

CHARACTERISTIC MATRICES FOR LINEAR PERIODIC DELAY DIFFERENTIAL EQUATIONS

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Abstract. Szalai et al. (SIAM J. on Sci. Comp. 28(4), 2006) gave a general construction for characteristic matrices for systems of linear delay-differential equations with periodic coefficients. First, we show that matrices constructed in this way can have a discrete set of poles in the complex plane, which may possibly obstruct their use when determining the stability of the linear system. Then we modify and generalize the original construction such that the poles get pushed into a small neighborhood of the origin of the complex plane.

AMS subject classifications. 34K06, 34K08, 34K20

Key words. delay differential equations, characteristic matrix, stability of periodic orbits

sec:intro

1. Introduction. Linear delay differential equations (DDEs) with coefficients periodic in time come up naturally when one analyses nonlinear DDEs: one of the most common and simplest invariant sets encountered in nonlinear systems are periodic orbits [3]. If one wants to find out if a periodic orbit is dynamically stable or unstable, or how its stability changes as one changes system parameters one has to linearize the nonlinear DDE system in the periodic orbit, which results in a linear DDE with time-periodic coefficients [1, 4]. This linear DDE is typically written in the form

$$\dot{x}(t) = L(t)x_t \tag{1.1}$$

eq:introdde

where the coefficients (hidden in the operator $L(t)$) are periodic in t . Without loss of generality we can assume that the time dependence of L is of period 1 (this corresponds to rescaling time and L). Then the exponential asymptotic stability of the periodic orbit in the full nonlinear system is determined by the eigenvalues of the time-1 map T generated by the linear DDE (1.1) [7]. For analytical purposes (and, possibly, for numerical purposes) it is useful to reduce the eigenvalue problem for the infinite-dimensional time-1 map T to an algebraic problem via a characteristic matrix $\Delta(\lambda) \in \mathbb{C}^{n \times n}$. One would expect that this matrix Δ should satisfy, for example, that

(i) $\Delta(\lambda)$ is regular if (and only if) $\lambda \in \mathbb{C}$ is in the resolvent set of T (that is, $\lambda I - T$ is an isomorphism),

(ii) λ is a root of $\det \Delta(\cdot)$ of order q if and only if λ is an eigenvalue of T of algebraic multiplicity q (see [5] or Lemma 4.2 in §4 how one can construct the Jordan chains of T from Δ), and

(iii) $\dim \ker \Delta(\lambda)$ is the geometric multiplicity of λ as an eigenvalue of T .

An immediate consequence of the existence of such a matrix Δ would be the upper limit n (the dimension of $\Delta(\lambda)$) on the geometric multiplicity of eigenvalues of T .

For time-invariant linear DDEs (where L does not depend on t in (1.1)) the existence of a characteristic matrix has been known and used for a long time. The general theoretical background and construction (extended to more general evolutionary systems such as, for example, neutral equations and some classes of hyperbolic partial differential equations) is given by Kaashoek and Verduyn-Lunel in [5].

For time-periodic linear DDEs the textbook of Hale and Verduyn-Lunel [4] has developed its theory only for single time delays equal to the period. This has been generalized to delays depending rationally on the period in order to derive analytical

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stability results for periodic orbits of the classical scalar delayed positive feedback problem [8].

Szalai et al. give a general construction for a characteristic matrix Δ for time-periodic DDEs in [9] that lends itself easily to robust numerical computation. Since one expects that $\Delta(\lambda)$ must have an essential singularity at $\lambda = 0$ (the eigenvalues of the compact time-1 map T accumulate at 0) the constructed matrix is rather of the form $\Delta(\mu)$ where $\mu = \lambda^{-1}$. We show (in §3) that the characteristic matrix $\Delta(\mu)$ as constructed by Szalai et al. typically has poles in the complex plane, making it unusable whenever eigenvalues of T of interest coincide with these poles. However, the set of these poles is discrete and accumulates only at ∞ . Moreover, in §4 we provide a modification $\Delta_k(\mu)$ of the original matrix $\Delta(\mu)$, which allows us to push the poles of Δ_k to the outside of any given ball of radius $R \geq 1$ in the complex plane by increasing the dimension of Δ_k . This modification reduces the eigenvalue problem for T to a root-finding problem for $\det \Delta_k(\mu)$ for all eigenvalues of T with modulus larger than $1/R$, which is useful because one is typically interested in the largest eigenvalues of T . To keep the notational overhead limited we develop our arguments for the case of a DDE with a single delayed argument and a constant delay $\tau < 1$. In Appendix B we show that the generalization to arbitrary delays (distributed and reaching arbitrarily far into the past) is straightforward.

One immediate application for the characteristic matrix came up in [10] where Yanchuk and Perlikowski consider periodic orbits of nonlinear DDEs with a single fixed delay τ but change the delay to $\tau + kP$ (where k is an integer and P is the period of the periodic orbit), and then study the stability of the periodic orbit (which does not change its shape) in the limit $k \rightarrow \infty$. The limit is, of course, a singular limit if one considers the time- P map T but the characteristic matrix Δ_k has a well-defined regular limit. This permitted the authors to draw conclusions about the linear stability properties of periodic orbits for sufficiently large delays from the properties of the so-called *pseudo-continuous spectrum* [10]. As [10] relies on the original construction from [9], which may have poles near the unit circle, our note closes a gap in the argument of [10].

sec:constr

2. Construction of characteristic matrix for a single fixed delay. Consider a periodic linear differential equation of dimension n with a single delay τ and continuous periodic coefficient matrices $A(t)$ and $B(t)$ (both of size $n \times n$):

$$\dot{x}(t) = A(t)x(t) + B(t)x(t - \tau) \quad (2.1)$$

eq:spdde

where we assume that the period of the time dependence of A and B equals 1 without loss of generality (this corresponds to rescaling time, τ , A and B). We also assume for simplicity of notation that $\tau < 1$. Let us denote the monodromy operator (also called the time-1 map) for (2.1) by $T : C([-1, 0]; \mathbb{C}^n) \mapsto C([-1, 0]; \mathbb{C}^n)$. $C([-1, 0]; \mathbb{C}^n)$ is the space of continuous functions on the interval $[-1, 0]$ with values in \mathbb{C}^n . That is, T maps an initial condition x in $C([-1, 0]; \mathbb{C}^n)$ to the solution of (2.1) after time 1 starting from the initial value $x(0)$ (the head point) and using the history segment $x(\cdot)$. For $x \in C([-1, 0]; \mathbb{C}^n)$ we define Tx precisely as the solution $y \in C([-1, 0]; \mathbb{C}^n)$ of

$$\dot{y}(t) = A(t)y(t) + B(t) \begin{cases} y(t - \tau) & \text{if } t \in [\tau - 1, 0] \\ x(1 + t - \tau) & \text{if } t \in [-1, \tau - 1] \end{cases} \quad (2.2)$$

$$y(-1) = x(0).$$

eq:tdef

The complex number λ is an eigenvalue of T (a *Floquet multiplier* for (2.1)) if one can find a non-zero x such that $Tx = \lambda x$. We know that T is compact [4] for $\tau < 1$ (and that, for arbitrary τ , we can find a power T^m that is compact). Hence, any

non-zero λ is either in the resolvent set of T (that is, $\lambda I - T$ is regular) or it is an eigenvalue of T with a finite algebraic multiplicity.

