Essays on Monetary Policy

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Abstract

This thesis consists of three essays on optimal monetary policy. In the first essay I study time-consistent monetary policy in an small open economy model with incomplete financial markets. I demonstrate the existence of two discretionary equilibria. The model is capable of explaining periods of different exchange rate volatilities as well as the transition between those regimes. Following a shock the economy can be stabilised either ‘quickly’ or ‘slow’, where both dynamic paths satisfy the conditions of optimality and time-consistency. I also show that a policy of partially targeting the exchange rate results in far worse welfare outcomes relative to a strict inflation targeting policy.

In the second essay, I analyse how a policy maker can avoid expectation traps and coordination failures. Using a framework developed by Schaumburg and Tambalotti (2007) and Debortoli and Nunes (2010) in which a policy maker may or may not default on past promises I show that already mild degrees of precommitment are sufficient to generate uniqueness of the Pareto-preferred equilibrium.

In the last chapter, I examine optimal monetary policy from an empirical perspective. I estimate a simple small open economy model separately for a policy maker acting under commitment and discretion and find that the data favours the commitment approach. Furthermore, the data suggest that the Bank of Canada did not target the nominal exchange rate in the inspected time period.
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Chapter 1

Introduction

Since the famous critique of Lucas (1977) the expectations of economic agents play a prominent role in any policy analysis in macroeconomic models. Following Kydland and Prescott (1977) a huge research program emerged studying the strategic interactions between policy makers and the private sector in a rational expectations framework with forward looking agents. Most of the debate on monetary policy design centered around the question if a policy maker should commit to a policy plan or if he should conduct monetary policy in a more flexible, discretionary way. A commitment policy implies that the policy maker always keeps his promises and can stick to the policy plan he or his predecessors announced in the past.\footnote{In more technical terms, under commitment a policy maker can coordinate all future actions of consequent policy makers, which allows him to choose once, and apply indefinitely, an intertemporal contingency plan.} In difference, under discretion the policy maker is allowed to adjust the policy plan in each period disregarding past promises. As first shown by Kydland and Prescott (1977) discretionary policy results in a lower level of social welfare compared to commitment. Crucial for their finding is the way private sector expectations affect the design of monetary policy. Under a commitment policy the policy maker has full control over the expectations of all economic agents and can steer them in a way to achieve a
socially desirable outcome. However, under discretion the policy maker has to take the expectation of the private sector as given. There is no room to manipulate them, because the forward looking private agents understand that in the future period the policy maker will reoptimize again.

Since the seminal works by Kydland and Prescott (1977) and Calvo (1978) it has been well understood that in models with rational expectations a commitment policy suffers from a credibility problem and is generally time-inconsistent. A policy is called time-inconsistent if the policy maker has an incentive to deviate in future periods from previously announced policy plans, which were optimal at the time of the announcement. As the private agents are forward looking and rational, they understand that the promised policy plan is not credible and adjust their expectation accordingly. Barro and Gordon (1983b) extended the analysis of Kydland and Prescott (1977) and developed a positive theory of monetary policy. In their model the policy maker faces a trade-off between stabilizing inflation and unemployment. They find that discretionary policy results in an \textit{inflation bias}, i.e. an inflation rate that exceeds its socially desired level, while a credible central banker can achieve a socially optimal inflation rate of zero. The inflation bias occurs, because the central banker tries to push unemployment below its natural level.

Recent research suggests that even in a low inflation environment and in the absence of overambitious policy goals the time inconsistency problem remains important. It has been shown e.g. by Clarida et al. (1999) and Woodford (2003a) that discretionary policy can also lead to a \textit{stabilization bias} and therefore to a lower social welfare outcome relative to commitment. Again, the expectations of the private sector are at the center of this result. A policy maker without a credible commitment technology cannot exploit the expectations of the private sector and faces a worse trade-off between his goal variables compared to the full commitment solution.\textsuperscript{2}

\textsuperscript{2}In the models used by Clarida et al. (1999) and Woodford (2003a) the stabilization bias arises, because a central banker faces a trade-off between stabilizing inflation and the output gap. In this branch of models discretionary policy leads to a higher volatility in the inflation rate and a stronger...
A long line of research exists aiming to resolve the time-inconsistency problem. In general one can broadly divide possible solutions to this problem along two different approaches. One approach is based on the theory of repeated games (see e.g. Barro and Gordon (1983a) and Backus and Drifill (1985)). In this line of research the policy maker bears a cost (e.g. loss in reputation), when reneging on its past promises. If this cost is high enough, it is possible to sustain equilibria that are close to the commitment solution. The second approach tries to alter the institutional setting to bring the policy behaviour under discretion closer to the commitment solution. Famous examples are to delegate monetary policy to a conservative central banker (Rogoff (1985)) or to set-up contracts/ rules that minimizes the incentive of a policy maker to exploit private sector expectations (Walsh (1995), Svensson (1997a)).

However, discretionary policy suffers from another potential drawback. As King and Wolman (2004) and Albanesi et al. (2003) show an optimal time consistent policy may lead to multiple equilibria in models with rational expectations.\(^3\) Under discretionary policy multiple equilibria may arise because the expectations of the private sector are shaped by anticipations about future policy behavior. Since coordination among subsequent discretionary policy makers is impossible, a policy maker cannot control these expectations.\(^4\) Hence, every policy maker faces expectations concerning his behavior which were set in the past by the private sector and which cannot be changed retrospectively. Those expectations may trap the policy maker into implementing a policy that validates them. The trap is closed and a coordination failure occurs if it is less costly for the policy maker to validate the beliefs of the private sector about future policy than to ignore those expectations and stick to his initial policy plan.

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\(^3\)Lockwood and Philippopoulos (1994) demonstrated the existence of multiple discretionary equilibria in a model with static expectations, while Blake and Kirsanova (2012) extended the analysis to LQ RE models.

\(^4\)In the terminolgy of Cooper and John (1988) there is a potential for strategic complementarities to arise due to the interaction among current and future policymakers and private sector expectations.
Although multiplicity has neat implications from an empirical perspective\(^5\), it imposes problems from the view of a central banker. First, the task of monetary policy becomes more complex. In the presence of multiple equilibria, a policy maker will not be solely focused on stabilizing the economy, but also on coordinating beliefs on the socially optimal equilibrium and on preventing costly switches among different equilibria. Secondly, some of the solutions to the time-inconsistency problem are relying on the uniqueness property, e.g. the delegation approach. Furthermore, as the welfare costs in the bad equilibria and the cost associated with the transition between different equilibria are likely to be higher than the cost associated with the time-inconsistency problem, it seems to be more beneficial to develop rules that are able to select the best equilibrium and help to avoid such fluctuations between different equilibria.

The aim of this thesis is to contribute to a better understanding of optimal monetary policy in linear quadratic rational expectation models. A particular focus will be on the implications of multiplicity for the design of monetary policy.

This thesis consists of three main chapters (2-4). In Chapter 2 we give an example of how discretionary policy may help to understand empirical observations about the exchange rate behaviour in developing countries. We show how discretionary policy may lead to different periods of exchange rate volatility. We use a small open economy New Keynesian model with incomplete financial markets and demonstrate the existence of two discretionary equilibria. Following a shock the economy can be stabilised either ‘quickly’ or ‘slow’, where both dynamic paths do satisfy optimality conditions and time-consistency. The model argues that tranquil and volatile exchange rate periods and the transition between them are a result of the interaction between the expectations of the private sector and monetary policy. We also show that delegating monetary policy to a central banker who is concerned

\(^5\)Models with multiple equilibria can e.g. explain periods of high and low inflation (Albanesi et al. (2003)) where the transition between between the different economic states can be a result of changes in the policy regime (see e.g. Davig and Leeper (2006)).
about exchange rate volatility results in far worse welfare outcomes compared to a strict inflation targeting regime.

Chapter 3 contains an analysis on how a policy maker can avoid expectation traps and coordination failures. We build upon the limited commitment framework developed by Schaumburg and Tambalotti (2007) and Debortoli and Nunes (2010). This framework allows us to study the intermediate cases between discretion and commitment. Under limited commitment every period a new policy maker arrives in office with an exogenous probability. After arrival the new policy maker will renege on the past policy plan of his predecessor and credible commit to a new policy plan that is optimal at this point in time. This policy plan will be in place till a new policy maker arrives. Expectation traps and coordination failures are likely to occur in this framework, because a policy maker can neither completely control the expectations of the private sector, nor can he coordinate the actions of all future policy makers. We demonstrate the existence of coordination failures and multiple equilibria in such a framework and identify the minimum degree of commitment which is needed to escape from expectation traps. We find that already a mild degree of precommitment is sufficient to generate uniqueness of the Pareto-preferred equilibrium.\(^6\)

In Chapter 3 we investigate the question which policy behavior, discretion or commitment, fits the data better. We estimate a simplified version of a small open economy model separately under commitment and discretion for the Canadian economy using Bayesian techniques. In difference to the previous literature, we find that the monetary policy in Canada is best described as a regime under commitment. Additionally, there is no evidence of (partial) exchange rate targeting by the Bank of Canada for the analysed time period.

\(^6\)The results vary depending on the underlying New Keynesian model, but we find that an average office tenure between 2-5 years is enough to ensure uniqueness.
Chapter 2

The Interest Rate - Exchange Rate Nexus: Exchange Rate Regimes and Policy Equilibria

2.1 Introduction

Fixing nominal exchange rates is frequently justified as a way to avoid excessive variability of economic variables, in particular in developing countries. The idea behind an exchange rate peg is that it will anchor inflation expectations, increase trade directly through lower uncertainty and smaller adjustment costs, and indirectly through its effect on the allocation of resources and government policies (see Côte (1994)). It may also encourage investment into long-term projects due to lower exchange rate risk/transaction costs and therefore has a positive economic impact (see Prasad et al. (2003)). However, being prone to speculative attacks hard pegs became less popular, especially after the Asian crisis of 1997. On the other hand recent evidence suggests that monetary authorities in many developing countries still see targeting the nominal exchange as their main priority, despite that they officially
claim to have floating regimes.\textsuperscript{1} Developing and emerging countries like Indonesia, Malaysia, Thailand, South Korea, Turkey, Russia adopted \textit{de jure} flexible exchange rate regimes, but \textit{de facto} the exchange rate remained one of the most important if not the only target of their monetary policy.\textsuperscript{2} Reinhart and Rogoff (2004) report that a crawling peg was the most common type of exchange rate arrangement in the Asian emerging countries between 1990 and 2001.

Despite a relatively tranquil post-1997 decade in most developing and emerging countries, the exchange rate volatility under these ‘soft pegs’ varied over time.\textsuperscript{3} There is a number of studies that document difficulties in explaining these sudden changes in ‘regimes’ between periods of high and low volatilities.\textsuperscript{4} Theoretical explanations for these different regimes include non-rational behaviour, non-linear decisions or heterogeneity of agents like the presence of ‘noise traders’ (see Jeanne and Rose (2002) for an important example).

The main aim of this paper is to present a much simpler model that can help to understand some of these empirical facts. We claim that the way \textit{how} monetary policy is conducted can be responsible for the existence of time periods with large difference in the volatility of macroeconomic variables. We employ a simple linear stochastic model that has become the workhorse model in monetary economics and abstract from many features that may characterize many developing or emerging countries, e.g. capital controls or incomplete exchange rate pass-through. However, we account for incomplete financial markets and study discretionary monetary policy. Both of these features are fairly common in developing countries: financial markets are incomplete and governments cannot precommit. The above set-up is

\textsuperscript{1}See e.g Levy-Yeyati and Sturzenegger (2005) and Calvo and Reinhart (2002).
\textsuperscript{2}See Rahmatsyah et al. (2002) for Thailand, Dogolnar (2002) for Turkey, Korea, Malaysia, Indonesia and Pakistan, and Arize et al. (2000) for 13 developing countries.
\textsuperscript{3}We do not take into account the recent financial crisis.
\textsuperscript{4}See e.g. Engel and Hamilton (1990), Clarida et al. (2003) or Chen (2006) who apply Markov-switching models to explain these changes. These models have also been employed to describe exchange rate behaviour in floating regimes. However, their succes is still a matter of current debate see e.g. again Clarida et al. (2003) and Engel et al. (2007).
sufficient to generate multiple policy equilibria that will also exist in more detailed models. Specifically, we demonstrate the existence of two rational expectations equilibria that are associated with different speeds of adjustment towards the steady state and therefore with different volatilities of all macroeconomic variables.

The keys to multiplicity are the time-consistency property of discretionary policy and dynamic complementarities. Under discretionary policy, the policy maker takes current and future economic conditions into account, but can only commit to current behavior. The economic conditions are affected by the response of the rational private sector and any response is based on a forecast of future economic conditions. As a consequence multiple equilibria may arise: A policy maker responds to a state that is at least partly determined by forecasts of his behaviour. Different sets of beliefs about the future policy generate different future courses for policy to follow. Therefore, if the economy is hit by a shock, it can follow one of several adjustment paths, where the volatility along these paths is different. In the presence of several equilibria, a coordination failure occurs: the agents can choose any of them and a sunspot decides which one will realise.

A necessary condition for multiple equilibria is the existence of strategic complementarities. We demonstrate that our model is able to generate such complementarities. Under conventional inflation targeting, following an increase in the interest rate, the effect of consumption on the terms of trade reinforces the effect of the interest rate on the terms of trade. Thus we have a strategic complementarity between consumption and the terms of trade in their effect on marginal cost, which are crucial for the control of inflation. The resulting two equilibria can be classified as ‘dry/patient’ and ‘wet/impatient’, based on the observed strength of the interest rate response.

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In the second part of this chapter, we look at a policy maker who introduces an additional positive weight in its policy objective that punishes the volatility of the nominal exchange rate (provided that the anchor country ensures price stability). In this case the ‘wet/impatient’ equilibrium becomes non-existent, while another equilibrium with even lower social welfare arises. If the policy maker solely targets the nominal exchange rate in the objective (‘currency peg’), he is able to stabilise the exchange rate only if there is a common belief that the nominal exchange rate remains stable. Additionally, depending on the forecasts of the private sector another equilibrium may prevail. If there is a common belief that the nominal exchange rate is going to depreciate/appreciate in the future – non-zero exchange rate volatility is consistent with targeting policy – it becomes optimal for a policy maker to validate these beliefs and generate the forecasted depreciation/appreciation. This is in line with empirical work by Engel et al. (2007) who find that short-run movements in exchange rates are primarily determined by changes in expectations.

Our paper is also related to the work on ‘optimal delegation’. For discretionary policy Rogoff (1985) shows that it may be desirable for a society to allow the objective function of the monetary authority and the social objective to differ in order to improve overall social welfare.\footnote{A non-exhaustive list includes Rogoff (1985) and Svensson (1997a) on inflation conservatism, Woodford (2003b) on interest rate smoothing policy and Vestin (2006) on price-level targeting.} We demonstrate that adding a exchange rate stabilisation target to the objective function can marginally improve social welfare, but only in the worst equilibrium; the additional exchange target is damaging for the best equilibrium. Hence, we suggest that at least in our framework a ‘soft peg’ is, generally speaking, undesirable.

The chapter is organized as follows. The next Section outlines the model. Section 2.3 discusses the two policy equilibria for an inflation targeting regime and Section 2.4 discusses them under nominal exchange rate targeting. Section 2.5 concludes.
2.2 A Small Open Economy Model with Incomplete Financial Markets

The framework is relatively standard and builds heavily on the small open economy model introduced by Galí and Monacelli (2005). We allow for a non-zero current-account balance by including incomplete financial markets following a framework proposed by Benigno (2001). There are two economies: the small open economy and the rest of the world. The economic performance and domestic policy decisions of the small open economy do not have any impact on the rest of the world. Both economies are populated by a continuum of infinity-living households, which consume two goods. One is produced domestically and the other good is imported from the rest of the world, which we treat as a single, large ‘foreign country’. The law of one price holds, but deviations from purchasing power parity (PPP) arise due to the existence of home bias in consumption. Production takes place in two stages. First, there is a continuum of intermediate goods firms, which produce a differentiated input. In the second stage final goods producers combine these inputs into output and sell them to households in both countries. Monopolistic competition and sticky prices are introduced to get a meaningful role for policy.

2.2.1 Households

Both economies, home ($H$) and foreign ($F$), consist of a continuum of infinity-living households and share identical preferences and technology. We assume that every household seeks to maximize the following utility function

$$
E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right]
$$

(2.1)

\(^8\)In a very similar model De Paoli (2009b) analyzes the welfare effects of incomplete financial markets under a Ramsey (precommitment) policy.
where $C_t$ denotes private consumption and $N_t$ hours of labour. $\beta$ is the subjective discount rate, $1/\varphi$ measures the Frisch-elasticity of labour supply and $E_0$ is the actuarial expectation at time $t = 0$. In more detail $C_t$ is a composite consumption index defined by
\[
C_t \equiv \left[ (1 - \alpha)^{\frac{1}{\gamma}} (C_{H,t})^{\frac{\gamma + 1}{\gamma}} + \alpha^\gamma (C_{F,t})^{\frac{\gamma + 1}{\gamma}} \right]^{\frac{\gamma}{\gamma + 1}}.
\]

Parameter $\eta > 0$ denotes the elasticity of substitution between domestic and foreign produced goods from the viewpoint of the domestic consumer. Parameter $\alpha \in [0, 1]$ is the weight of imported goods in private home consumption and is inversely related to the degree of home bias in preferences. Another interpretation for $\alpha$ is as a natural index of openness.

$C_{H,t}$ and $C_{F,t}$ are the Dixit-Stiglitz indexes of consumption of domestic and foreign goods given by the CES (constant elasticity of substitution) aggregators
\[
C_{H,t} = \left( \int_0^1 C_{H,t}(j)\frac{1}{\epsilon}dj \right)^{\frac{1}{1-\epsilon}}; \quad C_{F,t} = \left( \int_0^1 C_{F,t}(j)\frac{1}{\epsilon}dj \right)^{\frac{1}{1-\epsilon}}
\]
where $j \in [0, 1]$ denotes the good variety and $\epsilon > 1$ is the elasticity of substitution between varieties of goods produced within a given country. The appropriate consumption-based price index (CPI) that corresponds to the above specification is
\[
P_t = \left[ (1 - \alpha)P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta} \right]^{\frac{1}{1-\eta}}
\]
with
\[
P_{H,t} = \left( \int_0^1 P_{H,t}(j)\frac{1}{1-\epsilon}dj \right)^{\frac{1}{1-\epsilon}}; \quad P_{F,t} = \left( \int_0^1 P_{F,t}(j)\frac{1}{1-\epsilon}dj \right)^{\frac{1}{1-\epsilon}}
\]
where $P_{H,t}(j)$ is the price of domestic good $j$ and $P_{F,t}(j)$ denotes the price of variety $j$ imported from country $F$, where the latter is expressed in domestic currency. Assuming that the law of one price holds at all times and that all goods are traded we can write
\[
P_{H,t} = \mathcal{E}_t P^*_t; \quad P_{F,t} = \mathcal{E}_t P^*_t
\]
where $\mathcal{E}_t$ is the nominal exchange rate, given as the price of one unit of foreign currency in terms of home currency: The terms of trade are defined as the price
of foreign goods relative to the price of goods produced in country $H$ and given by $S_t = P_{F,t}/P_{H,t}$. The real exchange rate can be written as $Q_t = \mathcal{E}_t P_t^* / P_t$. Note that the purchasing power parity condition $P_t = \mathcal{E}_t P_t^*$ does not hold in the short run due to the presence of home bias in consumption ($\alpha < 1$).

The demand for good $j$ produced in a given country can be written as

$$C_{H,t}(j) = \left( \frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\epsilon} C_{H,t}; \quad C_{F,t}(j) = \left( \frac{P_{F,t}(j)}{P_{F,t}} \right)^{-\epsilon} C_{F,t}$$

for all $j \in [0, 1]$. Note that as $\epsilon$ rises, the individual goods become closer substitutes and therefore the individual firms have less market power.

Finally, the optimal condition of expenditures between domestic and imported goods is given by

$$C_{H,t} = (1 - \alpha) \left( \frac{P_{H,t}}{P_t} \right)^{-\eta} C_t; \quad C_{F,t} = \alpha \left( \frac{P_{F,t}}{P_t} \right)^{-\eta} C_t$$

Correspondingly we can write total consumption expenditures by domestic households as

$$P_t C_t = P_{H,t} C_{H,t} + P_{F,t} C_{F,t}$$

Households of country $H$ can trade in two nominal one-period, risk-free bonds. One bond is denominated in home currency, the other in foreign currency. We assume that the home currency denominated bonds is solely traded domestically, so only the foreign bond is traded internationally. Furthermore, households belonging to country $H$ have to pay an intermediation cost, if they want to trade in the foreign bond.\footnote{This cost of bond holding ensures stationarity, see Schmitt-Grohe and Uribe (2003).} Let $\mathcal{D}_t$ and $\mathcal{D}_t^*$ denote the nominal holdings in the home and foreign bond of all households belonging to country $H$, the latter is measured in foreign currency. Thus, the nominal intertemporal budget constraint for a household $i$ belonging to country $H$ is given by

$$P_t C_t^i + \frac{\mathcal{D}_t^i}{(1 + i_t)} + \frac{\mathcal{E}_t \mathcal{D}_t^{i*}}{(1 + i_t^*)} \leq \mathcal{D}_{t-1}^i + \mathcal{E}_t \mathcal{D}_{t-1}^{i*} + (1 - \Upsilon_t) (W_t N_t^i + \Pi_t) \quad (2.2)$$

$$- \frac{\chi P_t}{2 (1 + i_t^*)} \left( \frac{\mathcal{E}_t \mathcal{D}_t^{i*}}{P_t} - \bar{d} \right)^2 + P_{H,t} \bar{r}_t^i.$$
where $W_t$ is the nominal wage and $Tr_t$ denotes lump-sum taxes/transfers to household $j$, which include government transfers. $\Upsilon_t$ denotes a country specific tax on nominal income. Here, $D_t^*$ is denominated in foreign currency and $D_t$ are home currency denominated assets. Following Benigno (2009) and Schmitt-Grohe and Uribe (2003) there is a quadratic cost occurring when domestic households change their foreign asset position and trade in the foreign bond market $\bar{\delta}$, which is a constant and will be used to determine the steady state level of foreign assets. $\chi$ is a non-negative parameter that measures this cost in terms of units of the consumption index, which is rescaled by the factor $1/(1 + i_t^*)$ just for analytical convenience and without losing generality.\footnote{The cost of moving the holdings of foreign assets determines the steady-state value of the foreign-asset position in a zero-order approximation without the need of taking second-order approximations as in Devereux and Sutherland (2008).}

In characterizing the budget constraint, we also assume that all the households belonging to a country share the revenues from running the firms in equal proportion. In particular we assume that the households in the foreign country share their intermediation profits $K^*$ equally, where $K^*$ is defined analogous to Benigno (2001)

$$K^* = \int \frac{\chi}{2(1 + i_t^*)} \left( \frac{\xi_t D_t^{is} - \bar{\delta}}{P_t} \right)^2 d_j.$$

The foreign budget constraint of household $i$ is then given as

$$P_t^* c_t^{is} + \frac{D_t^{is}}{(1 + i_t^*)} = D_{t-1}^{is} + (1 - \Upsilon_t^*) \left( W_t^* N_t^{is} + \Pi_t^{is} \right) + P_{t,F_t}^* T_{r_t}^{is} + K^*.$$

We further assume that the initial level of wealth is the same across all households. Together with the assumption that households in a given country produce all goods and firm’s profits are equally shared, this implies that all households within a country face the same budget constraint. Hence, every household faces the same decision problem and we can consider a representative household for each country. However, consumption will not be necessarily risk shared at an international level, because although there is idiosyncratic risk pooling among consumers within the same country, there will not be necessarily risk sharing at an international level.
Maximizing (2.1) with respect to (2.2) yields the following FOCs:

$$E_t (M_{t,t+1} (1 + i_t)) = 1$$

which results from the small open economy optimal choice of bonds denominated in domestic currency and

$$E_t \left( M_{t,t+1} (1 + i_t^*) \frac{\xi_{t+1}}{\xi_t} \right) = 1 + \chi \left( \frac{E_t D_t^*}{P_t} - \bar{d} \right)$$

reflects trading in the bond denominated in foreign currency, where the nominal stochastic discount factor is defined as

$$M_{t,t+1} = \beta \frac{C_t^\sigma P_t}{C_t^{\sigma+1} P_{t+1}^{\sigma}}.$$  

A similar Euler equation holds for the rest or the world:\footnote{Because of Walras Law the foreign budget constraint is redundant (see Benigno (2001)).}

$$E_t M_{t,t+1}^* (1 + i_t^*) = 1.$$  

which reflects trading in foreign currency dominated bonds, where the nominal stochastic discount factor is again

$$M_{t,t+1}^* = \beta \left( \frac{C_t^*}{C_t^{*+1}} \right)^\sigma \frac{P_t^*}{P_{t+1}^*}.$$  

The aggregate budget constraint of country H is obtained by integrating the budget constraints of the households living in country H together with the government budget constraint defined below

$$\frac{E_t D_t^*}{(1 + i_t^*)} = E_t D^*_{t-1} + P_{H,t} (Y_t - G_t) - P_t C_t - \frac{\chi P_t}{2(1 + i_t^*)} \left( \frac{E_t D_t^*}{P_t} - \bar{d} \right)^2$$

where $G_t$ is total government spending.

### 2.2.2 Firms

#### Technology

There is a continuum of monopolistic competitive firms $j \in [0, 1]$ in both countries and each firm produces a differentiated good with a linear technology, represented
by the following production function

\[ Y_t(j) = A_t N_t(j), \]

where \( A_t \) is a exogenous, country-specific technology shock. The demand curve for each firm is given by

\[ Y_{H,t}(j) = \left( \frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\epsilon} Y_{H,t}, \]

where \( Y_{H,t} = \left( \int_0^1 Y_{H,t}^*(j) dj \right)^{\frac{\epsilon}{1-\epsilon}} \) is the aggregate output index of country \( H \).

The amount of labour hired is given by

\[ N_t \equiv \int_0^1 N_t(j) dj = \frac{Y_t Z_t}{A_t}, \]

where \( Z_t \equiv \int_0^1 \frac{Y_t(j)}{Y_t} dj \).

**Price setting**

Prices are set by monopolistic competitive firms in a staggered fashion using the framework proposed by Calvo (1983). There is a constant probability \( 1 - \theta \) for a firm to be allowed to change its price, while the other fraction of firms \( \theta \) have to keep their price unchanged. This probability does not depend on the history of past price changes. Hence, the expected time between price adjustments is \( 1/(1 - \theta) \). The parameter \( \theta \) measures the degree of nominal rigidity and a larger value of \( \theta \) implies a higher degree of price stickiness. Firms not allowed to change their price adjust their output to meet demand. Since the problem is symmetric, every firm faces the same decision problem and will choose the same optimal price \( P_{H,t} \), if it is allowed to reset in period \( t \).

The \( j^{th} \) intermediate firm maximizes the expected discounted sum of current and future profits

\[ \max_{P_{H,t}(j)} \sum_{s=0}^{\infty} \theta^s \mathbb{E}_t M_{t,t+s} \left[ Y_{t+s} \frac{P_{H,t}(j)}{P_{t+s}} Y(j)_{H,t+s} - \frac{W_{t+s} Y(j)_{H,t+s}}{P_{t+s} A_{H,t+s}} \right], \]
subject to
\[ Y_{H,t+s}(j) = \left( \frac{P_{H,t}(j)}{P_{H,t+s}} \right)^{-\epsilon} Y_{H,t+s} \]
where \( P_{H,t}(j) \) is the price set by firm \( j \) adjusting its price in the current period and \( M_{t,t+s} = \beta^e \mathbb{E}_t (C_t/C_{t+s})^\sigma (P_t/P_{t+s}) \) is again the subjective discount factor of the households.

The FOC gives the optimal price set in period \( t \) and can be written as\(^{12}\)
\[
P_{H,t} = \frac{\epsilon}{(\epsilon - 1)} \sum_{s=0}^{\infty} (\theta)^s \mathbb{E}_t M_{t,t+s} \left[ \frac{W_{t+s} P_{H,t+s} Y_{H,t+s}}{P_{t+s}} \right]
\]
\[
(1 - \gamma_t) P_{t+s} P_{H,t+s} Y_{H,t+s}
\]
To simplify we assume that the tax on nominal income is set in way to offset the distortions caused by monopolistic competition \((1 - \gamma_t) = (\epsilon - 1)/\epsilon\). Finally, under this price setting structure the domestic price index involves according to
\[
P_{H,t} \equiv \left[ \theta P_{H,t-1}^{1-\epsilon} + (1 - \theta) P_{H,t}^{1-\epsilon} \right]^{1/\epsilon}.
\]

### 2.2.3 Government

The government only provides goods and services which are produced in the domestic country. The public good aggregate of country \( H \) is given by
\[
G_{H,t} = \left( \int_0^1 G_{H,t}(j) \, dj \right)^{\epsilon - 1}, \quad \epsilon > 1
\]
where \( G_{H,t}(j) \) is the quantity of domestic good \( j \) purchased by the government. The demand schedule of government spending is also similar to the consumption case and can be written as
\[
G_{H,t}(j) = \left( \frac{P_{H,t}(j)}{P_t} \right)^{-\epsilon} G_{H,t}.
\]
The government budget constraint in the home economy is given by
\[
\frac{D_t}{1 + \tau_t} = D_{t-1} + P_{H,t}(G_t + T_t) - \gamma_t P_{H,t} Y_t, \quad (2.7)
\]
\(^{12}\)For the derivation of the FOC and the NKPC in an open economy see e.g. Galí and Monacelli (2005).
where $Y_t$ is a tax on the nominal income of the domestic households and $G_{H,t}$ is government spending, whereby both are exogenous and financed by lump-sum taxes/transfers $Tr_t$.

### 2.2.4 Market Clearing Conditions

The output of the small open economy can either be consumed domestically by the households or the government or can be exported. Assume that $C^*_H(j)$ is the world demand for domestic good $j$. Hence the market clearing for good $j$ requires

$$Y_{H,t}(j) = C_{H,t}(j) + C^*_H(j) + G_{H,t}(j)$$

$$= \left( \frac{P_{H,t}(j)}{P_H} \right)^{-\epsilon} \left( \frac{P_{H,t}}{P_t} \right)^{-\eta} \left[ (1 - \alpha)C_t + \alpha^* \left( \frac{1}{Q_t} \right)^{-\eta} C^*_t \right] + G_{H,t}$$

Plugging the previous equation into the definition of aggregate domestic output $Y_{H,t} = \int_0^1 Y_t(j) \frac{dY}{dj} \hat{t}$ yields

$$Y_{H,t} = \left( \frac{P_{H,t}}{P_t} \right)^{-\eta} \left[ (1 - \alpha)C_t + \alpha^* \left( \frac{1}{Q_t} \right)^{-\eta} C^*_t \right] + G_{H,t}.$$ (2.8)

### 2.2.5 Log-Linear Approximation of the Equilibrium Conditions

We linearize the system around a zero inflation steady state and denote log deviations of variable from its steady state value with a hat.\footnote{\textsuperscript{13}For details of all derivations see Appendix (A.1).}

Linearization of the aggregated budget constraint yields

$$\beta d_t = \beta \frac{D^*}{Y} \hat{y}_t + d_{t-1} - \frac{D^*}{Y} \hat{\pi}_t + (1 - \beta) \frac{D^*}{Y} (1 - \alpha) \hat{S}_t + \hat{Y}_t - \gamma \hat{G}_t - (1 - \gamma) \alpha \hat{S}_t - (1 - \gamma) \hat{C}_t$$ (2.9)
where we defined \( d_t = \frac{D_t}{Y} \) and \( \gamma = \frac{G}{Y} \) denotes the government spending share in the steady state.

