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*Seigniorage-maximizing inflation**

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ABSTRACT

What is the seigniorage-maximizing level of inflation? Four models formulae for the seigniorage maximizing inflation rate (SMIR) are compared. Two sticky-price models arrive at very different quantitative recommendations although both predict somewhat lower SMIRs than Cagan's formula and a variant of a .ex-price model due to Kimbrough (2006). The models differ markedly in how inflation distorts the labour market: The Calvo model implies that inflation and output are negatively related and that output is falling in price stickiness whilst the Rotemberg cost-of-price-adjustment model implies exactly the opposite. Interestingly, if our version of the Calvo model is to be believed, the level of inflation experienced recently in advanced economies such as the USA and the UK may be quite close to the SMIR.

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1. Introduction

What is the seigniorage-maximizing level of inflation? The answer provided by Cagan (1956) was to set the inflation tax rate equal to the inverse of the interest semi-elasticity of the demand for money. Modern general equilibrium treatments of the issue allow that question to be posed in richer theoretical environments. For example, Easterly et al. (1995), endogenized the money demand function via a cash-in-advance constraint resulting in a variable semi-interest elasticity of money demand. They argued that this qualifies in important ways the conclusions of the 'Cagan rule'. Recently, Kimbrough (2006) has argued that, in addition to modelling money demand, it is important to model the labour-leisure choice. Failing to do so ignores the real effects of inflation and hence its impact on the inflation-tax base. He develops a flexible-price, general equilibrium model with an endogenous labour-leisure choice and with a role for money in reducing transactions costs associated with consumption. He argues that inflation reduces the effectiveness of money and causes a substitution toward leisure. The combined effect (on money demand and labour supply) implies a maximizing level of seigniorage lower than prescribed by the Cagan rule.

This paper incorporates the potential for price stickiness to impact the seigniorage maximizing inflation rate (SMIR). Two versions of popular models of price stickiness are developed, the Calvo (1983) model and the Rotemberg (1982) quadratic-costs-of-price-adjustment model (see also Schmitt-Grohe and Uribe, 2004). Similar along some obvious dimensions, these models nevertheless differ in important ways on how inflation distorts the real economy. Consequently, they arrive at quantitatively different conclusions regarding the SMIR. The quantitative conclusions of the sticky-price models are compared with a close variant of the flex-price model developed by Kimbrough (2006) and with the Cagan rule. The key issue that is highlighted is how inflation distorts equilibrium in the labour market and so a careful decomposition is provided of the impact of inflation on the labour markets in the general equilibrium environments developed. Interestingly, the sticky-price models generally suggest a relatively low SMIR; in particular the Calvo model often recommends an inflation rate quite close to that seen recently in many advanced economies, such as the US and the UK.

The rest of the paper has the following structure. Section 2 sets up a model similar to Kimbrough (2006) but extended to incorporate monopolistically competitive producers. The section briefly derives the Cagan rule before modelling firms' price setting behavior following Calvo (1983). Section 3 introduces the

Rotemberg model. Section 4 contains a detailed comparison of the labour markets and of the relationships between inflation and output in the models that we develop. The seigniorage maximizing inflation rates across frameworks are calculated and compared in section 5. Section 6 offers some brief conclusions. Appendices contain a number of derivations and proofs referred to in the text.

2. The core model

The single good, flexible-price, general equilibrium model with an endogenous labour-leisure decision, of Kimbrough (2006) is generalized by introducing monopolistically competitive producers. That additional structure is to facilitate comparison across models as presently we shall incorporate price stickiness. First, the Calvo model is adopted where some firms are 'permitted' to change prices and some are not; in other words there is price-dispersion. Then a sticky price model is developed with no such price dispersion, but where changing prices is costly (the Rotemberg model).

There are a large number of identical agents in the economy who evaluate their utility in accordance with following criterion:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t(i)) = E_0 \sum_{t=0}^{\infty} \beta^t \left(\log(C_t) - \frac{\lambda}{1+v} \left(\int_i N_t(i) di \right)^{1+v} \right). \tag{2.1}$$

E_t denotes the expectations operator at time t, β is the discount factor, C_t is consumption and $N_t(i)$ is the quantity of labour supplied to firm i. $v \geq 0$ measures the labour supply elasticity while λ is a 'preference' parameter.

Consumption is defined over a Dixit-Stiglitz basket of goods

$$C_t = \left[\int_0^1 c_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}. \tag{2.2}$$

The price-level, P_t, is known to be

$$P_t = \left[\int_0^1 p_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}}. \tag{2.3}$$

The demand for each good is given by

$$Y_t(i) = \left(\frac{p_t(i)}{P_t} \right)^{-\theta} Y_t^d, \tag{2.4}$$

where $p_t(i)$ is the nominal price of the final good produced in firm i and Y_t^d denotes aggregate demand.

Agents face the following flow budget constraint

$$P_t C_t \left(1 + T(M_t/P_t/C_t)\right) + B_t + M_t = \left[1 + R_{t-1}\right] B_{t-1} + M_{t-1} + P_t w_t N_t + \Pi_t. \qquad (2.5)$$

Money is held because it helps reduce transaction costs. In order to purchase C units of consumption, the consumer need to spend $(1 + T(m_t))\, C$. The transaction cost function is given by

$$T = T(m_t),\ T' < 0,\ T'' > 0,$$

where $m_t = M_t/P_t/C_t$. A prime denotes $dT(m_t)/dm_t$. As all agents are identical, the only financial assets traded in equilibrium will be those issued by the fiscal authority. Here B_t denotes the nominal value at the end of date t of government bond holdings, $1+R_t$ denotes the nominal interest rate on this 'riskless' one-period nominal asset, w_t denotes the real wage in period t (all agents supply the same labour to each firm, and firms pay the same real wage to labour), and Π_t is a lump sum transfer from firms in the form of dividends.

The first-order conditions imply a labour supply equation (2.6)

$$-\frac{U_N'(C_t, N_t)}{U_C'(C_t, N_t)} = \lambda_t N_t^v C_t = \frac{w_t}{1 + T(m_t) - T'(m_t) m_t}; \qquad (2.6)$$

an (inverse) money demand equation

$$T'(m_t) = \frac{-R_t}{1 + R_t} = -i;$$

and a consumption Euler equation

$$E_t \left\{ \frac{\beta U_C'(C_{t+1}) P_t \left(1 + T(m_t) - T'(m_t) m_t\right)}{U_C'(C_t) P_{t+1} \left(1 + T(m_{t+1}) - T'(m_{t+1}) m_{t+1}\right)} \right\} = \frac{1}{1 + R_t} = 1 - i_t, \qquad (2.7)$$

where we define $i_t := \frac{R_t}{1+R_t}$.