If we introduce $\mu = 1/\lambda$ we may write the eigenvalue problem for T as follows:

$$\dot{x}(t) = A(t)x(t) + B(t) \begin{cases} x(t - \tau) & \text{if } t \in [\tau - 1, 0], \\ \mu x(1 + t - \tau) & \text{if } t \in [-1, \tau - 1], \end{cases} \quad (2.3) \quad \text{eq:bvp}$$

$$x(-1) = \mu x(0) \quad (2.4) \quad \text{eq:bc}$$

where $x \in C([-1, 0]; \mathbb{C}^n)$ is continuous. Reference [9] constructs a characteristic matrix $\Delta(\mu)$ for T based on the following hypothesis.

thm:sz

HYPOTHESIS 1 (Szalai'06). *The initial-value problem on the interval $[-1, 0]$*

$$\dot{x}(t) = A(t)x(t) + B(t) \begin{cases} x(t - \tau) & \text{if } t \in [\tau - 1, 0], \\ \mu x(1 + t - \tau) & \text{if } t \in [-1, \tau - 1], \end{cases} \quad (2.5) \quad \text{eq:ivp}$$

$$x(-1) = v \quad (2.6) \quad \text{eq:ic}$$

has a unique solution $x \in C([-1, 0]; \mathbb{C}^n)$ for all $\mu \in \mathbb{C}$ and $v \in \mathbb{C}^n$. Note that the problem (2.5)–(2.6) differs from the boundary-value problem (2.3)–(2.4) because we have replaced the boundary condition $x(-1) = \mu x(0)$ by an initial condition $x(-1) = v$. The formulation of the hypothesis in [9] is (only apparently) weaker: it claims the existence only for $\mu \in \mathbb{C}$ for which $1/\mu$ is either in the resolvent set of T or an eigenvalue of T of algebraic multiplicity one. If Hypothesis 1 is true then one can define the characteristic matrix $\Delta(\mu)$ via

$$\Delta(\mu)v = v - \mu x(0) \quad (2.7) \quad \text{eq:delta}$$

where $x(0)$ is the value of the unique solution of (2.5)–(2.6) at the end of the time interval $[-1, 0]$.

Integral equation for (2.5)–(2.6). For system (2.5)–(2.6) one has to clarify what it means for $x \in C([-1, 0]; \mathbb{C}^n)$ to be a solution. We call $x \in C([-1, 0]; \mathbb{C}^n)$ a solution of (2.5)–(2.6) if x satisfies the integral equation

$$x(t) = v + M_1(\mu)[x](t) \quad \text{where} \quad (2.8) \quad \text{eq:varp}$$

$$[M_1(\mu)x](t) = \int_{-1}^t A(s)x(s) + B(s) \begin{cases} x(s - \tau) & \text{if } s \in [\tau - 1, 0], \\ \mu x(1 + s - \tau) & \text{if } s \in [-1, \tau - 1] \end{cases} ds \quad (2.9) \quad \text{eq:varm}$$

for t on the interval $[-1, 0]$. The linear operator $M_1(\mu)$ maps $C([-1, 0]; \mathbb{C}^n)$ back into $C([-1, 0]; \mathbb{C}^n)$ and is compact. (M_1 will be generalized later to M_k for $k > 1$.)

thm:smallmu

LEMMA 2.1. *For μ sufficiently close to 0 the initial-value problem (2.5)–(2.6) has a unique solution for all v .*

Proof. For (2.8) to have a unique solution it is enough to show that $I - M_1(\mu)$ is invertible. Since the time-1 map T is well-defined and linear in $x = 0$ (that is, $Tx = 0$ for $x = 0$) we have that (2.2) has only the trivial solution for $x = 0$. This in turn implies that the corresponding integral equation

$$x = M_1(0)x \quad (2.10) \quad \text{eq:fixp0}$$

has only the trivial solution in $C([-1, 0]; \mathbb{C}^n)$. Thus, the kernel of $I - M_1(0)$ is trivial, and, since $I - M_1(0)$ is a compact perturbation of the identity, $I - M_1(0)$ is invertible. Moreover, the real-valued function $\mu \mapsto \|M_1(\mu) - M_1(0)\|$ (using the operator norm induced by the maximum norm of $C([-1, 0]; \mathbb{C}^n)$) is continuous in μ because M_1 depends continuously on μ (see definition of M_1 in (2.9)). This implies that $I - M_1(\mu)$ is also invertible for small μ , which in turn means that (2.8)–(2.9) has a unique solution for small μ . \square

The following lemma states that the function on the complex domain $\mu \mapsto [I - M_1(\mu)]^{-1}$, which has operators in $\mathcal{L}(C([-1, 0]; \mathbb{C}^n); C([-1, 0]; \mathbb{C}^n))$ as its values, is well-defined and analytic everywhere in the complex plane except, possibly, in a discrete set of poles.

thm:finpoles

LEMMA 2.2. *The operator-valued complex function $\mu \mapsto [I - M_1(\mu)]^{-1}$ can have at most a finite number of poles in any bounded subset of \mathbb{C} . All poles of $[I - M_1(\mu)]^{-1}$ (if there are any) are of finite multiplicity. If μ is not a pole then $\mu \mapsto [I - M_1(\mu)]^{-1}$ is analytic in μ .*

Proof. We can split the operator $M_1(\mu)$ into a sum of two operators:

$$M_1(\mu) = M_1(0) + \mu L \quad \text{where} \quad (2.11) \quad \text{eq:msplit}$$

$$[Lx](t) = \int_{-1}^t B(s) \begin{cases} 0 & \text{if } s \in [\tau - 1, 0], \\ x(1 + s - \tau) & \text{if } s \in [-1, \tau - 1] \end{cases} ds. \quad (2.12) \quad \text{eq:ldef}$$

The operator $I - M_1(0)$ is invertible (and a compact perturbation of the identity) and $L : C([-1, 0]; \mathbb{C}^n) \mapsto C([-1, 0]; \mathbb{C}^n)$ is compact. Thus, we can re-write $I - M_1(\mu)$ as

$$\begin{aligned} I - M_1(\mu) &= (I - M_1(0)) [I - \mu(I - M_1(0))^{-1}L] \quad \text{and, thus,} \\ [I - M_1(\mu)]^{-1} &= \lambda \cdot [\lambda I - (I - M_1(0))^{-1}L]^{-1} (I - M_1(0))^{-1} \end{aligned}$$

(keeping in mind that $\mu = 1/\lambda$). The operator on the right-hand side $[\lambda I - (I - M_1(0))^{-1}L]^{-1}$ is the standard resolvent of the compact operator $(I - M_1(0))^{-1}L$, which has only finitely many poles of finite multiplicity outside of any neighborhood of the complex origin according to [6], and is analytic everywhere else. \square

Lemma 2.2 carries over to the characteristic matrix $\Delta(\mu)$ defined in (2.7): since $x = [I - M_1(\mu)]^{-1}v$ (if we extend $v \in \mathbb{C}^n$ to a constant function), $\Delta(\mu)v = v - \mu x(0)$ is an analytic function with at most a discrete set of poles of finite multiplicity (possibly accumulating at ∞).

The characteristic matrix $\Delta(\mu)$ (where it is well-defined) has the properties one expects: a complex number μ in the domain of definition of Δ $1/\mu$ is a Floquet multiplier of the DDE (2.1) if and only if $\Delta(\mu)$ is singular ($\det \Delta(\mu) = 0$). Any vector v in its kernel is the value of an eigenfunction for $1/\mu$ at time $t = -1$. The full eigenfunction can be obtained as the solution of the initial-value problem (2.5)–(2.6). If $\Delta(\mu)$ is regular then $1/\mu$ is in the resolvent set of the eigenvalue problem (2.3)–(2.4). The Jordan chains associated to eigenvalues $1/\mu$ can be obtained by following the general theory described in [5]. We give a precise statement (see Lemma 4.2 in Section 4) about the Jordan chain structure after discussing Hypothesis 1.

