that different bifurcations occur when certain conditions on $F$ are satisfied. For example, a saddle-node bifurcation in a one-dimensional system occurs when [75, Ch. 3]

$$\frac{\partial F}{\partial x} \Big|_{(x_{eq}, p_{bif})} = 0, \quad \frac{\partial^2 F}{\partial x^2} \Big|_{(x_{eq}, p_{bif})} \neq 0 \quad \text{and} \quad \frac{\partial F}{\partial p} \Big|_{(x_{eq}, p_{bif})} \neq 0,$$

(2.33)

where $x_{eq}$ is an equilibrium satisfying (2.31), and $p_{bif}$ is the location of the saddle-node bifurcation. While continuing an equilibrium, one can use (2.33) to detect a saddle-node bifurcation. Once the bifurcation has been detected, it too can be continued in a higher dimensional parameter space.

A key advantage of numerical continuation over direct numerical integration is that it is possible to continue equilibria (or limit-cycles) even when they are unstable. Bifurcation diagrams can get extremely complex (sometimes with infinitely many bifurcation curves), so we need a way to pick out the most relevant bifurcation structures. For this we were
2.3.4 Lyapunov Exponents

Lyapunov exponents are the average (exponential) rates of expansion and contraction of nearby trajectories. In this section we give a technical definition of Lyapunov exponents. Let \( y(t) = x(t) - x^*(t) \) denote deviations from the trajectory \( x^*(t) \). Linearising (2.22) about the trajectory \( x^*(t) \) gives

\[
\frac{dy}{dt} = DF(x^*(t))y(t),
\]

(2.34)

where \( DF(x^*(t)) \) is the Jacobian of \( F \) evaluated along \( x^*(t) \). Integrating Eqn. (2.34) along the trajectory \( x^*(t) \) gives a tangent map \( M(t) \) which is an \( n \times n \), typically time varying, matrix. The time evolution of \( M(t) \) is given by [5]

\[
\frac{dM}{dt} = DF(x^*(t))M(t).
\]

(2.35)

Intuitively, the vector \( M(t)y(t) \) is a small variation in the trajectory of Eqs. (2.22) caused by a small change in initial conditions. Under certain conditions, given by the Oseledec theorem [94, 37], the following limit exists:

\[
L = \lim_{t \to \infty} \left( M(t)M^T(t) \right)^\frac{1}{2}.
\]

(2.36)

The Lyapunov numbers, \( \Lambda_j \), are the eigenvalues of \( L \) and the Lyapunov exponents are given by

\[
\mu_j = \log(\Lambda_j) \quad \text{for } j = 1, \ldots, n.
\]

(2.37)

The collection of Lyapunov exponents \( \{\mu_j\}_{j=1}^n \) is known as the Lyapunov spectrum and is usually ordered such that \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \).

The sketch in Fig. 2.10 illustrates that Lyapunov exponents quantify local expansion/contraction properties associated with invariants sets of a system’s phase space [96, Ch. 4.4]. In the illustration the invariant set is a saddle equilibrium whose stable and unstable eigendirections are \( x_2 = -x_1 \) and \( x_2 = x_1 \), respectively. Consider a ball (with radius \( r \)) of initial conditions centred around the saddle equilibrium with Lyapunov exponents \( \mu_2 < 0 < \mu_1 \) (Fig. 2.10(a)). As the system evolves forward in time, the ball of initial conditions deforms into an ellipse; the lengths of its principle axes are given by \( re^{\mu_j t} \) for \( j = 1, 2 \) (Fig. 2.10(b)). The example used in Fig. 2.10 is in fact trivial because the Lyapunov exponents of the saddle equilibrium are the same as the eigenvalues of its
Figure 2.10: A sketch illustrating how Lyapunov exponents quantify local expansion/contraction properties associated with invariant sets of a system’s phase space. (a) a ball of initial conditions centred at the saddle equilibrium (blue cross) whose Lyapunov exponents are $\mu_2 \leq 0 \leq \mu_1$. (b) after time $t$ the ball evolves under the flow generated by (2.22) to an ellipse.

The example shown in Fig. 2.10 considered the Lyapunov spectrum of an unstable equilibrium. We are usually interested in the Lyapunov spectrum associated with attractors of the system, in which case one positive Lyapunov exponent is an indication of chaotic dynamics. In fact, it is possible to distinguish between different attractor types from their Lyapunov spectrums [148]. If $\mu_1 < 0$ then the attractor is an equilibrium, if $\mu_1 = 0$ and $\mu_2 < 0$ then the attractor is a limit cycle, if $\mu_1, \ldots, T = 0$ and $\mu_{T+1} < 0$ then the attractor is a $T$-torus, and finally if $\mu_1 > 0$ then the attractor is chaotic. Note that this classification scheme should be used with caution in systems with continuous symmetry. For example, a ring of nonhyperbolic equilibria has the same Lyapunov spectrum as a limit cycle. Accounting for symmetries, this classification scheme is used throughout the thesis to partition a two-dimensional parameter plane $(p_1, p_2)$ into regions that contain different attractor types. The regions are then coloured according to Table 2.2. More specifically, we discretise the $(p_1, p_2)$ parameter plane into a grid of $800 \times 800$ points. For each fixed value of $p_2$ we sweep the parameter $p_1$ and calculate the Lyapunov exponents (see Appendix B) using the final point on the trajectory (slightly perturbed) as an initial condition for the subsequent value of $p_1$. The Lyapunov exponents are then used to identify the corresponding attractor type and the parameter plane is coloured according to Table 2.2. See Fig. 4.1 for an example of a Lyapunov diagram.
<table>
<thead>
<tr>
<th>Key</th>
<th>Attractor Type</th>
<th>Lyapunov Exponents $\mu_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilibrium ⇒ Phase-locking</td>
<td>$\mu_j &lt; 0$ for $j = 1, \ldots, \dim$</td>
<td></td>
</tr>
<tr>
<td>Limit cycle (weakly - strongly attracting)</td>
<td>$\mu_1 = 0, \mu_j &lt; 0$ for $j = 2, \ldots, \dim$</td>
<td></td>
</tr>
<tr>
<td>Torus (weakly - strongly attracting)</td>
<td>$\mu_{1,2} = 0, \mu_j &lt; 0$ for $j = 3, \ldots, \dim$</td>
<td></td>
</tr>
<tr>
<td>Chaotic attractor (slow - fast divergence)</td>
<td>At least one $\mu_j &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>Attractor not in Fix($\mathbb{Z}_2$)</td>
<td>Stable within Fix($\mathbb{Z}_2$) but transversally unstable</td>
<td>At least one $\mu_j^\perp &gt; 0$</td>
</tr>
</tbody>
</table>

Table 2.2: The colour coding used in Lyapunov diagrams for the classification of different attractor types. \( \dim \) is the dimension of the system under consideration, and Fix($\mathbb{Z}_2$) is an invariant manifold defined in (8.14) called the fixed-point subspace.

**Tangential and Transversal Lyapunov Exponents**

Flow-invariant sub-manifolds in the phase space of system (2.22) are important concepts that often arise when studying synchronisation [68]. Specifically, synchronisation conditions impose relationships between the components of the state vector, \( \mathbf{x} \in \mathbb{R}^n \), that define sub-manifolds of the phase space. For example, in a three-dimensional system \( x_1 = x_2 = x_3 \) defines a one dimensional sub-manifold of \( \mathbb{R}^3 \). If these sub-manifolds are invariant under time evolution of (2.22) then they are *synchronisation manifolds*. It is of interest to know what the dynamics are on a synchronisation manifold, and in particular, if they are stable to transverse perturbations. Transverse and tangential Lyapunov exponents can be used for this purpose.

Let \( \mathcal{M} \) denote a flow invariant sub-manifold of the phase space of the dynamical system (2.22). The dynamics on this manifold are governed by the restriction of system (2.22) to \( \mathcal{M} \), which we denote by \( \frac{d}{dt}|_{\mathcal{M}} \). Let \( A \) be an attractor of \( \frac{d}{dt}|_{\mathcal{M}} \). The full Lyapunov spectrum of \( A \) can be split into two sets:

\[
\{\mu_j\}_{j=1}^n = \{\mu_j^\parallel \geq \cdots \geq \mu_{\dim(\mathcal{M})}^\parallel\} \cup \{\mu_j^\perp \geq \cdots \geq \mu_{\codim(\mathcal{M})}^\perp\}. \tag{2.38}
\]

Here, \( \mu_j^\parallel \) are known as *tangential Lyapunov exponents* and correspond to the growth \( (\mu_j^\parallel > 0) \) or decay \( (\mu_j^\parallel < 0) \) of perturbations solely within \( \mathcal{M} \). \( \mu_j^\perp \) are known as *transverse Lyapunov exponents* and correspond to the growth \( (\mu_j^\perp > 0) \) or decay \( (\mu_j^\perp < 0) \) of perturbations solely transverse to \( \mathcal{M} \). Tangential Lyapunov exponents can be used in a similar fashion as \( \mu_j \) to determine attractor types of \( \frac{d}{dt}|_{\mathcal{M}} \). Whether attractors of \( \frac{d}{dt}|_{\mathcal{M}} \) are attractors for the full system (2.22) depends on the sign of the largest transverse Lyapunov exponent. If \( \mu_1^\perp(\mathbf{x}(0)) < 0 \) for \( \mathbf{x}(0) \in A \), then \( A \) is an attractor for the full system (2.22). If \( \mu_1^\perp(\mathbf{x}(0)) > 0 \) for \( \mathbf{x}(0) \in A \), then \( A \) is unstable in the full system (2.22). If \( A \) is a chaotic attractor of \( \frac{d}{dt}|_{\mathcal{M}} \) one needs to consider the transverse Lyapunov exponent.
for ‘typical’ $x(0) \in A$. We reserve this discussion to Ch. 10.1 where this distinction has important consequences.

### 2.3.5 Assembling a Puzzle

How does this all fit together? The systems of ODEs considered in this thesis have three bifurcation parameters, i.e. system (2.22) with $p \in \mathbb{R}^3$. We build two parameter Lyapunov exponent diagrams for a range of different but fixed values of the third parameter. These diagrams provide an overview of the different attractor types in system (2.22) and their parametric dependencies. If the system has a complex dynamical structure, then the Lyapunov exponent diagrams can be used to highlight interesting features in the system dynamics. Furthermore, since Lyapunov exponents provide information about the attractors of the system, anything that they reveal is physically relevant. Bifurcation analysis is then used to develop an understanding of the underlying mechanisms responsible for the structure revealed by the Lyapunov exponent diagrams. Equivariance of the vector field in (2.22) is used to facilitate numerical continuation and to distinguish between attractors with different symmetry properties. We also derive a number of analytical results that can be used to further verify and guide the numerical analysis. The final result being two parameter bifurcation diagrams for different fixed values of the third parameter that form the back bone of the system dynamics. Together with the Lyapunov exponent diagrams they provide a detailed ‘road map’ of the physically relevant dynamics in the system.
Part II

Shear-Induced Bifurcations and Chaos in Models of Three Coupled Lasers
Chapter 3

Introduction for Part II

In Part II we study nonlinear dynamics in a linear array of three coupled laser oscillators with rotational $\mathbb{S}^1$ and reflectional $\mathbb{Z}_2$ symmetry. The focus is on a coupled-laser model with dependence on the three parameters: laser coupling strength, $\kappa$, laser frequency detuning, $\Delta$, and degree of coupling between the magnitude and phase of each lasers electric field, $\alpha$, also known as shear or nonisochronicity. Numerical bifurcation analysis is used in conjunction with Lyapunov exponent calculations to study the different aspects of the system dynamics. Firstly, the shape and extent of regions with stable phase locking in the $(\kappa, \Delta)$ plane change drastically with $\alpha$. We explain these changes in terms of codimension-two and -three bifurcations of (relative) equilibria. Furthermore, we identify locking-unlocking transitions due to global homoclinic and heteroclinic bifurcations, and associated infinite cascades of local bifurcations. Secondly, vast regions of deterministic chaos emerge in the $(\kappa, \Delta)$ plane for nonzero $\alpha$. We give an intuitive explanation of this effect in terms of $\alpha$-induced stretch-and-fold action that creates horseshoes, and discuss chaotic attractors with different topologies. Similar analysis of the more accurate composite-cavity mode model reveals good agreement with the coupled-laser model on the level of local and global bifurcations as well as chaotic dynamics, provided that coupling between lasers is not too strong. The results give new insight into modelling approaches and methodologies for studying nonlinear behaviour of laser arrays.


This research has also been highlighted in the Dynamical Systems Magazine: http://www.dynamicalsystems.org/pi/fr/detail?item=118.
Chapter 4

Coupled-Laser Model

In the coupled-laser approach, laser coupling is completely neglected in solving the homogeneous wave equation (2.18), which is justified for weakly coupled lasers. The solutions $U_j(r)$ are then simply the passive optical modes of the individual uncoupled lasers with constant refractive index $n$. The coupling is accounted for by an additional source term in the right-hand side of the inhomogeneous wave equation for each individual laser [119, 120, 109, 41]. The resulting set of ODEs can be thought of as a space-discretised version of the original Maxwell–Bloch equations with adiabatically eliminated polarisation (class-B lasers [143]). Specifically, the three-laser system sketched in Fig. 2.4 can be modelled by rate equations for the normalised slowly-varying complex-valued electric fields\(^1\), $E_s$,

\[
\begin{align*}
\frac{dE_A}{dt} &= \beta \gamma (1 - i\alpha) N_A E_A - i\Delta_A E_A + i\kappa E_B, \\
\frac{dE_B}{dt} &= \beta \gamma (1 - i\alpha) N_B E_B - i\Delta_B E_B + i\kappa (E_A + E_C), \\
\frac{dE_C}{dt} &= \beta \gamma (1 - i\alpha) N_C E_C - i\Delta_C E_C + i\kappa E_B,
\end{align*}
\]

and the normalised real-valued population inversions, $N_s$,

\[
\frac{dN_s}{dt} = \Lambda - N_s - (1 + \beta N_s)|E_s|^2 \quad \text{for } s = A, B, C,
\]

where $t$ is the normalised time. Different derivations of Eqs. (4.1)–(4.2) from Eqn. (2.15) can be found in [119, 120, 109, 41].

The optical coupling between lasers is described by the last terms of Eqs. (4.1), where $\kappa$ is the normalised coupling strength. The coupled-laser model (4.1)–(4.2) can

\(^1\)Note that in Part II of this thesis different lasers are indicated by the subscripts $A, B, C$ instead of $1, 2, 3$. 53
describe both physical realisations of optical coupling from Fig. 2.4 provided that the coupling is weak enough and that there are no gaps between the face-to-face coupled lasers (Fig. 2.4(b))\(^2\). For the different realisations, the coupling strength, \(\kappa\), depends on different physical parameters. For side-to-side coupling (Fig. 2.4(a)), \(\kappa\) is a function of the distance, \(d\), between the lasers [125, 109, 41]:

\[
\kappa \sim e^{-d}.
\]

(4.3)

For face-to-face coupling (Fig. 2.4(b)), \(\kappa\) is a function of the transmission coefficient, \(T\), of the common coupling mirror between the lasers, and the length, \(L\), of the optical resonator [120]:

\[
\kappa \sim \frac{\sqrt{T}}{L\sqrt{1-T}}.
\]

(4.4)

The weak-coupling assumption implies a sufficiently large distance, \(d\), or a sufficiently low transmission coefficient, \(T\), of the coupling mirror separating the lasers.

\(\Delta_s = \Omega_s - \nu\) is the frequency detuning between the normalised natural frequency, \(\Omega_s\), of laser \(s\) and a conveniently chosen reference frequency, \(\nu\). We assume that the outer two lasers are identical and therefore \(\Omega_A = \Omega_C\), but allow for the middle laser to have a different natural frequency. This choice of natural frequencies has the advantage of reducing the dimension of the parameter space as we can set the reference frequency, \(\nu\), to be

\[
\nu = \Omega_B,
\]

and define the normalised frequency detuning between the middle and two outer lasers as

\[
\Delta = \Omega_B - \Omega_{A,C}.
\]

(4.5)

The symmetry-breaking effects of having different outer lasers has been studied in [42].

The parameters, \(\beta\), \(\gamma\), and \(\Lambda\), are the normalised gain coefficient, the ratio of field and population inversion decay rates, and the pump rate, respectively. The \(\alpha\)-parameter is the coupling between the magnitude, \(|E_s|\), and phase, \(\text{arg}(E_s)\), of each lasers electric field, \(E_s\), and is discussed in Ch. 2.1.3.

\(^2\)If there are gaps between face-to-face coupled lasers, one needs to include additional equations for the field dynamics within the gap (gap size comparable to the laser size) or time delay in the coupling terms satisfying appropriate boundary conditions (gap size larger than the laser size).
4.1 Symmetry Properties

Knowledge of the symmetries present in a system of ODEs can be used to facilitate their analysis. The coupled-laser model has $\mathbb{S}^1 \times \mathbb{Z}_2$ symmetry [51]. Here, we reduce the $\mathbb{S}^1$ symmetry to facilitate the bifurcation analysis and make use of $\mathbb{Z}_2$ symmetry to distinguish between solution types with different sets of generic bifurcations.

The (continuous) $\mathbb{S}^1$ symmetry is due to the equivariance of the vector field defined by Eqs. (4.1)–(4.2) under the transformation,

$$T_{\mathbb{S}^1} : (E_A, E_B, E_C, N_A, N_B, N_C)^T \rightarrow (e^{ia}E_A, e^{ia}E_B, e^{ia}E_C, N_A, N_B, N_C)^T \quad \forall a \in [0, 2\pi).$$

This transformation corresponds to the same phase shift in all the laser fields, $E_s$, and can be represented by the matrix,

$$R_{\mathbb{S}^1} = \begin{pmatrix} I_3 e^{ia} & 0 \\ 0 & I_3 \end{pmatrix},$$

where $I_3$ is the $3 \times 3$ identity matrix. As a result, the simplest nonzero solution of Eqs. (4.1)–(4.2) is a $\mathbb{S}^1$ group orbit or a relative equilibrium in the form of a limit cycle or a circle of non-hyperbolic equilibria. A group orbit reduction [24] greatly facilitates numerical bifurcation analysis as it allows, for example, periodic $\mathbb{S}^1$ group orbits to be studied as isolated equilibria in the group orbit space. To carry out the group orbit reduction we express the complex-valued electric fields, $E_s(t)$, in terms of their magnitudes, $|E_s(t)|$, and phases, $\varphi_s(t)$:

$$E_s(t) = |E_s(t)|e^{i\varphi_s(t)}.$$ (4.8)

Substituting (4.8) into the electric field Eqs. (4.1), introducing the phase differences,

$$\varphi_{BA} = \varphi_B - \varphi_A, \quad \varphi_{BC} = \varphi_B - \varphi_C,$$ (4.9)

and using (4.5) gives:
\[
\frac{d|E_A|}{dt} = \beta \gamma N_A |E_A| - \kappa |E_B| \sin \varphi_{BA},
\]
\[
\frac{d|E_B|}{dt} = \beta \gamma N_B |E_B| + \kappa |E_A| \sin \varphi_{BA} + \kappa |E_C| \sin \varphi_{BC},
\]
\[
\frac{d|E_C|}{dt} = \beta \gamma N_C |E_C| - \kappa |E_B| \sin \phi_{BC},
\]
\[
\frac{d\varphi_{BA}}{dt} = \kappa \left( \frac{|E_A|}{|E_B|} - \frac{|E_B|}{|E_A|} \right) \cos \varphi_{BA} + \kappa \frac{|E_C|}{|E_B|} \cos \varphi_{BC}
\quad - \alpha \beta \gamma (N_B - N_A) - \Delta,
\]
\[
\frac{d\varphi_{BC}}{dt} = \kappa \left( \frac{|E_C|}{|E_B|} - \frac{|E_B|}{|E_C|} \right) \cos \varphi_{BC} + \kappa \frac{|E_A|}{|E_B|} \cos \varphi_{BA}
\quad - \alpha \beta \gamma (N_B - N_C) - \Delta.
\] (4.10)

The electric field Eqs. (4.10) along with the population inversion Eqs. (4.2) give an eight-dimensional $S^1$-reduced system without $S^1$ symmetry. Clearly, an equilibrium for the $S^1$-reduced system corresponds to a limit cycle ($d\varphi_s/dt \neq 0$ for $s = A, B, C$) or a circle of non-hyperbolic equilibria ($d\varphi_s/dt = 0$ for $s = A, B, C$) for the original system (4.1)–(4.2). One drawback of this approach is that it introduces singularities in the phase difference equations when $|E_s| = 0$. Since a laser field is typically nonzero above the lasing threshold ($\Lambda > 0$), the $S^1$-reduced system (4.2) and (4.10) works well in practice.

In the $S^1$-reduced system the $Z_2$ symmetry is due to the equivariance of the vector field defined by Eqs. (4.2) and (4.10) under the linear transformation,

\[
T_{Z_2} : (|E_A|, |E_B|, |E_C|, \varphi_{BA}, \varphi_{BC}, N_A, N_B, N_C)^T \rightarrow (|E_C|, |E_B|, |E_A|, \varphi_{BC}, \varphi_{BA}, N_C, N_B, N_A)^T.
\] (4.11)

In physical terms, $T_{Z_2}$ corresponds to swapping the outer two lasers, and can be represented by the matrix,

\[
R_{Z_2} = \begin{pmatrix}
A_3 & 0 & 0 \\
0 & A_2 & 0 \\
0 & 0 & A_3
\end{pmatrix},
\] (4.12)

where $A_n$ is an $n \times n$ anti-diagonal identity matrix. $R_{Z_2}$ along with the identity matrix $I_8$ form a representation of the group $Z_2$. For the $S^1$-reduced system the fixed-point subspace [51] due to the $Z_2$ symmetry,

\[
\text{Fix}(Z_2) = \{(|E_A|, |E_B|, |E_C|, \varphi_{BA}, \varphi_{BC}, N_A, N_B, N_C)^T \in \mathbb{R}^8 : |E_A| = |E_C|, \varphi_{BA} = \varphi_{BC}, N_A = N_C\},
\] (4.13)
is a five-dimensional manifold that is invariant under the transformation (4.11) and under the flow given by Eqn. (4.2) and (4.10). A trajectory in the fixed-point subspace, \( x_1(t) \in \text{Fix}(Z_2) \), satisfies

\[
R_{Z_2}x_1(t) = x_1(t) \quad \text{for all } t \in \mathbb{R},
\]  

and is called a fixed trajectory. Fixed trajectories correspond to a situation where the outer lasers have the same field magnitude and phase, and population inversion. Clearly, a closed invariant set, \( X_1 \subset \text{Fix}(Z_2) \), is \( R_{Z_2} \)-invariant,

\[
R_{Z_2}X_1 = X_1,
\]  

and we call it a fixed closed invariant set. A trajectory not in the fixed-point subspace, \( x_1(t) \notin \text{Fix}(Z_2) \), satisfies

\[
R_{Z_2}x_1(t) = x_2(t) \neq x_1(t) \quad \text{for any } t \in \mathbb{R},
\]  

and we call it a conjugate trajectory. Conjugate trajectories corresponds to the situation where the outer lasers have different field magnitudes, phases, and inversions. There are two types of closed invariant sets that are not fixed. The first type is a closed invariant set, \( X_1 \notin \text{Fix}(Z_2) \), that is not \( R_{Z_2} \)-invariant and satisfies

\[
R_{Z_2}X_1 = X_2 \neq X_1,
\]  

where \( X_1 \) and \( X_2 \) are conjugate closed invariant sets. The second type is a closed invariant set, \( X_1 \notin \text{Fix}(Z_2) \), that is \( R_{Z_2} \)-invariant because it satisfies (4.15) rather than (4.17). We call such a set a symmetric closed invariant set. This classification is important in equivariant bifurcation theory where it is made in terms of spatial and spatio-temporal symmetries [51, Ch. 3]. Here, we have used similar terminology to Ref. [75, Ch. 7.4]. In the \( S^1 \)-reduced system (4.2) and (4.10), \( Z_2 \) symmetry places restrictions on the Jacobian of the system that leads to different generic bifurcations for fixed, conjugate, and symmetric closed invariant sets [75]. Bifurcations of conjugate closed invariant sets happen as in systems without symmetry. Also, a bifurcation of \( X_1 \) implies the same bifurcation of the symmetric counterpart \( X_2 \). In the two-parameter diagrams, we use a star (*) to indicate double bifurcations of conjugate equilibria and conjugate limit cycles. However, bifurcations of symmetric and fixed invariant sets are different. For example a pitchfork bifurcation is generic in system (4.2) and (4.10) but only for fixed equilibria as well as fixed and symmetric limit cycles. Also, symmetric limit cycles cannot have the simple
Floquet multiplier of $-1$ and period double (this is true for $\mathbb{Z}_k$ symmetry with even $k$) while fixed and conjugate limit cycles can [75, Ch. 7.4]. Note that a symmetric limit cycle corresponds to the situation where the two outer lasers exchange their role after half a period [75, Ch. 7.4].