Following Kimbrough we introduce the money demand function $k(i_t)$

$$m_t = k(i_t) = inv(-T'),$$

which implies that $k' = -\frac{1}{T''} < 0$. The complete markets assumption implies the unique nominal stochastic discount factor:

$$Q_{t,t+k} = \frac{\beta^k U_C'(C_{t+1}) P_t \left(1 + T(m_t) - T'(m_t) m_t\right)}{U_C'(C_t) P_{t+k} \left(1 + T(m_{t+k}) - T'(m_{t+k}) m_{t+k}\right)}, \qquad (2.8)$$

where

$$E_t \left\{ Q_{t,t+k} \right\} = E_t \prod_{j=0}^{k} \frac{1}{1 + R_{t+j}}.$$

2.1. Seigniorage maximization

The government maximizes the net present value of its revenue from seigniorage as measured by:

$$S_t = E_t \sum_{k=0}^{\infty} d_{t,t+k} \frac{M_{t+k} - M_{t+k-1}}{P_{t+k}}, \tag{2.9}$$

where $d_{t,t+k}$ is real discount factor defined as $d_{t,t+k} = \prod_{s=t}^{t+k} \frac{1}{1+r_s}$. Inverse inflation can be rewritten as $\frac{1}{\pi_t} = \left(\frac{1+r_t}{1+R_{t-1}}\right) = (1+r_t)(1-i_{t-1})$. Therefore, we may rewrite the preceding expression as

$$S_t = E_t \sum_{k=0}^{\infty} d_{t+k} \frac{M_{t+k}}{P_{t+k}} i_{t+k} - \frac{M_{-1}}{P_0} = E_t \sum_{k=0}^{\infty} d_{t+k} C_{t+k} k(i_{t+k}) i_{t+k} - \frac{M_{-1}}{P_0}. \tag{2.10}$$

In steady state, assuming that $\frac{1}{1+r_s} = \beta$, the inflation tax i is chosen to maximize seigniorage

$$S(i) = C(i)k(i)i. \tag{2.11}$$

2.2. Cagan's rule for the SMIR

Cagan's rule is very familiar to monetary economists. It is derived from (2.11) assuming that the real interest rate and consumption are 'givens'. Then the revenue maximization problem is simply equivalent to the maximization of $k(i_{t+k})i_{t+k}$ in each period. From that we immediately derive Cagan's rule:

$$i_{Cagan} = -\frac{k(i_{Cagan})}{k'(i_{Cagan})}, \quad \forall t.$$

If the money demand elasticity is given by γ, then the 'optimal' inflation tax is $i_{Cagan} = 1/\gamma$. Lucas (2000) suggests a plausible value for γ is in the region of $5-7$, implying the seigniorage maximizing inflation should be in the range $12-20\%$.

However, Kimbrough (2006) points out that high inflation may compromise the effectiveness of real money balances in reducing transaction costs. The upshot is a substitution into leisure and a contraction of the seigniorage tax base. Thus, the seigniorage maximizing inflation rate will be lower than predicted by the simple Cagan rule. We explore the quantitative difference below.

However, sticky price models may suggest additional negative real effects from inflation; other things constant, a higher inflation rate may mean that firms operate less efficiently, shrinking further the tax base. In the Calvo model that shrinkage is quite severe, but may be much less so in the cost-of-adjustment model. The following analysis works out the details of these models.

2.3. Representative firm: factor demand

Let us assume that labour is the only factor of production. Firms are monopolistic competitors who produce their distinctive goods according to the following technology

$$Y_t(i) = A_t \left[N_t(i)\right]^{1/\phi}, \tag{2.12}$$

where $N_t(i)$ denotes the amount of labour hired by firm i in period t, A_t is a productivity shifter. It is assumed that $\phi > 1$ so that there is diminishing returns; that assumption is important as discussed below.

The demand for output determines the demand for labour. Hence one finds that

$$N_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi} \left(\frac{Y_t}{A_t}\right)^{\phi}. \tag{2.13}$$

It follows that the total amount of labour demanded will be

$$N_t = \int N_t(i)di = \left(\frac{Y_t}{A_t}\right)^{\phi} \int \left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi} di = N_t^* \Delta_t, \tag{2.14}$$

where $\Delta_t \left\langle -\theta\phi \right\rangle$ is a measure of price dispersion:

$$\Delta_t \left\langle -\theta\phi \right\rangle \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\theta\phi} di. \tag{2.15}$$

In this simple set-up, were all firms given the chance to re-price at any instant in time, they would all choose the same price. In that case, with all prices similar, for a given level of output the labour supply is

$$N_t^* = \left(A_t^{-1} Y_t\right)^{\phi}. \tag{2.16}$$

Substituting (2.16) into (2.14) one receives

$$N_t = \Delta_t \left(\frac{Y_t}{A_t}\right)^{\phi}. \tag{2.17}$$

This corresponds to the amount of labour which would be employed to produce quantity Y_t should prices not be equal across firms. Finally, it follows from (2.6) that the equilibrium wage can be written as

$$w_t = (1 + T + ik)\, \lambda_t C_t \Delta_t^{v} \left(\frac{Y_t}{A_t}\right)^{\phi v}. \tag{2.18}$$

2.4. Costly price adjustment: The Calvo model

The Calvo (1983) approach to modelling price-stickiness is familiar. Each period a measure, $1-\alpha$, of firms is allowed to adjust prices. Those firms choose the nominal price which maximizes their expected profit given that they may have to charge the same price in $k-$periods time, with probability α^k.

Since firms are cost-takers, the price setting problem can be characterized as follows:

$$\max_{p'_t} E_t \sum_{k=0}^{\infty} \alpha^k d_{t,t+k} \left(Y_{t+k} \left(\frac{p'_t}{P_{t+k}} \right)^{1-\theta} - w_{t+k} A_{t+k}^{-\phi} Y_{t+k}^{\phi} \left(\frac{p'_t}{P_{t+k}} \right)^{-\theta\phi} \right), \tag{2.19}$$

where p'_t is the new price charged by all firms updating prices. w_t is the real wage, $w_t = W_t/P_t$, and $d_{t,t+k}$ is the real discount factor

$$d_{t,t+k} = \frac{\beta^k U_c(C_{t+k}) \left(1 + T(m_t) - T'(m_t)m_t\right)}{U_c(C_t) \left(1 + T(m_{t+k}) - T'(m_{t+k})m_{t+k}\right)}.$$

The first order condition with respect to $p'_t(i)$ implies

$$\left(\frac{p'_t}{P_t} \right)^{1+\theta(\phi-1)} = \frac{\theta\phi}{\theta-1} \frac{E_t \sum\limits_{k=0}^{\infty} (\alpha\beta)^k U_c(C_{t+k}) \left[w_{t+k} A_{t+k}^{-\phi} Y_{t+k}^{\phi} (\frac{P_t}{P_{t+k}})^{-\theta\phi} \right] \Gamma_{t+k}}{E_t \sum\limits_{k=0}^{\infty} (\alpha\beta)^k U_c(C_{t+k}) \left[Y_{t+k} (\frac{P_t}{P_{t+k}})^{1-\theta} \right] \Gamma_{t+k}}, \tag{2.20}$$

where $\Gamma_{t+k} \equiv \left(1 + T(m_{t+k}) - T'(m_{t+k})m_{t+k}\right)^{-1}$. The price index then evolves according to a law of motion

$$P_t = \left[(1-\alpha)\, p_t'^{1-\theta} + \alpha P_{t-1}^{1-\theta} \right]^{1/(1-\theta)}, \tag{2.21}$$

which implies the following relation between the optimal relative price and inflation

$$\frac{p'_t}{P_t} = \left(\frac{1 - \alpha\pi_t^{\theta-1}}{1-\alpha} \right)^{1/(1-\theta)}. \tag{2.22}$$

Because the relative prices of the firms that do not change their prices in period t falls by the rate of inflation, one derives a law of motion for the measure of price dispersion

$$\Delta_t = \alpha\Delta_{t-1}\pi_t^{\theta\phi} + (1-\alpha) \left(\frac{p'_t}{P_t} \right)^{-\theta\phi}. \tag{2.23}$$

In equilibrium market clearing requires

$$C_t \left(1 + T(m_t)\right) = Y_t \tag{2.24}$$

which together with (2.18) yields an expression for the real wage in terms of production:

$$w_t = \frac{(1+T_t+i_t k(i_t))}{1+T_t} Y_t \Delta_t^v \left(\frac{Y_t}{A_t}\right)^{\phi v}. \tag{2.25}$$

This expression for the real wage will prove useful in Section 4.