The construction can be extended in a straightforward manner to multiple fixed discrete delays and distributed delays. The only modification is that higher powers of μ have to be included for delays larger than 1. For example, if $\tau \in (1, 2]$ then the initial-value problem (2.5)–(2.6) is modified to

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t) \begin{cases} \mu x(1 + t - \tau) & \text{if } t \in [\tau - 2, 0], \\ \mu^2 x(2 + t - \tau) & \text{if } t \in [-1, \tau - 2], \end{cases} \\ x(-1) &= v. \end{aligned}$$

Thus, also for arbitrary delays $\Delta(\mu)$ always has at worst finitely many poles of finite multiplicity in any bounded subset of \mathbb{C} . Appendix B discusses the case where the functional accessing the history of x is a Lebesgue-Stieltjes integral.

sec:problem

3. The poles of the characteristic matrix. Hypothesis 1 is not true in its stated form (and neither in the form stated in [9]). Hypothesis 1 can only be true if

the homogeneous problem

$$\dot{x}(t) = A(t)x(t) + B(t) \begin{cases} x(t - \tau) & \text{if } t \in [\tau - 1, 0], \\ \mu x(1 + t - \tau) & \text{if } t \in [-1, \tau - 1), \end{cases} \quad (3.1) \quad \boxed{\text{eq:ivp0}}$$

$$x(-1) = 0 \quad (3.2) \quad \boxed{\text{eq:ic0}}$$

has only the trivial solution $x = 0$ for all μ : if (3.1)–(3.2) has a non-trivial solution x for some μ then for every v (otherwise $\Delta(\mu)v \neq 0$ for $v = 0$). Let us analyse the simple scalar example

$$\dot{x}(t) = x \left(t - \frac{1}{2} \right) \quad (3.3) \quad \boxed{\text{eq:cdde}}$$

on the interval $[-1, 0]$. Problem (3.1)–(3.2) re-stated for this example is equivalent to

$$\begin{aligned} \dot{x}_1(t) &= \mu x_2(t), & x_1(0) &= 0 \\ \dot{x}_2(t) &= x_1(t), & x_2(0) &= x_1(1/2) \end{aligned} \quad (3.4) \quad \boxed{\text{eq:exdde}}$$

where x_1 and x_2 are in $C([0, 1/2]; \mathbb{C})$, and $x_1(t) = x(t - 1)$ and $x_2(t) = x(t - 1/2)$ for t in the interval $[0, 1/2]$. The general solution of (3.4) (without regard to the boundary conditions) is of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \sqrt{\mu} \exp(\sqrt{\mu}t) & \sqrt{\mu} \exp(-\sqrt{\mu}t) \\ \exp(\sqrt{\mu}t) & -\exp(-\sqrt{\mu}t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

where $\sqrt{\mu}$ refers to the principal branch of the complex square root (the set of solutions is the same for both branches). This solution satisfies $x_1(0) = 0$ and $x_2(0) - x_1(1/2) = 0$ for a non-zero $(a, b)^T$ if

$$0 = 2 - \sqrt{\mu} \left[\exp\left(\frac{\sqrt{\mu}}{2}\right) - \exp\left(-\frac{\sqrt{\mu}}{2}\right) \right].$$

Thus, we see that for example (3.3) the homogeneous initial-value problem (3.1)–(3.2) has a nontrivial solution whenever μ is a root of the function

$$f(\mu) = 2 - \sqrt{\mu} \left[\exp\left(\frac{\sqrt{\mu}}{2}\right) - \exp\left(-\frac{\sqrt{\mu}}{2}\right) \right].$$

This function f is a globally defined real analytic function of μ (the expression on the right-hand-side is even in $\sqrt{\mu}$). It has an infinite number of complex roots accumulating at infinity. One of the roots is real: $f(0) = 2$, and $f(\mu) \rightarrow -\infty$ for $\mu \rightarrow +\infty$ and $\mu \in \mathbb{R}$ (let's call it μ_0 : $\mu_0 \approx 1.8535$). Consequently, example (3.3) provides a counterexample to Hypothesis 1. If one extends example (3.3) by a decoupled equation that has its Floquet multiplier at $1/\mu_0$ (for example, $\dot{y} = -[\log \mu_0]y$) then it also contradicts the hypothesis as stated in [9].

As we have discussed in Section 2, one finds that $\mu = 1/\lambda$ is a pole of $\Delta(\mu)$ if λ is a non-zero eigenvalue of the operator $[I - M_1(0)]^{-1}L$ where M_1 and L are defined in (2.9) and (2.12). Thus, whenever $[I - M_1(0)]^{-1}L$ has a non-zero spectral radius we must expect that $\Delta(\mu)$ has poles.

sec:fix

4. The extended characteristic matrix. The construction of $\Delta(\mu)$ suffers from the problem that the poles of $\Delta(\mu)$ may coincide with the inverse of Floquet multipliers of (2.1) of interest. A simple extension of the construction of $\Delta(\mu)$ permits one to push the poles to the outside of a circle of any desired radius R . Then the characteristic matrix can be used to find all Floquet multipliers outside of

the ball of radius $1/R$. We explain the extension in detail for a single delay $\tau < 1$ (the straightforward generalization to the general case is relegated to Appendix B).

The idea is based on the observation that the linear part of the right-hand-side in the integral equation formulation (2.8)–(2.9) of initial-value problem (2.5)–(2.6) has a norm less than 1 if the interval is sufficiently short (thus, making the fixed point problem (2.8)–(2.9) uniquely solvable).

Similar to a multiple shooting approach, we partition the interval $[-1, 0]$ into k intervals of size $1/k$:

$$J_i = [t_i, t_{i+1}) = \left[-1 + \frac{i}{k}, -1 + \frac{i+1}{k} \right) \text{ for } i = 0, \dots, k-1. \quad (4.1) \quad \boxed{\text{eq:jidef}}$$

Then we write down a system of k coupled initial value problems for a vector of k initial values $(v_0, \dots, v_{k-1})^T \in \mathbb{C}^{nk}$:

$$\dot{x}(t) = A(t)x(t) + B(t) \begin{cases} x(t-\tau) & \text{if } t \in [\tau-1, 0], \\ \mu x(1+t-\tau) & \text{if } t \in [-1, \tau-1], \end{cases} \quad (4.2) \quad \boxed{\text{eq:ivpext}}$$

$$x(t_i) = v_i \text{ for } i = 0, \dots, k-1 \quad (4.3) \quad \boxed{\text{eq:icext}}$$

where $t \in [-1, 0]$. System (4.2)–(4.3) corresponds to a coupled system of k differential equations for the k functions $x|_{J_i}$ in the time intervals J_i . More precisely, x is defined as a solution of the fixed point problem

$$x(t) = v_s(t) + \int_{a_k(t)}^t A(s)x(s) + B(s) \begin{cases} x(s-\tau) & \text{if } s \in [\tau-1, 0], \\ \mu x(1+s-\tau) & \text{if } s \in [-1, \tau-1] \end{cases} ds \text{ where} \quad (4.4) \quad \boxed{\text{eq:varpext}}$$

$$v_s(t) = v_i \text{ if } t \in J_i, \quad (4.5) \quad \boxed{\text{eq:varpextv}}$$

$$a_k(t) = t_i \text{ if } t \in J_i. \quad (4.6) \quad \boxed{\text{eq:varpexta}}$$

We point out that a solution x of (4.2)–(4.3) is not necessarily in $C([-1, 0]; \mathbb{C}^n)$

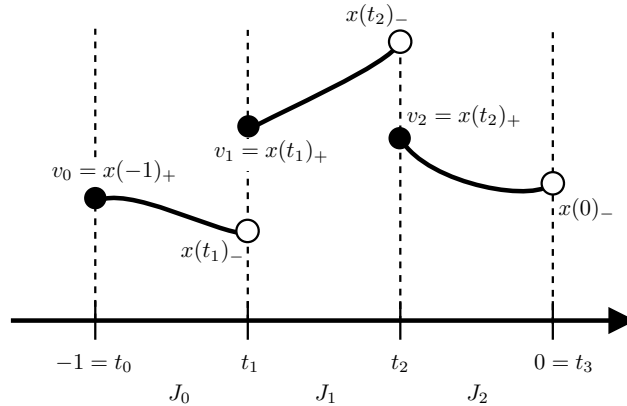


FIG. 4.1. Illustration of solution $x \in C_k$ for multiple-shooting problem (4.2)–(4.3) with $k = 3$ sub-intervals. The right-sided limits $x(t_i)_+$ are equal to the variables v_i . The gaps between the left-sided limits $x(t_i)_-$ and $x(t_i)_+$ are zero if $[v_0, \dots, v_k]$ is in the kernel of the characteristic matrix $\Delta_k(\mu)$. The equations for $x|_{J_i}$ on the sub-intervals are coupled due to the terms $t - \tau$ or $1 + t - \tau$ in (4.2). $\boxed{\text{fig:msproblem}}$