Due to the cost in trading foreign bonds the uncovered interest rate parity condition is not valid anymore

\[
\hat{i}_t = \hat{i}_t^* + \Delta \varepsilon_{t+1} - \chi Y d_t - \chi Y D_t^* (1 - \alpha) \hat{S}_t
\] (2.10)

There is a time varying risk-premium that depends on both the net foreign asset position in period \( t \), \( d_t \), and the constant cost of bond holdings \( \chi \). This risk premium can be either positive or negative depending on the domestic country being a borrower or a lender in the international asset market.

The log-linearized Euler equation and the Phillips curve (PC) of the small open economy are given as

\[
\hat{C}_t = \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} \left( \hat{i}_t - \pi_{t+1} \right)
\] (2.11)

and

\[
\pi_{Ht} = \beta \mathbb{E}_t \pi_{Ht+1} + \frac{(1 - \theta)(1 - \theta \beta)}{\theta} \left( \sigma \hat{C}_t + \varphi \hat{Y}_t + \alpha \hat{S}_t - (\varphi + 1) \hat{A}_t \right) + \eta_{t}. \] (2.12)

The only difference between the PC of the small open economy to its closed economy counterpart is effect of the terms of trade on marginal cost. The log-linearized aggregate demand equation is

\[
\hat{Y}_t = (1 - \alpha)(1 - \gamma) \hat{C}_t + \eta \alpha (2 - \alpha)(1 - \gamma) \hat{S}_t + \alpha (1 - \gamma) \hat{C}_t^* + \gamma \hat{G}_t
\] (2.13)

and states that domestic output is positive related to domestic and foreign consumption and government spending, but negatively to improvements in the terms of trade. To close this system we use the link between domestic and CPI inflation

\[
\pi_t = \pi_{Ht} + \alpha \Delta \hat{S}_t
\] (2.14)

and the relationship between the terms of trade and the nominal exchange rate:

\[
\Delta \hat{S}_t = \Delta \varepsilon_t + \pi_t^* - \pi_{Ht}
\] (2.15)
The final log-linearised system of first order conditions for the rest of the world can be written as:

\[ \hat{C}_t = \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} \left( i_t^* - \hat{\pi}_{t+1}^* \right), \quad (2.16) \]

\[ \hat{\pi}_t^* = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \frac{(1 - \theta) (1 - \theta \beta)}{\theta} \left( (\sigma + \varphi) \hat{C}_t^* - (\varphi + 1) \hat{A}_t \right) + \eta_t^*. \quad (2.17) \]

Note that the above system is not affected by domestic variables and therefore all foreign variables can be treated as exogenous from the perspective of the domestic economy. Nevertheless the way monetary policy in the rest of the world is conducted has important implications for the small open economy. We assume that the policy maker in the rest of the world solves a conventional inflation targeting problem and acts under commitment. Such a policy choice ensures price level stability in the foreign economy. Since we will analyze the implications of an exchange rate peg for the small open economy later in this chapter and such a peg is often used to ‘import’ foreign inflation, the above choice of a price-stable foreign economy is important to ensure stability in the small open economy.

### 2.2.6 Monetary Policy

#### Inflation Targeting

We assume that the central bank uses the nominal short-term interest rate gap \( i_t = i_t - \hat{\pi}_t^* \) as its instrument and that the social welfare function is well captured by the following discounted quadratic loss function:

\[ W_t^S = \frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} \left( \hat{\pi}_s^2 + \omega_y \left( \hat{Y}_s - \hat{Y}_n^* \right)^2 \right), \quad (2.18) \]

where \( \omega_y = \frac{(1-\theta)(1-\theta \beta)}{\delta e} \) and all variables in the loss function are written in gap form. \( \hat{Y}_n^* \) is the natural level output, i.e. the equilibrium level of output in the absence of nominal rigidities. This welfare function has been shown by Woodford (2003a),
Ch. 6, to approximate the aggregate of individual utility functions in a closed economy model with complete financial markets. In our model, this approximation will not hold up to the second order and so our policy objective function is to some degree *ad hoc*. However, as King and Wolman (2004) and Blake and Kirsanova (2012) argue, multiplicity under discretion is not a consequence of a particularly ‘unfortunate’ form of social welfare, but rather a general property of discretionary policy, as the private sector and the policy maker make decisions based on forecast of each other’s actions. From now on, we will simply refer to this objective as to the social objective. We also label the regime with the above social policy objective as ‘inflation targeting’. Note that we do this for convenience and not to take a stand on the optimality or the precise nature of inflation targeting regimes as practiced in real life.

**Nominal Exchange Rate Targeting under Discretion**

We study the implications of partial nominal exchange rate targeting using the following policy objective function

\[
\frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} \left( \pi_s^2 + \omega_y \left( \hat{Y}_s - \hat{\gamma}_s \right)^2 + \omega_e \hat{E}_s^2 \right),
\]

where we impose an additional weight \( \omega_e \) on the stabilisation of the nominal exchange rate around its steady state value.

14Note that we also abstract from the terms of trade externality (e.g. Obstfeld and Rogoff (1998), Corsetti and Pesenti (2001)). In an open economy the policymaker may have the incentive to influence the terms-of-trade in a way benificial to domestic households. Assuming fully optimal time-inconsistent policy, De Paoli (2009a) shows that in a small open economy an improvement in the terms-of-trade can increase the welfare of the households, if domestic and foreign goods are close substitutes. In this case domestic households consume more imported goods and can therefore reduce their labor effort without a corresponding fall in consumption levels. This derivation, however, is not suitable for our model with discretionary policy and we prefer to use a more traditional alternative. In the above specification the volatility of the terms of trade does affect welfare, but only indirectly through its affect on the output gap.
If $\omega_e = 0$ then the objective (2.19) reduces to (2.18) and we are again in the standard inflation targeting regime. If $\omega_e$ is infinitely large, the objective (2.19) is equivalent to a strict exchange rate targeting regime:

$$\frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} \hat{E}_s^2. \quad (2.20)$$

We label this scenario as ‘soft peg’ and study it as an extreme case of partial nominal exchange rate targeting. This targeting regime has some similarities with a fixed exchange rate regime. In particular, this regime assumes that the policy maker announces the target, perhaps within a corridor (which we do not model as binding in any way, so it does not affect expectations of the private sector) and uses the short term interest rate to keep the exchange rate on target. The exchange rate, however, is allowed to deviate from the target, although such deviations are costly. The above described regime is different from a ‘hard peg’ where the monetary policy maker is prepared to sell any quantity of reserves at a given price to keep the exchange rate exactly on target. The hard peg cannot be modelled with a quadratic loss function because any regime with quadratic loss function allows (costly) deviations from the parity. Another way to model a fixed exchange rate regime is to assume a simple interest rate rule that feeds back on exchange rate deviations from their target, like in Galí and Monacelli (2005). Such a rule allows for deviations from the exchange rate target, but requires credibility of the policy maker, which may be a problem in developing countries.

**The Ramsey allocation**

In the following, we will also report the results from the Ramsey allocation as a benchmark for our welfare evaluations. The Ramsey allocation requires commitment to policy plan, so we will term the solution to the Ramsey problem as the *commitment solution*. A Ramsey policy maker will the minimize the linear quadratic loss function (2.18) with respect to the constraints (2.9)-(2.15) for all $t \geq 0$. 
2.2.7 Calibration

The calibration of our model follows mainly Galí and Monacelli (2005) and Benigno (2009) and is relatively standard. We set the subjective discount rate $\beta = 0.99$, which implies a steady state real interest rate slightly above 4% (in a quarterly model). We set $\varphi = 3$, which implies a Frisch-elasticity of labour supply of $1/3$. Price contracts last on average for one year and hence $\theta$ is set to 0.75. In line with Galí and Monacelli (2005) the steady state markup equals $\mu = 1.2$, which implies that the elasticity of substitution of goods produced within a country $\epsilon$ is equal to 6. The government share of output $\gamma$ is set to 0.25. This value is in line with Gali and Monacelli (2008) and roughly consistent with European data. In the benchmark calibration we assume a unitary coefficient of risk sharing $\sigma$, implying a log utility function. Following Benigno (2009) $\delta$ equals 0.001 which implies a 10 basis point spread of the domestic interest rate over the foreign one. The intertemporal elasticity of substitution between domestic and foreign goods $\eta$ is set to 3, which lies in between the values assumed in the RBC literature (1-2) and recent studies who suggest a value of around 6 (e.g. Trefler and Lai (2002)). Finally, cost push shocks are assumed to be i.i.d. and the technology shocks in the domestic country and the rest of the world follow AR(1) processes with persistence parameters $\rho_n = \rho_{\sigma^*} = 0.8$. The standard deviation of a cost push shock is 0.005 and of a productivity shock it is 0.0075.

2.3 Discretionary Equilibria and Inflation Targeting

In the first part of this section we analyze the inflation targeting regime and discuss the arising discretionary equilibria. In the second part we study the extreme case of a ‘soft peg’ as this is the most simple setting to discuss problems that might arise if a country targets the exchange rate. We intentionally ignore all other possible
## Table 2.1: Calibration of the Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.99</td>
<td>Subjective discount rate</td>
</tr>
<tr>
<td>$\eta$</td>
<td>3</td>
<td>Intertemporal elasticity of substitution between domestic and foreign goods</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>3</td>
<td>Elasticity of substitution between domestic goods</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>1</td>
<td>Risk sharing coefficient</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>3</td>
<td>Labour supply elasticity</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.001</td>
<td>Intermediation cost</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.75</td>
<td>Average period between price adjustments</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.35</td>
<td>Index of openness</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.25</td>
<td>Government spending share of GDP</td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>0.8</td>
<td>Persistence of the technology shock</td>
</tr>
</tbody>
</table>

targets of the central bank for simplicity and clarity. Finally we investigate the intermediate case of ‘partial’ exchange rate targeting. In such a regime the central bank puts some weight in its loss function on exchange rate deviations additionally to its other targets.

### 2.3.1 Inflation Targeting Regime

In the inflation targeting regime the policy maker minimizes the social welfare function (2.18). The evolution of the economic system (2.9)-(2.17) can be written in
reduced form as:\textsuperscript{15}

\[ 
\beta \hat{d}_t = \hat{d}_{t-1} - \alpha (1 - \gamma) \hat{C}_t + \alpha (1 - \gamma) (\eta (2 - \alpha) - 1) \hat{S}_t \\
+ \alpha (1 - \gamma) \hat{C}_t^*, \\
\hat{i}_t = \mathbb{E}_t \hat{S}_{t+1} - \hat{S}_t + \mathbb{E}_t \hat{\pi}_{H,t+1} - \chi \hat{d}_t + \hat{i}_t^*, \\
\hat{C}_t = \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} \left( \hat{i}_t - \mathbb{E}_t \hat{\pi}_{H,t+1} - \alpha \left( \mathbb{E}_t \hat{S}_{t+1} - \hat{S}_t \right) \right), \\
\hat{\pi}_{H,t} = \beta \mathbb{E}_t \hat{\pi}_{H,t+1} + \lambda (\sigma + \varphi (1 - \alpha) (1 - \gamma)) \hat{C}_t + \lambda \varphi \alpha (1 - \gamma) \hat{C}_t^* \\
+ \lambda \varphi \gamma \hat{G}_t + \lambda \alpha (1 + \varphi \eta (2 - \alpha) (1 - \gamma)) \hat{S}_t - \lambda (\varphi + 1) \hat{A}_t + \eta_t, 
\]

where we substituted out all static equations and left only the dynamic relationships. Therefore, the private sector rational expectations equilibrium consists of the set of processes \{\hat{C}_t, \pi_{H,t}, \hat{d}_t, \hat{Y}_t, \hat{S}_t\} satisfying equations (2.21)–(2.24), given a policy \{\hat{i}_t\}, the exogenous processes \{\eta_t, \hat{A}_t, \hat{i}_t^*, \hat{A}_t^*\} (as \{\hat{C}_t^*, \hat{\pi}_{H,t}^*, \hat{i}_t^*\} are functions of \{\hat{A}_t^*, \eta_t^*\}) and initial conditions \(\hat{d}_0\).

\textbf{2.3.2 The Two Equilibria}

The baseline calibration leads to two discretionary equilibria under an inflation targeting regime. The upper panel of Figure 2.1 demonstrates, that following an initial current account deficit the economy can follow two different transition paths, both satisfying the first-order conditions for optimality and time-consistency. The corresponding adjustment paths are plotted using either solid or dashed lines.\textsuperscript{16}

When the economy starts out of the steady state with a negative position in net foreign assets the household will wish to adjust. The households will choose consumption and prices, taking into account the future paths of the interest rate as

\textsuperscript{15}For the rest of this chapter we assume for simplicity a symmetric steady state. This implies a zero net foreign asset position in the steady state.

\textsuperscript{16}The properties of the the two discretionary equilibria are discussed in more detail in Appendix A.3.
well as the state of the economy. Then, the policy maker has to choose an optimal interest rate, based on the forecasts and expectations of the private sector about his policy.

Choosing the adjustment paths the household can foresee that the negative position in net foreign assets can be closed either quickly or slowly. If the policy maker cuts the interest rate sharply output and inflation will increase. The terms of trade worsen and net foreign assets start to accumulate.\textsuperscript{17} The alternative adjustment path is to close the negative position in net foreign assets slowly by raising the interest rate slightly. This policy results in a small consequent fall in consumption and only a marginal adjustment in the terms of trade. Therefore, foreign bonds will accumulate slowly back to the steady state. Depending on whether the adjustment is expected to be slow or fast, the private sector will set corresponding expectations and appropriate prices. It will be optimal for the policy maker to validate the beliefs that will prevail despite their different welfare implications. Two equilibria arise.

These different adjustment paths can be explained by the multiplicity of policy-induced private sector equilibria. Essentially, for every policy there is more than one locally optimal response of the private sector, which is of course, conditional on the forecast of future policy. In order to understand how multiplicity arise we can look at the role which consumption and the terms of trade play in the determination of the law of motion for marginal costs. After some algebra, the deterministic component of the marginal cost can be written as:

\[
c_{t} = (\sigma + \varphi(1 - \alpha)(1 - \gamma))\dot{C}_t + \alpha(1 + \varphi(2 - \alpha)(1 - \gamma))\dot{S}_t.
\]

It is apparent that consumption and the terms of trade are strategic complements in the control of inflation. If a positive cost-push shock hits the economy, the policy maker raises the interest rate to reduce marginal cost via a cut in consumption. From the Euler equation (2.23) follows an improvement terms of trade due to the decrease in consumption. Additionally, the rise in the interest rate leads to an

\textsuperscript{17}The terms of trade dominate the effect of consumption on the evolution of net foreign assets in (2.21) for \( \eta > 1 \).
improvement of the terms of trade via the covered interest parity condition (2.10) and this improvement will reduce marginal costs even further. In other words, following an increase in the interest rate, the effect of consumption on the terms of trade reinforces the effect of the interest rate on the terms of trade via (2.10), so we have a strategic complementarity as defined, for example, in Cooper and John (1988). In such an environment multiplicity of the policy-induced private sector equilibria becomes a likely outcome. The private sector may choose to react in several possible ways – here they are ‘slow’ and ‘fast’ – each of them is consistent with a given forecast about future policy. The policy maker will react differently in response to different private sector expectations, if it is less costly for him to validate the expectations of the private sector than to stick to his initial policy plan. In presence of the two equilibria, coordination failure happens: the private sector can choose any of the two. A sunspot decides which one will realise as we discuss in more detail in the next section.

In order to illustrate the mechanism in a stochastic setting, we plot the impulse responses to a unit cost push shock in the home economy in the middle panel of Figure 2.1. In both equilibria the central bank fights inflation through an increase in the interest rate. Consumption and output decline and the terms of trade improve. Households use the current account as a risk-sharing tool and sell foreign assets to dampen the decline in consumption. Therefore the country will run a current account deficit. In subsequent periods output and consumption converge to their steady states as does the price level through periods of (a very small) deflation. However, in the slow equilibrium the policy maker is able to generate dynamics closer to the commitment solution. Due to the strong increase in the interest rate the nominal exchange rate initially appreciates and much of the adjustment process is done through the terms of trade. In the fast equilibrium the policy maker raises the interest rate only slightly. The dynamics are similar to the slow adjustment equilibrium, but of smaller magnitude except for inflation. The only difference is the evolution of the nominal exchange rate. Due to the small increase in the interest rate, the cost push shock is not very well accompanied by the policy maker and the
initial impact of the shock on inflation is strong. Additionally, the terms of trade fail to work as adjustment tool. Hence, the nominal exchange rate will depreciate. Note that the nominal exchange rate does not converge back to its initial value in any of the stochastic regimes. The terms of trade are stabilised, but both the price level and the nominal exchange rate are unit root variables. This is a common feature of discretionary policy and a result of the inability of the policy maker to anchor the expectations of the private sector. However, in the slow equilibrium the policy maker is able to coordinate private sector expectations to an equilibrium close to the commitment solution.

If the economy is hit by an external cost-push shock, the foreign interest rate is raised and foreign consumption falls. The value of foreign bonds increases and therefore domestic households start to accumulate foreign assets. The terms of trade worsen for the domestic country and home inflation increases. The central bank raises the interest rate in both regimes to counteract the "imported" inflation and therefore enforces a decline in consumption. Again, we observe two different equilibria with different speeds of adjustment. In the fast regime the interest rate is raised by much more, resulting in a strong decline in consumption and an appreciation of the currency. Due to the strong policy response the terms of trade marginally improve and counteract partly the effect of consumption on net foreign assets. In the slow regime the interest rate is raised only slightly. This results in a smaller decline of consumption, but an massive worsening of the terms of trade. Again households will accumulate net foreign assets. The worsening of the terms of trade dominates the negative effect of the decline in consumption on output and output increase. In subsequent periods the interest rate is moved below the baseline generating a small inflation and a small improvement of the terms of trade. The accumulated net foreign assets start to converge slowly back to the steady state in the second period and the consequent improvement terms of trade gradually reduces inflation back to the steady state.

The policy responses to a positive cost-push shock for both equilibria look famil-
iar to the classic problem of ‘dry’ and ‘wet’ policy makers, adapted for a dynamic setting (for the discussion of a conservative central bank proposal in the spirit of Rogoff (1985) in a dynamic setting see Clarida et al. (1999), p. 1677). In both scenarios, the policy maker rises the interest rate in response to the cost push shock, but in one scenario the increase is much stronger. As a result, and as in the ‘dry’ versus ‘wet’ example, in the first scenario inflation is lower at the cost of a more severe recession relative to the second scenario. However, rather differently from the classic problem, we have two locally optimal responses under identical policy objectives.

Based on our observations, the two discretionary solutions can also be seen as the product of ‘seemingly patient’ respectively ‘seemingly impatient’ policy makers. Using this interpretation the degree of patience is determined by the speed of the adjustment process of the economy back to the steady state. This distinction has nothing to do with the discount factor in the objective function, as they remain the same. As we argue next, households/ firms make decisions based on the forecast of future policy: They either decide to bear the cost of adjustment or they do not. The private sector, thus, chooses the equilibrium. The policy maker has no choice but to validate the forecast. We will investigate this in the following section in more detail.

2.3.3 Policy Traps and Equilibrium Selection

Table 2.2 reports the welfare losses under the baseline calibration for the two different regimes. We claim that despite there is a clear difference in welfare ranking, the discretionary policy maker is unable to choose the regime which yields the highest welfare. Multiplicity can only exist if there is a multiplicity of beliefs, shared by the private sector and the policy maker, about the future course of policy. The discretionary policy maker is unable to manipulate the private sector’s beliefs in order to choose the best equilibrium globally. To understand this, it is instructive
Figure 2.1: Inflation Targeting
Table 2.2: Social Welfare Loss, % of steady state consumption.

<table>
<thead>
<tr>
<th>Policymaker</th>
<th>Policy Equilibrium</th>
<th>Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>‘Seemingly Dry’ and ‘Patient’</td>
<td>With slow adjustment</td>
<td>0.278</td>
</tr>
<tr>
<td>‘Seemingly Wet’ and ‘Impatient’</td>
<td>With fast adjustment</td>
<td>3.419</td>
</tr>
<tr>
<td>Reputational</td>
<td>Fully Optimal</td>
<td>0.250</td>
</tr>
</tbody>
</table>

...to compare what is happening in commitment and discretionary equilibria. Under commitment the policy maker is able to manipulate the private sector’s expectations along the whole future path, and thus, by implication, is able to choose the best path for all variables including beliefs. The discretionary policy maker is only able to manipulate private sector beliefs within a single period $t$. This is because the discretionary policy maker acts as an intra-period Stackelberg leader (see Cohen and Michel (1988), sections 4 and 5). However, the policy choice in period $t$ has to be consistent with (or conditional on) beliefs set in previous periods if there are endogenous predetermined state variables in the model. These past-period beliefs cannot be changed retrospectively. Once they are set their effect is long-lasting and so at time $t$ the policy maker has to take into account the future evolution of the economy which has been affected by beliefs set prior to period $t$. (Again, unlike the case of commitment there is no ‘period 0’ when a policy maker has the power to ‘change everything’ irrespective of history.) In this sense, the private sector ‘traps’ the policy maker in a particular equilibrium.

In our model the net foreign asset position is an endogenous predetermined state variable. Their evolution is determined by consumption/output and the terms of trade (and thus also by price-setting behaviour). Consumption and prices are chosen based on the forecasted path of all variables including foreign assets. The representative household (who owns all firms too) is either willing or not to adjust prices and consumption in response to a shock. Its choice depends on beliefs about how quickly any adjustment will happen, and only on this. A household does not face such a choice in a world with complete financial markets, i.e. with
no predetermined endogenous variable. The history dependence of the system is a necessary condition for the existence of multiple equilibria in a linear-quadratic rational expectations framework. In the case without predetermined variables the economy once disturbed converges back to the steady state within a single period under discretionary policy. All monetary policy can do is to reduce the amplitude of the immediate reactions of economic variables to shocks. The feedback coefficient of the policy rule on the observed shocks is responsible for this reduction. If there is at least one predetermined variables in the system, then policy can also reduce the half-life of the effects of shocks which hit the economy in the past and are still in the system.

These two tasks - reducing the half-life of shocks already in the system and reducing the amplitude of shocks - are completely orthogonal to each other, i.e. two rules which only differ in the feedback coefficients on shocks will ensure the same half-life, and two rules which only differ by feedback coefficients on predetermined (dynamic) states will identically reduce the amplitude of concurrent shocks. The private sector can perceive the policy maker as being either ‘quick’ or ‘slow’ to stabilize the economy. These expectations decide about the first-period position in foreign assets. In turn, these expectation affect the economy more than one period into the future, as they are embedded in the dynamics of the evolution of net foreign assets. Any implied future dynamics of the economy are necessarily taken into account by future policy.

The impulse responses to a cost-push shock in Figure 2.1 certainly resemble a policy maker that is either ‘dry/patient’ or ‘wet/impatient’. However, this is because the policy maker has to use the initial movement in the interest rate to fulfill the perceptions of the private sector. Hence the speed of the adjustment process - either quick or slow - of the economy back to its steady state is determined by the expectations of the private sector. The policy maker has to validate these expectations and future policy makers have to stick to this policy.
2.4 Discretionary Equilibria and Exchange Rate Targeting

In this Section we analyze multiple equilibria in different exchange rate targeting regimes. In order to study nominal exchange rate targeting we need to rewrite the reduced from of the system as:

\[
\begin{align*}
\beta \hat{d}_t &= \hat{d}_{t-1} - \alpha (1 - \gamma) \hat{C}_t + \alpha (1 - \gamma) (\eta (2 - \alpha) - 1) (\check{E}_t + \check{p}_{H,t} - \check{p}^*_t) \\
&\quad + \alpha (1 - \gamma) \hat{C}^*_t, \\
\hat{i}_t &= \hat{i}^*_t + \mathbb{E}_t \check{E}_{t+1} - \check{E}_t - \chi \hat{d}_t, \\
\hat{C}_t &= \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} (\hat{i}_t - (1 - \alpha) \mathbb{E}_t \hat{\pi}_{H,t+1} - \alpha (\mathbb{E}_t \check{E}_{t+1} - \check{E}_t)), \\
\hat{\pi}_{H,t} &= \beta \mathbb{E}_t \hat{\pi}_{H,t+1} + \lambda (\sigma + \varphi (1 - \alpha) (1 - \gamma)) \hat{C}_t + \lambda \varphi \alpha (1 - \gamma) \hat{C}^*_t \\
&\quad + \lambda \varphi \gamma \hat{G}_t + \lambda \alpha (1 + \varphi \eta (2 - \alpha) (1 - \gamma)) (\check{E}_t + \check{p}_{H,t} - \check{p}^*_t) - \lambda (\varphi + 1) \hat{A}_t + \eta_t, \\
\check{p}_{H,t} &= \check{p}_{H,t-1} + \hat{\pi}_{H,t}.
\end{align*}
\]

Differently from the version in the previous section there are two predetermined endogenous state variables: net foreign assets and the price level, \( \hat{d}_{t-1} \) and \( \check{p}_{H,t-1} \). The non-predetermined variables are \( \hat{E}_t, \hat{\pi}_{H,t} \) and \( \hat{C}_t \). There are two shocks, \( \eta_t \) and \( \eta^*_t \), whereby \( \check{i}^*_t, \check{C}^*_t \) and \( \check{\pi}^*_t \) are again all functions of \( \eta^*_t \) and \( \check{A}^*_t \). Unlike under inflation targeting, we cannot substitute the nominal exchange rate and the price level into only one variable, the terms of trade. We have to account for the dynamics of the nominal exchange rate separately because it is the goal variable of the policy maker.

In Section 2.4.1 we analyze the extreme case of a ‘soft peg’. We completely ignore other targets, which may not be realistic, but the resulting regime is a useful simplification to illustrate our main point. We study partial nominal exchange rate targeting, which seems to be closer to reality for many developing countries, in Section 2.4.2.
2.4.1 The ‘Soft Peg’

Following most of the NOEM literature, we claim that it is possible to keep the exchange rate on target at all time under a ‘soft peg’ (see e.g. Galí and Monacelli (2005)). Suppose we are in the deterministic version of the model and the economy starts with an excessive level of foreign debt. This scenario is plotted in the upper panel of Figure 2.2. In order to steer the economy back to the steady state the policy maker moves the interest rate based on the forecast of the future net foreign asset position and of the expected exchange rate behaviour. If there is a common belief that the nominal exchange rate will remain in the steady state in the future, it will be optimal for the policy maker to raise the interest rate by little, to offset the effect net foreign assets have on the exchange rate via equation (2.26). There will be a slight fall in consumption and inflation. Due to the fall in consumption foreign bonds start to converge back to the steady state slowly. The adjustment process of the net foreign assets is supported by a small positive interest rate until the foreign bonds are back at their steady state level. The nominal exchange rate will remain in the steady state, i.e. the forecast will be validated by the policy maker.

However, the equilibrium with a stable nominal exchange rate is not the only one. The second possible adjustment path is plotted in the upper panel of Figure 2.2 with a dashed line. Following an initial negative position in foreign debt and if there is a commonly shared belief in a future appreciation of the nominal exchange rate, it is optimal for the central bank to reduce the interest rate sharply, causing an immediate currency depreciation. The cut in the interest rate increases consumption and inflation. The terms of trade worsen, which leads to a fast accumulation of net foreign assets. A higher level of foreign assets pushes the optimal interest rate to fall even further. In other words, there is a complementarity between the interest rate and foreign assets: a reduction in the interest rate raises foreign assets that

\[ \text{The procedure to find all equilibria is similar to the case of inflation targeting which we discuss in Appendix A.3.} \]
require again a lower equilibrium interest rate. With complementarities, multiplicity of equilibria becomes a likely outcome. Therefore it is not surprising that such an outcome has realised in this model.

The ‘soft peg’ requires that the price level returns to its initial level. Therefore we should observe inflation overshooting and, indeed, this is achieved in the second equilibrium by lowering the interest rate sharply. The decline in the interest rate generates an increase in consumption and a depreciation of the terms of trade which both generate an increase in the value of the stock of net foreign assets above its steady state level. When the interest rate is raised back to its baseline level, this higher value of net foreign assets creates an additional pressure on the terms of trade. The terms of trade overshoot, improve and stay below their baseline level for a number of periods. The effect on marginal cost generates inflation overshooting and price level stability. The nominal exchange rate is stabilised at its initial level.

The middle panel of Figure 2.2 plots the impulse responses to a domestic cost push shock. In the slow equilibrium inflation is accommodated: the interest rate is only marginally raised in response to the shock, and consumption falls slightly. The terms of trade improve and the value of foreign bonds decreases. The negative position in net foreign assets pushes the interest rate further up and consumption remains substantially below the baseline. This fall in consumption becomes large and long enough to reduce the marginal cost and hence inflation will fall sharply. The same mechanism then works to steer the economy back to the steady state. In subsequent periods a small deflation is needed to ensure price stability. This is again achieved by a worsening of the terms of trade, which additionally helps net foreign assets to converge back to the steady state.

In the fast equilibrium inflation the policy maker will cut the interest rate to accommodate the impact of the shock. This counterintuitive policy behaviour results from the role played by the perceptions of the private sector. If there is a common belief that the currency will appreciate in the future it is optimal for the policy
Figure 2.2: Exchange Rate Targeting
maker to lower the interest rate now, create an immediate depreciation and worsen the terms of trade, which all leads to a rise in the real value of foreign assets. Then the expected appreciation will drive private consumption down, marginal costs follow and thus also inflation is reduced below the baseline. The terms of trade improve and households start to decumulate net foreign assets. There will be less pressure to keep the interest rate low and the economy gradually converges back. A substantial inflation overshooting guarantees price level stability and the nominal exchange rate is on target.

Note that the existence of the second equilibrium does not depend on any credibility issues: discretionary policy is credible by construction, the policy maker has never promised to keep the nominal exchange rate on target. The policy maker has only promised to minimise the volatility around the target. Therefore, it is acceptable to have deviations from the target as long as it is known to everybody. The possibility of ‘future appreciations/ depreciations’, i.e. different speeds of convergence towards the steady state, generates the second equilibrium.

The lower panel in Figure 2.2 demonstrates the two equilibria if the economy is hit by an external cost push shock. Again, it is possible to offset the shock completely. If there is a common belief that the nominal exchange rate will remain on target, the interest rate can be raised in order to offset the effect of the foreign interest rate on the economy. Consumption will fall, savings increase and net foreign assets will rise. As the terms of trade worsen after the foreign cost-push shock, inflation will drive up in the second period. Consumption will rise and savings will fall, so net foreign assets start to decumulate. A consequent improvement in the terms of trade ensures a slow convergence back to the steady state.