2.4.1. Steady state

By combining (2.20) with (2.22), and substituting for the real wage (2.25), the relation between output and the inflation is recovered:

$$\frac{1-\alpha\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta-1}} \frac{1-\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta\phi}} \frac{1+T(i)}{1+T(i)+ik(i)} \frac{\theta-1}{\theta\phi} = \Delta Y^{\phi(v+1)}. \tag{2.26}$$

Price dispersion, Δ, can be easily computed using the steady state version of the law of motion (2.23)

$$\Delta = \frac{(1-\alpha)}{(1-\alpha\pi^{\theta\phi})} \left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{\theta\phi}{\theta-1}}.$$

Hence, (2.26) can be rewritten compactly as

$$Y^{\phi(v+1)} = \frac{h(\pi)\sigma(i)}{\Delta^{v+1}(\pi)}, \tag{2.27}$$

where we define $h(\pi) := \frac{1-\alpha\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta-1}} \frac{1-\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta\phi}}$, and $\sigma(i) := \frac{1+T(i)}{1+T(i)+ik(i)}$. It is straightforward to show that $h(\pi)$ increases with inflation and price stickiness. Hence, the function $h(\pi)$ represents the stimulating effect of inflation; since prices are sticky, the aggregate price level is lower than in the case of price flexibility. The function $\sigma(i)$ reflects the transaction costs which consumers have to pay; $\sigma(i)$ decreases in inflation and reduces demand. Finally the price dispersion term, $\Delta^{v+1}(\pi)$, captures the direct costs of production, which reduce supply as inflation rises. It is easy to show that $\Delta^{v+1}(\pi) \geq 1$ and increases in both inflation and price stickiness, α.

Proposition 2.1. *Output declines in the degree of price stickiness, α, for $\pi > 1$.*

A small increase in inflation has offsetting effects: On the one hand it reduces the monopolistic price distortion, tending to boost equilibrium output. However, higher inflation increases price dispersion, pushing equilibrium output lower (for reasons discussed later). By direct differentiation one can show that the negative effects associated with price dispersion dominate.

Proposition 2.2. *Inflation and output are negatively related in the Calvo model for $\pi_{Cagan} > \pi > \pi^*$, where π^* is defined by*

$$\frac{\left(1-\alpha\beta\pi^{\theta-1}\right)\left(1-\beta\pi^{\theta\phi-(\theta-1)}\right)}{\left(1-\alpha\beta\pi^{\theta\phi}\right)} = \frac{(\theta-1)(1-\beta)}{\theta\phi} \tag{2.28}$$

and $\pi^* < (1/\beta)^{1/(\theta\phi-\theta+1)}$, while π_{Cagan} represents the SMIR following Cagan's rule.

Both the upper and lower bounds in proposition 2.2 represent sufficient conditions. The upper bound corresponds to the SMIR in Cagan's model while the lower bound represents the inflation threshold which makes the negative effect from price dispersion dominate the positive effect of a reduction in the monopolistic distortion represented by $h(\pi)$. Using the calibration of Section 5, π^* is of the order of magnitude of one third of one percentage point. See appendix 9.2 for the proof of the propositions.

Moreover, by combining the expression for seigniorage, (2.11), with market clearing, one can derive an expression for the steady state level of seigniorage

$$S_t(i) = \frac{ik(i)}{1+T(i)}Y(i). \tag{2.29}$$

Since equilibrium output declines in price stickiness, α, seigniorage will also decline. Consequently, since the flexible-price model is nested to the Calvo model by setting $\alpha = 0$, one concludes that for any given level of inflation, seigniorage revenue in the Calvo model is lower than predicted by the flexible-price framework.

It follows that the SMIR is lower in the Calvo model than in a model with flexible prices. Combining (2.29) with (2.27) one establishes the following relation between seigniorage revenues

$$S\_Calvo(\pi) = S\_flex(\pi) \times \tau(\pi),$$

where $\tau(\pi) = \left(\frac{h(\pi)}{\Delta^{v+1}(\pi)}\right)^{1/\phi(v+1)}$ is a positive, declining function of π, for $\pi > \pi^*$ (See Appendix). Therefore at the the level of inflation which maximizes seigniorage in the flexible-price model, π_{flex}, seigniorage in the Calvo set-up is declining. Hence, by reducing inflation the government will be able to increase seigniorage revenue. That establishes the following proposition:

Proposition 2.3. *The seigniorage-maximizing inflation rate is smaller in the Calvo model than in the flexible price model for $\pi_{flex} > \pi^*$.*

Propositions 2.1, 2.2 and 2.3 do *not* hold in the costs-of-price-adjustment model, as we now show.

3. Costly price adjustment: The Rotemberg model

We follow Rotemberg (1992) and Schmitt-Grohe and Uribe (2004) and now introduce sluggish price adjustment by assuming that the firm faces a resource cost that is quadratic in the inflation rate of the good it produces:

$$\text{Price adjustment cost} = \frac{\rho}{2}\left(\frac{P_t(i)}{P_{t-1}(i)} - 1\right)^2,$$

where $\rho > 0$ is a measure of price stickiness.

The firm sets its price to maximize the net present value of future profits:

$$\max E_t \sum_{k=0}^{\infty} d_{t,t+k}\left(Y_{t+k}\left(\frac{P_{t+k}(i)}{P_{t+k}}\right)^{1-\theta} - w_{t+k}N_{t+k}(i) - \frac{\rho}{2}\left(\frac{P_{t+k}(i)}{P_{t+k-1}(i)} - 1\right)^2\right).$$

The first-order condition with respect to $\frac{P_t(i)}{P_t}$ can be derived in a straightforward way (as detailed in the appendix) and, since all the firms charge the same price, in equilibrium $\frac{P_t(i)}{P_t} = 1$. That gives rise to the Phillips relation:

$$(\theta - 1) Y_t - \theta\phi w_t A_t^{-\phi} Y_t^{\phi} + \rho\pi_t(\pi_t - 1) = E_t d_{t+1}(\rho\pi_{t+1}(\pi_{t+1} - 1)).$$

When wages are flexible, the average wage is given by

$$w_t = (1 + T + ik) C_t \left(\frac{Y_t}{A_t}\right)^{\phi v}. \tag{3.1}$$

The cost of price adjustment will be transferred to the consumer through a reduction in profit share. Consequently, the household budget constraint implies the following market clearing condition

$$C_t = \frac{Y_t - \frac{\rho}{2}(\pi_t - 1)^2}{(1 + T_t)}. \tag{3.2}$$

Let inflation be constant and let $d_{t,t+t} = \beta$. It follows that output is solved for using

$$\begin{aligned} & \theta\phi\lambda\frac{(1 + T(i) - k(i)i)}{(1 + T(i))}\left(Y - \frac{\rho}{2}(\pi - 1)^2\right) Y^{\phi(v+1)} - (\theta - 1) Y \\ = \; & \rho(1 - \beta)\pi(\pi - 1). \end{aligned} \tag{3.3}$$

By direct differentiation of (3.3), one can prove propositions 3.1 and 3.2 (see the appendix for the details). These imply that the relationships between output, inflation and price stickiness in the Rotemberg model differ in important ways from those described in Propositions 2.2 and 2.1 for the Calvo set up.

