# Applications of transfer operator methods to the dynamics of low-dimensional piecewise smooth maps

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I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

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# Abstract

This thesis primarily concentrates on stochastic and spectral properties of the transfer operator generated by piecewise expanding maps (PWEs) & piecewise isometries (PWIs). We also consider the applications of the transfer operator in thermodynamic formalism. The original motivation stems from studies of one-dimensional PWEs. In particular, any one dimensional mixing PWE admits a unique absolutely continuous invariant probability measure (ACIP) and this ACIP has a bounded variation density. The methodology used to prove the existence of this ACIP is based on a so-called functional analytic approach and a key step in this approach is to show that the corresponding transfer operator has a spectral gap. Moreover, when a PWE has Markov property this ACIP can also be viewed as a Gibbs measure in thermodynamic formalism.

In this thesis, we extend the studies on one-dimensional PWEs in several aspects. First, we use the functional analytic approach to study piecewise area preserving maps (PAPs) in particular to search for the ACIPs with multidimensional bounded variation densities. We also explore the relationship between the uniqueness of ACIPs with bounded variation densities and topological transitivity/ minimality for PWIs.

Second, we consider the mixing and corresponding mixing rate properties of a collection of piecewise linear Markov maps generated by composing $x \mapsto mx \mod 1$ with permutations in $S_N$. We show that typical permutations preserve the mixing property under the composition. Moreover, by applying the Fredholm determinant approach, we calculate the mixing rate via spectral gaps and obtain the max/min spectral gaps when $m, N$ are fixed. The spectral gaps can be made arbitrarily small when the permutations are fully refined.

Finally, we consider the computations of fractal dimensions for generalized Moran constructions, where different iteration function systems are applied on different levels. By using the techniques in thermodynamic formalism, we approximate the fractal dimensions via the zeros of the Bowen's equation on the pressure functions truncated at each level.

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