If, following an external cost push shock and a rise in the foreign interest rate, there is a common belief that the currency can depreciate in the future, then the policy maker has to raise the interest rate by more than if the currency is expected to be on target. This results in an immediate appreciation of the currency and
Table 2.3: Social welfare Loss under Exchange Rate Targeting, % of steady state consumption

<table>
<thead>
<tr>
<th>Policymaker</th>
<th>Policy Equilibrium</th>
<th>Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good (‘slow’ equilibrium)</td>
<td>With slow adjustment</td>
<td>1.457</td>
</tr>
<tr>
<td>Bad (‘fast’ equilibrium)</td>
<td>With fast adjustment</td>
<td>5.770</td>
</tr>
<tr>
<td>Commitment (soft peg)</td>
<td>Fully Optimal</td>
<td>1.457</td>
</tr>
</tbody>
</table>

an improvement in the terms of trade. The terms of trade effect dominates all other effects on the net foreign assets position – foreign assets lose in value. The appreciated terms of trade result in higher marginal costs and an higher inflation. The low real interest rate leads to an increase in consumption, so foreign assets start to accumulate and the economy eventually converges back to the steady state.

Despite it is commonly suggested that currency pegging is an efficient way to import low and stable inflation, it is apparent that in the case of a ‘soft peg’ the implied volatility of the nominal exchange rate and domestic inflation in the worst regime is higher than it is in the case of inflation targeting. This is not surprising: these are two ‘second-best’ scenarios, and there cannot be any a priori ranking between them. Also, the welfare minimisation in the ‘soft peg’ case assumes that both predetermined states (foreign assets and prices) can be out of the steady state so their volatility should be minimised ‘on average’, see Currie and Levine (1985). This is contrary to the inflation targeting regime where only one predetermined state exists.

Table 2.3 reports the social losses computed assuming that the economy starts in the steady state and is then hit by internal and external cost-push and productivity shocks. These shocks are distributed as explained in Section 2.2.7. The ‘slow’ discretionary equilibrium is able to replicate the commitment equilibrium under the ‘soft peg’. In the ‘fast’ equilibrium the loss is substantially higher.
2.4.2 Partial Exchange Rate Targeting

Suppose a policy maker under an inflation targeting regime

\[ W_t^{II} = \hat{\pi}_{Ht}^2 + \omega_y \left( \hat{Y}_t - \hat{Y}_t^n \right)^2 \]

decides to target additionally the nominal exchange rate (provided that the anchor country pursues price stability) and chooses a social welfare objective of the following form

\[ W_t = \hat{\pi}_{Ht}^2 + \omega_y \left( \hat{Y}_t - \hat{Y}_t^n \right)^2 + \omega_e \hat{E}_t^2. \]

We report in Table 2.4 four results. In every line we measure the implied social loss as a function of \( \omega_e \). Note that if \( \omega_e = 0 \), the fast equilibrium does not exist: we are in a regime of pure inflation targeting, with two different equilibria. However it is impossible to obtain the ‘wet/impatient’ equilibrium of 2.3 from the fast equilibrium under partial exchange rate targeting if \( \omega_e \) is tending to zero. We report the loss associated of the fast/wet equilibrium under inflation targeting, \( W^{II,F} \), in the fifth line for comparison. Note that the welfare losses for the fast equilibrium are higher in the partial exchange rate targeting regime than in the pure inflation targeting regime.

In contrast to a policy under commitment, adding additional targets to the social objective of a discretionary policy maker can improve the overall policy outcome. There are many known examples of ‘optimal delegation’, among others see e.g. Woodford (2003b) on interest rate smoothing, Walsh (2003) on speed-limit policy and Svensson (1999) and Vestin (2006) on price level targeting. Apparently, an additional exchange rate target does not play a similar role. Lines two and three in Table 2.4 report the social welfare for the two equilibria, slow and fast (\( S \) and \( F \)), under partial exchange rate targeting. The social loss monotonically rises with \( \omega_e \) in the best (slow) equilibrium, while the social loss in the bad (fast) equilibrium can be slightly reduced with an appropriate choice of \( \omega_e \) (\( \omega_e = 0.05 \)). However the improvement is only marginal and, even more important, the introduction of the
Figure 2.3: Partial Exchange Rate Targeting
Table 2.4: Welfare Loss under the Partial Exchange Rate Targeting, % of steady state consumption.

<table>
<thead>
<tr>
<th>$1/\omega_e$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.5</th>
<th>1.0</th>
<th>5.0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_e$</td>
<td>$\infty$</td>
<td>10</td>
<td>2</td>
<td>1.0</td>
<td>0.2</td>
<td>0.1</td>
<td>0.05</td>
<td>0.033</td>
<td>0.025</td>
<td>0.02</td>
<td>0</td>
</tr>
<tr>
<td>$W^S$</td>
<td>1.46</td>
<td>1.43</td>
<td>1.34</td>
<td>1.24</td>
<td>0.76</td>
<td>0.53</td>
<td>0.38</td>
<td>0.33</td>
<td>0.31</td>
<td>0.30</td>
<td>0.28</td>
</tr>
<tr>
<td>$W^F$</td>
<td>5.77</td>
<td>5.76</td>
<td>5.74</td>
<td>5.72</td>
<td>5.59</td>
<td>5.52</td>
<td>5.49</td>
<td>5.50</td>
<td>5.55</td>
<td>5.61</td>
<td>–</td>
</tr>
<tr>
<td>$W^{\Pi,F}$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>3.419</td>
</tr>
</tbody>
</table>

Loss under commitment in inflation targeting regime: 0.250

Exchange rate target into the otherwise social welfare function is the reason for the existence of the worst equilibrium. We therefore conclude that according to our model exchange rate targeting does not solve ‘the problem of optimal delegation’ and is not desirable.

Figure 2.3 plots the impulse responses of an external cost push shock under the discretionary regimes and commitment. The dotted line plots the commitment solution under pure inflation targeting (i.e. using the social welfare function). This is the best possible outcome. The solid and the dashed lines demonstrate responses under discretion when the policy maker imposes $\omega_e = 0.05$. The impulse responses are very similar to those plotted in the third panel of Figure 2.2, there are only small differences because of the inability to keep the exchange rate exactly on target.

2.5 Conclusion

This paper demonstrates how multiple equilibria can prevail in a mainstream open economy model with incomplete financial market under discretionary monetary policy. In the presence of dynamic complementarities the private sector can trap the policy maker into stabilizing the economy either slowly or quickly irrespective of the initial plan of the policy maker.
We demonstrate that the introduction of nominal exchange rate targeting into the policy objective of a discretionary policy maker, that is commonly adopted in developing and emerging countries, does not solve the ‘optimal delegation’ problem – it only leads to higher social losses.

We believe that the presented model is capable of explaining recent empirical evidence on exchange rate behaviour: there can be switches between regimes that are characterized by changes in the volatility of the nominal exchange rate. This can happen for a wide and realistic class of policy objectives, as long as the policy maker acts under discretion and there is at least one predetermined state variable in the system. A sufficiently complex model with these features will retain multiplicity of equilibria and should be able to replicate the observed volatilities of key macroeconomic variables, in particular the nominal exchange rate.
Chapter 3

Escaping Expectation Traps: How Much Commitment is Required?

3.1 Introduction

In this Chapter we study the existence and uniqueness properties of monetary policy in a limited commitment framework in the Blanchard and Kahn (1980) class of linear quadratic rational expectation models (LQ RE). This class of models is typically used to study aggregate fluctuations in macroeconomics. Building on research in Schaumburg and Tambalotti (2007) and Debortoli and Nunes (2010) we show the existence of multiple equilibria under limited commitment policy.\(^1\) Similar to the case of pure discretion, under limited commitment policy makers cannot manage private sector expectations which can lead to expectation traps and coordination failures. We investigate the question of how much precommitment is needed to escape such expectation traps and to coordinate on the Pareto-preferred equilibrium. We find that the necessary degree of precommitment to eliminate multiplicity is

\(^1\)Originally, their framework is based on Roberds (1987). Lohmann (1992) studied limited commitment policies in a one-period setting.
relatively small - from two to five years - which is consistent with tenure terms of monetary policy makers in many countries.

It is well known that in LQ models with rational expectations policies under commitment and discretion may imply very different dynamics for the economy. With full commitment the policy maker has complete control over the private sector’s expectations about future policy and steers them in a way that furthers his stabilization goals. The policy maker can coordinate all future actions of consequent policy makers, which allows him to choose once, and apply indefinitely, an intertemporal contingency plan (Kydland and Prescott (1977)). In linear quadratic models a commitment policy, if it exists, is always unique (Kwakernaak and Sivan (1972), Backus and Drifill (1986)).

With no commitment at all, i.e. under pure discretion, the policy maker does not control the expectations of the private sector and fails to coordinate the actions of consequent policy makers. Under discretion the policy maker optimizes in each period of time and the private sector knows that future policy makers will implement the same decision process in subsequent periods (see e.g. Oudiz and Sachs (1985), Backus and Drifill (1986), Currie and Levine (1993)). However, under pure discretionary policy expectation traps and multiple equilibria can arise because the expectations of the private sector are shaped by anticipations about future policy behavior. Since the policy maker cannot fully control private sector expectations, those expectations may trap the policy maker into implementing a policy that validates them. The trap is closed if it is less costly for the policy maker to validate the private sector beliefs about future policy than to ignore those expectations, see King and Wolman (2004).  

Under limited commitment a new policy maker arrives in office with an exogenous probability $\alpha$ every period, reneges on the past policy plan of his predecessor and

\footnote{Dynamic RE models with multiple discretionary equilibria are presented in King and Wolman (2004) and Blake and Kirsanova (2012). Lockwood and Philippopoulos (1994), Albanesi et al. (2003) give examples of multiplicity in models with static expectations.}
credibly commits to a new policy plan that is optimal at this point in time. Clearly, this framework has elements of both discretion and commitment. However, the policy maker can neither completely control the expectations of the private sector, nor can he coordinate the actions of all future policy makers. Therefore coordination failures between the sequence of policy makers and the private sector can occur and may result in multiple equilibria and expectation traps. Models with expectation traps can help us to explain the observed excess volatility of macroeconomic data. These models should also be used to improve macroeconomic policy to avoid such traps.

Our contribution is twofold. First, we demonstrate, by example, that similar to discretion expectation traps also exist under limited commitment. We use a simple NK model with government debt accumulation which describes an economic behavior that is familiar from the literature on the fiscal theory of the price level (see e.g. Leeper (1991)). Second, we obtain the minimum degree of policy precommitment that is required to select the best equilibrium. We demonstrate that a small degree of precommitment is enough to select the best equilibrium; a tenure of about 2-5 years is sufficient to eliminate all equilibria except the Pareto-preferred.

The chapter is organized as follows. In Section 3.2 we introduce the NK model with debt accumulation. We first review properties of discretion and commitment policies for this model and demonstrate the existence of expectation traps under quasi-commitment. Then we find the minimum length of precommitment that is required to select the best equilibrium in our model. Section 3.5 concludes. Finally, the Appendix presents a numerical algorithm to solve for the optimal policy under limited commitment.

3Discretionary policy with multiple equilibria generates data series which can be observed as satisfying a Markov-switching regime (Blake and Kirsanova (2012)). There is much empirical evidence on such regimes; for one example which uses a similar model as we study here see Davig and Leeper (2006).

4Schaumburg and Tambalotti (2007) term limited commitment ‘quasi-commitment’ and Debor-toli and Nunes (2010) use ‘loose commitment’. In this chapter we use these terms interchangeably.
3.2 The Model with Government Debt

This section demonstrates the existence of multiple equilibria under limited commitment by example. We present a simple NK model with government debt accumulation in the spirit of Leeper (1991). This model is well suited to use as an example to demonstrate the existence of expectation traps and to study the dynamic properties of an economy under monetary policy with limited commitment. First, unlike the model in Schaumburg and Tambalotti (2007) this model has an endogenous predetermined state variable, government debt, which is affected by policy. The presence of such a variable is crucial to generate multiple equilibria under discretionary policy in LQ RE models (Blake and Kirsanova (2012)). A necessary condition for multiplicity is the existence of strategic complementarities between the decisions of agents. An endogenous state variable ensures that the current policy maker reacts (indirectly) to the past actions of the private sector and his predecessors. Therefore the policy maker can be trapped into implementing an undesired policy, if it is less costly to validate the expectations formed in the past, than sticking to his initial policy plan. Second, the model is simple enough to derive most of our results analytically.\(^5\)

We adopt the model from Benigno and Woodford (2004).\(^6\) The economy consists of a representative household, a representative firm that produces the final good, a continuum of intermediate goods producing firms and a monetary and fiscal authority. The intermediate goods producing firms act under monopolistic competition and produce according to a production function that depends only on labor. Goods are combined via a Dixit and Stiglitz (1977) technology to produce aggregate output. Firms set their prices subject to a Calvo (1983) price rigidity. Households choose consumption and leisure and can transfer income through time through their

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\(^5\)Debortoli and Nunes (2010) use a non-linear model to illustrate a generalization of the quasi-commitment equilibrium concept to a non-linear setting. Their model is not suitable for our analysis because of the assumption of non-linearity.

\(^6\)It was also used in Blake and Kirsanova (2012) to investigate the properties of multiple equilibria under discretionary policy.
holdings of government bonds. All agents can observe and affect the accumulation of real government debt. The accumulation of government debt must depend on a fiscal stance. Hence, in the model there is a non-optimizing fiscal authority facing a stream of exogenous public consumption. These expenditures are financed by levying income taxes and by issuing one-period risk-free nominal bonds. We assume that the fiscal authority imposes a simple proportional rule for the tax rate: if real debt is higher (lower) than in the steady state the tax rate rises (falls). We shall refer to the tax rate as ‘taxes’ and to the parameter of the proportional rule as the ‘fiscal feedback’. The size of this fiscal feedback measures the strength of the fiscal stabilization of debt and, as we shall show, plays an important role in the model. The presence of the non-optimizing fiscal authority in the economy is captured by this single feedback parameter $\mu$.

We assume that all public debt consist of riskless one-period bonds. Accordingly, the nominal value of end-of-period public debt $B_t$ evolves according to the following law of motion:

$$B_t = (1 + i_{t-1}) B_{t-1} + P_t G_t - \Upsilon_t P_t Y_t,$$

(3.1)

where $\Upsilon_t$ represents the share of nominal income that the government taxes in period $t$. $G_t$ denotes government purchases which are exogenously given. The aggregate price level is denoted by $P_t$ and the nominal interest rate of government bonds is denoted by $i_t$. The national income identity yields

$$Y_t = C_t + G_t,$$

(3.2)

where $C_t$ is private consumption. For analytical convenience, we define $B_t = (1 + i_{t-1}) B_{t-1}/P_{t-1}$ as a measure of real government debt. Because $B_t$ is observed at the beginning of period $t$, (3.1) can be rewritten as

$$B_{t+1} = (1 + i_t) \left( B_t \frac{P_{t-1}}{P_t} - \Upsilon_t Y_t + G_t \right).$$

(3.3)

We assume that fiscal policy is conducted according to a simple mechanistic feedback rule that relates the tax rate, $\Upsilon_t$, to the stock of real debt, $B_t$

$$\Upsilon_t = \tilde{\Upsilon} \left( \frac{B_t}{\bar{B}} \right)^{\frac{\mu}{\nu}}.$$

(3.4)
Here and below the tilde denotes the steady-state value of the corresponding variable in the model’s zero-inflation non-stochastic steady state.

Log-linearization of equations (3.3) and (3.4) yields

\[ b_{t+1} = \frac{\bar{B}}{Y} t_t + \frac{1}{\beta} \left( \left( 1 - \mu \tilde{Y} \right) b_t - \frac{\tilde{C}}{Y} c_t - \frac{\bar{B}}{Y} \pi_t \right), \]

where \( b_t = \frac{\bar{B}}{Y} \ln \left( \frac{B_t}{\bar{B}} \right), c_t = \ln \left( \frac{C_t}{\bar{C}} \right) \) and \( \pi_t = \ln \left( \frac{1+i_t}{1+i} \right). \) The private sector’s discount factor, \( \beta, \) satisfies \( \beta = 1/(1+i). \) To make the model particularly simple we assume \( \bar{B} = 0, \) which eliminates the first-order effect of the interest rate and inflation on debt, and obtain the final version of linearized debt accumulation equation

\[ b_{t+1} = \rho b_t - \eta c_t, \] (3.6)

where the parameter \( \rho = \left( 1 - \mu \tilde{Y} \right) / \beta \) is a function of the tax rate, implying that with stronger fiscal feedback \( \mu \) the stock of real debt is stabilized more rapidly, and where the parameter \( \eta = \tilde{C} \tilde{Y} / \left( \beta \tilde{Y} \right) \) describes the sensitivity of debt to the tax base.

The derivation of the appropriate Phillips curve is standard (Benigno and Woodford (2004), Sec. A.5) and real marginal cost is a function of output and taxes. Log-linearizing the price-setting-firms’ pricing decision around the zero-inflation non-stochastic steady state yields the following New Keynesian Phillips curve

\[ \pi_t = \beta E_t \pi_{t+1} + \delta \left( \frac{1}{\sigma} + \frac{\theta}{\psi} \right) c_t + \frac{\tilde{Y}}{(1 - \tilde{Y})} \tau_t + u_t, \]

where \( \delta = \frac{(1-\gamma)(1-\gamma)\psi}{\gamma(\psi+\sigma)} \) is the slope of Phillips curve, \( \tau_t = \ln \left( \frac{\pi_t}{\pi} \right), \) \( \sigma \) is the inverse of the intertemporal elasticity of substitution, \( \psi \) is the elasticity of labour supply, \( \theta = \tilde{C} / \tilde{Y} \) is the steady state consumption to output ratio and \( u_t \) is an AR(1) cost push shock with persistence parameter \( \rho_u. \) \( E_t \) is the expectation operator conditional on information available at time \( t. \) Substituting the log-linearized equations (3.2) and (3.4) into the New Keynesian Phillips curve yields

\[ \pi_t = \beta E_t \pi_{t+1} + \kappa c_t + \nu b_t + u_t, \] (3.7)
where \( \nu = \mu \kappa \tilde{Y} / \left(1 - \tilde{Y}\right) \) and \( \kappa = \delta \left(1/\sigma + \theta/\psi\right) \).

In summary, the model is described by the debt accumulation equation (3.6) and the Phillips curve (3.7). The aggregate agents’ decision variable is inflation, \( \pi_t \), and the initial state, \( \tilde{b} \), is known to all agents. We assume that the policy maker chooses consumption \( c_t \). In contrast to the standard New Keynesian model (used in Schaumburg and Tambalotti (2007)) the predetermined state variable, \( b_{t+1} \), is affected by policy, \( c_t \).

The intertemporal welfare criterion of the policy maker is defined by the following quadratic objective

\[
L = \frac{1}{2} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left( \pi_t^2 + \lambda c_t^2 \right).
\]

This criterion is microfounded and derived under the assumption of a steady state labour subsidy, in the absence of productivity and taste shocks.\(^7\) Parameter \( \lambda \) is a function of model parameters, \( \lambda = \theta \kappa / \epsilon \), and \( \epsilon \) is the elasticity of substitution between any pair of monopolistically produced goods.

The policy maker knows the laws of motion (3.6)-(3.7) of the aggregate economy and takes them into account when formulating policy.

### 3.3 Preliminaries: Discretion and Commitment

We shall compare the dynamics of the model under quasi-commitment policy with dynamics under the two limiting cases, discretion and commitment.\(^8\) This Section gives all necessary definitions and presents solutions to these two limiting cases in a comparable form using the model above as an example.

---

\(^7\) For derivation see Kirsanova and Wren-Lewis (2011).

\(^8\) In this section we largely follow the approach and results in Blake and Kirsanova (2012) and in Kirsanova and Wren-Lewis (2011), but present the results in a form that is more convenient for our purposes.
3.3.1 Discretionary Policy

Under discretion there is a sequence of policy makers: each period a new policy maker arrives in office. The new policy maker chooses the best policy knowing that he stays in office for only one period and the next-period’s policy maker will re-optimize again.\(^9\) The law of motion of the aggregate economy (3.6)-(3.7) is known by the policy maker and taken into account when he formulates the optimal policy. Furthermore, the policy maker finds the best action every period and knows that future policy makers have the freedom to change policy, but will apply the same decision process. At every point in time \(t\) the decision rules of each agent are linear functions of the current state

\[
\begin{align*}
    c_t &= c_u u_t + c_b b_t, \quad (3.9) \\
    \pi_t &= \pi_u u_t + \pi_b b_t. \quad (3.10)
\end{align*}
\]

Note that from

\[
\begin{align*}
    \mathbb{E}_t \pi_{t+1} &\overset{eq.(3.10)}{=} \mathbb{E}_t(\pi_u u_{t+1} + \pi_b b_{t+1}) \\
    &\overset{eq.(3.6)}{=} \pi_u \rho_u u_t + \pi_b (\rho b_t - \eta c_t) \\
    &\overset{eq.(3.7)}{=} \frac{1}{\beta} \pi_t - \frac{\kappa}{\beta} c_t - \frac{\nu}{\beta} b_t - \frac{1}{\beta} u_t,
\end{align*}
\]

it follows that the private sector’s decision can also be written as

\[
\pi_t = (\beta \rho_u + 1) u_t + (\beta \rho \pi_b + \nu) b_t + (\kappa - \beta \eta \pi_b) c_t. \quad (3.11)
\]

The policy maker moves first within each period and the private sector observes the action of the policy maker. Thus, the private sector takes into account the ‘instantaneous’ influence of the policy choice measured by \((\kappa - \beta \eta \pi_b)\).

We can give now a more precise definition of discretionary policy: A policy determined by (3.9) is discretionary if the policy maker finds it optimal to follow it

\(^9\)Our definition of discretionary policy is standard and follows Oudiz and Sachs (1985), Backus and Drifill (1986) and Clarida et al. (1999).
in every period \( s > t \), given the private sector (i) observes the current policy, (ii) knows that future policy makers re-optimize and use the same decision process, and (iii) expects policy (3.9) will be implemented in all future periods.

We can write the criterion for optimality as

\[
S_{uu}u_t^2 + 2S_{ub}u_t b_t + S_{bb}b_t^2 = \min_{c_t} \left( (\pi_t^2 + \lambda c_t^2) + \beta \left( S_{uu}u_{t+1}^2 + 2S_{ub}u_{t+1}b_{t+1} + S_{bb}b_{t+1}^2 \right) \right),
\]

subject to constraints (3.6) and (3.11).

One can solve the problem using Lagrange multipliers. The expected Lagrangian can be written as

\[
\mathcal{L}_t^d = \frac{1}{2} (\pi_t^2 + \lambda c_t^2) + \frac{1}{2} \left( S_{uu}\rho_u^2 u_t^2 + 2S_{ub}\rho_u u_t b_{t+1} + S_{bb}b_{t+1}^2 \right) + \xi_{t+1} (\rho b_t - \eta c_t - b_{t+1}) + \phi_{t+1} (\pi_t - \kappa c_t - \nu b_t - u_t - \beta (\pi_u u_t + \pi_b b_{t+1})).
\]

This approach exploits the intertemporal representation (3.6)-(3.7) together with the underlying assumption that the private sector expectations about its own future decisions will be necessarily a function of the future state.

Only current period constraints matter for the policy maker and the first order conditions can be written as

\[
0 = \beta S_{bb}b_{t+1} + \beta S_{ub}\rho_u u_t - \xi_{t+1} - \beta \pi_b \phi_{t+1},
\]

\[
0 = \pi_t + \phi_{t+1},
\]

\[
0 = \lambda c_t - \eta \xi_{t+1} - \kappa \phi_{t+1},
\]

\[
0 = \rho b_t - \eta c_t - b_{t+1},
\]

\[
0 = \beta \pi_b b_{t+1} + \kappa c_t + \nu b_t - \pi_t + (1 + \beta \pi_u \rho_u) u_t.
\]
which collapses to

\[
\begin{align*}
\ell_{t+1} &= \frac{\lambda \rho + (\kappa \rho + \nu \eta) (\kappa + \beta \eta \pi_b) \ell_t}{(\kappa - \beta \eta \pi_b)^2 + \lambda + S_{bb} \beta \eta^2} \\
&\quad + \frac{\eta (\kappa - \beta \eta \pi_b + \beta \rho_u ((\kappa - \beta \eta \pi_b) \pi_u - \eta S_{ub})) \ell_t}{(\kappa - \beta \eta \pi_b)^2 + \lambda + S_{bb} \beta \eta^2} \\
\phi_{t+1} &= -\frac{\lambda (\nu + \beta \rho \pi_b) + \beta \eta (\nu + \kappa \rho) S_{bb} \ell_t}{(\kappa - \beta \eta \pi_b)^2 + \lambda + S_{bb} \beta \eta^2} \\
&\quad - \frac{(\lambda + \beta \eta^2 S_{bb}) (1 + \beta \rho_u \pi_u) + \beta \eta \rho_u S_{ub} (\kappa - \beta \eta \pi_b) \ell_t}{(\kappa - \beta \eta \pi_b)^2 + \lambda + S_{bb} \beta \eta^2}
\end{align*}
\] (3.19)

after straightforward substitutions of \(c_t, \xi_t+1\) and \(\pi_t\).

The optimal policy response can be written in the form of (3.9) with

\[
\begin{align*}
c_u &= -\frac{((\kappa - \beta \pi_b \eta) (\beta \pi_u \rho_u + 1) - \eta \beta S_{ub} \rho_u)}{(\beta \eta^2 S_{bb} + (\kappa - \beta \pi_b \eta)^2 + \lambda)} \\
&\quad - \beta \rho^2 S_{uu} - 2 \rho \eta S_{ub} c_u \eta + \eta^2 S_{bb} c_u^2, \\
c_b &= -\frac{((\kappa - \beta \pi_b \eta) (\beta \pi_b \rho + \nu) - \eta \beta S_{bb} \rho)}{(\beta \eta^2 S_{bb} + (\kappa - \beta \pi_b \eta)^2 + \lambda)} \\
&\quad - \beta \rho S_{uu} c_b + \beta S_{ub} \rho_u (\rho - \eta c_b) - \beta S_{bb} \eta c_u (\rho - \eta c_b), \\
S_{uu} &= ((\beta \pi_u \rho_u + 1) + (\kappa - \beta \pi_b \eta) c_u)^2 + \lambda c_u^2 + \beta (\rho^2 S_{uu} - 2 \rho \eta S_{ub} c_u + \eta^2 S_{bb} c_u^2), \\
S_{ub} &= ((\beta \pi_u \rho_u + 1) + (\kappa - \beta \pi_b \eta) c_u) ((\beta \pi_b \rho + \nu) + (\kappa - \beta \pi_b \eta) c_b) \\
&\quad + \beta \rho c_u c_b + \beta S_{ub} \rho_u (\rho - \eta c_b) - \beta S_{bb} \eta c_u (\rho - \eta c_b), \\
\ell_{t+1} &= (((\beta \pi_b \rho + \nu) + (\kappa - \beta \pi_b \eta) c_b)^2 + \beta S_{bb} (\rho - \eta c_b)^2 + \lambda c_b^2. \\
S_{bb} &= ((\beta \pi_b \rho + \nu) + (\kappa - \beta \pi_b \eta) c_b)^2 + \beta S_{bb} (\rho - \eta c_b)^2 + \lambda c_b^2. \\
S_{bb} &= ((\beta \pi_b \rho + \nu) + (\kappa - \beta \pi_b \eta) c_b)^2 + \beta S_{bb} (\rho - \eta c_b)^2 + \lambda c_b^2.
\end{align*}
\] (3.20)

This yields the following coefficients in (3.10)

\[
\begin{align*}
\pi_u &= \beta \pi_u \rho_u + 1 + (\kappa - \beta \pi_b \eta) c_u, \\
\pi_b &= \beta \pi_b \rho + \nu + (\kappa - \beta \pi_b \eta) c_b.
\end{align*}
\] (3.21)
The solution to (3.21)-(3.27) gives the discretionary equilibrium described by the set \{c_u, c_b, \pi_u, \pi_b, S_{uu}, S_{ub}, S_{bb}\}. Note that the discretionary equilibrium is fully characterized by the deterministic component of the solution, \{\pi_b, c_b, S_{bb}\}. Indeed, we can solve system (3.21)-(3.27) in a recursive way. We first solve (3.22), (3.25) and (3.27) for \{c_b, \pi_b, S_{bb}\} and then solve the rest of the system for the stochastic component of the solution. We use this well known fact to find all discretionary equilibria in the following simple and illustrative way.\(^{10}\)

Suppose the policy maker guesses the response of the private sector to the state, \pi_b. Then the optimal discretionary policy is given by the pair (3.22) and (3.25). We find \(c_b\) and therefore the optimal response \(\pi_b^*\) of the private sector is given by (3.27). Then, for every - not necessarily optimal - \(\pi_b\) we can compute a unique \(\pi_b^*\) and plot the dependence \(\pi_b^*(\pi_b)\), see the first panel in Figure 3.1, Panel I. Clearly, if \(\pi_b = \pi_b^*\) we have a solution to the discretionary problem.

In order to produce simulations we use the benchmark calibration which follows Schaumburg and Tambalotti (2007) and Blake and Kirsanova (2012) and is typical of those used in the literature. The model’s frequency is quarterly. The subjective discount rate \(\beta\) is set to 0.99, the government share of total output \(1 - \rho\) is 0.25. The elasticity of intertemporal substitution \(\sigma\) is 1/2, the Frisch elasticity of labor supply \(\varphi = 1/2\), and the elasticity of demand \(\epsilon\) is set to 7. The Calvo parameter \(\gamma = 0.75\). The most crucial parameter for our results is the fiscal feedback, \(\mu\). The recent empirical evidence suggests that, although the strength of fiscal feedback varies across countries and with time, the chosen value \(\mu = 0.05\) is realistic. See e.g. Leeper et al. (2010) who find a reaction of labour taxes to debt of about 0.05 percentage points for the post-1960 period in the US; see also Coenen and Straub (2005) and Forni et al. (2009) who estimate the response of taxes to debt for the Euro Area.

\(^{10}\)See Anderson et al. (1996) on certainty equivalence in this class of models and Blake and Kirsanova (2012) for explicit formulas for stochastic components as functions of deterministic components in discretionary models.
Figure 3.1: Multiple policy equilibria for different degrees of precommitment
For our baseline calibration the graph of $\pi^*_b (\pi_b)$ intersects the $45^\circ$ degree line in three points labelled $A$, $B$ and $C$, so we have three discretionary policy equilibria. A moderate inflation, set by the firms in response to a given debt level, $\pi_b$, increases the marginal return to a policy decision that increases consumption in response to this level of debt, $c_b$. Higher consumption raises demand and firms will increase their response to debt, $\pi^*_b$. This complementarity ensures the steepness of $\pi^*_b (\pi_b)$ and three equilibria arise.