Proposition 3.1. *In the model with costly price adjustment, output increases with price stickiness, for any given level of inflation.*

Proposition 3.2. *In the model with costly price adjustment, output increases with inflation, for any level of inflation smaller than Cagan's SMIR.*

Proof. See the appendix and the further discussion in Section 4. ■

Defining seigniorage as follows

$$S_t(i) = \frac{ik(i)}{1+T(i)} \left(Y(i) - \frac{\rho}{2} (\pi - 1)^2 \right),$$

one finds that, generally speaking, equilibrium output increases with inflation as well as with price stickiness. However, the effect on consumption and seigniorage is ambiguous. It depends crucially on the concavity of the production function, ϕ, and the elasticity of labour supply, v. Those two parameters are responsible for the convexity of the average labour costs. Indeed, from (3.1) one can conclude that the average labour cost is given by

$$Cost(Y) = wN = (1 + T + ik)\, C \left(\frac{Y}{A} \right)^{\phi(v+1)},$$

from which one calculates the cost elasticity of supply:

$$\frac{\partial Cost}{\partial Y} \frac{Y}{Cost} = \phi (v+1).$$

When that elasticity is small, the seigniorage-optimizing inflation rate may be *higher* in the Rotemberg model than in the flexible-price framework.

Proposition 3.3. *When the "convexity of average labour cost" is small,* $\phi (v+1) \simeq 1$*, consumption and seigniorage revenue increase with* ρ *for any given level of inflation.*

Proof. See the appendix. ■

However, for a reasonable calibration, the seigniorage maximizing inflation is indeed lower in the Rotemberg model than in our variant of Kimbrough's flex-price framework.

The key to understanding the results above turns, to a large degree, on how the different models perceive inflation to disrupt the workings of the labour market. In the following Section we spell out in detail some of the key effects.

4. Inflation and labour market distortions

Labour supply: Besides the direct effect of inflation on consumption via higher transaction costs, inflation tends to introduce significant distortions into the labour market in all three general equilibrium models. As is clear from Kimbrough (2006), workers demand higher wages for a given level of labour supply

$$w^s\_flex_t = (1 + T + ik(i))\, C_t N_t^v;$$

and for a given level of production, the wage is determined as

$$w^s\_flex_t = \sigma^{-1}(i) Y_t \left(\frac{Y_t}{A_t}\right)^{\phi v},$$

where $\sigma^{-1}(i)$ increases with inflation on the interval $[1, i_{cagan}]$.

By contrast, in the Calvo model with relative price distortion, more labour is required for the same production level as in the flexible-price environment:

$$N_t = \Delta_t \left(\frac{Y_t}{A_t}\right)^{\phi} > \frac{Y_t}{A_t}.$$

Hence, in the Calvo model, for given level of production, the wage demanded by labour increases in inflation:

$$\begin{aligned} w^s\_Calvo_t &= \frac{(1 + T + ik(i))}{1 + T} Y_t \Delta_t^v \left(\frac{Y_t}{A_t}\right)^{\phi v} = \sigma^{-1}(i) Y_t \Delta_t^v \left(\frac{Y_t}{A_t}\right)^{\phi v}; \\ w^s\_Calvo_t &= \Delta_t^v(i) \times w^s\_flex_t\,(Y_t) > w^s\_flex_t\,(Y_t), \end{aligned}$$

where $\Delta_t^v(i) > 1$ and is increasing in inflation.

However, in the Rotemberg model there is no relative price distortion, although consumers bear the costs of price adjustment. So,

$$\begin{aligned} w^s\_Rotemberg &= \sigma^{-1}(i) \left(\frac{Y_t}{A_t}\right)^{\phi v} \left(Y_t - \frac{\rho}{2}(\pi_t - 1)\right); \\ w^s\_Rotemberg &= w^s\_flex_t\,(Y_t) - \frac{\rho}{2}(\pi_t - 1)\,\sigma^{-1}(i) \left(\frac{Y_t}{A_t}\right)^{\phi v} < w^s\_flex. \end{aligned}$$

In other words, for a given level of output, wages are *lower* in the Rotemberg model as compared with the flexible-price model; the supply schedule of labour in wage-output space shifts to the right.

Demand for labour: In the flexible-price model of Kimbrough (2006), the real wage is simply the marginal product of labour, $w^d\_Kimbrough_t = \frac{1}{\phi} Y^{1-\phi}$.

In our version with monopolistic distortion and flexible prices the wage will be smaller $w^d\_flex = \frac{\theta-1}{\theta} w^d\_Kimbrough$.

In the Calvo model one has to use the Phillips relation to recover an expression for the steady-state real wage:

$$\begin{aligned} w^d\_Calvo &= \left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{1-\theta+\theta\phi}{1-\theta}} \frac{1-\alpha\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta-1}} \frac{1}{\phi} Y^{1-\phi} \frac{\theta-1}{\theta} \qquad (4.1) \\ &= w^d\_flex \times g(\pi) < w^d\_flex \end{aligned}$$

where $g(\pi) := \left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{1-\theta+\theta\phi}{1-\theta}} \frac{1-\alpha\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta-1}} < 1$. By direct differentiation, one can show that $g(\pi)$ is decreasing in α and decreasing in inflation for $\pi > \pi^*$ (see appendix 9.1). Therefore, in the Calvo framework, firms pay *lower* wages than in the flexible-price model.

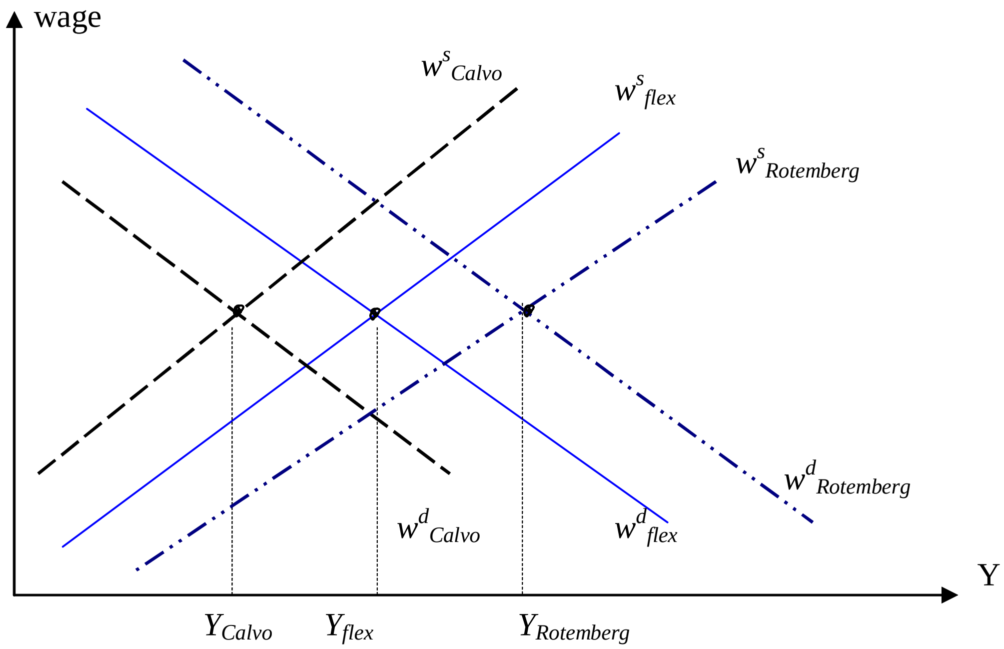
The steady state Phillips relation in the Rotemberg model yields

$$\begin{aligned} w^d\_Rotemberg &= \frac{1}{\phi}\frac{(\theta-1)}{\theta} Y_t^{1-\phi} + Y_t^{1-\phi}\frac{1}{\phi\theta}(1-\beta)\rho\pi_t(\pi_t-1) = \\ = \; & w^d\_flex + Y_t^{1-\phi}\frac{1}{\phi\theta}(1-\beta)\rho\pi_t(\pi_t-1) > w^d\_flex \end{aligned}$$

So in this set-up firms are willing to pay *higher* wages for producing a given level of output; the demand schedule of labour in wage-output space shifts to the right.

