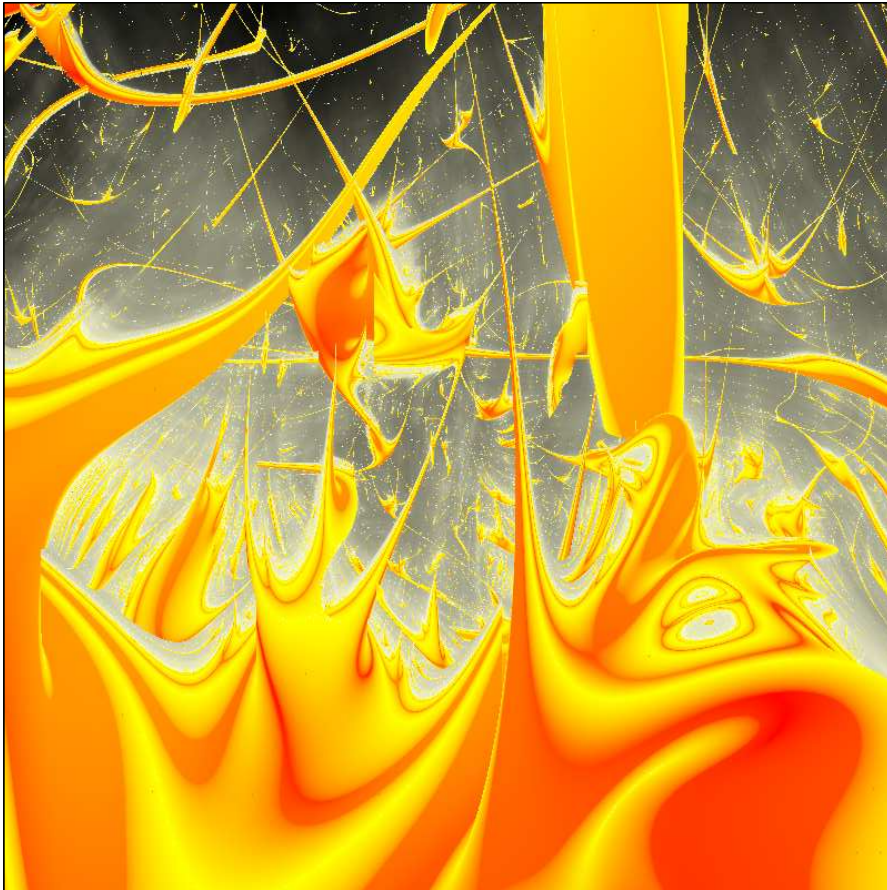


# A Journey Through the Dynamical World of Coupled Laser Oscillators



A thesis submitted to the University of Exeter  
by Nicholas Blackbeard

First Supervisor: Dr Sebastian Wieczorek  
Second Supervisor: Dr Hartmut Erzgräber

Image on front cover: Stability diagram for three coupled laser oscillators. Specifically, it is an expanded view from Fig. 4.9(b)–(c) with a different colour scale; yellow-red = periodic intensity fluctuations, and white-black = chaotic intensity fluctuations. This picture received a prize for the Engineering, Mathematics, and Physical Sciences Image Competition.

# A Journey Through the Dynamical World of Coupled Laser Oscillators

Submitted by Nicholas Blackbeard, to the University of Exeter as a thesis  
for the degree of Doctor of Philosophy in Mathematics, January 2012.

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# Abstract

The focus of this thesis is the dynamical behaviour of linear arrays of laser oscillators with nearest-neighbour coupling. In particular, we study how laser dynamics are influenced by laser-coupling strength,  $\kappa$ , the natural frequencies of the uncoupled lasers,  $\tilde{\Omega}_j$ , and the coupling between the magnitude and phase of each laser's electric field,  $\alpha$ . Equivariant bifurcation analysis, combined with Lyapunov exponent calculations, is used to study different aspects of the laser dynamics. Firstly, codimension-one and -two bifurcations of relative equilibria determine the laser coupling conditions required to achieve stable phase locking. Furthermore, we find that global bifurcations and their associated infinite cascades of local bifurcations are responsible for interesting locking-unlocking transitions. Secondly, for large  $\alpha$ , vast regions of the parameter space are found to support chaotic dynamics. We explain this phenomenon through simulations of  $\alpha$ -induced stretching-and-folding of the phase space that is responsible for the creation of horseshoes. A comparison between the results of a simple *coupled-laser model* and a more accurate *composite-cavity mode model* reveals a good agreement, which further supports the use of the simpler model to study coupling-induced instabilities in laser arrays. Finally, synchronisation properties of the laser array are studied. Laser coupling conditions are derived that guarantee the existence of synchronised solutions where all the lasers emit light with the same frequency and intensity. Analytical stability conditions are obtained for two special cases of such laser synchronisation: (i) where all the lasers oscillate in-phase with each other and (ii) where each laser oscillates in anti-phase with its direct neighbours. Transitions from complete synchronisation (where all the lasers synchronise) to optical turbulence (where no lasers synchronise and each laser is chaotic in time) are studied and explained through symmetry breaking bifurcations. Lastly, the effect of increasing the number of lasers in the array is discussed in relation to persistent optical turbulence.

## Acknowledgements

First and foremost I would like to thank my supervisors and friends, Sebastian and Hartmut, for their never-ending patience, support, and of course, much needed constructive criticism. I am extremely grateful that, no matter how busy they were, they could always find time for me, whether to provide a solution, point me in the right direction, admire some mathematics, or just have a chat. Without their unwavering guidance through the beautiful and complex world of dynamical systems I could never have finished this thesis. In addition, I would like to thank my examiners, Prof. Stuart Townley and Prof. Jonathan Dawes, who had the arduous task of reading this thesis in its entirety. I greatly appreciate their feedback and advice, and feel that the thesis is all the better for it.

Next I would like to thank everyone from the Mathematics department at Exeter who have contributed to the truly wonderful experience I have had overall. I am grateful to the Dynamical Systems community who always provided stimulating new subjects to discuss, which was at times hard work, but always very rewarding. I also owe half my sanity to Mark and Hartmut for our regular, always memorable, excursions to some of the best playgrounds in the UK, including the cliffs of the south-west, the rolling expanses of Dartmoor, and the rugged peaks of the Lake District and Snowdonia. Another welcome source of distraction came from playing squash with Özgür, Hartmut, Özkan, Sam, Jack, Richard and Amrita.

The other half of my sanity I owe to my dearest friends and climbing partners. There have been many over the years, but I would like to mention a few in particular: James, Tom, Gemma, Eva and Dan. Bonds built when attached to the end of a rope tend to be everlasting and I believe this to be true for all those mentioned. One in particular I have become rather fond of. This is my partner, Gemma, whom I am indebted to. She has been a constant source of happiness, particularly when times have been difficult. For this I am eternally grateful.

Finally, I would like to thank my Mum, Dad, sister, and brother for always being there, the life advice and support that they offer, and their unconditional love.

I would like dedicate this work to Guks, whose warm and positive outlook on life will be greatly missed.



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