Finally, there is the parameter symmetry:

$$ (\varphi_{BA}, \varphi_{BC}, \alpha, \Delta) \rightarrow (-\varphi_{BA} + \pi, -\varphi_{BC} + \pi, -\alpha, -\Delta). \quad (4.18) $$

It implies that if $\alpha = 0$ the bifurcation diagram in the $(\kappa, \Delta)$ plane has reflectional symmetry about the line $\Delta = 0$.

### 4.2 Phase Locking

We are interested in stability of phase-locked solutions,

$$ E_s(t) = |E^0_s|e^{-i(\omega^0 t + \varphi^0_s)}, $$

$$ N_s(t) = N^0_s, \quad \text{for } s = A, B, C, \quad (4.19) $$

where all three lasers oscillate with the same optical frequency $\omega^0$, have constant nonzero magnitudes $|E^0_s|$, constant phase shifts $\varphi^0_s$, and constant population inversions $N^0_s$. A phase-locked solution (4.19) with $\omega^0 \neq 0$ is a limit cycle for the original system (4.1)–(4.2) and an isolated equilibrium for the reduced system (4.2) and (4.10). In the Lyapunov and bifurcation diagrams, parameter regions with stable phase-locked solutions (4.19) are shaded in green.

### 4.3 Overview of the Dynamics in the Coupled-Laser Model

The two Lyapunov diagrams in Fig. 4.1, for $\alpha = 0$ and $\alpha = 1$, give a rough overview of different attractor types and are used to motivate more detailed (bifurcation) analysis. To calculate the Lyapunov diagrams we discretise the $(\kappa, \Delta)$ parameter plane according to the method outlined in Ch. 2.3.4 and use the original system (4.1)–(4.2) to avoid possible singularities at $E_s = 0$ in the vector field of the $S^1$-reduced system (4.2) and (4.10). Owing to the $S^1$ symmetry of the original system there is always one zero Lyapunov exponent that corresponds to the drift along the group orbit. The remaining Lyapunov exponents are used to distinguish between different attractors of the $S^1$-reduced system (4.2) and (4.10) as shown in Table 2.2.
Figure 4.1: Lyapunov diagrams for the $\mathbb{S}^1$-reduced system (4.2) and (4.10) in the $(\kappa, \Delta)$ parameter plane for (a) $\alpha = 0$, and (b) $\alpha = 1$, obtained by decreasing $\kappa$. For the colour coding see Table 2.2.
For $\alpha = 0$, the ($\kappa, \Delta$) plane is dominated by stable limit cycles, where the grey scale quantifies the convergence rate along the leading direction. Stable phase locking occurs in the two green bands around the lines $\kappa = \pm \Delta$, meaning that lasers can phase-lock for any coupling strength, $\kappa$, provided that the frequency detuning, $\Delta$, is sufficiently large. In Fig. 4.1(a) the green bands of phase locking are interrupted with grey intervals due to bistability. Interestingly, we find no stable tori nor chaotic attractors for $\alpha = 0$. Note that the Lyapunov diagram (Fig. 4.1(a)) has reflectional symmetry about the line $\Delta = 0$ in agreement with (4.18). In contrast, for $\alpha = 1$, the ($\kappa, \Delta$) plane is dominated by chaotic attractors, where the yellow-red scale quantifies the associated divergence rate. Furthermore, different regions with stable tori appear as indicated in blue. Stable phase locking is confined to a small parameter region around $\Delta = 0$ and small $\kappa$, meaning that above some critical coupling strength the lasers cannot phase-lock for any $\Delta$. Stable limit cycles are found mainly for small $\kappa$. As expected from the symmetry (4.18), the diagram in Fig. 4.1(b) does not have reflectional symmetry.

Clearly, there are a number of striking differences between Fig. 4.1(a) and (b). Firstly, there is a big difference in the shape and extent of the stable phase locking regions plotted in green. Secondly, we note the appearance of vast chaotic regions for nonzero $\alpha$. Thirdly, different attractor types found in the vicinity of the phase locking regions suggest rather different mechanisms underlying the locking-unlocking transitions. In the following sections we address these three points in more detail by combining Lyapunov exponent calculations, numerical bifurcation continuation [36], and simulations demonstrating $\alpha$-induced phase space stretching and folding.

4.4 Local Bifurcations and Locking

The different types of equilibria of Eqs. (4.2) and (4.10) correspond to different types of phase locking. A fixed equilibrium (4.15) satisfies

$$E_A = E_C \quad \text{and} \quad N_A = N_C,$$

and describes a situation where the outer lasers oscillate in phase with each other but typically out of phase with the middle laser. Fixed-locking regions contain stable fixed equilibria and are indicated by light green shading in the bifurcation diagrams. A conjugate equilibrium (4.17) satisfies

$$E_A \neq E_C \quad \text{and / or} \quad N_A \neq N_C,$$
and describes out-of-phase locking between all three lasers. *Conjugate-locking regions* contain stable conjugate equilibria and are indicated by dark green shading in the bifurcation diagrams. Bifurcations of equilibria define boundaries of the locking regions. Given the \( \mathbb{Z}_2 \) symmetry of Eqs. (4.2) and (4.10), the set of generic codimension-one bifurcations of equilibria includes saddle-node (S), pitchfork (P), and Hopf (H) bifurcations. These bifurcations are two-dimensional surfaces in the three-dimensional \((\kappa, \Delta, \alpha)\) parameter space and can be computed [36] as curves in the \((\kappa, \Delta)\) plane for different but fixed values of \(\alpha\). Crossings or tangencies between different codimension-one bifurcations of the same equilibrium typically give rise to codimension-two bifurcations. They are curves in the \((\kappa, \Delta, \alpha)\) parameter space that indicate changes in the type of the locking boundary. Codimension-two bifurcations include saddle-node-Hopf (SH), pitchfork-Hopf (PH), double-Hopf (HH), and Bogdanov–Takens (BT) bifurcations, and are marked with black dots in the \((\kappa, \Delta)\) plane. In the \((\kappa, \Delta, \alpha)\) parameter space, different bifurcation curves of codimension-two can merge at special points of codimension higher than two. Table 2.1 contains a summary of the bifurcation diagram coding.

### 4.4.1 Bifurcations of Codimensions One and Two

Bifurcation diagrams in the \((\kappa, \Delta)\) plane for different but fixed values of \(\alpha \in [0, 1]\) are shown in Figs. 4.2 and 4.3, where the thick curves indicate bifurcations of stable phase-locked solutions. For \(\alpha = 0\) (Fig. 4.2(a)), we restrict the discussion to the positive half plane owing to the parameter symmetry (4.18). Three different bifurcations of stable phase-locked solutions emerge from the origin, such that stable locking is found within the green-shaded band around \(\kappa = \Delta\). This is similar to the Lyapunov diagram in Fig 4.1(a). The upper boundary of the fixed-locking region (light green) starts at the origin as a saddle-node curve (S) representing bifurcations *within* the fixed-point subspace \(\text{Fix}(\mathbb{Z}_2)\), and switches to a Hopf curve (H) via a saddle-node Hopf (SH) bifurcation point. The lower boundary of the fixed-locking region starts at the origin as a pitchfork curve (P), undergoes a pitchfork-Hopf (PH) bifurcation, and also changes to a Hopf curve. Both Hopf curves extend to large \(\kappa\) where they become parallel to each other giving rise to an unbounded locking region. Hence, stable fixed-locking is possible for any coupling strength, \(\kappa\). In contrast, the conjugate-locking region (dark green) is entirely bounded by a pitchfork, saddle-node, and two Hopf curves, meaning that stable conjugate-locking cannot be achieved above some critical value of \(\kappa\). The supercritical part of the pitchfork curve, between the origin and \(\text{PS}^*\), forms the boundary between fixed and conjugate locking. At \(\text{PS}^*\), the upper boundary of the conjugate-locking region changes to a saddle-node curve, and the pitchfork bifurcation becomes subcritical. This gives rise to a small region of tristability between two conjugate equilibria and a fixed equilibrium.
Figure 4.2: Two-parameter bifurcation diagrams in the $(\kappa, \Delta)$ plane for different values of $\alpha$. Regions of locking are shaded in green. Light and dark shading correspond to different types of phase locking defined by Eqs. (4.20) and (4.21), respectively. Curves represent codimension-one bifurcations and black dots indicate where crossings and tangencies between different bifurcation curves give rise to codimension-two bifurcation points. For the labelling, colour coding, and shading see Table 2.1.

For $\alpha = 0.1$ (Fig. 4.2(b)) the symmetry of the bifurcation diagram is broken in agreement with (4.18). The left boundary of the upper ($\Delta > 0$) locking band is now given by a pitchfork saddle-node point (PS*) that has emerged from the origin, giving rise to an open interval of small $\kappa$ where the lasers can phase-lock only if $\Delta < 0$. The lower boundary of the upper conjugate-locking region involves an additional Hopf curve.
Figure 4.3: Continuation of Fig. 4.2 showing two-parameter bifurcation diagrams in the $(\kappa, \Delta)$ plane for different values of $\alpha$. Curves represent codimension-one bifurcations and black dots indicate where crossings and tangencies between different bifurcation curves give rise to codimension-two bifurcation points. For the labelling, colour coding, and shading see Table 2.1.

due to a codimension-three triple-Hopf bifurcation at $\alpha \approx 0.04$ that is discussed in more detail in Ch. 4.4.2. The lower ($\Delta < 0$) locking region remains qualitatively unchanged at small $\kappa$ but becomes bounded at large $\kappa$ as the two Hopf curves forming its boundary intersect at a double-Hopf point at $(\kappa, \Delta) \approx (92, 91)$ (not shown in the figure).

As $\alpha$ is increased, the pitchfork saddle-node point (PS) indicating the left boundary of the upper ($\Delta > 0$) locking region shifts further away from the origin and reaches
\((\kappa, \Delta) \approx (14.9, 15.1)\) for \(\alpha = 0.2\) (Fig. 4.2(c)). Due to a codimension-three bifurcation involving a double-Hopf (\(\text{HH}^{*}\)) and two saddle-node Hopf (\(\text{SH}^{*}\)) points, there are changes to the boundary of the upper conjugate-locking region at larger \(\kappa\). Namely, it consists of one fewer Hopf curve, one fewer double-Hopf point, and a different saddle-node Hopf point. Concurrently, the double-Hopf point (\(\text{HH}\)) indicating the right boundary of the lower \((\Delta < 0)\) locking region shifts towards the origin. As a result of these transitions, for \(\Delta < 0\) there is an increase in the width of the interval of \(\kappa\) in which stable phase locking is impossible, whereas for \(\Delta > 0\) there is a decrease in the width of the interval of \(\kappa\) in which stable phase locking is possible.

Between \(\alpha = 0.2\) and \(\alpha = 0.3\) the locking region for \(\Delta > 0\) splits into two parts as the \((\kappa, \Delta)\) plane becomes tangent to a minimum of a codimension-two pitchfork-Hopf curve in the \((\kappa, \Delta, \alpha)\) parameter space. This bifurcation is discussed in more detail in Ch. 4.4.2 but its effects can be seen for \(\alpha = 0.3\) (Fig. 4.2(d)). The striking new feature is an open interval of the coupling strength, at around \(25.6 < \kappa < 46.1\), where stable phase locking is no longer possible for any value of the frequency detuning, \(\Delta\). On the right side of this interval, the locking region (dominated by fixed-locking) is found only for \(\Delta > 0\). It is bounded by a pitchfork and two Hopf curves at lower \(\kappa\) but remains unbounded for increasing \(\kappa\). On the left side of this interval there are two separate locking regions. The larger locking region for \(\Delta < 0\) is restricted to smaller values of \(\kappa\) and \(|\Delta|\) than previously but its boundary remains qualitative the same. The smaller locking region for \(\Delta > 0\) is bounded by a pitchfork (\(P\)) and two Hopf curves (\(H^{*}\)), and it is almost vanishing for \(\alpha = 0.3\) (see the inset in Fig. 4.2(d)).

For \(\alpha = 0.425\), neither of the two locking regions for \(\Delta > 0\) are present in the bifurcation diagram (Fig. 4.3(a)). The unbounded locking region has moved to very large values of \(\kappa\) where the model is no longer valid, and the small locking region from the inset in Fig. 4.2(d) has shrunk and disappeared so that the only relevant locking region is the one for \(\Delta < 0\). Its conjugate-locking component forms a thin strip bounded by a pitchfork, saddle-node, and two Hopf curves. The fixed-locking component is much larger and its lower boundary involves a codimension-three bifurcation where a Hopf curve forms a cusp at the saddle-node Hopf bifurcation point. This bifurcation marks the final qualitative change in the boundary of the fixed-locking region for \(\alpha \leq 1\) and is discussed in more detail in Ch. 4.4.2.

While the fixed-locking regions in Fig. 4.3(b)–(d) are qualitatively the same, they still undergo some important quantitative transitions. In particular, for \(\alpha = 0.55\) (Fig. 4.3(b)), the fixed-locking region crosses the \(\Delta = 0\) line making stable phase locking possible for both signs of \(\Delta\) again (Fig. 4.3(c)–(d)). The conjugate-locking region is qualitatively similar to that from Fig. 4.3(a), but has contracted in the \(\Delta\) direction to a very thin strip.
that is hardly visible. For $\alpha = 0.6$ (Fig. 4.3(c)) it shrinks to a tiny region to the right of the Bogdanov–Takens point (BT$^*$) at $(\kappa, \Delta) \approx (22, -5)$, and vanishes for even higher values of $\alpha$ (Fig. 4.3(d)). The key bifurcations responsible for the disappearance of the conjugate-locking region are discussed in Ch. 4.4.2.

Finally, for $\alpha = 1$ (Fig. 4.3(d)) there is only one locking region, and it is of fixed type. It is found for low values of the coupling strength, around $0 < \kappa < 10$, and has a rather wide extent in the frequency detuning, $-15 < \Delta < 10$. Starting at the origin and going counter-clock-wise, its boundary consists of a saddle-node curve (S), two Hopf curves (H), and a pitchfork curve (P). The changes in the boundary type occur via saddle-node Hopf (SH), double-Hopf (HH), and pitchfork Hopf (PH) bifurcations. For even higher values of $\alpha$, the locking region expands along the $\Delta$ direction, and its right boundary shifts towards the origin.

In summary, the bifurcation analysis of phase-locked solutions (8.13), presented in Figs. 4.2 and 4.3, reveal a complicated web of codimension-one bifurcation curves in the $(\kappa, \Delta)$ plane from which we extracted those that form the backbone of the system dynamics and might be of interest for laser applications. The diagrams uncover an intricate structure of fixed- and conjugate-locking regions imposed by the $Z_2$ symmetry. Both locking types are possible for sufficiently small $\alpha$ but a number of higher codimension bifurcations occur for $0 < \alpha < 1$ that drastically modify the shape, extent and number of the locking regions. Ultimately, for sufficiently large $\alpha$, these bifurcations lead to just one (fixed) locking region where the outer lasers oscillate exactly in phase, but are typically out of phase with the middle laser.

### 4.4.2 Bifurcations of Codimension Higher-Than-Two

Following [54, Ch. 3.1], we define the *codimension of a bifurcation* as the lowest dimension of a parameter space that contains the bifurcation in a persistent way. In the $(\kappa, \Delta)$ plane, we identify two types of bifurcation points of codimension higher than two: genuine codimension-three bifurcations, and extrema of codimension-two bifurcation curves in the $(\alpha, \kappa, \Delta)$ parameter space that are tangent to the $(\kappa, \Delta)$ plane. The latter are artifacts of the particular two-dimensional cross section of the three-dimensional parameter space. The bifurcations are in fact codimension-two for a differently defined section. Nevertheless, such extrema result in qualitative changes to the locking regions in the $(\kappa, \Delta)$ plane that are important from the applications viewpoint.

A codimension-three triple-Hopf bifurcation for $\alpha \approx 0.0425$ alters the boundary of the locking region. The bifurcation diagram for $\alpha = 0.03$ in Fig. 4.4(a) is representative of the situation close to, but before the triple-Hopf bifurcation (HH$^*_{123}$). Three different curves of Hopf bifurcation ($H^*_1, H^*_2$ and $H^*_3$) involving conjugate equilibria intersect at three
Figure 4.4: Two-parameter bifurcation diagrams in the \((\kappa, \Delta)\) parameter plane showing an expanded view around the codimension-three triple-Hopf bifurcation point \((HH^*_1)\). Curves represent codimension-one bifurcations and black dots indicate where crossings and tangencies between different bifurcation curves give rise to codimension-two bifurcation points. For the labelling, colour coding, and shading see Table 2.1.

Figure 4.5: Two-parameter bifurcation diagrams in the \((\kappa, \Delta)\) parameter plane showing an expanded view around the codimension-three saddle-node Hopf bifurcation point \((SH)\). Curves represent codimension-one bifurcations and black dots indicate where crossings and tangencies between different bifurcation curves give rise to codimension-two bifurcation points. For the labelling and colour coding, and shading see Table 2.1.

different codimension-two double-Hopf points \((HH^*_1, HH^*_1, HH^*_2)\). In particular, \(H^*_2\) and \(H^*_3\) bound the conjugate-locking region and meet at the corner of this region at \(HH^*_2\). As \(\alpha\) is increased, the three double-Hopf bifurcations move towards each other and meet when \(\alpha \approx 0.0425\) at the triple-Hopf bifurcation \((HH^*_1)\) in Fig. 4.4(b)). At this point, each of the two conjugate equilibria involved has three distinct pairs of purely imaginary eigenvalues. Past the codimension-three bifurcation, the three double-Hopf points move apart so that all three curves, \(H^*_1, H^*_2\) and \(H^*_3\), bound the locking region with corners at \(HH^*_1, HH^*_1\) (Fig. 4.4(c)). A comparison with the bifurcation analysis in Fig. 4.2 reveals that Fig. 4.2(a) and Fig. 4.2(b) are either side of the triple-Hopf bifurcation from
Figure 4.6: Two-parameter bifurcation diagrams in the \((\kappa, \Delta)\) parameter plane showing an expanded view around the codimension-two-plus-one pitchfork-Hopf bifurcation point (PH). Curves represent codimension-one bifurcations and black dots indicate where crossings and tangencies between different bifurcation curves give rise to codimension-two bifurcation points. For the labelling and colour coding, and shading see Table 2.1. For clarity each panel contains an inset with a sketch.

Another codimension-three bifurcation that alters the shape of the locking region is a saddle-node Hopf cusp bifurcation for \(\alpha \approx 0.425\), where a Hopf curve (H) has a cusp at the tangency point with a saddle-node curve (S). In all three panels of Fig. 4.5 the locking region is bounded by saddle-node and Hopf bifurcation curves that are tangent at a saddle-node Hopf point (SH). A stable focus and a saddle-focus are created along the saddle-node curve on the boundary of the fixed-locking region. When moving in a clockwise direction about SH, the stable focus loses stability at a Hopf bifurcation. The locking boundary is smooth below the codimension-three point (Fig. 4.5(a)). However, it becomes piecewise smooth at the codimension-three bifurcation, where the thin and thick branches of H form a cusp in order to swap their relative position (Fig. 4.5(b)). The boundary remains smooth for higher values of \(\alpha\) (Fig. 4.5(c)). Such a codimension-three bifurcation is a transition between different unfoldings of a codimension-two saddle-node Hopf bifurcation (unfolding number four and three in Ref. [75, Fig. 8.17 and Fig. 8.16]) with different type and location of complicated non-stationary dynamics originating from SH. We note that this bifurcation was identified as an important organising centre in optically injected lasers [143] and two-laser systems [141, 41].

As \(\alpha\) is increased, the \((\kappa, \Delta)\) plane passes through extrema of codimension-two bifurcation curves that form part of the locking boundary in the \((\alpha, \kappa, \Delta)\) parameter space. Such transitions explain the splitting of the locking region into separate parts in the \((\kappa, \Delta)\) plane and are illustrated with three examples below. The first example is a minimum of a pitchfork-Hopf bifurcation curve at \(\alpha \approx 0.27\). Just below the minimum, a
stable fixed equilibrium from the fixed-locking region loses stability either via a Hopf or a supercritical pitchfork bifurcation along the curves H and P, respectively (Fig. 4.6(a)). Likewise, a pair of stable conjugate equilibria created by the pitchfork bifurcation along P lose stability via a Hopf bifurcation along H∗. Increasing α causes the curves, H, H∗ and P, to move closer together. When α ≈ 0.27, the curves become tangent so that the locking region is pinched at a pitchfork-Hopf point (PH in Fig. 4.6(b)). This point is a minimum of a pitchfork-Hopf curve in (α, κ, ∆) space. Past the minimum, the curve H∗ splits into two disjoint branches, H∗₁ and H∗₂ (Fig. 4.6(c)). These branches emanate from two pitchfork-Hopf bifurcations (PH) at the intersection points between P and H. Concurrently, the locking region (including its fixed and conjugate components) splits into two separate parts; compare with Fig. 4.2(c)–(d).

The second example is a minimum of a Bogdanov–Takens bifurcation curve at α ≈ 0.5699. Just below the minimum, there is a single locking region where two stable conjugate equilibria are simultaneously created at saddle-node bifurcations along S∗, and lose stability in a Hopf bifurcation along H∗ (Fig. 4.7(a)). When α ≈ 0.5669, the curves S∗ and H∗ become tangent at a Bogdanov–Takens point (BT∗ in Fig. 4.7(b)). This point is a minimum of a Bogdanov–Takens curve in (α, κ, ∆) space. Past the minimum, the curve H∗ splits into two disjoint branches, H∗₁ and H∗₂ (Fig. 4.7(c)). These branches emanate from two BT∗ points at the corners of two separate locking regions; compare with Fig. 4.3(b)–(c). Such a transition was also reported in a laser with time-delayed optical feedback [40].

Finally, we briefly describe an interesting bifurcation transition associated with the
disappearance of the conjugate component of the locking region shown in Fig. 4.3(a). For $\alpha = 0.57$, starting at the origin and moving counter-clockwise (Fig. 4.8(a)), the conjugate-locking component is bounded by a pitchfork curve (P), a pitchfork saddle-node point ($PS^*$), a saddle-node curve ($S^*$), a Bogdanov–Takens point ($BT^*$) (at larger $\kappa$ not shown), and a Hopf curve ($H^*$) extending between BT and the origin. With increasing $\alpha$, the points $PS^*$ and $BT^*$ move toward the origin along P and H respectively. The $PS^*$ point reaches the origin first, when $\alpha \approx 0.5773$ (Fig. 4.8(b)). The conjugate-locking component (too thin to be visible in Fig. 4.8(b)) vanishes as H shrinks due to the $BT^*$ point moving into the origin and colliding with its symmetric counterpart (found for $\kappa < 0$) at a maximum of a Bogdanov–Takens curve in ($\alpha, \kappa, \Delta$) space. Concurrently, the thin and thick branches of P switch their relative position and $PS^*$ moves away from the origin along the thin branch of P (Fig. 4.8(c)).

4.5 Global Bifurcations and Locking-Unlocking Transitions

So far, higher codimension bifurcations of equilibria have been described as evidence for changes in the type and shape of the locking boundary. Such bifurcations play another very important role—they act as \textit{organising centres} [142] linking different types of non-stationary behaviour that are usually found in the vicinity of the locking regions but can also coexist with a stable equilibrium [75]. A different approach to studying laser dynamics involves Lyapunov exponent calculations [23, 18, 80]. In this section, we combine bifurcation analysis and Lyapunov exponent calculations to study (global) bi-
Figure 4.9: (a) Bifurcation diagram, (b) Lyapunov diagram for increasing $\kappa$, and (c) both superimposed in the $(\kappa, \Delta)$ parameter plane for $\alpha = 2$. For the labelling and colour coding see Tables 2.1 and 2.2.
Figure 4.10: One-parameter bifurcation diagram showing the period, $T$, of limit cycles vs. $\Delta$ on approaching the homoclinic bifurcation. Panels (a)–(c) show examples of limit cycles projected onto the $(|E_A|, N_A)$ plane. The dots indicate first steps in the infinite cascade of saddle-node of limit cycles (SL) and period-doubling (PD) bifurcations. The parameter values are as follows: $\alpha = 2$, $\kappa = 3.229$, (a) $(\Delta, T) = (-24.046, 0.377)$, (b) $(\Delta, T) = (-19.580, 0.906)$, and (c) $(\Delta, T) = (-19.239, 2.25)$.

Bifurcations and ensuing multistability involved in the locking-unlocking transitions near certain codimension-two bifurcations. Because of our interest in modern laser applications and the effects of shear, we focus here on semiconductor lasers with $\alpha = 2$.

Figure 4.9 gives a broad overview of the system’s dynamics for weak coupling. The locking region boundary is formed by saddle-node (S), pitchfork (P), and Hopf (H) bifurcations (Fig. 4.9(a)), and it is qualitatively similar to the boundary found in Fig. 4.3(d) for $\alpha = 1$. A comparison between Fig. 4.9(a)–(b) shows that the Lyapunov exponent calculations did not recover all of the locking region obtained from the bifurcation analysis. This mismatch is a clear indication of bistability and can be explained by global homoclinic and heteroclinic bifurcations not studied in Ch. 4.4.