So, a useful way to sum up the results of this section is as follows: In the Calvo model the supply curve is shifted up and the demand curve is shifted down as inflation rises compared to the flexible price model. The unambiguous result is that equilibrium output is necessarily lower than in the flexible price model. However, in the Rotemberg model the opposite is true: both the supply and the demand curves are shifted to the right, as shown on figure 1.

Figure 1: Comparative analyses of the Labour Markets



5. The seigniorage maximizing inflation rates across frameworks

To get a feel for the SMIRs implied by the different models, we now turn to parameterize the models in a (hopefully) sensible way. We assume that $T(i)$ has the same functional form as in Kimbrough (2006):

$$\begin{aligned} T(i) &= \eta - \eta e^{-\gamma i}(1+\gamma i); \\ k(i) &= \eta\gamma e^{-\gamma i}. \end{aligned}$$

We take our calibrated parameters from existing work. Lucas (2000) suggests γ to be between 5 and 9 for the USA. So, for our benchmark we will adopt $\gamma = 7$, and $\eta = 0.3$; the latter is also consistent with Lucas's work. The other parameters we adopt are from Woodford (2003) and are standard for the calibration of the Calvo model. Price stickiness is given by $\alpha \in \{0.125, 0.25, 0.5, 0.6\}$, the elasticity of substitution is $\theta = 8$, and the labour supply elasticity is $v = 1.7$, while production elasticity of labour is $\phi = 1.6$.

We choose a value for the cost of price adjustment in the Rotemberg model, ρ, by requiring that its value equalize the slopes of the Phillips curves across the Rotemberg and Calvo models. Note that the value is significantly larger than the one used in Schmitt-Grohe and Uribe (2004), i.e., $\rho = 4.25$. Table 1 summarizes our calibration parameters.

Table 1: Calibration parameters

Symbol	**value**	**Description**
β	0.96	subjective discount factor
ρ	5.5, 15, 68, 125	Rotemberg cost of price adjustment
α	0.125, 0.25, 0.5, 0.75	Calvo parameters
θ	8	Competitiveness
ϕ	1.6	Production Elasticity of Labour
v	1.7	Labour Supply Elasticity
γ	3, 5, 7	Elasticity of money demand
η	0.3	Money demand parameter

The calculations of the seigniorage maximising inflation rates for different models are given in table 2. For a wide range of parameter values we conclude that the value of seigniorage maximizing inflation declines in ρ and α.

Table 2: SMIR for different models

	π_{Cagan}	π_{flex}	$\pi_{Rotemberg}$				π_{Calvo}			
$\gamma = 7$	12.0%	10.3%	9.2%	8.0%	5.4%	4.2%	7.4%	5.5%	2.8%	2.0%
α or ρ		0	5.5	15	68	125	0.125	0.25	0.5	0.6
$\gamma = 5$	20.0%	17.3%	13.8%	11.1%	6.5%	4.9%	9.1%	6.2%	3.0%	2.2%
α or ρ		0	5.5	15	68	125	0.125	0.25	0.5	0.6
$\gamma = 3$	44%	37%	20.9%	14.9%	7.7%	5.7%	10.5%	6.9%	3.3%	2.3%
α or ρ		0	5.5	15	68	125	0.125	0.25	0.5	0.6

Here π_{Cagan}, π_{flexi}, π_{Calvo}, $\pi_{Rotemberg}$ indicate the seigniorage maximizing inflation rate in our various models. One notes first of all that both the Cagan and the flex-price SMIRs imply high (double-digit) inflation rates across all calibrations. Furthermore, the difference between the Cagan and flex-price SMIRs is relatively small. As noted above, the Calvo model incorporates a distortion entirely absent from the other models in that there is a relative price distortion across firms. It turns out that distortion can be very costly; it implies that the economy with sub-optimal price dispersion has a lower 'optimal' inflation rate because (*i*) the labour input employed, (*ii*) the aggregate production costs, (*iii*) the disutility from labour and (*iv*) transactions costs are all increasing in price dispersion, for a given level of production. Moreover, the impact that price dispersion has is akin to, and of the order of magnitude of, a negative productivity shock, as demonstrated by Damjanovic and Nolan (2006).

The seigniorage maximizing inflation rate is lower in the Rotemberg framework than in the flexible price case, however it is higher than in Calvo setting. In the Rotemberg model, the cost of price adjustment results in larger production as inflation rises. However, the consumers ultimately have to pay that cost, resulting in lower consumption and therefore lower money demand.

In a sense these results confirm Kimbrough's assertion on the importance of the labour-leisure choice in calculating SMIRs. However, our findings suggest that the existence or otherwise of price inflexibility is of substantial qualitative and quantitative significance. Finally, it is interesting to note that the average SMIR in the Calvo framework is a little over 5% across all calibrations, and around 3.5% if one excludes those calibrations that induce double-digit inflation rates.

6. Conclusion

The paper compared four models' qualitative assessments of, and quantitative prescriptions for, the seigniorage maximizing inflation rate. Along with the form of the demand for money and the endogeneity of the labour-leisure choice, it was argued that the seigniorage maximizing inflation rate depends crucially on the existence of price rigidity, and on the form that rigidity takes. A detailed analysis of the reasons for these differences was provided and of the role of inflation in distorting the labour market. Sticky price models predict substantially lower SMIRs compared to the Cagan rule and a flexible-price model. If our version of the Calvo model (and our calibration) is to be believed, the level of inflation experienced recently in advanced economies such as the USA and the UK may be quite close to the SMIR.

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7. Appendix

7.0.2. Deriving the Rotemberg Phillips Curve

The Rotemberg model developed here is very similar to the version of the Calvo model except that all firms are identical; they post identical prices and there is no aggregate price dispersion, $\Delta = 1$. So, consumer preferences (2.1), labour supply (2.6) and saving decision (2.7), production technology (2.12) and the equilibrium in the goods market (2.24) are all identical to the Calvo set-up. However, the price setting dynamics are different and can be described as follows. Firms set prices maximizing the net present value of future profits:

$$\begin{aligned}
\Pi_t = E_t \sum_{k=0}^{\infty} d_{t+k} &\left(Y_{t+k} \left(\frac{P_{t+k}(i)}{P_{t+k}} \right)^{1-\theta} - w_{t+k} N_{t+k}(i) - \frac{\rho}{2} \left(\frac{P_{t+k}(i)}{P_{t+k-1}(i)} - 1 \right)^2 \right) \\
= \quad & E_t \sum_{k=0}^{\infty} \beta^k d_{t+k} Y_{t+k} \left(\frac{P_{t+k}(i)}{P_{t+k}} \right)^{1-\theta} - w_{t+k} A_{t+k}^{-\phi} Y_{t+k}^{\phi} \left(\frac{P_{t+k}(i)}{P_{t+k}} \right)^{-\theta\phi} \\
& - E_t \sum_{k=0}^{\infty} \beta^k d_{t+k} \frac{\rho}{2} \left(\frac{P_{t+k}(i)}{P_{t+k}} \frac{P_{t+k}}{P_{t+k-1}} \frac{P_{t+k-1}}{P_{t+k-1}(i)} - 1 \right)^2
\end{aligned}$$

where w_t is real wage $w_t = W_t / P_t$, defined in (2.25).

