DYNAMICS OF COUPLED CELL NETWORKS: SYNCHRONY, HETEROCLINIC CYCLES AND INFLATION

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ABSTRACT. We consider the dynamics of small networks of coupled cells. We usually assume asymmetric inputs and no global or local symmetries in the network and consider equivalence of networks in this setting; that is, when two networks with different architectures give rise to the same set of possible dynamics. Focusing on transitive (strongly connected) networks that have only one type of cell (identical cell networks) we address three questions relating the network structure to dynamics. The first question is how the structure of the network may force the existence of invariant subspaces (synchrony subspaces). The second question is how these invariant subspaces can support robust heteroclinic attractors. Finally, we investigate how the dynamics of coupled cell networks with different structures and numbers of cells can be related; in particular we consider the sets of possible "inflations" of a coupled cell network that are obtained by replacing one cell by many of the same type, in such a way that the original network dynamics is still present within a synchrony subspace. We illustrate the results with a number of examples of networks of up to six cells.

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Contents

1. Introduction

1.1. **Coupled dynamical systems.** Coupled dynamical systems are used as models in a wide range of applications such as neuronal networks, biological processes, communications systems, arrays of Josephson junctions and mechanical and electrical systems. If we regard the individual unit in these models as a "cell", with deterministic dynamics (specified by a vector field — continuous dynamics; or map — discrete dynamics), then a *coupled cell system* may be regarded as a set of individual but interacting dynamical systems.

Each cell can be thought of as having an *output* and a number of *inputs* coming from other cells in the system. A coupled cell system has a *network architecture* that can be represented by a directed graph with vertices corresponding to cells and each directed edge corresponding to a specific output–input connection. Different input types correspond to different edge types in the graph. At this level of abstraction, we use the term *coupled cell network* as a way of specifying and emphasizing the connection structure rather than the dynamics of the cells.

A central issue in the theory of coupled cell networks is understanding the extent to which the network architecture impacts on the dynamics. If the network contains subsets consisting of identical cells, then there is the possibility of non-trivial invariant subspaces consisting of groups of synchronized cells ('synchrony subspaces'). The classification of these patterns of synchrony is a significant step in understanding the dynamics forced by the network architecture. It is well–known from the study of equivariant dynamical systems and population dynamics that the presence of invariant subspaces can have a major impact on both global and local dynamics and lead to robust heteroclinic cycles, heteroclinic networks and bifurcation phenomena that would be non-generic in the absence of invariant subspaces.

In this paper, we develop a structured and intuitive approach to the dynamics of networks. We are interested in inferring dynamical properties of large networks in terms of properties of smaller networks determined, for example, by invariant subspaces forced by network architecture. Our approach is bottom-up (synthetic), rather than top-down (analytic). Our focus is on small networks and we give constructions that naturally lead to the appearance of robust heteroclinic cycles and (switching) heteroclinic networks. One of the most surprising results of this investigation is that robust heteroclinic cycles and switching heteroclinic networks can occur in very simple asymmetric architectures with just three or four cells, without any assumption of network symmetry or specific properties (for example, Lotka-Volterra dynamics) of the cells – we include some constructions of examples.

In many networks it is possible to talk about 'feedback loops' or 'feed-forward' subnetworks. Implicit in this terminology is the idea that in some sense the network has a direction – more precisely, a direction of information flow. We will consider directed networks that typically have no symmetries. By *directed* we mean that there is a definite direction of coupling. Each cell in the network will have an output that depends only on the internal state of the cell and the inputs. Such networks appear, for example, in information processing — whether electronic, biological or otherwise, and correspond in electrical circuits to processing elements that have "high impedance" inputs and "low impedance" outputs – so that the cell is not affected by the load on its outputs. For example, in neuronal networks with synaptic coupling, one neuron signals to another by a pulse (spike); the signalling neuron communicates with the receiving neuron in such a way that the signalling neuron is unaffected by the state of the receiving neuron. Of course, many networks do not have a natural directed structure in this sense. A network consisting entirely of cells where this is not the case (for example, where both inputs and outputs are of similar impedance) can be modelled by including two connections, one in each direction, in our formalism. This type of connection structure appears, for example, in the case of neural gap junction coupling.

Our implicit assumption that the network is directed will inform both our language and notational conventions. We have intentionally adopted a 'flow-chart' formalism similar to that used in electrical and computer engineering or control theory as it transparently encapsulates the idea of outputs that are "decoupled" from inputs. An important consequence of our assumption that outputs are "decoupled" from inputs is that it becomes natural to change the connection structure of the network and connect cells in new ways. In the papers [5, 6] it is shown that equivalence of networks under rearrangement of inputs is subtly different from the equivalence under rearrangement of outputs.

We start with relatively simple network structures and then connect, recombine and "inflate" them to understand more complex dynamical behavior, including synchrony and heteroclinic switching. The dynamics for cells may take a range of types, including discrete, continuous time or hybrid dynamics. However, for most our results we will assume dynamics is governed by smooth finite dimensional ordinary differential equations (ODEs). Note that even if the individual cells are modelled by ODEs, the connections between subnetworks may be modelled by discrete dynamics that in practice may be asynchronously updated.

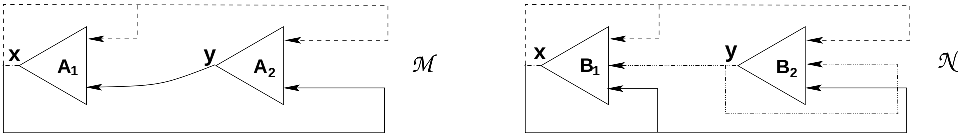
1.2. Approaches to modelling synchrony patterns in network dynamics.

Many methods are used for analysis of the dynamics of large complex networks. These methods range from numerical simulations to statistical graph theoretical studies of the interactions, assuming the network connections are distributed according to some known statistical law. For example, Restrepo *et al.* [52] give conditions on the adjacency matrix of a network that imply synchronization of the network. Strogatz [56] discusses synchronization in the presence of noise in phase oscillators while the monograph Pikovsky *et al.* [49] discusses a range of approaches to synchronization in coupled periodic and chaotic systems.

There are various approaches to modelling the nonlinear dynamics of exact synchrony in coupled cell networks. One powerful approach assumes that patterns of synchrony are associated with symmetries in the network. This has been widely used to understand coupled networks of *identical* oscillators where the symmetry group in the model may be implicitly [30] or explicitly [16] described. Examples in applications that have used symmetries to understand synchrony patterns include, for example, Golubitsky *et al.* [26] who use symmetry to model quadruped locomotion using a simple 8 cell system with symmetries. This network can be thought of as a model of a neural central pattern generator (CPG) embedded in a larger neural network that actuates muscle control. The symmetry group of a network is the group of permutations of cells that preserves the network structure. A necessary condition for this group to be non-trivial is that the network must contain subsets of *identical* cells. For example, the network symmetry group of a set of N identical cells with all-to-all coupling and symmetric inputs is S_N, the symmetric group on N symbols. If the network symmetry group is non-trivial, this can lead to a range of different synchrony subspaces that can be enumerated; see for example [16] for phase oscillators.

However, network symmetries are not necessary for synchrony subspaces to be present. Most of the networks we investigate in the paper have trivial network symmetry groups: symmetry plays a very minor role. In a series of papers, Stewart, Golubitsky and coworkers [55, 24, 28] have formulated a general theory for asymmetric networks of interacting dynamical systems; for a recent overview, see [27]. Their approach is algebraic in character and uses balanced equivalence relations, groupoid formalism, graphs and the idea of a quotient network to develop a number of results. The notation developed by Stewart *et al.* (in particular, their multi-arrow formalism [28]) can be seen as equivalent to the formalism here but because our conceptual approach, language and notation are somewhat different, we obtain different insights and results. Indeed, the groupoid formalism [27] elegantly encodes the local symmetry structure of a network, and many results have been obtained. For example, it has helped illuminate the appearence of multi-rhythms, the effect of local symmetries, and how network structure influences bifurcation theory for coupled cell networks (see [27, 25]). An alternative approach, closer to the approach of this paper, is discussed in [21].

We emphasize some aspects of our approach as a contrast with that of [27]. (a) We focus on the cells themselves as the building blocks, and the importance of distinguishing outputs and number and type of inputs *a priori.* (b) We consider a range of input types where the symmetry assumptions among permutations of inputs are explicit. (c) We consider processes that "inflate" cells into multiple copies within networks, as a way of designing new networks with desired properties. (d) We regard synchrony subspaces as primary quantities rather than the balanced equivalence relations and groupoids used in [27]. (e) We emphasize the importance of identifying equivalence of networks that produce the same dynamics.



$\dot{x} = f(x;x,y)$
$\dot{y} = f(y;x,x)$

$\dot{x} = g(x;x,y,x)$
$\dot{y} = g(y;x,y,x)$

(x,y denote the states of first and second cells)

FIGURE 1. Dynamically equivalent networks; for any f that represents the cell dynamics of the network on the left, there is a g that gives precisely the same dynamics for the network on the right, and vice-versa.

Concerning (e), we are not only interested in identifying when two different network architectures are dynamically equivalent, but we are also concerned with how specific dynamics with one architecture is realized in a different but equivalent architecture. This viewpoint is extensively developed in Agarwal and Field [5, 6] where dynamical equivalence is described in terms of input and output equivalence. Dias and Stewart [18] have developed the abstract ideas of ODE and linear equivalence in a restricted set of networks (linear phase space, smooth dynamics) but their approach does not easily yield explicit algorithms for obtaining dynamical equivalence (indeed, obtaining such algorithms was not the intent of their work).

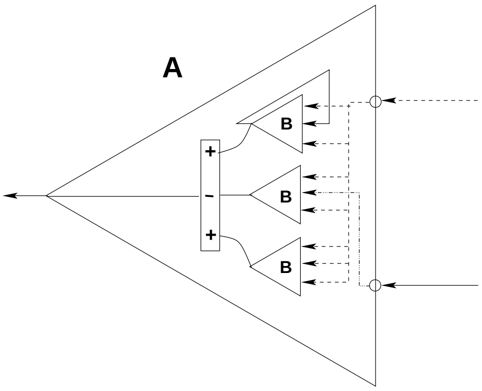


Figure 2. Building a new **A** cell from **B** cells. Using two of these new **A** cells in the network $\mathcal{M}$, we obtain identical dynamics to in the original network $\mathcal{N}$.

For example, the network architectures $\mathcal{M}$ and $\mathcal{N}$ in figure 1, with two identical cells and both with asymmetric inputs are dynamically equivalent (we give precise explanations of notation and terminology in subsequent sections). The dynamic equivalence of the network architectures follows easily from a simple algebraic condition on adjacency matrices [18, 5]. Moreover, it is shown in Agarwal and Field [5] that equivalence can be achieved by either input or output equivalence. What this means is that if we build a two input cell by linearly combining outputs of three **B** cells used to construct the network $\mathcal{N}$ — as in figure 2 — and connect these new cells according to the architecture $\mathcal{M}$, then the resulting coupled cell system (with architecture $\mathcal{M}$) has identical dynamics to that defined by the original **B** cells connected according to the architecture $\mathcal{N}$ (we also can achieve this result by linearly combining inputs). In this way, one can view the networks as having a *hierarchical* structure where we may wish to consider cells as a smaller subnetworks or modular *motifs* [37, 58].

1.3. Robust heteroclinic cycles in coupled cell networks.

For systems that have a nontrivial set of synchrony patterns, there is the possibility of robust heteroclinic cycles between different synchrony patterns. Heteroclinic attractors are not structurally stable for ‘typical’ smooth dynamical systems, but they can appear in a robust manner in the presence of simple constraints on the system that force certain subspaces to be invariant. Robust heteroclinic cycles are a well-known phenomenon in coupled differential equations modeling populations (see, for example, [31, 32, 33, 34, 45]) and in differential equations that are symmetric (or *equivariant*) with respect to a group of symmetries (see for example, [14, 20, 22, 23, 41, 42, 43, 46, 51]). In some systems, the invariant subspaces appear because of system symmetries; in others they appear for reasons such as the preservation of a 'zero-population' state in a Lotka-Volterra system. It is a natural question to ask whether robust heteroclinic cycles are possible in *asymmetric* coupled cell systems that do not use Lotka-Volterra type ODE models (cf the heteroclinic sequences of [2]) or, more generally, 'semilinear feedback' models [20, chapter 7].

The dos Reis/Busse-Heikes/Guckenheimer-Holmes system [29] is a symmetric system of ODEs that admits robust simple heteroclinic cycles on an approximately spherical (globally) attracting invariant set for the system [22]. This is an example of a *simple heteroclinic cycle*; that is, one in which the unstable manifold of each saddle point is one dimensional and contained in the stable manifold of another saddle point lying on an invariant subspace. The heteroclinic cycle can be attracting in the sense that an open set in a neighbourhood of the cycle is attracted to the cycle. The issue of asymptotic stability of heteroclinic cycles is complicated because, at least in equivariant dynamics (as opposed to Lotka-Volterra population models), the cycle is usually forced to be part of a larger network of connections generated by the action of all the symmetry group elements. Although we indicate some sufficiency results for asymptotic stability, we do not provide the technical details (which use straightforward and well-known techniques [31, 41, 42, 43, 53]). The Guckenheimer-Holmes system can be regarded as a (symmetrically) coupled system of three identical cells with 1-dimensional cell dynamics, with internal symmetries and coupling as in figure 14(a). The presence of symmetries constrains the type of robust simple heteroclinic cycles that may occur. In particular, heteroclinic cycles will typically connect equilibria with broken symmetries rather than synchronized — fully symmetric — states (we refer to [19, 20] for more examples where symmetric systems are treated as coupled cell systems).

There has been significant recent interest in robust heteroclinic cycles that appear in neural microcircuits. It has been suggested that they give a useful paradigm to explain the function of certain neural systems, because they give nonlinear models with "winnerless competition"; there is a local competition between different states but not necessarily a global winner [50]. They seem particularly useful for explaining sequence generation and spatio-temporal encoding and have been found in rate-based [2] and other models [48]. They can also be found in Hodgkin-Huxley-based models [30] or more general phase oscillator models [13]. In addition they can be used to perform finite-state computations [12]. Figure 3 shows a typical example of the appearance of a robust heteroclinic attractor in a simple coupled neuron system; in this case it shows a heteroclinic cycle for a microcircuit consisting of three coupled neurons [48]; some evidence of heteroclinic switching has been observed *in vivo* in Abeles *et al.* [1]. Analogous behaviour is also found in hybrid models of neural systems such as the networks of unstable attractors in systems of delay-pulse coupled oscillators [47] as well as in coupled chemical reaction systems [39].

We give conditions for the network architecture to support robust heteroclinic cycles as well as constructions that lead naturally to very simple networks supporting robust heteroclinic cycles. Symmetry will play no role in these constructions. We emphasize that we make no symmetry assumptions on the coupling, and no constraints on cell dynamics (as is the case if we assume, for example, Lotka-Volterra type dynamics [2]).

