On the Evolutionary Selection of Nash Equilibrium Components[1]

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Abstract

It is well known for the common multi-population evolutionary dynamics applied to normal form games that a pure strategy combination is asymptotically stable if and only if it is a strict equilibrium point. We extend this result to sets and show the following. For certain regular selection dynamics every connected and closed asymptotically stable set of rest points containing a pure strategy combination is a strict equilibrium set and hence a Nash equilibrium component. A converse statement holds for two person games, for convex strict equilibrium sets and for the standard replicator dynamic.

Keywords: evolutionary dynamics, replicator dynamic, regular selection dynamics, strict equilibrium set, Nash equilibrium component.

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1 Introduction

For the typical evolutionary dynamic of an asymmetric game a pure strategy combination is asymptotically stable if and only if it is a strict Nash equilibrium.

This observation has been made repeatedly (see, for instance, Eshel and Akin (1983), Ritzberger and Vogelsberger (1989), Samuelson and Zhang (1992) and Ritzberger and Weibull (1995)). The term 'typical' in the above statement refers to generalizations of the (standard) replicator dynamics for asymmetric normal form games. The restriction to pure strategy combinations is important. It is well-known that the Nash equilibrium in Matching Pennies (which is mixed) may or may not be asymptotically stable under very similar dynamics (see, e.g. Maynard Smith (1982), Appendix J, or Weibull (1995), p. 199 ff).

Many games, in particular extensive-form games, have components of Nash equilibria that do not consist of isolated equilibria. If a strategy combination is contained in such a component then it will not be asymptotically stable as Nash equilibria are rest points. We will hence study the asymptotic stability of entire sets of strategy combinations. To keep the focus on Nash equilibria, we limit attention to sets of rest points.

We are interested in results that hold under any evolutionary dynamics that shares some basic properties with the replicator dynamics. While some theoretical results single out the replicator dynamics as a central learning and imitation dynamics (e.g. Borgers and Sarin (1997), Gale, Binmore, and Samuelson (1995), Schlag (1998)), the replicator dynamics is a knife-edge case in the sense that very similar dynamics perform very differently. By focussing on basic properties we can hence focus on results that depend on these properties and not on specific functional forms.

Still, a necessary first step is to understand selection under the replicator dynamics. For this special case we obtain a very clear-cut characterization. We find that asymptotically stable sets of rest points are precisely the strict equilibrium sets of the game. A *strict equilibrium set* (short, *SE set*) is a set-valued generalization of the concept of a strict equilibrium. Recall that a Nash equilibrium is called *strict* if every player strictly loses by unilaterally deviating from the equilibrium. Following Balkenborg (1994) a set of Nash equilibria is called a *strict equilibrium set (SE set)* if for any element in the set, a player either loses strictly by unilaterally deviating or if his deviation leads to another equilibrium in the set. Thus the set of Nash equilibria as a whole is robust against deviations in the same sense as it is the case for a strict equilibrium. The proof of the asymptotic stability of SE sets relies on an alternative characterization of SE sets that is motivated by Thomas (1985) notion of an evolutionarily stable set.

Returning to more general evolutionary dynamics we work with the following framework. The dynamics considered apply to asymmetric normal form games resulting from asymmetric evolutionary conflicts between as many populations as there are players in the game.[1] The individuals in each population are assumed to play pure strategies. The dynamics governing the change in play are in continuous time. We require that the dynamics have the following qualitative features. They are *regular selection dynamics* as defined in Samuelson and Zhang (1992). This ensures that faces of the strategy simplex are invariant. Intuitively, such dynamics model processes where only mutation, not selection itself, can cause new strategies to be played. They have been extensively studied in the literature. However, many processes which assume more rationality of the agents like the best response dynamics (see Hofbauer (1995b)), fictitious play or perturbed versions of these dynamics (see, e.g., Fudenberg and Levine (1998) and Hopkins (1999)) are ruled out. In addition we require that the dynamics *reinforces best replies* which is defined through the

[1] In contrast, the classic concept of an ESS applies to settings where a symmetric game is played within a single population.

following four properties applied to each population separately. A) The growth rate of any best reply is non-negative. B) Unless all strategies currently played in the population are already best replies these growth rates are strictly positive. C) Non-best replies have a negative growth rate when all strategies currently played are best replies. D) Some strategy currently played has a positive growth rate if not all strategies currently played achieve the same payoff. The class of dynamics reinforcing best replies includes all sign preserving (Nachbar (1990)) and monotone dynamics (Samuelson and Zhang (1992)), in particular the standard and the adjusted replicator dynamics, all sophisticated imitation dynamics in Hofbauer and Schlag (2000) and many of the imitation dynamics considered in Hofbauer (1995a) and Weibull (1995).

Warned by the Matching Pennies example we need to restrict the kinds of sets we consider in order to achieve our goal of obtaining a single selection result for any dynamics that reinforces best replies. Singleton sets consisting of a pure strategy combination and SE sets should remain as candidates. We decide to focus on sets that contain at least one pure strategy combination in each of its components, a property that any SE set has. We show that any asymptotically stable set of rest points with this property is a SE set. In this respect dynamics that reinforce best replies do not differ from the replicator dynamics where the additional property is satisfied automatically.

While conversely any SE set is asymptotically stable under the replicator dynamics we cannot show this for any dynamics reinforcing best replies. We show this only when the SE set has the additional property that each element of the SE set is a best response to its best responses. This additional condition is automatically satisfied in two-player games or when the SE set is convex but it need not hold in general for SE sets.

Combining the above findings, we obtain for two player games a tight characterization of asymptotic stability in purely static, game theoretic terms. Namely a non-empty set of strategy combinations is an asymptotically stable sets of rest points for all regular selection dynamics reinforcing best replies if and only if the set is an SE set.

In a related study, Ritzberger and Weibull (1995) show for a slightly different class of evolutionary dynamics that a face is asymptotically stable if and only if it is *closed under better replies.* (There is no condition on how the dynamics behaves within the set.) Our findings complement their results. To be closed under better replies means hereby that after replacing the strategy of one or several players by a pure better reply to the current strategy combination the resulting strategy combination also belongs to the face. While they consider only faces we focus on sets containing a pure strategy combination in each component. They allow for cycling and other behavior within the set while we consider only sets of rest points. For this reason SE sets do not always exist, in contrast to faces that are closed under better replies. A SE set can be strictly contained in a face that is closed under better replies but not vice versa. SE sets are subsets closed under better replies only when they are convex in which case they equal an entire face. In fact, a face consisting of Nash equilibria is a SE set if and only if it is closed under better replies.[2]

This paper adds to the literature on set-wise solution concepts for evolutionary games. As shown in more detail in Balkenborg (1994) SE sets are the multi-population counterpart of evolutionarily stable sets defined for single population contests in Thomas (1985) (see also Balkenborg and Schlag (2001)). However, the use of explicit dynamics differentiates this paper from other investigations into the evolutionary stability of sets (e.g. Sobel (1993) or Swinkels (1992)) that remain in a static framework.

Our characterization result for the replicator dynamics also connects to recent research on strategic stability. Applying Corollary 1 in DeMichelis and Ritzberger (2000), we obtain that a connected SE set contains a strategically stable set if its Euler characteristic

[2]This follows immediately from Proposition 1 iii) and Lemma 2.

is non-zero.

The rest of the paper is organized as follows. SE sets are defined in Section 2 and their basic properties are studied. In Section 3 the connection between SE sets and the replicator dynamics is made using the concept of a direct evolutionarily stable set. Section 4 introduces the properties defining a dynamics that reinforces best replies. Sections 4 and 5 investigate the relationship between asymptotically stable sets of rest points and SE sets. Section 6 contains the conclusion.

2 Strict Equilibrium Sets

Before we introduce strict equilibrium sets we must first recall some basic game-theoretic terminology and indicate the notations we use. We closely follow Ritzberger and Weibull (1995).

For a (finite) *normal form game* Γ the finite set of players is denoted by $\mathcal{N} = \{1, 2, \ldots, n\}$. The finite set consisting of the K_i pure strategies s_i^k, $k = 1, 2, \ldots, K_i$, of player $i \in \mathcal{N}$ is denoted by $\mathcal{S}_i$. $\mathcal{S} = \times_{i \in \mathcal{N}} \mathcal{S}_i$ is the set of pure strategy combinations with generic element $s = (s_{1,} s_2, \ldots, s_n)$. The set of *mixed strategies* of player i is the $(K_i - 1)$-dimensional unit simplex $\Delta_i = \left\{ \sigma_i \in \Re_+^{K_i} \mid \sum_{k=1}^{K_i} \sigma_i^k = 1 \right\}$. Pure strategies s_i^k are identified with the corresponding unit vectors $e_i^k \in \Delta_i$. $\Delta = \times_{i \in \mathcal{N}} \Delta_i$ is the set of mixed strategy combinations with generic element $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. We will often ignore the order when describing an n-tupel provided it is clear from the indexation. For instance, we will write (σ_{-i}, τ_i) for the strategy combination played when player i uses strategy τ_i while all other players use their respective strategies in the strategy combination σ.

The *support* of a mixed strategy $\sigma_i \in \Delta_i$ is denoted by $\operatorname{supp}(\sigma_i) = \left\{ s_i^k \in \mathcal{S}_i \,|\, \sigma_i^k > 0 \right\}$.

The *support* of a mixed strategy combination $\sigma \in \Delta$ is the Cartesian product $\text{supp}(\sigma) := \times_{i\in\mathcal{N}} \text{supp}(\sigma_i)$. Every subset $T_i \subseteq \mathcal{S}_i$ of a player's pure strategy set spans a face $\mathfrak{F}(T_i) := \{\sigma_i \in \Delta_i \mid \text{supp}(\sigma_i) \subseteq T_i\}$ of his mixed strategy simplex Δ_i. The faces of the convex polyhedron Δ are the sets $\mathfrak{F}(T) = \times_{i\in\mathcal{N}}\mathfrak{F}(T_i)$ spanned by a Cartesian product of pure strategy sets $T = \times_{i\in\mathcal{N}} T_i$, $T_i \subseteq \mathcal{S}_i$ for $i \in \mathcal{N}$. Every mixed strategy $\sigma_i \in \Delta_i$ generates a face $\mathfrak{F}(\sigma_i) = \mathfrak{F}(\text{supp}(\sigma_i)) \subseteq \Delta_i$ and every mixed strategy combination $\sigma \in \Delta$ generates a face $\mathfrak{F}(\sigma) = \mathfrak{F}(\text{supp}(\sigma)) \subseteq \Delta$.