The first order condition with respect to $\frac{P_t(i)}{P_t}$ is

$$\begin{aligned}
&\left(- (1-\theta) Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\theta} + \theta\phi w_t A_t^{-\phi} Y_t^{\phi} \left(\frac{P_t(i)}{P_t} \right)^{-\theta\phi - 1} - \rho \pi_t \left(\frac{P_t(i)}{P_t} \pi_t - 1 \right) \right) \\
&+ E_t d_{t+1} \left(\rho \pi_{t+1} \left(\frac{P_t}{P_t(i)} \right)^{-2} \left(\frac{P_{t+1}(i)}{P_{t+1}} \pi_{t+1} \left(\frac{P_t}{P_t(i)} \right)^{-1} - 1 \right) \right). \qquad (7.1)
\end{aligned}$$

Since all the firms charge the same price, in equilibrium $\frac{P_t(i)}{P_t} = 1$ which gives us the Phillips curve:

$$- (1-\theta) Y_t + \theta\phi w_t A_t^{-\phi} Y_t^{\phi} - \rho \pi_t (\pi_t - 1) + E_t d_{t+1} (\rho \pi_{t+1} (\pi_{t+1} - 1)) = 0. \quad (7.2)$$

That completes the derivation of the Phillips curve.

7.1. Calibration for the costs of price adjustment, ρ

For calibration purposes we equalized the slope of the Phillips curves across our two sticky-price models (in the deterministic equilibrium with zero trend inflation). For the Rotemberg model we need to log linearize (7.2) assuming constant discounting, $d_{t+1} = \beta$; that implies (7.3)

$$\frac{\theta\phi}{(\theta-1)} \frac{w}{Y} Y^{\phi} \left(\widehat{w}_t + \phi \left(\widehat{Y}_t - \widehat{A}_t \right) \right) - \widehat{Y}_t = \frac{\rho}{\theta - 1} \frac{1}{Y} E_t \left(\widehat{\pi}_t - \beta \widehat{\pi}_{t+1} \right). \qquad (7.3)$$

For the Calvo model, combining (2.20), (2.21), and (2.22) we can derive representation (7.4)

$$\begin{aligned} & \frac{\theta\phi}{\theta-1} w_t A_t^{-\phi} Y_t^{\phi} - \left(\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{1+\theta(\phi-1)}{(1-\theta)})} Y_t \quad (7.4) \\ = \; & E_t \left[\left(\frac{1-\alpha\pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{1+\theta(\phi-1)}{(1-\theta)})} a\pi_{t+1}^{\theta-1} - a\pi_{t+1}^{\theta\phi} \left(\frac{1-\alpha\pi_{t+1}^{\theta-1}}{1-\alpha} \right)^{\frac{1+\theta(\phi-1)}{(1-\theta)})} \right] X_{t+1}, \end{aligned}$$

where X_{t+1} is defined as $X_{t+1} = ad_{t,t+1} \sum_{k=0}^{\infty} \alpha^k d_{t+1,t+1+k} \left[Y_{t+k+1} (\frac{P_{t+1}}{P_{t+k+1}})^{1-\theta} \right]$. That implies that in steady state $X = \frac{\alpha\beta}{1-\alpha\beta} Y$. The log-linearization of (7.4) results in (7.5)

$$\begin{aligned} & \frac{\theta\phi}{\theta-1} \frac{wY^{\phi}}{Y} \left(\widehat{w}_t + \phi \left(\widehat{Y}_t - \widehat{A}_t \right) \right) - \widehat{Y}_t \quad (7.5) \\ = \; & (1-\theta+\theta\phi) \frac{1}{1-\alpha\beta} \frac{\alpha}{1-\alpha} E_t \left[\widehat{\pi}_t - \beta \widehat{\pi}_{t+1} \right]. \end{aligned}$$

Clearly equations (**??**) and (7.5) are identical when it is the case that

$$\rho = Y(\theta-1)(1-\theta+\theta\phi) \frac{1}{1-\alpha\beta} \frac{\alpha}{1-\alpha}; \quad (7.6)$$

here, Y is the steady state level of output from either (7.2) or (7.4). Also,.as $Y = \frac{\theta\phi}{\theta-1} wY^{\phi}$ the steady state wage is defined from (2.25) as $w = \frac{(1+T+ik)}{(1+T)} Y^{\phi v+1}$, and $i = \frac{\beta-1}{\beta}$. Using formula (7.6) we obtain the following correspondence between price rigidity parameters in the Rotemberg and Calvo models.

α	0.125	0.25	0.5	0.6
$\rho(a)$	5.5	15	68	125

This calibration is used for the results reported in table 2.

8. Appendixes for Rotemberg model

8.1. Output

Proposition 3.1: *In the model with costly price adjustment, output increases with price stickiness, for any given level of inflation.*

Proof. The equilibrium level of output solves equation (8.1)

$$\begin{aligned} & \theta\phi\sigma(i)\left(Y-\frac{\rho}{2}(\pi-1)^2\right)Y^{\phi(v+1)}-(\theta-1)Y \\ = \;& \rho(1-\beta)\pi(\pi-1)\,. \end{aligned} \tag{8.1}$$

One can differentiate indirectly to receive

$$\begin{aligned} & \phi(v+1)\,\theta\phi\sigma(i)\left(Y-\frac{\rho}{2}(\pi-1)^2\right)Y^{\phi(v+1)}\frac{dY}{Y} \\ & +\theta\phi\sigma(i)Y^{\phi(v+1)}dY-(\theta-1)\,dY \\ = \;& d\rho\,(\pi-1)\,\pi\,(1-\beta)+\theta\phi\sigma(i)\left(\frac{1}{2}(\pi-1)^2\right)Y^{\phi(v+1)}d\rho. \end{aligned} \tag{8.2}$$

This expression may be rewritten as

$$\begin{aligned} & [\phi(v+1)-1](\theta-1)\,dY+\phi(v+1)\,\rho(\pi-1)(\pi-\beta)\frac{dY}{Y} \\ & +\theta\phi\sigma(i)Y^{\phi(v+1)}dY \\ = \;& (\pi-1)\,\pi\,(1-\beta)\,d\rho+\theta\phi\sigma(i)\left(\frac{1}{2}(\pi-1)^2\right)Y^{\phi(v+1)}d\rho. \end{aligned} \tag{8.3}$$

Since all coefficient are positive, one may conclude that output increases with price stickiness for $\partial Y/\partial\rho > 0$, which completes the proof. ■

Proposition 3.2: *In the model with costly price adjustment, output increases with inflation, for any level of inflation smaller than Cagan's SMIR.*