because it will typically have discontinuities at the restarting times t_i for $i = 1 \dots k-1$ as illustrated in Fig. 4.1. Thus, we should define the space in which we look for

solutions as the space of *piecewise continuous* functions:

$$C_k = \{x : [-1, 0] \mapsto \mathbb{C}^n : x \text{ continuous on each (half-open) subinterval } J_i \quad (4.7)$$

$$\text{and } \lim_{t \nearrow t_i} x(t) \text{ exists for all } i = 1 \dots k.\}$$

eq:ckdef

$$C_{k,0} = \{x \in C_k : x(t_i) = 0 \text{ for all } i = 0 \dots k-1\}. \quad (4.8)$$

eq:ck0def

We call $x \in C_k$ a solution of (4.4)–(4.6) if it satisfies (4.4) for every $t \in [-1, 0]$. Both spaces, C_k and $C_{k,0}$ are equipped with the usual maximum norm. For example, the piecewise constant function v_s as defined in (4.5) is an element of C_k . Several operations are useful when dealing with functions in C_k to keep the notation compact. Any element $x \in C_k$ has well defined one-sided limits at the boundaries t_i , which we denote by the subscripts $+$ and $-$:

$$x(t_i)_- = \lim_{t \nearrow t_i} x(t) \quad \text{for } i = 1 \dots k,$$

$$x(t_i)_+ = \lim_{t \searrow t_i} x(t) = x(t_i) \quad \text{for } i = 0 \dots k-1.$$

We also define the four maps

$$S : \mathbb{C}^{nk} \mapsto C_k \quad S[v_0 \dots v_{k-1}]^T(t) = v_i \text{ if } t \in [t_i, t_{i+1}) \text{ for } i = 0 \dots k-1,$$

$$\Gamma_+ : C_k \mapsto \mathbb{C}^{nk} \quad \Gamma_+[x(\cdot)] = [x(-1)_+, x(t_1)_+, \dots, x(t_{k-1})_+]^T,$$

$$\Gamma_-(\mu) : C_k \mapsto \mathbb{C}^{nk} \quad \Gamma_-(\mu)[x(\cdot)] = [\mu x(0)_-, x(t_1)_-, \dots, x(t_{k-1})_-]^T,$$

$$M_k(\mu) : C_k \mapsto C_{k,0} \quad \text{where}$$

$$[M_k(\mu)x](t) = \int_{a_k(t)}^t A(s)x(s) + B(s) \begin{cases} x(s-\tau) & \text{if } s \in [\tau-1, 0], \\ \mu x(1+s-\tau) & \text{if } s \in [-1, \tau-1) \end{cases} ds \quad (4.9)$$

eq:opsdef

(the piecewise constant function a_k was defined in (4.6) as $a_k(t) = t_i$ if $t \in [t_i, t_{i+1})$). The map S takes a tuple of vectors v_0, \dots, v_{k-1} and maps it to a piecewise constant function, assigning $x(t_i) = v_i$ and then extending with a constant to the subinterval J_i . For example, the function v_s in (4.4)–(4.5) is equal to Sv . The map Γ_+ takes the right-side limits of a piecewise continuous function (thus, Γ_+S is the identity in \mathbb{C}^{nk}). The map $\Gamma_-(\mu)$ takes the left-side limits at the interior boundaries t_i ($i \geq 1$), and in its first component it takes the left-side limit at $t_k = 0$ (the end of the interval) multiplied by μ . The map $M_k(\mu)$ is the generalization of M_1 defined in (2.9): in the definition of M_k the lower boundary in the integral is not -1 but $a_k(t)$ such that we can estimate its norm:

$$\|M_k(\mu)\|_\infty \leq \frac{1}{k} \left[\max_{t \in [-1, 0]} \|A(t)\|_\infty + |\mu| \max_{t \in [0, 1]} \|B(t)\|_\infty \right] =: \frac{C_*(|\mu|)}{k}. \quad (4.10)$$

eq:mknorm

According to the Theorem of Arzelá-Ascoli (see [2], page 266) the operator $M_k(\mu)$ is also compact because it maps any bounded set into a set with uniformly bounded Lipschitz constants. Using the above notation we can write the differential equation (4.2)–(4.3) (or, rather, the corresponding integral equation (4.4)) as a fixed point problem in C_k . For a given $v = [v_0, \dots, v_{k-1}] \in \mathbb{C}^{nk}$ and $\mu \in \mathbb{C}$ find a function $x \in C_k$ such that

$$x = Sv + M_k(\mu)x. \quad (4.11)$$

eq:fixpp

Due to the norm estimate (4.10) on $M_k(\mu)$ we can state the following fact about the existence of a fixed point x of (4.11) for a given v and μ :

thm:uniquext

LEMMA 4.1. *Let $R \geq 1$ be given. If we choose $k > C_*(R)$ then the system (4.2)–(4.3) has a unique solution $x \in C_k$ for all initial values $(v_0 \dots v_{k-1})^T$ and all μ satisfying $|\mu| < R$.*

Proof. The differential equation (4.2)–(4.3) is equivalent to fixed point problem (4.11). The norm estimate (4.10) implies that $I - M_k(\mu)$ is invertible such that the solution x of (4.11) is given by $[I - M_k(\mu)]^{-1}Sv$. \square

4.1. Extended characteristic matrix. We can use Lemma 4.1 to define the *extended* characteristic matrix $\Delta_k(\mu)$ for μ satisfying $C_*(|\mu|) < k$. This is a matrix in $\mathbb{C}^{(nk) \times (nk)}$, and is defined as

$$\Delta_k(\mu)v = \Delta(\mu) \begin{bmatrix} v_0 \\ \vdots \\ v_{k-1} \end{bmatrix} = \begin{bmatrix} v_0 - \mu x(0) \\ v_1 - x(t_1)_- \\ \vdots \\ v_{k-1} - x(t_{k-1})_- \end{bmatrix} \quad (4.12) \quad \text{eq:deltaedef}$$

where $x = [I - M_k(\mu)]^{-1}Sv \in C_k$ is the unique solution of the coupled system of initial-value problems, (4.2)–(4.3), with initial condition $v = (v_0, \dots, v_{k-1})^T \in \mathbb{C}^{nk}$. The first row in the right side of definition (4.12) corresponds to the boundary condition (2.4). The other $k - 1$ rows, $v_i - x(t_i)_-$, guarantee for $v \in \ker \Delta(\mu)$ that the right and the left limit of x at the restarting times t_i agree, making the gaps shown in Fig. 4.1 zero. Thus, that x is not only in C_k but in $C([-1, 0]; \mathbb{C}^n)$. In fact, if $\Delta_k(\mu)v = 0$ then the right-hand side of (4.2) for the solution $x = [I - M_k(\mu)]^{-1}Sv$ is continuous such that x is continuously differentiable and satisfies for every t the differential equation (2.3)–(2.4) defining the eigenvalue problem for T .

Let us list several useful facts about C_k , $C_{k,0}$, $\Delta_k(\mu)$, T and the quantities defined in (4.9).

1. We can express $\Delta_k(\mu)$ using the maps defined in (4.9):

$$\Delta_k(\mu) = I - \Gamma_-(\mu) [I - M_k(\mu)]^{-1} S \quad (4.13) \quad \text{eq:deltaeexp}$$

2. C_k is isomorphic to $\mathbb{C}^{nk} \times C_{k,0}$:

$$\begin{aligned} (v, \varphi) \in \mathbb{C}^{nk} \times C_{k,0} &\mapsto Sv + \varphi \in C_k \\ \psi \in C_k &\mapsto (\Gamma_+ \psi, \psi - S\Gamma_+ \psi) \in \mathbb{C}^{nk} \times C_{k,0}. \end{aligned} \quad (4.14) \quad \text{eq:isock}$$

item:dde

3. The differential equation

$$\begin{aligned} \dot{x}(t) &= A(t)x + B(t) \begin{cases} x(t - \tau) & \text{if } t \in [\tau - 1, 0], \\ 0 & \text{if } t \in [-1, \tau - 1), \end{cases} \\ x(-1)_+ &= 0 \\ x(t_i)_+ &= x(t_i)_- \quad \text{for } i = 1, \dots, k - 1 \end{aligned} \quad (4.15) \quad \text{eq:T0}$$

has only the trivial solution (it is identical to (2.5)–(2.6) for $\mu = 0$). Its integral formulation is $x = S\Gamma_-(0)x + M_k(0)x$. Thus, the operator $I - S\Gamma_-(0) - M_k(0) : C_k \mapsto C_k$ (which is of Fredholm index 0) is invertible.