4.5.1 Negative $\Delta$ and Homoclinic Bifurcations

It turns out that a part of the saddle-node bifurcation (S) bounding the locking region at negative $\Delta$ in Fig. 4.9(a) is of global type. It emanates from the origin as a saddle-node homoclinic bifurcation [75, Ch. 7], Shom in Fig. 4.9(c), where a codimension-one homoclinic orbit is tangent to the neutral eigendirection of a non-hyperbolic saddle-node equilibrium. In other words, the saddle-node bifurcation takes place on a limit cycle. At the codimension-two non-central saddle-node homoclinic bifurcation [25, 33] (NC), Shom
Figure 4.11: An expanded view of the \((\kappa, \Delta)\) parameter plane from Fig. 4.9 around the non-central saddle-node homoclinic bifurcation point (NC) showing (a) bifurcation diagram, (b) Lyapunov diagram for increasing \(\kappa\), and (c) both superimposed for \(\alpha = 2\). The plots highlight complicated dynamics near the locking region (green) due to a self-similar cascade of period-doublings and saddle-node of limit cycles associated with the homoclinic bifurcation (hom). For the labelling and colour coding see Tables 2.1 and 2.2.
changes to a local saddle-node bifurcation (S) and meets a codimension-one homoclinic orbit to a saddle-focus bifurcation (hom). Starting from NC, hom extends above S cutting through the tip of the locking region (Fig. 4.9(c)). On approaching the locking region from below, and to the right of the NC point, the saddle-node bifurcation occurs before the homoclinic bifurcation, i.e. it no longer takes place on a limit cycle. Inside the tip of the locking region, the stable equilibrium coexists with (complicated) non-stationary dynamics associated with infinite bifurcation cascades that accumulate onto hom and are described in more detail in Fig. 4.11. Homoclinic orbits Shom and hom are fixed because they are contained within \( \text{Fix}(Z_2) \).

The type of dynamics found near hom depends crucially on the saddle index [144, Ch. 3.2]:

\[
\delta \equiv \frac{\text{Re}(\lambda_s)}{\lambda_u},
\]

(4.22)

where \( \lambda_s \) and \( \lambda_u \) are the stable and unstable central eigenvalues of the saddle-focus, respectively. The part of hom between NC and \( \delta_{-1} \) is the ‘tame’ Shilnikov case with \( \delta < -1 \), where the homoclinic orbit bifurcates into a stable limit cycle found below hom. The point \( \delta_{-1} \) defined by \( \delta = -1 \) is a codimension-two Belyakov bifurcation [13, 48] that marks the transition between the ‘tame’ and ‘chaotic’ Shilnikov cases. In the ‘chaotic’ Shilnikov case, one expects complicated dynamics owing to the existence of infinitely many limit cycles of arbitrary period sufficiently close to hom. Figure 4.10 shows a one parameter bifurcation diagram for the ‘chaotic’ case with fixed \( \kappa = 3.229 \) and varied \( \Delta \), together with examples of limit cycles converging to the homoclinic orbit. For \( -1 < \delta < 0 \) the theory predicts infinitely many turning points along a branch of limit cycles, corresponding to saddle-node of limit cycle bifurcations (SL). As the period, \( T \), of the limit cycles tends to infinity, the characteristic ‘Shilnikov wiggles’ converge from each side to the homoclinic bifurcation. Furthermore, if \( -1 < \delta < -0.5 \), the node-cycles are born stable at SL but may lose stability via period doubling bifurcations (PD). The bifurcation structure in the \( (\kappa, \Delta) \) plane is shown in an expanded view around the bottom corner of the locking region in Fig. 4.11(a). We computed cascades of saddle-node of limit cycle (SL) and period doubling (PD) bifurcation curves that accumulate on to the ‘chaotic’ part of hom. In particular, codimension-two cusp points on the SL curves and folds of the PD curves accumulate onto the Belyakov point, \( \delta_{-1} \), as predicted theoretically in Ref. [48]. Furthermore, each of the PD curves involves (infinitely many) secondary period-doublings (not shown in the figure). Lyapunov exponent calculations in Fig. 4.11(b)–(c) reveal that such accumulating cascades of SL and PD curves give rise to accumulating regions of homoclinic chaos [8, 6] that extend relatively far away from hom (see Fig. 4.14(a)–(b) for
Figure 4.12: One-parameter bifurcation diagram showing the period, $T$, of limit cycles vs. $\Delta$ on approaching the heteroclinic bifurcation. Panels (a)--(c) show examples of limit cycles projected onto the $(N_A, N_C)$ plane. The dots indicate first steps in the infinite cascade of saddle-node of limit cycle (SL) and pitchfork of limit cycle (PL) bifurcations. The parameter values are as follows: $\alpha = 2$, $\kappa = 3.321875$, (a) $(\Delta, T) = (13, 1.286)$, (b) $(\Delta, T) = (12.497, 2.459)$, and (c) $(\Delta, T) = (12.470, 8)$.

It is worth noting that another region of chaos, found in the upper-right corner in Fig. 4.11(b)--(c), is associated with a break-up of an invariant torus. The torus bifurcation curve (T) extends between the saddle-node Hopf bifurcation point (SH) and the 1:2 resonance point. Along T, a stable limit cycle created at the Hopf bifurcation involved in SH, bifurcates to a stable torus. The Lyapunov exponent calculations in Fig. 4.11(b)--(c) reveal quasi-periodic oscillations, some of the infinitely many Arnold tongues [75, Ch. 7] with periodic dynamics, and large regions of chaos to the right of T.

### 4.5.2 Positive $\Delta$ and Heteroclinic Bifurcations

A considerable difference in the locking region between Fig. 4.9(a)--(b) is found for positive $\Delta$. This difference is due to a heteroclinic bifurcation [75, Ch. 6], het in Fig. 4.9(c). Along het, two codimension-one conjugate heteroclinic connections between two conjugate saddles form a non-robust and symmetric (i.e. $R_{\mathbb{Z}^2}$-invariant) heteroclinic cycle. An individual heteroclinic connection within the cycle is a relative homoclinic connection, meaning that it is a homoclinic connection to a saddle in the $\mathbb{Z}_2$ group orbit space. Hence, Shilnikov theorems for homoclinic bifurcations should apply here. The curve het emanates from the origin and involves simple saddles that become saddle-foci at $\kappa \approx 1$. 

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Figure 4.13: An expanded view of the $(\kappa, \Delta)$ parameter plane from Fig. 4.9 around the relative Shilnikov–Hopf point (ShH$^*$) showing (a) bifurcation diagram, (b) Lyapunov diagram for increasing $\kappa$, and (c) both superimposed for $\alpha = 2$. The plots highlight a difference in the locking region (green) revealed by the bifurcation analysis and the Lyapunov exponent computations which is due to the heteroclinic bifurcation (het). For the labelling and colour coding see Tables 2.1 and 2.2.
Along the ‘tame’ part of het, the homoclinic orbit bifurcates into a stable limit cycle that is found above het. The het curve changes its type from ‘tame’ to ‘chaotic’ through a codimension-two Belyakov bifurcation ($\delta_{-1}$) and meets a curve of Hopf bifurcations ($H^*$) at a codimension-two relative Shilnikov–Hopf bifurcation point (ShH*) [63, 22]. At ShH* both conjugate saddles within the heteroclinic cycle undergo a Hopf bifurcation giving rise to a heteroclinic cycle with four point-to-orbit connections.

Figure 4.12 shows a one parameter bifurcation diagram for the ‘chaotic’ case with fixed $\kappa \approx 3.322$ and varied $\Delta$, together with examples of limit cycles converging to the heteroclinic orbit. As the period of the limit cycles, $T$, tends to infinity, the characteristic ‘Shilnikov wiggles’ that involve a cascade of saddle-node of limit cycle bifurcations (SL) converge from each side to the heteroclinic bifurcation as in Fig. 4.10. The slower rate of convergence that decreases towards ShH* is a consequence of imaginary parts of the central stable complex conjugate eigenvalues being large compared to the real part of the leading unstable eigenvalue [144, Ch. 3.2]. An important difference from the homoclinic bifurcation is the absence of period doubling bifurcations on the branch of limit cycles plotted in Fig. 4.12. This is because the limit cycles created in the saddle-node bifurcations (SL) are symmetric, and according to equivariant bifurcation theory cannot period double [75, Ch. 7.4]. Such cycles can only undergo saddle-node, pitchfork, and torus bifurcations, and so the period doubling bifurcations from Fig. 4.10 are replaced by a cascade of pitchfork bifurcations in Fig. 4.12. In the ($\kappa, \Delta$) plane, we calculated cascades of saddle-node of limit cycle (SL) and pitchfork of limit cycle (PL) bifurcation curves that accumulate onto the ‘chaotic’ part of het (Fig. 4.13(a)). Lyapunov exponent calculations in Fig. 4.13(b)–(c) reveal that cascades of SL and PL curves are associated with a self-similar cascade of chaotic regions that too accumulate on het. In particular, chaos is found to the right of the PL curves in Fig. 4.13(c) where two conjugate limit cycles bifurcate from the symmetric limit cycle in a pitchfork bifurcation and then undergo a period-doubling cascade to heteroclinic chaos (see Fig. 4.14(c)–(d) for an example of such a chaotic attractor).

It is worth noting another chaotic region in Fig. 4.13(c) that does not seem to be related to het. This region has hardly visible periodic windows on the scale chosen here and its boundary aligns with bifurcations of limit cycles (Fig. 4.13(c)). Starting from the top, the boundary aligns with a subcritical pitchfork bifurcation (PL) and then changes to saddle-node bifurcation (SL) at the codimension-two pitchfork-saddle-node of limit cycle bifurcation (PSL*). Both boundary types correspond to an intermittent transition between periodic oscillations and chaos [96, Ch. 8.2].

The overall picture is that the two global bifurcation curves roughly split the locking region into three parts. The middle part is comprised of monostable locking with the ex-
ception of small regions near to the curves het and hom. The upper and lower parts are multistable, meaning that they involve additional attractor(s) that coexists with the stable equilibrium. In the vicinity of the locking region there are a number of chaotic regions associated with different bifurcation scenarios. Specifically, we identified accumulating cascades of period-doublings near hom and het, break-up of invariant tori born along the bifurcation curve originating from the saddle-node Hopf bifurcation on the locking boundary, and (intermittent) limit cycle to chaos transitions due to subcritical pitchfork and saddle-node bifurcations of limit cycles.

4.6 Coupling-Induced Chaotic Attractors

In the previous section we found that for $\alpha$ sufficiently large there are extensive regions of chaos in the $(\kappa, \Delta)$ parameter plane (Fig. 4.1). This is potentially attractive for a number of laser applications that require the lasers to be operating in a chaotic regime. It is also a technologically relevant setting which offers the possibility to study persistent [3, 2] (or robust) chaos. However, as we show in this section, different chaotic regions can be associated with topologically different chaotic attractors that originate from different routes to chaos. Figure 4.14 shows three examples of chaotic attractors with different topologies. The first example (Fig. 4.14(a)–(b)) is found close to a homoclinic bifurcation. Its shape resembles the homoclinic orbit and such an attractor is often referred to as homoclinic chaos. The shape of the chaotic attractor in the second example resembles the shape of the nearby heteroclinic orbit (Fig. 4.14(c)–(d)) and such an attractor is often referred to as heteroclinic chaos. The third example is quite different from the first two in that it does not seem to resemble any regular shape at all (Fig. 4.14(e)–(f)). It is clear that developing techniques for detecting bifurcations of chaotic attractors is important if we want to fully utilise the chaotic nature of lasers.

4.7 Finding Chaos in Larger Arrays of Coupled Laser Oscillators

Thus far we have concentrated on an array of three coupled laser oscillators. We found no regions of chaos in the $(\kappa, \Delta)$ plane for $\alpha = 0$, whereas for $\alpha > 0$ and sufficiently large, the $(\kappa, \Delta)$ plane is dominated by regions containing chaotic attractors (Fig. 4.1). This suggests, in line with previous studies [139], that $\alpha$ is a key parameter for the creation of chaotic attractors in laser systems. In this section we test whether the above observation holds for larger arrays of coupled laser oscillators. The three coupled-laser model (4.1)–
Figure 4.14: Examples of different chaotic attractors for the coupled-laser Eqs. (4.2) and (4.10) with $\alpha = 2$ shown as two-dimensional phase space projections (left column) and the corresponding time series of $|E_A|$ (right column). For (a)–(b) $(\kappa, \Delta) = (3.229, -19.2804)$; (c)–(d) $(\kappa, \Delta) = (3.321875, 12.5275)$; and (e)–(f) $(\kappa, \Delta) = (10, 0)$. Approximations to the largest Lyapunov exponents are 0.18, 1.22, and 10.7 respectively.
Figure 4.15: Lyapunov exponent diagrams in the $(\kappa, \Delta = \Omega_{in} - \Omega_{out})$ plane illustrating attractors that are chaotic (yellow-red) and those that are not (black) for (a)–(b) five lasers and (c)–(d) ten lasers. For (a) & (c) $\alpha = 0$ while for (b) & (d) $\alpha = 2$. 
(4.2) is easily generalised to $M$ coupled lasers (8.1)–(8.2). To allow for a comparison with the three coupled-laser model where we considered identical outer lasers, and to facilitate the presentation of our results, we assume that all inner lasers have the same natural frequency, $\Omega_{in}$, as do the pair of outer lasers, $\Omega_{out}$.

In Fig. 4.15 we present two-parameter Lyapunov exponent diagrams that indicate regions of the $(\kappa, \Delta = \Omega_{in} - \Omega_{out})$ plane containing chaotic attractors (yellow-red) and non-chaotic attractors (black) for (a)–(b) five and (c)–(d) ten coupled laser oscillators. For $\alpha = 0$ ((a)&(c)) the $(\kappa, \Delta)$ plane contains some regions of chaos. Emerging from the origin of the $(\kappa, \Delta)$ plane, and approximately along the lines $|\Delta| = \kappa$, are two bands that contain chaotic attractors. There is also a region of chaos centred about $\Delta = 0$ for $\kappa \approx 30$. This region is considerably larger for the array comprised of ten laser oscillators. The maximum largest Lyapunov exponent for five laser oscillators is $\max_{(\kappa, \Delta)} \mu_1 = 3.4$, while for ten laser oscillators it is $\max_{(\kappa, \Delta)} \mu_1 = 5.14$. For $\alpha = 2$ ((b)&(d)), the $(\kappa, \Delta)$ plane for five and ten coupled laser oscillators is dominated by regions containing chaotic attractors. In fact, regions containing non-chaotic attractors are confined to approximately $\kappa < 10$. The maximum largest Lyapunov exponent for five laser oscillators is $\max_{(\kappa, \Delta)} \mu_1 = 15.5$, while for ten laser oscillators it is $\max_{(\kappa, \Delta)} \mu_1 = 19.4$.

From these results we conclude that it is possible to find chaotic attractors for $\alpha = 0$ by increasing the array size (five was the smallest in which we found chaotic attractors). However, the extent and intensity of the chaos (indicated by the largest Lyapunov exponent) is much smaller than that found in arrays with larger $\alpha$. Further research could be conducted into exactly how chaos develops within the $(\kappa, \Delta)$ plane as $\alpha$ is increased. In the next section we provide insight as to why positive $\alpha$ is conducive to the creation of chaos.

4.8 Shear-Induced Chaos

The rate equations for a single-mode laser (2.1)–(2.2) in polar coordinates are given by

$$\frac{d|E|}{dt} = \beta \gamma N |E|,$$

$$\frac{d\varphi}{dt} = -\alpha \beta \gamma N - (\Omega - \nu),$$

$$\frac{dN}{dt} = \Lambda - N - (1 + \beta N)|E|^2,$$

and define a three-dimensional dynamical system with $\mathbb{S}^1$ symmetry. Since the $\mathbb{S}^1$ group orbit space is just two-dimensional (Eqn. (4.24) decouples from (4.23) and (4.25)) the single-laser model cannot admit chaotic solutions. The equilibrium, at the origin of the
complex $E$-plane, is globally stable for $-1 < \Lambda < 0$ and turns unstable via a supercritical Hopf bifurcation (if $\Omega - \nu \neq 0$) when $\Lambda = 0$. The stable limit cycle exists for $\Lambda > 0$ as a unique, asymptotically stable attractor (see Ch. 2.1.2).

However, it has been well established, both experimentally and theoretically, that a class-B laser can display a variety of instabilities and chaos in response to external perturbations [123]. Complex nonlinear dynamics and chaos have been reported for $\alpha$ large enough but very little or no chaos at all have been found for $\alpha \sim 0$ [74]. The same is true for the coupled laser system at hand, where for $\alpha$ sufficiently large, extensive regions of chaos emerge in the $(\kappa, \Delta)$ parameter plan (Fig. 4.15).

Previous studies and previous sections of this thesis focus on different approaches to quantify externally induced chaotic attractors (for example Lyapunov exponent calculations, bifurcation or asymptotic analysis), but little is understood as to why they appear. From an applications viewpoint, this question is of interest for the analysis of large arrays of semiconductor lasers, where the parameter space is largely dominated by chaotic attractors and their bifurcations. We now identify the special properties of the laser phase space and of the external perturbation that may produce chaos.

Figure 4.16: Time evolution of sets of initial conditions showing the creation of horseshoes in the phase space of a suitably kicked oscillator with no shear. The sets of initial conditions are (top) the stable red circle and (bottom) the blue and green sets containing parts of the circle. Shown are phase portraits (a) before and (b)–(c) after the first kick.
Figure 4.17: Snapshots at times (a) $t = 0$, (b) $t = 0.28$, (c) $t = 0.56$, and (d) $t = 1$ showing the time evolution (black curve) of 15000 initial conditions, initially distributed equally over the stable red circle $\Gamma$, in the solitary-laser model (4.23)–(4.25) with kicks applied at $t = 0, 0.25, 0.5, 0.75$. A comparison between $\alpha = 0$ and $\alpha = 2$ illustrates the $\alpha$-induced stretch-and-fold action in the laser phase space. See [17] or the multimedia part of [16] for the full-time simulation.
In their recent work, Wang & Young [135] prove that, when suitably perturbed, any stable hyperbolic limit cycle can be turned into ‘observable’ chaos (a strange attractor). This result is derived for periodic but discrete-time perturbations (kicks) that deform the stable limit cycle of the unkedicked system. The key concept is the creation of Smale horseshoes [117] via a stretch-and-fold action due to an interplay between the kicks and the local geometry of the phase space. Depending on the degree of shear, quite different kicks are required to create a stretch-and-fold action and horseshoes. Intuitively, it can be described as follows. In systems without shear, where points in phase space rotate with the same angular frequency independent of their distance from the origin, the kick alone has to create the stretch-and-fold action. This is demonstrated in Fig. 4.16. Horseshoes are formed ((a2)–(c2)) as the system is suitably kicked in both radial and angular directions ((b1) & (b2)) and then relaxes back to the attractor (red curve) of the unkedicked system ((c1) & (c2)). Repeating this process reveals chaotic invariant sets, however proving whether or not a specific kick results in ‘observable’ chaos is a non-trivial task requiring the techniques developed in [134]. In the presence of shear, where points in phase space rotate with different angular frequencies depending on their distance from the origin, the kick does not have to be so specific or carefully chosen. In fact, it may be sufficient to kick non-uniformly in the radial direction alone, and rely on natural forces of shear to provide the necessary stretch-and-fold action.

These effects are illustrated in a single laser model (4.23)–(4.25) with non-uniform kicks in the radial direction alone for $\alpha = 0$ (no shear) and $\alpha = 2$ (shear) in Fig. 4.17. When $\nu = \Omega$, there is a stable circle of non-hyperbolic equilibria which we plot in red and refer to as $\Gamma$. Kicks modify the electric field magnitude, $|E|$, by a factor of $0.8 \sin(4\varphi)$ at times $t = 0, 0.25, 0.5, \text{and} 0.75$ but leave the phase, $\varphi$, unchanged. For $\alpha = 0$ each point on the black curve spirals onto $\Gamma$ in time but remains in the same radial plane defined by a constant phase $\varphi = \varphi(0)$. Hence, the black curve does not have any folds at any time. However, for $\alpha = 2$, points on the black curve with larger magnitudes $|E(t)|$ rotate with larger angular frequencies. This gives rise to an intricate stretch-and-fold action that is strongly enhanced by the spiralling transient motion about $\Gamma$ [139]. Folds and horseshoes are formed under the evolution of the flow even though the kicks are in the radial direction alone; the full-time simulation can be viewed at [17] or in the multimedia part of [16].

It is important to note that the rigorous results for turning stable limit cycles into chaotic attractors are derived for periodic and discrete-time perturbations. Coupled laser systems have continuous-time perturbations that may not be periodic, meaning that the analysis in [135] cannot be directly applied to our problem. Nonetheless, it gives a new valuable insight as to why vast regions of chaos appear for $\alpha$ sufficiently large.
Additionally, it shows that creating ‘observable’ chaos for $\alpha = 0$ may be difficult but not impossible because lasers are perturbed in both radial and angular directions (see Eqs. (4.10)). In fact, small regions of chaos occur in a linear arrays of five or more coupled laser oscillators with $\alpha = 0$ (Fig. 4.15).
Chapter 5

Composite-Cavity Mode Model

In the previous chapter we discussed the coupled-laser model (4.1)–(4.2) which is based on the field expansion (2.16) in terms of passive optical modes of the uncoupled lasers. Such an ODE model is relatively simple, independent of the physical coupling realisation (side-to-side or face-to-face), but limited to weak coupling between lasers. This chapter discusses the composite-cavity mode model that approximates more accurately the electromagnetic wave equation (2.15) by fully taking into account the coupling between lasers in the spatial part of the problem [113, 27]. As such, it serves as a benchmark against which the simpler coupled-laser model can be compared.

In the composite-cavity mode approach, laser coupling is fully included in solving the homogeneous wave equation (2.15). The solutions $U_j(r)$ are then the passive optical modes for the entire coupled-laser system, and are referred to as composite-cavity modes [26, 112, 113, 141, 41] or supermodes [69]. Nonlinear interactions between composite-cavity modes rather than individual lasers is the main conceptual difference from the coupled-laser approach. The composite-cavity mode approach can describe both physical realisations of optical coupling from Fig. 2.4 [28, Ch. 7-6]. However, each physical realisation has to be treated as a separate problem because it will involve a different refractive-index function, $n(r)$, and different boundary conditions.

Here, we consider three single-mode lasers coupled side-to-side (Fig. 2.4(a)). The lasers have the shape of rectangular bars of width $w_A = w_C = 4 \, \mu m$ and $w_B = (4 + \Delta w) \, \mu m$, and are placed a distance, $d$, apart in the $x$ direction. They have the same height, $h$, in the $y$ direction, and the same length, $L$, in the $z$ direction [42]. To simplify the analysis, we follow [28, Ch. 7-6] and use the effective index approximation to give

$$U_j(r) = X_j(x) Y(x, y) Z(z),$$

where $Y$ has a weak dependence on $x$. We assume a standing wave solution in the $z$
direction,
\[ Z(z) = \sin(k_z z) \quad \text{for} \quad z \in [0, L], \]
where \( L = j\pi/k_z \), \( j \) is an integer, and \( k_z = 5\pi \times 10^6 \text{ m}^{-1} \) is the \( z \)-component of the total wavevector,
\[ k_j = \frac{\tilde{\Omega}_j}{c} = \sqrt{k_{x,j}^2 + k_y^2 + k_z^2}. \]

Then, we focus on a one-dimensional Helmholtz equation for the \( x \) direction,
\[
\left[ \frac{\partial^2}{\partial x^2} + n_{\text{eff}}^2(x) \frac{\tilde{\Omega}_j^2}{c^2} - k_z^2 \right] X_j(x) = 0, \tag{5.1}
\]
that follows from Eqn. (2.19). The effective refractive index, \( n_{\text{eff}}(x) \), differs slightly from the actual refractive index, \( n(x) \), and we assume a piecewise constant function
\[
n_{\text{eff}}(x) = \begin{cases} 
3.61 & \text{in lasers A, B and C}, \\
3.6 & \text{outside the lasers}. 
\end{cases} \tag{5.2}
\]

The electromagnetic theory requires that \( X_j(x) \), and its first derivative with respect to \( x \), are continuous at each laser boundary (defined by a discontinuity in the effective refractive index). Additionally, we require that \( X_j(x) \) tends to zero as \( x \to \pm\infty \). As in Ref. [42], we seek analytical solutions \( X_j(x) \) to (5.1) in the form of sine and cosine functions within the lasers and exponential decays outside the lasers. Given such boundary conditions, the solutions \( X_j(x) \) satisfy the orthogonality relation,
\[
\int_{-\infty}^{\infty} n_{\text{eff}}^2(x) X_j(x) X_{j'}(x) dx = \delta_{jj'} N_x,
\]
where we choose the normalisation constant
\[
N_x = \frac{n_g^2}{2} (3w_0 + 2d_0),
\]
with \( w_0 = 4\mu\text{m} \) and \( d_0 = 4\mu\text{m} \). The component \( Y(x, y) \) is obtained separately for a laser bar and passive sections outside lasers (hence the weak dependence on \( x \)), and \( Y(x, y) \) tends to zero as \( y \to \pm\infty \) [28].