Proof. The equilibrium level of output solves equations (8.1). Again, one can differentiate indirectly to receive

$$\begin{aligned} & \phi(v+1)\,\theta\phi\sigma(i)\left(Y-\frac{\rho}{2}(\pi-1)^2\right)Y^{\phi(v+1)}\frac{dY}{Y} \\ & +\theta\phi\sigma(i)Y^{\phi(v+1)}dY-(\theta-1)\,dY \\ = \;& \rho(1-\beta)(2\pi-1)\,d\pi \\ & +2\theta\phi\sigma(i)\frac{\rho}{2}(\pi-1)\,Y^{\phi(v+1)}d\pi \\ & -\theta\phi\sigma(i)\left(Y-\frac{\rho}{2}(\pi-1)^2\right)Y^{\phi(v+1)}\frac{\partial\sigma(i)}{\partial i}\frac{\partial i}{\partial\pi}d\pi. \end{aligned}$$

This expression may be rewritten as

$$\begin{aligned} & \phi(v+1)\left(\rho(1-\beta)\,\pi\,(\pi-1)\right)\frac{dY}{Y} \\ & +\left(\phi(v+1)-1\right)dY+\theta\phi\sigma(i)Y^{\phi(v+1)}dY \\ = \;& \rho(1-\beta)(2\pi-1)\,d\pi++2\theta\phi\sigma(i)\frac{\rho}{2}(\pi-1)\,Y^{\phi(v+1)}d\pi \\ & -\theta\phi\sigma(i)\left(Y-\frac{\rho}{2}(\pi-1)^2\right)Y^{\phi(v+1)}\frac{\partial\sigma(i)}{\partial i}\frac{\partial i}{\partial\pi}d\pi. \end{aligned}$$

All coefficients are positive when $\frac{\partial\sigma(i)}{\partial i} < 0$, which is true for any level of inflation smaller that Cagan's SMIR. One may conclude that output increases with price inflation for $\partial Y/\partial\pi > 0$, which completes the proof. ■

8.2. Consumption and Seigniorage

Here we provide a proof for Proposition 3.3

Proposition *3.3: When the "convexity of labour" is small, $\phi(v+1) \simeq 1$, consumption and seigniorage revenue increase with ρ for any given level of inflation, $\pi > 1$.*

Proof. Recall that

$$C = \frac{1}{(1+T(i))}\left(Y - \frac{\rho}{2}(\pi-1)^2\right)$$

Hence,

$$\begin{aligned} dC &= \frac{1}{(1+T(i))}\left(dY - \frac{1}{2}(\pi-1)^2 d\rho\right); \\ dS &= \frac{ik}{(1+T(i))}\left(dY - \frac{1}{2}(\pi-1)^2 d\rho\right). \end{aligned}$$

Using formula (8.3)

$$\begin{aligned} & [\phi(v+1)-1](\theta-1)\left(dY - \frac{1}{2}(\pi-1)^2 d\rho\right) \\ & +\phi(v+1)\rho(\pi-1)(\pi-\beta)\frac{1}{Y}\left(dY - \frac{1}{2}(\pi-1)^2 d\rho\right) \\ & +\theta\phi\sigma(i)Y^{\phi(v+1)}\left(dY - \frac{1}{2}(\pi-1)^2 d\rho\right) \\ = & -[\phi(v+1)-1]\pi(\pi-1)(1-\beta)\,d\rho \\ & -\frac{1}{2}[\phi(v+1)-1](\theta-1)(\pi-1)^2 d\rho \\ & +\phi(v+1)\pi(\pi-1)(1-\beta)\left(1-\frac{1}{Y}\frac{\rho}{2}(\pi-1)^2\right)d\rho. \end{aligned}$$

If $\phi(v+1) = 1$ the right hand side is positive, which implies $\frac{dC}{d\rho} > 0$, $\frac{dS}{d\rho} > 0$. ■

9. Appendixes for Calvo Model

9.1. Labour demand

In this appendix it is shown that for a given level of production firms pay lower wages in the Calvo model than with flexible prices (i.e., our variant of Kimbrough

(2006) which includes monopolistic competition). Recall equation (4.1)

$$w^d\_calvo = w^d\_flex \times g(\pi);$$

where $g(\pi) = \left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{1-\theta+\theta\phi}{1-\theta}} \frac{1-\alpha\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta-1}}$. Therefore to analyze the partial equilibrium one needs to investigate how $g(\pi)$ changes with π and α. The necessary analysis is provided by direct differentiation. That is,

$$\begin{aligned}
\pi\frac{\partial \log g(\pi)}{\partial \pi} &= -\frac{1-\theta+\theta\phi}{1-\theta}\frac{(\theta-1)\,\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} - \frac{\theta\phi\alpha\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta\phi}} + \frac{(\theta-1)\,\alpha\beta\pi^{\theta-1}}{1-\alpha\beta\pi^{\theta-1}} \\
&= -\frac{(\theta-1)(1-\beta)\,\alpha\pi^{\theta-1}}{(1-\alpha\beta\pi^{\theta-1})(1-\alpha\pi^{\theta-1})} + \theta\phi\left(\frac{\alpha\pi^{\theta-1}\left(1-\beta\pi^{\theta\phi-(\theta-1)}\right)}{(1-\alpha\pi^{\theta-1})(1-\alpha\beta\pi^{\theta\phi})}\right),
\end{aligned}$$

which is negative for $\pi > \pi^*$, where $\pi^* < (1/\beta)^{\frac{1}{\theta\phi-(\theta-1)}}$, and solves equation $\pi\frac{\partial \log g(\pi)}{\partial \pi} = 0$, which is equivalent to (2.28)

For any inflation level, $\pi > 1$, the wage offered by firms declines in price stickiness. Similarly, it follows from (9.1)

$$\frac{\partial \log g(\pi)}{\partial \alpha} = -\frac{(1-\theta+\theta\phi)\left(\pi^{\theta-1}-1\right)}{(\theta-1)(1-\alpha\pi^{\theta-1})(1-\alpha)} - \frac{\beta\left(\pi^{\theta\phi}-\pi^{\theta-1}\right)}{1-\alpha\beta\pi^{\theta\phi}} < 0. \tag{9.1}$$

9.2. Output in Calvo model

Proposition 2.2: *Inflation and output are negatively related in the Calvo model for* $\pi_{Cagan} > \pi > \pi^*$

Proof. We recall formula (2.27)

$$(Y/A)^{\phi(v+1)} = \frac{h(\pi)\sigma(i)}{\Delta^{v+1}(\pi)},$$

where $h(\pi) := \frac{1-\alpha\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta-1}}\frac{1-\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta\phi}}$; $\sigma(i) = \frac{1+T(i)}{1+T(i)+ik(i)}$; and $\Delta = \frac{(1-\alpha)}{\left(1-\alpha\pi^{\theta\phi}\right)}\left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{\theta\phi}{\theta-1}}$.