The three equilibria, whose characteristics are presented in Table 3.2 result in qualitatively and quantitatively different dynamics of the economy. Figure 3.2, which shows the responses of key variables to a unit markup shock for equilibria $A$ and $C$ (as equilibrium $B$ is similar to equilibrium $A$ for the benchmark calibration) using dotted lines with markers.\(^{11}\) Focusing first on equilibrium $A$, inflation rises following the markup shock and the policy response is to defer consumption (by raising the nominal interest rate sufficiently high, this is implicit in our model). The decline in consumption lowers output and government tax revenues, which leads to a rise in government debt. In subsequent periods, although interest rates are lowered to stimulate the economy and bring it out of recession, government debt is brought back to baseline predominantly through (primary) fiscal surpluses, rather than through a decline in the cost of financing government debt. In equilibrium $C$ monetary policy responds to the markup shock by stimulating consumption and output, raises real marginal costs, and causes inflation to rise by more than it otherwise would. This monetary policy causes tax revenues to rise and leads to a decline in government debt. To stabilize government debt, future policy makers raise the cost of financing government debt, which causes consumption, output, and real marginal costs to decline and places downward pressure on inflation. In the spirit of Leeper (1991) monetary policy can be thought of as being active in equilibria $A$ and $B$ and passive in equilibrium $C$. Table 3.2 reveals this trade-off between the response to government debt and the response to the markup shock: The more ‘actively’ the policy maker behaves, the stronger is the policy-induced recession in response to the

\(^{11}\)These impulse responses are identical in each panel.
Table 3.1: Properties of Discretionary Equilibria in the NK Model with Debt Accumulation

<table>
<thead>
<tr>
<th>Characteristics of Discretionary Policy Equilibria</th>
<th>Eq. A</th>
<th>Eq. B</th>
<th>Eq. C</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Policy Reaction $[c_\eta \ c_b]$</td>
<td>-4.8 -0.02</td>
<td>-4.5 -0.01</td>
<td>-0.4 1.9</td>
</tr>
<tr>
<td>(2) Private Sector $[\pi_\eta \ \pi_b]$</td>
<td>0.7 0.01</td>
<td>0.8 0.02</td>
<td>1.0 0.3</td>
</tr>
<tr>
<td>Reaction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3) Normalized Loss $L$</td>
<td>1.3326</td>
<td>1.3872</td>
<td>1.8283</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Characteristics of Commitment Policy Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4) Policy Reaction $[c_\eta \ c_b \ c_\phi]$</td>
</tr>
<tr>
<td>(5) Private Sector $[\pi_\eta \ \pi_b \ \pi_\phi]$</td>
</tr>
<tr>
<td>Reaction</td>
</tr>
<tr>
<td>(6) Normalized Loss $L$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Degree of Precommitment Required to Select the Best Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>(7) Duration of commitment period to select Eq. A (quarters)</td>
</tr>
</tbody>
</table>

Table 3.2

mark-up shock.

3.3.2 Commitment Policy

Under the full commitment policy the policy maker optimizes only once, in the initial moment. He chooses a contingency plan, which is than applied indefinitely but can be implemented sequentially. If there is a change of policy makers, the subsequent policy maker continues the policy of its predecessor; therefore we can assume that there is only one policy maker which takes office in period zero and stays infinitely. When optimizing, the policy maker internalizes the effect of its choice on private
sector’s expectations and solves the following Lagrangian
\[ \mathcal{L}^c = \sum_{t=0}^{\infty} \beta^t \left( \frac{1}{2} (\pi_t^2 + \lambda c_t^2) + \xi_{t+1} (\rho b_t - \eta c_t - b_{t+1}) + \phi_{t+1} (\pi_t - \kappa c_t - \nu b_t - u_t - \beta \pi_{t+1}) \right). \]

The corresponding first order conditions are:

\[ 0 = -\xi_t + \rho \beta \xi_{t+1} - \nu \beta \phi_{t+1}, \quad (3.28) \]
\[ 0 = \pi_t + \phi_{t+1} - \phi_t, \quad (3.29) \]
\[ 0 = \lambda c_t - \eta \xi_{t+1} - \kappa \phi_{t+1}, \quad (3.30) \]
\[ 0 = \rho b_t - \eta c_t - b_{t+1}, \quad (3.31) \]
\[ 0 = \beta \pi_{t+1} + \kappa c_t + \nu b_t + u_t - \pi_t, \quad (3.32) \]

for \( t \geq 0 \); with initial conditions \( b_0 = \bar{b} \) and \( \phi_0 = 0 \), and the transversality condition \( \lim_{t \to \infty} b_t < \infty \).

The solution to the commitment problem can be written in the following linear form (see Appendix B.1)

\[ \pi_t = \pi_u u_t + \pi_b b_t + \pi_\phi \phi_t, \quad (3.33) \]
\[ c_t = c_u u_t + c_b b_t + c_\phi \phi_t, \quad (3.34) \]
\[ \xi_t = \xi_u u_t + \xi_b b_t + \xi_\phi \phi_t. \quad (3.35) \]

Writing the solution in this way allows us to compare it with the discretionary solution. Again, suppose the response of the private sector to debt, \( \pi_b \), is given. We can guess the other feedback coefficients in the system (3.33)-(3.35) and iterate the Riccati equation as suggested in Appendix B.1, but do not update \( \pi_b \). If the procedure converges, we have obtained the optimal response of the policy maker to the private sector decision, provided that the private sector responds to the Lagrange multiplier (set by the policy maker) in an optimal way. Then, we iterate the Riccati equation once again to obtain \( \pi_b^* \). A solution to the commitment problem implies \( \pi_b^* = \pi_b \). The graph of \( \pi_b^* (\pi_b) \) intersects the 45° degree line in one point labelled
A, see the second panel in Figure 3.1, and we can verify with standard methods (Söderlind (1999)) that this point is, indeed, a solution. For the baseline calibration the economy is stabilized by the policy maker in the unique equilibrium $A$.

Figure 3.2 reports the responses of all variables to a positive unit cost push shock. Under commitment (the blue dotted line with x-markers) the policy maker engineers a fall in private consumption, which will dampen marginal costs. Although the dynamics of the economy is very similar to the one in discretionary equilibrium $A$, in contrast to this discretionary equilibrium, the policy maker keeps consumption below the steady state for several periods. Such a policy allows the policy maker to lower expected future inflation and ensures price stability in the long run. Government debt initially increases due to the fall in consumption, but is brought back to the steady state with higher taxes.

### 3.4 Quasi-Commitment Policy

This Section studies monetary policy within a limited commitment framework. We discuss the continuum of intermediate cases between commitment and discretion. We want to understand (i) how a ‘quasi-commitment bridge’ links the economy under commitment and under discretion when multiple equilibria exist, and (ii) how effectively quasi-commitment helps to select the best equilibrium.

#### 3.4.1 Existence of Multiple Policy Equilibria

A quasi-commitment policy, as introduced in Schaumburg and Tambalotti (2007), also assumes sequential policy making. A new policy maker is appointed with a constant and exogenous probability $\alpha$ every period. When a new policy maker arrives in office, he reneges on the promises of his predecessor and commits to a new policy plan that is optimal at the time of the change. All agents understand the
Figure 3.2: Impulse Responses to a 1% cost push shock in the model with government debt
possibility and the nature of this change and form expectations accordingly. The private sector knows that a new policy maker will re-optimize, therefore it doubts the reliability of outstanding promises.

As in Schaumburg and Tambalotti (2007) and Debortoli and Nunes (2010) we assume that the policy maker’s tenure in office depends on a sequence of exogenous i.i.d. Bernoulli signals \( \{\Omega_t\}_{t \geq 0} \) with \( \mathbb{E}[\Omega_t] = \alpha \). If \( \alpha = 1 \) the policy authority acts under full discretion and every period a new policy maker arrives in office and re-optimizes the planning problem. If \( \alpha = 0 \) the policy maker stays in office infinitely long and keeps his promises.

Schaumburg and Tambalotti (2007) and Debortoli and Nunes (2010) demonstrate that the optimization problem under limited commitment can be expressed by the following Lagrangian

\[
L^{qc} = \sum_{t=0}^{\infty} (\beta (1 - \alpha))^{t} \left( \frac{1}{2} \left[ \frac{\pi_t^2 + \lambda c_t^2}{1 - \alpha} \right] + \phi_t + \nu b_t - u_t - \beta (1 - \alpha) \pi_{t+1} + \beta \alpha \pi_0 b_{t+1} - \beta \alpha \pi_0 u_{t+1} \right)
\]

for \( 0 < \alpha < 1 \). The first order conditions are

\[
0 = \beta \alpha S^a u_t + \beta \alpha S^a b_t - \pi_t + \rho \beta (1 - \alpha) \xi_{t+1}
\]

\[
-\nu \beta (1 - \alpha) \phi_{t+1} - \beta \alpha \pi_0 \phi_t,
\]

\[
0 = \pi_t + \phi_{t+1} - \phi_t,
\]

\[
0 = \lambda c_t - \eta \xi_{t+1} - \kappa \phi_{t+1},
\]

\[
0 = \rho b_t - \eta c_t - b_{t+1},
\]

\[
0 = \beta (1 - \alpha) \pi_{t+1} + \beta \alpha \pi_0 b_{t+1} + \kappa c_t + \nu b_t + (\beta \alpha \pi_0 \rho_0 + 1) u_t - \pi_t,
\]

for \( t \geq 0 \), with initial conditions \( b_0 = \tilde{b} \) and \( \phi_0 = 0 \), and the transversality condition \( \lim_{t \to \infty} b_t < \infty \). These first order conditions are similar to those for commitment,
but depend additionally on the parameters \( \{\pi^\alpha_b, S^\alpha_{bb}, S^\alpha_{ub}\} \). When the policy maker defaults on the previous promise (with probability \( \alpha \)) then the firms’ decisions can be written as

\[
\pi_t = \pi^\alpha_u u_t + \pi^\alpha_b b_t
\]

and parameters \( S^\alpha_{bb}, S^\alpha_{ub} \) solve the corresponding Riccati equation (see Appendix B.2 for details).

We can plot the solutions to this system using the same approach as in Section 3.3.2. The solution to (3.37)-(3.41) can be written in form of (3.33)-(3.35). The resulting matrix Riccati equation is similar to the one under commitment, but its coefficients depend obviously also on \( \{\pi^\alpha_b, S^\alpha_{bb}, S^\alpha_{ub}\} \). We can use a similar solution method to find the number of equilibria: suppose we guess the response of the private sector to the state variable, \( \pi^\alpha_u \). Then, we can solve the policy maker’s problem and find the ‘first guess’ of \( S^\alpha \). In the next step we iterate the Riccati equation, but do not update \( \pi^\alpha_b \). If the procedure has converged, we iterate it once to obtain the update \( \pi^\alpha_b^* \). Solutions to the system (3.37)-(3.41) will be among the points where \( \pi^\alpha_b^* = \pi^\alpha_b \).

For the baseline calibration of \( \alpha = 1/2 \) (which implies an average regime duration of two quarters) the graph of \( \pi_b^{1/2} \left( \frac{\pi_b^{1/2}}{1/2} \right) \) intersects the 45° degree line in three points labelled A, B and C, see the third panel in Figure 3.1.\(^\text{12} \) Therefore, if we move from pure discretionary policy to a policy maker who stays in office on average for two periods all three equilibria survive. The survival of all discretionary equilibria under some degree of precommitment is not obvious. Note that if \( \alpha = 1 \), the policy maker defaults with certainty every period. Then, the Lagrangian (3.36) takes the form of (3.13), which describes the discretionary optimization problem. However, the first order conditions for the limited commitment optimization problem (3.37)-(3.41) are left-discontinuous at point \( \alpha = 1 \). System (3.37)-(3.41) does not collapse to (3.14)-

\(^\text{12} \) Again, we can verify with standard methods (based on Oudiz and Sachs (1985) and Backus and Drifill (1986), and discussed in Appendix B.3) that these are indeed solutions to the optimization problem.
(3.18), because the past-period constraints still bind for any $\alpha < 1$. Therefore, taking the limit $\alpha \to 1$ in system (3.37)-(3.41) does not eliminate the Lagrange multiplier on the previous-period constraint $\phi_t$ in equation (3.38). Because for any $\alpha < 1$ the private sector does not expect the occurrence of default with certainty in the next period, this property holds at the limit and implies discontinuity of the first order conditions.

**Proposition 1** There exists a $0 < \bar{\alpha} < 1$ such that if $\alpha \in [\bar{\alpha}, 1)$ then there are as many quasi-commitment policy equilibria as under pure discretionary policy.

**Proof.** Appendix B.2 presents the algebraic Riccati equations (B.11)-(B.12) which determine parameters of solution to the limited commitment problem. Taking the limit $\alpha \to 1$ we obtain system (B.13)-(B.25).

System (B.13)-(B.19) is equivalent to system (3.21)-(3.27) for $\{\pi^1_u, \pi^1_b, c^1_u, c^1_b, S^1_{uu}, S^1_{ub}, S^1_{bb}\}$. The other parameters, $\{\pi^1_\phi, c^1_\phi, \xi^1_u, \xi^1_b, \xi^1_\phi, U^1_{uu\phi}, U^1_{ub\phi}, U^1_{\phi\phi}\}$ are explicit functions of $\{\pi^1_u, \pi^1_b, c^1_u, c^1_b, S^1_{uu}\}$ see (B.20)-(B.25). System (3.21)-(3.27) has three solutions and all of them also solve system (B.13)-(B.25).

System (B.13)-(B.25) is a limiting case of system (B.11)-(B.12); system (B.11)-(B.12) is a polynomial system in $\{\pi_k, c_k\}_{k \in \{a, b, \phi\}}$, in components of $S$ and $U$, which coefficients are polynomial functions of $\alpha$. Therefore, all solutions to (B.11)-(B.12) at $\alpha = 1$ are continuous functions in $\alpha$. For any solution $j \in \{1, 2, 3\}$ to system (B.11)-(B.12) for $\alpha = 1$ there exists $\alpha^j < 1$ such that solution $j$ is a continuous function of $\alpha$ for $\alpha \in [\alpha^j, 1]$. The three solutions for $\alpha = 1$ exist for $\alpha \in [\bar{\alpha}, 1]$ where $\bar{\alpha} = \max_{j \in \{1, 2, 3\}} \{\alpha^j\}$. These three solutions determine three paths which solve (3.37)-(3.41) for $\alpha \in [\bar{\alpha}, 1]$.

We plot the case $\alpha \to 1$ in the fourth panel in Figure 3.1. The $\pi^*_{b0}$ ($\pi^1_{b}$) line intersects the 45° degree line in three points, which are the same points as under pure discretion.\(^{13}\)

\(^{13}\)The shape of $\pi^*_{b0}$ ($\pi^1_{b}$) is different than in Panel I because we take into account the Lagrange
Note that the system (3.37)-(3.41) describes the dynamics of the economy in which, although it is expected that new policy makers arrive in office with probability $\alpha$ and renege on the promises of their predecessors, defaults never happen in the realized history and therefore the Lagrange multiplier $\phi_s$ is never reset to zero for $s > t$. The left-discontinuity of the first order conditions at $\alpha = 1$ arise because for any $\alpha < 1$ the realized reoptimization may never happen, but it happens with certainty for $\alpha = 1$. However, because the consequent policy makers do reset $\phi_s$ to zero with probability $\alpha$, the dynamic properties of the economy are left-continuous at point $\alpha = 1$.

In Figure 3.2 we show the responses of all variables to a positive 1% cost push shock under a quasi-commitment policy. We set $\alpha = 1/2$, which implies an average regime duration of two quarters. We also demonstrate impulse responses under commitment and discretion (equilibria A and C).

Panel I of Figure 3.2 shows the impulse response functions of Type (i).14 These impulse responses demonstrate the evolution of the economy if no reoptimization happens over the horizon of interest, while the private sector expects them to happen every period with probability 1/2. In this scenario a central banker stays in office unexpectedly long, which becomes more and more unrealistic over time. To generate impulse responses we use the transition matrix given by the conditions (3.37)-(3.41). Similar to discretion we plot the two quasi-commitment equilibria A and C. We use solid and dash-dotted lines correspondingly. Compared to the full commitment policy, quasi-commitment policy in the active monetary policy equilibrium A delivers a stronger and longer lasting decrease in consumption. As reoptimizations are expected to happen the price setters expect future policy makers to increase consumption and therefore expect a high inflation in the future. Therefore, if the policy maker wants to exploit private sector expectations he has to pay a higher cost in from of a stronger recession. In the absence of reoptimizations this

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14The categorization of the impulse response functions follows Schaumburg and Tambalotti (2007).
results in stronger future deflation and higher debt, compared to commitment.

Type (i) impulse responses under quasi-commitment policy in equilibrium \( C \) are explosive. In this case the ‘passive’ monetary policy is not able to stabilize inflation, while trying to keep debt under control. After the shock occurred the policy maker cannot move consumption by much, since he has to avoid excessive debt accumulation. This is a similar behaviour as in discretionary equilibrium \( C \). Because the private sector expects defaults in the future and hence high future inflation, inflation can only be controlled with low demand. However, lower consumption would result in excessive debt accumulation. Therefore the reduction in consumption unwinds the attempt of the central bank to ensure fiscal solvency and the economy exhibits explosive behavior. As the fourth chart in the first panel shows, the Lagrange multiplier \( \phi_t \) which measures the shadow price of controlling the private sector inflation expectations is much higher in equilibrium \( C \) and explodes with time.\footnote{This Lagrange multiplier is set on the Phillips curve in the optimization problem of the policy maker.} The result is not surprising, given that the monetary policy maker has to control debt in the passive equilibrium. This task becomes incompatible with inflation stabilization if expected defaults do not happen.

Impulse responses of Type (ii) in Panel II of Figure 3.2 characterize a more typical behavior of the economy under quasi-commitment. Suppose reoptimizations happen in periods 2, 3, 6 and 8 after the initial shock. In each of these periods the reoptimizing policy maker reneges on the plan of its predecessor. When the policy maker defaults on the promises of his predecessor, he resets the predetermined Lagrange multiplier to zero. The policy maker takes this opportunity to end the promised recession of his predecessor and raises consumption back to its initial level. The increase in consumption also leads to a faster reduction of government debt.

Type (iii) impulse responses (Panel III in Figure 3.2) are the ex ante averages of all the possible conditional IRFs integrated over the distribution of the corresponding
reoptimization draws. Therefore they demonstrate the expected evolution of the
system following the initial shock. Naturally, they are in between the IRF of the
respective discretionary equilibria and the IRF under full commitment.

3.4.2 Equilibrium Selection

Proposition 2 There exists a $0 < \alpha \leq 1$ such that if (i) $\alpha \in (0, \alpha]$ and (ii) a
quasi-commitment equilibrium exists, than the equilibrium is unique.

Proof. Under commitment the policy equilibrium in LQ RE models, if it exists, is
unique, see e.g. Backus and Drifill (1986). Appendix B.2 presents the algebraic
Riccati equations (B.11)-(B.12) which determine the parameters of the solution to
the limited commitment problem. Equation (B.12) collapses to a symmetric discrete
algebraic Riccati equation for the value function if $\alpha \to 0$; this equation is known
to have a unique symmetric positive semi-definite solution, see e.g. Lancaster and
Rodman (1995). Equation (B.11) collapses to the Riccati equation (B.8) if $\alpha \to 0$; if
a solution to this equation exists, it is unique. System (B.11)-(B.12) is a polynomial
system in $\{\pi_k, c_k\}_{k \in \{u, b, \phi\}}$ and in components of $S$ and $U$, which coefficients are
again polynomial functions of $\alpha$. Therefore, all solutions to (B.11)-(B.12) at $\alpha = 0$
are continuous functions in $\alpha$. If a solution to system (B.11)-(B.12) exists for $\alpha = 0$
there exists $\underline{\alpha} > 0$ such that the solution is a continuous functions of $\alpha$ for $\alpha \in [0, \underline{\alpha}]$.
These three solutions determine three paths which solve (3.28)-(3.32) for $\alpha \in (0, \underline{\alpha}]$.

By continuity the selected equilibrium is always Pareto-optimal because the com-
mitment equilibrium, to which the selected equilibrium converges at the limit, de-

livers the lowest loss. The value of $\alpha$ which selects the unique equilibrium can be
smaller than the value of $\overline{\alpha}$ which ensures the same number of equilibria as under
discretion. How big are these values for our model? In particular, what is the

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16A commitment policy which stabilises the economy may not exist, see Appendix B.1.
sufficient degree of precommitment $\alpha$ such that only one equilibrium survives?

Panel I in Figure 3.3 plots the expected welfare loss for each equilibrium as a function of the average duration of the period of precommitment $1/\alpha$ for a given fiscal feedback parameter $\mu = 0.05$. In case of pure discretion ($\alpha = 1$) we have three equilibria denoted by triangular markers. With higher degrees of precommitment all three equilibria survive. The losses in the corresponding equilibria are marked with crosses. Panel I suggests that for the benchmark value of the fiscal feedback parameter the worst and the middle equilibria are eliminated if $\bar{\tau} = \alpha = 1/8$, as we also report in row (7) of Table 3.2. If the policy maker stays in the office only for two years on average this will guarantee the unique equilibrium under quasi-commitment policy.\(^{17}\)

To summarize, only a relatively small degree of commitment is required to select the best equilibrium. If a limited commitment technology is available, then it is a more powerful selection mechanism than a formation of a coalition of consequent discretionary policy makers, see Dennis and Kirsanova (2010). If consequent policy makers form coalitions and reoptimize under discretion only in the first period of each coalition tenure, sticking to the same time-consistent policy between reoptimizations, it requires a tenure period of three years to select the best equilibrium in this model.\(^{18}\) An access to the limited commitment technology reduces the necessary tenure period which is required to avoid falling into an expectation trap. Panel I in Figure 3.3 shows that for $\mu = 0.05$ multiplicity is eliminated, if a policy maker can commit on average for 2 years.\(^{19}\) Moreover, and more generally, Proposition 2 claims that there is some sufficient degree of commitment which will certainly select the Pareto-

\(^{17}\)Panel I in Figure 3.3 also demonstrates that the welfare loss is quickly reduced for a higher degree of precommitment. The initial gap between the loss in the best discretionary equilibrium $A$ and commitment is nearly halved after one year of precommitment. A further reduction in $\alpha$ demonstrates that the gains from even minimal levels of credibility are substantial.\(^{18}\)For the base line calibration of the model with $\mu = 0.05$.\(^{19}\)Our numerical experiments with different (and more complex) models show consistently that the best equilibrium is selected only after a few periods of precommitment.
Figure 3.3: Equilibrium Selection
preferred equilibrium (if the corresponding commitment equilibrium exists), but no coalition of discretionary policy makers might exist to select it (Dennis and Kirsanova (2010)).

Panel II in Figure 3.3 investigates the robustness of the above result for different values of the fiscal feedback parameter $\mu$, which is crucial for multiplicity. We concentrate on the range of the fiscal feedback $\mu$ which generates multiplicity of quasi-commitment equilibria for a given average regime duration, $1/\alpha$. For every (discrete) regime duration the square marker denotes the minimum level of $\mu$ above which there is a unique equilibrium characterized by an ‘active’ monetary policy. The area below the round markers displays unique equilibria characterized by a ‘passive’ monetary policy. In the area between the two markers we observe multiplicity of policy equilibria. Panel II demonstrates that with longer periods of precommitment the area of multiplicity shrinks very quickly; if the average period of precommitment is more than five years then expectation traps only exist for very small and empirically irrelevant values of the fiscal feedback $\mu$.

Parameter $\mu$ is crucial for multiplicity and, as we argued in Section 3.3.1, the range of fiscal feedbacks in Panel II in Figure 3.3 is empirically relevant. The baseline value of $\mu = 0.05$ creates multiplicity under pure discretionary policy. However, it is enough for a policy maker to stay in the office for two years in order to select the best equilibrium.

Our results are robust to different calibrations of other parameters of the model. Two parameters were found to affect the quantitative results most. More myopic agents (i.e. lower $\beta$) put higher relative weight on stabilization of the economy in the first periods after the shock. This eliminates the good equilibrium for low values of the fiscal feedback parameter $\mu$, because in this equilibrium the adjustment is relatively slow. Differently, a higher degree of price stickiness (bigger Calvo parameter $\gamma$) slows the adjustment process down and eliminates the bad equilibrium for high values of $\mu$. However, the shape of the curve in Panel I of Figure 3.3 stays in both
cases the same and the quantitative differences are not very large.

### 3.5 Conclusion

In this paper we study monetary policy in a limited commitment framework using a simple New Keynesian model with government debt. We show by example the existence of multiple equilibria under quasi-commitment policy using a model with government debt accumulation. We demonstrate the existence of expectation traps similar to those under pure discretionary policy. Because the private sector expects eventual re-optimizations to happen the current policy maker formulates its policy based on the forecast of the private sector about future policy makers’ behaviour. We find that there can be at least as many limited commitment policy equilibria as in the corresponding discretionary policy problem.

Although the previously developed equilibrium selection mechanism may suggest that economic agents are likely to coordinate on the best equilibrium, our example demonstrates that a limited commitment technology helps a policy maker to avoid falling into an expectation trap even if the degree of precommitment is very small.

In this paper we also provide an algorithm for computing quasi-commitment equilibria in the general class of LQ RE models with endogenous state variables and with exogenous probability of default. We leave the numerical investigation of properties of quasi-commitment policy in a wider class of non-linear dynamic models for future research. This research might investigate how much commitment is required to select the best equilibrium in a King and Wolman (2004) type of model with multiple discretionary equilibria. Once a robust algorithm to solve non-linear models is developed, future research will be able to endogenize the probability of default along the lines suggested in Debortoli and Nunes (2010).
Chapter 4

Estimating Central Bank Preferences in a Small Open Economy

4.1 Introduction

The literature on optimal monetary policy distinguishes between two policy behaviours known as commitment and discretion. The two approaches differ in the way the policy maker can deal with the expectations of the private sector. Under full commitment the policy maker has complete control over the private sector’s expectations about future policy and steers them in a way that furthers his stabilization goals. The policy maker can coordinate all future actions of consequent policy makers, which allows him to choose once, and apply indefinitely, an intertemporal contingency plan (Kydland and Prescott (1977)). However, this policy is time-inconsistent which means that future policy maker will always have an incentive to renege on promises made in the past.

Under discretion, the policy maker does not control the expectations of the
private sector and fails to coordinate the actions of consequent policy makers. The policy maker optimizes in each period of time and the private sector knows that future policy makers will implement the same decision process in subsequent periods (see e.g. Oudiz and Sachs (1985), Backus and Drifill (1986), Currie and Levine (1993)). Hence, the policy is credible by construction, because the private sector is expecting re-optimizations to happen every period.

The empirical literature has been relatively quiet on the question which policy is a better description of reality. Statements from most central banks suggest that their policy approach is closer to commitment (see e.g. Svensson (2009) for the Riksbank and Papademos (2006) for the ECB). Bernanke (2003) and King (1997) describe the policy framework of the Fed respectively the Bank of England as one of 'constraint discretion'. However, despite the label discretion, both central bankers argue that this approach is much closer to commitment and that both banks are able to control the expectations of the private sector (Bernanke (2003) asserts that the main task of the Fed is to "[...] establish a strong commitment to keeping inflation low and stable"). On the other hand, existing alternative views about the conduct of monetary policy vary from claiming that the US monetary regime is best described as discretionary (Bernanke and Mishkin (1997)) to none of the two approaches is a reasonable description of actual monetary policy design (McCallum (1999)).

In this paper we estimate a simplified version of the Galí and Monacelli (2005) small open economy model under different policy regimes for Canadian data. We estimate the weights of the loss function jointly with the structural parameters separately under commitment and discretion and for different policy objectives. The aim is to get some insight about the policy objective used by the Bank of Canada and to estimate which policy behaviour fits the data better.

In the late 1980s the Bank of Canada started targeting the inflation rate, after abandoning money targeting in 1981. Today's policy is described by the bank itself as one of commitment. However, similar to the "constraint discretion" approach of the
Fed and the Bank of England, the Bank of Canada has some flexibility and is allowed to deviate from the inflation target band for a short time period if the economy is hit by large shocks (Carney (2011)). Nevertheless this flexibility is limited, as Governor Mark Carney states, "There are limits to this flexibility. The Bank’s scope to exercise it is founded on the credibility built up through its demonstrated success in achieving the inflation target." (Carney (2011), page 6). However, the implied flexibility could also be interpreted as an element of discretion.

Our approach follows a growing literature that estimates policy preferences in a rational expectations framework. Ilbas (2010) and Adolfson et al. (2011) use Bayesian methods to estimate large-scale DSGE models conditional on the assumption that monetary policy is set under full commitment, while discretionary policy was studied by Castelnuovo (2006) and Dennis (2006). Closest related to our work are the studies by Kirsanova and Le-Roux (2012) and Givens (2012). Kirsanova and Le-Roux (2012) investigate empirically the monetary and fiscal policy interactions in the UK under fiscal leadership and find that discretionary monetary policy fits the data in the post 1997-period much better. Similarly, Givens (2012) finds that the monetary policy regime for the US in the Volcker-Greenspan era is best described as discretionary. Our estimations suggest a different result using Canadian data from 1987-2007. We find that the policy of the Bank of Canada is best described as a commitment policy. Moreover, we carry out estimations for different policy objectives and find in line with the announcements of the Bank of Canada little evidence for partial exchange rate targeting.

This Chapter is organized as follows. In the next Section we outline the model. Section 3 explains the theoretical framework for the two policy regimes. Section 4 gives a brief introduction to Bayesian methodology and discusses the choice of priors and the data set. Our results are presented in Section 5 and Section 6 concludes.
4.2 A Small Open Economy Model

In this section we lay out our model. Following Lubik and Schorfheide (2007) the model is a simplified version of the small open economy model by Galí and Monacelli (2005).

4.2.1 Households

In the domestic country there is a continuum of infinitely lived households in the unit interval. Across time, the representative household maximizes the following utility function

\[ E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{(C_t/A_{Wt})^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right] \]

where \( C_t \) denotes private consumption and \( N_t \) hours of labour. \( \beta \) is the subjective discount rate and \( E_0 \) is the actuarial expectation at time \( t = 0 \). \( \sigma \) and \( \varphi \) are the inverses of the intertemporal elasticities of consumption and labour. \( A_{Wt} \) is a non-stationary world-wide technology shock, and we define \( z_t = A_{Wt}/A_{Wt-1} \) as the growth rate of technological progress.\(^1\) \( C_t \) is a composite consumption index defined by

\[ C_t \equiv \left[ (1-\alpha)^{\frac{1}{\eta}} (C_{H,t})^{\frac{2-\eta}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{F,t})^{\frac{2-\eta}{\eta}} \right]^{\frac{1}{1-\eta}}. \]

Parameter \( \eta > 0 \) denotes the elasticity of substitution between domestic and foreign produced goods from the viewpoint of the domestic consumer. \( \alpha \in [0,1] \) is the weight of imported goods in private home consumption and is inversely related to the degree of home bias in preferences. Another interpretation for \( \alpha \) is as a natural index of openness. \( C_{H,t} \) and \( C_{F,t} \) are the Dixit-Stiglitz indexes of consumption of domestic and foreign goods given by the CES functions.