1.4. **Summary of the paper.** In this paper, we show how to find network architectures that can support robust heteroclinic cycles and heteroclinic networks [38, 20, 14] where the cells and the invariant subspaces are forced by the network architecture but not by symmetry or constraints on the ODE models. We describe global dynamics that can occur on ‘small’ networks of cells. One of the unexpected conclusions of our work is that small networks of three or four cells governed by one-dimensional dynamics can exhibit very rich dynamics. Moreover, using these results we obtain larger networks exhibiting heteroclinic dynamics. For that, we introduce the concept of network *inflation* developing a method for constructing large networks using simple small networks.

Most of the results in this paper focus on the case of networks that are *identical cell networks* (where there is only one *class* or ‘type’ of cell in the network) and cells have *asymmetric inputs* (the different inputs to each cell are distinguished and cannot be permuted without changing the network). Dynamically one can realize these in a number of ways, but for the most part we will consider a coupled cell network realized as a system of coupled ordinary differential equations (ODEs) with smooth dependence on variables. Robustness will be with reference to this class of systems. We only consider networks that are irreducible

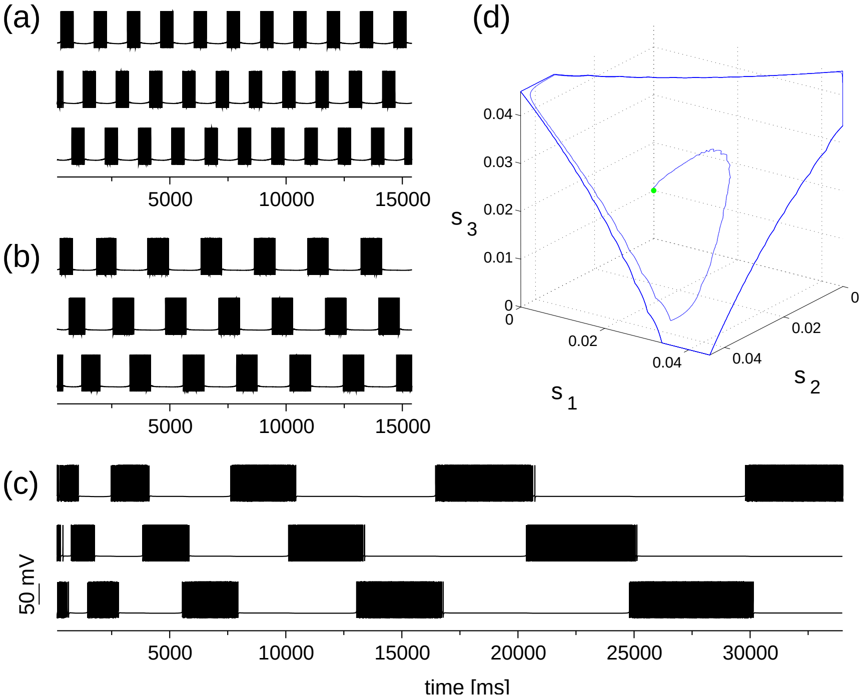


Figure 3. Bifurcation to a robust attracting heteroclinic cycle in a model system of three coupled neurons, reproduced from [48] with kind permission of the authors, T. Nowotny and M. Rabinovich. As a parameter – a stimulus current in this case – is progressively increased from (a)–(c), the time series of the membrane potentials V_i of the three neurons shows the appearance in (c) of an attracting heteroclinic cycle, showing the typical slowing down of switching between unstable phases. (d) shows the firing rates s_i of the three neurons plotted against each other as the system approaches the cycle. This system has the connection structure shown in figure 14(a).

in the sense that they contain no smaller subnetworks (equivalently, have no slaved or forced subnetworks [21]) by restricting attention to networks that are *transitive*. We remark that coupled cell systems with symmetric inputs are typically more degenerate and it is usually hard to find attracting (simple) heteroclinic cycles between fully synchronized states if one assumes symmetric inputs. However, we give one example of a network supporting robust attracting simple heteroclinic cycles that combines both symmetric and asymmetric inputs.

The paper is organized as follows: Section 2 introduces the formalism for coupled cell networks and the dynamics of the related coupled cell systems. We allow for continuous, discrete and hybrid models for the cells[1]. Section 3 discusses properties of coupled cell networks, including equivalence of coupled cell networks, synchrony subspaces, and *inflations* of coupled cell networks. Inflation provides a natural operation on networks that can generate many new invariant subspaces and can be seen as the inverse of forming a *quotient* network.

In section 4 we discuss two cell systems; we demonstrate that they have some surprises in terms of equivalence, but they do not support robust simple heteroclinic cycles. Section 5 discusses three cell networks. We show that there are networks of three identical coupled cells that support robust heteroclinic cycles. We give three new examples of three-cell ODE networks that have heteroclinic cycles in section 5.1, and briefly discuss how these three cell networks are likely to support switching heteroclinic networks such as those discussed by Homburg and Knobloch [35]. Section 6 examines networks with four or more cells and section 7 discusses sufficient conditions for the existence of robust heteroclinic cycles. We consider a model six cell example in some detail and show that it can in principle support a variety of heteroclinic cycles and networks. Some remarks on generalizations are included as part of the discussion in section 8, including comments on both multiple cell classes, symmetric inputs, and heteroclinic bifurcation.

2. Coupled cell networks: cells and connections

We distinguish between a *coupled cell network* which is an abstract arrangement of cells and connections, and a *coupled cell system* which is a particular realization of a coupled cell network as a system of coupled dynamical equations (in our case typically a set of coupled ordinary differential equations).

2.1. Structure of coupled cell networks.

Roughly speaking, a cell is a 'dynamical black box' that admits various types of input and that has an output which is uniquely determined by the inputs and the initial state. Two cells are regarded as being of the same *class* if the same inputs and initial state always result in the same output. We use

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[1] Hybrid systems are a mix of continuous, discrete and passive cells and are appropriate for modelling various types of system including, for example, fast-slow systems where we replace an ODE with multiple timescales by a singular system that can exhibit switching as well as motion on a slow manifold. Examples where passive cells are used appear in [5, 6].

the term 'class' rather than 'type' so as to distinguish clearly between cells and inputs.

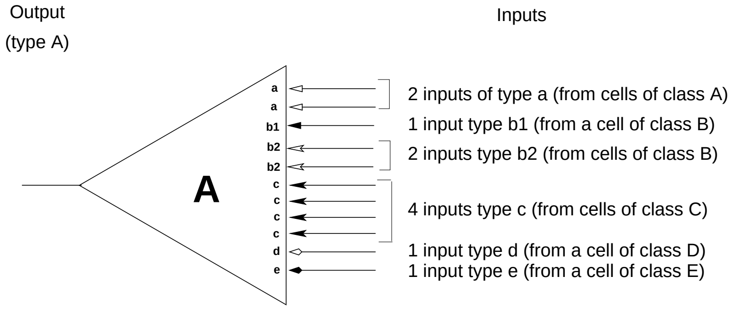

FIGURE 4. A cell with multiple inputs and one output. Inputs of the same type (for example the four inputs of type *c*) can be permuted without affecting the behaviour of the output.

Figure 4 shows a cell which accepts multiple inputs of different *types* from cells of different classes. Suppose that the displayed cell is of class **A** (we put **A** inside the symbol — in this case a triangle — to emphasize that the cell is of class **A**). The cell can accept inputs from cells of classes **A**, **B**, **C**, **D** and **E**. Observe that we may allow more than one type of input from a cell of a given class. For example, the displayed cell receives two different types of input from cells of class **B**. In addition to inputs, the cell has a single output which is of type **A**. Inputs of the same type can be permuted without affecting the behaviour of the output. We may think of cells as being coupled together using 'patchcords' (see [21] and note that the use of this terminology is suggestive of the combinatorics of coupled cell networks — we may repatch the connections and so vary the network architecture). Each patchcord goes from the output of a cell to an admissible input of the same or another cell. We show three simple examples in figure 5.

In figure 5(i), we show cells **X** and **Y** of class **A** and **B** respectively. The cell **X** of class **A** has two inputs, one of type **a**, the other of type **b**. Both inputs of cell **X** are filled — the **a** input by the output of cell **X**, the **b**-input by the output of cell **Y**. Cell **Y** has three inputs, two of type **a** and one of type **b**. The input of type **b** is not filled and only one of the type **a** inputs is filled (by the output of **X**).

Remarks 2.1. (1) Note that in figure 5 we have used different graphical representations of connections (patchcords) to show different types of input. For example, the connection for a type **a** input is represented by

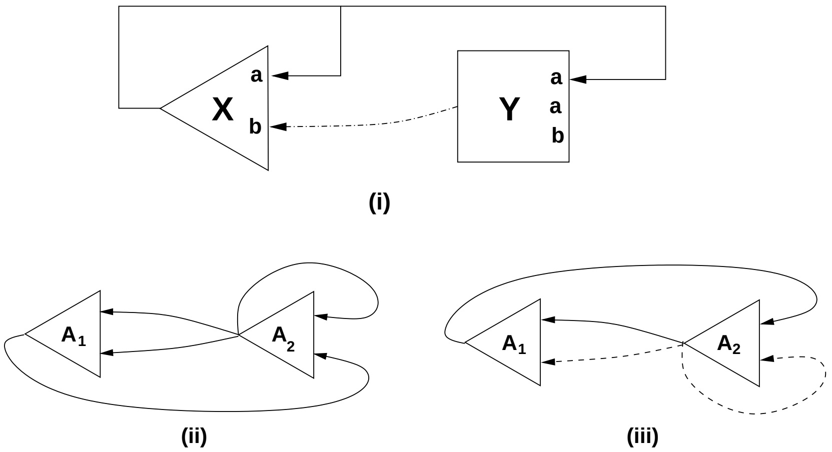

Figure 5. Three coupled cell network examples: (i) shows an incomplete network with two unfilled inputs to cell **Y**, (ii) shows a two identical cell network with inputs of the same type (symmetric inputs), (iii) shows a different two identical cell network whose inputs are of different type (as indicated by the different line types); we say the cells have asymmetric inputs.

a continuous line whereas the type **b** input is represented by a broken line. We may also use different arrowheads to distinguish between different types of input (as in figure 4). We can use different graphical symbols to differentiate between the class of a cell: a triangle for a cell of class **A** and a square for a cell of class **B**.

To improve clarity, we mostly use a specific Roman letter to identify the class of a cell – as in figure 5(ii,iii) – and use subscripts as labels if there is more than one cell of a given class. If there is only one cell of a given class we generally regard cell label and class label as synonymous. (2) As we indicated in the introduction, the output of a cell may represent the state of the cell (this is generally assumed in the setup of Stewart *et al.* [27]) or may be a scalar or vector quantity (see [5]).

In figure 5(ii), we show two cells $\mathbf{A}_1$, $\mathbf{A}_2$, both of class **A**. Each cell has two identical inputs of type **a** and all inputs are filled. When all the inputs are filled, as they are here, we refer to the set of cells and connections as a *network of coupled cells*. Finally in figure 5(iii), we have interchanged the inputs of $\mathbf{A}_2$ and made the inputs of different type. This network is *not* identical to that of figure 5(ii).

Remark 2.2. In this paper, we mainly consider cells which have *asymmetric* inputs. By this we mean that for each class of cell in the network, all the inputs are of different type. In particular, for a multi-input cell $\mathbf{X}$, with inputs from cells of class $\mathbf{A}$, it will matter into which type $\mathbf{A}$ input we plug the output of a class $\mathbf{A}$ cell. The network shown in figure 5(ii) has symmetric inputs (the cells have inputs of identical type) and if we interchange the inputs of $\mathbf{A}_1$, this will not change the network. On the other hand, interchanging the inputs of $\mathbf{A}_1$ will change the network shown in figure 5(iii).

Let $\mathfrak{C} = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots\}$ be a finite set of (distinct) cell classes. We assume that each $\mathbf{X} \in \mathfrak{C}$ has a finite (non-zero) number of inputs (and therefore a finite number of input types).

Definition 2.3. Suppose that $\mathfrak{C}$ is a set of cell classes.

(a) $\mathfrak{C}$ is *consistent* if for every $\mathbf{X} \in \mathfrak{C}$, each input type of $\mathbf{X}$ comes from a cell class in $\mathfrak{C}$.
(b) $\mathfrak{C}$ is *indecomposable* if $\mathfrak{C}$ is consistent and we cannot write $\mathfrak{C}$ as a disjoint union of two non-empty consistent sets.

Suppose that $\mathcal{N}$ is a coupled cell network modelled on $\mathfrak{C}$. Let $k = k(\mathcal{N})$ denote the number of cells in the network, and $c = c(\mathfrak{C}) = c(\mathcal{N})$ denote the number of cell classes.

Remark 2.4. Let $\mathfrak{C}$ be consistent and indecomposable. The indecomposability of $\mathfrak{C}$ implies that $\mathcal{T}$ is not a union of two disjoint coupled cell networks. If $\mathcal{N}$ is modelled on $\mathfrak{C}$ and $k(\mathcal{N}) > c(\mathfrak{C})$, then there will be more than one way the cells in $\mathcal{N}$ can be patched together to make a network. This leads to a combinatorics of repatching a network – we refer to [21] for more details and examples.

In this paper we will only consider networks built from an indecomposable (and therefore consistent) set of cell classes. To avoid trivialities, we always assume cells have at least one input.

Definition 2.5. Let $\mathfrak{C}$ be an indecomposable set of cell classes. A *coupled cell network* $\mathcal{N}$ modelled on $\mathfrak{C}$ will consist of a finite number of cells with cell classes in $\mathfrak{C}$ such that

(a) The cells are patched together according to the input-output rules described above.
(b) There are no unfilled inputs.

Remarks 2.6. (1) There are no restrictions on the number of outputs we take from a cell. If a cell is connected to a total of k inputs (possibly of different types), we say the cell has k *outputs*. We do not, however,

distinguish between outputs according to the type of the input to which the output is connected.
(2) If a cell has multiple inputs of the *same* type, it is immaterial which input the patchcord (connection) is plugged into. More precisely, if a cell $\mathbf{X}$ in the network has $k > 1$ inputs of the same type from cells of class $\mathbf{A}$ then permutation of the k connections from class $\mathbf{A}$ cells to these inputs is allowed and will not change the network structure. For the graphical representation of networks, we always represent inputs to cells of the same class in the same order. If there are symmetric inputs, these are always grouped together (as in figures 4 and figure 5(i)).
(3) Clearly a coupled cell network (such as those in figure 5) determines an associated directed graph where the nodes of the graph are the cells and there is a directed edge from cell $\mathbf{X}$ to cell $\mathbf{Y}$ if and only if cell $\mathbf{Y}$ receives an input from cell $\mathbf{X}$. Different input types will correspond to different edge types in the graph. If there are t different input types, then there will be t different types of edge in the associated graph. Noting remark (2) above we have an obvious definition of isomorphism of directed graphs determined by a coupled cell network. At this level of abstraction, we regard a coupled cell network as a structure — basically a connection structure satisfying definition 2.5.