The mapping $u : \mathcal{S} \to \mathfrak{R}^n$ defines for pure strategy combinations the *payoff* to each player. The multilinear expected payoff function $U : \Delta \to \mathfrak{R}^n$ and the mixed *best reply correspondence* $\tilde{\beta} = \times_{i\in\mathcal{N}}\tilde{\beta}_i : \Delta \to \Delta$ are defined in the usual manner. $\sigma \in \Delta$ is a Nash equilibrium when $\sigma \in \tilde{\beta}(\sigma)$, it is a strict Nash equilibrium when $\{\sigma\} = \tilde{\beta}(\sigma)$. The pure best reply correspondence $\beta = \times_{i\in\mathcal{N}}\beta_i : \Delta \to \mathcal{S}$ is defined by $\beta(\sigma) = \tilde{\beta}(\sigma) \cap \mathcal{S}$. We recall that $\rho \in \tilde{\beta}(\sigma)$ iff $\text{supp}(\rho) \subseteq \beta(\sigma)$. As done here we often write "iff" for "if and only if".

Definition 1 *A non-empty subset $G \subseteq \Delta$ is a* STRICT EQUILIBRIUM SET *(SE set) if for all $\sigma \in G$ and all $\tau_i \in \Delta_i$ the inequality*

$$U_i(\sigma_{-i}, \tau_i) \leq U_i(\sigma)$$

holds whereby equality implies $(\sigma_{-i}, \tau_i) \in G$.

In fact, if G is an SE set and σ, τ_i are as above then $U_i(\sigma_{-i}, \tau_i) = U_i(\sigma)$ *if* and only if $(\sigma_{-i}, \tau_i) \in G$. Notice that a SE Set consists of Nash equilibria. Moreover, a singleton set $\{\sigma\}$ is a SE set if and only if σ is a strict Nash equilibrium. Every set of strict Nash equilibria is a SE set, but not every strict equilibrium set is a set of strict Nash equilibria.

It is immediate from the definition that every union and every non-empty intersection of SE sets is a SE set.

Lemma 1 *A strict equilibrium set contains a strategy combination iff it contains its support.*

Proof. Fix a strategy combination $\sigma \in \Delta$. Fix $i \in \{1, .., n\}$ and $(r_1, \cdots, r_{i-1}, r_i) \in \times_{1 \le j \le i} \operatorname{supp}(\sigma_j)$. Let $\rho = (r_1, \cdots, r_{i-1}, r_i, \sigma_{i+1}, \cdots, \sigma_n)$.

Suppose (ρ_{-i}, σ_i) belongs to the SE set. Since (ρ_{-i}, σ_i) is a Nash equilibrium we have $U_i(\rho_{-i}, s_i) = U_i(\rho_{-i}, \sigma_i)$ for all $s_i \in \operatorname{supp}(\sigma_i)$. The definition of a SE set implies that all strategy combinations (ρ_{-i}, s_i), $s_i \in \operatorname{supp}(\sigma_i)$ belong to the SE set.

Suppose, conversely, that all strategy combinations (ρ_{-i}, s_i) with $s_i \in \operatorname{supp}(\sigma_i)$ belong to the SE set. Since $r_i \in \operatorname{supp}(\sigma_i)$ we have $U_i(\rho_{-i}, s_i) = U_i(\rho_{-i}, r_i)$ for all $s_i \in \operatorname{supp}(\sigma_i)$. (We obtain "$\ge$" and "$\le$" because the strategy combinations appearing on both sides of the equation belong to the SE set). Therefore $U_i(\rho_{-i}, \sigma_i) = U_i(\rho_{-i}, r_i)$ and hence (ρ_{-i}, σ_i) belongs to the SE set.

For the statements defined for all $0 \le i \le n$

(Si) "The set $(\times_{1 \le j \le i} \operatorname{supp}(\sigma_j)) \times (\times_{i+1 \le j \le n} \{\sigma_j\})$ is contained in the SE set."

we have hence shown that statement (Si) is equivalent to statement (S$(i-1)$). It follows by induction from $i = 0$ to $i = n$ and vice versa that statement (S0) is equivalent to (Sn). Since statement (S0) means that σ belongs to the SE set and statement (Sn) that the support of σ is contained in the SE set the lemma is proven. ■

Proposition 1 *i) Every SE set contains a pure strategy combination.*

ii) Every SE set is a finite union of faces and hence closed.

iii) A SE set is convex iff it is a face. Moreover, if a face $\mathfrak{F}$ *is a SE set then* $\tilde{\beta}(\sigma) = \mathfrak{F}$ *for all* $\sigma \in \mathfrak{F}$.

iv) Every SE set is a finite union of connected SE sets.

v) A SE set is connected iff it is a minimal SE set in the sense that it does not properly contain another SE set.

Proof. (i) and (ii) are obvious given Lemma 1.

Let G be a SE set and hence by (ii) a finite union of faces of Δ. Assume G is convex. Then there exists a maximal face $\mathfrak{F}(\sigma)$ contained in G, i.e. a face contained in G that is not a proper subset of another face contained in G. Assume that $\tau \in G$. Because G is convex it contains $\frac{1}{2}\sigma + \frac{1}{2}\tau$. The lemma implies then that G contains the face $\mathfrak{F}\left(\frac{1}{2}\sigma + \frac{1}{2}\tau\right)$. Since $\mathfrak{F}(\sigma)$ is a maximal face contained in G we must have $\mathfrak{F}(\sigma) = \mathfrak{F}\left(\frac{1}{2}\sigma + \frac{1}{2}\tau\right)$ and therefore $\tau \in \mathfrak{F}(\sigma)$. It follows that G is a subset of $\mathfrak{F}(\sigma)$ and hence that $\mathfrak{F}(\sigma) = G$.

Now suppose that the face $\mathfrak{F} = \times_{i \in \mathcal{N}} \mathfrak{F}_i$ is a SE set. Let $\sigma \in \mathfrak{F}$ and $i \in \mathcal{N}$. The definition of a SE set implies for every $\tau_i \in \tilde{\beta}_i(\sigma)$ that $\tau_i \in \mathfrak{F}_i$ and hence $\tilde{\beta}(\sigma) \subseteq \mathfrak{F}$. Conversely, if $\tau_i \in \mathfrak{F}_i$ then both σ and (σ_{-i}, τ_i) are Nash equilibria as elements of $\mathfrak{F}$ and therefore $\tau_i \in \tilde{\beta}_i(\sigma)$ holds. This proves $\mathfrak{F} \subseteq \tilde{\beta}(\sigma)$ and hence (iii).

As a finite union of faces a SE set G is a finite union of connected components, i.e. of maximally connected subsets of G. Consider such a connected component G' of G. Let $\sigma \in G'$ and let τ_i be a strategy of player $i \in \mathcal{N}$. Then $u_i(\sigma_{-i}, \tau_i) \leq u_i(\sigma)$. Suppose $u_i(\sigma_{-i}, \tau_i) = u_i(\sigma)$. Setting $\tau_i^\alpha = (1-\alpha)\sigma_i + \alpha\tau_i$ we find that $u_i(\sigma_{-i}, \tau_i^\alpha) = u_i(\sigma)$ holds for all $0 \leq \alpha \leq 1$. Hence $\{(\sigma_{-i}, \tau_i^\alpha)\}_{0 \leq \alpha \leq 1}$ is contained in G. Together with the fact that G' is a connected component and that $\sigma \in G'$ we obtain that $\{(\sigma_{-i}, \tau_i^\alpha)\}_{0 \leq \alpha \leq 1}$ is contained in G'. In particular (σ_{-i}, τ_i) belongs to G' which proves that G' is a SE set. Hence (iv) holds.

Given (iv) a minimal SE set must be connected. To prove the converse, assume that a SE set G' is properly contained in a connected SE set G. Since G is a connected union of faces it must contain a face $\mathfrak{F} = \times_{i \in \mathcal{N}} \mathfrak{F}(T_i)$ $(T_i \subseteq \mathcal{S}_i)$ that intersects G', but is not contained in G'. Since G' is also a union of faces, there exists a face $\mathfrak{F}' = \times_{i \in \mathcal{N}} \mathfrak{F}(T_i')$ that is maximal with respect to the property of being contained in the boundary of $\mathfrak{F}$ and being contained in G'. For some player $i \in \mathcal{N}$ there is a pure strategy $r_i \in T_i \backslash T_i'$. Hence there must exist a pure strategy combination $s \in \times_{i\in\mathcal{N}}\mathfrak{F}(T_i') \in G'$ with $(s_{-i}, r_i) \notin G'$ since otherwise a face larger than $\mathfrak{F}'$ in the boundary of $\mathfrak{F}$ would be contained in G'. We have $u_i(s) = u_i(s_{-i}, r_i)$ since both s and (s_{-i}, r_i) are contained in G. However, since $s \in G'$ and $(s_{-i}, r_i) \notin G'$ this contradicts the assumption that G' is a SE set. This concludes the proof of (v). ■

For an arbitrary subset, not necessarily a Cartesian product, $T \subseteq \mathcal{S}$ of pure strategy combinations we define the *n-convex hull* of T as[3]

$$\mathcal{G}(T) := \{\sigma \in \Delta \mid \operatorname{supp}(\sigma) \subseteq T\}.$$

$\mathcal{G}(T)$ is the (finite) union of all faces $\mathfrak{F}(\sigma)$ with $\operatorname{supp}(\sigma) \subseteq T$. In this terminology Lemma 1 states that a SE set is the n-convex hull of the pure strategy combinations it contains. We show now that in order to verify that a set of type $\mathcal{G}(T)$ is a SE set it suffices to verify the SE set conditions for pure strategy combinations only.