Given \( U_j(r) \), one can substitute the expansion (2.16) into the inhomogeneous wave equation (2.15) and use (2.21) to obtain a set of ODEs for the time evolution of the
normalised slowly-varying composite-cavity mode fields, \( E_j(t) \) [41]:

\[
\frac{dE_j}{dt} = - [i(\Omega_j - \nu) + \gamma \sum_{j'} \left\{ \sum_s K_{jj'}^s [1 + \beta N_s] - i \alpha \beta (1 + N_s) \right\}] E_{j'}. \tag{5.3}
\]

The normalised population inversion, \( N_s \), in laser \( s = A, B, C \), evolves accordingly by [41]

\[
\frac{dN_s}{dt} = \Lambda - (N_s + 1) - \sum_{j,j'} K_{jj'}^s (1 + \beta N_s) \text{Re} [E_j E_{j'}]. \tag{5.4}
\]

Clearly, Eqs. (5.3)–(5.4) describe nonlinear interactions of composite-cavity modes rather than individual lasers. The model has no limitations on the coupling strength as different composite-cavity modes become the modes of a single large-area laser in the limit of maximum coupling given by \( d \to 0 \). Another difference from the coupled-laser model is that physical coupling parameters such as the laser distance, \( d \), and the laser width difference, \( \Delta w \), enter Eqs. (5.3)–(5.4) implicitly via the 18 modal integrals

\[
K_{jj'}^s = \frac{n_l^2}{N} \int_s U_j(r)U_{j'}(r)dr = \Gamma \frac{n_l^2}{N_x} \int_x X_j(x)X_{j'}(x)dx. \tag{5.5}
\]

Here, the integration extends over the volume \((dr)\) or width \((dx)\) of the respective laser \( s \) with the effective refractive index, \( n_l \). The confinement factor, \( \Gamma \), quantifies the normalised overlap between a passive composite-cavity mode and a laser \( s \) in the \( y \) and \( z \) directions. From a physics viewpoint, a diagonal element, \( K_{jj'}^s \), quantifies the spatial contribution from laser \( s \) to the amplification and \( \alpha \)-induced frequency shift of mode \( j \). An off-diagonal element, \( K_{j\neq j'}^s \), quantifies the spatial contribution to nonlinear interactions between composite-cavity modes \( j \) and \( j' \) within laser \( s \) (resulting in coupling-induced frequency shift or competition for example).

Bifurcation analysis of the composite-cavity mode model is more complicated than of the coupled-laser model owing to the implicit dependence of Eqs. (5.3)–(5.4) on the bifurcation parameters \( d \) and \( \Delta w \). In fact, one needs to set up a continuation problem where solutions to Eqs. (5.3)–(5.4) are continued simultaneously with solutions to Eqs. (5.1) and modal integrals (5.5).
5.1 Passive Composite-Cavity Modes

For our system of three coupled lasers with $w_A = w_C$, it follows from Eqn. (5.1) that the $x$ component of the composite-cavity mode $X_j(x)$, is either symmetric

$$X_j(x) = X_j(-x),$$

(5.6)

or anti-symmetric

$$X_j(x) = -X_j(-x).$$

(5.7)

Henceforth, we study nonlinear interactions of three composite-cavity modes: two with a symmetric $x$ component, namely $X_1(x)$ and $X_3(x)$, and one with an anti-symmetric $x$ component, namely $X_2(x)$. The three functions $X_j(x)$, where $j = 1, 2$ and $3$, are shown in Fig. 5.1 for: $\Delta w = -0.05 \mu m$, where the two outer lasers are wider than the middle laser ($(a1)-(a3)$); $\Delta w = 0 \mu m$, where all lasers are identical ($(b1)-(b3)$); and $\Delta w = 0.05 \mu m$, where the middle laser is wider than the two outer lasers ($(c1)-(c3)$). Figure 5.1(d) shows the nonlinear dependence of the composite-cavity mode frequencies, $\tilde{\Omega}_j$, on the laser distance, $d$, and the width difference, $\Delta w$. For small $d$, the frequency separation between different composite-cavity modes is large and weakly dependent on $\Delta w$. As $d$ decreases, the separation becomes smaller. Specifically, for $\Delta w > 0$ composite-cavity modes 1 and 2 have a similar frequency, whereas for $\Delta w < 0$ composite-cavity modes
Figure 5.2: Modal integrals (a) $K_{11}^A$, (b) $K_{12}^A$, and (c) $K_{13}^A$ as a function of the laser distance, $d$, and the laser width difference, $\Delta w$.  

2 and 3 have a similar frequency. Changes in $\Delta w$ have a much stronger effect on the frequency of composite modes 1 and 3 than on the frequency of composite mode 2. This is because composite mode 2 has vanishing amplitude in the middle laser for all $\Delta w$.

The spatial symmetries specified by Eqs. (5.6) and (5.7) impose the following relations between $K_{jj'}^s$:

\[ K_{12}^B = K_{23}^B = 0, \]
\[ K_{11}^A = K_{22}^C, \quad K_{22}^A = K_{22}^C, \quad K_{33}^A = K_{33}^C, \]
\[ K_{12}^A = -K_{12}^C, \quad K_{23}^A = -K_{23}^C, \quad K_{13}^A = K_{13}^C, \]  

(5.8)

that lead to just 11 (10 nonzero and one zero) independent modal integrals. For small laser distance, $d$, $K_{jj'}^s$ have a weak dependence on $\Delta w$. However, $K_{jj'}^s$ become strongly nonlinear functions of $\Delta w$ at large $d$. Figure 5.2 illustrates this behaviour with the three integrals for composite-cavity mode 1 in laser $A$. In particular, $K_{11}^A$ large for $\Delta w > 0$, and vanishing for $\Delta w < 0$ indicates that mode 1 is localised in lasers $A$ and $C$ for $\Delta w > 0$ (Fig. 5.2(a)). The integral $K_{22}^A$ is almost constant and $K_{33}^A$ and $K_{11}^B$ are similar to $K_{11}^A$ when $\Delta w$ is replaced with $-\Delta w$ (not shown). Also, $K_{33}^B$ is similar to $K_{11}^A$, and $K_{22}^B$ remains close to zero (not shown). The integral $K_{12}^A$ indicates an abrupt transition from strong to weak spatial coupling between modes 1 and 2 as $\Delta w$ decreases through zero (Fig. 5.2(b)). Similar behaviour is found for $K_{23}^A$ when $\Delta w$ is replaced with $-\Delta w$ (not shown). A strong spatial coupling between modes 1 and 3 in lasers $A$ and $C$ for $\Delta w \approx 0$ decays rapidly to zero as $|\Delta w|$ increases (Fig. 5.2(c)). Finally $K_{13}^A$ is similar to $-K_{13}^B$.  

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5.2 Symmetry Properties

Similarly to the coupled-laser model, the composite-cavity mode model (5.1)–(5.4) has $S^1 \times Z_2$ symmetry.

The (continuous) $S^1$ symmetry is due to the equivariance of the vector field defined by the Eqs. (5.3)–(5.4) under the transformation,

$$ T_{S^1} : (E_1, E_2, E_3, N_A, N_B, N_C)^T \rightarrow (e^{ia}E_1, e^{ia}E_2, e^{ia}E_3, N_A, N_B, N_C)^T \quad \forall a \in [0, 2\pi). $$

(5.9)

This transformation corresponds to the same phase shift in the fields of all the composite-cavity modes, $E_j$. To facilitate numerical bifurcation analysis, we need to carry out a group orbit reduction. However, we cannot use the same reduction approach as in Ch. 4.1 for the coupled-laser model. This is because the composite-cavity mode fields, $E_j(t)$, vanish on open sets of parameters, where algebraic singularities in the equations for the phase differences prevent numerical bifurcation analysis. An alternative way to reduce $S^1$ symmetry is to introduce new coordinates, namely $Z_{11}, Z_{22}, Z_{33}, Z_{12}, Z_{13},$ and $Z_{23}$, which are defined by

$$ Z_{jj'} = E_j \overline{E}_{j'}, $$

(5.10)

and form a set of $S^1$-invariant monomials [52, 24] (where the bar denotes complex conjugation). This allows periodic $S^1$ group orbits of (5.3)–(5.4) to be studied as isolated equilibria of the the reduced system, that is obtained by replacing the $dE_j/dt$ equations with

$$ \frac{dZ_{jj'}}{dt} = \frac{dE_j}{dt} \overline{E}_{j'} + E_j \frac{d\overline{E}_{j'}}{dt}, $$

(5.11)

and replacing $E_j \overline{E}_{j'}$ with $Z_{jj'}$ in Eqs. (5.4). Such a reduced system does not have any algebraic singularities and greatly facilitates numerical continuation. However, one drawback of this approach is that the monomials (5.10) are not all independent and the reduced system is of higher dimension than the original system. This gives rise to additional eigenvalues and the possibility of ‘bogus’ bifurcations, that are not present in the original system, when these additional eigenvalues cross through zero.

The (discrete) $Z_2$ symmetry is due to the equivariance of the vector field defined by Eqs. (5.3)–(5.4) under the linear transformation,

$$ T_{Z_2} : (E_1, E_2, E_3, N_A, N_B, N_C)^T \rightarrow (E_1, -E_2, E_3, N_C, N_B, N_A)^T, $$

(5.12)

which corresponds to swapping the two outer lasers. This equivariance is made more transparent by substituting relations (5.8) into Eqs. (5.3)–(5.4).
In the composite-cavity mode model, the representation of the $\mathbb{Z}_2$ symmetry is given by the identity matrix $I_9$ and

$$R_{\mathbb{Z}_2} = \begin{pmatrix}
I_2 & 0 & 0 & 0 \\
0 & -I_2 & 0 & 0 \\
0 & 0 & I_2 & 0 \\
0 & 0 & 0 & A_3
\end{pmatrix},$$

(5.13)

where $A_3$ is the $3 \times 3$ anti-diagonal matrix. The fixed-point subspace due to the $\mathbb{Z}_2$ symmetry,

$$\text{Fix}(\mathbb{Z}_2) = \{(E_1, E_2, E_3, N_A, N_B, N_C)^T \in \mathbb{R}^9 : \ E_2 = 0, \ N_A = N_C\},$$

(5.14)

is a 6-dimensional manifold that is invariant under the transformation (5.12) and under the flow given by Eqs. (5.3)–(5.4). Given that $U_2(r) \neq 0$ in lasers A and C, it follows that lasers A and C have identical electric fields if and only if $E_2 = 0$. Hence, (5.14) is equivalent to (4.13). As in Ch. 4.1 for the coupled laser model, we can distinguish between fixed solutions, that lie within the fixed-point subspace, Fix($\mathbb{Z}_2$), and conjugate solutions that are not in the fixed-point subspace. Specifically, fixed solutions for the composite-cavity mode model have $E_2(t) = 0$, meaning that there are only contributions from the two symmetric composite modes 1 and 3, and the electric fields in the outer lasers are identical. For conjugate solutions, the anti-symmetric composite mode has a nonzero field, $E_2(t) \neq 0$, and the electric fields in the outer lasers are different.

### 5.3 Phase Locking

In the composite-cavity model, locking of all three lasers to the same optical frequency can be represented by two solution types. One type is a single composite-cavity mode solution. Another type is a multi-composite-cavity mode solution where all the lasing modes are phase-locked to a common frequency $\omega^0$ with a possible constant phase shift $\varphi_j^0$:

$$E_j(t) = |E_j^0|e^{-i(\omega^0 t + \varphi_j^0)}, \quad \text{for } j = 1, 2, 3,$$

$$N_s(t) = N_s^0, \quad \text{for } s = A, B, C.$$

(5.15)

In the bifurcation and Lyapunov diagrams, the locking regions are shaded in green.
5.4 Dynamics of the Composite-Cavity Mode Model

Figure 5.3 shows the bifurcation diagrams and the locking regions in the \((\kappa, \Delta w)\) plane for four different values of the \(\alpha\)-parameter. To facilitate a comparison with the coupled-laser model from Ch. 4, we introduce the normalised coupling strength,

\[ \kappa = Ce^{-pGd}, \]

where the coupling rate, \(C = 420\), and the inverse coupling length, \(p_G = 0.98 \mu m^{-1}\), were calculated in Ref. [41] for weakly coupled lasers. The laser width difference, \(\Delta w\), is related to the frequency detuning, \(\Delta\), between the middle and two outer lasers.

For \(\alpha = 0\) there are two locking bands off the line \(\Delta w = 0\) (Fig. 5.3(a)). Inside the fixed-locking region (light green), the anti-symmetric composite-cavity mode has zero magnitude, \(|E_2(t)| = 0\), and composite modes 1 and 3 are phase-locked to a single frequency. Such a two-composite-mode locking means that the outer lasers \(A\) and \(C\) have identical electric fields. When crossing the pitchfork bifurcations (P) from a light green to a dark green region, an additional solution with \(|E_2(t)| > 0\) bifurcates off the fixed-point subspace defined by (5.14). In other words, the anti-symmetric composite-cavity mode 2 moves from below to above its lasing threshold. Hence, in the conjugate locking regions (dark-green), none of the phase-locked lasers oscillates in phase with another laser.

As the \(\alpha\)-parameter is increased, the locking regions undergo a number of qualitative and quantitative changes. Specifically, the locking region at \(\Delta w > 0\) disappears (Fig. 5.3(b)) at around \(\alpha = 0.2\). The locking region at \(\Delta w < 0\) becomes bounded towards increasing \(\kappa\), and its conjugate component disappears before \(\alpha\) reaches 0.5 (Fig. 5.3(c)). For \(\alpha > 0.5\), there is only a single (fixed) locking region bounded by a saddle-node curve, two Hopf curves, and a pitchfork curve. This locking region shifts towards \(\Delta w = 0\) and expands along the \(\Delta w\) direction with increasing \(\alpha\) (Fig. 5.3(d)).

On the level of bifurcations of (relative) equilibria, the simpler coupled-laser model shows an impressive qualitative and good quantitative agreement with the composite-cavity model, provided that the analysis is restricted to weak coupling between lasers and that the appropriate normalisation of the coupling strength is known. The locking regions of both models have very similar structures, are bounded by the same bifurcation types, and have codimension-two points occurring for similar values of \(\kappa\). Nonetheless, the agreement is not as ‘perfect’ as for the two-laser system studied in Ref. [41]. We note that the bifurcation diagram in Fig. 5.3(a) for \(\alpha = 0\) does not have a reflectional symmetry about \(\Delta w = 0\). In particular, the upper locking band does not emerge from the origin and the lower locking band is noticeably larger. At any value of the \(\alpha\)-parameter, the lower locking band remains relatively larger and extends to higher values of \(\kappa\) than
Figure 5.3: Two-parameter bifurcation diagrams in the \((\kappa, \Delta w)\) plane. Regions of locking are shaded in green. Light and dark shading correspond to fixed and conjugate locking (5.15), respectively. From (a)–(d) \(\alpha = 0, 0.2, 0.5\) and 1. Compare with Figs. 4.2–4.3 for the coupled-laser model (4.2) and (4.10). For the labelling and colour coding see Table 2.1.
Figure 5.4: Superposition of the Lyapunov diagram and bifurcation diagram for the composite-cavity mode model (5.3)–(5.4) with $\alpha = 2$. Compare with Fig. 4.9(c) for the coupled-laser model (4.2) and (4.10).
in the simpler coupled-laser model.

The results obtained from the composite-cavity mode model depend on whether the total width of the system remains constant. Here, the total width of the system varies with \( \Delta w \) and Eqs. (5.3)–(5.4) do not have the parameter symmetry under the transformation \( \Delta w \to -\Delta w \). In particular, there is stronger mode amplification owing to larger total laser volume for negative \( \Delta w \). The coupled-laser model neglects these spatial effects which results in the parameter symmetry (4.18).

A comparison between Fig. 5.4 and Fig. 4.9(c) shows that the agreement between the two models holds as well for the global bifurcations described in Ch. 4.5, and for complicated solutions outside the locking region. Specifically, the upper locking boundary in the Lyapunov diagram in Fig. 5.4 coincides with the curve of heteroclinic bifurcations (het) that extends between the origin and the codimension-two Shilnikov-Hopf bifurcation point (ShH*). This heteroclinic bifurcation gives rise to a region of multistability with locked and unlocked dynamics coexisting between het and the upper pitchfork bifurcation curve (P). Towards decreasing \( \Delta w \), the locking region is bounded by the saddle-node homoclinic bifurcation (Shom) that meets the homoclinic bifurcation (hom) at the codimension-two non-central saddle-node homoclinic point (NC). Furthermore, there is a great similarity on the level of quasi-periodic and chaotic dynamics including the intricate structure in the Lyapunov diagram associated with bifurcation cascades accumulating onto hom and het.
Chapter 6

Conclusion for Part II

We studied nonlinear dynamics of three linearly coupled laser oscillators using $S^1$ symmetry reduction, complementary methods for stability analysis, multi-parameter study, and different modelling approaches. Preliminary studies suggest that the three-laser system is the smallest array size that shares many features with larger arrays. Understanding its nonlinear behaviour is therefore a step towards understanding nonlinear behaviour in larger arrays.

In a three-parameter study of the coupled-laser model, Lyapunov and bifurcation diagrams in the plane of the laser-coupling strength, $\kappa$, and frequency detuning, $\Delta$, were calculated for different but fixed values of the third parameter, $\alpha$, which quantifies the coupling between the magnitude, $|E_s|$, and phase, $\text{arg}(E_s)$, of each laser’s complex-valued field, $E_s$. In this way, we uncovered two striking results with increasing $\alpha$: severe changes to the shape and extent of the locking regions (where all three lasers oscillate at the same frequency), and emergence of vast regions of chaos. On the one hand, bifurcation analysis explained the intricate and changing shape of phase locking regions in terms of codimension-two and -three bifurcations. The analysis also highlighted the importance of global homoclinic and heteroclinic bifurcations associated with multistability and complicated locking-unlocking transitions. On the other hand, the $\alpha$-induced stretch-and-fold action creating horseshoes in the laser phase space gave an intuitive explanation for the appearance of vast regions of chaos. The emerging complicated web of codimension-one bifurcations and their infinite cascades linked together via bifurcations of higher codimension, often called organizing centres, provided the backbone of the coupled-laser dynamics.

The coupled-laser model was then compared with a conceptually different but more accurate composite-cavity model. Such a comparison was motivated by a need to further understand whether the simpler coupled-laser model captures accurately the essential (optical) nonlinearities of laser arrays. The main discrepancies in the results from the two models are the size of the locking regions for $\alpha < 0.5$ and the lack of reflectional symmetry.
in the \((\kappa, \Delta)\) plane for \(\alpha = 0\) (compare Fig. 4.2 with Fig. 5.3). These discrepancies are due to the fact that the coupled-laser model does not accurately take into account the spatial structure of the laser array. Despite this we found a very good agreement on the level of locking regions, local and global bifurcations, and chaotic dynamics (compare Fig. 4.9 with Fig. 5.4). Our results support the ‘simple’ coupled-laser approach to modelling arrays of weakly coupled lasers. They also show that understanding large semiconductor laser systems will require (bifurcation) analysis of chaotic attractors that occupy much of the parameter space.
Part III

From Locking to Optical Turbulence in the Coupled-Laser Model
Chapter 7

Introduction for Part III

In Part III we propose a definition for laser synchronisation and then study the synchronisation properties of a linear array of coupled laser oscillators with nearest-neighbour coupling. Our focus is on different synchronisation types with dependence on the number of lasers, $M$, the laser-coupling strength, $\kappa$, the laser frequency detunings, $\Delta_j$, and the coupling between the magnitude and phase of each lasers electric field, $\alpha$. Conditions are derived that guarantee the existence of synchronised solutions where all the lasers phase-lock and have the same intensity. Stability conditions are obtained analytically for two special cases of such synchronisation: (i) where all the lasers oscillate in-phase with each other, and (ii) where all the lasers oscillate in anti-phase with their direct neighbours. We then focus on an array of three semiconductor lasers (with $\alpha = 2$) and study the transitions from complete synchronisation (where all the lasers synchronise), to partial synchronisation (where only a subset of the lasers synchronise), to optical turbulence (where no lasers synchronise and each laser is chaotic). The bifurcations responsible for these transitions are identified—many of which are symmetry breaking bifurcations. In particular, when the lasers are chaotic and there is partial synchronisation we find that blowout bifurcations are responsible for changes in laser synchronisation. Finally, we investigate the effect of adding more lasers to the array, and discuss its consequence in relation to persistent optical turbulence.

Chapter 8

Coupled-Laser Model for $M$ Laser Oscillators

To study laser synchronisation we extend the coupled-laser model to $M$ lasers. The justification for this extension is the good agreement shown between the coupled-laser model (Ch. 4) and the more accurate composite-cavity mode model (Ch. 5). Rate equations for the normalised slowly-varying complex-valued electric fields, $E_j$, are given by

$$\frac{dE_j}{dt} = \beta \gamma (1 - i \alpha) N_j E_j - i \Delta_j E_j + i \kappa (E_{j-1} + E_{j+1}) \quad \text{for} \quad j = 2, \ldots, M-1, (8.1)$$

and for the normalised real-valued population inversions, $N_j$,

$$\frac{dN_j}{dt} = \Lambda - N_j - (1 + \beta N_j) |E_j|^2 \quad \text{for} \quad j = 1, \ldots, M, (8.2)$$

where $t$ is the normalised time. The coupling strength, $\kappa$, is assumed to be of equal strength between neighbouring lasers and can be scaled (Eqs. (4.3)-(4.4)) to represent both coupling configurations in Fig. 2.4. The frequency detuning, $\Delta_j = \Omega_j - \nu$, is the frequency difference between the normalised natural frequency, $\Omega_j$, of laser $j$ and a suitably chosen reference frequency, $\nu$. The $\alpha$-parameter is the coupling between the magnitude, $|E_j|$, and phase, $\arg(E_j)$, of each laser’s electric field, $E_j$ (see Ch. 2.1.3). The remaining parameters are the normalised gain coefficient, $\beta$, the ratio of photon and population inversion decay rates, $\gamma$, and the pump rate, $\Lambda$, which are kept constant at typical values given in Table A.1 of the Appendix.
8.1 Coupled Oscillator Representation

For the following analysis it is convenient to write Eqs. (8.1)–(8.2) as the following system of ordinary differential equations, which we refer to as the \( M \) coupled-laser model:

\[
\frac{dx_j}{dt} = f_j(x_j) + \kappa \sum_{j'=1}^{M} G_{jj'}Hx_{j'},
\]

where the laser index \( j = 1, \ldots, M \). The three components of the state vector, \( x_j(t) \), are the real and imaginary parts of the normalised complex-valued slowly-varying electric field, \( E_j(t) = A_j(t) + iB_j(t) \), and the real-valued normalised population inversion within the laser active medium, \( N_j(t) \). The vector field \( f_j \) is given by

\[
f_j(x_j(t)) = \begin{pmatrix}
\beta \gamma N_j(t)(A_j(t) + \alpha B_j(t)) + \Delta_j B_j(t) \\
\beta \gamma N_j(t)(B_j(t) - \alpha A_j(t)) - \Delta_j A_j(t) \\
\Lambda - N_j(t) - (1 + \beta N_j(t))(A_j(t)^2 + B_j(t)^2)
\end{pmatrix},
\]

and the coupling is through the same linear combination of the laser variables,

\[
H = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

We would like to remark that other forms of \( H \) have been used in the literature, some of which even lead to unbounded solutions as pointed out in Ref. [41]. Here, we use the form that has been shown in Part II of this thesis and in [109, 41] to be consistent with the solution of Maxwell’s equations and the correct boundary conditions. The coupling configuration is described by the elements of the \( M \times M \) connectivity matrix, \( G_{jj'} \). For the chain configuration with nearest-neighbour coupling found in commercially available laser arrays (Fig. 2.4),

\[
G_{jj'} = \begin{cases}
1 & \text{if } |j - j'| = 1 \\
0 & \text{otherwise}
\end{cases}.
\]

It is worth noticing that the \( M \) coupled-laser model (8.3) can be easily transformed into the widely studied form of diffusive coupling by replacing \( \Delta_j \) with

\[
\hat{\Delta}_j = \begin{cases}
\Delta_j - \kappa & \text{if } j = 1, M \\
\Delta_j - 2\kappa & \text{if } 1 < j < M
\end{cases}.
\]
and replacing $G_{jj'}$ with the *diffusive connectivity matrix*,

$$
\hat{G}_{jj'} = \begin{cases} 
-1 & \text{if } j = j' = 1, M \\
-2 & \text{if } 1 < j = j' < M \\
G_{jj'} & \text{otherwise}
\end{cases}
$$

(8.7)

### 8.2 Motivating Example

The $M$ coupled-laser model (8.3) supports two contrasting behaviour types (see Fig. 8.1). On the one hand, in Fig. 8.1(a) an array of 50 lasers shows strong synchronisation where the magnitudes of all the laser fields, $|E_j(t)|$, are identical and given by a constant. Such solutions are usually studied for identical lasers with different coupling geometries for example in a ring [76, 116, 77, 71], globally coupled [76, 116, 4], and linear arrays (or chains) [146, 1, 45, 20, 42, 16]. An important difference in this study is that we allow each laser oscillator to have a different natural frequency, $\tilde{\Omega}_j$, thereby providing wider possibilities of finding synchronised solutions like that shown in Fig. 8.1(a).