2.2. **Adjacency matrices.** Suppose that $\mathfrak{C} = \{\mathbf{C}\}$ and so $c(\mathfrak{C}) = 1$. Let $\mathcal{N}$ be a coupled cell network modelled on $\mathfrak{C}$. Label cells in $\mathcal{N}$ so that $\mathcal{N} = \{\mathbf{C}_1, \ldots, \mathbf{C}_k\}$. Let $t = t(\mathbf{C})$ denote the number of input types in $\mathbf{C}$. In this case, all input types come from cells of the same class. Let $\ell \in \{1, \ldots, t\}$. We define a $k \times k$ matrix $M_\ell = [m_{ij}]$ by setting m_{ij} to be the number of inputs of type ℓ that $\mathbf{C}_j$ receives from $\mathbf{C}_i$. The matrix M_ℓ is called the *adjacency matrix of type ℓ*. We also define $M = M_1 + \cdots + M_t$ and refer to M as *the* adjacency matrix of the network. The set $M_1, \cdots, M_t$, M of adjacency matrices plays a crucial role in the classification theory of coupled cell networks (see [18, 8, 5, 6]). Observe that inputs of type ℓ are asymmetric if and only if there is exactly one non-zero entry in each column of M_ℓ and this entry is equal to one. We will make use of this observation later. If $c(\mathfrak{C}) > 1$, we define adjacency matrices exactly as above — each adjacency matrix will be a $k \times k$-matrix, where $k = k(\mathcal{N})$.

Definition 2.7. (see [21, Definition 5.12])

(a) A coupled cell network is *connected* if the associated graph is connected.
(b) A coupled cell network is *(strongly) transitive* (or *strongly connected*) if given any ordered pair of nodes there exists a chain of directed edges connecting the first node to the second node.

Remarks 2.8. (1) Obviously a transitive coupled cell network is connected. All the coupled cell networks we consider will be transitive.
(2) A coupled cell network is transitive if and only if the adjacency matrix M of the network is *irreducible* [40] : for each $1 \leq i, j \leq k$, there exists p such that $(M^p)_{ij} \neq 0$.

Example 2.9. A network can be decomposed into transitive and *slaved* subnetworks. In figure 6, we show a five cell network $\mathcal{M}$ with two cell classes. The subnetwork $\mathcal{M}^\star$ comprised of the cells $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ and $\mathbf{B}_2$ is a transitive *network* (we remove the outputs to $\mathbf{B}_1$). The subnetwork comprised of the single cell $\mathbf{B}_1$ is slaved — in this case to the transitive network $\mathcal{M}^\star$.

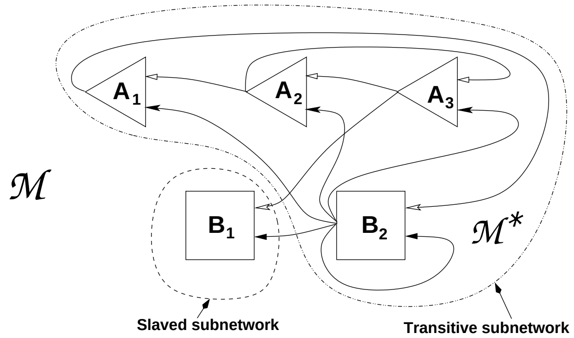

FIGURE 6. A network with slaved and transitive subnetworks.

2.3. **Networks and coupled cell system dynamics.** We now consider coupled dynamical systems (coupled cell systems) that reflect the structure of a given coupled cell network. In this paper, we regard connections between cells as dynamically neutral assignments of outputs to inputs.[2] We shall consider three contexts in which a network can be realized as a dynamical system.

If $\mathcal{N}$ denotes a coupled cell network, then by $\mathcal{F} \in \mathcal{N}$ we mean that $\mathcal{F}$ is a specific coupled cell system with connection and cell type structure determined by $\mathcal{N}$. If $\mathcal{N}$ is a coupled cell network, we emphasize that the number of cells $k = k(\mathcal{N})$, the number of cell classes $c = c(\mathcal{N})$, the number of input types $t = t(\mathcal{N})$ are the same for all systems $\mathcal{F} \in \mathcal{N}$.

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[2] One can regard certain dynamical effects, such as delay, as being incorporated in the connection (patchcord) structure (that is, as a 'delay line'). This approach is appropriate in some applications such as hybrid computing or modelling motion detection in vision. Of course, in the mathematical model one can always assume that delays are included as part of the input structure of the cell, at the expense of increasing the number of input types.

Continuous dynamics modelled by ordinary differential equations. For continuous dynamics, we assume that cell outputs (and therefore inputs) depend continuously on time. The canonical model for this situation is where each cell is modelled by an (autonomous) ordinary differential equation. We usually suppose that the evolution in time of cells depends on their internal state (not just their initial state). For simplicity, we further assume that the phase space for a cell is $\mathbb{R}^N$, where $N \geq 1$. However, all of what we say applies if the phase space is a differential manifold (see [5, 6]). We also assume that the output of a cell is given by the state of the cell (see [5] for more general classes of outputs).

If the phase space of a cell $\mathbf{X}$ is $\mathbb{R}^N$, then the output of $\mathbf{X}$ at time t is a vector $\mathbf{x}(t) \in \mathbb{R}^N$. We identify $\mathbf{x}(t)$ with the internal state of the cell. In particular if the cell has no inputs, then the ordinary differential equation model for the cell would simply be $\mathbf{x}' = F(\mathbf{x})$, where F is a vector field on $\mathbb{R}^N$.

Now suppose that we are given a coupled cell system with a finite cell set $\mathcal{C} = \{\mathbf{C}_1, \mathbf{C}_2, \ldots\}$. Let $\mathbf{X} \in \mathcal{C}$ and suppose that $\mathbf{X}$ has p-inputs. Then the dynamics of $\mathbf{X}$ will be given by a differential equation

$$\mathbf{x}' = F(\mathbf{x}; \mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \cdots, \mathbf{x}_{i_p}).$$

Observe that we write the internal variable $\mathbf{x}$ as the first variable of vector field F. If there is no dependence on the internal variable, we omit the initial $\mathbf{x}$ and write

$$\mathbf{x}' = F(\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \cdots, \mathbf{x}_{i_p}).$$

The remaining variables, following the semi-colon, give the inputs to the cell. Thus by $\mathbf{x}_{i_j}$ we mean an input from the cell $\mathbf{C}_{i_j}$. We do not require that $\{i_1, \ldots, i_p\}$ consists of distinct integers: they may all be equal. However, we do group inputs of the same type together. In particular, if the function is invariant under permutation of some of the inputs we signify this by an overline. For example, if $\mathbf{X}$ has p inputs and $\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \cdots, \mathbf{x}_{i_{p-1}}$ are all of the same input type, and of a different input type from $\mathbf{x}_{i_p}$, we write

$$\mathbf{x}' = F(\mathbf{x}; \overline{\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \cdots, \mathbf{x}_{i_{p-1}}}, \mathbf{x}_{i_p}).$$

We look at two simple examples that illustrate our conventions.

Examples 2.10. (1) We define a system with continuous dynamics with (identical) coupled cell structure given by figure 5(ii). This system has symmetric inputs. Allowing for dependence on the internal state of cells, we take any smooth map (C^1 usually suffices)

$f : \mathbb{R}^N \times \left(\mathbb{R}^N\right)^2 \to \mathbb{R}^N$ which is symmetric in the second and third variables. The ordinary differential equation model for the system is

$$\begin{aligned} \mathbf{x}_1' &= f(\mathbf{x}_1; \overline{\mathbf{x}_2, \mathbf{x}_2}), \\ \mathbf{x}_2' &= f(\mathbf{x}_2; \overline{\mathbf{x}_2, \mathbf{x}_1}). \end{aligned}$$

The evolution of the system is uniquely determined by the initial states $\mathbf{x}_1(0), \mathbf{x}_2(0)$ and if $\mathbf{x}_1(0) = \mathbf{x}_2(0)$ then $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ for all $t \in \mathbb{R}$.
(2) We define a system with continuous dynamics with coupled cell structure given by figure 5(iii). This system has asymmetric inputs. Let $f : \mathbb{R}^N \times \left(\mathbb{R}^N\right)^2 \to \mathbb{R}^N$ be any smooth map. Then the ordinary differential equation model is

$$\begin{aligned} \mathbf{x}_1' &= f(\mathbf{x}_1; \mathbf{x}_2, \mathbf{x}_2), \\ \mathbf{x}_2' &= f(\mathbf{x}_2; \mathbf{x}_1, \mathbf{x}_2). \end{aligned}$$

As before, the evolution of the system is uniquely determined by the initial states $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ and if $\mathbf{x}_1(0) = \mathbf{x}_2(0)$ then $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ for all $t \in \mathbb{R}$.

Discrete time dynamics. We define a discrete time coupled cell system to be a system of coupled maps updated at regular time intervals. As in the continuous time case, we model a cell $\mathbf{X}$ at time n using a phase space variable $\mathbf{x}(n) \in \mathbb{R}^N$ (for more general phase spaces see [5]) and then update by

$$\mathbf{x}(n+1) = f(\mathbf{x}(n); \mathbf{x}_{i_1}(n), \mathbf{x}_{i_2}(n), \cdots, \mathbf{x}_{i_p}(n)) \tag{2.1}$$

where f is a continuous function depending on the internal state $\mathbf{x}(n)$ together with the p inputs to cell $\mathbf{X}$.

Example 2.11. Given a continuous function $f : \mathbb{R}^N \times \left(\mathbb{R}^N\right)^2 \to \mathbb{R}^N$, we may define a system with discrete dynamics that realizes the network of figure 5 (iii). Specifically, if we let $\mathbf{x}_1(n), \mathbf{x}_2(n) \in \mathbb{R}^N$ denote the states at time n of $\mathbf{A_1}, \mathbf{A_2}$ respectively, then

$$\begin{aligned} \mathbf{x}_1(n+1) &= f(\mathbf{x}_1(n); \mathbf{x}_2(n), \mathbf{x}_2(n)), \\ \mathbf{x}_2(n+1) &= f(\mathbf{x}_2(n); \mathbf{x}_1(n), \mathbf{x}_2(n)). \end{aligned}$$

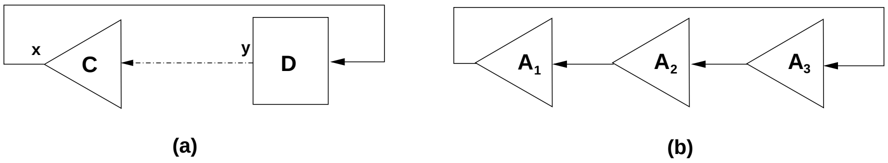
In this case the evolution depends on the internal state of the cells. If instead we consider the network of figure 5 (ii) (symmetric inputs) and assume evolution is independent of internal state, then

$$\begin{aligned} \mathbf{x}_1(n+1) &= f(\overline{\mathbf{x}_2(n), \mathbf{x}_2(n)}), \\ \mathbf{x}_2(n+1) &= f(\overline{\mathbf{x}_1(n), \mathbf{x}_2(n)}), \end{aligned}$$

where f is symmetric. In either case, the evolution of the system is uniquely determined by the initial state $\mathbf{x}_1(0), \mathbf{x}_2(0)$. If $\mathbf{x}_1(0) = \mathbf{x}_2(0)$, then $\mathbf{x}_1(n) = \mathbf{x}_2(n)$ for all $n \geq 1$.

Hybrid dynamics. Some models combine both discrete and continuous dynamics in a coupled cell system. This is particularly the case for models of fast-slow systems (prevalent for example in neural systems); we can model the effect of the fast dynamics as a slaving to slow variables except for occasional rapid transitions of the fast variables. We may also consider systems with thresholds controlling switching in cells where the dynamics are, for the most part, governed by (smooth) differential equations.

We give a simple example below (see figure 7(a)) of a two cell hybrid system where one cell is modelled by continuous dynamics, the other by discrete dynamics.



(a) (b)

FIGURE 7. Simple oscillatory hybrid and switching networks.

We suppose the cells shown in figure 7(a) have one dimensional dynamics. We denote the outputs of $\mathbf{C}, \mathbf{D}$ by $x, y \in \mathbb{R}$ respectively. Let the cell $\mathbf{C}$ be governed by the ordinary differential equation $x' = 1$. (There is no internal variable.) For discrete dynamics on $\mathbf{D}$, we fix $x_0 < X \in \mathbb{R}$. Let the output of $\mathbf{D}$ be given by the map $y : \mathbb{R} \to \mathbb{R}$ defined by

$$y(x) = \begin{cases} x, & \text{if } x < X \\ x_0, & \text{if } x \geq X. \end{cases}$$

For this system, whenever $x(t) < X$, $x(t)$ is given as a solution to the ODE $x' = 1$. When $x(t) = X$, the ODE is reset with initial condition x_0. Necessarily, there is a discontinuity in the solution $x(t)$. The system $\mathbf{C}, \mathbf{D}$ models a relaxation oscillator with output a sawtooth wave.

The example network shown in figure 7(b) produces an oscillator state if realised by three cells with threshold switching as follows. Suppose each cell has phase space $[0, 1]$ and satisfies $\dot{x}_i = f(x_i; x_{i+1})$ on the ring, where

$$f(x;y)=\begin{cases} ax+a, & \text{if } y=1,\ x\neq 0,\\ Ax+A, & \text{if } y<1,\ x\neq 1,\\ 0, & \text{otherwise,}\end{cases}$$

where $a \ll 0 < A$ (and so $a+A<0$). For this choice of f, the system has a unique attracting limit cycle of period $\frac{3}{A}\log 2$. One can check that the oscillatory state is such that each cell will be $\pm 2\pi/3$ out of phase with the remaining cells: a simple instance of how the network architecture may organize the dynamics.

In section 5.1, we propose the use of networks built from simple cells of this type, but with multiple inputs, as one way of modelling certain heteroclinic phenomena.

3. Properties of coupled cell networks

3.1. **Equivalence of coupled cell networks.**

In this section we give a notion of equivalence between network architectures that enables us to identify different network architectures that define the same dynamics. What we say is closely related to the ideas of network equivalence developed by Dias & Stewart [18]; however, the conceptual framework is different and, as indicated in the introduction, this leads to somewhat different questions and results [5, 6].

Let $\mathcal{F}$ and $\mathcal{G}$ be two coupled cell systems with the same number k of cells. We say that $\mathcal{F}$, $\mathcal{G}$ have identical dynamics if we can label the cells of $\mathcal{F}$ and $\mathcal{G}$, say as $\{\mathbf{A}_1,\dots,\mathbf{A}_k\}$ and $\{\mathbf{B}_1,\dots,\mathbf{B}_k\}$ respectively, so that

(1) The cells $\mathbf{A}_i$, $\mathbf{B}_i$ have the same phase space, $1\leq i\leq k$.
(2) The time evolution of both systems is identical.