Lemma 2 *Suppose $T \subseteq \mathcal{S}$ is a non-empty subset of pure strategy combinations such that for all $s \in T$ and all $t_i \in \mathcal{S}_i$ the inequality*

$$u_i(s_{-i}, t_i) \leq u_i(s) \tag{1}$$

holds whereby equality implies $(s_{-i}, t_i) \in T$. Then the n-convex hull $\mathcal{G}(T)$ of T is a SE set.

Proof. Suppose $\tau_i \in \Delta_i$ and $\sigma \in \mathcal{G}(T)$, i.e. $\operatorname{supp}(\sigma) \subseteq T$. The multilinearity of the expected payoff function then implies $U_i(\sigma_{-i}, \tau_i) \leq U_i(\sigma)$ because (1) applies to every

[3]This terminology is appropriate. Generalizing the notion of a biconvex set in Aumann and Hart (1986) we call a subset $R \subseteq \Delta$ *n-convex* if it contains with any σ, $(\sigma_{-i}, \rho_i) \in R$ the strategy combinations $(\sigma_{-i}, (1-\alpha)\sigma_i + \alpha\rho_i)$ for all $0 \leq \alpha \leq 1$. It is not difficult to show that the set $\mathcal{G}(T)$ defined in the text is the smallest n-convex set containing T.

$s \in \text{supp}(\sigma)$ and $t_i \in \text{supp}(\tau_i)$. $U_i(\sigma_i, \tau_i) = U_i(\sigma)$ implies equality in (1) and thereby $(s_{-i}, t_i) \in T$ for all $s \in \text{supp}(\sigma)$ and $t_i \in \text{supp}(\tau_i)$, which implies $(\sigma_{-i}, \tau_i) \in \mathcal{G}(T)$. ■

Remark 1 For a given set of players $\mathcal{N}$ and their pure strategy sets $\mathcal{S}_i$ every set $\mathcal{G}(T)$ $(\emptyset \neq T \subseteq \mathcal{S})$ occurs as a SE set for suitably chosen payoff functions. For instance, define a game with identical interests (i.e. a game with $u_i(s) = u_j(s)$ for all $s \in \mathcal{S}$, $i, j \in \mathcal{N}$) by

$$u_i(s) = \begin{cases} 1 & \text{for} \quad s \in T \\ 0 & \text{for} \quad s \notin T \end{cases}.$$

Lemma 2 implies immediately that $\mathcal{G}(T)$ is a SE set of this game.

Remark 2 As unions of faces SE sets are fairly simple objects. Still, their topological properties can be interesting. To illustrate this, suppose that all N players have the same number K of strategies, so $\mathcal{S}_i = \{s_1^1, \cdots, s_i^K\}$. Consider the game with identical interest defined as in the previous remark when T is the set of pure strategy combinations where not all players choose the strategy with the same index k, i.e.

$$T = \mathcal{S} \setminus \{(s_1^k, \cdots, s_N^k) \mid 1 \leq k \leq K\}$$

When there are three players each with two strategies the SE set $\mathcal{G}(T)$ is a cycle. The space of mixed strategy combinations Σ can then be visualized as a cube, as in Figure 1. The eight pure strategy combinations correspond to the eight vertices of the cube. The cycle $\mathcal{G}(T)$ indicated by the fat lines consists of six edges of the cube connecting the six vertices other than (s_1^1, s_2^1, s_3^1) and (s_1^2, s_2^2, s_3^2). Similarly, the SE set $\mathcal{G}(T)$ is a topological cycle when there are two players with three strategies each. For general N and K $\mathcal{G}(T)$ can be shown to be a topological sphere.

Remark 3 Not every finite union of faces can occur as a SE set. Consider, for instance, the boundary G of the space of mixed strategy combinations Δ. It is the union of all

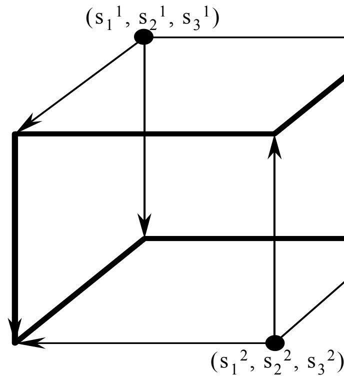

Figure 1: A SE set can be a cycle.

proper faces of Δ. G cannot be a SE set because it is not the n-convex hull of the pure strategy combinations it contains. (G contains the set of all pure strategy combinations $\mathcal{S}$, but $G \neq \mathcal{G}(\mathcal{S}) = \Delta$.)

Remark 4 The set of strategy combinations that maximize the sum of the player's payoffs in a game with identical interests (or the potential in a weighted potential game, see Monderer and Shapley (1996)) is a SE set. In particular, SE sets exists for these types of games. Of course SE sets often do not exist as, for instance, in Matching Pennies.

3 Evolutionary stability and the replicator dynamics

In order to understand the relevance of SE sets for the replicator dynamics it is necessary to demonstrate the equivalence between the notion of SE sets and a set-valued concept of evolutionary stability motivated by Thomas (1985). The results in this section were first proven in Balkenborg (1994), where the connection to Thomas' original concept is discussed in detail.

Definition 2 *A* DIRECT EVOLUTIONARILY STABLE SET *(direct ES set) is a closed, non-empty set G of strategy combinations where every strategy combination σ in G has a* *neighborhood $V(\sigma)$ such that*

$$\sum_{i\in\mathcal{N}} U_i(\rho) \leq \sum_{i\in\mathcal{N}} U_i(\rho_{-i}, \sigma_i) \tag{2}$$

holds for all $\rho \in V(\sigma)$ whereby equality implies $\rho \in G$.

We borrow the term "direct" from Selten (1980) to emphasize the difference to Thomas' notion of an evolutionarily stable set.

Proposition 2 *A set of strategy combinations is a direct ES set iff it is a SE set.*

Proof. Let σ be a strategy combination in a direct ES set G. Let τ_i be any strategy of player i, let $\varepsilon \in (0,1)$ and let $\rho = (\sigma_{-i}, (1-\varepsilon)\sigma_i + \varepsilon\tau_i)$ where ε is sufficiently small such that $\rho \in V(\sigma)$. Then inequality (2) must hold for this particular ρ. However, since $\rho = (\rho_{-j}, \sigma_j)$ for all $j \neq i$, this inequality simplifies to

$$U_i(\sigma_{-i}, (1-\varepsilon)\sigma_i + \varepsilon\tau_i) \leq U_i(\sigma)$$

$$\text{or} \qquad U_i(\sigma_{-i}, \tau_i) \leq U_i(\sigma). \tag{3}$$

i.e. σ is a Nash equilibrium. Moreover, equality in (3) implies $(\sigma_{-i}, (1-\varepsilon)\sigma_i + \varepsilon\tau_i) \in G$.[4] In order to conclude that (σ_{-i}, τ_i) is itself in G we must use in addition that G is a closed set. If we have equality in (3) we obtain $u_i(\sigma_{-i}, \tau_i) = u_i(\sigma_{-i}, \rho_i^\alpha)$ for all $\rho_i^\alpha = (1-\alpha)\sigma_i + \alpha\tau_i$, $0 \leq \alpha \leq 1$. Let δ be the supremum of all α with $(\sigma_{-i}, \rho_i^\alpha) \in G$. Closedness implies $(\sigma_{-i}, \rho_i^\delta) \in G$ and hence $u_i(\sigma_{-i}, \tau_i) = u_i(\sigma_{-i}, \rho_i^\delta)$ implies $(\sigma_{-i}, (1-\hat{\varepsilon})\rho_i^\delta + \hat{\varepsilon}\tau_i) \in G$ for small $\hat{\varepsilon} > 0$. This contradicts the definition of δ unless $\delta = 1$. Therefore $(\sigma_{-i}, \tau_i) \in G$. Consequently, every direct ES set is a SE set.

In order to prove the converse it is not enough to consider strategy combinations ρ that coincide with an element σ of the SE set except for the strategy of a single player. For arbitrary ρ close to σ we use multilinearity to obtain a Taylor expansion of the

[4] This part of the argument mimics the proof in Selten (1980) that a direct ESS is a pure strategy.

difference $\sum_{i\in\mathcal{N}}\left(u_i\left(\rho\right)-u_i\left(\rho_{-i},\sigma_i\right)\right)$ around σ. We show that the lowest nonzero terms in the expansion correspond to the payoff losses of players who deviate unilaterally from strategy combinations in the SE set. These are hence negative and it turns out that higher order terms can be ignored if ρ is sufficiently close to σ.