4. The time-1 map T of the linear DDE $\dot{x}(t) = A(t)x(t) + B(t)x(t - \tau)$, defined in (2.2) for initial history segments in $x \in C([-1, 0]; \mathbb{C}^n)$, can be easily extended to initial history segments in C_k . In fact, the extension of T to C_k can be defined by

$$T = [I - S\Gamma_-(0) - M_k(0)]^{-1} [S(\Gamma_-(1) - \Gamma_-(0)) + M_k(1) - M_k(0)],$$

which maps C_k into $C([-1, 0]; \mathbb{C}^n)$, such that

$$I - \mu T = [I - S\Gamma_-(0) - M_k(0)]^{-1} [I - S\Gamma_-(\mu) - M_k(\mu)], \quad (4.16) \quad \text{eq:texp}$$

which maps C_k into C_k (see Appendix A for a detailed decomposition of expression (4.16)). Since the extension of T maps into $C([-1, 0]; \mathbb{C}^n)$ (the original domain of the non-extended T) its spectrum is identical to the spectrum of the original T .

Lemma 4.1 and the above facts permit us to apply the general theory developed in [5], relating the extended characteristic matrix $\Delta_k(\mu) \in \mathbb{C}^{(nk) \times (nk)}$ to spectral properties of the time-1 map T : eigenvalues, eigenvectors, and the length of their Jordan chains.

thm:multiplicity

LEMMA 4.2 (Jordan chains). *Assume that $0 < |\mu_*| < R$ and $k > C_*(R)$ where $C_*(R)$ is defined by (4.10). If $\Delta_k(\mu_*)$ is regular then $1/\mu_*$ is in the resolvent set of the monodromy operator T . If $\Delta_k(\mu_*)$ is not regular then $\lambda_* = 1/\mu_*$ is an eigenvalue of T . The Jordan chain structure for λ_* as an eigenvalue of T is also determined by Δ_k :*

1. *the dimension l_0 of $\ker \Delta_k(\mu_*)$ is the geometric multiplicity of λ_* ,*
2. *the order p of the pole μ_* of $\Delta_k(\mu)^{-1}$ is the length of the longest Jordan chain associated to λ_* ,*
3. *the order of μ_* as a root of $\det \Delta_k(\mu)$ is the algebraic multiplicity of λ_* ,*
4. *the Jordan chains of length less than or equal p can be found as non-trivial solutions (y_0, \dots, y_{p-1}) of the linear system*

$$\begin{aligned}
0 &= \Delta_k^{(0)} y_0 \\
0 &= \Delta_k^{(1)} y_0 + \Delta_k^{(0)} y_1 \\
0 &= \frac{\Delta_k^{(2)}}{2} y_0 + \Delta_k^{(1)} y_1 + \Delta_k^{(0)} y_2 \\
&\vdots \\
0 &= \frac{\Delta_k^{(p-1)}}{(p-1)!} y_0 + \dots + \Delta_k^{(1)} y_{p-2} + \Delta_k^{(0)} y_{p-1}
\end{aligned} \tag{4.17}$$

eq:jcderiv

where $\Delta_k^{(j)}$ is the j th derivative of $\Delta_k(\mu)$ in μ_* and $\Delta_k^{(0)} = \Delta_k(\mu_*)$.

System (4.17) is equivalent to the requirement

$$\Delta_k(\mu) [y_0 + (\mu - \mu_*)y_1 + \dots + (\mu - \mu_*)^{p-1}y_{p-1}] = O((\mu - \mu_*)^p)$$

for all $\mu \approx \mu_*$. We note that the solutions of (4.17) are all Jordan chains, defined as (finite) sequences $(y_k)_{k=0}^{p-1}$ ($k < p$) satisfying $y_0 = \mu T y_0$, this is not a unique construction). In a maximal Jordan chain all y_j are non-trivial.

Proof. The proof extends and modifies the construction used by [9], and makes the general theory of [5] applicable to time-periodic delay equations.

First, a brief summary of the relevant statement of [5]. Let $G : \Omega \mapsto \mathcal{L}(X_1; Y_1)$ and $H : \Omega \mapsto \mathcal{L}(X_2; Y_2)$ be two functions which are holomorphic on the open subset Ω of the complex plane \mathbb{C} , and map into the space of bounded linear operators from one Banach space $X_{1,2}$ into another Banach space $Y_{1,2}$. The functions G and H are called *equivalent* if there exist two functions $E : \Omega \mapsto \mathcal{L}(X_2; X_1)$ and $F : \Omega \mapsto \mathcal{L}(Y_1; Y_2)$ (also holomorphic on Ω) whose values are isomorphisms such that

$$H(\mu) = F(\mu)G(\mu)E(\mu) \text{ for all } \mu \in \Omega.$$

If two operators G and H are equivalent then there is a one-to-one correspondence between the Jordan chains of the eigenvalue problems $G(\mu)v = 0$ and $H(\mu)v = 0$.

We construct the equivalence initially only for the case of a single delay $\tau < 1$. The necessary modifications for arbitrary delays can be found in Appendix B (the underlying idea is identical but there is more notational overhead). Also, we restrict ourselves here to merely stating what the operators H , F , G and E are, relegating the detailed checks and calculations to Appendix A.

The subset of permissible μ , $\Omega \subset \mathbb{C}$, is the ball of radius R around 0, where the initial-value problem (4.2)–(4.3) has a unique solution. In our equivalence the

operator functions G and H are

$$\begin{aligned} G(\mu) &= I - \mu T : & X_1 &= C_k & \mapsto & Y_1 = X_1, \\ H(\mu) &= \begin{pmatrix} \Delta_k(\mu) & 0 \\ 0 & I \end{pmatrix} : & X_2 &= \mathbb{C}^{nk} \times C_{k,0} & \mapsto & Y_2 = X_2. \end{aligned}$$

Note that the spaces X_1 and X_2 are isomorphic via relation (4.14). The eigenvalue problem $G(\mu)v = 0$ is the eigenvalue problem for T (re-formulated for $\mu = \lambda^{-1}$). The eigenvalue problem $H(\mu)[v, \varphi] = 0$ is equivalent to $\Delta_k(\mu)v = 0$, $\varphi = 0$. Thus, establishing equivalence between G and H proves Lemma 4.2. We define the isomorphisms E and F as:

$$\begin{aligned} E(\mu)[v, \varphi] &= [I - M_k(\mu)]^{-1} [Sv + \varphi] \\ F(\mu)\psi &= \begin{bmatrix} [\Gamma_+ - \Gamma_-(0)]\psi + \Gamma_-(\mu) [I - M_k(\mu)]^{-1} [I - S\Gamma_+ - M_k(0)]\psi \\ [I - S\Gamma_+ - M_k(0)]\psi \end{bmatrix} \end{aligned} \tag{4.18}$$

eq:efdef

See (4.13) and (4.16) for expressions giving $\Delta_k(\mu)$ and $I - \mu T$, defining G and H in terms of S , Γ_{\pm} and M_k . The inverse of $I - M_k(\mu)$ is guaranteed to exist by Lemma 4.1. The only choice we make is the definition of E , clearly an isomorphism from X_2 to X_1 (because $(v, \varphi) \mapsto Sv + \varphi$ is an isomorphism from X_2 to X_1). The definition of F is then uniquely determined if we want to achieve the equivalence $F(\mu)G(\mu)E(\mu) = H(\mu)$, and follows from a straightforward but technical calculation, given in Appendix A. \square

sec:conc

5. Conclusions. By proving Hypothesis 1 (in its weakened form of Lemma 4.1) we have fully justified the construction of a characteristic matrix for linear DDEs with time-periodic coefficients proposed by Szalai et al. [9] (in a slightly weaker form).