\(^1\)The shock is introduced to ensure that the model has a balanced growth path (see Lubik and Schorfheide (2005))
\[ C_{H,t} = \left( \int_0^1 C_{H,t}(j)^{\frac{\epsilon-1}{\epsilon}} \, dj \right)^{\frac{\epsilon}{\epsilon-1}} \text{ and } C_{F,t} = \left( \int_0^1 C_{F,t}(j)^{\frac{\epsilon-1}{\epsilon}} \, dj \right)^{\frac{\epsilon}{\epsilon-1}} \]

where \( j \in [0, 1] \) denotes the good variety and \( \epsilon > 1 \) is the elasticity of substitution between varieties of goods produced within a given country.

The nominal intertemporal budget constraint at time \( t \) for the representative household belonging to country \( H \) is given by

\[
\int_0^1 [P_{H,t}(j)C_{H,t}(j) + P_{F,t}(j)C_{F,t}(j)] \, dj + E_t \{ Q_{t,t+1}D_{t+1} \} \\
\leq D_t + (1 - \tau_t) (W_tN_t + \Pi_t) + P_{H,t}T_t
\]

where \( P_{H,t}(j) \) is the price of domestic good \( j \) and \( P_{F,t}(j) \) denotes the price of variety \( j \) imported from country \( F \), where the latter is expressed in domestic currency. \( W_t \) is the nominal wage and \( T_t \) denotes lump-sum taxes/transfers. \( \tau_t \) denotes a country specific tax on nominal income. \( Q_{t,t+1} \) is the stochastic discount factor.

The cost minimization problem of the household gives the following demand functions for good \( j \) produced at home or abroad

\[ C_{H,t}(j) = \left( \frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\epsilon} C_{H,t}; \quad C_{F,t}(j) = \left( \frac{P_{F,t}(j)}{P_{F,t}} \right)^{-\epsilon} C_{F,t} \]

for all \( j \in [0, 1] \), where \( P_{H,t} = \left( \int_0^1 P_{H,t}(j)^{1-\epsilon} \, dj \right)^{\frac{1}{1-\epsilon}} \) and \( P_{F,t} = \left( \int_0^1 P_{F,t}(j)^{1-\epsilon} \, dj \right)^{\frac{1}{1-\epsilon}} \) are the price indexes for domestic and imported goods, whereby the latter is expressed in domestic currency.

Finally, the optimal condition of expenditures between domestic and imported goods is given by

\[ C_{H,t} = (1 - \alpha) \left( \frac{P_{H,t}}{P_t} \right)^{-\eta} C_t; \quad C_{F,t} = \alpha \left( \frac{P_{F,t}}{P_t} \right)^{-\eta} C_t \]

where \( P_t = \left[ (1 - \alpha)P_{H,t}^{-\eta} + \alpha P_{F,t}^{-\eta} \right]^{\frac{1}{1-\eta}} \) is the consumer price index (CPI) in country \( H \). Correspondingly we can write total consumption expenditures by domestic
households as $P_t C_t = P_{H,t} C_{H,t} + P_{F,t} C_{F,t}$. The aggregated budget constraint can therefore be rewritten as

$$P_t C_t + E_t \{Q_{t,t+1} D_{t+1}\} = D_t + W_t N_t + \Pi_t + T_t. \quad (4.3)$$

The first order conditions of the households optimization problem are

$$\frac{W_t}{P_t} = A W_t \frac{N_t^\sigma}{(1 - \tau_t) (C_t/A W_t)^{-\sigma}} \quad (4.4)$$

$$\frac{1}{1 + i_t} = \beta E_t \left\{ \left( \frac{C_{t+1}/A W_{t+1}}{C_t/A W_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\} \quad (4.5)$$

where $1 + i = \frac{1}{\beta \delta Q_{t,t+1}}$ is the gross return on a riskless one-period discount bond paying off one unit of domestic currency in $t + 1$.

### 4.2.2 Uncovered Interest Parity, Real Exchange Rate and the Terms of Trade

The terms of trade are defined as the price of domestic goods relative to the price of goods produced in the foreign country

$$S_t = \frac{P_{H,t}}{P_{F,t}}.$$

Note that in this framework the purchasing power parity (PPP) does not hold, because of the presence of home bias in consumption. Although, we assume that in a symmetric steady state the PPP condition $P_{H,t} = P_{F,t}$ holds. Following Galí and Monacelli (2005) the foreign price index $P_{t}^* = P_{F,t}$. This results from the definition of the rest of the world as closed economy, implying that country $H$ goods production is a negligible fraction of the world’s consumption basket. Hence it follows that $\pi_t^* = \pi_{F,t}$ for all $t$, where $\pi_t = \frac{P_t}{P_{t-1}}$.

Under the assumption of free trade in all goods the law of one price holds for all individual goods at all times and implies

$$P_{F,t}(j) = \mathcal{E}_t P_{F,t}^*(j),$$
for all \( j \in [0, 1] \). \( \mathcal{E}_t \) is the nominal exchange rate and \( P_{F,t}^*(j) \) is the price of a foreign good expressed in foreign currency. Aggregating across all goods implies

\[
P_{F,t} = \mathcal{E}_t P_{F,t}^*.
\]

The real exchange rate – the ratio of CPI inflations, expressed in domestic currency – is defined as

\[
Q_t = \frac{\mathcal{E}_t P_t^*}{P_t}.
\]

Under the assumption of complete financial markets agents share their risk with the rest of the world. Analogous to 4.5 a similar FOC has to hold in the rest of the world.

\[
\frac{1}{1 + i_t} = \beta \mathbb{E}_t \left\{ \left( \frac{C_t^*}{C_{t+1}^*} \right)^\sigma \mathcal{E}_t P_t^* \frac{A_{Wt}}{P_{t+1}^* A_{Wt+1}} \right\} \tag{4.6}
\]

Combining (4.5) and (4.6) gives a version of the uncovered interest parity condition

\[
\mathbb{E}_t \left\{ \left( \frac{C_t^*}{C_{t+1}^*} \right)^\sigma \frac{P_t^*}{P_{t+1}^* A_{Wt}} A_{Wt} \right\} = \mathbb{E}_t \left\{ \left( \frac{C_{t+1}/A_{Wt+1}}{C/A_{Wt}} \right)^{-\sigma} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \frac{P_{t+1}}{P_t} \right\}
\]

### 4.2.3 Domestic Producers

**Technology**

There is a continuum of monopolistic competitive firms \( j \in [0, 1] \) in both countries and each firm produces a differentiated good with a linear technology, represented by the production function

\[
Y_{H,t}(j) = A_{Wt} N_t(j), \tag{4.7}
\]

where \( Y_{H,t}(j) \) is the amount of demand for output produced by firm \( j \) in period \( t \) and \( N_t(j) \) is the amount of labour employed by firm \( j \) in period \( t \). The demand curve for each firm is given by

\[
Y_{H,t}(j) = \left( \frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\sigma} (C_{H,t} + C_{H,t}^*), \tag{4.8}
\]

where \( Y_{H,t} = \left( \int_0^1 Y_{H,t}^*(j) dj \right)^{\frac{1}{\sigma}} \) is the aggregate output index of country \( H \).
Price Setting

The prices are set by monopolistic competitive firms facing price stickiness in a framework proposed by Calvo (1983). In each period, there is a constant probability $1 - \theta$ for a firm to adjust its price. This probability does not depend on the history of past price changes, but only on the random signal $\theta$. The expected time between price adjustments is $1/(1 - \theta)$. If the law of large numbers holds this implies that the fraction of firms not setting prices in period $t$ is $\theta$. The parameter $\theta$ measures the degree of nominal rigidity and a larger $\theta$ implies a higher degree of price stickiness.

Firms not changing the price adjust their output to meet demand. Since the problem is symmetric, every firm faces the same decision problem and will choose the same optimal price $\overline{P}_{H,t}$, if it is allowed to reset in period $t$.

The $j^{th}$-intermediate firm maximizes the expected discounted sum of current and future profits

$$\max_{\overline{P}_{H,t}(j)} \sum_{s=0}^{\infty} \theta^s \mathbb{E}_t Q_{t,t+s} \left[ \frac{\overline{P}_{H,t}(j)}{P_{t+s}} Y(j)_{H,t+s} - \frac{W_{t+s} Y(j)_{H,t+s}}{A_{W,t+s}} \right],$$

subject to

$$Y_{H,t+s}(j) = \left( \frac{\overline{P}_{H,t}(j)}{\overline{P}_{H,t+s}} \right)^{-\epsilon} Y_{H,t+s}$$

where $\overline{P}_{H,t}(j)$ is the price set by firm $j$ adjusting its price in the current period and $\mathbb{E}_t Q_{t,t+s} = \beta^s \mathbb{E}_t (C_{t}/C_t)^{\alpha}(P_t/P_{t+s})$ is the subjective discount factor of the households. The FOC gives the optimal price set in period $t$ and can be written as

$$\overline{P}_{H,t} = \frac{\sum_{s=0}^{\infty} (\theta)^s \mathbb{E}_t Q_{t,t+s} \left[ \frac{W_{t+s}}{P_{t+s}} P_{H,t+s}^{\epsilon} Y_{H,t+s} A_{W,t+s}^{\epsilon} \right]}{\sum_{s=0}^{\infty} (\theta)^s \mathbb{E}_t Q_{t,t+s} \left[ (\epsilon - 1)(1 - \tau_{t+s}) P_{t+s}^{-1} P_{H,t+s}^{\epsilon} Y_{H,t+s} \right]}$$

To simplify we assume that the tax on nominal income is set in way to offset the distortions caused by monopolistic competition $(1 - \tau_t) = (\epsilon - 1)/\epsilon$.

Finally, under this price setting structure the domestic price index involves according to

$$P_{H,t} \equiv \left[ \theta P^{1-\epsilon}_{H,t-1} + (1 - \theta) \overline{P}_{H,t}^{1-\epsilon} \right]^{1/(1-\epsilon)}$$

For the derivation of the FOC and the NKPC see e.g. Galí and Monacelli (2005).
4.2.4 General Equilibrium

The output of the small open economy can either be consumed domestically by the households or can be exported. Assume that $C_{H,t}^*(j)$ is the world demand for domestic good $j$. Hence the market clearing for good $j$ requires

$$Y_{H,t}(j) = C_{H,t}(j) + C_{H,t}^*(j)$$

$$= \left( \frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\varepsilon} (S_t Q_t)^{-\eta} [(1 - \alpha)C_t + (1 - \alpha^*)Q_t^{-\eta}C_t^*]$$

Plugging the pervious equation into the definition of aggregate domestic output $Y_t = \int Y_t(j)^{\frac{\xi - 1}{\tau}} dj$ yields

$$Y_t = (S_t Q_t)^{-\eta} [(1 - \alpha)C_t + (1 - \alpha^*)Q_t^{-\eta}C_t^*]$$

4.2.5 Log-linearization of the Model

We log-linearise the model around the balanced growth path. Before log-linearising we detrend all variables affected by the non-stationary component in form of the world-wide productivity shock $A_{Wt}$. In the following we denote by lower-case letters the stationary transformation of the corresponding variable, $y_t = \frac{Y_t}{A_{Wt}}$, $c_t = \frac{C_t}{A_{Wt}}$, $w_t = \frac{W_t}{A_{Wt}}$. The steady state in this model is well defined (see e.g. DelNegro and Schorfheide (2009)). All variables are in log-deviations from the zero inflation steady state, where $\bar{x}_t = \log X_t - \log X$. Following Lubik and Schorfheide (2007) we assume that $\eta = 1$, $\tau = 1/\sigma$, $\varphi = 0$ and specify the terms of trade by an exogenous law of
motion.\textsuperscript{3} Then the reduced form system can be written as

\begin{align}
\hat{y}_t &= \mathbb{E}_t y_{t+1} - (\tau + \lambda) \left( \hat{i}_t - \mathbb{E}_t \pi_{t+1} - \rho_z \hat{z}_t \right) \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \alpha \beta \Delta \hat{s}_{t+1} - \alpha \Delta \hat{s}_t + \frac{\kappa}{\tau + \lambda} \left( \hat{y}_t + \frac{\lambda}{\tau} \hat{y}^*_t \right) \\
\Delta \hat{s}_t &= \pi_t - (1 - \alpha) \Delta s_t - \pi^*_t \\
\hat{z}_t &= \rho_z \hat{z}_{t-1} + \varepsilon_{zt} \\
\pi^*_t &= \rho_{\pi^*} \pi^*_{t-1} + \varepsilon_{\pi^*t} \\
y^*_t &= \rho_{y^*} y^*_{t-1} + \varepsilon_{y^*t} \\
\Psi_t &= \Delta \hat{y}_t + \hat{z}_t
\end{align}

where \( \Psi_t \) is the growth rate of output and \( \lambda = \alpha (2 - \alpha) (1 - \tau) \). Treating the rest of the world as exogenous relaxes the potentially tight cross-equation restrictions embedded in such a small-scale model, especially the inclusion of \( \pi^*_t \) allows for deviations from PPP. The shocks \( \varepsilon_{st}, \varepsilon_{zt}, \varepsilon_{\pi^*t} \) and \( \varepsilon_{y^*t} \) are i.i.d. processes.

\textbf{4.2.6 The Loss Function}

We estimate the model for two different, ad hoc policy objectives. In both cases we assume that the policy maker uses the nominal interest rate \( i_t \) as its instrument. The first objective is relatively standard and widely used in the empirical literature (see e.g. Soderstrom et al. (2005), Castelnuovo (2006), Givens (2012)).

\textsuperscript{3}Lubik and Schorfheide (2007) find that determining the terms of trade endogenously results in a too tightly restricted model leading to unrealistic parameter estimates. This approach has become common when estimating small open economy models (see e.g. DelNegro and Schorfheide (2009) and Chen and MacDonald (2012)).
\[ L_t = \frac{1}{2} E_s \sum_{s=t}^{\infty} \beta^{s-t} \left( \pi_s^2 + \lambda_y (y_s - \overline{y}_s)^2 + \lambda_{\Delta i} (i_s - i_{s-1})^2 \right). \] (4.17)

Inflation is stabilized around its zero steady state level and its weight is normalized to one. \( \lambda_y \geq 0 \) is the weight on output-gap stabilization, where the gap is the difference between output and its potential level, which is defined as the equilibrium level of output in the absence of nominal rigidities. The inclusion of an interest smoothing term has been common in the empirical literature (see e.g. Rudebusch and Svensson (1999), Ilbas (2010)). The introduction of such a target helps to capture the inertial behaviour of policy observed in the data.\(^4\)

From a theoretical point of view it is well known that under discretion the inclusion of additional target variables into the social objective may improve overall welfare. The reasoning is that additional weights might move the outcome of the discretionary policy closer to the commitment solution and therefore reduce the inflation/stabilization bias.\(^5\) Hence, we also estimate the model assuming an objective in which the policy maker is additionally concerned with fluctuations in the nominal exchange rate

\[ L_t = \frac{1}{2} E_s \sum_{s=t}^{\infty} \beta^{s-t} \left( \pi_s^2 + \lambda_y (y_s - \overline{y}_s)^2 + \lambda_{\Delta \epsilon} \epsilon_s^2 + \lambda_{\Delta i} (i_s - i_{s-1})^2 \right). \] (4.18)

In the theoretical literature there is no agreement whether to include an exchange rate target in the welfare function or not. Ball (1999) argues that monetary policy should react to exchange rate fluctuations, while Gali and Monacelli (2005) derive a microfounded welfare function of the above model, which is similar to its closed economy counterpart, i.e. the welfare objective does not include open economy variables like the terms of trade or the exchange rate. However, De Paoli (2009a)

\(^4\)On theoretical grounds one can argue that this term captures concerns about financial markets stability (Lowe and Ellis (1997)), or uncertainty regarding incoming data (Orphanides (2003)).

\(^5\)Among many other proposals include e.g. speed-limit policies (Walsh (2003)) and price-level targeting (Svensson (1997b)).
shows that their result depends crucially on specific parameter assumptions. Relaxing these assumptions, monetary policy becomes concerned with movements in the real exchange rate. Engel (2011) argues that policy makers should consider the exchange rate in their objective due to currency misalignments. Since exchange rates are mainly driven by news about the future and not by current factors, markets are not able to deliver external balance due to nominal rigidities. Hence, Engel (2011) argues that the central bank should minimize currency misalignments by targeting the exchange rate. Although the Bank of Canada does not officially intervene in the foreign exchange rate market since 1997, testimonies from policy makers suggest that the Bank takes into account the effect of currency movements when formulating its policy (Freedman (1995)). Furthermore, Lubik and Schorfheide (2007) estimate policy rules for four small open economies and find that only the Bank of Canada does include the nominal exchange rate in its rule.\(^6\)

### 4.3 Policy Behaviour

The presented model belongs to the Blanchard and Kahn (1980) class of linear quadratic rational expectations models. The dynamic equations of the model can be written in general form as

\[
\begin{bmatrix}
y_{t+1} \\
x_{t+1}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \begin{bmatrix}
y_t \\
x_t
\end{bmatrix} + \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} [u_t] + \begin{bmatrix}
C \\
0
\end{bmatrix} \varepsilon_{t+1},
\]

where \(y_t\) is an \(n_1 \times 1\) vector of predetermined (state) variables with \(y_0\) given and \(x_t\) is an \(n_2 \times 2\) \((n = n_1 + n_2)\) vector of "forward looking" (jump) variables. \(u_t\) is an \(k \times 1\) vector of policy variables. \(A, B\) and \(C\) are \(n \times n, n \times k\) and \(n_1 \times n_\varepsilon\) matrices of coefficients. Normalizing the covariance matrix of the white noise innovations \(\varepsilon_t\) to \(I\), the covariance matrix of the shocks to the state variables \(y_t\) is \(CC'\). We assume that \(A_{22}\) is invertible.

\(^6\)However, Justiniano and Preston (2010b) find less evidence of exchange rate targeting using a medium-scale DSGE model.
The objective of the policy maker is given as

\[ L_t = \frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} g_s' Q g_s = \frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} (z_s' Q z_s + 2z_s' P u_s + u_s' R u_s). \]  \hspace{1cm} (4.20)

where \( g_s \) is a vector of goal variables of the policy maker and \( z_t = \begin{bmatrix} y_t & x_t \end{bmatrix}' \).

### 4.3.1 Commitment

A commitment policy requires minimisation of the loss function, \( L_t \), once-and-for-all in period \( t = 0 \) subject to (4.19) for \( t \geq 0 \) and \( y_0 = \overline{y}_0 \), where \( \overline{y}_0 \) is known in period \( t = 0 \). The corresponding Lagrangian is given as

\[
\mathcal{L} = \mathbb{E}_0 \sum_{s=0}^{\infty} \beta^s \begin{pmatrix}
(z_s' Q z_s + y_s' Q y_s + x_s' Q x_s) + (z' P u_s + y_s' P u_s) + (x' P u_s + y_s' P u_s)' + u_s' R u_s \\
+ \mu'_{y+s+1}(A_{11} y_s + A_{12} x_s + B_1 u_s + C \epsilon_{t+1} - y_{s+1}) \\
+ \mu'_{x+s+1}(A_{21} y_s + A_{22} x_s + B_2 u_s - x_{s+1})
\end{pmatrix}
\]

where \( \mu_{y} \) and \( \mu_{x} \) are the Lagrange multipliers associated with predetermined and non-predetermined variables. The solution of the optimization problem can be written in the following form\(^7\)

\[
\begin{pmatrix}
y_{s+1} \\
\mu_{y+s+1} \\
x_{s+1} \\
\mu_{x+s+1} \\
u_{s+1}
\end{pmatrix} = M^c \begin{pmatrix} y_s \\
\mu_{y+s} \\
x_s \\
\mu_{x+s} \\
u_s
\end{pmatrix} + \begin{pmatrix} C \\
0
\end{pmatrix} \epsilon_{t+1} \hspace{1cm} (4.21)
\]

\[
\begin{pmatrix}
y_{s+1} \\
\mu_{y+s+1} \\
x_{s+1} \\
\mu_{x+s+1} \\
u_{s+1}
\end{pmatrix} = F^c \begin{pmatrix} y_s \\
\mu_{y+s} \\
x_s \\
\mu_{x+s} \\
u_s
\end{pmatrix} \hspace{1cm} (4.22)
\]

Note that matrices \( M^c \) and \( F^c \) are functions of the system matrices \( A \) and \( B \) as well as of the policy objective \( Q \). We solve the above system using a solution algorithm based on Söderlind (1999).

\(^7\)See Appendix (C.1) for details.
4.3.2 Discretion

Under discretion the policy maker optimizes in each period of time and the private sector knows that future policy makers will implement the same decision process in subsequent periods. The policy maker has no impact on private sector’s expectations and takes them as given. Hence, expected future variables are taken as given and have to be a function of the current state $E x_{t+1} = N_{t+1} E t y_{t+1}$. As a consequence the decision rule and the current aggregate decision of the private sector can be written as linear functions of the current state

$$u_t = -F^d y_t$$

$$x_t = -J y_t - K u_t = -J y_t + K F^d y_t = -N y_t$$

The policy maker minimize the loss function (4.20) with respect to (4.19) taking into account (4.23)-(4.24). It can be shown that the solution to the optimization problem can be written as

$$y_{t+1} = M y_t + C e_{t+1}$$

$$x_t = -N y_t$$

$$u_t = -F^d y_t$$

where $M = A_{11} - A_{12} N - B_1 F^d$, $F^d = (R^* + \beta B^* SB^*)^{-1} (P^* + \beta B^* SA^*)$ and $N = (N_{t+1} A_{12} x - A_{22})^{-1} \left[ (A_{21} - B_2 F^d) - N_{t+1} (A_{11} - B_1 F^d) \right]$. Hence the solution to the discretionary problem can be brought in a similar form as the commitment solution.

Proposition 3 A solution to the commitment and discretionary problem can be

---

8See Appendix (A.2).
written in the following dynamic form

\[ g_{t+1}^{C,D} = M g_t \]  
\[ h_t^{C,D} = V g_t \]  

where for discretion \( g_t^D = [y_t] \) and under commitment \( g_t^C = [y_t, \mu_t^x] \). \( h_t^{C,D} = [x_t, u_t, \mu_t^y] \) and the matrices \( M \) and \( F \) are functions of the system matrices \( A \) and \( B \) as well as of the policy objective \( Q \).

4.4 Estimation Strategy and Empirical Implementation

We estimated our model using a Bayesian approach. This methodology has become widely applied for the estimation and evaluation of DSGE models. In the first part of this section we provide a brief overview about Bayesian methods. In the second part we discuss the data and our choice of the priors.

4.4.1 Bayesian Methodology

The solution of the system (4.25)-(4.26) can be expressed in state space form:

\[ s_t = \Phi_1 (\theta) s_{t-1} + \Phi_\epsilon (\theta) \epsilon_t. \]  

\( s_t \) denotes the vector of state variables and \( \epsilon_t \) stacks the innovations for the structural shocks. \( \Phi_1 \) and \( \Phi_\epsilon \) are system matrices \( (M, V) \) that are functions of the model’s structural parameters. The state variables \( s_t \) are linked to the vector of observables.

\(^9\)For detailed surveys about the benefits and shortcomings of Bayesian estimation see e.g. An and Schorfheide (2007) and Del Negro and Schorfheide (2011).
$Y_t$ through the following measurement equation

$$Y_t = Y_1(\theta) s_t + Y_\epsilon(\theta) \epsilon_t,$$

(4.28)

where $Y_t$ contains the observable series for $\{\pi_t, \Psi_t, i_t, \Delta s_t, \Delta \epsilon_t\}$. The measurement errors in (4.28) are assumed to be Gaussian white noise processes and therefore the Kalman filter can be used to evaluate the likelihood of the model. The Likelihood function $\mathcal{L}(Y_t|\theta)$ is defined as the joint density of the observables conditional on the parameters from equation (4.27). The initialization of the Kalman filter starts usually with a date-0 estimate of the initial state, which causes a problem for the estimation under commitment. A predetermined Lagrange multiplier $\mu^\tau_0 = 0$ implies that the policy maker is not bounded by past commitments in the initial period and can exploit the state of the economy at $t = 0$. To solve this problem in our estimation we follow the timeless-perspective approach advocated by Woodford (2003a) and assume a burn-in phase for the Kalman filter of 20 quarters. Ilbas (2010) shows that this phase is long enough to eliminate the initial effects of the multipliers on the estimation.\(^{10}\)

The Bayesian framework combines the information from the likelihood function with the information of the prior $p(\theta)$. The prior is a probability density function containing non data information about the model parameters $\theta$. Reweighing the Likelihood function $\mathcal{L}(Y_t|\theta)$ with the prior density $p(\theta)$ one obtains the posterior density $P(\theta|Y_t)$. The joint probability of $p(Y_t|\theta)$ can be calculated as

$$p(Y_t|\theta) = \mathcal{L}(Y_t|\theta) p(\theta)$$

(4.29)

or

$$p(Y_t|\theta) = P(\theta|Y_t) p(Y_t).$$

(4.30)

\(^{10}\)In subsequent periods the predetermined Lagrange multipliers, $\mu^\tau_t$; are updated through the Kalman filter just as the other state variables, see again Ilbas (2010). However, Adolfson et al. (2011) demonstrate that the difference between this approach and starting the estimation at $\mu^\tau_0 = 0$ is only minor.
Equating (4.29) with (4.30) and solving for \( P(\theta|\Omega_t) \) yields Bayes Theorem:

\[
P(\theta|Y_t) = \frac{\mathcal{L}(Y_t|\theta)p(\theta)}{p(Y_t)} \propto \mathcal{L}(Y_t|\theta)p(\theta).
\]

(4.31)

The posterior distribution \( P(\theta|Y_t) \) assigns probabilities to alternative values of \( \theta \) conditional on the observations \( Y \) and the prior \( p(\theta) \). The denominator \( p(Y_t) = \int_{\Theta_A} p(\theta; Y_T|A) d\Theta_A \) is the marginal density of the data conditional on the model and constant from the point of view of the distribution of \( \theta \). The estimation of the posterior distribution is analytically intractable and numerical methods have to be used. The draws for the posterior are obtained by using a Markov-Chain Monte-Carlo (MCMC) sampling method. In a first step we maximize (4.31) with respect to \( \theta \) to obtain an approximation to the mode of the posterior. Then we use a Random Walk Metropolis-Hastings algorithm to simulate the posterior draws. We run two Metropolis Hastings (MH) chains based on 50000 draws where the scaling factor is chosen to guarantee an acceptance rate between 35 and 40 percent. The diagnosis for the MH sampling algorithm as described in Brooks and Gelman (1998) are shown in Appendix C.2.

### 4.4.2 Data

The empirical specification of the estimated system consists of the equations (4.25)-(4.26). The sample of our observations covers the period from 1982:1 to 2007:4, which coincides with the abandonment of monetary targeting by the Bank of Canada. All the data are seasonally adjusted and measured at quarterly frequencies.\(^{11}\) Output growth is the log difference of real GDP, multiplied by 100. The inflation series is the consumer price index defined as log differences and multiplied by 100 to obtain quarterly percentage rates. The terms of trade are measured as the ratio of the relative price of exports to imports and converted to log differences. The nominal depreciation rate is the log difference of the effective exchange rate index (RCPI), multiplied by 100. All data series are demeaned prior to estimation and estimations

\(^{11}\)All data was obtain from the OECD data base.
are carried out using the DYNARE toolkit (Adjemian et al. (2011)).

### 4.4.3 Calibration and Choice of Priors

We calibrate a subset of parameters, which determine the steady state. The discount rate $\beta$ is set to 0.99, which implies an annual steady state interest rate of 4%. Note that the above log-linearized version of the model implies an elasticity of substitution between home and foreign goods of 1 and $\varphi = 0$. The choice of the prior distributions follows closely Lubik and Schorfheide (2007) and Adolfson et al. (2011). All prior distributions are assumed to be independent and we also impose size restrictions, such as non-negativity, on the parameters. In difference to Lubik and Schorfheide (2007) we center the slope coefficient in the Phillips curve $\kappa$ relatively tightly at 0.15. This ensures consistency with the values found in the New Keynesian literature, while Lubik and Schorfheide (2007) find unrealistically low degrees of price stickiness using a wider prior. The priors for the autoregressive coefficients of the shocks follow again closely Lubik and Schorfheide (2007).

The priors of the coefficients in the loss function are consistent with the ones used by Ilbas (2010) and Givens (2012). Following this literature we do not restrict the weight on the interest rate smoothing target $\lambda_{\Delta i}$ to be smaller than one. $\lambda_{\Delta i}$ measures the importance of interest rate smoothing relative to inflation stabilization. In principle the weight on interest rate smoothing should be below one, because in most central banks the primary concern of monetary policy is the stabilization of the inflation rate. However, estimations of optimal policy in New Keynesian models consistently find that the weight on interest rate smoothing dominates the weight on inflation stabilization (see e.g. Dennis (2006) and Soderstrom et al. (2005)). We put a fairly loose prior on the weight on the exchange rate target in the objective function with mean 0.25. The standard deviations of the shocks are assumed to be distributed as inverse Gamma distributions with 4 degrees of freedom and mean 1. All prior distributions are summarized in Table (4.1).
Table 4.1: Prior Distributions

<table>
<thead>
<tr>
<th>Domain</th>
<th>Density</th>
<th>Mean</th>
<th>st.dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Structural Parameters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tau$</td>
<td>[0,1)</td>
<td>Beta</td>
<td>0.5</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>[0,1)</td>
<td>Gamma</td>
<td>0.15</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>[0,1)</td>
<td>Beta</td>
<td>0.25</td>
</tr>
<tr>
<td>$\rho_{\pi_F}$</td>
<td>[0,1)</td>
<td>Beta</td>
<td>0.6</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>[0,1)</td>
<td>Beta</td>
<td>0.4</td>
</tr>
<tr>
<td>$\rho_{yF}$</td>
<td>[0,1)</td>
<td>Beta</td>
<td>0.9</td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>[0,1)</td>
<td>Beta</td>
<td>0.3</td>
</tr>
<tr>
<td>$\lambda_y$</td>
<td>$\mathbb{R}^+$</td>
<td>Gamma</td>
<td>0.5</td>
</tr>
<tr>
<td>$\lambda_{\Delta_i}$</td>
<td>$\mathbb{R}^+$</td>
<td>Gamma</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda_E$</td>
<td>$\mathbb{R}^+$</td>
<td>Gamma</td>
<td>0.5</td>
</tr>
<tr>
<td>Standard Deviation of Shocks</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{yF}$</td>
<td>$\mathbb{R}^+$</td>
<td>Invg</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{\pi_F}$</td>
<td>$\mathbb{R}^+$</td>
<td>Invg</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_S$</td>
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</tr>
<tr>
<td>$\sigma_a$</td>
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</tr>
<tr>
<td>$\sigma_R$</td>
<td>$\mathbb{R}^+$</td>
<td>Invg</td>
<td>1</td>
</tr>
</tbody>
</table>

4.5 Estimation Results

4.5.1 Structural Shocks and Parameters

In Table 4.2 we report the estimated posterior distributions for both policy regimes under inflation targeting and partial exchange rate targeting. The estimates are based on a joint estimation of the structural coefficients of the model and the policy weights. Additionally, the prior distributions of the parameters (solid line) are plotted together with the corresponding posterior distributions (bars) obtained through
the Metropolis-Hastings simulations in Figures 4.1 and 4.2.