More specifically, fix a strategy combination σ in the SE set G. It is well-known and immediately verified that the sets

$$V_{\tilde{\varepsilon}}\left(\sigma\right)=\left\{\rho\in\Delta\mid\text{for all } i\in\mathcal{N}\text{: } \rho_i=\left(1-\varepsilon_i\right)\sigma_i+\varepsilon\tau_i \text{ with } \tau_i\in\Delta_i \text{ and } 0\leq\varepsilon_i\leq\tilde{\varepsilon}\right\}$$

form for $0<\tilde{\varepsilon}<1$ a basis of neighborhoods of σ in Δ since Δ is a convex polyhedron.[5] We need the following notation. For a subset of players $I\subseteq\mathcal{N}$, let $\mathcal{S}^I=\times_{i\in I}\mathcal{S}_i$, $\varepsilon_I=\times_{i\in I}\varepsilon_i$, $\tau\left(s_I\right)=\times_{i\in I}\tau_i\left(s_i\right)$. $\left(\sigma_{\mathcal{N}\setminus I},\tau_I\right)$ denotes the strategy combination where the players in I use their strategies in τ while the others use their strategies in σ. For $\rho=\left(1-\varepsilon\right)\sigma+\varepsilon\tau\in V_{\tilde{\varepsilon}}\left(\sigma\right)$ we obtain the multilinear expansion

$$\begin{aligned}
&\sum_{i\in\mathcal{N}}\left(u_i\left(\rho\right)-u_i\left(\rho_{-i},\sigma_i\right)\right) &&(4)\\
&=\sum_{i\in\mathcal{N}}\varepsilon_i\left(u_i\left(\rho_{-i},\tau_i\right)-u_i\left(\rho_{-i},\sigma_i\right)\right) &&(5)\\
&=\sum_{i\in\mathcal{N}}\sum_{J\subseteq\mathcal{N}\setminus\{i\}}\varepsilon_i\left(1-\varepsilon\right)_{\mathcal{N}\setminus(J\cup\{i\})}\varepsilon_J\left[u_i\left(\sigma_{\mathcal{N}\setminus(J\cup\{i\})},\tau_{J\cup\{i\}}\right)-u_i\left(\sigma_{\mathcal{N}\setminus J},\tau_J\right)\right]\\
&=\sum_{\emptyset\neq I\subseteq\mathcal{N}}\left(1-\varepsilon\right)_{\mathcal{N}\setminus I}\varepsilon_I\left[\sum_{i\in I}\left(u_i\left(\sigma_{\mathcal{N}\setminus I},\tau_I\right)-u_i\left(\sigma_{\mathcal{N}\setminus(I\setminus\{i\})},\tau_{I\setminus\{i\}}\right)\right)\right] &&(6)\\
&=\sum_{\emptyset\neq I\subseteq\mathcal{N}}\sum_{s_I\in\mathcal{S}^I}\left(1-\varepsilon\right)_{\mathcal{N}\setminus I}\varepsilon_I\tau\left(s_I\right)\left[\sum_{i\in I}\left(u_i\left(\sigma_{\mathcal{N}\setminus I},s_I\right)-u_i\left(\sigma_{\mathcal{N}\setminus(I\setminus\{i\})},s_{I\setminus\{i\}}\right)\right)\right] &&(7)
\end{aligned}$$

For $\emptyset\neq I\subseteq\mathcal{N}$ we call $s_I\in\mathcal{S}^I$ *trivial* if $u_i\left(\sigma_{\mathcal{N}\setminus I},s_I\right)=u_i\left(\sigma_{\mathcal{N}\setminus(I\setminus\{i\})},s_{I\setminus\{i\}}\right)$ holds for all $i\in I$. We define recursively for any set of players $\emptyset\neq I\subseteq\mathcal{N}$ the sets S_0^I, S_-^I, $S_+^I\subseteq\mathcal{S}^I$ as follows. $s_I\in\mathcal{S}^I$ belongs to S_0^I if s_I is trivial and if either $I=\{i\}$ for some $i\in\mathcal{N}$ or if

[5] For arbitrary closed convex sets a corresponding statement would not necessarily be true.

$s_{I\setminus\{i\}} \in S_0^{I\setminus\{i\}}$ for all $i \in I$. $s_I \in \mathcal{S}^I$ belongs to S_-^I if s_I is *not* trivial and if either $I = \{i\}$ for some $i \in \mathcal{N}$ or if $s_{I\setminus\{i\}} \in S_0^{I\setminus\{i\}}$ for all $i \in I$. Otherwise $s_I \in S^I$ belongs to S_+^I.

Since $\sigma \in G$ it follows by induction for all $I \subseteq \mathcal{N}$ that $\left(\sigma_{\mathcal{N}\setminus I}, s_I\right) \in G$ for all $s_I \in S_0^I$. For $s_I \in S_-^I$ we obtain hence $u_i\left(\sigma_{\mathcal{N}\setminus I}, s_I\right) \leq u_i\left(\sigma_{\mathcal{N}\setminus(I\setminus\{i\})}, s_{I\setminus\{i\}}\right)$ for all $i \in I$ whereby a strict inequality must hold for at least one (and actually, all) $i \in I$ because s_I is not trivial. Because there are only finitely many strategy combinations of this type we can find a strictly positive number A such that

$$\sum_{i\in I}\left(u_i\left(\sigma_{\mathcal{N}\setminus I}, s_I\right) - u_i\left(\sigma_{\mathcal{N}\setminus(I\setminus\{i\})}, s_{I\setminus\{i\}}\right)\right) < -A$$

holds for all $\emptyset \neq I \subseteq \mathcal{N}$, $s_I \in S_-^I$. Similarly, there exists a positive constant C such that

$$\sum_{i\in I}\left(u_i\left(\sigma_{\mathcal{N}\setminus I}, s_I\right) - u_i\left(\sigma_{\mathcal{N}\setminus(I\setminus\{i\})}, s_{I\setminus\{i\}}\right)\right) < C$$

holds for all $\emptyset \neq I \subseteq \mathcal{N}$, $s_I \in S_+^I$. Moreover, our construction implies for all $\emptyset \neq I \subseteq \mathcal{N}$ and $s_I \in S_+^I$ that there exists a proper, non-empty subset J of I such that $s_I = \left(s_J, s_{I\setminus J}\right)$ with $s_J \in S_-^J$. Hence we obtain the following upper bound for the expression in (7):

$$\sum_{\emptyset\neq I\subseteq\mathcal{N}}\ \sum_{s_I\in S_-^I} \varepsilon_I \tau\left(s_I\right)\left[-A\left(1-\tilde{\varepsilon}\right)^{n-\#I} + \tilde{\varepsilon}WC\right]$$

where W is an appropriate positive constant. We can now choose $\tilde{\varepsilon} > 0$ so small that the terms in square brackets (which do depend on the choice of σ but not on the choice of τ) are strictly negative. It follows then that (4) is not positive for any τ. Moreover, if there exists $\emptyset \neq I \subseteq \mathcal{N}$ and $s_I \in S_-^I$ such that $\varepsilon_I\tau\left(s_I\right) > 0$ then (4) is strictly negative. Suppose hence that $\varepsilon_I\tau\left(s_I\right) = 0$ for all $\emptyset \neq I \subseteq \mathcal{N}$ and $s_I \in S_-^I$. We must then also have $\varepsilon_I\tau\left(s_I\right) = 0$ for all $\emptyset \neq I \subseteq \mathcal{N}$ and $s_I \in S_+^I$ since each such s_I takes the form $\left(s_J, s_{I\setminus J}\right)$ with $s_J \in S_-^J$. Thus all terms in the sum of (7) and of (6) are zero. It then follows by induction that all strategy combinations $\left(\sigma_{\mathcal{N}\setminus I}, \tau_I\right)$, $I \subseteq \mathcal{N}$, are in the SE set. By Lemma 1 G contains the support of all these strategy combinations and hence also the support of ρ. Again by Lemma 1, ρ belongs to the SE set. Thus G is a direct ES set. ■

We can now begin our investigation of evolutionary dynamics. We need the following terminology and notation. Again, we follow closely Ritzberger and Weibull (1995).

A *regular selection dynamics* on Δ is a system of ordinary differential equations

$$\dot{\sigma}_i^k = \sigma_i^k f_i^k(\sigma), \quad \forall k = 1, \ldots, K_i, \ \forall i \in \mathcal{N}$$

with continuous functions $f_i^k : \Delta \to \Re$ for $i \in \mathcal{N}$ describing the growth rates of pure strategies s_i^k such that all $f_i^k(\sigma)\sigma_i^k$ are Lipschitz continuous and such that $\sum_k f_i^k(\sigma)\sigma_i^k = 0$ holds for all $\sigma \in \Delta$, $i \in \mathcal{N}$. Unless explicitly stated otherwise, the term "dynamics" will always refer to a regular selection dynamics. For every $\sigma^0 \in \Delta$ the system of differential equations has a unique solution $\{\sigma^t\}_{t \geq 0} \subset \Delta$ starting in σ^0. This trajectory satisfies $\operatorname{supp}(\sigma^t) = \operatorname{supp}(\sigma^0)$ for all $t \geq 0$. In particular, faces of Δ are invariant under the dynamics.

An important example of a regular selection dynamics is the (standard) *replicator dynamics* (Taylor (1979)) $\dot{\sigma}_i^k = \sigma_i^k \left[U_i\left(\sigma_{-i}, s_i^k\right) - U_i(\sigma)\right]$. Borgers and Sarin (1997) show how the replicator dynamics arises from a model of learning and Schlag (1998) shows how it is derived from an "optimal" imitation rule.

The following concepts refer to a given dynamics. $\sigma \in \Delta$ is a *rest point* if $\dot{\sigma}_i^k = 0$ for all $i \in \mathcal{N}$ and $1 \leq k \leq K_i$. A set $G \subseteq \Delta$ is *forward invariant* if all trajectories starting in this set remain in this set. A *neighborhood* of a set $G \subseteq \Delta$ is a set containing an open set which contains G. G is *(Lyapunov) stable* if every neighborhood U of G contains a neighborhood V of G such that any trajectory starting in V never leaves U. ρ is an *ω-limit point* of a trajectory $\{\sigma^t\}_{t \geq 0}$ if every neighborhood of ρ contains points σ^t of the trajectory with arbitrarily large t. The *ω-limit set* of a trajectory is the set of all its ω-limit points. A closed non-empty set G is an *attractor* if it has a neighborhood U such that the ω-limit sets of trajectories starting in U are contained in G. A stable attractor is called *asymptotically stable.* A single strategy combination $\sigma \in \Delta$ is *(asymptotically)* *stable* if the singleton set $\{\sigma\}$ is (asymptotically) stable. In this paper we are primarily interested in asymptotically stable sets of rest points. Such sets are clearly isolated sets of rest points. A set of rest points is *isolated* if it contained in an open set that contains no other rest points. Asymptotically stable sets or rest points are clearly isolated.