Remarks 3.1. (1) By 'identical dynamics' we mean exactly that; we do not mean topological equivalence.
(2) If $\mathcal{F}$ and $\mathcal{G}$ have identical dynamics both systems must have the same number of cells ($c(\mathcal{F})=c(\mathcal{G})$) but the input structures of corresponding cells can be quite different.

We define a partial ordering on coupled cell networks, and an associated equivalence relation.

Definition 3.2. Let $\mathcal{N},\mathcal{M}$ be coupled cell networks with $k(\mathcal{N})=k(\mathcal{M})=k$.

(a) We write $\mathcal{N}\prec\mathcal{M}$ if there exist respective orderings $\{\mathbf{A}_1,\dots,\mathbf{A}_k\}$ and $\{\mathbf{B}_1,\dots,\mathbf{B}_k\}$ of the cells of $\mathcal{N}$, $\mathcal{M}$ such that given any

$\mathcal{F} \in \mathcal{N}$, there exists an $\mathcal{F}^\star \in \mathcal{M}$ such that the systems $\mathcal{F}$ and $\mathcal{F}^\star$ have identical dynamics.

(b) We say $\mathcal{N}$ and $\mathcal{M}$ are *equivalent*, $\mathcal{N} \sim \mathcal{M}$, if $\mathcal{N} \prec \mathcal{M}$ and $\mathcal{M} \prec \mathcal{N}$.

Clearly $\prec$ is a transitive relation on coupled cell networks and the relation $\sim$ is an equivalence relation on coupled cell networks.

Remarks 3.3. (1) Equivalence of two coupled cell networks $\mathcal{N}$, $\mathcal{M}$ implies that given a system in the class $\mathcal{N}$, we can replace it by an equivalent system in the class $\mathcal{M}$, and conversely. Both systems will have exactly the same dynamics though, of course, the structure of the networks $\mathcal{N}$ and $\mathcal{M}$ may appear quite different.
(2) If $\mathcal{F}, \mathcal{F}' \in \mathcal{N}$ have the same dynamics, it does not follow that $\mathcal{F} = \mathcal{F}'$. That is, the $\mathcal{F}$- and $\mathcal{F}'$-cells can be quite different but when connected according to the architecture $\mathcal{N}$ may give rise to identical dynamics. An example of a non-trivial self-equivalence is given in [6, Example 4.11].

We give examples of the use of these relations in the next section. For now we present a simple example to illustrate the ideas.

Example 3.4. In figure 8 we show two coupled cell networks, both consisting of identical cells. In the first network $\mathcal{N}$ each cell has two inputs of different types (the cell has *asymmetric* inputs). In the second network $\mathcal{M}$, each cell has two inputs of the same type (inputs are symmetric). Assume the phase space for cells is $\mathbb{R}^N$. We claim that $\mathcal{N} \sim \mathcal{M}$. Suppose first that $\mathcal{F} \in \mathcal{N}$. There exists $f : \left(\mathbb{R}^N\right)^3 \rightarrow \mathbb{R}^N$ such that dynamics on the cells $\mathbf{A}_1$, $\mathbf{A}_2$ are given by

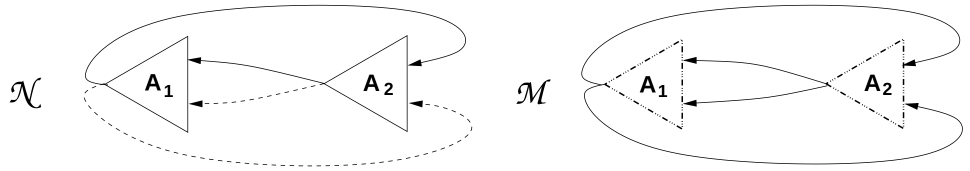


FIGURE 8. $\mathcal{N}$, $\mathcal{M}$: two input identical cell networks. Network $\mathcal{N}$ has asymmetric inputs; network $\mathcal{M}$ has symmetric inputs.

$$\begin{aligned} \mathbf{x}_1' &= f(\mathbf{x}_1; \mathbf{x}_2, \mathbf{x}_2), \\ \mathbf{x}_2' &= f(\mathbf{x}_2; \mathbf{x}_1, \mathbf{x}_1). \end{aligned}$$

There are two ways we can define $g : \left(\mathbb{R}^N\right)^3 \rightarrow \mathbb{R}^N$ to establish the equivalence: we can vary the input structure or the output structure.

These approaches give rise to the ideas of input and output equivalence and are developed in [5, 6]. We vary input structure by defining

$$g(\mathbf{x};\mathbf{y},\mathbf{z}) = f\left(\mathbf{x};\frac{\mathbf{y}+\mathbf{z}}{2},\frac{\mathbf{y}+\mathbf{z}}{2}\right).$$

We vary output structure by taking

$$g(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{1}{2}\left(f(\mathbf{x};\mathbf{y},\mathbf{z}) + f(\mathbf{x};\mathbf{z},\mathbf{y})\right).$$

Observe that in either case $g(\mathbf{x};\mathbf{y},\mathbf{z})$ is symmetric in $\mathbf{y},\mathbf{z}$ and so defines a coupled cell system $\mathcal{F}^\star \in \mathcal{M}$ which has the same dynamics as $\mathcal{F}$ (since $g(\mathbf{x};\mathbf{y},\mathbf{y}) = f(\mathbf{x};\mathbf{y},\mathbf{y})$, for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$). Hence $\mathcal{N} \prec \mathcal{M}$. Conversely suppose $\mathcal{F}^\star \in \mathcal{M}$ is defined by $g : \left(\mathbb{R}^N\right)^3 \to \mathbb{R}^N$. Then we may take $f = g$ so as to define a coupled cell system $\mathcal{F} \in \mathcal{N}$ (one may disregard the symmetry in the second and third variable). Hence $\mathcal{M} \prec \mathcal{N}$ and $\mathcal{N} \sim \mathcal{M}$. For this example one can check that $\mathcal{N}$ and $\mathcal{M}$ are equivalent to the two identical cell coupled cell network for which each cell has a single input coming from the other cell. For the remainder of this article, we realize equivalence using input equivalence as this is particularly simple for cells with asymmetric inputs [5]. This approach may fail if there are symmetric inputs or if the phase space is a differential manifold, such as a torus (we refer to [5, 6] for examples).

Remarks 3.5. (1) If we had disallowed internal variables, then we would still have $\mathcal{M} \sim \mathcal{N}$, by the same argument. Note that both networks have a symmetry defined by interchanging cells. The assumption of identical inputs does not force any extra symmetry. However, either network with internal variables is definitely not equivalent to the corresponding network without internal variables. For example, if we deny internal variables and outputs from a cell into itself, then the resulting flow is divergence free and so volume preserving (cf. Dias and Lamb [17]).

(2) In the setting of symmetry groupoids, Dias and Stewart [18] have considered the problem of equivalence in coupled cell networks with Euclidean phase space and continuous C^∞ dynamics. Their main result shows that coupled cell networks are equivalent if and only if they are equivalent when we restrict to linear vector fields on cells with 1-dimensional dynamics.

(3) In terms of adjacency matrices, two network architectures (with k cells) are equivalent if and only we can choose an ordering of the cells of both networks so that the adjacency matrices span the same subspace of $k \times k$ matrices (if we assume internal variables, the identity matrix is included in the spanning set). We refer to [18, 5, 6] for details.

Aguiar and Dias [8] have considered the problem of finding minimal models for coupled cell networks where by 'minimal' is meant finding representative(s) of the equivalence class with the smallest number of connections.
(4) Suppose that we consider the set Net(k) of networks with k identical cells. We assume asymmetric inputs. We can construct a maximal or *universal* network $\mathcal{U}$ in Net(k) for the order $\prec$. That is, for every $\mathcal{F} \in \mathcal{N} \in \text{Net}(k)$, there exists $\mathcal{F}^\star \in \mathcal{U}$ such that $\mathcal{F} \prec \mathcal{F}^\star$. Using the algebraic condition for equivalence described in (3) above, one can check that $\mathcal{U}$ can be chosen to have at most $k^2 - k + 1$ inputs (necessarily asymmetric). We will give specific examples later for the case $k = 2$. In general, even if $\mathcal{U}$ is universal and has the minimal number of inputs, $\mathcal{U}$ will not be unique (even up to isomorphism of associated graph structure).

3.2. **Synchrony subspaces.** Synchrony subspaces are discussed in some detail in [21] as well as in works that develop the balanced equivalence relation approach to coupled networks (for example, [55, 24, 28, 54, 9]).

Let $\mathcal{N}$ be a coupled cell network modelled on $\mathfrak{C}$. Let $k = k(\mathcal{N})$ be the number of cells and $c = c(\mathcal{N})$ the number of cell classes.

Suppose that $\mathcal{X} = \{X^j \mid 1 \leq j \leq p\}$ is a partition of the set of cells of $\mathcal{N}$ such that

1. Each X^j consists of cells of the same class.
2. Each X^j contains at least one cell and at least one X^j contains two or more cells.

Let $d(j) = |X^j|$ be the number of cells in X^j, $1 \leq j \leq p$.

If $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ denotes the state of the system, then we may group vectors according to the partition $\mathcal{X}$ and write

$$\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^p),$$

where $\mathbf{x}^j = (\mathbf{x}_{j_1}, \dots, \mathbf{x}_{d(j)})$ will denote the state of the $d(j)$ cells in X^j. Since all the cells in X^j are of the same class, the cells in X^j will all have the same underlying phase space, irrespective of whether we are looking at a continuous, discrete or hybrid system. Define

$$\Delta_j = \{\mathbf{x}^j \mid \mathbf{x}_{j_1} = \cdots = \mathbf{x}_{d(j)}\}, \ \ 1 \leq j \leq p$$

and

$$\Delta(\mathcal{X}) = \{\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^p) \mid \mathbf{x}^j \in \Delta_j, \ \ 1 \leq j \leq p\}.$$

Definition 3.6. The partition $\mathcal{X}$ is a *synchrony class* for the network $\mathcal{N}$ if the subspace $\Delta(\mathcal{X})$ is invariant for every realization of $\mathcal{N}$ as a coupled cell system. If $\mathcal{X}$ is a synchrony class then we say $\Delta(\mathcal{X})$ is a *synchrony subspace* (or a *polydiagonal subspace* in the terminology of [27]).

With a slight abuse of notation we will sometimes refer to a synchrony subspace of a network rather than a system.

Remarks 3.7. (1) Notice that if $d(j) = 1$, then there are no restrictions placed on the evolution of $\mathbf{x}^j$. Instead of allowing for partitions of the set of cells, we might equally have looked at families of mutually disjoint subsets of the set of cells with each subset consisting of at least two cells of the same type.
(2) If we partition the set of cells by class, then the corresponding subspace is always a synchrony subspace (excluding the trivial case $k(\mathcal{N}) = c(\mathcal{N})$). We call this partition the *maximal synchrony class* and denote it by $\mathbf{S} = \mathbf{S}(\mathcal{N})$. The associated space to this is the *maximal synchrony subspace* $\mathbf{S} = \Delta(\mathbf{S})$. Note that the maximal synchrony subspace is the synchrony subspace of minimal dimension for any given realization of a coupled cell network. In particular, it is a subspace of every synchrony subspace. Following [21], we use the term *null* (or *minimal*) synchrony class for the space of desynchronized cells — that is, the null synchrony subspace is the complement of the union of all synchrony subspaces. In terms of partitions, it is defined by the partition into singletons. The null synchrony subspace will be invariant for the dynamics of all invertible systems in $\mathcal{N}$ (that is, systems defined by continuous ODEs or invertible maps).
(3) Many examples of synchrony subspaces described using our approach may be found in [21, 4]. For examples based on the approach of Stewart *et al.* see [55, 24, 28].
(4) The simplest way of finding synchrony subspaces is to look at initial conditions. If we can find a coupled cell system realizing a network and can choose initial conditions in the proposed synchrony subspace determined by the partition $\mathcal{X}$ that lead to evolution out of the subspace then the subspace cannot be a synchrony subspace. In particular, if we can choose (identical) initial conditions for the cell(s) in a partition set $X^j \in \mathcal{X}$ that lead to *different* inputs on cells in the partition set $X^i \in \mathcal{X}$ (possibly equal to X^j), then the subspace cannot be a synchrony subspace. (The sufficiency of this condition for continuous dynamics follows by uniqueness of solutions differential equations — assuming that equations are Lipschitz.) We give a simple application of this observation in the proof of the next theorem.

Given distinct partitions $\mathcal{X}_j$, $j = 1, 2$ we denote by $\mathcal{X}_{12}$ the partition

$$\{X_1^i \cap X_2^j \mid X_1^i \in \mathcal{X}_1, X_2^j \in \mathcal{X}_2\}.$$

Theorem 3.8. *Let $\mathcal{N}$ be a coupled cell network with asymmetric inputs. Suppose that $\Delta(\mathcal{X}_j)$ are synchrony subspaces associated to the partitions $\mathcal{X}_j$, $j = 1, 2$. Then the partition $\mathcal{X}_{12}$ defines a synchrony subspace of $\mathcal{N}$ provided that there exist i, j such that $|X_1^i \cap X_2^j| > 1$.*

Proof. Observe first that $X_1^i \cap X_2^j = \emptyset$ unless X_1^i, X_2^j consist of cells of the *same* class. Suppose that $X_1^i \cap X_2^j \neq \emptyset$ and that $|X_1^i \cap X_2^j| = d > 1$ (there is nothing to prove if $d = 1$). Choose two different cells $\mathbf{C}_\alpha, \mathbf{C}_\beta \in X_1^i \cap X_2^j$. We consider the initial conditions on $\mathbf{C}_\alpha, \mathbf{C}_\beta$ at time $t = 0$ under the assumption that the cell states in $X_1^\ell \cap X_2^m$ are equal at $t = 0$, for all $X_1^\ell \cap X_2^m \in \mathcal{X}_{12}$. Let $\mathbf{u}$, $\mathbf{v}$ denote the initial condition on the qth input of $\mathbf{C}_\alpha$, $\mathbf{C}_\beta$ respectively, $q \geq 1$. Since $X_1^i \cap X_2^j \subset X_1^i \in \mathcal{X}_1$, there exists $X_1^\ell \in \mathcal{X}_1$ such that $\mathbf{u}$, $\mathbf{v}$ are outputs of cells in X_1^ℓ. Similarly, there exists $X_2^m \in \mathcal{X}_2$ such that $\mathbf{u}$, $\mathbf{v}$ are outputs of cells in X_2^m. Therefore, $\mathbf{u}$, $\mathbf{v}$ are outputs of cells in $X_1^\ell \cap X_2^m$ and so are equal. This holds for all cell pairs and all inputs. Hence $\mathcal{X}_{12}$ is a synchrony subspace by remarks 3.7(4). ☐

Remark 3.9. Theorem 3.8 is *false* if we allow symmetric inputs. For an example, see [21, Example 6.21] and also section 7.1.

