Nonlinear solutions of the amplitude equations governing fluid flow in rotating spherical geometries.

Submitted by

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I certify that all material in this thesis which is not my own work has been identified and that no material is included for which a degree has previously been conferred upon me.

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Abstract

We are interested in the onset of instability of the axisymmetric flow between two concentric spherical shells that differentially rotate about a common axis in the narrow-gap limit. The expected mode of instability takes the form of roughly square axisymmetric Taylor vortices which arise in the vicinity of the equator and are modulated on a latitudinal length scale large compared to the gap width but small compared to the shell radii. At the heart of the difficulties faced is the presence of phase mixing in the system, characterised by a non-zero frequency gradient at the equator and the tendency for vortices located off the equator to oscillate. This mechanism serves to enhance viscous dissipation in the fluid with the effect that the amplitude of any initial disturbance generated at onset is ultimately driven to zero.

In this thesis we study a complex Ginzburg–Landau equation derived from the weakly nonlinear analysis of Harris, Bassom and Soward [D. Harris, A. P. Bassom, A. M. Soward, Global bifurcation to travelling waves with application to narrow gap spherical Couette flow, Physica D 177 (2003) p. 122–174] (referred to as HBS) to govern the amplitude modulation of Taylor vortex disturbances in the vicinity of the equator. This equation was developed in a regime that requires the angular velocities of the bounding spheres to be very close. When the spherical shells do not co-rotate, it has the remarkable property that the linearised form of the equation has no non-trivial neutral modes. Furthermore no steady solutions to the nonlinear equation have been found.

Despite these challenges Bassom and Soward [A. P. Bassom, A. M. Soward, On finite amplitude subcritical instability in narrow-gap spherical Couette flow,

J. Fluid Mech. 499 (2004) p. 277–314] (referred to as BS) identified solutions to the equation in the form of pulse-trains. These pulse-trains consist of oscillatory finite amplitude solutions expressed in terms of a single complex amplitude localised as a pulse about the origin. Each pulse oscillates at a frequency proportional to its distance from the equatorial plane and the whole pulse-train is modulated under an envelope and drifts away from the equator at a relatively slow speed. The survival of the pulse-train depends upon the nonlinear mutual-interaction of close neighbours; as the absence of steady solutions suggests, self-interaction is inadequate.

Though we report new solutions to the HBS co-rotation model the primary focus in this work is the physically more interesting case when the shell velocities are far from close. More specifically we concentrate on the investigation of BS-style pulse-train solutions and, in the first part of this thesis, develop a generic framework for the identification and classification of pulse-train solutions.

Motivated by relaxation oscillations identified by Cole [S. J. Cole, Nonlinear rapidly rotating spherical convection, Ph.D. thesis, University of Exeter (2004)] whilst studying the related problem of thermal convection in a rapidly rotating self-gravitating sphere, we extend the HBS equation in the second part of this work. A model system is developed which captures many of the essential features exhibited by Cole's, much more complicated, system of equations. We successfully reproduce relaxation oscillations in this extended HBS model and document the solution as it undergoes a series of interesting bifurcations.

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