The relevance of direct ES sets for the replicator dynamics becomes apparent if we use Lyapunov functions similar to those considered in Zeeman (1980). Namely, for a mixed strategy combination σ and mixed strategies combinations $\rho \in \Delta$ with $\operatorname{supp}(\sigma) \subseteq \operatorname{supp}(\rho)$ define the function

$$L^{\sigma}(\rho) = \sum_{i \in \mathcal{N}} \sum_{s_i^k \in \operatorname{supp}(\sigma_i)} \sigma_i^k \ln\left(\rho_i^k\right).$$

It is a well-known straight forward exercise to show that L^{σ} is strictly concave with a unique maximum at σ. Hence the sets $V_{\delta}(\sigma) = \{\rho \in \mid L^{\sigma}(\rho) \geq L^{\sigma}(\sigma) - \delta\}$ $(\delta > 0)$ form a basis of neighborhoods of σ. Let $(\rho^t)_{t \geq 0}$ be the trajectory under the replicator dynamics where $\operatorname{supp}(\sigma) \subseteq \operatorname{supp}(\rho^0)$. Then

$$\begin{aligned}
\dot{L}^{\sigma}\left(\rho^t\right) = \frac{dL^{\sigma}(\rho^t)}{dt} &= \sum_{i \in \mathcal{N}} \sum_{s_i^k \in \operatorname{supp}(\sigma_i)} \frac{\partial L^{\sigma}}{\partial \rho_i^k} \frac{d\left(\rho^t\right)_i^k}{dt} \\
&= \sum_{i \in \mathcal{N}} \sum_{s_i^k \in \operatorname{supp}(\sigma_i)} \frac{\sigma_i^k}{\left(\rho^t\right)_i^k} \left[\left(\rho^t\right)_i^k \left(U_i\left(\rho_{-i}^t, s_i\right) - U_i\left(\rho^t\right)\right)\right] \\
&= \sum_{i \in \mathcal{N}} \left(U_i\left(\rho_{-i}^t, \sigma_i\right) - U_i\left(\rho^t\right)\right).
\end{aligned}$$

This calculation shows that a closed, non-empty set G of strategy combinations is by definition a direct ES set if and only if for each strategy combination $\sigma \in G$ and all sufficiently small $\delta > 0$ the neighborhoods $V_{\delta}(\sigma)$ are forward invariant under the replicator dynamics and contain only rest points in G.

A direct ES set G consists hence of stable rest points and, being compact, is itself stable. The following argument taken from Thomas (1985), proof of Theorem 2, shows that any trajectory $(\rho^t)_{t \geq 0}$ starting in a sufficiently small neighborhood $V_{\delta}(\sigma)$ of $\sigma \in G$

converges to a rest point $\hat{\rho}$ in G. Let $\hat{\rho} \in V_\delta(\sigma)$ be a ω-limit point of the trajectory. Then L^σ increases along the trajectory, hence $\dot{L}^\sigma(\hat{\rho}) = 0$ (Lemma 2.6.1 in Hofbauer and Sigmund (1998)) and so $\hat{\rho} \in G$ by the definition of a direct ES set. Thus the trajectory enters all sufficiently small neighborhoods $V_\delta(\hat{\rho})$ and never leaves them, i.e. it converges to $\hat{\rho}$. Using again the compactness of G, it follows that G is an attractor. To summarize, we have shown:

Lemma 3 *Under the replicator dynamics a direct ES set is asymptotically stable and consists of stable rest points.*

For the replicator dynamics our major aim is to establish the following result.

Theorem 1 *A non-empty set of strategies is a SE set iff it is an asymptotically stable set of rest points under the replicator dynamics. Moreover, for this dynamics every element of a SE set is Lyapunov stable.*

Proposition 2 and Lemma 3 above imply the "only if" and the "moreover" statement of this result. The proof of the "if" statement requires a different line of arguments which we develop in the next section.

SE sets consist of Nash equilibria and the latter are always rest points of the replicator dynamics. Since asymptotically stable sets of rest points are isolated Proposition 2 and Lemma 3 also imply the following result.

Corollary 1 *A SE set is a finite union of Nash equilibrium components.*

As an application of Corollary 1 in DeMichelis and Ritzberger (2000) we obtain, in their terminology, from Lemma 3 (iv) and Proposition 1 the following connection between SE sets and the refinement concept of strategic stability.

Corollary 2 *A SE set with non-zero Euler characteristic contains an M-stable set.*

4 When are asymptotically stable sets SE sets?

In the following we consider more general regular selection dynamics that satisfy the following intuitive properties.

Definition 3 *A regular selection dynamics* REINFORCES BEST REPLIES *if the following conditions hold for any player $i \in \mathcal{N}$, any strategy combination $\sigma \in \Delta$ and any pure strategy $s_i^k \in S_i^k$.*

A) The inequality $f_i^k(\sigma) \geq 0$ holds whenever $s_i^k \in \beta_i(\sigma)$.

B) The inequality $f_i^k(\sigma) > 0$ holds whenever $s_i^k \in \beta_i(\sigma)$ and $\sigma_i \notin \tilde{\beta}_i(\sigma)$.

C) The inequality $f_i^k(\sigma) < 0$ holds whenever $s_i^k \notin \beta_i(\sigma)$ and $\sigma_i \in \tilde{\beta}_i(\sigma)$.

D) If for some $s_i^k \in \operatorname{supp}(\sigma)$, $i \in \mathcal{N}$, $U_i\left(\sigma_{-i}, s_i^k\right) \neq U_i(\sigma)$ then σ is not a rest point.

Property (A) states that best replies are not selected against. Since $\sum_{s_i^k \in \operatorname{supp}(\sigma_i)} \sigma_i^k f_i^k(\sigma) = 0$ holds for all $\sigma \in \Delta$ and $i \in \mathcal{N}$ it implies that all Nash equilibria are rest points. Property (B) means literally that best replies are reinforced (i.e. have positive growth rates) whenever possible.[6] When applied to a best reply s_i^k that is currently not played, i.e. $s_i^k \notin \operatorname{supp}(\sigma_i)$, it has implications for what would happen if a small mutation would lead to the introduction of this strategy. We need it primarily for the following lemma.[7]

Lemma 4 *a) A pure strategy combination belonging to a stable set of rest points for a regular selection dynamics satisfying Property (B) is a Nash equilibrium.*

b) A stable set of rest points for a differentiable regular selection dynamics satisfying Property (B) consists of Nash equilibria.

[6] Notice that Property B) does not imply Property A). Consider the 2×2 game where all payoffs are identically zero. The replicator dynamics for matching pennies vacuously satisfies Property B) for this game, but it does not satisfy Property A).

[7] We conjecture that part b) of the lemma does not hold without differentiability.

Proof. Suppose σ belongs to a stable set of rest points, but is not a Nash equilibrium. Since σ is not a Nash equilibrium there exists by Property (B) a pure best reply s_i^k with $f_i^k(\sigma) > 0$. Since $\sigma_i^k f_i^k(\sigma) = 0$ holds at a rest point, s_i^k is not in the support of σ_i.

a) If $\sigma = s$ is a pure strategy combination then the strategy combinations $\sigma^\alpha = \left(s_{-i}, (1-\alpha) s_i + \alpha s_i^k\right)$ $(0 \leq \alpha \leq 1)$ form a one-dimensional face of Δ which is invariant under the dynamics. By continuity, $f_i^k(\sigma^\alpha) > 0$ holds for all sufficiently small α. Consequently, there is a $0 < \beta < 1$ with $f_i^k\left(\sigma^\beta\right) > 0$ such that any trajectory starting in σ^α with $\alpha > 0$ sufficiently small will reach σ^β in finite time. The set $V = \left\{\rho \in \Delta \mid \rho_i^k f_i^k(\rho) < \frac{1}{2}\beta f_i^k\left(\sigma^\beta\right)\right\}$ is a neighborhood of the set of all rest points that does not contain σ^β. Since trajectories starting arbitrarily close to s leave this neighborhood, s cannot belong to a stable set of rest points.

b) We can assume without loss of generality that $s_j^{K_j} \in \operatorname{supp}(\sigma_j)$ for all $j \in \mathcal{N}$. The linearization of the vectorfield around σ is then described by a matrix

$$A = \left(\frac{\partial \rho_i^k f_i^k}{\partial \rho_j^l}_{|\rho=\sigma}\right)_{\substack{i\in\mathcal{N},k\in\{1,..,K_i-1\} \\ j\in\mathcal{N},l\in\{1,..,K_j-1\}}}.$$

The product rule yields

$$\frac{\partial \rho_i^k f_i^k}{\partial \rho_j^l}_{|\rho=\sigma} = \begin{cases} \left(\rho_i^k \frac{\partial f_i^k}{\partial \rho_j^l}\right)_{|\rho=\sigma} & = \quad 0 \qquad \text{for} \quad j \neq i \text{ or } l \neq k \\ \left(f_i^k(\rho) + \rho_i^k \frac{\partial f_i^k}{\partial \rho_i^k}\right)_{|\rho=\sigma} & = \quad f_i^k(\sigma) \quad \text{for} \quad j = i \text{ and } l = k \end{cases}$$

The matrix A has hence a row where the diagonal element is $f_i^k(\sigma) > 0$ and where all other entries are zero. This implies that the linearization of the vectorfield around σ has a positive Eigenvalue. It is well known that σ cannot be a stable rest point under this condition (see e.g. Hirsch and Smale (1974)).

The proof given by Hirsch and Smale (1974) in fact shows more. In Hirsch and Smale (1974) the state space is transformed so that the rest point under consideration is the origin $x = 0$. Then a compact neighborhood U of the rest point and a closed cone C are constructed with the following three properties. i) There are no rest points other than $x = 0$ in $C \cap U$. ii) Trajectories starting in $C \cap U$ remain in C as long as they remain in U. iii) Trajectories which start in $C \cap U$ but not at the rest point $x = 0$ leave the neighborhood U in finite time. Notice that (i) follows from part (a) of the lemma on page 188 and that (iii) emerges from the discussion following the lemma on pages 189 – 190.