The next lemma will be useful later.

Lemma 3.10. *Let $\mathcal{N}$ be a transitive coupled cell network. Suppose that there exist distinct partitions $\mathcal{X}_1 = \{X_1^1, X_1^2\}$, $\mathcal{X}_2 = \{X_2^1, X_2^2\}$ defining synchrony subspaces (we always assume $X_j^i \neq \emptyset$, $i, j = 1, 2$). Then the partition $\mathcal{X}_{12}$ contains either three or four sets.*

Proof. If there exists i, j such that $X_1^i \subset X_2^j$ then $\mathcal{X}_{12}$ contains three sets. Otherwise $\mathcal{X}_{12}$ contains four sets. ☐

Definition 3.11. A synchrony subspace $\Delta(\mathcal{X})$ is *basic* if there is exactly one subset $X^j \in \mathcal{X}$ which contains more than one element. A basic synchrony subspace is *atomic* if it is not contained in any other synchrony subspace.

Remark 3.12. A basic synchrony subspace corresponds to a set of synchronized identical cells. If $\mathcal{F}$ is a coupled cell network consisting of identical cells then the maximal synchrony subspace is always basic and is atomic if and only if it is the only synchrony subspace of $\mathcal{N}$.

3.3. **Dynamics on a synchrony subspace.**

Let $\mathcal{N}$ be a coupled cell network and suppose that $\Delta(\mathcal{X})$ is the synchrony subspace determined by the synchrony class $\mathcal{X} = \{X^j \mid 1 \leq j \leq p\}$. Associated to $\mathcal{X}$ we may naturally define a coupled cell network $\mathcal{N}_{\mathcal{X}}$ in such a way that for every coupled cell system $\mathcal{F} \in \mathcal{N}$, there is a coupled cell system $\mathcal{F}_{\mathcal{X}} \in \mathcal{N}_{\mathcal{X}}$

which realizes the dynamics of $\mathcal{F}$ restricted to the synchrony subspace $\Delta(\mathcal{X})$.

We briefly recall the construction of this network using the repatching approach of [21]. We start by defining the cells of $\mathcal{N}_{\mathcal{X}}$. We choose exactly one cell $\mathbf{D}_j$ from each set $X^j \in \mathcal{X}$, $1 \leq j \leq p$. Observe that the only case where there is any choice is when $d(j) > 1$. It remains to define connections. If $\mathbf{D}_k$ receives a total of ℓ inputs of a given type from the cells in X^j, then we require that $\mathbf{D}_k$ receives ℓ inputs of that type from $\mathbf{D}_j$. Note that if $d(j) = 1$, then the connections from $\mathbf{D}_j$ to $\mathbf{D}_k$ are exactly the same as in $\mathcal{N}$.

The construction of $\mathcal{N}_{\mathcal{X}}$ is presented in [28] using a different terminology and the network $\mathcal{N}_{\mathcal{X}}$ is referred to as a 'quotient' network.

Notation for synchrony classes and subspaces. Let $\Delta(\mathcal{X})$ be the synchrony subspace determined by the synchrony class $\mathcal{X} = \{X^j \mid 1 \leq j \leq p\}$. Let $\mathcal{X}' = \{X^j \mid d(j) > 1\}$. Each $X^j \in \mathcal{X}'$ corresponds to a group of at least two cells. If $\mathcal{X}' = \{X^{j_1}, \ldots, X^{j_k}\}$, we denote the synchrony subspace (or class) by $\{X^{j_1} \| X^{j_2} \| \ldots \| X^{j_k}\}$. Typically, we expand each X^{j_i} to identify the individual cells. For example, suppose that $\mathcal{X}' = \{X^2\}$ and $X^2 = \{\mathbf{C}_1, \mathbf{C}_4\}$. Then we denote the associated synchrony subspace by $\{\mathbf{C}_1, \mathbf{C}_4\}$. If instead, $\mathcal{X}' = \{X^1, X^2\}$ and $X^1 = \{\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_5\}$, $X^2 = \{\mathbf{C}_3, \mathbf{C}_4\}$, then we denote the associated synchrony subspace by $\{\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_5 \| \mathbf{C}_3, \mathbf{C}_4\}$. In the first case the notation indicates that it is possible for the cells $\mathbf{C}_1, \mathbf{C}_4$ to be synchronized in any coupled cell system with this network structure. In the second example, the notation indicates that it is possible for the groups of cells $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_5$ and $\mathbf{C}_3, \mathbf{C}_4$ to be synchronized, but does not imply that $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4, \mathbf{C}_5$ are all synchronized (the cells need not even be of the same class).

3.4. Inflations of coupled cell networks.

In this section we describe a construction that can be regarded as an inverse to the construction of the 'quotient' network associated to a synchrony class. As we see later, this construction is very useful as an aid to constructing architectures which can support heteroclinic cycles and heteroclinic networks.

Let $\mathcal{N}$ be a coupled cell network with cell set $\mathcal{C} = \{C_1, \cdots, C_k\}$. A coupled cell network $\mathcal{M}$ is an *inflation* of $\mathcal{N}$ if there exists a map $\Pi : \mathcal{M} \rightarrow \mathcal{N}$ sending cells to cells, connections to connections (preserving type) such that

1. $\{\Pi^{-1}(C_1) \| \Pi^{-1}(C_2) \| \cdots \| \Pi^{-1}(C_k)\}$ is a synchrony subspace of $\mathcal{M}$.
2. Π maps the set of connections in $\mathcal{M}$ onto the set of connections in $\mathcal{N}$. More precisely, there is a connection of type ℓ from

C_i to C_j, if and only if there exist cells $C_{i\alpha} \in \Pi^{-1}(C_i)$ and $C_{j\beta} \in \Pi^{-1}(C_j)$ such that there is a connection of type ℓ from $C_{i\alpha}$ to $C_{j\beta}$.

Remarks 3.13. (1) It is trivial that if $\mathcal{M}$ is an inflation of $\mathcal{N}$, then $\mathcal{N}$ is the quotient network $\mathcal{M}_{\{\Pi^{-1}(C_1)\|\Pi^{-1}(C_2)\|\cdots\|\Pi^{-1}(C_k)\}}$ of $\mathcal{M}$.
(2) In this paper we always require that an inflation of a transitive network is transitive. Sometimes we emphasize this by saying 'transitive inflation'.

Of particular interest are what we term *simple* inflations. Let $p \geq 2$ and let $\mathbf{C}_i \in \mathcal{C}$. We say a coupled cell network $\mathcal{M}$ is a *(p-fold) simple inflation* of $\mathcal{N}$ at $\mathbf{C}_i$ if

1. The set of cells of $\mathcal{M}$ is equal to $\mathcal{C} \setminus \{\mathbf{C}_i\} \cup \{\mathbf{C}_{i1} \ldots, \mathbf{C}_{ip}\}$, where each cell $\mathbf{C}_{ij}$ has the same class as $\mathbf{C}_i$.
2. $\{\mathbf{C}_{i1}, \ldots, \mathbf{C}_{ip}\}$ is a synchrony class of $\mathcal{M}$ and the quotient network structure on the synchrony subspace $\{\mathbf{C}_{i1}, \ldots, \mathbf{C}_{ip}\}$ is equal to that of $\mathcal{N}$,

Remarks 3.14. (1) We give a number of examples of network inflation in section 6. For the present we emphasize that even a simple inflation need not be uniquely determined up to equivalence and that the number of p-fold simple inflations can grow very rapidly with the size of the network (see [4]).
(2) The work in [10] develops a systematic method, given a (quotient) identical cell network with one input type, to enumerate all networks that inflate from that quotient. That is, it describes the shape of the adjacency matrices of all networks that can have that quotient. Here we use inflation to construct bigger networks from smaller networks that have interesting dynamics.
(3) If we restrict to transitive networks, then single input unidirectional networks with more than one cell can *never* be simply inflated. A necessary and sufficient condition for $k \geq 2$ input transitive networks $\mathcal{N}$ to admit a p-fold simple inflation to a transitive network at a cell $\mathbf{A}$ is that either $\mathbf{A}$ has a self-input or $\mathbf{A}$ has at least p outputs. In particular, if $\mathbf{A}$ has no self input and only one cell receives an output from $\mathbf{A}$, then there are no simple inflations of $\mathcal{N}$ at $\mathbf{A}$.
(4) If two coupled cell networks $\mathcal{N}$, $\mathcal{M}$ are equivalent and $\mathcal{N}'$ is a simple inflation of $\mathcal{N}$, then it does not follow that there is simple inflation of $\mathcal{M}$ which is equivalent to $\mathcal{N}'$. That is, the set of equivalence classes of simple inflations of $\mathcal{N}$ can be quite different from the set of equivalence classes of simple inflations of $\mathcal{M}$ even though $\mathcal{N}$ and $\mathcal{M}$ are equivalent.

4. Networks of two identical cells

In this and the following section we consider the structure of (and equivalence between) transitive networks consisting of two or three identical cells with no more than three inputs, all of which are asymmetric. Our focus will be on possible configurations of synchrony subspaces and the existence of heteroclinic cycles. The standing assumption of a transitive network means there will be no slaved subnetworks. We will always realize networks as coupled cell systems modelled by ordinary differential equations.

Using the algebraic criterion for equivalence given in terms of the space spanned by the adjacency matrices (remarks 3.5(3)), it is simple to verify that, up to equivalence, there are exactly two classes of transitive networks with identical cells and asymmetric inputs. We show representative networks in figure 9.

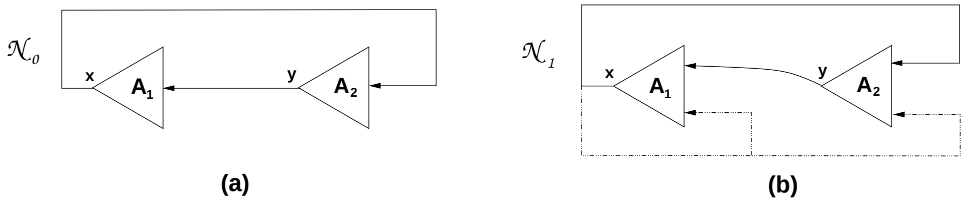


Figure 9. Equivalence classes of transitive 2-cell asymmetric input networks.

In figure 9(a) we show the unidirectional 2-cell network $\mathcal{N}_0$. This network is symmetric (under interchange of cells) and so cannot be equivalent to $\mathcal{N}_1$ which is asymmetric with respect to interchange of cells. If we regard $\mathcal{N}_0$ as a coupled cell system, modelled by an ordinary differential equation, and denote the outputs by $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, then $\mathbf{x}, \mathbf{y}$ will satisfy the differential equations

$$\mathbf{x}' = f(\mathbf{x};\mathbf{y}), \tag{4.2}$$

$$\mathbf{y}' = f(\mathbf{y};\mathbf{x}). \tag{4.3}$$

Here $f : \left(\mathbb{R}^N\right)^2 \to \mathbb{R}^N$ is an arbitrary smooth map. Like all two identical cell systems, there is exactly one synchrony subspace: the maximal synchrony subspace defined by $\Delta(\mathbf{S}) = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} = \mathbf{y}\}$.

The cells in $\mathcal{N}_0$ have just one input. In figure 10, we show two identical cell networks equivalent to $\mathcal{N}_0$ which are built from cells with two inputs.

Although these networks appear more complicated than the network $\mathcal{N}_0$ shown in figure 9(a), it is clear that all three networks should be

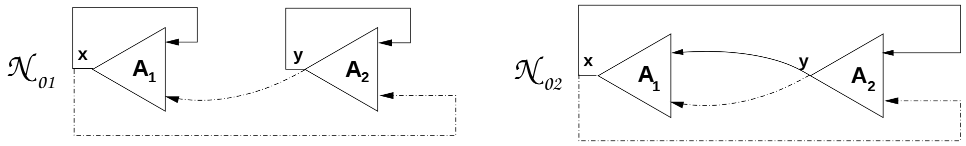


FIGURE 10. The networks $\mathcal{N}_{01}$, $\mathcal{N}_{02}$: two asymmetric input identical cell networks.

equivalent. In particular, for the network $\mathcal{N}_{01}$, we can absorb the cell feedback loops in the internal variables and for $\mathcal{N}_{02}$, we can replace the double connections by a single connection.

We now consider the network of figure 9(b). This network is a universal network for 2 identical cell networks. Specifically, the dynamics of any two identical cell network, with symmetric or asymmetric inputs, can be realized in the architecture of the network $\mathcal{N}_1$ (in particular, $\mathcal{N}_0 \prec \mathcal{N}_1$). In figure 11(a), we show a two cell network $\mathcal{M}$ where cells have three symmetric inputs. If we define a new two (asymmetric) input cell as shown in figure 11(b) and connect this cell according to the architecture $\mathcal{N}_1$ then the resulting system will have dynamics identical to that defined by the original cell in the architecture $\mathcal{M}$.

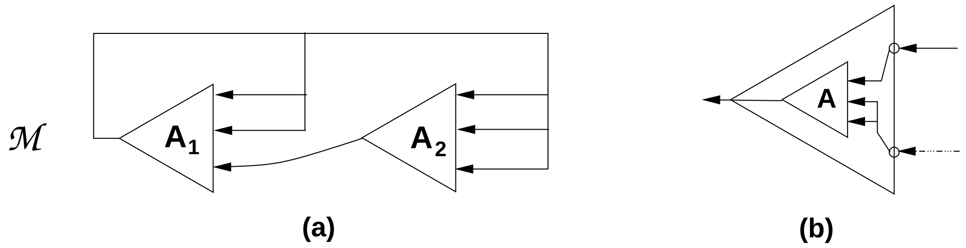


FIGURE 11. The network $\mathcal{M}$.

The network $\mathcal{N}_1$ is not unique amongst universal networks for two identical input cells with the minimal number of inputs. The network $\mathcal{N}_2$ shown in figure 12 is equivalent to $\mathcal{N}_1$ and therefore is also universal.

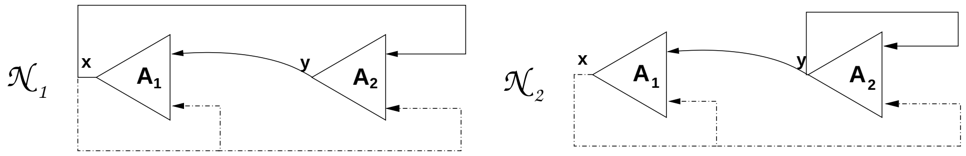


FIGURE 12. The networks $\mathcal{N}_1$, $\mathcal{N}_2$: equivalent 2 cell networks.