On the other hand, in Fig. 8.1(b), again for an array of 50 lasers, each laser is chaotic in time and there is no apparent synchronisation between any of the lasers. This is an example of what we refer to as optical turbulence. Such solutions are potentially useful for laser radars [78], chaos-based secure communication [131, 59, 7], and ultra-fast random number generation [129, 103]. All of these applications may benefit from different properties of the underlying chaotic attractors.

Before we explore the two contrasting behaviour types shown in Fig. 8.1, and the transitions between them, we first need to fix a suitable definition of laser synchronisation.

### 8.3 Definition of Laser Synchronisation

When lasers are coupled together it is natural to ask if some form of synchronisation is possible. We say that lasers $j$ and $j'$ are *synchronised* if two conditions are satisfied. Firstly, that there exists a stable, fixed-in-time relationship between their slowly-varying electric fields, such that

$$
 c_j E_j(t) - c_{j'} E_{j'}(t) = 0,
$$

(8.8)

for some constants $c_j, c_{j'} \in \mathbb{C}$. Secondly, that this relationship is attractive meaning that

$$
 \lim_{t \to \infty} (c_j E_j(t) - c_{j'} E_{j'}(t)) = 0,
$$

(8.9)
Figure 8.1: Contrasting dynamics in an array of 50 lasers with nearest-neighbour coupling. The distribution of natural frequencies, $\Delta_j$, is such that $\Delta_{1,50} = \Delta_{\text{out}}$ and $\Delta_{2,\ldots,49} = \Delta_{\text{in}}$. (a) Complete intensity synchronisation where each laser is in anti-phase with its nearest neighbour(s) ($\alpha = 2$, $\kappa = 1$ and $\Delta_{\text{in}} - \Delta_{\text{out}} = -1$). (b) Optical turbulence where no lasers are synchronised and each laser is chaotic ($\alpha = 5$, $\kappa = 30$ and $\Delta_{\text{in}} - \Delta_{\text{out}} = 0$).

for all initial conditions within some $3M$-dimensional subset of the phase space.

A particular type of synchronisation called phase locking is often considered in the literature. To study phase locking one needs to define a suitable phase variable of an oscillator [99, Ch. 2]. The phase of a laser oscillator can be defined straightforwardly as $\phi_j(t) = \arg(E_j(t))$. We say that lasers $j$ and $j'$ are phase-locked if there is a stable, fixed-in-time relationship between their phases, and this relationship is attractive:

$$\lim_{t \to \infty} (\phi_{jj'}(t) \equiv \phi_j(t) - \phi_j'(t)) = \text{const. mod } 2\pi .$$

Equation (8.10) is consistent with the previously introduced phase-locked solutions (4.19) in Ch. 4.2. Notice that if two lasers are synchronised according to Eqs. (8.8)–(8.9) then they are also phase-locked (8.10).

Furthermore, we distinguish between different types of synchronisation depending on the number of lasers that are synchronised, and the relationships between their phases and magnitudes. If all the lasers are synchronised we speak of complete synchronisation, but if only a subset of the lasers are synchronised we have partial synchronisation. Furthermore, if (8.8)–(8.9) are satisfied we speak of in-phase synchronisation if $\phi_{jj'} = 0$, anti-phase synchronisation if $\phi_{jj'} = \pi$, and out-of-phase synchronisation otherwise. Finally, if the
electric field magnitudes for lasers \( j \) and \( j' \) are identical, \(|E_j(t)| = |E_{j'}(t)|\), then we say that the lasers are \textit{intensity synchronised}.

### 8.4 Symmetries and Their Role in Synchronisation

The vector field \( f_j \) describing the uncoupled dynamics of an individual laser is rotationally symmetric (or equivariant see Ch. 2.3.2) under phase shifting the electric field, \( E_j(t) \), by a constant amount, \( \theta_j \),

\[
T : \quad E_j \rightarrow e^{i\theta_j} E_j \quad \text{for} \quad \theta_j \in [0, 2\pi).
\]

This transformation can be represented by the \( 3 \times 3 \) matrix,

\[
R_j = \begin{pmatrix}
\cos \theta_j & -\sin \theta_j & 0 \\
\sin \theta_j & \cos \theta_j & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

and we refer to \( R_j \) as an \textit{internal symmetry} of laser \( j \). If \( \kappa = 0 \) then the \( M \) coupled-laser model (8.3) has \( \prod_{j=1}^{M} S^1 \) symmetry due to the \( M \) rotational symmetries of the individual lasers. This is represented by the \( 3M \times 3M \) matrix,

\[
R = \begin{pmatrix}
R_1 & 0 & \cdots & 0 \\
0 & R_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & R_M
\end{pmatrix}.
\]

However, if \( \kappa > 0 \) then the \( M \) coupled-laser model (8.3) has only \( S^1 \) symmetry which is also represented by the matrix transformation \( R \) but with \( \theta_j = \theta \) for all \( j \). In this case the simplest nonzero asymptotic solutions lie on \( S^1 \)-group orbits and are typically limit cycles. Such solutions are given by

\[
E_j(t) = |E_j^0| e^{i(\omega t + \varphi_j^0)},
\]

\[
N_j(t) = N_j^0,
\]

for \( j = 1, \cdots, M \), where \(|E_j^0|\), \( \varphi_j^0 \), \( \omega \), and \( N_j^0 \) are real constants. When these solutions are stable they represent a form of complete synchronisation and are also referred to as \textit{locking} (compare with Ch. 4.2) since all the lasers phase-lock (8.10) to the same optical frequency \( \omega \). As in Ch. 4.1, a group orbit reduction can be used to obtain a \( S^1 \)-\textit{reduced}
Table 8.1: Dynamical properties associated with the fixed-point subspace. Column two: colours indicate which lasers (circles) are necessarily intensity and in-phase synchronised if a solution of the full system is contained in Fix($\mathbb{Z}_2$). Column three: coupling structure for the governing equations (8.3) restricted to Fix($\mathbb{Z}_2$).

<table>
<thead>
<tr>
<th>Odd number of lasers $M=2L-1$</th>
<th>Solution of full system contained in Fix($\mathbb{Z}_2$)</th>
<th>Governing equations restricted to Fix($\mathbb{Z}_2$)</th>
</tr>
</thead>
</table>
| $\begin{array}{c}
1 \\
\cdots \\
L \\
\cdots \\
2L-1 \\
\end{array}$ | $\begin{array}{c}
1 \\
\cdots \\
L \\
\cdots \\
2L-1 \\
\end{array}$ | $\begin{array}{c}
1 \\
\cdots \\
L \\
\end{array}$ |
| Even number of lasers $M=2L$ | $\begin{array}{c}
1 \\
\cdots \\
L \\
\end{array}$ | $\begin{array}{c}
1 \\
\cdots \\
L \\
\end{array}$ |

system with the $\mathbb{S}^1$ symmetry removed [16]. Synchronised solutions (8.13) are studied as isolated equilibria in the $\mathbb{S}^1$-reduced system which greatly facilitates analytical and numerical stability analysis.

We will consider cases where the $M$ coupled-laser model (8.3) has additional $\mathbb{Z}_2$ symmetry because its right-hand side is equivariant under the transformation,

$$
T_{\mathbb{Z}_2} : E_j \rightarrow E_{M+1-j},
$$

$$
N_j \rightarrow N_{M+1-j},
$$

for $j = 1, \ldots, M$. The $\mathbb{Z}_2$ symmetry implies the existence of an invariant manifold called the fixed-point subspace [51],

$$
\text{Fix}(\mathbb{Z}_2) = \{(x_1, \cdots, x_M)^T \in \mathbb{R}^M : x_j = x_{M-j+1} \text{ for } j = 1, \ldots, M\}.
$$

On this manifold pairs of lasers are synchronised with each other. More specifically, identical colours in Table 8.1 show which lasers (circles) are necessarily intensity and in-phase synchronised if a solution of the full system in contained in Fix($\mathbb{Z}_2$). There is a subtle difference between arrays of odd and even lasers. For $M = 2L - 1$ odd, there is a unique middle laser $j = L$ which need not be synchronised with any other laser, whereas for $M = 2L$ even, no such laser exists. This difference shows up when the governing equations (8.3) are restricted to the 3L-dimensional fixed-point subspace by setting $x_j = x_{M-j+1}$ for $j = 1, \ldots, M$. In both cases the restricted equations describe the dynamics of $L$ coupled lasers but there is a difference in the coupling terms. For $M$ odd, the laser $L - 1$ receives twice as much coupling from laser $L$, whereas for $M$ even, laser $L$ is coupled to itself. In the third column of Table 8.1 this difference in the coupling terms is indicated by red arrows.
Chapter 9

Complete Intensity Synchronisation

In this chapter we study complete intensity synchronisation where the electric field magnitudes of all the lasers are identical,

\[
|E_1(t)| = \cdots = |E_M(t)| = |E_s(t)|. \tag{9.1}
\]

The analysis is performed in two steps. First we derive conditions for the existence of different synchronised solutions. Then we give conditions for the synchronised solutions to be stable, and identify mechanisms responsible for synchronisation-desynchronisation transitions.

9.1 Existence of Synchronised Solutions and Manifolds

The aim of this section is to find all solutions of the \( M \) coupled-laser model (8.3) that satisfy the complete intensity synchronisation condition (9.1), meaning that the \( c_j \)'s in (8.8)–(8.9) become \( c_j = e^{i\theta_j} \) for \( j = 1, \cdots, M \). Let us assume a *synchronised solution* of the form

\[
e^{i\theta_1}E_1(t) = \cdots = e^{i\theta_M}E_M(t) = E_s(t) \neq 0, \tag{9.2}
\]

for all time \( t \). Differentiating the above equation with respect to time shows that (9.2) implies

\[
e^{i\theta_j} \frac{dE_j}{dt} = e^{i\theta_j} \frac{dE_j'}{dt} = \frac{dE_s}{dt}, \tag{9.3}
\]
for \( j, j' \in \{1, \ldots, M\} \) and all \( t \). Using (8.3) for the first two components of \( x_j \) and \( f_j(x_j) \), substituting (9.2), and taking the real and imaginary parts of (9.3) gives

\[
N_j(t) - N_j'(t) = \frac{-1}{\beta \gamma (1 - \alpha^2)} (\alpha (\Delta_j - \Delta_j') - \kappa (\alpha a_{jj'} + b_{jj'})),
\]

(9.4)

\[
0 = \frac{1}{\beta \gamma (1 - \alpha^2)} ((\Delta_j - \Delta_j') - \kappa (a_{jj'} - \alpha b_{jj'})),
\]

where

\[
a_{jj'} + ib_{jj'} = \sum_{j''=1}^{M} (G_{jj''} e^{i\theta_{jj''}} - G_{j'j''} e^{i\theta_{j'j''}}),
\]

and \( \theta_{jj'} = \theta_j - \theta_{j'} \). Furthermore, noting that the right-hand side of (9.4) does not vary in time, comparing the time derivative of (9.4) with (8.3) for the third component of \( x_j \) and \( f_j(x_j) \), and using (9.2) gives

\[
-(N_j(t) - N_j'(t)) - \beta |E_s(t)|^2 (N_j(t) - N_j'(t)) = 0,
\]

(9.5)

for \( j, j' \in \{1, \ldots, M\} \) and all \( t \). Equation (9.5) holds if and only if

\[
N_1(t) = \cdots = N_M(t) = N_s(t),
\]

(9.6)

for all \( t \). Note that (9.6) implies that

\[
\frac{dN_1}{dt} = \cdots = \frac{dN_M}{dt} = \frac{dN_s}{dt},
\]

(9.7)

for all \( t \), which is satisfied assuming (9.2) and (9.6). So far, we have shown by taking the real part of (9.3) that complete intensity synchronisation (9.2) implies equal levels of population inversion in all the lasers (9.6). The next step is to derive the corresponding conditions for \( \theta_j \) and \( \Delta_j \) that need to be satisfied for synchronised solutions (9.2) to exist.

To make the analysis more general and facilitate the presentation of results we use the internal symmetries \( R_j \) to define new variables,

\[
y_j = R_j x_j,
\]

and rewrite the \( M \) coupled-laser model (8.3) as

\[
\frac{dy_j}{dt} = f_j(y_j) + \kappa \sum_{j'=1}^{M} G_{jj'} R_j H R_{j'}^{-1} y_{j'},
\]

(9.8)
for \( j = 1, \cdots, M \). Furthermore, we rewrite the synchronised solution (9.2) and (9.6) as

\[
y_1 = \cdots = y_M = y_s, \tag{9.9}
\]

and conditions (9.3) and (9.7) as

\[
\frac{dy_1}{dt} = \cdots = \frac{dy_M}{dt} = \frac{dy_s}{dt}. \tag{9.10}
\]

It now becomes clear that if (9.9) is satisfied for some initial time \( t_0 \) and (9.10) is satisfied for all \( t \), then the synchronised solution \( y_s \) defined by (9.9) exists for all \( t \). To satisfy (9.10) for all \( t \) we require that

\[
f_j(y_s) - f_{j'}(y_s) = \kappa \sum_{j''=1}^{M} \left( G_{jj''} R_{j''} - G_{j''j} R_{j} R_{j''}^{-1} \right) y_s, \tag{9.11}
\]

for \( j, j' \in \{1, \cdots, M\} \). Setting \( j' = 1 \), using matrices (8.5) and (8.6), condition (9.11) can be expressed in terms of \((M - 1)\) equations:

\[
(\Delta_1 - \Delta_j) H y_s = \kappa \left( R_1 R_2^{-1} - R_j R_{j-1}^{-1} - \hat{\delta}_j R_j R_{j+1}^{-1} \right) H y_s, \tag{9.12}
\]

where \( j = 2, \cdots, M \) and \( \hat{\delta}_j = 1 - \delta_j \). Alternatively, (9.12) can be viewed as \((M - 1)\) eigenvalue problems:

\[
\lambda_j v = A_j v,
\]

where \( \lambda_j = (\Delta_1 - \Delta_j)/\kappa \), \( v \in \mathbb{R}^2 \), \( A_j \) is the \( 2 \times 2 \) matrix,

\[
A_j = \begin{pmatrix}
\cos \theta_{1,2} - \cos \theta_{j,j-1} - \delta_{j} \cos \theta_{j,j+1} & \sin \theta_{1,2} + \sin \theta_{j,j-1} + \delta_{j} \sin \theta_{j,j+1} \\
\sin \theta_{1,2} - \sin \theta_{j,j-1} - \delta_{j} \sin \theta_{j,j+1} & \cos \theta_{1,2} - \cos \theta_{j,j-1} - \delta_{j} \cos \theta_{j,j+1}
\end{pmatrix},
\]

and \( \theta_{j,j'} = \theta_j - \theta_{j'} \). The corresponding \((M - 1)\) eigenvalues are given by

\[
\frac{\Delta_1 - \Delta_j}{\kappa} = \cos \theta_{1,2} - \cos \theta_{j,j-1} - \delta_{jM} \cos \theta_{j,j+1} + i(\sin \theta_{1,2} - \sin \theta_{j,j-1} - \delta_{jM} \sin \theta_{j,j+1}), \tag{9.13}
\]

with \( j = 2, \cdots, M \). Taking the imaginary part of (9.13) gives

\[
\hat{\delta}_j = \sin \theta_{j,j+1} = \sin \theta_{1,2} - \sin \theta_{j,j-1}, \tag{9.14}
\]
for \( j = 2, \ldots, M \). One can show, using induction for \( j = 2, \ldots, M - 1 \), that (9.14) implies

\[
\sin \theta_{1,2} = \begin{cases} 
\frac{1}{j} \sin \theta_{j,j+1} & \text{for } j = 2, \ldots, M - 1 \\
\sin \theta_{M,M-1} & \text{for } j = M.
\end{cases}
\] (9.15)

By setting \( j = M - 1 \) in (9.15) one can see that \( \sin \theta_{1,2} = 0 \), which then implies that

\[
\theta_j - \theta_{j+1} = K_j \pi \mod 2\pi,
\] (9.16)

for \( K_j \in \{0, 1\} \) and \( j = 1, \ldots, M - 1 \). Taking the real part of (9.13) and using (9.16) gives

\[
\Delta_1 - \Delta_j = \kappa \left[ ( -1)^{K_1} - ( -1)^{K_{j-1}} - \hat{\delta}_{jM}( -1)^{K_j} \right].
\] (9.17)

for \( j = 2, \ldots, M \). Note that each synchronised solution is characterised by a different choice of \( K_j \)'s in (9.16), and the corresponding condition (9.17). These results mean that complete intensity synchronisation (9.2) requires (i) a difference of an integer multiple of \( \pi \) between the phases of any two neighbouring lasers (9.16), and (ii) the corresponding relation between the frequencies of individual lasers (9.17).

The synchronised solution conditions (9.2) and (9.6) (or (9.9)) define a three-dimensional invariant manifold:

\[
\mathcal{M}_s = \{ \mathbf{y} = (y_1, \ldots, y_M)^T \in \mathbb{R}^M : y_1 = \cdots = y_M = y_s \},
\] (9.18)

that is referred to as the synchronisation manifold [68]. Depending on the \( \theta_j \)'s in (9.16) and the corresponding \( \Delta_j \)'s in (9.17), a synchronised solution (9.2) may have certain symmetries. In particular, if both the \( \theta_j \)'s and \( \Delta_j \)'s satisfy the pattern in the second column of Table 8.1, then solution (9.2) has \( \mathbb{Z}_2 \) symmetry and \( \mathcal{M}_s \subseteq \text{Fix}(\mathbb{Z}_2) \). However, (9.16) shows that this need not be the case: the chain configuration depicted in Fig. 2.4 allows for synchronised solutions (9.2) without \( \mathbb{Z}_2 \) symmetry [39]. Furthermore, within the synchronisation manifold (9.8) reduces to the solitary-laser equations (2.1)–(2.2) with a modified frequency detuning. The concept of a synchronisation manifold facilitates stability analysis that is performed in the next sections.

### 9.2 Stability of Special Synchronised Solutions

We are interested in the stability of two special cases that are found in numerical simulations, are potentially interesting for applications, and can be treated analytically. The first case, given by setting \( K_j = K = 0 \) for all \( j \) in (9.16), describes a situation when all
lasers are intensity synchronised in-phase with each other,

\[ E_1(t) = \cdots = E_M(t) = E_s(t). \]  

(9.19)

Using (9.17), such solutions require that

\[ \Delta_1 = \Delta_M = \Delta_{\text{out}}, \]  

(9.20)

\[ \Delta_j = \Delta_{\text{in}}, \]  

(9.21)

for \( j = 2, \ldots, M - 1 \), and

\[ \Delta_{\text{in}} - \Delta_{\text{out}} = \kappa. \]  

(9.22)

The second case, given by setting \( K_j = K = 1 \) for all \( j \) in (9.16), describes a situation when every laser is intensity synchronised anti-phase with its neighbours,

\[ E_1(t) = -E_2(t) = \cdots = (-1)^{M-1}E_M(t) = E_s(t). \]  

(9.23)

Using (9.17), this requires that (9.20) and (9.21) are satisfied but with

\[ \Delta_{\text{in}} - \Delta_{\text{out}} = -\kappa. \]  

(9.24)

The conditions on the \( \Delta_j \)'s namely (9.20)–(9.21) and (9.22), or (9.20)–(9.21) and (9.24) , guarantee the existence of synchronised solutions of the forms (9.19) and (9.23) respectively. However, they do not imply laser synchronisation as defined in (8.8)–(8.9). For this, synchronised solutions need to be stable in the directions transverse to the corresponding synchronisation manifold.

With \( \theta_j \)'s satisfying (9.16) for \( K_j = K \in \{0, 1\} \), Eqs. (9.8) can be written as

\[ \frac{dy_j}{dt} = f_j(y_j) + \kappa \sum_{j'=1}^M G_{jj'} R_K H_j y_{j'}, \]  

(9.25)

where we used (9.16) to introduce

\[ R_K = R_j R_{j+1}^{-1} = \begin{pmatrix} (-1)^K & 0 & 0 \\ 0 & (-1)^K & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]  

(9.26)
Equations (9.25) can be rewritten as

\[
\frac{dy_j}{dt} = f_j(y_j) + \kappa n_j R_K H y_j + \kappa \sum_{j'=1}^{M} \hat{G}_{jj'} R_K H y_{j'}, \tag{9.27}
\]

where \(\hat{G}_{jj'}\) are elements of the diffusive connectivity matrix (8.7), and \(n_j = \sum_{j'=1}^{M} G_{jj'}\) is the number of inputs that laser \(j\) receives from its neighbours. Furthermore, we introduce

\[
f_K(y_j) = f_j(y_j) + \kappa n_j R_K H y_j, \tag{9.28}
\]

for \(j = 1, \ldots, M\), and rewrite (9.27) as

\[
\frac{dy_j}{dt} = f_K(y_j) + \kappa \sum_{j'=1}^{M} \hat{G}_{jj'} R_K H y_j, \tag{9.29}
\]

for \(j = 1, \ldots, M\). On the synchronisation manifold Eqs. (9.29) reduce to

\[
\frac{dy_s}{dt} = f_K(y_s), \tag{9.30}
\]

because \(\sum_{j'} \hat{G}_{jj'} = 0\). One can check that for \(\Lambda > 0\) there is a limit cycle,

\[
y_s(t) = (\sqrt{\Lambda} \cos(\varphi_K t), \sqrt{\Lambda} \sin(\varphi_K t), 0)^T, \tag{9.31}
\]

where \(\varphi_K = -\Delta_{out} + \kappa(-1)^K\). This limit cycle is stable within the synchronisation manifold \(M_s\) (9.18). Next we want to know whether the limit cycle is stable to perturbation transverse to \(M_s\). For this it is convenient to write (9.29) in matrix notation

\[
\frac{dY}{dt} = F(Y) + \kappa (\hat{G} \otimes R_K H) Y, \tag{9.32}
\]

where \(Y = (y_1, \ldots, y_M)^T \in \mathbb{R}^{3M}\), \(F(Y) = (f_K(y_1), \ldots, f_K(y_M))^T\), and \(\otimes\) is the Kronecker product. A synchronised solution, \(y_j(t) = y_s(t)\) for \(j = 1, \ldots, M\), is written in matrix notation as \(Y_s(t) = (y_s(t), \ldots, y_s(t))^T \in \mathbb{R}^{3M}\). Let \(\delta(t) = (\delta_1(t), \ldots, \delta_M(t))^T \in \mathbb{R}^{3M}\) denote an arbitrary perturbation vector. Making the substitution \(Y(t) = Y_s(t) + \delta(t)\), Taylor expanding (9.32) about \(Y_s(t)\), and neglecting higher order terms gives the linearised variational equation,

\[
\frac{d\delta}{dt} = \left( I_M \otimes Df_K|_{y_s(t)} + \kappa \hat{G} \otimes R_K H \right) \delta(t), \tag{9.33}
\]

where \(Df_K|_{y_s(t)}\) is the Jacobian of \(f_K\) evaluated at \(y_s(t)\), and \(I_M\) is the \(M \times M\) identity
matrix. To identify perturbations in directions transverse and tangential to $\mathcal{M}_s$ we diagonalise $\hat{G}$ by introducing ([52, pg. 394–396] and [98])

$$\xi(t) = (T \otimes I_3)\delta(t),$$  \hspace{1cm} (9.34)

where $T$ is a $3 \times 3$ matrix whose columns are the eigenvectors of $\hat{G}$, and rewrite (9.33) as

$$\frac{d\xi_j}{dt} = (Df_K|_{y_s(t)} + \kappa \lambda_j R_K H) \xi_j(t),$$  \hspace{1cm} (9.35)

for $j = 1, \cdots, M$. Note that $\xi_j \in \mathbb{R}^3$ and

$$\lambda_j = -4\sin^2\left(\frac{(j-1)\pi}{2M}\right),$$

for $j = 1, \cdots, M$, are the eigenvalues of the diffusive connectivity matrix $\hat{G}$ [38, 151]. A comparison of (9.35) with (9.30) shows that the first eigenvalue of $\hat{G}$, $\lambda_1 = 0$, gives the variational equation for the perturbations within $\mathcal{M}_s$, while the remaining $\lambda_j \neq 0$ correspond to the time evolution of perturbations transverse to $\mathcal{M}_s$. In general, the synchronised solution, $y_s(t)$, is time varying and hence (9.35) needs to be solved numerically [98]. However, if $y_s(t)$ is an equilibrium then it may be possible to determine the stability exactly. Owing to the rotational $S^1$ symmetry of laser oscillators, setting $\Delta_{out} = \kappa(-1)^K$ ‘freezes’ the limit cycle (9.31) into a circle of nonhyperbolic equilibria. We can choose any of these equilibria to linearise about; for convenience we take $y_s(t) = (\sqrt{\Lambda}, 0, 0)^T$. Then, the characteristic equation for (9.35) is

$$\lambda^3 + \lambda^2(1 + \beta \Lambda) + \lambda(\lambda_j^2 \kappa^2 + 2\beta \gamma \Lambda) + \lambda_j^2 \kappa^2(1 + \beta \Lambda) + 2\lambda_j \alpha \beta \gamma \kappa \Lambda (-1)^K = 0.$$  \hspace{1cm} (9.36)

Synchronised solutions (9.31) are transversally stable if all roots $s$ of Eqn. (9.36) with $j = 2, \cdots, M$ lie in the negative half-plane. Using the Routh–Hurwitz criterion [145, Ch. 1] this requires that

$$\kappa \lambda_j (-1)^K < \frac{1 + \beta \Lambda}{\alpha},$$  \hspace{1cm} (9.37)

$$\kappa > \frac{-(-1)^K 2\alpha \beta \gamma \Lambda}{\lambda_j (1 + \beta \Lambda)},$$  \hspace{1cm} (9.38)

for $j = 2, \cdots, M$.