The estimates of the structural parameters are fairly invariant to the specification of the way monetary policy is conducted and fall within plausible ranges. The estimates for the intertemporal elasticity of substitution $1/\tau$ are relatively robust across policy regimes, whereby we find slightly lower elasticity in the inflation targeting regime under discretion. All our estimates are between the ones found by Ilbas (2010), who estimates an intertemporal elasticity of substitution of 1.33 and Givens (2012) who finds values of above 100. Note that the shape of $1/\tau$ also depends on the calibration of $\eta$, which we assumed to be one. Justiniano and Preston (2010a) and Adolfson et al. (2011) estimated medium-scale open economy models and found values for $1/\tau$ of around 1.5. Estimates of $\kappa$ are within the range of values found in the theoretical literature and due to our prior specification lower than the ones obtained in Lubik and Schorfheide (2007). Similar to the previous literature our estimate for the import share $\alpha$ is much lower than the one observed by the data. Moreover, the corresponding subplots in Figures 4.1 and 4.2 show that for both commitment and discretion the posterior distribution has shifted to the left.\textsuperscript{12} The estimates for the stochastic processes are relatively robust across both policy regimes. As in Lubik and Schorfheide (2007) the persistence in the system is mainly driven by the high autocorrelation coefficient of the world output shock. The persistence of the shocks differs regarding the policy objective. Under inflation targeting the persistence of the world inflation shock and the world productivity shock is higher compared to partial exchange rate targeting for both policy regimes, while the world productivity shock and the ToT shock is relatively unaffected by the form of the objective.

The estimates of the standard deviations for all structural shocks are very close to those obtained in Lubik and Schorfheide (2007). The large standard deviation

\textsuperscript{12}This is a standard result in this strand of literature (Lubik and Schorfheide (2007), Justiniano and Preston (2010a)) as the likelihood function is not informative with respect to this prior and low estimates of $\alpha$ lowers the restrictions imposed by the open economy features of the model.
of the world inflation shock is a result of the inability of the model to capture the exchange rate volatility observed in the data.

### 4.5.2 Monetary Policy Preference Parameters

Regarding the preference parameters of the policy maker, we find across all policy regimes and objective functions a higher weight on output stability as predicted by theoretical models. Also most of the empirical literature finds values slightly below 0.1. However, Adolphson et al. (2011) estimate a medium-scale open economy model and find a weight on output gap stabilization of above 1. The results for $\lambda_{\Delta i}$ clearly suggest that interest rate smoothing is the most important target of the central bank. This is in line with most empirical studies (see e.g. Givens (2012), Dennis (2006)).\(^{13}\) Castelnuovo (2006) argues that the economically unrealistic high values for $\lambda_{\Delta i}$ capture model misspecification and are needed to generated the persistence in the interest rate observed in the data. Overall, the policy maker puts higher weights on all stabilization targets under discretionary policy.

Nominal exchange rate targeting plays only a minor role in the preference specification of the policy maker. As the plots in Figure 4.2 demonstrate, the posterior distributions have shifted extremely close to zero for both commitment and discretion suggesting that the Bank of Canada does not intervene in the foreign exchange market.

\(^{13}\)Ilbas (2010) and Adolphson et al. (2011) find an estimate for $\lambda_i$ of smaller than 1. However, they reach this result due to their prior specification.
### Table 4.2: Optimal Monetary Policy under Inflation Targeting and Partial Exchange Rate Targeting

<table>
<thead>
<tr>
<th>Structural Parameters</th>
<th>Discretion</th>
<th>Commitment</th>
<th>Discretion</th>
<th>Commitment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Inflation Targeting</td>
<td>Exchange Rate Targeting</td>
<td></td>
<td></td>
</tr>
<tr>
<td>World Infl. ( \rho_{\piT} )</td>
<td>0.4561</td>
<td>0.4486</td>
<td>0.3874</td>
<td>0.4221</td>
</tr>
<tr>
<td>World Outp. ( \rho_y )</td>
<td>0.9606</td>
<td>0.9935</td>
<td>0.9582</td>
<td>0.9939</td>
</tr>
<tr>
<td>W.Product. ( \rho_z )</td>
<td>0.3778</td>
<td>0.3234</td>
<td>0.2858</td>
<td>0.2273</td>
</tr>
<tr>
<td>ToT ( \rho_s )</td>
<td>0.5031</td>
<td>0.4737</td>
<td>0.5077</td>
<td>0.4724</td>
</tr>
<tr>
<td>Slope PC. ( \kappa )</td>
<td>0.1727</td>
<td>0.1020</td>
<td>0.1430</td>
<td>0.0987</td>
</tr>
<tr>
<td>Openness ( \alpha )</td>
<td>0.1599</td>
<td>0.1832</td>
<td>0.1565</td>
<td>0.1830</td>
</tr>
<tr>
<td>Cons.elast ( \tau )</td>
<td>0.3094</td>
<td>0.2302</td>
<td>0.2199</td>
<td>0.2104</td>
</tr>
</tbody>
</table>

| Monetary Policy Objectives | |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| Output Gap. \( \lambda_y \) | 0.3514 | 0.1488 | 0.3779 | 0.1633 |
| Ex. Rate \( \lambda_E \) | – | – | 0.0494 | 0.0160 |
| Int.Sm. \( \lambda_{\Delta i} \) | 2.2426 | 1.3493 | 2.5263 | 1.5605 |

| Standard Deviation of Shocks | |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| W.prod. \( \sigma_z \) | 0.3746 | 0.4199 | 0.4130 | 0.4687 |
| W. Infla. \( \sigma_{\piT} \) | 2.5767 | 2.6950 | 2.5940 | 2.7508 |
| W.Outp. \( \sigma_y \) | 0.5380 | 0.7320 | 0.3548 | 0.6500 |
| Mon.Pol. \( \sigma_R \) | 0.3023 | 0.3026 | 0.297 | 0.2924 |
| ToT \( \sigma_S \) | 1.4052 | 1.4588 | 1.4131 | 1.4478 |
| Data density | –457.85 | –397.63 | –472.48 | –413.18 |
Figure 4.1: Prior and Posterior Distributions under Inflation Targeting

Figure 4.2: Priors and Posteriors of Estimated Parameters under Partial Exchange Rate Targeting
4.5.3 Model Comparison

Marginal Likelihood Comparison

The marginal likelihood can be written as

\[ p(Y_t|M_i) = \int L(Y_t, \theta_A|M_i) p(\theta_A|M_i) d\theta_A \]

which describes the marginal data density conditional of the model \( M_i \), but unconditional on the parameters of the model. It is a measure of how well the model is able to explain the data. In Table (4.2) we report the marginal likelihood levels associated with the different policy regimes and objectives. In difference to the previous studies (Givens (2012), Kirsanova and Le-Roux (2012)), we find for both specifications of the loss function that commitment policy fits the data much better than discretion. However, our analysis differs in important aspects from the previous literature. First, we detrend our data along the lines suggested by Lubik and Schorfheide (2005).\(^{14}\) Secondly, compared to Givens (2012) our estimates seem to be more in line with the empirical data. Givens (2012) uses very wide priors and finds estimates of the intertemporal elasticity of substitution between 100 and 5000 and his estimates for \( \kappa \) imply that prices remain constant for above 4 years. Thirdly, our policy objective maybe misspecified and policy makers use different objectives than the ones studied above. The inclusion of additional weights, e.g. a price level target, will most likely improve the fit for discretionary policy. Comparing the marginal likelihood levels, we can also conclude that an inflation targeting regime strictly dominates a monetary policy regime that puts some weight on exchange rate stabilization. This is in line with official statements of the Bank of Canada as well as with our theoretical analysis in Chapter (2.4.2).

\(^{14}\)Our own experiments show that detrending significantly improves the fit of the model.
Figure 4.3: Bayesian Impulse Response Functions under Inflation Targeting
Bayesian Impulse Response Analysis

Figures 4.3 and 4.4 show the impulse response functions under inflation targeting respectively partial exchange rate targeting over a period of 20 quarters. The solid lines are the mean impulse responses while the shaded area depicts the 10% – 90% posterior interval.

A positive technology shocks increases output and therefore marginal costs. The central bank will react to the increase in the inflation rate by moving the real interest rate up. The currency will depreciate. Under discretion the shock has a larger impact on inflation due to the less pronounced policy reaction. An improvement in the terms of trade appreciates the currency. Inflation will decline and output increases. The central bank cuts the interest rate to end the deflation. A shock to foreign demand will lower domestic output, because households will shift a part of their consumption towards foreign goods and the exchange rate depreciates. Inflation will increase since the increase in foreign output dominates the effect of the reduction in domestic output on marginal costs. A discretionary policy maker raises the interest rate to offset the inflationary pressure and inflation converges slowly back to the steady state. In difference, under commitment the policy maker cuts the interest rate marginally. The economy still suffers a stronger recession relative to the discretionary case, because of the higher persistence in the foreign output shock under commitment. In subsequent periods the interest rate is raised back to its steady state. Despite this policy the adjustment process of inflation and the nominal exchange rate back to the steady state is still faster under commitment. Contractionary monetary policy decreases output and inflation. Due to the fall in inflation the currency appreciates.

Since the weight on exchange rate stabilization is very low, the size and magni-

\footnote{This results hinges crucially on our estimate of \( \tau \). The smaller \( \tau \) the stronger is the impact of foreign output on marginal costs.}

\footnote{DelNegro and Schorfheide (2009) find a similar response of the interest rate after a foreign demand shock estimating optimal Taylor rules for Chile.}
tude of all shocks do not differ very much between the two specifications of the loss function. The only difference are the dynamics after a shock to world inflation. This shock has only an (indirect) effect on other variables, if the central bank targets the exchange rate. A world inflation shock appreciates the currency. The policy maker will respond by loosening its monetary stance, which increases inflation and output.

Figure 4.4: Bayesian Impulse Response Functions under Partial Exchange Rate Targeting

Furthermore, Figure 4.5 reports the historic and the one-step ahead predicted
data under the two policy regimes under inflation targeting.\footnote{The in-sample fit for monetary policy under partial exchange rate targeting look similar.} It is apparent that both policy regimes fail badly to capture the growth rate of output, while commitment and discretion are rather successful in predicting the evolution of the interest rate and inflation. Note that commitment yields better in-sample fits for the interest rate and inflation relative to discretionary policy. The figure also shows that predictions for the volatility in the nominal exchange rate are quite good, but this result is mainly driven by the high standard deviation of shocks to foreign inflation. It is a well known fact that this sort of model is not able to generate the observed fluctuations in the nominal exchange rate intrinsically (see e.g. Bergin (2006)).

\subsection*{4.6 Conclusion}

In this paper we used a small open economy model along the lines of Lubik and Schorfheide (2007) and estimated it separately under commitment and discretion.
Additionally, we allowed for two different policy objectives. In the first model the policy maker had a standard loss function focusing on inflation stabilization, while for the second estimated model we added a nominal exchange rate target to the policy objective. We find that the estimates of the weights in the objective vary across policy regimes, but not as much between the two different specifications of the loss function. Generally, monetary authorities put a much higher weight on output stabilization and interest rate smoothing under discretionary policy relative to commitment. We find that commitment is a much better description of monetary policy in Canada for the time period between 1987-2007. The marginal likelihood of the models also indicate that the Bank of Canada was not concerned with movements in the nominal exchange rate and followed a strict inflation targeting regime.
Chapter 5

Conclusion

There has been a long tradition in macroeconomics analysing the impacts of monetary policy on economic activity. Starting with the rational expectations revolution in the 1970s, one important line of research has been the study of the interactions between policy makers and private agents. It is well known that optimal monetary policy under discretion can result in expectation traps and multiple equilibria due to the policy maker’s inability to control the expectation of the private sector. This thesis contributes to the research agenda by providing a better understanding of the interdependencies between optimal monetary policy and multiple equilibria.

In Chapter 2 we demonstrated how multiple equilibria may help to understand some empirical observations in developing countries. There are a number of studies which report difficulties in explaining the exchange rate behaviour in such countries. We argue that one possible interpretation of these switches in the volatility of the exchange rate could be the way monetary policy is conducted. We show that discretionary monetary policy in a small open economy model with incomplete financial markets can lead to multiple equilibria. Associated with these different equilibria are different paths of adjustment and hence different volatilities in all macroeconomic variables. Moreover, we show that policy delegation to a central banker who is ad-
ditionally concerned about exchange rate fluctuations does not improve the welfare fit of the model.

Chapter 3 studies a framework which helps to avoid such costly expectation traps and coordination failures. Building on recent research we demonstrate the existence of multiple equilibria in a framework where the policy maker has only access to a limited commitment technology. We identify the minimum degree of commitment which is needed to escape from expectation traps and find that already a mild degree of precommitment is sufficient to generate uniqueness of the Pareto-preferred equilibrium.

Chapter 3 uses a Bayesian procedure to investigate which policy design is validated by the data. We estimate a small open economy model of the Canadian economy separately for commitment and discretion and find that the data favours the commitment approach. Furthermore, the data suggest that the Bank of Canada did not target the nominal exchange rate in the inspected time period.
Appendices
Appendix A

Appendix Chapter 2

A.1 Model Derivations

A.1.1 Large Closed Economy

We assume that the policy maker in the rest of the world acts under commitment. This will guarantee price level stability in the foreign country and gives us a convenient benchmark. The problem is standard, see e.g. Woodford (2003a). The policy maker will choose the interest rate gap $i_t^* = \hat{i}_t - \hat{i}_t^{*n}$ in order to minimise the following social loss function

$$
\sum_{t=0}^{\infty} \beta^t \left( (\pi_t^*)^2 + \frac{1}{\varepsilon} (c_t^*)^2 \right),
$$

where $\pi_t^* = \hat{\pi}_t - \hat{\pi}_t^{*n}$ is the inflation gap and $c_t^* = \hat{C}_t - \hat{C}_t^{*n}$ is the consumption (output) gap. The natural rates are given as

$$
\hat{\pi}_{t}^{*n} = 0, \quad \hat{C}_{t}^{*n} = \frac{(\varphi + 1)}{(\sigma + \varphi)} \hat{A}_{t}^{*}, \quad i_{t}^{*n} = -\frac{\sigma (1 - \rho_{a}^{*}) (\varphi + 1)}{(\sigma + \varphi)} \hat{A}_{t}^{*}.
$$
The Lagrangian of the optimization problem can be written as

\[
L = \sum_{t=0}^{\infty} \beta^t \left( \left( \pi_t^* \right)^2 + \frac{\lambda}{\epsilon} \left( c_t^* \right)^2 + \zeta_{1t} \left( \frac{1}{\sigma} \left( \pi_{t+1}^* + \theta \left( \frac{1 - \theta}{\theta} \right) \left( \sigma + \varphi \right) c_t^* + \eta_t^* - \pi_t^* \right) \right) + \zeta_{2t} \left( c_{t+1}^* - \frac{1}{\sigma} \left( \eta_t^* - \pi_{t+1}^* \right) - c_t^* \right) \right)
\]

After solving for the FOCs the evolution of the economy under control can be written as:

\[
\begin{align*}
\eta_{t+1}^* &= \rho_\eta \eta_t^*, \\
\zeta_{1t} &= z_{10} \eta_t^* + z_{11} \zeta_{1t-1} + z_{12} \zeta_{2t-1}, \\
\zeta_{2t} &= z_{20} \eta_t^* + z_{21} \zeta_{1t-1} + z_{22} \zeta_{2t-1}, \\
\pi_t^* &= n_\pi_0 \eta_t^* + n_\pi_1 \zeta_{1t-1} + n_\pi_2 \zeta_{2t-1}, \\
c_t^* &= n_c \eta_t^* + n_c_1 \zeta_{1t-1} + n_c_2 \zeta_{2t-1}, \\
i_t^* &= f_0 \eta_t^* + f_1 \zeta_{1t-1} + f_2 \zeta_{2t-1}.
\end{align*}
\]

And the actual variables evolve as

\[
\begin{align*}
\hat{\eta}_{t+1}^* &= \rho_\eta \eta_t^*, \\
\hat{\zeta}_{1t} &= z_{10} \hat{\eta}_t^* + z_{11} \hat{\zeta}_{1t-1} + z_{12} \hat{\zeta}_{2t-1}, \\
\hat{\zeta}_{2t} &= z_{20} \hat{\eta}_t^* + z_{21} \hat{\zeta}_{1t-1} + z_{22} \hat{\zeta}_{2t-1}, \\
\hat{\pi}_t^* &= n_\pi_0 \hat{\eta}_t^* + n_\pi_1 \hat{\zeta}_{1t-1} + n_\pi_2 \hat{\zeta}_{2t-1}, \\
\hat{c}_t^* &= n_c \hat{\eta}_t^* + n_c_1 \hat{\zeta}_{1t-1} + n_c_2 \hat{\zeta}_{2t-1} + \frac{\varphi + 1}{\sigma + \varphi} \hat{A}_t^*, \\
\hat{i}_t^* &= f_0 \hat{\eta}_t^* + f_1 \hat{\zeta}_{1t-1} + f_2 \hat{\zeta}_{2t-1} - \frac{\sigma (1 - \rho_a^*) (\varphi + 1)}{\sigma + \varphi} \hat{A}_t^*.
\end{align*}
\]
A.1.2 Small Open Economy

Households

The Lagrangian of the representative household in the small open economy can be written as

\[
L = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} - N_t^{\frac{1+\varphi}{1+\varphi}} + \mathcal{D}_{t-1} + \mathcal{E}_t \mathcal{D}_{t-1}^* + (1 - \Upsilon) (W_t N_t + \Pi_t) 
\right.
\]

\[
- \frac{\chi P_t}{2(1+i_t^*)} \left( \frac{\mathcal{E}_t \mathcal{D}_t^*}{P_t} - \tilde{d} \right)^2 + P_{H,t} T_r - P_t C_t - \frac{D_t}{(1+\alpha_t)} - \frac{\mathcal{E}_t \mathcal{D}_t^*}{(1+i_t^*)}
\]

The resulting FOCs are

\[
\frac{\partial L}{\partial C_t} = \beta^t \left( C_t^{1-\sigma} - \Lambda_t P_t \right)^{\frac{1}{1-\sigma}} = 0
\]

\[
\frac{\partial L}{\partial N_t} = \beta^t \left( -N_t^\varphi + \Lambda_t (1 - \Upsilon) W_t \right)^{\frac{1}{\varphi}} = 0
\]

\[
\frac{\partial L}{\partial D_t} = -\beta^t \Lambda_t \frac{1}{(1+i_t)} + \beta^{t+1} \Lambda_{t+1}^{\frac{1}{1+\varphi}} = 0
\]

\[
\frac{\partial L}{\partial \mathcal{D}_t^*} = \beta^{t+1} \Lambda_{t+1} \mathcal{E}_t - \beta^t \Lambda_t \left( \frac{\mathcal{E}_t \mathcal{D}_t^*}{P_t} - \tilde{d} \right) - \beta^t \Lambda_t \frac{\mathcal{E}_t}{(1+i_t^*)} = 0
\]

\[
\frac{\partial L}{\partial \Lambda_t} = \mathcal{D}_{t-1} + \mathcal{E}_t \mathcal{D}_{t-1}^* - P_t C_t^j - \frac{D_t^j}{(1+i_t)} - \frac{\mathcal{E}_t \mathcal{D}_t^*}{(1+i_t^*)}
\]

\[
+(1 - \Upsilon) (W_t N_t^i + \Pi_t) - \frac{\chi P_t}{2(1+i_t^*)} \left( \frac{\mathcal{E}_t \mathcal{D}_t^*}{P_t} - \tilde{d} \right)^2 + P_{H,t} T_r \]

and rearranging yields

\[
\beta E_t \left\{ \frac{C_t^\varphi}{C_{t+1}^\varphi} \right\} = E_t \left\{ \frac{P_{t+1}}{P_t (1+i_t)} \right\}
\]

\[
(1+i_t^*) E_t \left\{ \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right\} = 1 + \chi \left( \frac{\mathcal{E}_t \mathcal{D}_t^*}{P_t} - \tilde{d} \right)
\]

(A.1) \quad \text{(A.2)}
Appendix Chapter 2

\[ C_t^R N_t^R = \frac{(1 - Y_t) W_t}{P_t} \]  \hspace{3cm} (A.3)

\[ \frac{\mathcal{E}_t D_t^R}{(1 + i_t^*)} = \mathcal{E}_t D_{t-1}^R + P_{H,t} (Y_t - G_t) - P_tC_t \]  \hspace{3cm} (A.4)

\[ -\frac{\chi P_t}{2(1 + i_t^*)} \left( \frac{\mathcal{E}_t D_t^R}{P_t} - \bar{d} \right)^2 \]

The steady state

In a zero-inflation steady state \( M = \beta \) (see (2.3)). From (2.5) we obtain that \( 1 + i = 1 + i^* = 1/\beta \). Then equation (A.2) determines the steady state value of real asset holdings as

\[ \mathcal{E}D^* \quad \frac{P}{P} = \bar{d}. \]

The flow budget constraint (2.6) determines the steady state level of consumption

\[ C = (1 - \beta) \bar{d} + \frac{P_{H}}{P} (Y - G). \]

The steady state relation of relative prices is

\[ (1 - \alpha) \left( \frac{P_{H}}{P} \right)^{1-\eta} + \alpha \left( \frac{P_{F}}{P} \right)^{1-\eta} = 1 \]

and from (2.8) follows

\[ Y - G = \left( \frac{P_{H}}{P} \right)^{-\eta} \left[ (1 - \alpha)C + \alpha^s \left( \frac{1}{Q} \right)^{-\eta} C^s \right] . \]

Loglinearization

The log-linear deviations of all variables are defined as \( \hat{X}_t = \log X_t - \log X \). We log-linearize around an asymmetric steady state, i.e. we allow for a non-zero net foreign asset position in the steady state.
Current account:

\[ d_t = \beta \frac{D^*}{Y} \hat{i}^*_t + d_{t-1} - \frac{D^*}{Y} \pi^*_t + (1 - \beta) \frac{D^*}{Y} \hat{Q}_t + \hat{Y}_t - \gamma \hat{G}_t - (1 - \gamma) \alpha \hat{S}_t - (1 - \gamma) \hat{C}_t \]  
(A.5)

where we defined \( d_t = \left[ \frac{D^*}{Y} \hat{D}^*_t \right] \). In the case of a symmetric steady, i.e. which implies a zero-steady state net foreign assets position \( D^* = 0 \) the above equation simplifies to (2.21) used in the main text. Note that in a symmetric steady state net foreign assets are defined as \( d_t = \hat{D}^*_t \).

UIP:

\[ \mathbb{E}_t \Delta \varepsilon_{t+1} + \hat{i}^*_t - \hat{i}_t = \chi D^* \hat{Q}_t + \chi Y d_t \]  
(A.6)

IS curve:

\[ \hat{C}_t = \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} (\hat{i}_t - \mathbb{E}_t \pi_{t+1}) \]  
(A.7)

Phillips curve:

\[ \pi_{Ht} = \beta \mathbb{E}_t \pi_{H,t+1} + \frac{(1 - \theta) (1 - \theta \beta)}{\theta} \left( \sigma \hat{C}_t + \varphi \hat{Y}_t + \alpha \hat{S}_t - (\varphi + 1) \hat{A}_t \right) + \eta_t \]  
(A.8)

ToT relationship

\[ \Delta \hat{S}_t = \Delta \varepsilon_t + \pi^*_t - \pi_{H,t} \]  
(A.9)

The final log-linearised system can be written as

\[
\begin{align*}
\beta \hat{d}_t &= \hat{d}_{t-1} - \alpha (1 - \gamma) \hat{C}_t + \alpha (1 - \gamma) (\eta (2 - \alpha) - 1) \hat{S}_t + \alpha (1 - \gamma) \hat{C}^*_t \\
\hat{i}_t &= \hat{i}_t^* + \hat{S}_{t+1} - \hat{S}_t + \pi_{H,t+1} - \pi_{t+1}^* - \chi d_t \\
\hat{C}_t &= \mathbb{E}_t \hat{C}^*_t - \frac{1}{\sigma} \left( \hat{i}_t - \mathbb{E}_t \hat{\pi}_{H,t+1} - \alpha \left( \mathbb{E}_t \hat{S}_{t+1} - \hat{S}_t \right) \right) \\
\hat{\pi}_{Ht} &= \beta \mathbb{E}_t \hat{\pi}_{H,t+1} + \lambda \left( \left( \sigma + \varphi (1 - \alpha) (1 - \gamma) \right) \hat{C}_t + \alpha (1 + \varphi \eta (2 - \alpha) (1 - \gamma)) \hat{S}_t \right) \\
&\quad + \lambda \varphi \alpha (1 - \gamma) \hat{C}^*_t + \lambda \varphi \gamma \hat{G}_t - \lambda (\phi + 1) \hat{A}_t + \eta_t \\
\hat{C}^*_t &= \mathbb{E}_t \hat{C}^*_t - \frac{1}{\sigma} (\hat{i}_t^* - \hat{\pi}_{t+1}^*) \\
\hat{\pi}_t &= \beta \mathbb{E}_t \hat{\pi}_t^* + \lambda \left( \left( \sigma + \varphi \right) \hat{C}^*_t - (\varphi + 1) \hat{A}^*_t \right) + \eta_t^*.
\end{align*}
\]
Substitute the expressions for $i_t^*, C_t^*, \pi_t^*$ into the above system gives

\[
\begin{align*}
\zeta_{1t} &= z_{10} \eta_t^* + z_{11} \zeta_{1t-1} + z_{12} \zeta_{2t-1} \\
\zeta_{2t} &= z_{20} \eta_t^* + z_{21} \zeta_{1t-1} + z_{22} \zeta_{2t-1} \\
\beta \hat{d}_t &= \hat{d}_{t-1} - \alpha (1 - \gamma) \hat{C}_t + \alpha (1 - \gamma) (\eta (2 - \alpha) - 1) \hat{S}_t + \alpha (1 - \gamma) n_{c0} \eta_t^* \\
&\quad + \alpha (1 - \gamma) n_{c1} \zeta_{1t-1} + \alpha (1 - \gamma) n_{c2} \zeta_{2t-1} + \alpha (1 - \gamma) \frac{(\varphi + 1)}{(\sigma + \varphi)} \hat{A}_t^*, \\
i_t &= \mathbb{E}_t \hat{S}_{t+1} - \hat{S}_t + \mathbb{E}_t \pi_{H,t+1} - \chi d_t - \frac{\sigma (1 - \rho^*) (\varphi + 1)}{(\sigma + \varphi)} \hat{A}_t^* \\
&\quad + (f_{c0} - n_{\pi0} \rho^* - n_{\pi1} z_{10} - n_{\pi2} z_{20}) \eta_t^* \\
&\quad + (f_{c1} - n_{\pi1} z_{11} - n_{\pi2} z_{21}) \zeta_{1t-1} + (f_{c2} - n_{\pi1} z_{12} - n_{\pi2} z_{22}) \zeta_{2t-1} \\
\hat{C}_t &= \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} \left( i_t - \mathbb{E}_t \hat{\pi}_{H,t+1} - \alpha \left( \mathbb{E}_t \hat{S}_{t+1} - \hat{S}_t \right) \right) \\
\hat{\pi}_{Ht} &= \beta \mathbb{E}_t \hat{\pi}_{H,t+1} + \lambda \left( \left( \sigma + \varphi (1 - \alpha) (1 - \gamma) \right) \hat{C}_t + \alpha (1 + \varphi \eta (2 - \alpha) (1 - \gamma)) \hat{S}_t \right) \\
&\quad + \lambda \varphi \gamma \hat{C}_t + \lambda \varphi \alpha (1 - \gamma) n_{c0} \eta_t^* + \lambda \varphi \alpha (1 - \gamma) n_{c1} \zeta_{1t-1} + \lambda \varphi \alpha (1 - \gamma) n_{c2} \zeta_{2t-1} \\
&\quad + \lambda \varphi \alpha (1 - \gamma) \frac{(\varphi + 1)}{(\sigma + \varphi)} \hat{A}_t^* - \lambda (\varphi + 1) \hat{A}_t + \eta_t \\
\end{align*}
\]

The above system has four independent shocks $\eta_t, \eta_t^*, \hat{A}_t, \hat{A}_t^*$, which are assumed to be independent AR(1) processes.
A.1.3 Small Open Economy: Natural Rates

The dynamic system for the flexible price equilibrium can be written as

\[
\beta \hat{d}_t^n = \hat{d}_{t-1}^n - \alpha (1 - \gamma) \hat{C}_t^n - \alpha (1 - \gamma) (1 - \eta (2 - \alpha)) \hat{S}_t^n + \alpha (1 - \gamma) \hat{C}_t^{sn},
\]

\[
i_t^n = i_t^{sn} + \hat{S}_{t+1}^n - \hat{S}_t^n - \delta d_t^n,
\]

\[
\hat{C}_t^n = \hat{C}_{t+1}^n - \frac{1}{\sigma} \left( i_t^n - \alpha \left( \hat{S}_{t+1}^n - \hat{S}_t^n \right) \right),
\]

\[
0 = \frac{(\sigma + \varphi (1 - \alpha) (1 - \gamma)) \hat{C}_t^n + \alpha (1 + \varphi (2 - \alpha) (1 - \gamma)) \hat{S}_t^n + \varphi (1 - \gamma) \hat{C}_t^{sn} + \phi \gamma \hat{G}_t - (\varphi + 1) \hat{A}_t.}
\]

We substitute foreign natural rates and, after some manipulations, obtain:

\[
\beta \hat{d}_t^n = \hat{d}_{t-1}^n + \xi_d \hat{S}_t^n + u_t, \quad (A.10)
\]

\[
\xi_s \hat{S}_{t+1}^n = \frac{\delta}{\sigma} d_t^n + \xi_s \hat{S}_t^n + v_t, \quad (A.11)
\]

where the parameters \( \xi_d \) and \( \xi_s \) are defined as

\[
\xi_d = \alpha (1 - \gamma) \left( \frac{\alpha (1 + \varphi \eta (2 - \alpha) (1 - \gamma))}{(\sigma + \varphi (1 - \alpha) (1 - \gamma))} - (1 - \eta (2 - \alpha)) \right),
\]

\[
\xi_s = \left( \frac{\alpha (1 + \varphi \eta (2 - \alpha) (1 - \gamma))}{(\sigma + \varphi (1 - \alpha) (1 - \gamma))} + \frac{1 - \alpha}{\sigma} \right),
\]

and shocks \( u_t \) and \( v_t \) are composite shocks:

\[
u_t = \frac{\alpha (1 - \gamma) \left( \varphi \gamma \hat{G}_t - (\varphi + 1) \hat{A}_t + \frac{(\sigma + (1 - \gamma) \varphi (\varphi + 1) \hat{A}_t^s)}{(\sigma + \varphi (1 - \alpha) (1 - \gamma))} \right)}{(\sigma + \varphi (1 - \alpha) (1 - \gamma))},
\]

\[
\nu_t = \frac{\left( \varphi (1 - \eta) \hat{G}_t - (1 - \rho_u) (\varphi + 1) \hat{A}_t + \frac{(1 - \rho_u) \varphi (\varphi + 1) (\sigma + (1 - \gamma) \varphi) \hat{A}_t^s}{(\sigma + \varphi (1 - \alpha) (1 - \gamma))} \right)}{(\sigma + \varphi (1 - \alpha) (1 - \gamma))}.\]
A solution to system (A.10)-(A.11) can be written as:

\[
\begin{align*}
\hat{d}_t^n &= \rho_d \hat{d}_{t-1}^n + \xi_u u_t + \xi_v v_t, \\
\hat{S}_t^n &= s_d \hat{d}_{t-1}^n + \kappa_u u_t + \kappa_v v_t.
\end{align*}
\]

Knowing \(\hat{d}_t^n\) and \(\hat{S}_t^n\), \(\hat{n}_t\) and \(\hat{C}_t^n\) can be easily restored.