Now let D be the boundary of U intersected with C. D is a closed set containing no rest points. Its complement N is an open neighborhood of the set of all rest points. By construction there are trajectories starting arbitrarily close to the rest point $x = 0$ in $C \cap U$ which enter D and hence leave N. This implies that $x = 0$ cannot belong to a stable set of rest points. ■

Thus Property (B) serves to make rest points that are not Nash equilibria unstable. Property (C) symmetrically serves the purpose of stabilizing Nash equilibria in the sense of not allowing trajectories to move in the direction of non-best responses. It will be used in the next section.

Properties (A) and (B) do not impose any restrictions on the dynamics in points where no player is choosing a pure best response, for instance when all players use only strictly dominated strategies. Property (D) requires that some selection is still at work in such situations. It states that the selection process does not come to a halt as long as not all pure strategies currently used are equally good. The assumption has one crucial implication. It implies that all rest points of the dynamics considered must also be rest points for the replicator dynamics. Moreover, we can describe the potential rest points σ in purely game-theoretic terms as "partial equilibria" in the sense that the Nash equilibrium condition $U_i(\sigma) \geq U_i\left(\sigma_{-i}, s_i^k\right)$ (effectively "=") is satisfied with respect to all strategies s_i^k in the support of σ_i, $i \in \mathcal{N}$.

Since a SE set is an isolated set of rest points for the replicator dynamics by Lemma 3 we obtain:

Lemma 5 *A SE set is an isolated set of rest points for any regular selection dynamics satisfying Properties (A) and (D).*

Classes of regular selection dynamics frequently discussed in the literature and to which our results apply are the sign preserving (Nachbar (1990)) and the monotone (Samuelson and Zhang (1992)) dynamics.

A dynamics is *monotone* if $\text{sign}\left(f_i^k(\sigma) - f_i^l(\sigma)\right) = \text{sign}\left(U_i\left(\sigma_{-i}, s_i^k\right) - U_i\left(\sigma_{-i}, s_i^l\right)\right)$ holds for all $\sigma \in \Delta$, $i \in \mathcal{N}$, $s_i^k, s_i^l \in \mathcal{S}_i$. Nachbar calls a dynamics *sign preserving* if $\text{sign}\, f_i^k(\sigma) = \text{sign}\left(U_i\left(\sigma_{-i}, s_i^k\right) - U_i(\sigma)\right)$ holds for all $\sigma \in \Delta$, $i \in \mathcal{N}$, $s_i^k \in \mathcal{S}_i$. (Weibull (1995) speaks of a "payoff positive dynamics".) It is immediate to verify that a sign preserving dynamics in the sense of Nachbar and a monotone dynamics reinforces best replies.[8] Notice, however, that Ritzberger and Weibull (1995) already call a dynamics sign preserving if $f_i^k(\sigma) < 0$ holds whenever $U_i\left(\sigma_{-i}, s_i^k\right) < U_i(\sigma)$ holds and vice versa. This weaker notion does not imply our Property (B).

The replicator dynamics is both monotone and sign preserving.

Recall that an asymptotically stable set of rest points is isolated.

Proposition 3 *Suppose the regular selection dynamics under consideration satisfies Properties (A) and (B) and is either differentiable or satisfies Property (D). Then an isolated and stable set of rest points of the dynamics for which every connected component contains a pure strategy combination is a SE set.*

[8]A detailed proof for monotone dynamics is given in Balkenborg and Schlag (2003).

Note that the statement is not correct without the restriction to sets that contain a pure strategy combination in each connected component. Matching Pennies is a counterexample. Its unique equilibrium is interior and is hence not contained in a SE. However, it is asymptotically stable under the "normalized" replicator dynamics (see Maynard Smith (1982), Appendix J, or Weibull (1995), p. 199ff) or various imitation dynamics (see Hofbauer and Schlag (2000)).

Proof. Let T be the non-empty set of all pure strategy combinations in a set of rest points R with the required properties. We will first apply Lemma 2 to show that $\mathcal{G}(T)$ is a SE set. By Lemma 4 (a) T consists of Nash equilibria. Hence it remains to be shown for all $s \in T$ and $s_i^k \in \mathcal{S}_i$ that $u_i\left(s_{-i}, s_i^k\right) = u_i(s)$ implies $\left(s_{-i}, s_i^k\right) \in T$. If this equality holds then both s_i and s_i^k are best replies to s. Property (A) implies that all strategy combinations σ^α $(0 \leq \alpha \leq 1)$ defined as in the proof of Lemma 4 (a) are rest points. Since R is an isolated set of rest points it must contain all σ^α and in particular the strategy combination $\sigma^1 = \left(s_{-i}, s_i^k\right)$, which was to be proven.

Thus $\mathcal{G}(T)$ is a SE set, consists of Nash equilibria and hence of rest points. Since every connected component of $\mathcal{G}(T)$ intersects the isolated set of rest points R, $\mathcal{G}(T)$ is contained in R.

If the dynamics is differentiable we know from Lemma 4 (b) that R consists of Nash equilibria. Because $\mathcal{G}(T)$ is a finite union of Nash equilibrium components (by Corollary 1) we can find a neighborhood of $\mathcal{G}(T)$ that contains no Nash equilibria outside $\mathcal{G}(T)$. Hence every connected component of R that intersects $\mathcal{G}(T)$ is contained in $\mathcal{G}(T)$. Therefore R is contained in $\mathcal{G}(T)$ and we conclude $\mathcal{G}(T) = R$.

If the dynamics satisfies Property (D) we can instead argue as follows. By Lemma 5 $\mathcal{G}(T)$ is an isolated set of rest points. Since every connected component of R intersects $\mathcal{G}(T)$ by assumption it must hence be contained in $\mathcal{G}(T)$. Again we obtain $R \subseteq \mathcal{G}(T)$

and hence $R = \mathcal{G}(T)$. ■

We can now prove the "if" statement in Theorem 1.

Proof. Given Proposition 3 all we have to show is that every connected component of an asymptotically stable set of rest points under the replicator dynamics contains a pure strategy combination. We first show that each of the connected components is itself asymptotically stable. The rest points of the replicator dynamics are precisely the "partial equilibria" described earlier. These form a closed semi-algebraic set and are therefore a finite union of closed, connected components. Given the definition of asymptotic stability it follows that each component is asymptotically stable.

Next we show that any given connected asymptotically stable set R contains a pure strategy combination. Consider a strategy combination σ contained in this set with minimal support. Let $\mathfrak{F}$ be the face spanned by the support of this strategy combination. The intersection $R \cap \mathfrak{F}$ is asymptotically stable for the replicator dynamics restricted to $\mathfrak{F}$, which is the replicator dynamics of the game restricted to this face. By minimality, σ is a pure strategy combination or $R \cap \mathfrak{F}$ is contained in the relative interior of $\mathfrak{F}$. Proposition 6 in Ritzberger and Weibull (1995) states that the latter cannot be true and hence σ is a pure strategy combination. ■

5 When are SE sets asymptotically stable?

For dynamics other than the replicator dynamics we can prove a converse statement to Proposition 3 only under an additional restriction which is always satisfied in two-player games even if the SE set is a cycle as in Remark 2 in Section 2 when $N = 2$ and $K = 3$. It is not always satisfied in games with more than three players as an example below will show. It still holds for the cycle in Remark 2 in Section 2 when $N = 3$ and $K = 2$ and for every convex SE set.

Lemma 6 *Suppose G is a SE set in a two-player game or a convex SE set. Then G satisfies the property:*

Every strategy combination σ in G has a neighborhood $V(\sigma)$ such that

$$\sigma \in G \text{ and } \rho \in V(\sigma) \cap \tilde{\beta}(\sigma) \text{ implies } \sigma \in \tilde{\beta}(\rho). \quad (*)$$

Proof. Choose $V(\sigma) = \Delta$. Consider $\sigma \in G$ and $\rho \in \tilde{\beta}(\sigma)$. Assume that there are only two players. Then $(\sigma_{-i}, \rho_i) \in G$ for $i = 1, 2$ and hence $\sigma \in \tilde{\beta}(\rho)$. Alternatively, assume that G is convex. Then $\sigma \in \tilde{\beta}(\rho)$ by Proposition 1 (iii). In both cases Property (*) holds for $V(\sigma) = \Delta$. ■

Proposition 4 *Every SE set satisfying Property (*) is an asymptotically stable set of stable rest points for any regular selection dynamics satisfying Properties (A), (B) and (C).*

Notice that if G is an asymptotically stable set of stable rest points then all trajectories starting sufficiently close to G converge to an element G. Thus Property (*) causes trajectories starting near to a SE set to converge to elements of the set.

Although the proof of Proposition 4 is complicated by the fact that we do not impose differentiability the intuition behind it is simple. Because of Property (C) all non-best replies die out sufficiently fast. Consequently, the local behavior of the dynamics near an element σ in the SE set is characterized by its behavior on the face $\tilde{\beta}(\sigma)$.[9] For trajectories starting in a neighborhood of σ in $\tilde{\beta}(\sigma)$, Properties (A) and (*) imply that the frequency

[9]When the dynamics is differentiable $\tilde{\beta}(\sigma)$ can be shown to be a center manifold. If it is twice differentiable this part of the argument follows then directly from the reduction principle for center manifolds, see Anosov and (Eds.) (1988) Part I, §4.3.

of every pure strategy in the support of σ cannot decrease over time. This causes σ to be stable. Property (B) can then be used to ensure that trajectories converge to Nash equilibria near σ which by Corollary 1 must be in the SE set.

Proof. Fix a strategy combination σ in the SE set G. σ is a rest point as σ is a Nash equilibrium. Let $\mathfrak{F}(\sigma) = \times_{i \in \mathcal{N}} \mathfrak{F}(\sigma_i) \subseteq G$ be the face generated by σ.