Now the two special cases, $K = 0$ and $K = 1$, are treated separately. For $K = 0$, inequality (9.37) is always satisfied and the necessary and sufficient condition for (9.31)
to be transversally stable comes from (9.38) with $j = 2$:

$$\kappa > \kappa_{\text{in}} = \frac{\alpha \beta \gamma \Lambda}{2(1 + \beta \Lambda)} \frac{1}{\sin^2 \left( \frac{\pi}{2M} \right)}.$$  \hspace{1cm} (9.39)

For $K = 1$, inequality (9.38) is always satisfied and the necessary and sufficient condition for (9.31) to be transversally stable comes from (9.37) with $j = M$:

$$\kappa < \kappa_{\text{anti}} = \frac{1 + \beta \Lambda}{4\alpha} \frac{1}{\cos^2 \left( \frac{\pi}{2M} \right)}.$$  \hspace{1cm} (9.40)

### 9.3 Summary of Analytical Results

We have shown that complete intensity synchronisation (9.2) is represented by solutions to (8.3) in the form of rotationally symmetric limit cycles (8.13) or (9.31). The in-phase synchronisation (9.19) where all lasers oscillate in-phase requires (9.20)–(9.22), is given by

$$E_1(t) = E_2(t) = \cdots = E_M(t) = \sqrt{\Lambda} e^{i(-\Delta_{\text{out}}+\kappa)t},$$
$$N_1(t) = N_2(t) = \cdots = N_M(t) = 0,$$  \hspace{1cm} (9.41)

and is stable if and only if the parameters satisfy (9.39). The anti-phase synchronisation (9.23) where each laser oscillates in anti-phase with its neighbours requires (9.20)–(9.21) and (9.24), is given by

$$E_1(t) = -E_2(t) = \cdots = (-1)^{M-1} E_M(t) = \sqrt{\Lambda} e^{i(-\Delta_{\text{out}}-\kappa)t},$$
$$N_1(t) = N_2(t) = \cdots = N_M(t) = 0,$$  \hspace{1cm} (9.42)

and is stable if and only if the parameters satisfy (9.40).

Shading in Fig. 9.1 indicates stability of these two synchronised solutions in the parameter plane of shear within an individual laser, $\alpha$, and the coupling strength, $\kappa$. For $\alpha = 0$, both synchronised solutions are stable for all $\kappa$, independent of the number of lasers, $M$. However, this is no longer the case for $\alpha > 0$. There is a lower bound on $\kappa$ for the in-phase solution such that it is stable for $\kappa > \kappa_{\text{in}}$, and an upper bound on $\kappa$ for the anti-phase solution such that it is stable for $\kappa < \kappa_{\text{anti}}$. Furthermore, both stability regions shrink with increasing either $\alpha$ or $M$.

These results are in agreement with numerical bifurcation diagrams for three coupled lasers in Ch. 4.4 (Fig. 4.2 and 4.3) and Ref. [16, Fig. 3 and 4], where synchronised solutions of the form (8.13) are shaded in green. For example, in Fig. 4.2(a) for $\alpha = 0$ synchronised
solutions of the form (8.13) are found in two green bands centred about the lines $\kappa = |\Delta|$. Increasing $\alpha$ causes the green-shaded parameter regions to change in a nontrivial manner. In particular, the upper green-shaded region moves towards increasing $\kappa$ and the lower green-shaded region becomes bounded to small $\kappa$ and $|\Delta|$ (Fig. 4.3(d)). Expressions (9.39)–(9.40) provide analytical conditions for some of the bifurcations responsible for these transitions. More precisely, $\kappa_{\text{in}}$ in (9.39) gives the analytical condition for the pitchfork bifurcation at $\kappa = \Delta_{\text{in}} - \Delta_{\text{out}} \equiv \Delta$ where the in-phase synchronised solution destabilises, and $\kappa_{\text{anti}}$ in (9.40) gives the analytical condition for the Hopf bifurcation at $\kappa = \Delta_{\text{out}} - \Delta_{\text{in}} = -\Delta$ where the anti-phase synchronised solution destabilises. Thus these results provide new insight into the strong effects of $\alpha$ on the green-shaded parameter regions of stable (relative) equilibria reported in Ch. 4.4 (Fig. 4.2 and 4.3) and Ref. [16, Fig. 3 and 4].
Chapter 10

Transitions From Complete Synchronisation to Optical Turbulence

Here, we investigate transitions between complete synchronisation in the form of (8.13), examples of which were studied in the previous chapter, and optical turbulence which we define as a chaotic attractor for (8.3) such that no lasers are synchronised. More specifically, we consider laser arrays with identical outer lasers, \( \Delta_1 = \Delta_M = \Delta_{\text{out}} \), and identical inner lasers, \( \Delta_j = \Delta_{\text{in}} \) for \( j = 2, \ldots, M-1 \), and focus on the relevant transitions in the \((\kappa, \Delta)\) parameter plane, where \( \Delta = \Delta_{\text{in}} - \Delta_{\text{out}} \). The analysis is performed in three steps. In the first step, we quantify the degree of synchronisation found in the \((\kappa, \Delta)\) plane by defining

\[
D(A) = \frac{2(M-2)!}{M!} \sum_{j=1}^{M-1} \sum_{j'=j}^{M} I(j, j'),
\]

for an attractor \( A \), where \( I(j, j') = 1 \) if lasers \( j \) and \( j' \) are synchronised (8.8)–(8.9) and 0 otherwise. \( D = 1 \) corresponds to complete synchronisation, \( 0 < D < 1 \) corresponds to partial synchronisation, and \( D = 0 \) corresponds to no lasers being synchronised. In the second step, Lyapunov exponent calculations reveal different attractor types in the \((\kappa, \Delta)\) plane, and determine transverse stability of the corresponding synchronisation manifolds. In the third step, we conduct bifurcation analysis to identify the underlying mechanisms responsible for changes between different types of synchronisation and for the complete loss of synchrony. To facilitate bifurcation analysis we rewrite (8.3) in polar coordinates, so that, in the resulting \( S^1 \)-reduced system, synchronised solutions (8.13) can be studied as isolated equilibria. Henceforth, we refer to attractors of the \( S^1 \)-reduced system.
first analyse an array of three lasers and then study larger arrays.

### 10.1 An Array of Three Coupled Laser Oscillators

In the eight-dimensional phase space of the $S^1$-reduced system for an array of three lasers, Eqs. (4.2) and (4.10), each point represents the three magnitudes of the electric fields, $|E_1|, \ |E_2|$ and $|E_3|$; the two phase differences between the electric fields, $\varphi_2 - \varphi_1$ and $\varphi_2 - \varphi_3$; and the population inversion within each laser, $N_1, N_2$ and $N_3$. Figure 10.1 shows regions of complete synchronisation (white), partial synchronisation (shades of grey), and no synchronisation (black/red). To calculate Fig. 10.1 we discretised the $(\kappa, \Delta)$ plane as outlined in Ch. 2.3.4 for Lyapunov diagrams. The red regions represent optical turbulence, where in addition to no lasers being synchronised, the corresponding attractor has at least one positive Lyapunov exponent. The region of complete synchronisation is confined to $\kappa < 3.5$, and is mostly surrounded by regions of partial synchronisation. Also, large regions of no synchronisation are occupied by optical turbulence.

The Lyapunov diagram in Fig. 10.2(a) gives an overview of the different attractor types in the $(\kappa, \Delta)$ parameter plane. To calculate Fig. 10.2(a) we used the same discretisation of the $(\kappa, \Delta)$ parameter plane and the same parameter sweep as for determining the degree of synchronisation. For small $\kappa$ the parameter plane is dominated by equilibria (green) and limit cycles (grey). There are also small regions of tori (blue). Towards increasing $\kappa$ the system undergoes various bifurcations so that for $\kappa > 5$ the parameter plane is dominated by chaotic attractors. The bifurcations that make up the boundary of the green region have been studied in detail in [16]. Comparing Fig. 10.1 with 10.2(a) reveals that the region of complete synchronisation corresponds to equilibria, whereas regions of partial and no synchronisation, both comprise of limit cycles, tori, and chaotic attractors.

In Fig. 10.2(b) we shade the regions of Fig. 10.2(a) if the corresponding attractor is not contained in the fixed-point subspace, $\text{Fix}(Z_2)$. The key result is that non-shaded regions are the same as the combined regions of complete and partial synchronisation in Fig. 10.1. We can thus identify $\text{Fix}(Z_2)$ as the synchronisation manifold responsible

<table>
<thead>
<tr>
<th>Key</th>
<th>Synchronisation Type</th>
<th>Quantified by</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>complete</td>
<td>$D = 1$</td>
</tr>
<tr>
<td></td>
<td>partial</td>
<td>$1 &gt; D &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>none</td>
<td>$D = 0$</td>
</tr>
<tr>
<td></td>
<td>none (optical turbulence)</td>
<td>$D = 0$ and $\mu_1 &gt; 0$</td>
</tr>
</tbody>
</table>

Table 10.1: The colour coding for degree of synchronisation and optical turbulence diagrams. $D$ is the degree of synchronisation defined in Eqn. (10.1) and $\mu_1$ is the largest Lyapunov exponent.
Figure 10.1: The \((\kappa, \Delta)\) parameter plane for \(\alpha = 2\) partitioned into regions that contain complete synchronisation (white), partial synchronisation (grey), no synchronisation (black/red), and optical turbulence (red) i.e. the lasers are chaotic and not synchronised (8.8)–(8.9).

for the regions of complete and partial synchronisation. The region of complete synchronisation in Fig. 10.1 corresponds to equilibria contained in Fix\((Z_2)\), therefore the outer two lasers are intensity synchronised in-phase, but typically out-of-phase with the middle laser (except for the two special cases studied analytically in Ch. 9.2). Furthermore, regions of partial synchronisation in Fig. 10.1 comprise of periodic, quasi-periodic, and chaotic oscillations where the outer lasers are intensity synchronised in-phase. We can distinguish between two types of boundaries. Genuine boundaries coincide with bifurcations and are responsible for transitions between attractors within and off the fixed-point subspace. Artificial boundaries do not coincide with any bifurcations but are related to bistability.

10.1.1 Bifurcations of Non-Chaotic Attractors

Starting from the origin in Fig. 10.2(b) and moving clockwise we identify different types of boundaries. Along the curve of heteroclinic bifurcations (het) that emerges from the origin and ends at the Shilnikov–Hopf bifurcation (ShH), a symmetric limit cycle [75, Ch. 7.4] is destroyed as it collides with two symmetrically related saddle equilibria to
form a heteroclinic cycle. For \((\kappa, \Delta)\) below the curve of heteroclinic bifurcations, the system settles down to a stable equilibrium that corresponds to complete synchronisation. At the Shilnikov–Hopf bifurcation [63] both equilibria involved in the heteroclinic cycle simultaneously undergo a Hopf bifurcation. It is thought that additional bifurcations emerging from ShH form the boundary connecting ShH and the curve of pitchfork of limit cycle bifurcations (PL) (see the inset). The next part of the boundary is formed by PL. Along PL, between the pitchfork-torus bifurcation (PT) and the first pitchfork-saddle-node bifurcation (PSL), a stable limit cycle contained in the fixed-point subspace collides with two unstable limit cycles. The yellow region to the right of the subcritical part of PL is due to an intermittent transition [96, Ch. 8.2] between periodic oscillations and chaos not in the fixed-point subspace. Along PL, between the two PSL bifurcations, a stable limit cycle contained in the fixed-point subspace loses stability and two stable limit cycles bifurcate out of the fixed-point subspace, hence this part of PL is supercritical. The grey region to the right of the supercritical part of PL corresponds to one of the stable limit cycles that bifurcated out of the fixed-point subspace along PL. The boundary near the lower pitchfork-saddle-node bifurcation (PSL) is an artificial boundary. It is a consequence of the nearby cusp bifurcation in the curve of saddle-node of limit cycle bifurcations (SL). For parameter values near the cusp and to the right of the SL curve, two stable limit cycles are very close together, one of these loses stability at the PL curve. As we increasing \(\kappa\), the system jumps to the stable limit cycle that is not involved with the pitchfork of limit cycle bifurcations along PL. The next boundary is due to subsequent bifurcations of the two limit cycles created along the supercritical part of PL. They first undergo a period doubling bifurcation, and then the two stable period doubled limit cycles lose stability at a supercritical Torus bifurcation. This is the source of the blue region of quasi-periodic dynamics. The remaining boundary is blurred and not clear cut. Since this boundary corresponds to bifurcations of chaotic attractors, it cannot be analysed using existing numerical continuation techniques. Instead, one needs to calculate Lyapunov exponents in the directions tangential and transverse to the synchronisation manifold \(\text{Fix}(Z_2)\). We first give a short description of tangential Lyapunov exponents and then explain the blurred synchronisation boundary.

Figure 10.2(c) contains a tangential Lyapunov exponent diagram where the colours (see Table 2.2) indicate different attractors of the \(S^1\)-reduced system restricted to \(\text{Fix}(Z_2)\). In Fig. 10.2(d) regions of the \((\kappa, \Delta)\) plane are shaded if the attractor within the synchronisation manifold \(\text{Fix}(Z_2)\) has at least one positive transverse Lyapunov exponent, meaning that it is unstable in the full system. Notice that the non-shaded region in Fig. 10.2(d) is larger than in Fig. 10.2(b) indicating bistability between attractors within and off the synchronisation manifold.
Figure 10.2: Two-parameter Lyapunov exponent diagrams in the $(\kappa, \Delta)$ plane for $\alpha = 2$. (a) attractors of the $S^1$-reduced system. (b) regions of (a) are shaded darker if their corresponding attractor $\not\subset \text{Fix}(\mathbb{Z}_2)$. (c) attractors of the $S^1$-reduced system restricted to $\text{Fix}(\mathbb{Z}_2)$. (d) regions of (c) are shaded darker if their corresponding attractor is transversally unstable. Also included in (b) and (d) are key bifurcation curves. For the labelling and colour coding see Table 2.1 and 2.2.

As before, boundaries of the shaded regions in Fig. 10.2(d) are of two types: genuine synchronisation boundaries that coincide with bifurcations, and artificial boundaries that do not. Starting from the origin and moving clockwise, genuine synchronisation boundaries are formed by the curves of pitchfork (P), Hopf (H), Torus (T) and pitchfork of limit cycle (PL) bifurcations shown in Fig. 10.2(d). The pitchfork bifurcations along P from the origin to the pitchfork–Hopf (PH) bifurcation are subcritical. Hence, a stable
equilibrium in the fixed-point subspace loses stability as it collides with two unstable equilibria not in the fixed-point subspace. The curve of Hopf bifurcations, between the pitchfork–Hopf (PH) bifurcation and the Hopf–Hopf (HH) bifurcation, is supercritical and corresponds to a stable equilibrium in the fixed-point subspace losing transverse stability via a collision with a stable limit cycle off the fixed-point subspace. Emanating from the Hopf–Hopf (HH) bifurcation is a curve of Torus (T) bifurcations which mark the boundary between HH and the pitchfork-torus (PT) bifurcation. Along this part of the Torus curve a stable limit cycle within the fixed-point subspace undergoes a Torus bifurcation and turns unstable. The final non-blurred boundary is the curve of pitchfork of limit cycles (PL) bifurcations discussed previously.

10.1.2 Bifurcations of Chaotic Attractors Along the ‘Blurred’ Synchronisation Boundaries

The largest transverse Lyapunov exponent, $\mu^\perp$, approximates the average rate of exponential growth/decay in a transverse direction to the synchronisation manifold $\text{Fix}(\mathbb{Z}_2)$, for typical initial conditions on a chaotic attractor $A^*$ of the restriction of the full system to $\text{Fix}(\mathbb{Z}_2)$. The ‘blurred’ boundaries of shaded regions in Fig. 10.2(d) correspond to blowout bifurcations [97, 11, 93] where $\mu^\perp$ crosses through zero and $A^*$ loses stability transverse to $\text{Fix}(\mathbb{Z}_2)$. To describe the interesting dynamical phenomena that take place near blowout bifurcations we also need to consider $\hat{\mu}^\perp = \max_{x_0 \in A} \mu^\perp(x_0)$ [96, Ch. 10], where $\mu^\perp$ is the largest transverse Lyapunov exponent for any initial condition $x_0$ on $A^*$ (not just the typical ones). The set of initial conditions such that $\mu^\perp(x_0) \neq \mu^\perp$, occupies
zero volume of the synchronisation manifold $\text{Fix}(\mathbb{Z}_2)$ and typically corresponds to unstable periodic orbits embedded in $A^*$. The sketch in Fig. 10.3 illustrates the relationship between $\mu^*_{\perp}$ and $\hat{\mu}^\perp$ as a parameter is varied across a blowout bifurcation. If $\hat{\mu}^\perp < 0$ then by definition $\mu^*_{\perp} < 0$, and $A^*$ is an exponentially stable attractor for the full system. As a parameter is increased further, a periodic orbit embedded in $A^*$ loses transverse stability and $\hat{\mu}^\perp$ crosses through zero in a bubbling bifurcation [10, 132]. Now every neighbourhood of $A^*$ contains trajectories that at some point leave that neighbourhood. If the bubbling bifurcation is supercritical then those trajectories return and eventually converge to $A^*$. However, if it is subcritical then those trajectories never return to the neighbourhood of $A^*$. In either case, $A^*$ is no longer an exponentially stable attractor, in fact it is not even Lyapunov stable. However, it is still a Milnor attractor [90, 91], meaning that its basin of attraction,

$$B(A) = \{x(0) \in \mathbb{R}^n : x(t) \to A \text{ as } t \to \infty\},$$

occupies a positive volume of the phase space. On increasing a parameter further, more and more periodic orbits embedded in $A^*$ lose transverse stability causing $\mu^*_{\perp}$ to cross through zero in a blowout bifurcation. Past this bifurcation, the basin of attraction of $A^*$ takes up zero volume of the phase space and $A^*$ is not even a Milnor attractor. If the bubbling and blowout bifurcations are supercritical then there is a new, larger attractor $A$ that intersects $\text{Fix}(\mathbb{Z}_2)$ and contains $A^*$ [11].

**Bubbling Bifurcation and its Criticality**

In general it is difficult to calculate the parameter values for which bubbling bifurcations occur. The exact periodic orbits that lose transverse stability are unknown and could even change in a non-trivial fashion as parameters vary—Hunt and Ott [65] conjectured that they are periodic orbits with a low period. Nonetheless, one can observe effects of bubbling bifurcations. These effects depend on the criticality of the bubbling bifurcation and we illustrate the two different cases by starting from a point in $A^*$, applying a perturbation, and studying the resulting trajectory. In the supercritical case we only perturb a very small amount in the transverse direction, $\text{Re}(E_1 - E_3)/2 \sim 10^{-6}$. Between the bubbling and blowout bifurcations, the resulting trajectory makes a number of large excursions away from the synchronisation manifold $\text{Fix}(\mathbb{Z}_2)$ but eventually converges back to $A^*$ (not shown). The subcritical case of the bubbling bifurcation results in $A^*$ having a riddled basin of attraction [118]. In between the bubbling and blowout bifurcations, any neighbourhood of an initial condition belonging to the basin of attraction of $A^*$ will contain a positive volume of phase space that belongs to the basin of attraction of a
different attractor. Figure 10.4 contains a two-dimensional slice of a riddled basin in the three coupled laser model for parameter values close to blowout bifurcation, \((\alpha, \kappa, \Delta) = (2, 5.642, 6.44)\). In Fig. 10.4, each point in the \((\text{Re}(E_1 - E_2)/2, \text{Re}(E_1 - E_3)/2)\) plane represents one perturbation; the points are coloured white if the resulting trajectory converges to \(A^*\), or black if it converges to a different attractor. Since \(A^*\) is still an asymptotically stable attractor for the restriction of the full system to \(\text{Fix}(Z_2)\), any perturbation solely within the synchronisation manifold \(\text{Fix}(Z_2)\) converges to \(A^*\) and hence there is a white line at \(\text{Re}(E_1 - E_3)/2 = 0\) in Fig. 10.4(a).

**Blowout Bifurcation and On-Off or In-Out Intermittency**

Past the blowout bifurcation, one expects intermittent dynamics on the new attractor \(A\) that intersects \(\text{Fix}(Z_2)\). To facilitate the discussion, it is useful to define \(A_0 = A \cap \text{Fix}(Z_2)\); note that \(A_0\) and \(A \cap \text{Fix}(Z_2)^C\) (where \(C\) denotes the complement) are both non-empty. Furthermore, if \(A\) is an asymptotic attractor then \(A_0\) must contain a Milnor attractor for the restriction of the full system to \(\text{Fix}(Z_2)\) [12], which we denote by \(A^*\).

If \(A^* = A_0\), then the attractor \(A\) for the full system displays *on-off intermittency*. Figure 10.5(b) contains a sketch illustrating the fact that repelling and attracting sets responsible for on-off intermittency are intermingled in \(A_0\). In Fig. 10.5(c)–(f) we show an example believed to be on-off intermittency in the three coupled laser model. The time
Figure 10.5: (a) Largest tangential (red) and transversal (blue dashed) Lyapunov exponents for a parameter sweep in $\kappa$. (b) A sketch illustrating on-off intermittency due to repulsion from and attraction towards the same chaotic attractor, $A^*$, of the restriction to $\text{Fix}(\mathbb{Z}_2)$. (c)–(f) Time series indicating on-off intermittency in the three coupled-laser model. (c) Chaotic attractor of the restriction to $\text{Fix}(\mathbb{Z}_2)$. (d) On-off intermittent chaotic attractor. (e)–(f) A variable transverse to $\text{Fix}(\mathbb{Z}_2)$. The $y$-axis in (f) has a log scale. ($\alpha, \kappa, \Delta$) = (2, 8.89, 0).

series (Fig. 10.5(c)) for the chaotic attractor $A_0$ of the synchronisation manifold $\text{Fix}(\mathbb{Z}_2)$ is similar to that of the on-off intermittent attractor $A$ of the full system (Fig. 10.5(d)). Intermittent behaviour shows up as bursting away from $A_0$ (Fig. 10.5(e)). However, the same figure with the $y$-axis on a log scale (Fig. 10.5(f)) shows no evidence of distinctively different mechanisms responsible for repulsion away from and attraction towards $A_0$. This observation is in line with the sketch in Fig. 10.5(b).

If $A^* \subset A_0$, then $A$ displays in-out intermittency. For in-out intermittency different mechanisms are responsible for the repulsion away from and attraction towards $A_0$. Figure 10.6(b) shows a sketch with one attractive and one repulsive mechanism. A trajectory on the attractor $A$ of the full system is repelled from $A_0$ along the unstable manifold of a limit cycle that is contained in $A_0$ and stable to perturbations within $\text{Fix}(\mathbb{Z}_2)$—this is the out phase. The trajectory is then globally reinjected towards $A_0$, and the in phase begins when it approaches $A_0$ along a stable manifold of a chaotic saddle contained in $A_0$. The chaotic saddle is repelling within $\text{Fix}(\mathbb{Z}_2)$, meaning that the trajectory will eventually approach the limit cycle again, then move away from $A_0$, etc. In Fig. 10.6(c)–(f) we show an example of in-out intermittency in the three-coupled laser model. On the one hand,
transverse and tangential Lyapunov exponents in Fig. 10.6(a) along with the time series in Fig. 10.6(c) confirm that there exists a limit cycle that is unstable for the full system but stable within $\text{Fix}(\mathbb{Z}_2)$. During the intermittent (bursting) dynamics (Fig. 10.6(d)–(e)), this limit cycle is clearly visible in the time series for the attractor of the full system (Fig. 10.6(d)). What is more, ‘spells’ of periodic-like oscillations due to this limit cycle coincide with out phases during which the system is repelled from $A_0$ (Fig. 10.6(d)–(f)). On the other hand, it is clear from Fig. 10.6(d)–(f) that quite different mechanism(s) are responsible for attraction towards $A_0$. A diffusive character of the time series during the in phase indicated in Fig. 10.6(f) strongly suggest the influence of a chaotic saddle. While the in and out phases indicated in the mid-part of the time series are in line with the sketch in Fig. 10.6(a), there appears to be other attracting mechanisms visible at the beginning and end of that time series. This suggest a possibility of generalised in-out intermittency in which there are many forms of ‘in’-dynamics and/or many forms of ‘out’-dynamics [122].
Figure 10.7: The \((\kappa, \Delta)\) plane with \(\alpha = 2\) for \(M = 4\) and \(M = 5\) partitioned according to: (a) and (c), degree of synchronisation and optical turbulence; and (b) and (d), attractor types quantified by their tangential and transversal Lyapunov exponents. Colour coding in Tables 10.1 and 2.2.