A system with network architecture $\mathcal{N}_1$ is modelled by the equations

$$\mathbf{x}' = g(\mathbf{x};\mathbf{y},\mathbf{x}), \tag{4.4}$$

$$\mathbf{y}' = g(\mathbf{y};\mathbf{x},\mathbf{x}), \tag{4.5}$$

and a system with network architecture $\mathcal{N}_2$ by the equations

$$\mathbf{x}' = h(\mathbf{x};\mathbf{y},\mathbf{x}), \tag{4.6}$$

$$\mathbf{y}' = h(\mathbf{y};\mathbf{y},\mathbf{x}). \tag{4.7}$$

We give a direct verification that $\mathcal{N}_1 \sim \mathcal{N}_2$ (using ideas based on input equivalence [5]).

We first prove $\mathcal{N}_2 \prec \mathcal{N}_1$. Suppose h determines the dynamics of the network $\mathcal{N}_2$ according to (4.6,4.7). Define $g : \left(\mathbb{R}^N\right)^3 \to \mathbb{R}^N$ by

$$g(\mathbf{x};\mathbf{y},\mathbf{z}) = h(\mathbf{x};\mathbf{y}-\mathbf{z}+\mathbf{x},\mathbf{z}).$$

We have $g(\mathbf{x};\mathbf{y},\mathbf{x}) = h(\mathbf{x};\mathbf{y},\mathbf{x})$ and $g(\mathbf{y};\mathbf{x},\mathbf{x}) = h(\mathbf{y};\mathbf{y},\mathbf{x})$. Hence the dynamics of the network $\mathcal{N}_1$ defined by g is identical to that of the network $\mathcal{N}_2$ defined by h, proving $\mathcal{N}_2 \prec \mathcal{N}_1$. The proof of the reverse relation $\mathcal{N}_1 \prec \mathcal{N}_2$ is equally simple.

Remarks 4.1. (1) The coupled cell network $\mathcal{N}_2$ is *not* equivalent to $\mathcal{N}_1$ if we assume both networks have *symmetric* inputs. Indeed, we then have $\mathcal{N}_1 \not\prec \mathcal{N}_2$ and $\mathcal{N}_2 \not\prec \mathcal{N}_1$.
(2) At first sight it may seem surprising that the coupled cell networks $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent. This equivalence holds quite generally; for example, if we assume dynamics defined on a manifold or for many networks determined by discrete dynamics. The best way of understanding the equivalence is through the ideas of input and output equivalence and we refer the reader to [5, 6] for more examples and results.

4.1. Robust heteroclinic cycles for two identical cell networks.

Let $\mathcal{N}$ be a two identical cell network and note that we can take $\mathcal{N} \sim \mathcal{N}_0$ or $\mathcal{N} \sim \mathcal{N}_1$. As previously noted, $\mathcal{N}$ has precisely one synchrony subspace — the maximal synchrony subspace, $\mathbf{S}$.

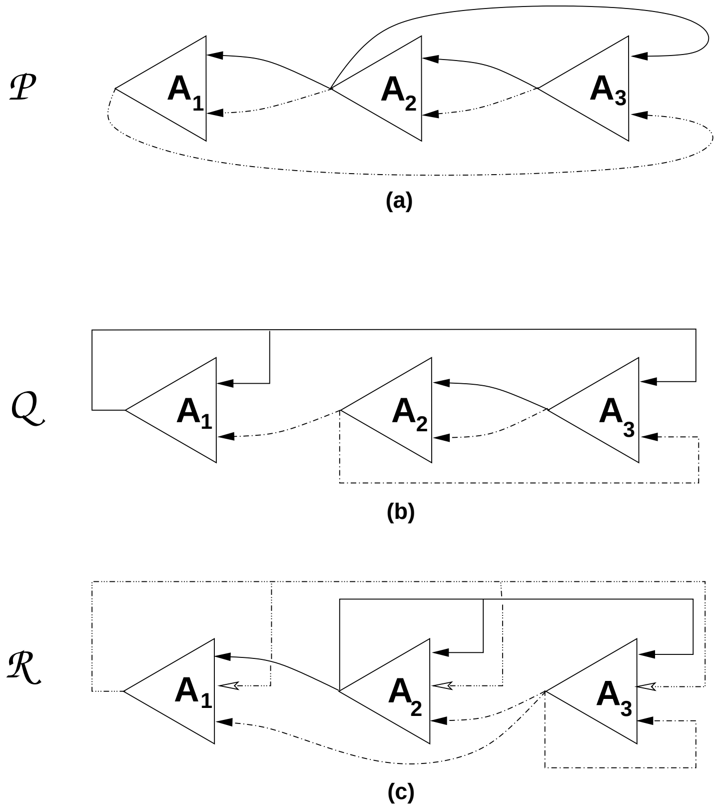
We claim that the ODE coupled cell systems associated with $\mathcal{N}$ do not support robust *simple* heteroclinic cycles between equilibria. Indeed, in view of the robustness requirement, we can assume that equilibria are hyperbolic and that intersections of stable and unstable manifolds of equilibria are transverse off $\mathbf{S}$. Consequently, the only way we can generate robust non-transverse intersections is if equilibria lie on $\mathbf{S}$. We look at the case where there are two equilibria $\mathbf{p},\mathbf{q} \in \mathbf{S}$ and we assume one-dimensional cell dynamics and so $\dim(\mathbf{S}) = 1$ (the general case is no more difficult). A simple local computation (assuming $\mathcal{N} \sim \mathcal{N}_0$ or $\mathcal{N} \sim \mathcal{N}_1$), shows we can assume $\mathbf{p},\mathbf{q}$ are saddles and that there is a one-dimensional saddle connection from $\mathbf{p}$ to $\mathbf{q}$ within $\mathbf{S}$. But now the unstable manifold of $W^u(\mathbf{q})$, and $W^s(\mathbf{p})$ will both be one-dimensional and transverse to $\mathbf{S}$. Generically, we have $W^u(\mathbf{q}) \pitchfork W^s(\mathbf{p})$ and so, since the phase space for $\mathcal{N}$ is two-dimensional, $W^u(\mathbf{q}) \cap W^s(\mathbf{p}) = \emptyset$. Hence there can be no robust simple heteroclinic cycle.

Remarks 4.2. (1) The same argument shows that if $\mathcal{N}$ is a coupled cell network consisting of N-identical cells, $N > 2$, such that the only synchrony subspace of $\mathcal{N}$ is the maximal synchrony subspace, then the coupled cell systems associated with $\mathcal{N}$ do not support robust simple heteroclinic cycles between equilibria. For those familiar with the theory of equivariant dynamics, we emphasize that robustness here is relative to the set of all smooth ODE coupled cell systems realizing the architecture $\mathcal{N}$. In particular, we do not assume any extra structure or limit the number of inputs of the individual cells.
(2) If we drop the requirement that the heteroclinic cycle is simple, then there may exist robust heteroclinic cycles connecting equilibria on the maximal synchrony subspace even when there are no other synchrony subspaces. We indicate an example in the next section.

5. Networks of three identical cells

In this section we consider transitive coupled cell networks with three identical cells and up to three asymmetric inputs. As we shall see, there are two equivalence classes of three identical cell networks that admit robust simple heteroclinic cycles. Conversely, by remark 4.2(1), if a coupled cell network has only the maximal synchrony subspace then it does not admit robust simple heteroclinic cycles. In particular, no coupled cell systems associated with networks of identical cells with single input can admit robust simple heteroclinic cycles. Figure 13(a-c) shows three examples of three cell networks. The only synchrony subspace for network $\mathcal{P}$ is the maximal synchrony subspace $\mathbf{S} = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$. Network $\mathcal{Q}$ has the non-maximal synchrony subspace $\{\mathbf{A}_1, \mathbf{A}_3\}$. Finally, network $\mathcal{R}$ has non-maximal synchrony subspaces $\{\mathbf{A}_1, \mathbf{A}_2\}$, $\{\mathbf{A}_1, \mathbf{A}_3\}$, and $\{\mathbf{A}_2, \mathbf{A}_3\}$. All of these synchrony classes are basic and atomic. Since the number of synchrony classes is obviously an invariant of equivalence, these networks are inequivalent.

Remark 5.1. Unless we assume symmetric inputs, there are no three cell networks with two-input identical cells that admit $\{\mathbf{A}_1, \mathbf{A}_2\}$, $\{\mathbf{A}_1, \mathbf{A}_3\}$, and $\{\mathbf{A}_2, \mathbf{A}_3\}$ as synchrony classes. In figure 14(a) we show an example of a two input identical three cell network that has no non-maximal synchrony subspaces if inputs are regarded as asymmetric but which has synchrony subspaces $\{\mathbf{A}_1, \mathbf{A}_2\}$, $\{\mathbf{A}_1, \mathbf{A}_3\}$, and $\{\mathbf{A}_2, \mathbf{A}_3\}$ if inputs



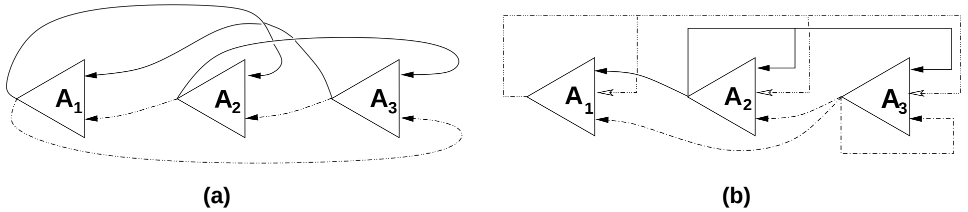
(a)

(b)

(c)

FIGURE 13. Three inequivalent three cell networks with different synchrony subspaces; see text.

are symmetric. In figure 14(b) we show an example of a three cell networks with three asymmetric inputs that has synchrony subspaces $\{\mathbf{A}_1, \mathbf{A}_2\}$, $\{\mathbf{A}_1, \mathbf{A}_3\}$, and $\{\mathbf{A}_2, \mathbf{A}_3\}$.



(a)

(b)

FIGURE 14. 3 cell networks with three non-maximal synchrony subspaces.

Although we can sometimes get more synchrony subspaces with symmetric inputs, the assumption of symmetric inputs typically forces multiple eigenvalues of the linearization at synchronized equilibria. This can make it hard (or impossible) to construct robust *simple* heteroclinic cycles between fully synchronized equilibria. With or without symmetric inputs, the networks of figure 14 do not support robust simple heteroclinic cycles (indeed, any type of heteroclinic cycle in the case of 1-dimensional cell dynamics). Nonetheless, such cycles may be supported if there are higher dimensional phase spaces - an example of this for the network in Figure 14(a) is shown in Figure 3.

Since equilibria for a robust simple heteroclinic cycle must lie on the maximal synchrony subspace, this claim can be verified by computing the linearization of network compatible vector fields at equilibrium points on the maximal synchrony subspace. We find that in the direction transverse to the maximal synchrony subspace, there is always a *two-dimensional* eigenspace which is either contracting or expanding for the flow.

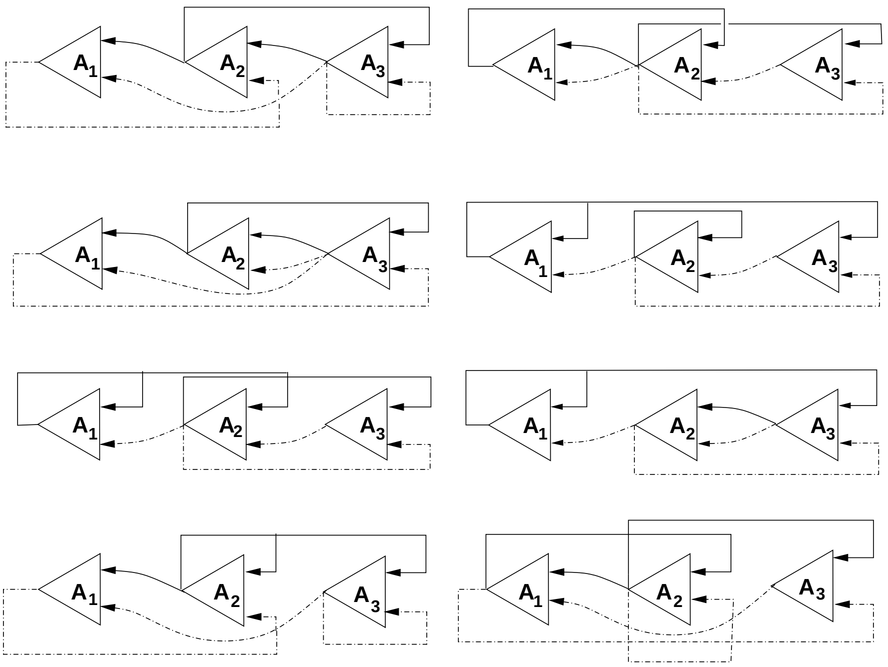


FIGURE 15. Eight inequivalent networks with one non-maximal synchrony class $\{\mathbf{A}_1, \mathbf{A}_3\}$.

Networks with one non-maximal synchrony class. Using the same argument presented in the previous section, it is straightforward to show that a transitive network of three identical cells which has exactly one non-maximal synchrony subspace admits no robust simple heteroclinic cycles. For reference we show the complete set of equivalence classes of two input three cell identical cell networks with one non-maximal synchrony class in figure 15. Observe that each of these networks has $\{\mathbf{A}_1, \mathbf{A}_3\}$ as non-maximal synchrony class. The straightforward verification of this classification, together with that of networks with two non-maximal synchrony subspaces presented below, uses the linear equivalence results of Dias and Stewart [18] and starts off from the classification of one- and two-input three cell homogeneous networks given by Leite and Golubitsky [44]. Alternatively, we can apply simple inflation to the networks $\mathcal{N}_{01}, \mathcal{N}_{02}, \mathcal{N}_1$ and $\mathcal{N}_2$.

Networks with two non-maximal synchrony subspaces. In figure 16, we show (up to equivalence) the only two input three cell identical cell networks that admit two non-maximal synchrony classes.

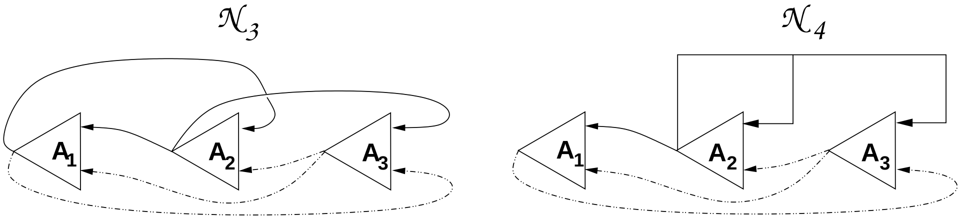


FIGURE 16. The two inequivalent networks with non-maximal synchrony classes $\{\mathbf{A}_1, \mathbf{A}_2\}$, $\{\mathbf{A}_1, \mathbf{A}_3\}$.