We define for $\rho \in \Delta$

$$\gamma(\rho) := \sum_{i \in \mathcal{N}} \sum_{s_i^k \notin \beta_i(\sigma)} \rho_i^k \qquad \text{and} \qquad \dot{\gamma}(\rho) := \sum_{i \in \mathcal{N}} \sum_{s_i^k \notin \beta_i(\sigma)} \rho_i^k f_i^k(\rho).$$

With these definitions $\frac{d\gamma(\rho^t)}{dt} = \dot{\gamma}(\rho^t)$ holds along every trajectory $\{\rho^t\}$. Since the dynamics satisfies Property (C) we have $f_i^k(\sigma) < 0$ for all $s_i^k \notin \beta_i(\sigma)$. Because each $f_i^k(\sigma)$ is continuous there exists a constant $c > 0$ such that $f_i^k(\rho) < -c$ holds for all $i \in \mathcal{N}$ and all $s_i^k \notin \beta_i(\sigma)$ in a sufficiently small neighborhood of σ. Therefore

$$\dot{\gamma}(\rho) \leq -c\gamma(\rho) \leq 0 \tag{8}$$

holds in such a neighborhood.

Since the dynamics is Lipschitz continuous we can find a constant $L > 0$ such that the inequality

$$\sum_{i \in \mathcal{N}} \sum_{s_i^l \in \mathcal{S}_i} \left| \rho_i^l f_i^l(\rho) - \hat{\rho}_i^l f_i^l(\hat{\rho}) \right| \leq L \sum_{i \in \mathcal{N}} \sum_{s_i^l \in \mathcal{S}_i} \left| \rho_i^l - \hat{\rho}_i^l \right| \tag{9}$$

holds for all $\rho, \hat{\rho} \in \Delta$ sufficiently close to σ.

We now show that the sets

$$V_\delta(\sigma) = \left\{ \rho \in \Delta \;\middle|\; \begin{array}{c} \gamma(\rho) < \delta \quad \text{and} \\ \rho_i^k > \sigma_i^k + \frac{2L}{c}\gamma(\rho) - \frac{2L}{c}\delta \quad \text{for all } i \in \mathcal{N},\ s_i^k \in \operatorname{supp}(\sigma_i) \end{array} \right\}$$

indexed by $\delta > 0$ form a basis of neighborhoods of σ. Notice that each $V_\delta(\sigma)$ is an open set containing σ and hence a neighborhood of σ. The claim then follows once we show that every neighborhood $U_\varepsilon(\sigma) = \left\{\rho \in \Delta \mid \left|\rho_i^k - \sigma_i^k\right| < \varepsilon\right\}$ with $\varepsilon > 0$ contains a neighborhood $V_\delta(\sigma)$. Given $\rho \in V_\delta(\sigma)$ we have for each $i \in \mathcal{N}$ that $\rho_i^k - \sigma_i^k > -\frac{2L}{c}\delta$ holds for all $s_i^k \in \operatorname{supp}(\sigma_i)$ and also for all $s_i^k \notin \operatorname{supp}(\sigma_i)$ since then $\sigma_i^k = 0$. This implies for all $s_i^k \in \mathcal{S}_i$

$$\rho_i^k - \sigma_i^k = \sum_{l \neq k} \left(\sigma_i^l - \rho_i^l\right) < (K_i - 1)\frac{2L}{c}\delta.$$

Therefore $V_\delta(\sigma) \subseteq U_\varepsilon(\sigma)$ whenever $\delta \leq \frac{c}{2L(K-1)}\varepsilon$ where $K = \max_{i \in \mathcal{N}} K_i$.

Next we establish a useful inequality. Represent each $\rho_i \in \Delta_i$ as $\rho_i = (1-\lambda_i)\tau_i + \lambda_i \nu_i$ where $\lambda_i \in [0,1]$, $\tau_i \in \tilde{\beta}_i(\sigma)$ and $\operatorname{supp}(v_i) \cap \beta_i(\sigma) = \emptyset$. For this representation

$$\sum_{i \in \mathcal{N}} \sum_{s_i^l \in \mathcal{S}_i} \left|\rho_i^l - \tau_i^l\right| = 2\sum_{i \in \mathcal{N}} \lambda_i = 2\gamma(\rho). \tag{10}$$

Fix i and k such that $s_i^k \in \operatorname{supp}(\sigma_i)$. Assumption (*) implies for $\tau = (\tau_i)_{i \in \mathcal{N}} \in \tilde{\beta}(\sigma)$ that each s_i^k is a best reply to τ. Hence $f_i^k(\tau) \geq 0$ holds since the dynamics satisfies Property (A). (9) and (10) yield

$$-\rho_i^k f_i^k(\rho) \leq -\rho_i^k f_i^k(\rho) + \tau_i^k f_i^k(\tau) \leq L \sum_{j \in \mathcal{N}} \sum_{s_j^l \in \mathcal{S}_j} \left|\rho_j^l - \tau_j^l\right| = 2L\gamma(\rho).$$

Together with (8) we obtain

$$\rho_i^k f_i^k(\rho) \geq \frac{2L}{c}\dot{\gamma}(\rho)$$

for all ρ sufficiently close to σ.

Now we are ready to show that $V_\delta(\sigma)$ is forward invariant for sufficiently small $\delta > 0$ which means that σ is Lyapunov stable. Let $\delta > 0$ be sufficiently small such that (8) and (9) hold for all $\rho \in V_\delta(\sigma)$. Let $\{\rho(t)\}_{t \geq 0}$ be a trajectory with $\rho(0) \in V_\delta(\sigma)$. Let $T > 0$ be the first time where the trajectory leaves $V_\delta(\sigma)$. We must have $\gamma(\rho(T)) < \delta$ since $\dot{\gamma}(\rho) \leq 0$ holds for all $\rho \in V_\delta(\sigma)$. Moreover, for all $s_i^k \in \operatorname{supp}(\sigma_i)$, $i \in \mathcal{N}$,

$$\rho_i^k(T) - \rho_i^k(0) = \int_0^T \rho_i^k(t) f_i^k(\rho(t))\,dt \geq \frac{2L}{c}\int_0^T \dot{\gamma}(\rho(t))\,dt = \frac{2L}{c}\left(\gamma(\rho(T)) - \gamma(\rho(0))\right)$$

and hence

$$\rho_i^k(T) \geq \rho_i^k(0) + \frac{2L}{c}\gamma(\rho(T)) - \frac{2L}{c}\gamma(\rho(0)) > \sigma_i^k + \frac{2L}{c}\gamma(\rho(T)) - \frac{2L}{c}\delta$$

so that $\rho(T) \in V_\delta(\sigma)$, a contradiction.

Finally, we have to show that G is asymptotically stable which follows once we show for all sufficiently small $\delta > 0$ that the ω-limit set of any trajectory $\{\rho(t)\}_{t\geq 0}$ starting in $V_\delta(\sigma)$ is contained in G. Let $\hat{\rho} \in V_\delta(\sigma)$ be an ω-limit point of a trajectory $\{\rho(t)\}_{t\geq 0}$ starting in $V_{\delta/2}$. By Corollary 1 we can assume that the only Nash equilibria in $V_\delta(\sigma)$ belong to G. We can also assume that $\operatorname{supp}(\sigma) \subseteq \operatorname{supp}(\rho)$ for all $\rho \in V_\delta(\sigma)$. $\gamma(\rho(t))$ is non-increasing along the trajectory and hence (see Hofbauer and Sigmund (1998) Theorem 2.6.1) $\dot{\gamma}(\hat{\rho}) = 0$. Because $\dot{\gamma}(\rho) < 0$ for all $\rho \in V_\delta(\sigma) \setminus \tilde{\beta}(\sigma)$ we conclude $\hat{\rho} \in \tilde{\beta}(\sigma)$. By (*) it follows that each $s_i^k \in \operatorname{supp}(\sigma)$ is a best reply to $\hat{\rho}$. If $\hat{\rho}$ were not a Nash equilibrium we could find by Property (B) of the dynamics a player $i \in \mathcal{N}$ such that $f_i^k(\hat{\rho}) > 0$ and hence $\hat{\rho}_i^k f_i^k(\hat{\rho}) > 0$ would hold for all $s_i^k \in \operatorname{supp}(\sigma)$. Then $\rho_i^k(t)$ would be strictly increasing for sufficiently large t. Theorem 2.6.1 in Hofbauer and Sigmund (1998) would yield the contradiction $\hat{\rho}_i^k f_i^k(\hat{\rho}) = 0$. Consequently $\hat{\rho}$ is a Nash equilibrium and therefore in G. ■

Central to our proof is our use of Property (*) to show that every strategy combination in a SE set is stable. We derive asymptotic stability of the set as a consequence. We do not know whether SE sets are still asymptotically stable under any dynamics that reinforces better replies if (*) is not satisfied. However, the following example of a SE set in a $2 \times 2 \times 2$-game shows that one can no longer expect stability of each strategy combination in the set. The dynamics we construct satisfies all our conditions and is actually *aggregate monotonic* in the sense of Samuelson and Zhang (1992).

In the game in Figure 2 player 1 chooses one of the rows U or D, player 2 one of the columns L or R and player 3 one of the matrices F or B. Let $x_1 = \operatorname{prob}(D)$, $x_2 = \operatorname{prob}(R)$ and $x_3 = \operatorname{prob}(B)$. We describe mixed strategy combinations by vectors (x_1, x_2, x_3).

F

	L	R
U	0 0 0	0 1 0
D	1 0 1	1 0 1

B

	L	R
U	1 0 1	1 0 1
D	1 0 1	1 0 1

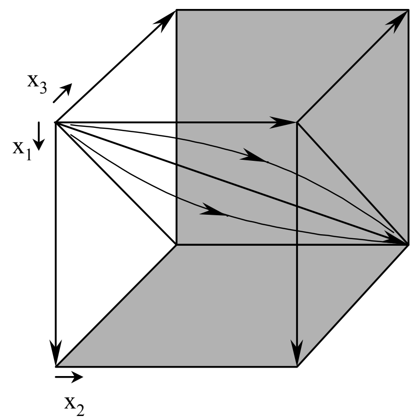

Figure 2: A game with an SE set not satisfying Property (*) where not all Nash equilibria are stable.