10.2 Larger Arrays of Coupled Laser Oscillators

For larger laser arrays there are more possible synchronisation manifolds corresponding to different types and degrees of intensity synchronisation. An interesting question arises: is the fixed-point subspace \(\text{Fix}(\mathbb{Z}_2)\) still a stable synchronisation manifold? To answer this question we partitioned the \((\kappa, \Delta)\) plane into regions with different degree of synchronisation using (10.1), different attractor types within \(\text{Fix}(\mathbb{Z}_2)\), and their transverse stability. In this way, we analysed system (8.3) with \(M = 4, 5, 6\) and 7, and identified
Figure 10.8: (a) Dependence of the largest Lyapunov exponent, \( \mu_1 \), on array size \( M \) and coupling strength \( \kappa \). (b) Dependence of the nine largest Lyapunov exponents on \( \kappa \) for an array with \( M = 7 \). (c) Number of positive Lyapunov exponents and (d) normalised Lyapunov dimension as functions of \( M \) for \( \kappa = 30 \) (●) and \( \kappa = 75 \) (▲). (\( \alpha, \Delta \)) = (5, 0).

In all cases that we considered, the \((\kappa, \Delta)\) plane features one region of complete synchronisation (stable equilibria) which is confined to small \( \kappa \) and centred around \( \Delta = 0 \), only one degree of partial synchronisation, and vast regions of optical turbulence. We spotted two main differences between laser arrays with \( M \) odd and \( M \) even. Firstly, the proportion of the \((\kappa, \Delta)\) plane with an attractor corresponding to partial synchronisation is significantly smaller for \( M \) even than for \( M \) odd. Secondly, the relationship between

recurring patterns that are illustrated in Fig. 10.7 for \( M = 4 \) and \( M = 5 \).
partially synchronised lasers is different. For arrays with an even number of lasers, \( M = 2L \), we did not find any attractors contained in \( \text{Fix}(\mathbb{Z}_2) \) (see Fig. 10.7(b)). Rather, the regions of complete and partial synchronisation are due to attractors contained in the synchronisation manifold given by:

\[
E_1 = -E_{2L}, \quad E_2 = -E_{2L-1}, \quad \cdots \quad E_L = -E_{L+1}, \\
N_1 = N_{2L}, \quad N_2 = N_{2L-1}, \quad \cdots \quad N_L = N_{L+1},
\]

which corresponds to pairs of lasers being intensity synchronised in anti-phase. In contrast to this, for \( M \) odd, regions of complete and partial synchronisation correspond to attractors contained in \( \text{Fix}(\mathbb{Z}_2) \), meaning that pairs of lasers are intensity synchronised in-phase. Finally, note that for \( M \geq 5 \) and odd, there are only small parameter regions where we find partially synchronised chaos (see Fig. 10.7(c)–(d)).

### 10.3 Properties of Optical Turbulence

For \( \alpha \) sufficiently large the \((\kappa, \Delta)\) parameter plane is dominated by optical turbulence. In this section we focus on the properties of the underlying chaotic attractors. The properties that we are concerned with are determined by the Lyapunov spectrum\(^1\), \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{3M-1} \), of the chaotic attractor. The intensity of the chaos [100] is quantified by \( \mu_1 \), the number of unstable directions is given by the number of \( \mu_i > 0 \), and the amount of phase space that the chaotic attractor explores is given by the Lyapunov dimension [47]

\[
\mu_D = I + \sum_{i=1}^{I} \frac{\mu_i}{|\mu_{i+1}|},
\]

where \( I \) is the highest, integer number of Lyapunov exponents such that \( \sum_{i=1}^{I} \mu_i > 0 \). When studying the dependence of the Lyapunov dimension on array size, it is useful to look at the normalised Lyapunov dimension, \( \mu_D/(3M-1) \), expressed as a percentage of the total phase space. For the following analysis we set \( \alpha = 5 \) since it is known that large \( \alpha \) is conducive to the creation of chaos (see Ch. 4.8 and Ref. [139, 17]). Over the range of \( \Delta \) that we considered (\(|\Delta| < 30\)) the results did not change drastically, so we present here the case of identical lasers given by \( \Delta = 0 \).

Figure 10.8(a), shows the dependence of \( \mu_1 \) on the number of lasers, \( M \), and the coupling strength, \( \kappa \). For each \( M \), \( \mu_1 \) increases rapidly, reaches a maximum around \( \kappa \approx 30 \), and then gradually decreases. Fixing \( \kappa \) and increasing \( M \) results in \( \mu_1 \) converging to a \( \kappa \)-dependent constant. We would like to stress that, already for \( M = 4 \), we did not

\(^1\)There are \( 3M-1 \) Lyapunov exponents since we exclude the one associated with the \( S^1 \) symmetry.
detect any periodic windows despite having 10,000 points within the plotted interval. The lack of periodic windows suggest that, from a practical point of view, chaos generated by laser arrays is persistent under parameter changes.

Figure 10.8(b) shows the dependence of the nine largest Lyapunov exponents on $\kappa$ for $M = 7$. There is a large interval of $\kappa$ with eight positive Lyapunov exponents. To explore the dependence of the number of positive Lyapunov exponents on array size, $M$, in more extent we fix $\kappa = 30 (\bullet)$ and $\kappa = 75 (\triangle)$; these values of $\kappa$ are also indicated by blue ($\kappa = 30$) and black ($\kappa = 75$) curves in Fig. 10.8(a). For each value of $\kappa$ we calculate the Lyapunov spectrum with increasing array size, $M$, and plot the number of positive Lyapunov exponents and the normalised Lyapunov dimension in Fig. 10.8(c) and (d), respectively. For both values of $\kappa$ the number of positive Lyapunov exponents increases monotonically with $M$ (Fig. 10.8(c)). The normalised Lyapunov dimension increases rapidly at first, but then saturates at $\approx 95.4\%$ for $\kappa = 30$, and at $\approx 95.5\%$ for $\kappa = 75$ (Fig. 10.8(d)).

From these observations we conclude that the intensity of the chaos as characterised by $\mu_1$, and the normalised Lyapunov dimension, $\mu_D/(3M-1)$, saturate to a constant value for low $M$. Our results together with recent findings—see Ref. [100] where there is an optimal number of globally coupled phase oscillators above which $\mu_1$ decreases—highlight important differences between nearest-neighbour and globally coupled oscillators. The dependence of the Lyapunov spectrum on $\kappa$ and $M$ reported here for amplitude-and-phase oscillators coupled in a linear array shows good qualitative agreement with results in [3, 2]. Those papers reported persistent chaos in high dimensional dynamical systems and conjectured that, as the dimension of a typical dissipative dynamical system is increased, the number of positive Lyapunov exponents increases monotonically and the number of windows with periodic behaviour decreases. It was also conjectured that at large coupling all the Lyapunov exponents become negative again. In the $M$ coupled laser model (8.3) this occurs but at values of $\kappa$ that are beyond the validity of the model.
Chapter 11

Conclusion for Part III

We studied an array of $M$ linearly coupled laser oscillators, all of which are identical with the exception of different natural frequencies detunings, $\Delta_j$. Particular focus was placed on different types of laser synchronisation (8.8)–(8.9) and its dependence on the coupling strength, $\kappa$, frequency detuning, $\Delta_j$, amount of shear or amplitude-phase coupling, $\alpha$, and the number of lasers, $M$.

It was found, by analytical calculations, that the lasers can be specifically detuned so that synchronised solutions corresponding to complete intensity synchronisation exist, and that depending on the specific values of $\Delta_j$, pairs of lasers are either synchronised in-phase or in anti-phase with each other. Using this result we found that the inner lasers ($\Delta_j = \Delta_{in}$ for $j = 2, \ldots, M - 1$) have to be detuned from the pair of outer lasers ($\Delta_1 = \Delta_M = \Delta_{out}$) in order to achieve two special cases of complete intensity synchronisation. The first is where each laser is in-phase with every other laser, and the second is where each laser is in anti-phase with its neighbours. We were able to analytically obtain stability conditions for these two special cases of synchronisation. For semiconductor lasers (typically $\alpha \in [1, 10]$), intensity synchronisation where all the lasers oscillate in phase with each other is stable for coupling strengths above a critical value, $\kappa_{in}(M)$, which increases with $M$. However, intensity synchronisation where lasers oscillate in anti-phase with their neighbours is stable below a critical value, $\kappa_{anti}(M)$, which decreases with $M$.

In addition to the two special cases of synchronisation studied here, there are complete intensity synchronised solutions that warrant further investigation. In particular, for the purpose of high-powered light generation [29, 19] it could be that there are physically accessible parameter values, which support stable synchronised solutions where most of the lasers are in-phase and only a few are in anti-phase.

Through a combination of (equivariant) bifurcation analysis and Lyapunov exponent calculations we were able to provide an in depth study of transitions in the $(\kappa, \Delta_{in} - \Delta_{out})$ parameter plane between attractors corresponding to different types of synchronisation.
More specifically, for an array of three (semiconductor) lasers we found: one region of complete synchronisation, where the outer two lasers are intensity synchronised in-phase but typically out-of-phase with the middle laser; regions of partial synchronisation comprising of periodic, quasi-periodic and chaotic oscillations; and large regions of optical turbulence. The synchronisation-desynchronisation transitions were identified as bifurcations. In particular, blowout bifurcations were found to be responsible for the destabilisation of synchronised chaos. Interestingly, regions of synchronised chaos decrease for larger arrays of laser oscillators; instead the \((\kappa, \Delta_{in} - \Delta_{out})\) parameter plane is dominated by optical turbulence. Finally, from the Lyapunov spectrum of the underlying chaotic attractors we found that the intensity of chaos increases rapidly then remains nearly constant (saturates) with increasing array size.

From the results of Part III we are better able to locate the contrasting behaviours shown in Fig. 8.1. Complete intensity synchronisation, where each laser is in anti-phase with its neighbours (Fig. 8.1(a)), can be found for any array size, \(M \geq 2\), provided that the lasers are appropriately detuned and \(\kappa < (1 + \beta \Lambda)/(4\alpha)\). In contrast, optical turbulence (Fig. 8.1(b)) is found for \(M \geq 2\) by choosing large \(\alpha\) and moderate coupling strengths. Furthermore, we explained possible transitions from one behaviour type to the other.
Part IV

Overall Summary
The purpose of this thesis was to investigate the nonlinear dynamics arising in linear arrays of coupled laser oscillators with nearest-neighbour coupling. In Part I we used techniques from dynamical systems theory to study a solitary (class-B) laser and highlighted two well-known features of its dynamics. Firstly, increasing the pump strength causes the laser to undergo a Hopf bifurcation. For pump strengths above the Hopf bifurcation (laser threshold) the laser emits light with a constant intensity and is a self-sustained nonlinear oscillator. Secondly, by studying the isochrones of the laser oscillator we showed that the linewidth enhancement factor, \( \alpha \), determines the amount of shear (amplitude-phase coupling) in the system. Then we introduced two physical setups that lead to linear arrays of laser oscillators and discussed the two modelling approaches that we took in this thesis: the coupled-laser approach and the composite-cavity mode approach. Part I was then concluded with an overview of the tools and techniques from dynamical systems theory that were to be used throughout the thesis.

In Part II, we conducted a detailed analysis of three coupled laser oscillators by focusing on a three-dimensional parameter space which comprised of the laser coupling strength, \( \kappa \), the laser frequency difference between the outer and inner lasers, \( \Delta \), and the coupling, \( \alpha \), between the magnitude, \( |E_j| \), and phase, \( \arg(E_j) \), of a single laser’s complex-valued electric field, \( E_j \). To determine the ability of the lasers to phase-lock we studied solutions of the form

\[
E_j(t) = |E_j^0| e^{-i(\omega^0 t + \varphi_j^0)},
\]

\[
N_j(t) = N_j^0, \quad \text{for } j = 1, 2, 3,
\]

where \( |E_j^0| > 0, \omega^0, N_j^0 \) are real-valued constants. We found that phase locking can be achieved for \( \alpha = 0 \) with any coupling strength provided that the lasers are appropriately detuned. In contrast, for \( \alpha \) sufficiently large (\( \alpha > 1 \)), locking (11.1) is confined to small \( \kappa \) and \( |\Delta| \). A surprising result from this research was the emergence of vast regions of chaos in the \((\kappa, \Delta)\) parameter plane for \( \alpha > 1 \). For larger arrays, we found that although it is possible to find chaos for \( \alpha = 0 \), the extent and intensity (indicated by the largest Lyapunov exponent) of this chaos is dramatically enhanced by increasing \( \alpha \). While previous studies have focused on reporting instabilities and chaos caused by \( \alpha \), we proposed an explanation for the underlying mechanism of these phenomena, in terms of shear-induced stretching and folding of a single laser’s phase space. There is potential for future research to strengthen this proposal, for example by investigating the relationship between the shear, \( \alpha \), and the largest Lyapunov exponent in the kicked solitary laser model (4.23)–(4.25). We also undertook a comparison of results from the simple coupled-laser model with those from the more accurate composite-cavity mode
model. The coupled-laser model does not fully take into account the spatial structure of the laser array, and as a result some discrepancies arose between the results from the two models. In particular, the volume of active laser medium is larger for $\Delta < 0$ than it is for $\Delta > 0$. In the composite-cavity model this leads to a large locking region for $\Delta < 0$ and a small locking region for $\Delta > 0$ (Fig. 4.2(a)). The coupled-laser model does not capture this effect (Fig. 5.3)(a)). However, overall, we found a very good agreement on the level of locking regions, local and global bifurcations, and chaotic dynamics (compare Fig. 4.9 with Fig. 5.4). Our results therefore support the use of the simple coupled-laser model to accurately capture the coupling-induced instabilities in (moderately coupled) laser arrays.

In Part III, we extended the coupled-laser model to an array of $M$ lasers, and studied synchronisation properties of the array. Our approach differs from previous studies in that we let each laser have its own natural frequency, $\Omega_j$, thereby providing more possibilities for finding synchronised solutions. We proved the existence of intensity synchronisation,

$$|E_1(t)| = \cdots = |E_M(t)|,$$

(11.2)

where pairs of lasers are phase-locked to an integer multiple of $\pi$,

$$\arg(E_j) - \arg(E_{j+1}) = K_j \pi \mod 2\pi,$$

(11.3)

for $j = 1, \cdots, M - 1$, and $K_j \in \{0, 1\}$. Such solutions exist provided that the natural frequencies of the lasers satisfy

$$\Omega_1 - \Omega_j = \kappa \left[ (-1)^{K_1} - (-1)^{K_{j-1}} - \delta_{jM}(-1)^{K_j} \right],$$

(11.4)

for $j = 1, \cdots, M - 1$. We were then able to calculate stability conditions for two special types of synchronisation: (i) where the phase difference between all the lasers is zero, $K_j = K = 0$ for all $j$ in Eqs. (11.3)–(11.4); and (ii) where each laser is in anti-phase with its direct neighbours, $K_j = K = 1$ for all $j$ in Eqs. (11.3)–(11.4). If $\alpha = 0$, then either of the aforementioned synchronisation types is stable for any value of $\kappa$, provided that the lasers are detuned appropriately, i.e. $\Omega_{in} - \Omega_{out} = (-1)^K$. However, if $\alpha > 0$, and $\Omega_{in} - \Omega_{out} = \kappa$, then $\kappa$ needs to be increased in order to stabilise in-phase intensity synchronisation, whereas if $\Omega_{in} - \Omega_{out} = -\kappa$, then there is a maximum value of $\kappa$ beyond which anti-phase intensity synchronisation is unstable.

In-phase synchronisation is desirable for high-powered light generation because it maximises the intensity of the laser array [29, 19]. For this purpose we can make three suggestions from the results of this thesis. The first suggestion is to fabricate semiconduc-
tor lasers with $\alpha$ as small as possible. This would reduce the coupling strength required to stabilise in-phase synchronisation. The second suggestion concerns modifying the physical setup of the laser array (i.e. $\Omega_j$’s) in order to find synchronised solutions where most of the lasers are synchronised in-phase. We have shown that the natural frequencies can be chosen such that these synchronised solutions exist, but their stability requires further investigation. Finally, one could make use of the anti-phase synchronised solutions by diverting light from every other laser in the array.

For the remainder of Part III we focused on the transitions from locking (where all the lasers are synchronised) to optical turbulence (where no lasers are synchronised and each laser is chaotic in time). Once again the main focus was placed on an array of three coupled laser oscillators. We identified the fixed-point subspace $\text{Fix}(\mathbb{Z}_2)$ as a synchronisation manifold, such that if solutions in this manifold are transversally stable, then the outer two lasers are intensity synchronised in-phase, $E_1(t) = E_3(t)$. The dynamics within $\text{Fix}(\mathbb{Z}_2)$ reduce to that of two coupled lasers with uneven coupling (see Table 8.1). It is therefore constructive to compare our results for the dynamics within $\text{Fix}(\mathbb{Z}_2)$ with the case of two identically coupled lasers studied in Ref. [41]. In the case of two identically coupled lasers there is a parameter symmetry which corresponds to exchanging the natural frequencies of the two lasers. For two unevenly coupled lasers this symmetry is not present, as evidenced by the lack of reflectional symmetry in Fig. 10.2(c). However, in both cases there is a locking region for weak coupling between the lasers. For two identically coupled lasers a second locking region was found for large coupling. It would be interesting to see if this locking region also exists for two lasers with uneven coupling. If it does, there could be another region of complete synchronisation in the three coupled-laser model (4.1)–(4.2). Our research also uncovered large regions of chaos in the synchronisation manifold $\text{Fix}(\mathbb{Z}_2)$. Transverse Lyapunov exponent calculations revealed that some of this chaos is stable in the three coupled-laser model (4.1)–(4.2), and hence corresponds to partially synchronised. We found that blowout bifurcations are responsible for interesting transitions from this partially synchronised chaos to optical turbulence.

Our results indicate that optical turbulence is a generic property of laser arrays (with $\alpha$ sufficiently large). We thus focused on properties of optical turbulence by considering the Lyapunov spectrum of the underlying chaotic attractors and their dependence on the laser coupling strength, $\kappa$, and the number of lasers, $M$. In particular, we found that for physically realistic parameter values there is an array size, $M$, which if exceeded, offers little significant gain to both the intensity of the chaos, $\mu_1$, and the normalised Lyapunov dimension, $\mu_D/(3M - 1)$. Our results differed from that of a previous study carried out on globally coupled phase oscillators [100]. Further research is needed in order to ascertain
if this difference is due to the coupling structure (nearest-neighbour vs. global), the effect of nonisochronicity (or shear), or a combination of both.
Appendix A

Physical Model of a Solitary laser

The total electric field inside the laser is a real-valued dynamical variable that depends on space and time. It is convenient to separate the spatial and temporal dependence and write the electric field as a sum of products of the space-dependent optical modes $U_j(r)$ and time-dependent complex-valued electric fields $A_j(t)$:

$$
E(r, \tilde{t}) = \frac{1}{2} \sum_j [U_j(r)A_j(\tilde{t}) + \bar{U}_j(r)\bar{A}_j(\tilde{t})].
$$

The index $j$ denotes possible modes of the passive optical resonator. Furthermore, it is useful to rewrite

$$
A_j(\tilde{t}) = \tilde{E}_j(\tilde{t}) e^{-i\hat{\nu}_j \tilde{t}},
$$

where $\tilde{E}_j(\tilde{t}) = |\tilde{E}_j(\tilde{t})|e^{-i\hat{\varphi}_j(\tilde{t})}$ is the slowly-varying complex-valued electric field, and $\hat{\nu}_j \tilde{t}$ is the fast-varying phase. The optical frequencies $\hat{\nu}_j \approx 10^{14}$ s$^{-1}$ are high, and the terms $e^{-i\hat{\nu}_j \tilde{t}}$ are fast-varying. The terms $\tilde{E}(\tilde{t}), \hat{\varphi}_j(\tilde{t}) \sim 10^{10}$ s$^{-1}$ are slow-varying compared to $e^{-i\hat{\nu}_j \tilde{t}}$. The lasing frequency, $\nu_{\text{las}}$, of the $j$-th optical mode is calculated as the time derivative of the total phase of the $j$-th component of the total electric field $E(r, \tilde{t})$,

$$
\nu_{\text{las}} = \hat{\nu}_j + \frac{d\hat{\varphi}_j(\tilde{t})}{d\tilde{t}}.
$$

Furthermore, the mode intensity, in units of Watts over meter squared, can be calculated as

$$
I_j(\tilde{t}) = \frac{1}{2} c\varepsilon_0 n_0^2 |\tilde{E}_j(\tilde{t})|^2,
$$

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where $c$ is the speed of light (units of m/s$^2$), $\epsilon_0$ is the permittivity of free space (dimensionless), and $n_b$ is the index of refraction of the passive semiconductor material (dimensionless). For this reason the magnitude squared of the complex-valued electric field, $|\tilde{E}(\tilde{t})|^2$, is often referred to as the intensity.

### A.1 Rate Equations of Single-Mode Laser

Laser rate equations can be derived in the framework of semiclassical laser theory. The calculations start from the classical Maxwell’s equations and quantum-mechanical Schrödinger equation, and are rather complicated. We skip the intermediate steps and present the final result. Also, we focus on the case where only a single optical mode is amplified, this is called single-mode theory. The time evolution of the single-mode electric field is governed by

\[
\frac{dA}{dt} = -\frac{1}{2}\gamma_c A + \Gamma \frac{c}{n_b} g(\tilde{N}) A - i \left[ \Omega - \Gamma \frac{\nu}{n_b} \delta n(\tilde{N}) \right] A, \quad (A.3)
\]

where $\tilde{N}$ is the number of excited atoms, molecules, or electrons per unit volume and is known as the population inversion (units of m$^{-3}$); $\gamma_c$ is the optical resonator decay rate (units of s$^{-1}$); and $\Gamma$ is the confinement factor (dimensionless) which quantifies the spatial overlap of the optical mode with the semiconductor active medium. The active medium gain ($\tilde{N} > N_{tr}$) or absorption ($\tilde{N} < N_{tr}$),

\[
g(\tilde{N}) = g_{th} + \xi (\tilde{N} - N_{th}), \quad (A.4)
\]

is the amount of light (number of photons) produced or absorbed per unit length (units of m$^{-1}$). Population inversion at transparency is denoted by $N_{tr}$ and defined through

\[
g(N_{tr}) \equiv 0.
\]

Population inversion at threshold is denoted by $N_{th}$,

\[
g_{th} \equiv g(N_{th}) = \gamma_c n_b / (2c\Gamma)
\]

denotes gain at threshold (units of m$^{-1}$), and $\xi$ is the gain coefficient (units of m$^2$). While $N_{th}$ depends on the optical resonator design/parameters, $N_{tr}$ is an inherent property of the active medium, and the two are related via

\[
N_{th} = N_{tr} + n_b \gamma_c / (2c\xi\Gamma).
\]
The resonant frequencies of the passive optical resonator are given by

\[ \tilde{\Omega} \equiv \tilde{\Omega}(\tilde{N} = 0) = j \frac{\pi c}{n_b L}, \]

where \( L \) is the resonator length. In the single-mode theory we consider just one fixed \( j \). Supplying energy to the active medium causes some of its members to be excited to higher energy states. This results in a change of the refractive index which is given by

\[ \delta n(\tilde{N}) = -\frac{c}{\nu} \alpha \xi \tilde{N}, \quad (A.5) \]

where \( \alpha \) is a very important parameter known as the \textit{linewidth enhancement factor} (see Ch. 2.1.3). Physically, \( \alpha \) quantifies the dependence of the refractive index, and hence the laser resonant frequency, on the population inversion \( \tilde{N} \). More specifically, the dependence of the resonant frequency on population inversion is given by

\[ \tilde{\Omega}(\tilde{N}) = j \frac{\pi c}{n_r(\tilde{N}) L}, \quad (A.6) \]

where

\[ n_r(\tilde{N}) = n_b + \Gamma \delta n(\tilde{N}) = n_b - \Gamma \frac{c}{\nu} \alpha \xi \tilde{N}. \quad (A.7) \]

Equations (A.5)–(A.7) with the assumption that \( \delta n << n_b \) lead to the approximation

\[ \tilde{\Omega}(\tilde{N}) = \tilde{\Omega} + \Gamma \frac{c}{n_b} \alpha \xi \tilde{N}, \quad (A.8) \]

where we have used that \( \tilde{\Omega} \) and \( \tilde{\nu} \) are both \( \sim 10^{14} \). Using Eqn. (A.8) we can calculate the resonant frequency at threshold to be

\[ \Omega_{th} \equiv \tilde{\Omega}(N_{th}) = \tilde{\Omega} + \Gamma \frac{c}{n_b} \alpha \xi N_{th}. \]

Substitute Eqs. (A.4) and (A.5) into Eqn. (A.3), and using the formula for \( g_{th} \) leads to

\[ \frac{dA}{dt} = \Gamma \frac{c}{n_b} \xi (\tilde{N} - N_{th})(1 - i\alpha)A - i\Omega_{th}A. \quad (A.9) \]

Next, use Eqn. (A.1) to get

\[ \frac{d\tilde{E}}{dt} = -i(\Omega_{th} - \tilde{\nu})\tilde{E} + \Gamma \frac{c}{n_b} \xi (\tilde{N} - N_{th})(1 - i\alpha)\tilde{E}. \quad (A.10) \]
From Eqn. (A.10) it is clear that $\alpha$ quantifies the amount of coupling between the magnitude, $|\tilde{E}|$, and phase, $\tilde{\varphi}$, of the laser oscillator.