### A.1.4 Small Open Economy: System in Gaps

Similar to the rest of the world, we assume that monetary policy in the small open economy uses the interest rate gap \(i_t = \hat{i}_t - \hat{i}_t^n\) in order to minimise its social loss

\[
\sum_{t=0}^{\infty} \beta^t \left( \pi_{Ht}^2 + \frac{\lambda}{\varepsilon} y_t^2 \right)
\]

where \(\pi_t = \hat{\pi}_{Ht} - \hat{\pi}_{Ht}^n\) is the inflation gap and \(y_t = \hat{Y}_t - \hat{Y}_t^n\) denotes the output gap.

The evolution of the small open economy in gap form can be written as:

\[
\begin{align*}
\zeta_{1t} &= z_{10} \eta_t^* + z_{11} \zeta_{1t-1} + z_{12} \zeta_{2t-1} \\
\zeta_{2t} &= z_{20} \eta_t^* + z_{21} \zeta_{1t-1} + z_{22} \zeta_{2t-1} \\
\beta d_{t} &= d_{t-1} - \alpha (1 - \gamma) c_t + \alpha (1 - \gamma) (\eta (2 - \alpha) - 1) s_t \\
&\quad + \alpha (1 - \gamma) n_c \eta_t^* + \alpha (1 - \gamma) n_c \zeta_{1t-1} + \alpha (1 - \gamma) n_c \zeta_{2t-1} \\
i_t &= \mathbb{E}_t \Delta s_{t+1} + \mathbb{E}_t \pi_{H,t+1} - \delta d_t + (f_{c0} - n_{\pi0} \rho_y - n_{\pi1} \pi_{10} - n_{\pi2} \pi_{20}) \eta_t^* \\
&\quad + (f_{c1} - n_{\pi1} \pi_{11} - n_{\pi2} \pi_{21}) \zeta_{1t-1} + (f_{c2} - n_{\pi1} \pi_{12} - n_{\pi2} \pi_{22}) \zeta_{2t-1} \\
c_t &= c_{t+1} - \frac{1}{\sigma} (i_t - \pi_{H,t+1} - \alpha \Delta s_{t+1}) \\
\pi_{Ht} &= \beta \mathbb{E}_t \pi_{H,t+1} + \lambda ((\sigma + \varphi (1 - \alpha)) (1 - \gamma)) c_t + \alpha (1 + \varphi \eta (2 - \alpha) (1 - \gamma)) s_t \\
&\quad + \lambda \varphi \alpha (1 - \gamma) n_c \eta_t^* + \lambda \varphi \alpha (1 - \gamma) n_c \zeta_{1t-1} + \lambda \varphi \alpha (1 - \gamma) n_c \zeta_{2t-1} + \eta_t
\end{align*}
\]
and the goal variable $y_t$ is given as

$$y_t = (1 - \alpha)(1 - \gamma)c_t - \eta \alpha(\alpha - 2)(1 - \gamma)s_t + \alpha(1 - \gamma)c_t^* + \alpha(1 - \gamma)n_{c0}\eta_t^* + \alpha(1 - \gamma)n_{c1}\zeta_{t-1} + \alpha(1 - \gamma)n_{c2}\zeta_{2t-1}$$

### A.2 Discretionary Policy in LQ RE Models

In this short section, we describe the general class of problems, and the types of equilibria that arise, formally. We assume a non-singular linear stochastic rational expectations model of the type described by Blanchard and Kahn (1980). The linearized model equations can be represented in general from by the following dynamic system

$$
\begin{bmatrix}
  y_{t+1} \\
  E_t x_{t+1}
\end{bmatrix} =
\begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
  y_t \\
  x_t
\end{bmatrix} +
\begin{bmatrix}
  B_1 \\
  B_2
\end{bmatrix}
\begin{bmatrix}
  u_t \\
  0
\end{bmatrix} +
\begin{bmatrix}
  \varepsilon_{t+1}
\end{bmatrix},
$$

(A.12)

where $y_t$ is an $n_1$-vector of predetermined variables with initial conditions $y_0$ given, $x_t$ is $n_2$-vector of forward-looking (or jump) variables, and $u_t$ is a $k$-vector of policy instruments (control variables). $\varepsilon_{t+1}$ is a $n_1 \times 1$ white noise process with covariance matrix $\Sigma$. For notational convenience we define the $n$-vector $z_t = (y_t', x_t')'$ where $n = n_1 + n_2$. $E_t$ denotes the expectations operator, conditional on information available at time $t$. The matrices $A$ and $B$ are constant functions of structural model parameters.

The intertemporal loss function of the policy maker can be written as

$$W_t = \frac{1}{2}E_t \sum_{s=1}^{\infty} \beta^{s-t} (z_s'Qz_s + 2z_s'Pu_s + u_s'Ru_s).$$

(A.13)

---

1 This section follows closely Backus and Drifill (1986), Söderlind (1999) and Blake and Kirsanova (2012).
The matrix $Q$ and $R$ are assumed to be symmetric and positive semi-definite. Under discretion the policy maker optimizes in each period of time taking private sector expectations as given. The private sector knows that future policy makers will implement the same decision process in subsequent periods. Hence, expected future variables are taken as given and at any time $t$ the policy maker and the private sector respond only to the current state.

\[
\mathbb{E}_t x_{t+1} = \mathbb{E}_t N_{t+1} y_{t+1},
\]
\[
\mathbb{E}_t u_{t+1} = \mathbb{E}_t F_{t+1} y_{t+1}
\]

where $N$ and $F$ are assumed to be known in period $t$ and determined by the decision problem in period $t + 1$.

At time $t$ the policy maker will choose an optimal policy $u_t$ to minimize the intertemporal loss function:

\[
\min_{u_t} W_t, \tag{A.14}
\]

subject to system (A.12) and a pair of linear rules

\[
u_t = -F^d y_t \tag{A.15}
\]
\[
x_t = -J y_t - Ku_t = -J y_t + K F^d y_t = -Ny_t \tag{A.16}
\]

where $N = J - K F^d$. This follows from

\[
\mathbb{E}_t x_{t+1} = \mathbb{E}_t N_{t+1} y_{t+1} = N_{t+1} (A_{11} y_t + A_{12} x_t + B_1 u_t)
\]
\[= A_{21} y_t + A_{22} x_t + B_2 u_t
\]
\[\Rightarrow N_{t+1} A_{12} x_t - A_{22} x_t = A_{21} y_t - N_{t+1} A_{11} y_t + B_2 u_t - N_{t+1} B_1 u_t
\]
\[\Rightarrow (N_{t+1} A_{12} - A_{22}) x_t = (A_{21} - N_{t+1} A_{11}) y_t + (B_2 - N_{t+1} B_1) u_t
\]
\[\Rightarrow x_t = (N_{t+1} A_{12} - A_{22})^{-1} (A_{21} - N_{t+1} A_{11}) y_t + (B_2 - N_{t+1} B_1) u_t
\]
Hence we can write

\[ y_{t+1} = A_{11} y_t + A_{12} (-J_t y_t - K_t u_t) + B_1 u_t + \varepsilon_{t+1} \]
\[ = (A_{11} - A_{12} J_t) y_t + (B_1 - A_{12} K_t) u_t + \varepsilon_{t+1} \]
\[ = A_t^* y_t + B_t^* u_t + \varepsilon_{t+1} \]  \hspace{1cm} (A.17)

The discretionary optimization problem of the policy maker can then be written as

\[ H_t (y_t) + v_t = \min_{u_t} \ y_t' Q^* y_t + 2 y_t' P^* u_t + u_t R^* u_t + \beta \left( y_{t+1}' H_{t+1} y_{t+1} + v_{t+1} \right) \]

Because of the quadratic nature of the per-period objective in (A.14) and because policy and private sector decisions are both linear in the state (A.15)-(A.16), the solution in \( t + 1 \) gives a value function which is quadratic in the state

\[ H_t (y_t) = y_t' S y_t + v_t. \]

Hence the Bellman equation can be rewritten as

\[ y_t' S y_t + v = \min_{u_t} \ y_t' Q^* y_t + 2 y_t' P^* u_t + u_t R^* u_t + \beta \left( y_{t+1}' S y_{t+1} + v \right) \]

where (A.18) is minimized subject to (A.17) and

\[ Q^* = Q_{11} - Q_{12} J - J' Q_{21} + J' Q_{22} J; \quad P^* = J' Q_{22} K - Q_{12} K + P_1 - J' P_2; \]

\[ R^* = K' Q_{22} K + R - K' P_2 - P_2' K; \quad A_t^* = A_{11} - A_{12} J; \quad B^* = B_1 - A_{12} K. \]  \hspace{1cm} (A.19)

The resulting FOCs are

\[ \frac{\partial H_t}{\partial u_t} = 2 P^* y_t + 2 R^* u_t + 2 \beta B^* S B^* u_t + 2 \beta B^* S A^* y_t = 0 \]  \hspace{1cm} (A.21)

which gives the decision rule of the policy maker

\[ u_t = -(R^* + \beta B^* S B^*)^{-1} (\beta B^* S A^* + P^*) y_t \]
\[ = -F^d y_t \]
where $F^d_i$ is $k \times n_1$. Inserting the reaction function of the policy maker back into (A.18) gives

$$y_t S y_t + (1 - \beta) v = y_t Q^* y_t - 2y_t P^* F^d y_s + y_t R^* F^d y_t$$

$$+ \beta \mathbb{E} \left( \left( A^* y_t - B^* F^d y_t + C \varepsilon_{t+1} \right)^\prime S \left( A^* y_t - B^* F^d y_t + C \varepsilon_{t+1} \right) \right)$$

$$= y_t^\prime \begin{bmatrix}
    Q^* - 2P^* F^d + F^d R^* F^d + \beta A^* S A^* \\
    + F^d (B^* S F^d + \beta F^d B^* S F^d + \beta F^d B^* S A^*)
\end{bmatrix} y_t + \beta \mathbb{E} \varepsilon_{t+1}^\prime \varepsilon_{t+1}$$

If we find $F$ and $N$ then the evolution of the dynamic system under control can be written as: $y_{t+1} = (A_{11} - A_{12} N - B_1 F^d) y_t + \varepsilon_{t+1} = M y_t + \varepsilon_{t+1}$, and the economy will be stable if all eigenvalues of transition matrix $M$ are inside the unit circle.

**Definition 4** The system of first order conditions to optimization problem (A.12)-(A.13) for matrices \{N, S, F^d\} can be written in the following form:

$$S = Q^* - 2P^* F^d + F^d R^* F^d + \beta A^* S A^* + \beta A^* S B^* F^d$$

(A.22)

$$F^d = (R^* + \beta B^* S F^d)^{-1} \left( P^* + \beta B^* S A^* \right),$$

(A.23)

$$N = (A_{22} + \bar{N} A_{12})^{-1} ((A_{21} - B_2 F^d) + \bar{N} (A_{11} - B_1 F^d)),$$

(A.24)

$$v = \frac{\beta}{1 - \beta} \text{trace} (S \Sigma)$$

(A.25)

$$Q^* = Q_{11} - Q_{12} J - J^\prime Q_{21} + J^\prime Q_{22} J,$$

(A.26)

$$P^* = J^\prime Q_{22} K - Q_{12} K + P_1 - J^\prime P_2,$$

(A.27)

$$R^* = K^\prime Q_{22} K + R - K^\prime P_2 - P_2^\prime K,$$

(A.28)

$$A^* = A_{11} - A_{12} J,$$

(A.29)

$$B^* = B_1 - A_{12} K,$$

$$J = (A_{22} + NA_{12})^{-1} (A_{21} + NA_{11}),$$

(A.30)

$$K = (A_{22} + NA_{12})^{-1} (B_2 + NB_1).$$

(A.31)

There is a one-to-one mapping between equilibrium trajectories and \{y_s, x_s, u_s\}_s=0^\infty and the triplet $T = \{N, S, F^d\}$, so it is convenient to continue with definition of pol-
icy equilibrium in terms of $T$, not trajectories. Hence, in what follows it is convenient to use the following definition. The following two results were shown in the literature (see e.g. Blake and Kirsanova (2012)).

**Definition 5** A triplet $T = \{N, S, F^d\}$ is a discretionary equilibrium if it satisfies the system of FOCs (A.22)-(A.31).

**Proposition 6** Suppose $N$ is given. The following two results hold.

1. There is a unique symmetric solution to (A.22) if matrix pair $(A^*, B^*)$ is controllable, i.e. if the controllability matrix, $[B^*, A^* B^*, A^{*2} B^*, ..., A^{*n-1} B^*]$, has full row rank.

2. Policy $F$, which is uniquely determined from (A.23) if $S$ is given, is stabilising, i.e. all eigenvalues of matrix $\Omega(F(S))$ that defines the evolution of the dynamic system under control

\[
y_{t+1} = A_{11} y_t + A_{12} x_t + B_1 u_t + \varepsilon_{t+1} = (A_{11} - A_{12} J) y_t + (B_1 - A_{12} K) u_t + \varepsilon_{t+1} \\
= (A^* - B^* F^d) y_t + \varepsilon_{t+1} = \Omega(F(S)) y_t + \varepsilon_{t+1} \tag{A.32}
\]

are strictly inside the unit circle.

It is apparent from system (A.22)-(A.31) that matrices $N$, $S$ and $F^d$ satisfy the three quadratic algebraic matrix equations (Riccati equations) (A.22)-(A.24), where the coefficients in these equations are also non-linear functions of model matrices and matrix $N$. This makes the whole system (A.22)-(A.31) non-linear and it is not surprising that it may have many solution triplets $T^J = \{N^J, S^J, F^{d,J}\}$, $J = 1, ..M$ where $M$ is the total number of solutions.$^2$

$^2$Blake and Kirsanova (2012) investigate the general properties of these discretionary solutions and Dennis and Kirsanova (2010) discuss different equilibrium selection mechanisms.
A.3 Discretionary Equilibria in the Inflation Targeting Regime

Following Appendix A.2 we demonstrate next how we find the discretionary equilibria in the small open economy model under the inflation targeting regime.

A.3.1 The System in Matrix Form

For the inflation targeting regime it is convenient to rewrite the dynamic system in the gap form using the notation $z_t = \hat{Z}_t - \hat{Z}_t^n$ for any variable $Z_t$. We can substitute out all static variables in order to arrive at the following reduced form optimization problem, written in a matrix form. We only work with the deterministic part as the model is certainty equivalent and the stochastic part can be added later in a unique way (Anderson et al. (1996)).

The objective function can be rewritten as:

$$\frac{1}{2} \sum_{s=t}^{\infty} \beta^{s-t} \begin{bmatrix} s_t \\ \pi_{Ht} \\ c_t \end{bmatrix}' \begin{bmatrix} \alpha^2 \eta^2 \omega (2 - \alpha)^2 (1 - \gamma)^2 & 0 & \alpha \eta \omega (1 - \alpha) (1 - \gamma)^2 \\ 0 & 1 & 0 \\ \alpha \eta \omega (1 - \alpha) (2 - \alpha) (1 - \gamma)^2 & 0 & \omega (1 - \alpha)^2 (1 - \gamma)^2 \end{bmatrix} \begin{bmatrix} s_t \\ \pi_{Ht} \\ c_t \end{bmatrix}$$

We optimize the above objective function with respect to the dynamic system:

$$\begin{bmatrix} d_t \\ s_{t+1} \\ \pi_{Ht+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & \frac{\alpha}{\beta} (1 - \gamma) (\eta (2 - \alpha) - 1) \\ \frac{\delta}{\beta} & \frac{\alpha \lambda \phi \eta (2 - \alpha) (1 - \gamma) + 1}{\beta} + \frac{\alpha \delta (1 - \gamma) (\eta (2 - \alpha) - 1)}{\beta} + 1 \\
0 & \frac{\lambda (\sigma + \phi (1 - \alpha) (1 - \gamma))}{\beta} - \frac{\lambda (1 - \alpha) \sigma \phi}{\beta} + 1 \\
\frac{1}{\beta} & \frac{1}{\beta} \frac{\alpha \lambda \phi \eta (2 - \alpha) (1 - \gamma) + 1}{\beta} & \frac{\alpha \delta (1 - \gamma) (\eta (2 - \alpha) - 1)}{\beta} + 1 \end{bmatrix} \begin{bmatrix} d_{t-1} \\ s_t \\ \pi_{Ht} \\ c_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ (1 - \alpha) \end{bmatrix} [\epsilon_t]$$
The variables \( c_t, \pi_{Ht} \) and \( s \) are non-predicted, \( d_{t-1} \) is a predetermined variable and \( i_t \) is the instrument of the policy maker. The system matrices of our model needed for computation are:

\[
A_{11} = \begin{bmatrix} 1 \\ \beta \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \gamma (2 - \alpha) (\eta (2 - \alpha) - 1) & 0 & -\frac{\gamma}{\beta} (1 - \gamma) \end{bmatrix}, \quad B_1 = [0],
\]

\[
A_{21} = \begin{bmatrix} \frac{\delta}{\beta} \\ 0 \\ -\frac{\alpha \delta}{\sigma \beta} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ \frac{(1-\alpha)}{\sigma} \end{bmatrix},
\]

\[
A_{22} = \begin{bmatrix}
\alpha \lambda (\phi \eta (2-\alpha)(1-\gamma)+1)+\alpha \delta (1-\gamma)(\eta (2-\alpha)-1)+\beta \\
-\alpha \lambda (\phi \eta (2-\alpha)(1-\gamma)+1) \\
\alpha \lambda (\phi \eta (2-\alpha)(1-\gamma)+1)-\alpha^2 \delta (1-\gamma)(\eta (2-\alpha)-1)
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\beta} \\
\frac{1}{\beta} \\
\frac{1}{\sigma \beta}
\end{bmatrix}
-\frac{\gamma}{\beta} (1 - \gamma)
-\frac{\lambda (\sigma + \phi (1-\alpha)(1-\gamma)) - \alpha \delta (1-\gamma)}{\beta}
-\frac{\lambda (\sigma + \phi (1-\alpha)(1-\gamma))}{\beta}
-\frac{\alpha^2 \delta (1-\gamma)+\lambda (1-\alpha)(\sigma + \phi (1-\alpha)(1-\gamma)) + \sigma \beta}{\sigma \beta}
\]

\[
Q_{11} = [0], \quad Q_{12} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},
\]

\[
Q_{21} = Q_{12}' \quad \text{and} \quad R = [0], \quad P_1 = [0],
\]

\[
Q_{22} = \begin{bmatrix}
\alpha \eta \omega (2 - \alpha) (1 - \gamma)^2 & 0 & \alpha \eta \omega (1 - \alpha) (2 - \alpha) (1 - \gamma)^2 \\
0 & 1 & 0 \\
\alpha \eta \omega (1 - \alpha) (2 - \alpha) (1 - \gamma)^2 & 0 & \omega (1 - \alpha)^2 (1 - \gamma)^2
\end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]
A.3.2 The Policy Maker’s Reaction Function and the Value Function

Suppose the reaction of the private sector can be written in the following linear form
where the coefficients of matrices $J$ and $K$ are indeterminate:

\[
\begin{bmatrix}
 s_t \\
\pi_{Ht} \\
c_t
\end{bmatrix} = \left( \begin{bmatrix}
 J_s \\
\pi_H \\
J_c
\end{bmatrix} \right) [d_{t-1}] - \left( \begin{bmatrix}
 K_s \\
K_x \\
K_c
\end{bmatrix} \right) [i_t],
\]

We can compute the following scalars (see equations (A.26)-(A.31)):
\[ Q^* = J_c^2 + \omega J_c^2 (1 - \alpha)^2 (1 - \gamma)^2 + \alpha \eta \omega J_c J_s (1 - \alpha) (2 - \alpha) (-\gamma + 1)^2 + \alpha \eta \omega J_c (1 - \alpha) (2 - \alpha) (1 - \gamma)^2 + \alpha^2 \eta^2 \omega J_s^2 (2 - \alpha)^2 (1 - \gamma)^2, \]

\[ P^* = J_c K_c + K_c \left( \omega J_c (1 - \alpha)^2 (1 - \gamma)^2 + \alpha \eta \omega J_s (1 - \alpha) (2 - \alpha) (1 - \gamma)^2 \right) + K_s \left( \alpha \eta \omega J_c (1 - \alpha) (2 - \alpha) (1 - \gamma)^2 + \alpha^2 \eta^2 \omega J_s (2 - \alpha)^2 (1 - \gamma)^2 \right), \]

\[ R^* = K_c^2 + K_c \left( \omega K_c (1 - \alpha)^2 (1 - \gamma)^2 + \alpha \eta K_s (1 - \alpha) (2 - \alpha) (-\gamma + 1)^2 \right) + K_s \left( \alpha \eta \omega K_c (1 - \alpha) (2 - \alpha) (1 - \gamma)^2 + \alpha^2 \eta^2 \omega K_s (2 - \alpha)^2 (1 - \gamma)^2 \right), \]

\[ A^* = \frac{1}{\beta} + \frac{\alpha}{\beta} J_c (1 - \gamma) - \frac{\alpha}{\beta} J_s (1 - \gamma) (\eta (2 - \alpha) - 1), \]

\[ B^* = \frac{\alpha}{\beta} K_c (1 - \gamma) - \frac{\alpha}{\beta} K_s (1 - \gamma) (\eta (2 - \alpha) - 1) \]

Because the model has only one state variable, we can substitute the steady state version of equation (A.23) into (A.22) and obtain a quadratic equation for the scalar variable \( S \) that can be written as:

\[
S = Q^* + \beta S A^2 - \frac{(P^* + \beta S B A)^2}{(R^* + \beta S B^2)} \Rightarrow \beta B^2 S^2 + \left( R^* - \beta \left( Q^* B^2 - 2P^* B^* A^* + R^* A^*^2 \right) \right) S = 0 \tag{A.33}
\]

It is easy to show that the product of the two eigenvalues of the quadratic equation is negative:

\[
P^*^2 - Q^* R^* = -\omega (\gamma - 1)^2 \left( -J_c K_c + J_c K_c + J_c K_s \alpha - J_c K_s \alpha \right) - 2J_c K_s \alpha \eta + 2J_s K_c \alpha \eta + J_c K_s \alpha^2 \eta - J_s K_s \alpha^2 \eta \right)^2 < 0.
\]

It also immediately follows that the determinant of this equation is positive:

\[
D = \left( R^* - \beta \left( Q^* B^*^2 - 2P^* B^* A^* + R^* A^*^2 \right) \right)^2 - 4\beta B^2 \left( P^*^2 - Q^* R^* \right) > 0,
\]

as the second term is negative.
Therefore, the two eigenvalues of (A.33) are always real and of different signs. Since we are looking for a positive value function $S$, the solution is unique. We can easily find it with conventional methods for solving quadratic equations. Having found $S$ we can uniquely determine the optimal discretionary policy as the reaction function:

$$F^d = \frac{P^* + \beta B^* A^* S}{R^* + \beta B^2 S}.$$  \hspace{1cm} (A.34)

Note that in equations (A.33) and (A.34) all coefficients depend on $J_s$, $J_\pi$, $J_c$, $K_s$, $K_\pi$, and $K_c$. Equations (A.30)-(A.31) suggest that all $J$ and $K$ are, in their turn, functions of $N_s, N_\pi$ and $N_c$. Therefore $F = F(S(N_s, N_\pi, N_c)) = F(N_s, N_\pi, N_c)$ depends on the three coefficients of the reaction function of the private sector.

### A.3.3 The Private Sector’s Reaction Function

This section completes the analytical characterization of the solution to the problem. Here we derive the private sector’s reaction function and construct the equilibria of the private sector.

The pair of linear rules (A.15)-(A.16) yields the following reaction function for the policy maker:

$$i_t = -F^d d_{t-1},$$

and for the private sector the reaction function is:

$$\begin{bmatrix}
  s_t \\
  \pi_t \\
  c_t
\end{bmatrix} = -\begin{bmatrix}
  N_s \\
  N_\pi \\
  N_c
\end{bmatrix} [d_{t-1}].$$

The private sector’s reaction function solves (A.24), which simplifies for our model
to

\[ 0 = NA_{11} + A_{21} - B_2 F^d - NA_{12}N - A_{22}N \]
\[ = \begin{bmatrix} N_s \\ N_\pi \\ N_c \end{bmatrix} A_{11} + A_{21} - B_2 F^d - \begin{bmatrix} N_s \\ N_\pi \\ N_c \end{bmatrix} A_{12} - A_{22} \begin{bmatrix} N_s \\ N_\pi \\ N_c \end{bmatrix} \]

This system consists of three quadratic equations in \( N_s, N_\pi \) and \( N_c \):

\[ 0 = \frac{1}{\beta} N_s + \frac{\delta}{\beta} - F^d - \left( \frac{\alpha}{\beta} N_s^2 (1 - \gamma) (\eta (2 - \alpha) - 1) - \frac{\alpha}{\beta} N_c N_s (1 - \gamma) \right) \] (A.35)
\[ - \left( N_c \left( -\frac{\alpha}{\beta} \delta (1 - \gamma) + \frac{\lambda}{\beta} (\sigma + \varphi (1 - \alpha) (1 - \gamma)) \right) - \frac{1}{\beta} N_\pi \right) \]
\[ - N_s \left( \frac{\alpha \delta (1 - \gamma) (\eta (2 - \alpha) - 1)}{\beta} + \frac{\alpha \lambda (\varphi (2 - \alpha) (1 - \gamma) + 1)}{\beta} \right) \]
\[ 0 = \frac{1}{\beta} N_\pi - \left( -\frac{\alpha}{\beta} N_\pi N_c (1 - \gamma) + \frac{\alpha}{\beta} N_\pi N_s (1 - \gamma) (\eta (2 - \alpha) - 1) \right) \] (A.36)
\[ - \left( \frac{1}{\beta} N_\pi - \frac{\lambda (\sigma + \varphi (1 - \alpha) (\gamma + 1))}{\beta} - \frac{\alpha \lambda (\varphi (2 - \alpha) (1 - \gamma) + 1)}{\beta} \right) N_s \]
\[ 0 = \frac{1}{\beta} N_c - \frac{\alpha \delta}{\sigma \beta} - \frac{(1 - \alpha)}{\sigma} F^d \] (A.37)
\[ - \left( \frac{\alpha}{\beta} N_c N_s (1 - \gamma) (\eta (2 - \alpha) - 1) - \frac{\alpha}{\beta} N_c^2 (1 - \gamma) \right) \]
\[ - N_s \left( \frac{\alpha \lambda (1 - \alpha) (\varphi (2 - \alpha) (1 - \gamma) + 1)}{\sigma \beta} - \frac{\alpha^2 \delta (1 - \gamma) (\eta (2 - \alpha) - 1)}{\sigma \beta} \right) \]
\[ - N_c \left( \frac{\sigma \alpha^2 \delta (1 - \gamma)}{\sigma \beta} + \frac{\lambda (1 - \alpha) (\sigma + \varphi (1 - \alpha) (1 - \gamma))}{\sigma \beta} + 1 \right) \].

Note that equation (A.36) is linear in \( N_\pi \) and does not contain \( F^d \). Hence we can write the equation as \( N_\pi = N_\pi (N_s, N_c) \). We can substitute out \( N_\pi \) from the first and the third equation and the above system simplifies to a two-equation polynomial system (A.35) and (A.37), that can be solved numerically. We plot solutions to each equation in the top panel of Figure A.1. When these lines intersect we have a solution to the system (A.35) and (A.37). They intersect at five points, denoted by circles and diamonds. Having found all solutions for \( \{N_s(F^d), N_c(F^d)\} \) we can also compute \( N_\pi = N_\pi (N_s(F^d), N_c(F^d)) = N_\pi (F^d) \) for each of them.
Given $F^d$ we can substitute $N$ into the dynamic system (A.32) and check that only two solutions, denoted by circles in the top panel of Figure A.1, ensure stability of the system; so all eigenvalues of the matrix $M$ are inside the unit circle. The other three solutions are unstable. Although the stability properties can change if we vary $F^d$, we have checked that this is not the case. We will ignore unstable solutions in what follows.

To summarize, for a given $F^d$ we can find a solution triplet $\{N_c(F^d), N_s(F^d), N_c(F^d)\}$, which describes the equilibrium response of the private sector to a policy action. We have found five private sector equilibria, as shown in the top panel of Figure A.1, but only two of them are stable.

### A.3.4 Discretionary Solutions

A discretionary solution, by definition, is a triplet of matrices $\{N, S, F^d\}$ which is an asymptotically stable steady state of system (A.22)-(A.24). In other words, all solutions can be described as points of intersection of solutions of the system (A.22)-(A.23) with solutions of equation (A.24).

The top panel of Figure A.1 plots the solution pairs $\{N_c(F^d), N_s(F^d)\}$ for a given value of $F^d$. If we vary $F^d$ then each of the five points becomes a one-dimensional curve in the three-dimensional space with coordinates $N_c, N_s$ and $F^d$. Similarly, equations (A.33) and (A.34) define $F^d$ for every triplet $\{N_s, N_\pi, N_c\}$ and, from $N_\pi = N_s(N_s, N_c)$ follows $F^d = F^d(N_s, N_c)$. This determines a two-dimensional surface in the same three-dimensional space. Discretionary solutions exist where these curves intersect the surface. We plot the surface and the two curves, that correspond to stable private sector equilibria as described above, in the bottom panel of Figure A.1. They intersect in four points so there are four discretionary equilibria. We disregard two of them as they cannot be located neither by backward induction as discussed in Blake and Kirsanova (2012) nor are they Iterative Expectations-
Panel I: Reaction of the Private Sector $N_c, N_s$ to Policy $F$

Discretionary Equilibria

Figure A.1: Discretionary Solution
stable under joint learning as discussed in Dennis and Kirsanova (2010). We discuss the economic properties of the remaining equilibria in the paper.
B.1 Commitment FOCs in the Form of a Riccati Equation

When optimizing, the policy maker internalizes the effect of its choice on private sector’s expectations and solves the following Lagrangian

$$L^c = \sum_{t=0}^{\infty} \beta^t \left( \frac{1}{2} (\pi_t^2 + \lambda c_t^2) + \xi_{t+1} (\rho b_t - \eta c_t - b_{t+1}) + \phi_{t+1} (\pi_t - \kappa c_t - \nu b_t - u_t - \beta \pi_{t+1}) \right)$$

The corresponding first order conditions are:

0 = $-\xi_t + \rho \beta \xi_{t+1} - \nu \beta \phi_{t+1}$, \hspace{1cm} (B.1)
0 = $\pi_t + \phi_{t+1} - \phi_t$, \hspace{1cm} (B.2)
0 = $\lambda c_t - \eta \xi_{t+1} - \kappa \phi_{t+1}$, \hspace{1cm} (B.3)
0 = $\rho b_t - \eta c_t - b_{t+1}$, \hspace{1cm} (B.4)
0 = $\beta \pi_{t+1} + \kappa c_t + \nu b_t + u_t - \pi_t$, \hspace{1cm} (B.5)

for $t \geq 0$; with initial conditions $b_0 = \bar{b}$ and $\phi_0 = 0$, and the transversality condition $\lim_{t \to \infty} b_t < \infty$. 