The set $G = \{x_1 = 1\} \cup \{x_3 = 1\}$ is a SE set of the game. It is shaded in the graphic. This SE set does not satisfy property (*) because any strategy combination $(x_1, x_2, x_3) = (1-\varepsilon, 0, 1-\varepsilon)$ with $0 < \varepsilon < 1$ is a best reply to $(D, L, B) \in G$ while (D, R, B) is the unique best reply to (x_1, x_2, x_3).

Consider the following dynamics:

$$\dot{x}_1 = x_1 (1 - x_1) (1 - x_3)^2$$

$$\dot{x}_2 = x_2 (1 - x_2) (1 - x_1) (1 - x_3)$$

$$\dot{x}_3 = x_3 (1 - x_3) (1 - x_1)^2$$

It is obtained from the replicator equations by squaring the terms $(1 - x_3)$ and, respectively, $(1 - x_1)$ in the equations for $\dot{x}_1$ and $\dot{x}_3$ so that the speed of a trajectory in the x_1

and x_3 direction is slowed down while the speed in the x_2 direction is unaltered. It is not difficult to see that G is an asymptotically stable set of rest points. We claim that the points $(1, x_2, 1) \in G$ with $x_2 < 1$ are not stable.

By symmetry, the plane $\{x_1 = x_3\}$ is invariant. Consider any trajectory $x(t)$ starting in the relative interior of this plane. Then $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_3(t) = 1$. Separation of variables shows that

$$x_2(t) = \frac{cx_1(t)}{1 - x_1(t) + cx_1(t)}$$

holds for an appropriate constant $c > 0$. Therefore, $\lim_{t\to\infty} x_2(t) = 1$. Thus we can find trajectories starting arbitrarily close to any $(1, x_2, 1)$ which converge to $(1, 1, 1)$.

Bringing together Propositions 3, 4 and Lemma 6 we obtain the following characterization results for dynamics other than the replicator dynamics:

Theorem 2 *Consider a regular selection dynamics reinforcing best replies and a non-empty set G for which every connected component contains a pure strategy combination.*

a) If the game is a two-player game then G is an asymptotically stable set of stable rest points iff G is a strict equilibrium set.

b) If G is convex then G is an asymptotically stable set of rest points iff G is a strict equilibrium set.

Moreover, we obtain for two-player games or convex sets that elements of asymptotically stable sets of rest points are stable. This means that trajectories starting sufficiently close to the set converge to a point in the set. Starting nearby to the set a learning process will not only lead to convergence to the set (which does not exclude the possibility of endless cycling) but to convergence towards a single form of behavior.

6 Conclusion

Conditions for meeting the requirements of a SE set are easily verified in specific examples which makes the results of this paper very applicable. For illustration we point out two evolutionary investigations where the existence of SE sets is now easily verified. The set of Nash equilibria inducing the forward induction outcome for the twice repeated battle-of-the-sexes game (between two players) selected in van Damme (1989) is a SE set and thus asymptotically stable under any dynamics that reinforces best replies. The set of efficient equilibria in the action commitment game between two players studied by van Damme and Hurkens (1996) is also a SE set. In this latter case the notion of a SE set refines their solution concept which also included inefficient outcomes. Further examples of SE sets in repeated games are contained in Balkenborg (1995), for cheap talk games see Schlag (1994).

Unfortunately SE sets often do not exist. In such cases, following Proposition 3, there will be no asymptotically stable set of rest points that contains a pure strategy in each component. In order to investigate whether other asymptotically stable sets of rest points exist, following Theorem 1, more information about the specific dynamics is needed. Moreover, it may be necessary to impose weaker dynamic conditions on the sets predicted, e.g. to abandon the restriction to rest points as in Ritzberger and Weibull (1995), or to consider asymptotic stability only with respect to interior trajectories as in Binmore and Samuelson (1994), Cressman (1996) and Cressman and Schlag (1997).

We conjecture that SE sets are asymptotically stable even if Property (*) is not satisfied and hence the SE set does not have to consist of stable rest points. However, our method of proof cannot be used to show this because it used the stability of the rest points to show the asymptotic stability of the entire set. Additional interesting topics for future research include investigating necessary conditions for asymptotic stability combined with point stability and analyzing how dynamics behave close to a SE set when payoffs are perturbed.

References

Anosov, D., and V. A. (Eds.) (1988): *Dynamical Systems*. Springer Verlag, Berlin, Heidelberg, New York.

Aumann, R. J., and S. Hart (1986): "Bi-Convexity and Bi-Martingales," *Israel Journal of Mathematics*, 54, 159 – 180.

Balkenborg, D. (1994): "Strictness and Evolutionary Stability.," Disc. Paper 52, The Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, http://www.exeter.ac.uk/∼dgbalken.

——— (1995): "Strictness, Evolutionary Stability and Repeated Games with Common Interests," SFB Disc. Paper B-305, University of Bonn, http://www.exeter.ac.uk/∼dgbalken/index.html.

Balkenborg, D., and K. Schlag (2003): "On the Evolutionary Selection of Nash Equilibrium Components: Supplementary Material," http://www.exeter.ac.uk/∼dgbalken/papers, pdf file.

Balkenborg, D., and K. H. Schlag (2001): "Evolutionarily Stable Sets," *International Journal of Game Theory*, 29, 571 – 595.

Binmore, K. G., and L. Samuelson (1994): "Drift," *European Economic Review*, 38, 851 – 867.

Borgers, T., and R. Sarin (1997): "Learning Through Reinforcement and Replicator Dynamics," *Journal of Economic Theory*, 77, 1 – 14.

Cressman, R. (1996): "Evolutionary Stability in the Finitely Repeated Prisoner's Dilemma Game," *Journal of Economic Theory*, 68, 234 –248.

Cressman, R., and K. H. Schlag (1997): "The Dynamic (In)Stability of Backwards Induction," *Journal of Economic Theory*, 26, 260 – 285.

DeMichelis, S., and K. Ritzberger (2000): "From Evolutionary to Strategic Stability," mimeo, forthcoming in the Journal of Economic Theory.

Eshel, I., and E. Akin (1983): "Coevolutionary Instability of Mixed Nash Solutions.," *Journal of Mathematical Biology*, 118, 123 – 133.

Fudenberg, D., and D. K. Levine (1998): *The Theory of Learning in Games*. MIT Press, Cambridge, Massachussetts.

Gale, J., K. Binmore, and L. Samuelson (1995): "Learning to Be Imperfect — the Ultimatum Game," *Games and Economic Behaviour*, 8, 56 – 90.

Hirsch, M. W., and S. Smale (1974): *Differential Equations, Dynamical Systems and Linear Algebra*. Academic Press, New York, San Francisco, London.

Hofbauer, J. (1995a): "Imitation Dynamics for Games," Collegium Budapest, Hungary.

——— (1995b): "Stability for the Best Response Dynamic," mimeo.

Hofbauer, J., and K. H. Schlag (2000): "Sophisticated Imitation in Cyclic Games," *Journal of Evolutionary Economics*, 10, 523 – 543.

Hofbauer, J., and K. Sigmund (1998): *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge.

Hopkins, E. (1999): "A Note on the Best Response Dynamics," *Games and Economic Behavior*, 29, 138 – 150.

Maynard Smith, J. (1982): *Evolution and the theory of games.* Cambridge University Press, Cambridge.

Monderer, D., and L. Shapley (1996): “Potential Games,” *Games and Economic Behavior*, 14, 124–143.

Nachbar, J. H. (1990): ““Evolutionary” Selection Dynamic in Games: Convergence and Limit Properties,” *International Journal of Game Theory*, 19, 59 – 90.

Ritzberger, K., and K. Vogelsberger (1989): “The Nash Field,” mimeo.

Ritzberger, K., and J. W. Weibull (1995): “Evolutionary Selection in Normal Form Games.,” *Econometrica*, 63, 1371 –1399.

Samuelson, L., and J. Zhang (1992): “Evolutionary Stability in Asymmetric Games,” *J. Econ. Theory*, 57, 363 – 391.

Schlag, K. H. (1994): “When Does Evolution Lead to Efficiency in Communication Games?,” SFB Disc. Paper B-299, University of Bonn, ftp://ftp.econ2.uni-bonn.de/papers/1994/b/bonnsfb299.pdf.

——— (1998): “Why Imitate, and If so, How? A Boundedly Rational Approach to Multi-Armed Bandits,” *Journal of Economic Theory*, 78, 130 – 156.

Selten, R. (1980): “A Note on Evolutionarily Stable Strategies in Asymmetric Animal Conflicts,” *J. Theor. Biol.*, 84, 93 – 101.

Sobel, J. (1993): “Evolutionary Stability and Efficiency,” *Economic Letters*, 42, 301 – 312.

Swinkels, J. (1992): “Evolutionary Stability with Equilibrium Entrants,” *Journal of Economic Theory*, 57, 306–332.

Taylor, P. D. (1979): “Evolutionary stable strategies with two types of players.,” *J. Appl. Prob.*, 16, 76 – 83.

Thomas, B. (1985): "On Evolutionarily Stable Sets," *J. Math. Biology*, 22, 105 – 115.

van Damme, E. (1989): "Stable Equilibria and Forward Induction.," *Journal of Economic Theory*, 48, 467 –496.

van Damme, E., and S. Hurkens (1996): "Commitment Robust Equilibria and Endogeneous Timing," *Games and Economic Behavior*, 15, 290 – 311.

Weibull, J. W. (1995): *Evolutionary Game Theory.* MIT Press, Cambridge, Massachusetts.

Zeeman, E. C. (1980): "Population Dynamics from Game Theory," in *Global Theory of Dynamical Systems. Lecture Notes in Math. 819*, ed. by Z. Nitecki, and C. Robinson, pp. 471 – 497, Berlin, Heidelberg, New York. Springer-Verlag.