These terms were discussed in the main text. By direct differentiation one can prove that the positive effect on output of a rise in inflation by reducing the monopolistic distortion is smaller that the negative effect due to the rise in price distortion consequent on the same increase in inflation:

It is easy to see that

$$\phi\,(v+1)\,\frac{\partial \log(Y)}{\partial \pi} = \left(\frac{\partial \log h}{\partial \pi} - \frac{\partial\,(\log \Delta)}{\partial \pi}\right) - v\frac{\partial\,(\log \Delta)}{\partial \pi} + \frac{\partial \log \sigma(i)}{\partial \pi}. \tag{9.2}$$

Below we establish that $\frac{d(\log\Delta)}{d\pi} > 0$, $\left(\pi\frac{d\log h(\pi)}{d\pi} - \pi\frac{d(\log\Delta)}{d\pi}\right) < 0$ and $\frac{d\log(\sigma)}{di} < 0$, for $\pi_{Cagan} > \pi > \pi^*$.

$$\begin{aligned}\pi\frac{d(\log\Delta)}{d\pi} &= \frac{\theta\phi\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta\phi}} - \frac{\theta\phi}{\theta-1}\frac{(\theta-1)\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} = \frac{\theta\phi\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta\phi}} - \frac{\theta\phi\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} = \\ &= \theta\phi\alpha\frac{\pi^{\theta\phi}-\pi^{\theta-1}}{(1-\alpha\pi^{\theta\phi})(1-\alpha\pi^{\theta-1})} > 0.\end{aligned} \tag{9.3}$$

$$\begin{aligned}\pi\frac{d\log h(\pi)}{d\pi} &= \frac{\theta\phi\alpha\pi^{\theta\phi}}{1-\alpha\pi^{\theta\phi}} - \frac{\theta\phi\alpha\beta\pi^{\theta\phi}}{1-\beta\alpha\pi^{\theta\phi}} + \frac{(\theta-1)\alpha\beta\pi^{\theta-1}}{1-\beta\alpha\pi^{\theta-1}} - \frac{(\theta-1)\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} = \\ &= \theta\phi\frac{(1-\beta)\alpha\pi^{\theta\phi}}{(1-\alpha\pi^{\theta\phi})(1-\beta\alpha\pi^{\theta\phi})} - (\theta-1)\frac{(1-\beta)\alpha\pi^{\theta-1}}{(1-\beta\alpha\pi^{\theta-1})(1-\alpha\pi^{\theta-1})} > 0.\end{aligned}$$

Therefore:

$$\begin{aligned}&\pi\frac{d\log h(\pi)}{d\pi} - \pi\frac{d(\log\Delta)}{d\pi} \\ &= -\frac{\theta\phi\alpha\beta\pi^{\theta\phi}}{1-\beta\alpha\pi^{\theta\phi}} + \frac{(\theta-1)\alpha\beta\pi^{\theta-1}}{1-\beta\alpha\pi^{\theta-1}} - \frac{(\theta-1)\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} + \frac{\theta\phi\alpha\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} = \\ &= -\theta\phi\alpha\frac{\beta\pi^{\theta\phi}-\pi^{\theta-1}}{(1-\beta\alpha\pi^{\theta\phi})(1-\alpha\pi^{\theta-1})} - (\theta-1)\alpha\frac{\pi^{\theta-1}(1-\beta)}{(1-\beta\alpha\pi^{\theta-1})(1-\alpha\pi^{\theta-1})},\end{aligned} \tag{9.4}$$

which is negative for $\pi > \pi^*$, where π^* is defined as in above.

$$\begin{aligned}\frac{\partial\log(\sigma)}{\partial i} &= \frac{T'}{1+T(i)} - \frac{T'(i)+ik'(i)+k(i)}{1+T(i)+ik(i)} = \frac{-k(i)}{1+T(i)} - \frac{ik'(i)}{1+T(i)+ik(i)} \\ &= -\frac{k(i)+ik'(i)}{(1+T(i))(1+T(i)+ik(i))},\end{aligned} \tag{9.5}$$

which is negative for $i < i_c = 1/\gamma$. Combining (9.4), (9.5) and (9.3) with (9.2) one concludes that in the Calvo model output declines with inflation for $\pi > \pi^*$. That completes the proof of proposition 2.2. ■

Proposition 2.1: *Output declines in the degree of price stickiness, α, for $\pi > 1$.*

Proof. To prove proposition 2.1 we shall use similar logic and show that equilibrium output declines with price stickiness

$$\phi(v+1)\frac{\partial\log(Y)}{\partial\alpha} = \left(\frac{\partial\log h}{\partial\alpha} - \frac{\partial(\log\Delta)}{\partial\alpha}\right) - v\frac{\partial(\log\Delta)}{\partial\alpha}. \tag{9.6}$$

First we establish that $\frac{\partial(\log\Delta)}{\partial\alpha} > 0$:

$$\frac{d\log\Delta}{d\alpha} = \frac{\theta\phi-\theta+1}{\theta-1}\frac{1-\pi^{\theta-1}}{\left(1-\alpha\pi^{\theta-1}\right)(1-\alpha)} + \frac{\left(\pi^{\theta\phi}-\pi^{\theta-1}\right)}{\left(1-\alpha\pi^{\theta-1}\right)\left(1-\alpha\pi^{\theta\phi}\right)} > 0. \tag{9.7}$$

The proof of that statement is slightly more subtle. First one needs to recognize that $\frac{d\log\Delta}{d\alpha}_{(\pi=1)} = 0$; then one computes the second derivative

$$\pi\frac{d^2\log\Delta}{\partial\alpha\partial\pi} = \theta\phi\left(\frac{\pi^{\theta\phi}}{\left(1-\alpha\pi^{\theta\phi}\right)^2} - \frac{\pi^{\theta-1}}{\left(1-\alpha\pi^{\theta-1}\right)^2}\right) > 0,$$

which is positive for $\alpha\pi^{\theta\phi} < 1$. Since $\frac{d\log\Delta}{d\alpha}$ is a strictly increasing function in π and $\frac{d\log\Delta}{d\alpha}(1) = 0$, it follows that $\frac{d\log\Delta}{d\alpha}(\pi) > 0$ for $\pi > 1$. Next

$$\begin{aligned} \frac{d\log h}{d\alpha} &= \frac{-\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta\phi}} + \frac{\beta\pi^{\theta-1}}{1-\alpha\beta\pi^{\theta-1}} - \frac{\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}} + \frac{\pi^{\theta\phi}}{1-\alpha\pi^{\theta\phi}}; \\ &= \frac{(1-\beta)\,\pi^{\theta\phi}}{\left(1-\alpha\pi^{\theta\phi}\right)\left(1-\alpha\beta\pi^{\theta\phi}\right)} - \frac{(1-\beta)\,\pi^{\theta-1}}{\left(1-\alpha\beta\pi^{\theta-1}\right)\left(1-\alpha\pi^{\theta-1}\right)} > 0. \end{aligned}$$

To some extent the function h represents the beneficial impact of inflation in the Calvo model. However the positive effect on output of the reduction in the monopolistic distortion is completely offset by the negative effect from the relative price distortion:

$$h(\pi)/\Delta = \frac{1-\alpha\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta-1}}\left(\frac{1-\alpha\pi^{\theta-1}}{1-\alpha}\right)^{\frac{\theta-1-\theta\phi}{\theta-1}}.$$

Hence,

$$\begin{aligned} & \frac{d\log h}{d\alpha} - \frac{d\log\Delta}{d\alpha} = \\ & \frac{-\beta\pi^{\theta\phi}}{1-\alpha\beta\pi^{\theta\phi}} + \frac{\beta\pi^{\theta-1}}{1-\alpha\beta\pi^{\theta-1}} + \frac{\theta-1-\theta\phi}{\theta-1}\left(\frac{1}{1-\alpha} - \frac{\pi^{\theta-1}}{1-\alpha\pi^{\theta-1}}\right) < 0. \end{aligned} \tag{9.8}$$

Again, combining (9.7) and (9.8) with (9.6) one concludes that in the Calvo model output declines with price stickiness for any $\pi > 1$. That completes the proof of proposition 2.2 ■

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