If the system (3.6)-(3.7) is controllable, there always exists a unique path \( \{c_t, \pi_t, b_t\}_{t \geq 0} \) which (i) satisfies system (B.1)-(B.5) and the initial conditions and (ii) all eigenvalues of the resulting transition matrix are less than \( 1/\sqrt{\beta} \) in modulus (see, e.g. Kwakernaak and Sivan (1972), Backus and Driffill (1986)). For the rest of this paper we use the following definition: The economy is stabilized by a policy if all eigenvalues of the transition matrix are inside the unit circle. If the economy is stabilized by a policy we call such a policy stabilizing. In general, because \( \beta < 1 \) a stabilizing commitment policy may not exist for all problems in the LQ RE class.

One way to solve the system (B.1)-(B.5) is to use the Schur decomposition, see e.g. Söderlind (1999). Alternatively, and more convenient for our purpose, we can also solve the system using an iterative scheme.

System (B.1)-(B.5) can be written as

\[
\begin{bmatrix}
0 & 0 & \eta \\
0 & \beta & 0 \\
0 & 0 & \rho \beta
\end{bmatrix}
\begin{bmatrix}
c_{t+1} \\
\pi_{t+1} \\
\xi_{t+1}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & -\kappa \\
-1 & -\nu & 0 \\
0 & 0 & \nu \beta
\end{bmatrix}
\begin{bmatrix}
u_t \\
b_t \\
\phi_t
\end{bmatrix}
+ \begin{bmatrix}
\lambda & \kappa & 0 \\
-\kappa & 1 & 0 \\
0 & -\nu \beta & 1
\end{bmatrix}
\begin{bmatrix}
c_t \\
\pi_t \\
\xi_t
\end{bmatrix}
\]

(B.6)

\[
\begin{bmatrix}
u_{t+1} \\
b_{t+1} \\
\phi_{t+1}
\end{bmatrix}
= \begin{bmatrix}
\rho_\alpha & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_t \\
b_t \\
\phi_t
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
-\eta & 0 & 0 \\
0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
c_t \\
\pi_t \\
\xi_t
\end{bmatrix}
\]

(B.7)

Substituting (3.33)-(3.35) into both sides of (B.6) and use (B.7) to substitute out \( u_{t+1}, b_{t+1}, \phi_{t+1} \) to get

\[
\begin{bmatrix}
\lambda + \eta^2 \xi_b & \kappa + \eta \xi_\phi & 0 \\
\beta \eta \pi_b - \kappa & \beta \pi_\phi + 1 & 0 \\
\beta \eta \rho \xi_b & -\beta (\nu - \rho \xi_\phi) & 1
\end{bmatrix}
\begin{bmatrix}
c_t \\
\pi_t \\
\xi_t
\end{bmatrix}
= \begin{bmatrix}
\eta \xi_u \rho_\alpha & \eta \rho \xi_b & \kappa + \eta \xi_\phi \\
\beta \rho_\alpha \pi_u + 1 & \nu + \beta \rho \pi_b & \beta \pi_\phi \\
\beta \rho \xi_u \rho_\alpha & \beta \rho_\alpha \xi_b & -\beta (\nu - \rho \xi_\phi)
\end{bmatrix}
\begin{bmatrix}
u_t \\
b_t \\
\phi_t
\end{bmatrix}
\]

Appendix Chapter 3
Substitution of (3.33)-(3.35) yields the following matrix algebraic Riccati equation:

\[
\begin{bmatrix}
   c_u & c_b & c_\phi \\
   \pi_u & \pi_b & \pi_\phi \\
   \xi_u & \xi_b & \xi_\phi
\end{bmatrix}
\begin{bmatrix}
   \lambda + \eta^2 \xi_b & \kappa + \eta \xi_\phi & 0 \\
   \beta \eta \xi_b - \kappa & \beta \pi_\phi + 1 & 0 \\
   \beta \eta \xi_b - \beta (\nu - \rho \xi_\phi) & 1
\end{bmatrix}
= \begin{bmatrix}
   \eta \xi_u \rho_u & \eta \rho \xi_b & \kappa + \eta \xi_\phi \\
   \beta \rho_u \pi_u + 1 & \nu + \beta \rho \pi_b & \beta \pi_\phi \\
   \beta \rho_u \xi_u \rho_u & \beta \rho^2 \xi_b & -\beta (\nu - \rho \xi_\phi)
\end{bmatrix}
\]

We can guess all feedback coefficients in (3.33)-(3.35) and thus in the right hand side of the equation above. Then, the Riccati equation gives an update of these coefficients: in the next step we update the right hand side of it and iterate until convergence. The algorithm will converge (Lancaster and Rodman (1995)).

Although the baseline calibration delivers a stabilizing solution, note that if the fiscal feedback is weak, \(0 < \mu < \mu^*\), the economy is not stabilized by a policy. The optimal monetary policy still delivers a finite value of the loss function (4.18), but all variables exhibit slow explosion with a rate of explosion less than \(1/\sqrt{\beta}\). However, this solution should be disregarded as it violates the assumption of a finite working week.

Finally, note that equation (B.2) implies price stability: if \(\phi_t = 0\) and \(\lim_{t \to \infty} \phi_t = 0\) it follows that \(\sum_{t=0}^{\infty} \pi_t = 0\).

---

1Kirsanova and Wren-Lewis (2011) show that in our model \(\mu^* = \left(1 - \tilde{T}\right) \left(1 - \beta\right) \kappa / \left(\tilde{\gamma} \left(\left(1 - \tilde{T}\right) \kappa - \zeta \theta \tilde{T}\right)\right)\).

2This result was shown in a similar model in Schmitt-Grohe and Uribe (2004) and in Kirsanova and Wren-Lewis (2011).
B.2 Limited Commitment and Riccati Equations

System (3.37)-(3.41) can be written as

\[
\begin{bmatrix}
0 & 0 & \eta \\
0 & \beta (1 - \alpha) & 0 \\
0 & \rho \beta (1 - \alpha) & 0
\end{bmatrix}
\begin{bmatrix}
c_{t+1} \\
\pi_{t+1} \\
\xi_{t+1}
\end{bmatrix}
= \begin{bmatrix}
\lambda & \kappa & 0 \\
\beta \alpha \pi_b^\alpha \eta - \kappa & 1 & 0 \\
0 & -\nu \beta (1 - \alpha) & 1
\end{bmatrix}
\begin{bmatrix}
c_t \\
\pi_t \\
\xi_t
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
-\beta \alpha \pi_u^\alpha \rho_u - 1 & -\beta \alpha \pi_b^\alpha \rho - \nu & 0 \\
-\beta \alpha S_{ub}^\alpha & -\beta \alpha S_{bb}^\alpha & \beta (\nu (1 - \alpha) + \alpha \pi_b^\alpha)
\end{bmatrix}
\begin{bmatrix}
u_t \\
b_t \\
\phi_t
\end{bmatrix}
\]

(B.9)

\[
\begin{bmatrix}
u_{t+1} \\
b_{t+1} \\
\phi_{t+1}
\end{bmatrix}
= \begin{bmatrix}
\rho_u & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_t \\
b_t \\
\phi_t
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
-\eta & 0 & 0 \\
0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
c_t \\
\pi_t \\
\xi_t
\end{bmatrix}
\]

(B.10)

Substitute (3.33)-(3.35) into both sides of (B.9) and use (B.10) to substitute out \(u_{t+1}, b_{t+1}, \phi_{t+1}\). We obtain the following matrix discrete algebraic Riccati equation:

\[
\begin{bmatrix}
c_u^\alpha & c_b^\alpha & c_\phi^\alpha \\
\pi_u^\alpha & \pi_b^\alpha & \pi_\phi^\alpha \\
\xi_u^\alpha & \xi_b^\alpha & \xi_\phi^\alpha
\end{bmatrix}
= \begin{bmatrix}
\lambda + \eta^2 \xi_b^\alpha & \kappa + \eta \xi_\phi^\alpha & 0^1 \\
\beta \eta \pi_b^\alpha - \kappa & (1 - \alpha) \beta \pi_\phi^\alpha + 1 & 0 \\
\beta \eta \pi_b^\alpha (1 - \alpha) & \beta (\rho \xi_\phi^\alpha - \nu) (1 - \alpha) & 1
\end{bmatrix}
\]

(B.11)

\[
\times \begin{bmatrix}
\eta \xi_u^\alpha \rho_u & \eta \rho \xi_b^\alpha & \kappa + \eta \xi_\phi^\alpha \\
\beta \rho_u \pi_u^\alpha + 1 & \nu + \beta \rho \pi_b^\alpha & \beta \pi_\phi^\alpha (1 - \alpha) \\
\beta (\alpha S_{bu}^\alpha + (1 - \alpha) \rho \xi_u^\alpha \rho_u) & \beta (\alpha S_{bb}^\alpha + (1 - \alpha) \rho \pi_b^\alpha) & \beta ((1 - \alpha) (\rho \xi_\phi^\alpha - \nu) - \alpha \pi_\phi^\alpha)
\end{bmatrix}
\]

Unlike under commitment, we need to know parameters \(S_{bu}^\alpha\) and \(S_{bb}^\alpha\) to solve the problem. They are the components of the value function; to obtain them we write (3.36) in the form of Bellman equation and substitute (3.33)-(3.35). We obtain the
The following matrix equation:

\[
\begin{bmatrix}
    S_{aa}^\alpha & S_{ab}^\alpha & U_{a0}^\alpha \\
    S_{ab}^\alpha & S_{bb}^\alpha & U_{b0}^\alpha \\
    U_{a0}^\alpha & U_{b0}^\alpha & U_{00}^\alpha
\end{bmatrix}
\begin{bmatrix}
    \pi_{aa}^\alpha + \lambda \varepsilon_{aa}^\alpha \\
    \pi_{bb}^\alpha + \lambda \varepsilon_{bb}^\alpha \\
    \pi_{00}^\alpha
\end{bmatrix}
= \begin{bmatrix}
    \pi_{aa}^\alpha + \lambda \varepsilon_{aa}^\alpha \\
    \pi_{bb}^\alpha + \lambda \varepsilon_{bb}^\alpha \\
    \pi_{00}^\alpha
\end{bmatrix} + \beta (1 - \alpha) \times \left( \begin{bmatrix}
    \rho_a - \eta \varepsilon_a^\alpha & -\pi_a^\alpha \\
    0 & \rho - \eta \varepsilon_b^\alpha & -\pi_b^\alpha \\
    0 & 0 & 1 - \pi_a^\alpha
\end{bmatrix} \begin{bmatrix}
    S_{aa}^u & S_{aa}^u \\
    S_{bb}^u & S_{bb}^u \\
    U_{a0}^u & U_{b0}^u
\end{bmatrix} \right) \cdots \times \left( \begin{bmatrix}
    \rho_a & -\eta \varepsilon_a^\alpha & 0 \\
    -\eta \varepsilon_a^\alpha & \rho - \eta \varepsilon_b^\alpha & -\eta \varepsilon_b^\alpha \\
    -\pi_a^\alpha & -\pi_b^\alpha & 1 - \pi_a^\alpha
\end{bmatrix} \right)
\]

Equations (B.11) and (B.12) solve the limited commitment problem.

The limiting case of \( \alpha \to 1 \) leads to the following system:

\[
S^1_u = (\pi^1_u)^2 + \lambda (c^1_u)^2 + \beta \left( \rho^2_u S^1_u + \eta^2 (c^1_u)^2 S^1_{bb} - 2\eta \rho_u c^1_u S^1_{bu} \right), \quad (B.13)
\]
\[
S^1_{ab} = \pi^1_{ab} + \lambda c^1_{ab} + \beta \rho_u (\rho - \eta c^1_b) S^1_{ab} + \beta \eta c^1_u (\eta c^1_b - \rho) S^1_{bb}, \quad (B.14)
\]
\[
S^1_{bb} = (\pi^1_b)^2 + \lambda (c^1_b)^2 + \beta (-\rho + \eta c^1_b)^2 S^1_{bb}, \quad (B.15)
\]
\[
\eta \beta S^1_{bu} \rho_u = \kappa \pi^1_u + \lambda c^1_u + \eta \beta S^1_{bb} c^1_u - \eta \beta \pi^1_u p^1_u, \quad (B.16)
\]
\[
\eta \rho S^1_{bb} = \kappa \pi^1_b + \lambda c^1_b + \eta \beta S^1_{bb} c^1_b - \eta \beta \pi^1_b p^1_b, \quad (B.17)
\]
\[
\beta \rho_u p^1_u + 1 = \pi^1_u - \kappa c^1_u + \beta \pi^1_u c^1_u, \quad (B.18)
\]
\[
\nu + \beta \rho p^1_b = \pi^1_b - \kappa c^1_b + \beta \pi^1_b c^1_b, \quad (B.19)
\]
\[ \xi^1_u = \beta S^1_{bu}, \quad \xi^1_b = \beta S^1_{bb}, \quad \xi^1_\phi = -\beta \pi^1_b. \quad \tag{B.20} \]

\[ \pi^1_\phi = \frac{(\kappa - \eta \beta \pi^1_b)^2}{(\kappa - \eta \beta \pi^1_b)^2 + \lambda + \eta^2 \beta S^1_{bb}}, \quad \tag{B.21} \]

\[ c^1_\phi = \frac{\kappa - \eta \beta \pi^1_b}{(\kappa - \eta \beta \pi^1_b)^2 + \lambda + \eta^2 \beta S^1_{bb}}, \quad \tag{B.22} \]

\[ U^1_\phi = \frac{(\kappa - \eta \beta \pi^1_b)^2}{(\kappa - \eta \beta \pi^1_b)^2 + \lambda + \eta^2 \beta S^1_{bb}}, \quad \tag{B.23} \]

\[ U^1_u = \left( \kappa - \eta \beta \pi^1_b \right) \frac{\pi^1_u (\kappa - \eta \beta \pi^1_b) + c^1_b (\lambda + \beta \eta^2 S^1_{bb}) - \beta \eta \pi^1_u S^1_{bu}}{(\kappa - \eta \beta \pi^1_b)^2 + \lambda + \eta^2 \beta S^1_{bb}}, \quad \tag{B.24} \]

\[ U^1_b = \left( \kappa - \eta \beta \pi^1_b \right) \frac{\pi^1_b (\kappa - \eta \beta \pi^1_b) + c^1_b (\lambda + \beta \eta^2 S^1_{bb}) - \beta \eta \pi^1_b S^1_{bb}}{(\kappa - \eta \beta \pi^1_b)^2 + \lambda + \eta^2 \beta S^1_{bb}}. \quad \tag{B.25} \]

### B.3 Limited Commitment Policy in General LQ RE Framework

We assume a non-singular linear deterministic rational expectations model, augmented by a vector of control instruments. Specifically, the evolution of the economy is explained by the linear system

\[
\begin{bmatrix}
  y_{t+1} \\
  \mathbb{E}_t x_{t+1}
\end{bmatrix} =
\begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
  y_t \\
  x_t
\end{bmatrix} +
\begin{bmatrix}
  B_1 \\
  B_2
\end{bmatrix} u_t +
\begin{bmatrix}
  \xi_{t+1} \\
  0
\end{bmatrix},
\]

where \( y_t \) is an \( n_1 \)-vector of predetermined variables with initial conditions \( y_0 \) given, \( x_t \) is \( n_2 \)-vector of non-predetermined (or jump) variables with \( \lim_{t \to \infty} x_t = 0 \), \( u_t \) is a \( k \)-vector of policy instruments of the policy maker, and \( \xi_t \) is a vector of i.i.d. shocks with covariance matrix \( \Sigma \). For notational convenience we define the \( n \)-vector \( z_t = (y'_t, x'_t)' \) where \( n = n_1 + n_2 \). We assume \( A_{22} \) is non-singular.

The inter-temporal policy maker’s welfare criterion is defined by the quadratic loss function

\[ L_0 = \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t g'_t Q g_t = \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{t-1} (z'_t Q z_t + 2 z'_t P u_t + u'_t R u_t). \quad \tag{B.27} \]
The elements of vector $g_s$ are the goal variables of the policy maker, $g_t = C(z'_t, u'_t)'. Matrix $Q$ is assumed to be symmetric and positive semi-definite.\footnote{It is standard to assume that $R$ is symmetric positive definite (see Anderson et al. (1996), for example). However, since many economic applications involve a loss function that places no penalty on the control variables, we note that the requirement of $Q$ being positive definite can be weakened to $Q$ being positive semi-definite if additional assumptions about other system matrices are met (Clements and Wimmer (2003)). The analysis in this paper is valid for $R \equiv 0$.}

Schaumburg and Tambalotti (2007) and then Debortoli and Nunes (2010) demonstrate that the optimization problem can be written as

$$
\min \mathbb{E}_0 \sum_{t=0}^{\infty} (\beta (1 - \alpha))^t \left( z'_t Q z_t + 2 z'_t Pu_t + u'_t Ru_t + \beta \alpha y_{t+1}' S y_{t+1} \right) \quad \text{(B.28)}
$$

subject to

$$
y_{t+1} = A_{11} y_t + A_{12} x_t + B_1 u_t + C \xi_{t+1}
$$

$$
(1 - \alpha) \mathbb{E}_t x_{t+1} + \alpha H y_{t+1} = A_{21} y_t + A_{22} x_t + B_2 u_t
$$

where $H$ and $S$ are components of the solution to the corresponding discretionary problem, $x_t = Hy_t$ and the loss is $L_t (y_t) = \frac{1}{2} y_t' S y_t$.

The first order conditions to the appropriate Lagrangian

$$
\mathcal{L}^c = \sum_{t=0}^{\infty} (\beta (1 - \alpha))^t \left( z'_t Q z_t + 2 z'_t Pu_t + u'_t Ru_t + \beta \alpha y_{t+1}' S y_{t+1} \right)
$$

$$
+ 2 \nu_{t+1}' \left( A_{21} y_t + A_{22} x_t + B_2 u_t - (1 - \alpha) x_{t+1} - \alpha H y_{t+1} \right)
$$

$$
+ 2 \nu_{t+1}' \left( A_{11} y_s + A_{12} x_s + B_1 u_s + \xi_{t+1} - y_{s+1} \right)
$$
can be written as
\[
\begin{bmatrix}
I & 0 & 0 & 0 & 0 \\
0 & \beta A'_{22} & 0 & 0 & \beta A'_{12} \\
0 & B'_{2} & 0 & 0 & B'_{1} \\
\alpha H & 0 & 0 & (1 - \alpha) I & 0 \\
0 & \beta (1 - \alpha) A'_{21} & 0 & 0 & \beta (1 - \alpha) A'_{11}
\end{bmatrix}
\begin{bmatrix}
y_{t+1} \\
\varphi_{t+1} \\
x_{t+1} \\
\psi_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
A_{11} & 0 & B_{1} & A_{12} & 0 \\
-\beta Q'_{12} & I & -\beta P_{2} & -\beta Q_{22} & 0 \\
-P'_{1} & 0 & -R & -P'_{2} & 0 \\
A_{21} & 0 & B_{2} & A_{22} & 0 \\
-\beta ((1 - \alpha) Q_{11} + \alpha S) & \alpha H' & -\beta (1 - \alpha) P_{1} & -\beta (1 - \alpha) Q_{12} & I
\end{bmatrix}
\begin{bmatrix}
y_{t} \\
\varphi_{t} \\
x_{t} \\
\psi_{t}
\end{bmatrix}
\tag{B.29}
\]

Solution to this system (using Schur decomposition, for example, or iteration Riccati equation as we do in the text) can be written in the form
\[
\begin{bmatrix}
u_{t} \\
x_{t} \\
\psi_{t} \\
y_{t+1} \\
\varphi_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
X_{uy} & X_{u\varphi} \\
X_{xy} & X_{x\varphi} \\
X_{\psi y} & X_{\psi\varphi}
\end{bmatrix}
\begin{bmatrix}
y_{t} \\
\varphi_{t}
\end{bmatrix}
, 
\tag{B.30}
\]
\[
\begin{bmatrix}
y_{t+1} \\
\varphi_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
M_{yy} & M_{y\varphi} \\
M_{\psi y} & M_{\psi\varphi}
\end{bmatrix}
\begin{bmatrix}
y_{t} \\
\varphi_{t}
\end{bmatrix}
, 
\tag{B.31}
\]
\[
W_{t}(y_{t}, \varphi_{t}) = \frac{1}{2} \left( \begin{bmatrix}
y_{t} \\
\varphi_{t}
\end{bmatrix}^{T} \begin{bmatrix} U_{11} & U_{12} \\
U_{21} & U_{22} \end{bmatrix} \begin{bmatrix}
y_{t} \\
\varphi_{t}
\end{bmatrix} \right)
\]

Equation (B.28) yields
\[
\begin{bmatrix}
U_{11} & U_{12} \\
U_{21} & U_{22}
\end{bmatrix}
= 
\begin{bmatrix}
I & 0 & 0 & 0 \\
X_{xy} & X_{x\varphi} & Q'_{12} & Q_{22} \\
X_{uy} & X_{u\varphi} & P'_{1} & P_{2} \\
\end{bmatrix}
\begin{bmatrix}
Q_{11} & Q_{12} & P_{1} \\
Q_{12} & Q_{22} & P_{2} \\
P_{1}' & P_{2}' & R \\
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
X_{xy} & X_{x\varphi} \\
P'_{1} & P'_{2} \\
\end{bmatrix}
\begin{bmatrix}
I \\
X_{uy} & X_{u\varphi}
\end{bmatrix}
+ M'
\begin{bmatrix}
\beta (1 - \alpha) U_{11} + \beta \alpha S & \beta (1 - \alpha) U_{12} \\
\beta (1 - \alpha) U_{21} & \beta (1 - \alpha) U_{22}
\end{bmatrix}
M
\tag{B.32}
\]
A possible iterative scheme is (different order of updates is possible):

1. Guess $M, X, U$, as part of them we have $H = X_{xy}, S = U_{11}$

2. Compute an update of $U$ using (B.32)

3. Solve (B.29) using Schur decomposition (with stability threshold as $1/\sqrt{\beta (1 - \alpha)}$) to find an update for $X$ and $M$.

Once the procedure has converged, we can find the loss using the standard approach. Assume that the social welfare loss is given by

$$L^S = \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t g_t Q^S g_t = \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{s-t} \begin{bmatrix} y_t \\ x_t \\ u_t \end{bmatrix}' \begin{bmatrix} Q_{11}^S & Q_{12}^S & P_1^S \\ Q_{12}^S & Q_{22}^S & P_2^S \\ P_{11}^S & P_{22}^S & R^S \end{bmatrix} \begin{bmatrix} y_t \\ x_t \\ u_t \end{bmatrix}$$

where

$$\hat{Q}^S = \begin{bmatrix} I & 0 \\ X_{xy} & X_{x\varphi} \\ X_{uy} & X_{u\varphi} \end{bmatrix} \begin{bmatrix} Q_{11}^S & Q_{12}^S & P_1^S \\ Q_{12}^S & Q_{22}^S & P_2^S \\ P_{11}^S & P_{22}^S & R^S \end{bmatrix} \begin{bmatrix} I & 0 \\ X_{xy} & X_{x\varphi} \\ X_{uy} & X_{u\varphi} \end{bmatrix}$$

and $Q^S$ is not necessarily the same as $Q$ in (B.28) because the policy maker’s objectives are not necessarily social.\(^4\) Matrix $\hat{P}$ can be found from

$$vec(\hat{P}) = \left( I - \beta \left( \hat{M} \otimes \hat{M} \right) \right)^{-1} vec \left( \frac{\beta}{1 - \beta} V + Z_0 \right).$$

\(^4\)This algorithm is more naturally suited to deal with the case of $Q^S \neq Q$ than the algorithm given in Schaumburg and Tambalotti (2007).
where \( \hat{M} = (1 - \alpha) M + \alpha \begin{bmatrix} M_{yy} & 0 \\ 0 & 0 \end{bmatrix} \), \( V = \mathbb{E}_0 \begin{bmatrix} \xi_{t+1} \\ 0 \end{bmatrix} \begin{bmatrix} \xi_{t+1} \\ 0 \end{bmatrix}' \) and \( Z_0 = \begin{bmatrix} y_0 \\ \varphi_0 \end{bmatrix}' \).
Appendix C

Appendix Chapter 4

C.1 Commitment

A commitment policy requires minimisation of the loss function, $L_t$, once-and-for-all in period $t = 0$ subject to (B.26) for $t \geq 0$ and $y_0 = \overline{y}_0$ where $\overline{y}_0$ is known in period $t = 0$. The corresponding Lagrangian is given as

$$L = E_0 \sum_{s=t}^{\infty} \beta^{s-t} \left( \begin{array}{c} (x'_{s}Q_{22}x_{s} + y'_{s}Q_{12}y_{s} + x'_{s}Q_{21}y_{s} + y'_{s}Q_{11}y_{s} \\ + x'_{s}P_{2}u_{s} + y'_{s}P_{1}u_{s} + (x'_{s}P_{2}u_{s} + y'_{s}P_{1}u_{s})' + u'_{s}Ru_{s} \\ + \mu'_{s+1}(A_{11}y_{s} + A_{12}x_{s} + B_{1}u_{s} + \varepsilon_{t+1} - y_{s+1}) \\ + \mu'_{s+1}(A_{21}y_{s} + A_{22}x_{s} + B_{2}u_{s} - x_{s+1}) \end{array} \right)$$

where Lagrange multipliers $\lambda^y$ are non-predetermined (as those on predetermined variables) with terminal conditions and $\lambda^x$ are predetermined with initial conditions.
The first order conditions are:

\[
\frac{\partial L}{\partial x_s} = \beta^{s-1}(Q_{22} x_s + Q_{21} y_s + P_2 u_s) + (\mu_{s+1}' A_{12})' + (\mu_{s+1}' A_{22})' - (\mu_s')' = 0
\]

\[
\frac{\partial L}{\partial y_s} = \beta^{s-1}(Q_{12} x_s + Q_{11} y_s + P_1 u_s) + (\mu_{s+1}' A_{11})' + (\mu_{s+1}' A_{21})' - (\lambda_{y_s}')' = 0
\]

\[
\frac{\partial L}{\partial u_s} = (\lambda_{s+1}' B_1)' + (\mu_{s+1}' B_2)' + \beta^{s-1}(P_1 y_s + P_2' x_s + R u_s) = 0
\]

\[
\frac{\partial L}{\partial \mu_{s+1}'} = A_{11} y_s + A_{12} x_s + B_1 u_s + C \varepsilon_{t+1} - y_{s+1} = 0!
\]

\[
\frac{\partial L}{\partial \mu_{s+1}'} = A_{21} y_s + A_{22} x_s + B_2 u_s - x_{s+1} = 0!
\]

which can be summarized as

\[
\begin{bmatrix}
I & 0 & 0 & 0 & 0 \\
0 & \beta A_{12} & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & \beta A_{11}' & 0 & \beta A_{11} & 0 \\
0 & \beta B_1' & 0 & \beta B_1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
y_{s+1} \\
x_{s+1} \\
y_{s+1} \\
x_{s+1} \\
y_{s+1} \\
\end{bmatrix}
= 
\begin{bmatrix}
A_{11} & 0 & A_{12} & 0 & B_1 \\
0 & 0 & 0 & 0 & 0 \\
A_{21} & 0 & A_{22} & 0 & B_2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
y_s \\
x_s \\
y_s \\
x_s \\
y_s \\
\end{bmatrix}
\]

We can denote this as:

\[
G \begin{bmatrix} K_{s+1} \\ L_{s+1} \end{bmatrix} = D \begin{bmatrix} K_s \\ L_s \end{bmatrix}
\]

where \( K_s = (\mu_{s}', \lambda_{s}'') \) is predetermined variable, and \( L_s = (u_{s}', x_{s}', \mu_{s}') \) is non-predetermined variable. Matrix \( G \) can be singular, so using singular form decomposition (see Söderlind (1999), Klein (2000)), we can find the solution of the system in the form:

\[
\begin{bmatrix}
y_{s+1} \\
x_s \\
y_{s+1} \\
x_{s+1} \\
y_s \\
\end{bmatrix}
= 
Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} \begin{bmatrix}
y_s \\
x_s \\
y_{s+1} \\
x_{s+1} \\
y_s \\
\end{bmatrix}
+ Z_{11}^{-1} \varepsilon_{t+1}
\]

\[
\begin{bmatrix}
x_s \\
y_{s+1} \\
x_{s+1} \\
x_{s+1} \\
y_s \\
\end{bmatrix}
= 
Z_{21} Z_{11}^{-1} \begin{bmatrix}
y_s \\
x_s \\
y_{s+1} \\
x_{s+1} \\
y_s \\
\end{bmatrix}
\]

We can denote this as:

\[
G \begin{bmatrix} K_{s+1} \\ L_{s+1} \end{bmatrix} = D \begin{bmatrix} K_s \\ L_s \end{bmatrix}
\]

where \( K_s = (\mu_{s}', \lambda_{s}'') \) is predetermined variable, and \( L_s = (u_{s}', x_{s}', \mu_{s}') \) is non-predetermined variable. Matrix \( G \) can be singular, so using singular form decomposition (see Söderlind (1999), Klein (2000)), we can find the solution of the system in the form:
Equations (C.1) and (C.2) give a complete description of the evolution of the system together with the initial values \( y_0 \) and \( \mu_0^x = 0_{n_2 \times 1} \)

Denote \( M^c = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1} \) and \( F^d = Z_{21}Z_{11}^{-1} \) and system (C.1)-(C.2) can be written in form of (4.21)-(4.22).
C.2 Metropolis-Hastings Convergence Diagnostics

Table C.1: Univariate MH convergence diagnostics (Inflation Targeting under Commitment)
Table C.2: Univariate MH Convergence Diagnostics (Partial Exchange Rate Targeting under Commitment)
Table C.3: Univariate MH Convergence Diagnostics (Inflation Targeting under Discretion)
Table C.4: Univariate MH Convergence Diagnostics (Partial Exchange Rate Targeting under Discretion)
Bibliography