5.1. **Robust simple heteroclinic cycles: one dimensional cell dynamics.** We show that the network architectures $\mathcal{N}_3, \mathcal{N}_4$ in figure 16 support the existence of robust attracting simple heteroclinic cycles. We start by verifying this for identical cells governed by one dimensional dynamics (the lowest dimensional case is the hardest on account of consistency conditions forced by the identical cell structure). Specifically, we show that if cells have one dimensional dynamics, then it is possible to find cell dynamics such that there exists a robust asymptotically attracting simple heteroclinic cycle with two equilibria $\mathbf{p}, \mathbf{q}$ both lying on the maximal synchrony subspace. The connections between $\mathbf{p}$ and $\mathbf{q}$ will lie in the non-maximal synchrony subspaces $\{\mathbf{A}_1, \mathbf{A}_2\}$ and $\{\mathbf{A}_1, \mathbf{A}_3\}$. Referring to figure 17, the equilibria $\mathbf{p}, \mathbf{q} \in \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\} = \{\mathbf{A}_1, \mathbf{A}_2\} \cap \{\mathbf{A}_1, \mathbf{A}_3\}$ and are both assumed to be hyperbolic saddle points with 1-dimensional unstable manifolds. We assume $W^u(\mathbf{p}) \subset \{\mathbf{A}_1, \mathbf{A}_3\}$ and $W^u(\mathbf{q}) \subset \{\mathbf{A}_1, \mathbf{A}_2\}$. Since synchrony subspaces are flow invariant, the conditions that the equilibria lie in the maximal synchrony subspace and 1-dimensional unstable manifolds lie in a non-maximal synchrony subspace are preserved under all sufficiently small C^1-perturbations of the vector fields consistent with the network architecture.

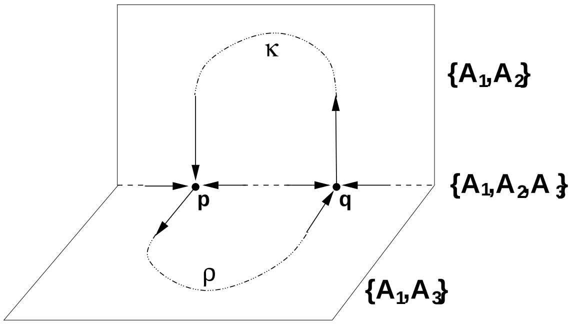


Figure 17. A simple heteroclinic cycle connecting $\mathbf{p}, \mathbf{q}$.

Before we start a detailed analysis we give a heuristic explanation as to why heteroclinic cycles can be expected in coupled cell systems with the architecture $\mathcal{N}_3$ (the argument for $\mathcal{N}_4$ is broadly similar). Consider first a trajectory on the $\{\mathbf{A}_1, \mathbf{A}_2\}$ synchrony subspace. Since both cells $\mathbf{A}_1$ and $\mathbf{A}_2$ receive the same input from $\mathbf{A}_3$, it is possible for the cell $\mathbf{A}_3$ to 'drive' the trajectory towards a fully synchronous equilibrium — for this $\mathbf{A}_3$ only needs one of the inputs from $\mathbf{A}_1$ and $\mathbf{A}_2$ as a reference. The same argument shows that $\mathbf{A}_2$ can drive a trajectory on $\{\mathbf{A}_1, \mathbf{A}_3\}$ towards a fully synchronous equilibrium. This shows that it is reasonable for there to exist connections in $\{\mathbf{A}_1, \mathbf{A}_2\}$ and $\{\mathbf{A}_1, \mathbf{A}_3\}$ between equilibria on the trivial synchrony subspace. For the architecture $\mathcal{N}_3$, the cell $\mathbf{A}_1$ plays a special role in that there are two connections of *different* type from $\mathbf{A}_1$; one to $\mathbf{A}_2$, the other to $\mathbf{A}_3$. If we regard $\mathbf{A}_1$ as a 'controller' cell, this allows the possibility of periodic switching between approximately $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_1, \mathbf{A}_3$ synchronized states. It is straightforward to realize these possibilities using hybrid or threshold dynamics. However, there are some subtleties involved if we work with one dimensional cell dynamics governed by an ordinary differential equation.

Turning now to the details, there are two issues we have to address in the construction of a robust heteroclinic cycle of the type shown in figure 17. The first problem is local: given a point $\mathbf{u} \in \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$, can we choose a vector field consistent with the given coupled cell structure so that $\mathbf{u}$ is a hyperbolic saddle with a one dimensional unstable manifold lying either in $\{\mathbf{A}_1, \mathbf{A}_2\}$ *or* $\{\mathbf{A}_1, \mathbf{A}_3\}$? The second problem is global: given a vector field defined on a neighbourhood of $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$

and with saddle equilibria as shown in figure 17, can we extend the vector field consistent with the coupled cell structure so that we obtain saddle connections between $\mathbf{p}, \mathbf{q}$ defining a simple heteroclinic cycle? We outline below the solution to these problems for the coupled cell network $\mathcal{N}_3$ (a similar analysis applies to $\mathcal{N}_4$).

5.2. **Heteroclinic orbits: local construction.** Differential equations modelling the network $\mathcal{N}_3$ are given by

$$\begin{aligned} \mathbf{x}' &= f(\mathbf{x}; \mathbf{y}, \mathbf{z}), \\ \mathbf{y}' &= f(\mathbf{y}; \mathbf{x}, \mathbf{z}), \\ \mathbf{z}' &= f(\mathbf{z}; \mathbf{y}, \mathbf{x}). \end{aligned} \tag{5.8}$$

For 1-dimensional dynamics, we write x, y, z rather than $\mathbf{x}, \mathbf{y}, \mathbf{z}$. We assume $f : \mathbb{R}^3 \to \mathbb{R}$ is a general C^1-map. Let X_f denote the vector field on $\mathbb{R}^3$ determined by (5.8). Let $\mathbf{p} = (v, v, v)$ lie on the synchrony space $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$. If $f(\mathbf{p}) = 0$, then $\mathbf{p}$ is an equilibrium of X_f. The matrix $J(\mathbf{p})$ of the linearization of X_f at $\mathbf{p}$ is

$$J(\mathbf{p}) = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{bmatrix},$$

where $\alpha = \frac{\partial f}{\partial x}(\mathbf{p})$, $\beta = \frac{\partial f}{\partial y}(\mathbf{p})$, and $\gamma = \frac{\partial f}{\partial z}(\mathbf{p})$. A simple computation verifies that the eigenvalues and eigenlines of $J(\mathbf{p})$ are

$$\begin{array}{lllll} \mu_1 &=& \alpha + \beta + \gamma, & x = y = z & (= \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}), \\ \mu_2 &=& \alpha - \beta, & y = -(1 + \gamma\beta^{-1})x, \ x = z & (\subset \{\mathbf{A}_1, \mathbf{A}_3\}), \\ \mu_3 &=& \alpha - \gamma, & z = -(1 + \beta\gamma^{-1})x, \ x = y & (\subset \{\mathbf{A}_1, \mathbf{A}_2\}). \end{array}$$

It is clear that we can choose α, β, γ so that μ_1, μ_2, μ_3 take any preassigned values.

Remarks 5.2. (1) If $\beta\gamma \neq 0$, then the eigenlines corresponding to μ_2, μ_3 have slopes of opposite sign if and only if $\beta\gamma < 0$,
(2) Similar computations and results hold for the network $\mathcal{N}_4$ of figure 16 (the eigenvalue $\alpha - \beta$ is replaced by α).
(3) If we assume p-dimensional cell dynamics, $p > 1$, then we replace α, β, γ by general $p \times p$ matrices A, B, C respectively. Eigenvalues are then given by the eigenvalues of $A + B + C$, $A - B$, $A - C$. We may choose A, B, C so as to achieve any preassigned set $\mu_1, \dots, \mu_{3p}$ of eigenvalues (real or complex). In particular, we can always choose A, B, C so that there is exactly one eigenvalue $\mu > 0$, with eigenspace in either $\{\mathbf{A}_1, \mathbf{A}_3\}$ or $\{\mathbf{A}_1, \mathbf{A}_2\}$, and all other eigenvalues have negative real part strictly less than $-\mu$.

We start by looking at dynamics near the maximal synchrony subspace $x = y = z$. In figure 18, we show two possible configurations of eigenlines in $\{\mathbf{A}_1, \mathbf{A}_2\}$ and $\{\mathbf{A}_1, \mathbf{A}_3\}$ that are consistent with the stability assignments in figure 17. In both cases $\mu_1 < 0$ at $\mathbf{p}, \mathbf{q}$. In the left hand figure, the $\{\mathbf{A}_1, \mathbf{A}_3\}$-eigenline lies above the $\{\mathbf{A}_1, \mathbf{A}_2\}$ eigenline at $\mathbf{q}$ and below the $\{\mathbf{A}_1, \mathbf{A}_2\}$-eigenline at $\mathbf{p}$. In the right hand figure, $\{\mathbf{A}_1, \mathbf{A}_3\}$-eigenline lies above the $\{\mathbf{A}_1, \mathbf{A}_2\}$ eigenline at $\mathbf{p}$ and $\mathbf{q}$. It is easy to realize either configuration of eigenlines by taking f to be a cubic polynomial.

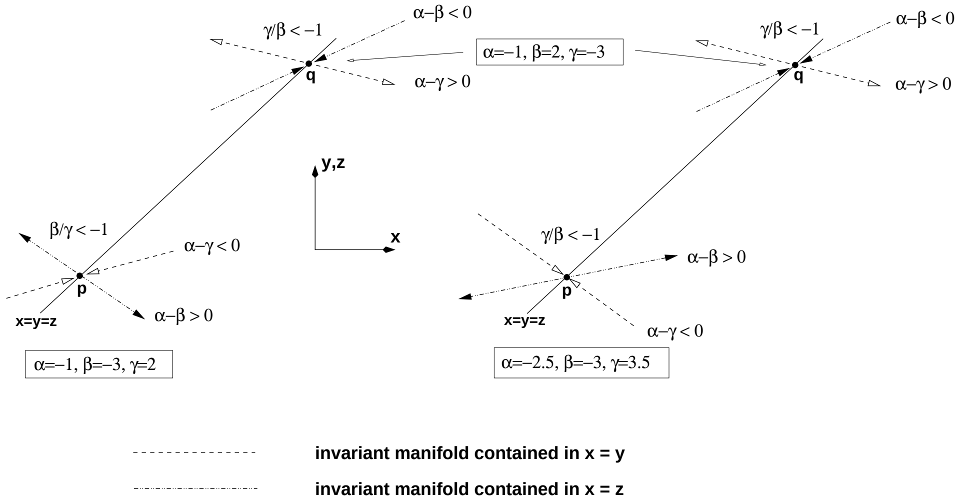


FIGURE 18. Possible configurations of eigenlines — the synchrony subspaces $x = y$ and $x = z$ are projected onto the same plane.

5.3. **Heteroclinic orbits: global construction.** In order to complete our construction we need to show that we can choose f so that there are 1-dimensional connections between $\mathbf{p}, \mathbf{q}$ (the connections ρ, κ shown in figure 17). Since unstable manifolds are 1-dimensional and disconnected by removing the equilibrium, we need to construct four connections to give the maximal number of heteroclinic connections between $\mathbf{p}$ and $\mathbf{q}$.

Let $Y = (Y_x, Y_y)$, $Z = (Z_x, Z_z)$ denote the vector fields obtained from X_f by restriction to the (x, y)-space $\{\mathbf{A}_1, \mathbf{A}_3\}$ and (x, z)-space $\{\mathbf{A}_1, \mathbf{A}_2\}$ respectively. Then Y, Z have the same vertical components: $Y_y(t, s) = Z_z(t, s)$, for all $(t, s) \in \mathbb{R}^2$. Outside of a neighbourhood of the diagonal $\mathbf{S}$, there are no other constraints on Y, Z. We have already shown that there are no obstructions on eigenvalues and eigendirections along the diagonal. If we assume the left hand configuration shown in figure 18, there are no obstructions to extending f outside of a neighbourhood U of the diagonal so as to obtain a pair of connections — it suffices to ensure the connections in the $(x,y)=(x,z)$-space (that is, identifying y and z) do not cross (see figure 19 and note that, following the conventions of figure 17, we let ρ_1,ρ_2 denote the connections from $\mathbf{p}$ to $\mathbf{q}$ and κ_1,κ_2 denote the connections from $\mathbf{q}$ to $\mathbf{p}$).

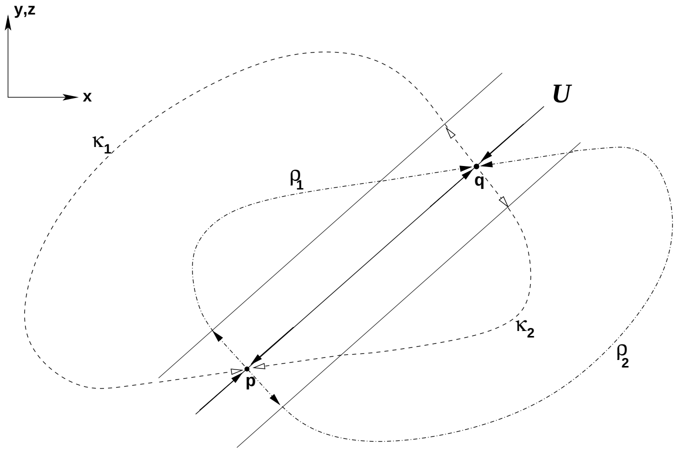


Figure 19. Non-crossing configuration of connections.

If the second configuration of figure 18 holds, the connections must cross in the $(x,y)=(x,z)$-space. Where the connections cross, the vertical components of Y, Z coincide. We can construct an explicit smooth vector field f satisfying these conditions using bump functions. However, it is not so easy to construct a low degree polynomial which gives all four connections. An example of a quartic polynomial vector field f which has a single simple heteroclinic cycle (two connections) which is attracting in the quadrant of $\mathbb{R}^3$ defined by $z\geq x\geq y$ is given by

$$\begin{aligned} f(x;y,z) &= x(1-x^2-y^2-z^2)+0.13x(y-z) \\ &\quad +\beta x^2(y+z)+0.18x^3(y-z)+0.52(y^2-z^2) \\ &\quad +0.17(y+z)+0.25x^2(y^2-z^2), \end{aligned} \tag{5.9}$$

where $\beta=0.498$. In figure 20 we show the time series for the x, y and z which are characteristic of a heteroclinic cycle (initial conditions are $x=0.6313$, $y=0.631$, $z=0.64$). The cycle connects the equilibria $\pm\mathbf{p}=(p,p,p)$, where $p\approx 0.8175$. Heteroclinic cycles exist for these initial conditions for β lying between 0 and 0.4987. For slightly larger values of β, the cycle ceases to be asymptotically attracting and, after a long periodic transient close to the cycle, trajectories converge to an attracting equilibrium[3].