The time evolution for the population inversion is governed by

$$
\frac{d\tilde{N}}{dt} = \tilde{\Lambda} - \gamma_N \tilde{N} - \frac{\varepsilon_0 n_B c}{h\nu} g(\tilde{N})|\tilde{E}|^2. \tag{A.11}
$$

Substituting Eqn. (A.4) into Eqn. (A.11) gives

$$
\frac{d\tilde{N}}{dt} = \tilde{\Lambda} - \gamma_N \tilde{N} - \frac{\varepsilon_0 n_B c}{h\nu} \left[ g_{th} + \xi (\tilde{N} - N_{th}) \right] |\tilde{E}|^2, \tag{A.12}
$$

where $\tilde{\Lambda}$ is the carrier pump rate and $\gamma_N$ is the carrier decay rate. The threshold pump rate is defined as $\Lambda_{th} = \gamma_N N_{th}$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>Ratio of field and population decay rates</td>
<td>100</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Normalised gain coefficient</td>
<td>5.16</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>Normalised pump rate</td>
<td>2</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Normalised coupling strength</td>
<td>[0, 100]</td>
</tr>
<tr>
<td>$\Delta = \Omega_{in} - \Omega_{out}$</td>
<td>Normalised frequency detuning</td>
<td>[-60, 60]</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Linewidth enhancement factor</td>
<td>[0, 5]</td>
</tr>
</tbody>
</table>

Table A.1: Laser parameters and their values [141].

### A.2 Nondimensionalisation

To facilitate numerical computations and presentation of our results, we introduce the normalised variables and time,

$$
E = \frac{\tilde{E}}{|E_0|}, \quad N = \frac{\tilde{N} - N_{th}}{N_{th}}, \quad t = \gamma_N \tilde{t},
$$

where $|E_0| = \sqrt{2h\nu\Gamma_N N_{th}/(\varepsilon_0 n_B^2 c)}$ is the field magnitude of a solitary laser at twice the threshold ($\Lambda = 2\Lambda_{th}$). After nondimensionalising, Eqs. (A.10) and (A.12) can be rewritten in dimensionless form as

$$
\frac{dE}{dt} = \beta \gamma (1 - i\alpha) N E - i(\Omega - \nu) E, \tag{A.13}
$$

$$
\frac{dN}{dt} = \Lambda - N - (1 + \beta N)|E|^2. \tag{A.14}
$$
All of the calculations in this thesis are carried out using the nondimensionalised form of the laser rate equations. The normalised parameters are defined as,

\[ \beta = 1 + \frac{2c\Gamma}{n_b\gamma_c} N_{tr}, \quad \gamma = \frac{\gamma_c}{2\gamma_N}, \quad \Omega = \frac{\Omega_{th}}{\gamma_N}, \]

\[ \nu = \frac{\bar{\nu}}{\gamma_N}, \quad \Lambda = \frac{\bar{\Lambda}}{\gamma_N N_{th}} - 1 = \frac{\bar{\Lambda}}{\Lambda_{th}} - 1. \]

The parameter values used in this thesis are representative of typical semiconductor lasers and are listed in Table A.1. More details of the model derivation and normalisations can be found in [141, 41].
Appendix B

Computation of Lyapunov Exponents in the $M$ Coupled-Laser Model

In general analytical expressions for Lyapunov exponents are not easily achievable. To combat this, numerical methods for their calculation have been developed [114, 14, 15, 31, 102]. Initially, from the definition of Lyapunov exponents (Ch. 2.3.4), one might try integrating the original set of ODEs (2.22), along with Eqs. (2.35) for the time evolution of $M(t)$. Approximations to the Lyapunov exponents could then be obtained from the eigenvalues of $M^T(t)M(t)$ (see Eqs. (2.36)–(2.37)). However there are problems with this approach. Firstly, the magnitudes of some of the eigenvalues of $M^T(t)M(t)$ can be very large, causing this method to fail due to limitations in computing memory [37, 34]. Secondly, Lyapunov exponent calculations are computationally intensive, especially when dealing with large dimensional dynamical systems. Since it is not always necessary to know the entire Lyapunov spectrum of an attractor, a method that only computes a subset of the spectrum would save time. For example, only the four largest Lyapunov exponents (out of nine) were needed to calculate the Lyapunov diagram in Fig. 4.1.

For the calculations in this thesis we used the method developed in Refs. [114, 14, 15]. The main concept is to monitor the time evolution of a set of (originally orthonormal) vectors, whose base is the same trajectory. The vectors stretch in some directions and contract in others. Lyapunov exponents are obtained from the volume of the resulting parallelepiped. The first problem mentioned above is circumvented by periodically re-orthonormalising the set of vectors. The second problem is circumvented by only monitoring as many vectors as the required number of Lyapunov exponents. Good explanations of Lyapunov exponents and their numerical calculations can be found in Refs. [148] and [96, Ch. 4.4]. For results regarding the errors present in Lyapunov exponent calculations
see Refs. [35, 34]. Here, we outline the main steps for calculating Lyapunov exponents, including tangential and transversal, in the \( M \) coupled-laser model (8.3).

To compute the largest Lyapunov exponent (LE) we numerically integrate an extended system which comprises the original system (8.3), and its linearisation that governs the time evolution of a perturbation vector, \( \xi(t) = (\xi_1(t), \cdots, \xi_M(t))^T \in \mathbb{R}^{3M} \),

\[
\frac{d\xi_j}{dt} = Df_j(x_j)\xi_j + \kappa \sum_{j' = 1}^{M} G_{jj'}H\xi_{j'},
\]

where \( Df_j(x_j) \) is the Jacobian of (8.4) evaluated along a trajectory of the original system. Equations (8.3) and (B.1) are numerically integrated subject to initial conditions

\[(x_1(0), \cdots, x_M(0))^T \in A \quad \text{and} \quad \xi(0) = v,\]

where \( A \) is an attractor of interest, and \( v \) has length one and points in arbitrary direction. The vector \( \xi(t) \) is renormalised to length one at times \( t_k = k \tau \), where \( k = 1, \cdots, n \). The largest LE of \( A \) is approximated by

\[
\mu_1 = \frac{1}{n\tau} \sum_{k=1}^{n} \ln |\xi(t_k)|,
\]

for suitably chosen \( n \) and \( \tau \), and using \( \xi(t_k) \) before renormalisation. To compute the remaining LEs we need to know the evolution of \( l = 2, \cdots, 3M \) vectors under (B.1). These vectors must be initially orthonormal, and are orthonormalised at times \( t_k \) using Gram-Schmidt orthonormalisation [14]. The remaining LEs are obtained from

\[
\sum_{i=1}^{l} \mu_i = \frac{1}{n\tau} \sum_{k=1}^{n} \ln(V(l, t_k)),
\]

where \( V(l, t_k) \) is the \( l \)-dimensional volume of the phase-space spanned by the \( l \) vectors before orthonormalisation.

### B.1 Tangential and Transverse Lyapunov Exponents for Attractors Within the Invariant Synchronisation Manifold Fix(\( \mathbb{Z}_2 \))

The Lyapunov spectrum of an attractor contained in an invariant synchronisation manifold, such as \( \text{Fix}(\mathbb{Z}_2) \), splits into two sets. Elements of the first set are known as tangential
LEs, they correspond to perturbations solely within \( \text{Fix}(\mathbb{Z}_2) \), and are denoted by \( \mu_j^\parallel \). Elements of the second set are known as transverse LEs, they correspond to perturbations solely in a direction transverse to \( \text{Fix}(\mathbb{Z}_2) \), and are denoted by \( \mu_j^\perp \).

**B.1.1 Tangential Lyapunov Exponents**

To compute tangential LEs for \( M = 2L - 1 \) odd and \( M = 2L \) even, we numerically integrate two systems. The first system is the restriction of (8.3) to \( \text{Fix}(\mathbb{Z}_2) \). It is obtained by equating variables of lasers marked with the same colour in the second column of Table 8.1, as illustrated in the third column of Table 8.1. The second system is the linearisation of the restricted system and governs the time evolution of the perturbation vector \( \xi(t) = (\xi_1(t), \cdots, \xi_L(t)) \in \text{Fix}(\mathbb{Z}_2) \). The resulting sets of equations have the same general form as (8.3) and (B.1), but with \( j, j' = 1, \cdots, L \), and

\[
G_{jj'} = \begin{cases} 
1 & \text{if } |j - j'| = 1 \text{ for } j = 1, \cdots, L - 1 \text{ and } j' = 1, \cdots, L, \\
2 & \text{if } j = L \text{ and } j' = L - 1, \\
0 & \text{otherwise}, 
\end{cases} \quad (B.4)
\]

for \( M = 2L - 1 \) odd, and

\[
G_{jj'} = \begin{cases} 
1 & \text{if } |j - j'| = 1, \\
1 & \text{if } j = j' = L, \\
0 & \text{otherwise}, 
\end{cases} \quad (B.5)
\]

for \( M = 2L \) even. Tangential LEs are obtained by replacing \( \xi(t_k) \) with \( \xi(t_k) \) in (B.2)–(B.3).

**B.1.2 Transverse Lyapunov Exponents**

To compute transverse LEs, we numerically integrate the restriction of (8.3) to \( \text{Fix}(\mathbb{Z}_2) \) (given in Ch. B.1.1), together with linear equations governing the time evolution of a perturbation vector, \( \xi(t) \), solely transverse to \( \text{Fix}(\mathbb{Z}_2) \). For \( M = 2L - 1 \) odd, the transverse perturbation vector is given by

\[
\begin{pmatrix} 
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_L \\
\xi_{L-1}
\end{pmatrix} = \begin{pmatrix} 
\xi_M - \xi_1 \\
\xi_{M-1} - \xi_2 \\
\vdots \\
\xi_{L+1} - \xi_{L-1}
\end{pmatrix},
\]

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and \( d\xi^j_j / dt \) has the same general form as (B.1), but with \( j, j' = 1, \ldots, L - 1 \), and

\[
G_{jj'} = \begin{cases} 
1 & \text{if } |j - j'| = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

For \( M = 2L \) even, the transverse perturbation vector is given by

\[
\begin{pmatrix}
\xi^1_1 \\
\xi^2_2 \\
\vdots \\
\xi^L_L
\end{pmatrix} = \begin{pmatrix}
\xi_M - \xi_1 \\
\xi_{M-1} - \xi_2 \\
\vdots \\
\xi_{L+1} - \xi_L
\end{pmatrix},
\]

and \( d\xi^j_j / dt \) has the same general form as (B.1), but with \( j, j' = 1, \ldots, L \), and

\[
G_{jj'} = \begin{cases} 
1 & \text{if } |j - j'| = 1, \\
-1 & \text{if } j = j' = L \\
0 & \text{otherwise}.
\end{cases}
\]

Transverse LEs are obtained by replacing \( \xi(t_k) \) with \( \xi^\perp(t_k) \) in (B.2)–(B.3).

### B.2 Convergence of Lyapunov Exponent Calculations

In this section we demonstrate the convergence rate of Lyapunov exponent (LE) calculations in the three coupled-laser model (4.1)–(4.2). More specifically, the method outlined above is used to calculate the four largest LEs, using 30 different sets of randomly distributed initial conditions, for both a non-chaotic attractor (Fig. B.1) and a chaotic attractor (Fig. B.2). The integration step size was taken to be 0.0015 and we reorthonormalised every 200 time steps for a total of 25,000 iterations, i.e. \( \tau = 0.3 \) and \( n = 25,000 \) in Eqn. (B.2).

As an example of a non-chaotic attractor we consider parameter values \( (\alpha, \kappa, \Delta) = (2, 4, -1) \) for which the three-coupled laser model (4.1)–(4.2) has a stable torus (Fig. B.1(a)). In the \( S^1 \)-reduced system this torus corresponds to a limit cycle (Fig. B.1(b)). In Fig B.1(c) we show the convergence of the four largest LEs for just one initial condition. Along the \( x \)-axis is the number of reorthonormalisations on a log-scale. For this particular initial condition the LE calculations converge well, yielding \( \mu_1 = 0.000272, \mu_2 = 0.000332, \mu_3 = -2.33298, \) and \( \mu_4 = -2.33306. \) The results presented in Fig B.1(d) are the envelopes of 30 different LE calculations with randomly distributed initial conditions. The envelope for each LE is initially large and then rapidly decreases with the number of
Figure B.1: (a) projection of a torus from the three coupled-laser model (4.1)–(4.2). (b) in an appropriate coordinate system the torus from (a) is a limit cycle. (c) convergence plot of the four largest LEs. (d) envelopes of the LE calculations for 30 different sets of initial conditions. Parameter values: \((\alpha, \kappa, \Delta) = (2, 4, -1)\).

reorthonormalisations. From the 30 different sets of initial conditions, the range of values for each LE was as follows: 
- \(-0.000190 \leq \mu_1 \leq 0.000346\), 
- \(-0.0000415 \leq \mu_2 \leq 0.000361\), 
- \(-2.33321 \leq \mu_3 \leq -2.33238\), and 
- \(-2.33343 \leq \mu_4 \leq -2.33267\).

As an example of a chaotic attractor (Fig. B.2(a)) we consider the three-coupled laser model (4.1)–(4.2) with parameters \((\alpha, \kappa, \Delta) = (2, 10, -1)\). In Fig. B.2(b) we plot the envelopes obtained from 30 different LE calculations with randomly distributed initial conditions. Again, the envelope for each LE starts large and then decreases with the number of reorthonormalisations. While the LEs appear to be converge, they do not converge as well as in the non-chaotic example. The range of approximations for each LE is as follows: 
- \(10.7158 \leq \mu_1 \leq 10.7821\), 
- \(2.9958 \leq \mu_2 \leq 3.0721\), 
- \(0.000108 \leq \mu_3 \leq 0.000971\),
We can give an indication of the accuracy of our LE calculations by identifying properties of the Lyapunov spectrum that agree with theory. For example, in all of our calculations there is always one zero LE which corresponds to the rotational $S^1$ symmetry of the $M$ coupled-laser model (Ch. 8.4). We note that caution needs to be exercised when interpreting results from the LE calculations alone. They become less reliable when the attractor is close to bifurcation, as here the attraction can be very weak and the fixed time that we integrate for may not accurately capture the slow decay. In addition, if the true value of a LE is within the tolerance interval of what we consider to be zero then it is possible to incorrectly classify the attractor type. An example of this occurs in Fig. 4.9(b)–(c), where the blue region close to zero $\kappa$ is in fact a very weakly stable limit cycle (in the $S^1$-reduced system (4.2) and (4.10)) and hence should be coloured grey, not blue. Nevertheless, when used in conjunction with numerical continuation, and alongside well developed dynamical systems theory, LE calculations are a valuable tool for nonlinear analysis.
Appendix C

Numerical Continuation Code

The majority of the bifurcation diagrams in this thesis were computed using a numerical continuation program called AUTO [36]. AUTO requires two user-supplied files. The first is an equations-file named xxx.f90 (if using Fortran90), where xxx is a user supplied name. This file should contain several Fortran routines, and which of these routines is non-empty depends on the required task. The second file is a constants-file named c.xxx. This is where the user specifies error tolerances, the types of bifurcations that are detected, the parameters to vary, whether the starting point is an equilibrium or a limit cycle, etc. It is important to set the correct constants for the desired task. All the meanings of the different constants along with worked examples are given in the AUTO manual.

C.1 A Worked Example for Three Coupled Lasers

In this section we provide a demonstration of how AUTO was used for the research carried out in this thesis. We do so by locating a codimension-two saddle-node Hopf bifurcation on the locking region’s boundary of three coupled-lasers with $\alpha = 2$ (Fig. 4.9(a)). The equations-file and constants-file for this purpose are given in Appendix C.3 and C.4 respectively. Note that only two routines in the equations-file are non-empty: FUNC contains the right-hand side of Eqs. (4.10) and (4.2), and STPNT contains details of a numerically obtained equilibrium. To run AUTO one has to type commands into AUTO’s command line user interface. Below is a screen-shot of the first continuation (or run) in $\Delta$: 153
The first line is a command issued to AUTO. It tells AUTO that the equations-file is tcl.f90 and that the constants-file is c.tcl. Next comes the output from AUTO. As the equilibrium is continued AUTO detects bifurcations and prints out their type (column headed TY) and the parameter values at which they occur. Aside from the two EPs which stand for endpoints, AUTO detected a saddle-node bifurcation (LP for limit-point) and a branch point (BP), which in this case is a pitchfork bifurcation. The final command is a plotting command which generates the one-parameter bifurcation diagram shown in Fig. C.1(a). Fig. C.1(a) summarises the data generated from this continuation. The stable equilibrium that we started with loses stability via a saddle-node bifurcation. In run 2 we continue the saddle-node bifurcation in two parameters:

The first argument in the run command tells AUTO to start at the LP point from the first run r1. ICP=[1,2] specifies the two continuation parameters, in this case 1 corresponds to κ, and 2 to Δ. ISW=2 selects two-parameter continuation mode. Since we want to detect further codimension-two bifurcations we set ISP=2. DS=0.01 is the start step size and NMX tells AUTO to continue for a maximum of 1000 steps. In this run AUTO detects a saddle-node Hopf bifurcation (ZH for zero Hopf) and then terminates at a user specified point (UZ), which was set to be κ = 10 in the c.tcl file. Figure C.1(b) shows the computed curve of saddle-node bifurcations in the (κ, Δ)-parameter plane.
Proceeding in a similar way, i.e. by detecting bifurcations that affect stable objects and then continuing them, allows us to build up a two-parameter bifurcation diagram. In Fig. C.1(c) we show the two-parameter bifurcation diagram after subsequent runs that detected and continued a curve of Hopf bifurcations. As expected, the saddle-node and Hopf curve are tangent at the saddle-node Hopf bifurcation [75, Ch. 8]. From bifurcation theory we know that there are four possible types of saddle-node Hopf bifurcations [75, Ch. 8], some of which have curves of torus bifurcations emanating from them. In a final step we detect and continue a Torus bifurcation curve which does indeed emanate from the saddle-node Hopf bifurcation point (Fig. C.1(d)).
C.2 Verifying the Accuracy of AUTO’s Results

In this section we use the analytical results obtained in Ch. 9 to verify the numerical results produced by AUTO. To do this we focus on the locking region for three coupled lasers with \( \alpha = 0.3 \) (Fig. 4.2(d)). For \( \Delta = -\kappa \) there exists locked solutions (4.19) of the form (9.42). From Eqn. (9.40) we know that these solutions are stable if

\[
\kappa < \kappa_{\text{anti}} = 12.5861869753 \quad \text{(to 10 decimal places).} \tag{C.1}
\]

Modifying the equations-file in Appendix C.3 by setting \( \Delta = -\kappa \), allows us to start AUTO from a stable locked solution, continue in \( \kappa \) (and hence \( \Delta \)), and detect the parameter values at which the locked solution loses stability. The results from AUTO are

\[
\begin{array}{cccccccc}
\text{BR} & \text{PT} & \text{TY} & \text{LAB} & \text{PAR(1)} & \text{L2-NORM} & U(1) & U(2) & U(3) \\
1 & 1 & EP & 1 & 2.560000E+00 & 5.07335E+08 & 1.41417E+88 & 1.41417E+88 & 1.41417E+88 \\
1 & 105 & HB & 2 & 1.25862E+01 & 5.07338E+08 & 1.41421E+88 & 1.41421E+88 & 1.41421E+88 \\
1 & 200 & EP & 3 & 2.20682E+01 & 5.07338E+08 & 1.41421E+88 & 1.41421E+88 & 1.41421E+88 \\
\end{array}
\]

Total Time 0.763E-01
	
tcl ... done

The Hopf bifurcation that AUTO detects agrees up to 6 decimal places with the analytically obtained \( \kappa_{\text{anti}} \) from Eqn. (C.1). In fact, AUTO does better than this because the output above has been truncated for the computer screen. Looking up the number in one of AUTO’s diagnostic files (fort.7) leads to the even better approximation of 12.586187979.

C.3 Equations-File

```
! ---------------------------------------------------------
!
! tcl : S1 reduced system of Three Coupled Lasers
!
!
!
SUBROUTINE FUNC(NDIM,U,ICP,PAR,IJAC,F,DFDU,DFDP)
!
! Evaluates the algebraic equations or ODE right hand side
!
```

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! Input arguments :
!   NDIM : Dimension of the ODE system
!   U : State variables
!   ICP : Array indicating the free parameter(s)
!   PAR : Equation parameters
!
! Values to be returned :
!   F : ODE right hand side values
!
! Normally unused Jacobian arguments : IJAC, DFDU, DFDP (see manual)
!
IMPLICIT NONE
INTEGER NDIM, IJAC, ICP(*)
DOUBLE PRECISION U(NDIM), PAR(*), F(NDIM), DFDU(NDIM,*), DFDP(NDIM,*)
DOUBLE PRECISION A,B,C,phiBA,phiBC,NA,NB,NC
DOUBLE PRECISION kappa, Delta, alpha
!
REAL, PARAMETER :: c_light = 2.99792458E08
REAL, PARAMETER :: Gama = 0.1
REAL, PARAMETER :: xi = 2.5E-20
REAL, PARAMETER :: n_b = 3.6
REAL, PARAMETER :: gammaN = 1E9
REAL, PARAMETER :: Ntr = 2E24
REAL, PARAMETER :: gammac = 2E11
REAL, PARAMETER :: gamma = gammac/(2*gammaN)
REAL, PARAMETER :: beta = 1 + (2*c_light*Gama*xi*Ntr)/(n_b*gamma)
REAL, PARAMETER :: Lambda = 2
!
A = U(1)
B = U(2)
C = U(3)
phiBA = U(4)
phiBC = U(5)
NA = U(6)
NB = U(7)
NC = U(8)
!
! kappa = PAR(1)
    Delta = PAR(2)
    alpha = PAR(3)
!
F(1) = beta*gamma*NA*A - kappa*B*sin(phiBA)
F(2) = beta*gamma*NB*B + kappa*A*sin(phiBA) + kappa*C*sin(phiBC)
F(3) = beta*gamma*NC*C - kappa*B*sin(phiBC)
\[ F(4) = \kappa \times \left( \frac{A}{B} - \frac{B}{A} \right) \cos(\phi_{BA}) + \kappa \times \left( \frac{C}{B} \right) \cos(\phi_{BC}) - \alpha \times \beta \times \gamma \times (NB - NA) - \Delta F(5) = \kappa \times \left( \frac{C}{B} - \frac{B}{C} \right) \cos(\phi_{BC}) + \kappa \times \left( \frac{A}{B} \right) \cos(\phi_{BA}) - \alpha \times \beta \times \gamma \times (NB - NC) - \Delta 
\]

\[ F(6) = \Lambda - NA - (1 + \beta \times NA) \times A \times A \]
\[ F(7) = \Lambda - NB - (1 + \beta \times NB) \times B \times B \]
\[ F(8) = \Lambda - NC - (1 + \beta \times NC) \times C \times C \]

```
SUBROUTINE STPNT(NDIM, U, PAR, T)

! Input arguments :
! NDIM : Dimension of the ODE system
!
! Values to be returned :
! U : A starting solution vector
! PAR : The corresponding equation-parameter values
! T : Not used here
!
IMPLICIT NONE
INTEGER NDIM
DOUBLE PRECISION U(NDIM), PAR(*), T
!
! Initialize the equation parameters
PAR(1) = 1.2
PAR(2) = -5.
PAR(3) = 2.
!
! Initialize the solution
U(1) = 1.4192877
U(2) = 1.403990
U(3) = 1.4192877
U(4) = -2.560978
U(5) = -2.560978
U(6) = -0.001260995
U(7) = 0.0025772
U(8) = -0.001260995
!
!
RETURN
```
The following subroutines are not used here, but they must be supplied as dummy routines:

```fortran
SUBROUTINE BCND
    RETURN
END

SUBROUTINE ICND
    RETURN
END

SUBROUTINE FOPT
    RETURN
END

SUBROUTINE PVLS
    RETURN
END
```

C.4 Constants-File

```
NDIM=  8, IPS =  1, IRS =  0, ILP =  1
ICP = [2]
NTST=  50, NCOL=  4, IAD =  3, ISP =  1, ISW =  1, IPLT=  0, NBC=  0, NINT=  0
NMX = 100, NPR=  200, MXBF=  0, IID =  2, ITMX=  8, ITNW=  5, NWTN=  3, JAC=  0
EPSL= 1e-07, EPSU = 1e-07, EPSS = 1e-05
DS = -0.01, DSMIN= 0.0001, DSMAX= 0.1, IADS=  1
NPAR=  3, THL = {11: 0.0}, THU = {}
UZR = {-1:0,-1:10}
```
Bibliography


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