An open question is how our extended matrix is related to the characteristic matrix introduced in the textbook of Hale and Verduyn-Lunel [4] for periodic DDEs with delay identical to the period. An equivalent of a small delay expansion should show what happens to poles of $\Delta(\mu)$ as the delay approaches 1 (the period).

sec:check

Appendix A. Details of the equivalence in the proof of Lemma 4.2.

Inverse of $E(\mu)$. Let $[v, \varphi] = [(v_0, \dots, v_{k-1})^T, \varphi] \in \mathbb{C}^{nk} \times C_{k,0}$ be an element of the space X_2 . Then the definition of $x = E(\mu)[v, \varphi]$ in (4.18) means that x is the solution of the fixed point problem

$$x = Sv + \varphi + M_k(\mu)x. \tag{A.1}$$

eq:efixpp

Consequently, we can recover v by applying Γ_+ to (A.1) ($\Gamma_+\varphi = 0$ since $\varphi \in C_{k,0}$ and $\Gamma_+M_k(\mu) = 0$ since M_k maps into $C_{k,0}$). Then φ can be recovered as $\varphi = x - S\Gamma_+x - M_k(\mu)x$ such that the inverse of E is

$$E^{-1}(\mu)x = \begin{bmatrix} \Gamma_+x \\ [I - S\Gamma_+ - M_k(\mu)]x \end{bmatrix}. \tag{A.2}$$

eq:einvdef

Expression (4.16) for $I - \mu T$. Before finding an expression for $F(\mu)$ we check that $G(\mu) = I - \mu T$ is indeed given by the expression (4.16). If we denote the image of x under $G(\mu)$ by y then y has the form $x - \tilde{y}$ where $\tilde{y} = \mu Tx$. By definition (2.2) of the monodromy operator T , \tilde{y} is the unique solution of the inhomogeneous initial-value problem on $[-1, 0]$

$$\dot{\tilde{y}}(t) = A(t)\tilde{y}(t) + B(t) \begin{cases} \tilde{y}(t - \tau) & \text{if } t \in [\tau - 1, 0] \\ \mu x(1 + t - \tau) & \text{if } t \in [-1, \tau - 1] \end{cases} \tag{A.3}$$

eq:tde

$$\begin{aligned} \tilde{y}(-1) &= \mu x(0) \\ \tilde{y}(t_i)_+ &= \tilde{y}(t_i)_- \text{ for } i = 1 \dots k - 1. \end{aligned} \tag{A.4}$$

eq:tic

This solution \tilde{y} is in $C([-1, 0]; \mathbb{C}^n)$. We note that the initial conditions in (A.4) mean that $\Gamma_+ \tilde{y} = \Gamma_-(0) \tilde{y} + [\Gamma_-(\mu) - \Gamma_-(0)]x$. Thus, the integral equation corresponding to (A.3)–(A.4) is

$$\tilde{y} = S[\Gamma_-(0) \tilde{y} + [\Gamma_-(\mu) - \Gamma_-(0)]x] + M_k(0) \tilde{y} + [M_k(\mu) - M_k(0)]x. \quad (\text{A.5})$$

eq:tvoc

Inserting $\tilde{y} = x - y$ into (A.5) gives a fixed point problem for $y = G(\mu)x = x - \mu T x$ equivalent to expression (4.16):

$$y = S\Gamma_-(0)y + M_k(0)y + x - S\Gamma_-(\mu)x - M_k(\mu)x. \quad (\text{A.6})$$

eq:tfixpp

The factor $I - S\Gamma_-(0) - M_k(0)$ (which is in front of y if we want to isolate y in (A.6)) is invertible as explained below (4.15) in the list of useful facts in §4.

Determination of $F(\mu)$. From the fixed point equation (A.6) it becomes clear how to choose the other isomorphism $F(\mu)$ such that $F(\mu)G(\mu)E(\mu)[v, \varphi] = H(\mu)[v, \varphi]$. We observe that (A.6) implies (by applying Γ_+ to both sides)

$$\Gamma_+ y = \Gamma_-(0)y + \Gamma_+ x - \Gamma_-(\mu)x. \quad (\text{A.7})$$

eq:gygx

We apply S to this identity and subtract it from the fixed point equation (A.6) defining y :

$$y - S\Gamma_+ y = M_k(0)y + x - S\Gamma_+ x - M_k(\mu)x.$$

If $x = E(\mu)[v, \varphi] = [I - M_k(\mu)]^{-1}[Sv + \varphi]$ then $\Gamma_+ x = v$ such that

$$y - S\Gamma_+ y - M_k(0)y = \varphi. \quad (\text{A.8})$$

eq:f2def

Thus, $[I - S\Gamma_+ - M_k(0)]G(\mu)E(\mu)[v, \varphi] = \varphi$ for all $(v, \varphi) \in X_1$, which justifies our choice of the second component of $F(\mu)$ in (4.18). Inserting the expression $E(\mu)[v, \varphi] = [I - M_k(\mu)]^{-1}[Sv + \varphi]$ for x and $\Gamma_+ x = v$ into (A.7) we obtain that

$$\Gamma_+ y - \Gamma_-(0)y = v - \Gamma_-(\mu)[I - M_k(\mu)]^{-1}[Sv + \varphi] \quad (\text{A.9})$$

eq:gyvphi1

$$\begin{aligned} &= [I - \Gamma_-(\mu)[I - M_k(\mu)]^{-1}S]v - \Gamma_-(\mu)[I - M_k(\mu)]^{-1}\varphi \\ &= \Delta_k(\mu)v - \Gamma_-(\mu)[I - M_k(\mu)]^{-1}\varphi, \end{aligned} \quad (\text{A.10})$$

eq:gyvphi2

where we used the definition (4.13) of the characteristic matrix $\Delta_k(\mu)$. Since we know that φ can be recovered from y via (A.8) we obtain from (A.10) the first component of the definition of $F(\mu)$ in (4.18):

$$\Gamma_+ y - \Gamma_-(0)y + \Gamma_-(\mu)[I - M_k(\mu)]^{-1}[I - S\Gamma_+ - M_k(0)]y = \Delta_k(\mu)v$$

Inverse of $F(\mu)$. It remains to be checked that F is invertible. Given $v \in \mathbb{C}^{nk}$ and $\varphi \in C_{k,0}$ such that

$$v = \Gamma_+ y - \Gamma_-(0)y + \Gamma_-(\mu)[I - M_k(\mu)]^{-1}\varphi, \quad (\text{A.11})$$

eq:vy

$$\varphi = y - S\Gamma_+ y - M_k(0)y, \quad (\text{A.12})$$

eq:phiy

how do we recover y ? Applying S to (A.11) and adding the result to (A.12) gives

$$Sv + \varphi = S\Gamma_-(\mu)[I - M_k(\mu)]^{-1}\varphi + [I - S\Gamma_-(0) - M_k(0)]y, \quad (\text{A.13})$$

eq:vphiy

which can be rearranged for y because $I - S\Gamma_-(0) - M_k(0)$ is invertible. Consequently,

$$F(\mu)^{-1}[v, \varphi] = [I - S\Gamma_-(0) - M_k(0)]^{-1}[Sv + \varphi - S\Gamma_-(\mu)[I - M_k(\mu)]^{-1}\varphi]. \quad (\text{A.14})$$

eq:finvdef

sec:gen

Appendix B. Characteristic matrices for general periodic delay equations. Consider a linear DDE with coefficients of time period 1 and with maximal delay less than or equal to an integer m :

$$\dot{x}(t) = \int_0^m d_\theta \eta(t, \theta) x(t - \theta) \quad (\text{B.1}) \quad \text{eq:genlong}$$

where $\eta(t, \theta)$ is bounded, measurable and periodic in its first argument t (with period 1), and of bounded variation in its second argument θ . We choose $\eta(t, \cdot) \in NBV(\mathbb{R}; \mathbb{C}^{n \times n})$, that is, $\eta(t, \cdot) = 0$ on $(-\infty, 0]$, $\eta(t, \cdot)$ is continuous from the right on $(0, m)$, and $\eta(t, \cdot)$ is constant on $[m, \infty)$. In addition we assume that the total variations of all $\eta(t, \cdot)$ have a common upper bound:

$$V_0^m \eta(t, \cdot) \leq \bar{V} \quad (\text{B.2}) \quad \text{eq:etabound}$$

where $V_0^m f$ is the total variation of f [1]. In formulation (B.1) the dependent variable is the function x on the history interval $[-m, 0]$. The time-1 map Tx of (B.1) for $x \in C([-m, 0]; \mathbb{C}^n)$ is defined as the solution $y \in C([-m, 0]; \mathbb{C}^n)$ of the equation

$$t > -1 : \quad \dot{y}(t) = \int_0^m d_\theta \eta(t, \theta) \begin{cases} y(t - \theta) & \text{if } t - \theta \geq -1 \\ x(1 + t - \theta) & \text{if } t - \theta < -1 \end{cases} \quad (\text{B.3}) \quad \text{eq:tgende}$$

$$t \leq -1 : \quad y(t) = x(1 + t). \quad (\text{B.4}) \quad \text{eq:tgenshift}$$

Only the part of y on the interval $(-1, 0]$ is the result of a differential equation (namely (B.3)). The remainder of y , $y|_{[-m, -1]}$, is merely a shift of x , given by (B.4). Note that we have included the initial condition for (B.3) into (B.4), namely $y(-1) = x(0)$. Correspondingly, the eigenvalue problem for T reads

$$t > -1 : \quad \dot{x}(t) = \int_0^m d_\theta \eta(t, \theta) \begin{cases} x(t - \theta) & \text{if } t - \theta \geq -1 \\ \mu x(1 + t - \theta) & \text{if } t - \theta < -1 \end{cases} \quad (\text{B.5}) \quad \text{eq:tevgende}$$

$$t \leq -1 : \quad x(t) = \mu x(1 + t). \quad (\text{B.6}) \quad \text{eq:tevgenshift}$$

The spaces C_k and $C_{k,0}$, and the maps S , Γ_\pm and $M_k(\mu)$ defined in (4.7), (4.8) and (4.9) for a single delay can be extended in a straightforward way. We partition the interval $[-m, 0]$ into sub-intervals of length $1/k$, $J_i = [t_i, t_{i+1}] = [-1 + i/k, -1 + (i+1)/k]$ ($i = k - mk \dots k - 1$; note that i can become negative), and define the spaces

$$C_k = \{x : [-m, 0] \mapsto \mathbb{C}^n : x \text{ continuous on all (half-open) subintervals } J_i, \\ \text{and } \lim_{t \nearrow t_i} x(t) \text{ exists for all } i = k - mk + 1 \dots k\}$$

$$C_{k,0} = \{x \in C_k : x(t_i) = 0 \text{ for all } i = 0 \dots k - 1\}.$$

$$(\text{B.7}) \quad \text{eq:ckgndef}$$

Note that non-negative indices i of t_i correspond to times t_i in the interval $[-1, 0]$ whereas negative indices i correspond to times t_i in the interval $[-m, -1)$. These points of discontinuity are treated differently in the definition of $C_{k,0}$: to be an element of $C_{k,0}$ the function x is only required to be zero at t_i with non-negative i . The space C_k consists of piecewise continuous functions where the points of discontinuity are the times $t_i \in [-m, 0]$. For functions in C_k we define the maps

(again analogous to the definitions in (4.9) for the single delay)

$$\begin{aligned}
S : \mathbb{C}^{nk} &\mapsto C_k & S[v_0 \dots v_{k-1}]^T(t) &= \begin{cases} v_i & \text{if } t \in [t_i, t_{i+1}) \text{ for } i = 0 \dots k-1, \\ 0 & \text{if } t \in [-m, -1), \end{cases} \\
\Gamma_+ : C_k &\mapsto \mathbb{C}^{nk} & \Gamma_+ x &= [x(-1)_+, x(t_1)_+, \dots, x(t_{k-1})_+]^T, \\
\Gamma_-(\mu) : C_k &\mapsto \mathbb{C}^{nk} & \Gamma_-(\mu)x &= [\mu x(0)_-, x(t_1)_-, \dots, x(t_{k-1})_-]^T, \\
M_k(\mu) : C_k &\mapsto C_{k,0} & \text{where} & \\
[M_k(\mu)x](t) &= & \begin{cases} \int_{a_k(t)}^t \int_0^m d_\theta \eta(s, \theta) \begin{cases} x(s-\theta) & \text{if } s-\theta \geq -1 \\ \mu x(1+s-\theta) & \text{if } s-\theta < -1 \end{cases} & \text{for } t > -1, \\ \mu x(1+t) & & \text{for } t \leq -1. \end{cases} & \tag{B.8}
\end{aligned}$$

eq:opsgendef

The map S extends a tuple of vectors into a piecewise constant function using the the elements of the tuple as the values on the subintervals $J_i \subset [-1, 0]$, and setting the function to 0 on $[-m, -1)$. Γ_+ and Γ_- are defined in exactly the same way as in (4.9) (again, the notation $x(t_i)_\pm$ refers to left sided and right sided limits of x at t_i). The definition of $M_k(\mu)$ is identical to the single delay case for $t > -1$. For $t \leq -1$ we define $M_k(\mu)$ as a shift multiplied by μ . In the same manner as for the single delay we consider the extended initial-value problem

$$t > -1 : \quad \dot{x}(t) = \int_0^m d_\theta \eta(t, \theta) \begin{cases} x(t-\theta) & \text{if } t-\theta \geq -1, \\ \mu x(1+t-\theta) & \text{if } t-\theta < -1, \end{cases} \tag{B.9}$$

eq:ivpxgende

$$x(t_i)_+ = v_i \quad \text{for } i = 0 \dots k-1, \tag{B.10}$$

eq:ivpxgenic

$$t < -1 : \quad x(t) = \mu x(1+t). \tag{B.11}$$

eq:ivpxgenshift

Note that the initial values v_i are only given at the discontinuity points $t_i \in [-1, 0]$ ($i \geq 0$). The solution $x \in [-m, -1)$ is defined by a *backward* shift using (B.11). The precise meaning of the initial-value problem is given by its corresponding integral equation, which is a fixed point problem in C_k , and can be expressed as

$$x = Sv + M_k(\mu)x \tag{B.12}$$

eq:fixppgen

using the maps S and $M_k(\mu)$. The following lemma gives a sufficient condition for the existence of a unique solution $x \in C_k$ of (B.12) and, thus, the invertibility of $I - M_k(\mu)$. It is a straightforward generalization of Lemma 4.1. The only difference in the proof is that we can achieve $\|M_k(\mu)\| < 1$ only in a weighted maximum norm $\|\cdot\|_R$ (which is nevertheless equivalent to $\|\cdot\|_\infty$).

thm:uniquegen

LEMMA B.1 (Unique solution for initial-value problem). *Let $R > 1$ be fixed. Then the fixed point problem (B.12) has a unique solution x for all $v \in \mathbb{C}^{nk}$ and for all complex μ satisfying $|\mu| < R$ if the spacing of the sub-intervals is chosen such that*

$$R^{1/k} < 1 + \bar{V}^{-1} \frac{R - |\mu|}{R^{m+2}} \log R. \tag{B.13}$$

eq:genkbound

Note that \bar{V} is the upper bound on the total variation of η . The left-hand side of (B.13) approaches 1 for $k \rightarrow \infty$ whereas the right-hand side is larger than 1 if $R > 1$ and $R > |\mu|$.

Proof. We define the norm

$$\|x\|_R = \max_{t \in [-m, 0]} |R^t x(t)|$$

on the spaces C_k and $C_{k,0}$. This norm is equivalent to the standard maximum norm such that (B.12) has a unique solution in C_k whenever $\|M_k(\mu)\|_R < 1$. The norms of two operators that appear in $M_k(\mu)$ are:

$$\begin{aligned} \text{shift by } \theta \text{ (padded with zero)} : & \quad \|x(\cdot - \theta)\|_R \leq R^\theta \|x\|_R, \\ \text{integration from } a_k(t) \text{ to } t \text{ for } t \in [-1, 0] : & \quad \left\| \int_{a_k(t)}^t x(s) ds \right\|_R \leq \frac{R^{1/k} - 1}{\log R} \|x\|_R, \end{aligned}$$

In order to find the $\|\cdot\|_R$ -norm of the integrand in the definition of $M_k(\mu)$, (B.8), we have to estimate the $\|\cdot\|_R$ -norm for a Stieltjes sum over an arbitrary partition $0 \leq \theta_0 < \dots < \theta_N \leq m$ (with arbitrary intermediate values $\tilde{\theta}_j \in [\theta_j, \theta_{j+1}]$):

$$\begin{aligned} & \left\| \sum_{j=0}^{N-1} [\eta(t, \theta_{j+1}) - \eta(t, \theta_j)] \begin{cases} x(t - \tilde{\theta}_j) & \text{if } t - \tilde{\theta}_j \geq -1 \\ \mu x(1 + t - \tilde{\theta}_j) & \text{if } t - \tilde{\theta}_j < -1 \end{cases} \right\|_R \\ & \leq \max_{t \in [-1, 0]} R^t \sum_{j=0}^{N-1} |\eta(t, \theta_{j+1}) - \eta(t, \theta_j)| R \begin{cases} |x(t - \tilde{\theta}_j)| & \text{if } t - \tilde{\theta}_j \geq -1 \\ |x(1 + t - \tilde{\theta}_j)| & \text{if } t - \tilde{\theta}_j < -1 \end{cases} \\ & \leq \max_{t \in [-1, 0]} \sum_{j=0}^{N-1} |\eta(t, \theta_{j+1}) - \eta(t, \theta_j)| R^{\tilde{\theta}_j + 1} \|x\|_R \\ & \leq R^{m+1} \|x\|_R \max_{t \in [-1, 0]} V_0^m \eta(t, \cdot) = R^{m+1} \|x\|_R \bar{V}. \end{aligned}$$

Combined with the norm estimate for integration from $a_k(t)$ to t we obtain that

$$\| [M_k(\mu)x] |_{[-1, 0]} \|_R < R^{m+1} \bar{V} \frac{R^{1/k} - 1}{\log R} \|x\|_R. \quad (\text{B.14}) \quad \boxed{\text{eq:genmkbound01}}$$

The part of $M_k(\mu)x$ on the interval $[-m, -1)$ has a norm less than $\|x\|_R$ by choice of R :

$$\| [M_k(\mu)x] |_{[-m, -1)} \|_R \leq |\mu| R^{-1} \|x\|_R. \quad (\text{B.15}) \quad \boxed{\text{eq:genmkbound1m}}$$

Adding up the inequalities (B.14) and (B.15) we obtain an upper bound for the total norm $\|M_k(\mu)x\|_R$:

$$\|M_k(\mu)x\|_R \leq \left[R^{m+1} \bar{V} \frac{R^{1/k} - 1}{\log R} + \frac{|\mu|}{R} \right] \|x\|_R, \quad (\text{B.16}) \quad \boxed{\text{eq:proofgenmkbound}}$$

which is less than $\|x\|_R$ if k satisfies inequality (B.13) required in the statement of the lemma. \square

The invertibility of $I - M_k(\mu)$ permits us to choose exactly the same constructions for the characteristic matrix $\Delta_k(\mu)$ and for the isomorphisms E and F proving Lemma 4.2 in the same way as for the single delay case (see (4.13), (4.16) and

(4.18):

$$\begin{aligned}
X_1 &= C_k, \\
X_2 &= \mathbb{C}^{nk} \times C_{k,0}, \\
G(\mu) &= I - \mu T = [I - ST_-(0) - M_k(0)]^{-1} [I - ST_-(\mu)M_k(\mu)], \\
H(\mu) &= \begin{pmatrix} \Delta_k(\mu) & 0 \\ 0 & I \end{pmatrix}, \\
\Delta_k(\mu) &= I - \Gamma_-(\mu) [I - M_k(\mu)]^{-1} S \in \mathbb{C}^{nk} \times \mathbb{C}^{nk}, \\
E(\mu) &= [I - M_k(\mu)]^{-1} [Sv + \varphi], \\
F(\mu) &= \begin{bmatrix} \Gamma_+ - \Gamma_-(0) + \Gamma_-(\mu) [I - M_k(\mu)]^{-1} [I - ST_+ - M_k(0)], \\ I - ST_+ - M_k(0). \end{bmatrix}
\end{aligned}$$

With these constructions the maps $E(\mu) : X_2 \mapsto X_1$ and $F(\mu) : X_1 \mapsto X_2$ are isomorphisms for $|\mu| < R$ (see (A.2) and (A.14) for the inverses). The relation $H(\mu) = F(\mu)G(\mu)E(\mu)$ makes the infinite-dimensional eigenvalue problem of the time-1 map, $[I - \mu T]x = 0$, equivalent to the finite-dimensional eigenvalue problem $\Delta_k(\mu)v = 0$.

We remark that the invertibility of $I - ST_-(0) - M_k(0)$ is equivalent to the statement that the initial-value problem (B.3)–(B.4) has only the trivial solution for $x = 0$ (the operator is a compact perturbation of the identity), in other words, it is equivalent to the statement that the time-1 map Tx is well-defined (and equal to zero) in $x = 0$.

REFERENCES

- DGLW95** [1] O. DIEKMANN, S. A. VAN GILS, S. M. VERDUYN LUNEL, AND H.-O. WALTHER, *Delay equations*, vol. 110 of Applied Mathematical Sciences, Springer-Verlag, New York, 1995. Functional, complex, and nonlinear analysis.
- DS58a** [2] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators Part I: General Theory*, Pure and Applied Mathematics, Interscience Publishers, New York, 1958.
- GH83** [3] J. GUCKENHEIMER AND P. HOLMES, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, vol. 42 of Applied Mathematical Sciences, Springer-Verlag, New York, 1990. Revised and corrected reprint of the 1983 original.
- HL93** [4] J. K. HALE AND S. M. VERDUYN LUNEL, *Introduction to functional-differential equations*, vol. 99 of Applied Mathematical Sciences, Springer-Verlag, New York, 1993.
- KL92** [5] M. A. KAASHOEK AND S. M. VERDUYN LUNEL, *Characteristic matrices and spectral properties of evolutionary systems*, Trans. Amer. Math. Soc., 334 (1992), pp. 479–517.
- K66** [6] T. KATO, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- RS07** [7] D. ROOSE AND R. SZALAI, *Continuation and bifurcation analysis of delay differential equations*, in Numerical Continuation Methods for Dynamical Systems: Path following and boundary value problems, B. Krauskopf, H. Osinga, and J. Galán-Vioque, eds., Springer-Verlag, Dordrecht, 2007, pp. 51–75.
- SW06** [8] A. SKUBACHEVSKII AND H.-O. WALTHER, *On the Floquet multipliers of periodic solutions to nonlinear functional differential equations*, J. Dynam. Diff. Eq., 18 (2006), pp. 257–355.
- SSH06** [9] R. SZALAI, G. STÉPÁN, AND S. HOGAN, *Continuation of bifurcations in periodic delay differential equations using characteristic matrices*, SIAM Journal on Scientific Computing, 28 (2006), pp. 1301–1317.
- YP09** [10] S. YANCHUK AND P. PERLIKOWSKI, *Delay and periodicity*, Physical Review E, 79 (2009), p. 046221.