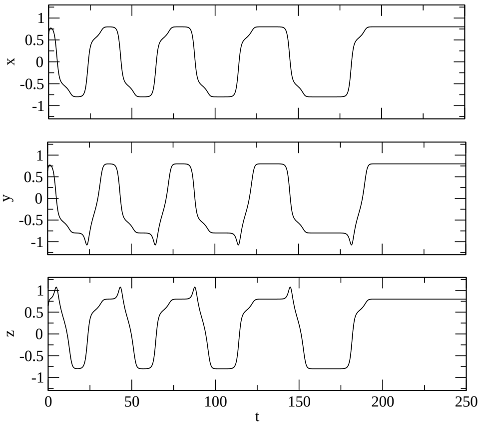


FIGURE 20. Time series x, y, z of (5.8) with f defined by (5.9), giving an attracting simple heteroclinic cycle. Note that there is a switching between two states that are both of from (x, x, x), and a slowing down associated with approach the heteroclinic cycle.

Alternatively, we may introduce a threshold in the cells so that if the state of any of the cells reaches a critical level T, then there is a jump in the state of the cell. We can arrange this so as to produce a connection. Note that we only require a jump in state for the isolated *unsynchronized cell*, not for the pair of synchronized cells. An explicit example can be built using the hybrid dynamics described in section 2.3. In particular, we suggest it should be relatively easy to design a simple three identical cell electronic circuit realizing this network that exhibits a simple attracting heteroclinic cycle.

5.4. **Example: two-dimensional cell dynamics.** For the example we continue to assume the architecture $\mathcal{N}_3$. With two-dimensional cell dynamics, the constructions are simpler as 1-dimensional connections generically do not intersect in the 4-dimensional synchrony subspaces $\mathbf{x} = \mathbf{y}$ and $\mathbf{x} = \mathbf{z}$. Complex eigenvalues may occur for linearizations at equilibria in the maximal synchrony subspace and with associated eigenspaces transverse to the maximal synchrony subspace $\mathbf{x} = \mathbf{y} = \mathbf{z}$. Since the synchrony subspaces $\mathbf{x} = \mathbf{y}$ and $\mathbf{x} = \mathbf{z}$ are of codimension

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[3] Computations were performed using *xppaut* [57] with a fourth order Runge-Kutta integrator and time step 0.05.

two, there is also the possibility of *heteroclinic switching* and *forward switching* [35, 7].

We give some very brief details on a recent example constructed by Agarwal that at least hints at the rich dynamics that can occur in this simple network architecture. Full details will be presented elsewhere and what we reproduce here is with the kind permission of Nikita Agarwal.

Agarwal's equations. Consider the equations (5.8) on $\left(\mathbb{R}^2\right)^3$ for the cubic $f = (f_1, f_2)$ given by

$$\begin{aligned} f_1(\mathbf{x};\mathbf{y},\mathbf{z}) = {}& x_1(1-x_1^2) - 0.4(y_1^3+z_1^3) + x_1\left(\frac{1}{6}(y_1-z_1) + \frac{3}{2}(y_2-z_2)\right) \\ & - \frac{1}{4}(2x_2-y_2-z_2)(y_1^2+z_1^2) + y_1^2 - z_1^2 + y_2^3 + z_2^3 \\ f_2(\mathbf{x};\mathbf{y},\mathbf{z}) = {}& -x_2 + 0.4(y_2+z_2) - 3x_1(y_1-z_1) + \frac{5}{6}x_1(y_2-z_2) \\ & + y_1^2 - z_1^2. \end{aligned}$$

Agarwal shows that if we use f as a model in the architecture $\mathcal{N}_3$, then there are three hyperbolic equillibria $\pm\mathbf{p}$ and $\mathbf{O}$ on the 2-dimensional maximal synchrony subspace $\{\mathbf{x}=\mathbf{y}=\mathbf{z}\}$ given by

$$\mathbf{p} = \left(\sqrt{\frac{1}{2.6}}, 0, \sqrt{\frac{1}{2.6}}, 0, \sqrt{\frac{1}{2.6}}, 0\right), \quad \mathbf{O} = (0,0,0,0,0,0).$$

The equilibria $\pm\mathbf{p}$ have 1-dimensional unstable manifolds as well as complex contracting eigenvalues with eigenspace contained in one of the non-maximal synchrony subspaces and transverse to $\{\mathbf{x}=\mathbf{y}=\mathbf{z}\}$. In the four-dimensional space $\mathbf{x}=\mathbf{y}$, there are connections γ_1, γ_2 from $-\mathbf{p}$ to $\mathbf{p}$. Similarly, in the four-dimensional space $\mathbf{x}=\mathbf{z}$, there are connections γ_3, γ_4 from $\mathbf{p}$ to $-\mathbf{p}$. The heteroclinic cycle consisting of these four connections is robust under perturbations preserving the network architecture and is asymptotically stable.

We indicate an example of the dynamics that can occur using this vector field in figures 21 and 22.

5.5. **Example: phase dynamics.** As a final example, we consider the following three cell coupled system with the same architecture as above, where each cell is described by a phase - i.e. is parametrized by

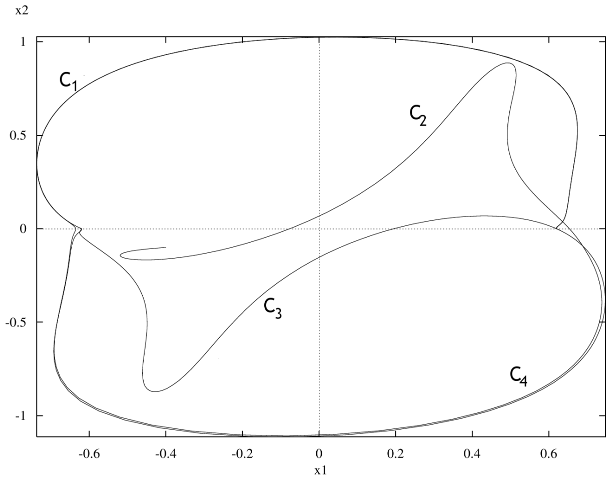


FIGURE 21. Plot showing time evolution of $\mathbf{x} = (x_1, x_2)$. The labels C_i indicate which connection the trajectory $\mathbf{x}(t)$ is tracking in the (x_1, x_2)-plane – each C_i is close to the projection of γ_i in the (x_1, x_2)-plane.

a single periodic variable $\theta_i \in [0, 2\pi)$:

$$\begin{aligned} \theta_1' &= f(\theta_1; \theta_2, \theta_3) \\ \theta_2' &= f(\theta_2; \theta_1, \theta_3) \\ \theta_3' &= f(\theta_3; \theta_2, \theta_1) \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} f(x; y, z) = &p_0 + p_1 \sin(2x) + p_2 \sin(y - z) \sin(x + p_3) \\ &+ p_4 \sin(2(y - z)) \sin(x + p_5) + p_6 \sin(y + z + p_7) \end{aligned} \tag{5.11}$$

and we choose parameters

$$p_0 = 2.0 \; p_1 = 1.9 \; p_2 = 1.5 \; p_3 = -1 \; p_4 = 1.3 \; p_5 = 5 \; p_6 = 0.7 \; p_7 = -0.5.$$

For the system (5.10,5.11) with these parameters, there is an attracting robust heteroclinic cycle that is shown in figure 23 (a) in a simulation for small amount of added noise - (b) shows the schematic structure of the heteroclinic cycle where the two connections from the saddles wind around the torus in a nontrivial way. For other choices of f we suspect there may be "heteroclinic ratchets" in this system where only one branch of each unstable manifold winds nontrivially on the torus [36].

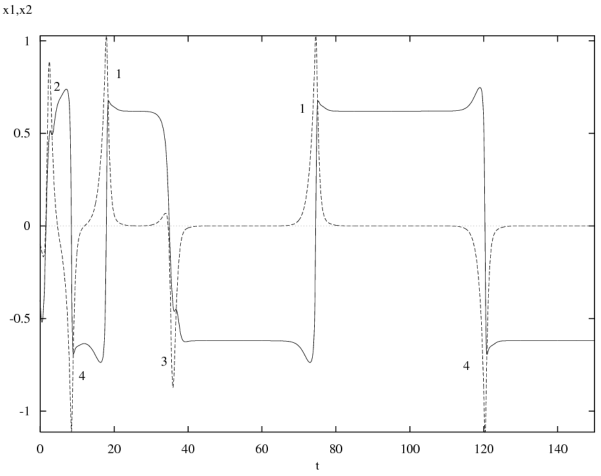
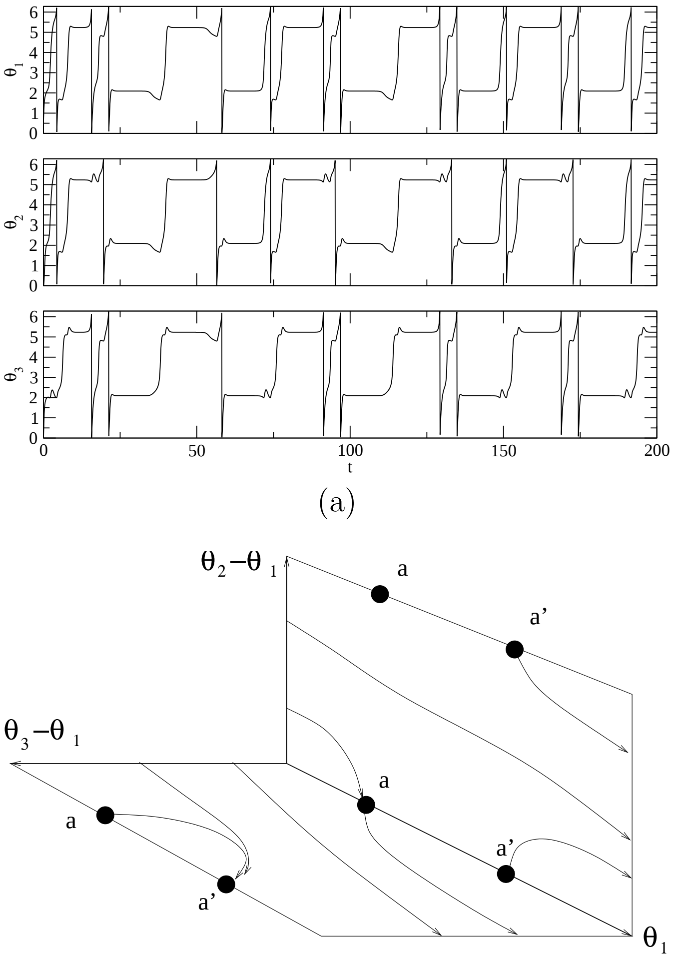


FIGURE 22. Plots of components of corresponding variables $x_1(t), x_2(t)$ from three two-dimensional cells coupled as (5.8) from Agarwal's example. Observe the switching between connections and the clear "slowing down" - a sign of asymptotic stability of the heteroclinic cycle.

Remark 5.3. Although we have emphasized robust simple cycles in this section, it is possible for systems with three identical cells, with symmetric or asymmetric inputs and one-dimensional cell dynamics, to support robust non-simple heteroclinic cycles. This is the case even if the only synchrony class is the maximal synchrony class. For example, the network $\mathcal{P}$ of figure 13 supports a coupled cell system with one-dimensional cell dynamics such that there is a robust non-simple heteroclinic cycle connecting equilibria $\mathbf{p}, \mathbf{q} \in \mathbf{S}$. For this example, eigenvalues of the linearizations at $\mathbf{p}, \mathbf{q}$ transverse to $\mathbf{S}$ can be required to be complex. This type of heteroclinic cycle may lead to complex dynamical behaviour such as switching [7].

6. NETWORKS WITH FOUR OR MORE IDENTICAL CELLS

In this section we look at transitive coupled cell networks with more than three cells with a view to identifying network architectures that can support heteroclinic cycles. Even though the number of configurations grows superexponentially in the number of cells, we are able to



(a)

(b)

FIGURE 23. Attracting heteroclinic network for the system (5.10,5.11). (a) The time series in the presence of low amplitude noise. (b) The cycle relative to the invariant subspaces $\theta_1 = \theta_3$ and $\theta_1 = \theta_2$, shown schematically on $\mathbb{T}^3$. Note that the equilibria marked a are identified on the torus by a change in phase by 2π. It can be seen in that the cycle connects saddle equilibria a, a' that are fully synchronized.

identify certain classes of coupled cell networks whose associated coupled cell systems can support simple heteroclinic cycles and networks. We will show that inflation gives a very effective and systematic way of identifying network architectures that can support heteroclinic cycles.

In figure 24, we show a simple example of a configuration that has non-maximal synchrony classes: $\{\mathbf{A}_1, \mathbf{A}_2\}$, $\{\mathbf{A}_3, \mathbf{A}_4\}$, $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_4\}$, and $\{\mathbf{A}_1, \mathbf{A}_2 \| \mathbf{A}_3, \mathbf{A}_4\}$. We recall that the last notation means that $\mathbf{A}_1 = \mathbf{A}_2$ and $\mathbf{A}_3 = \mathbf{A}_4$. The maximal synchrony class is a *proper* subspace of $\{\mathbf{A}_1, \mathbf{A}_2 \| \mathbf{A}_3, \mathbf{A}_4\}$. If we assume one-dimensional cell dynamics (the multidimensional case is no harder), it is straightforward to show that this configuration supports hyperbolic saddle points on $\{\mathbf{A}_1, \mathbf{A}_2 \| \mathbf{A}_3, \mathbf{A}_4\}$ which have a one-dimensional unstable manifold contained in either $\{\mathbf{A}_1, \mathbf{A}_2\}$ or $\{\mathbf{A}_3, \mathbf{A}_4\}$. Along similar lines to those we used to study simple heteroclinic cycles in coupled cell systems associated to networks with three identical cells, we can construct simple attracting heteroclinic cycles connecting two equilibria in $\{\mathbf{A}_1, \mathbf{A}_2 \| \mathbf{A}_3, \mathbf{A}_4\}$.

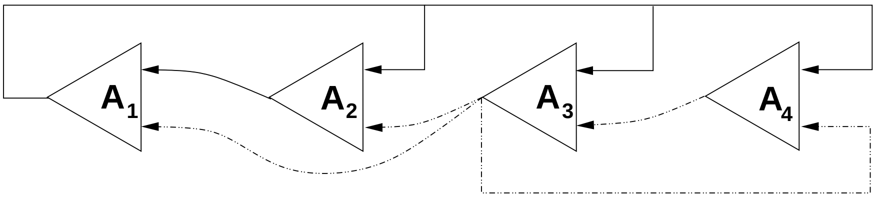


FIGURE 24. A four cell network supporting a heteroclinic cycle.

Remarks 6.1. (1) If we assume identical (symmetric) inputs in figure 24, there are no robust simple heteroclinic cycles.
(2) Assuming one-dimensional cell dynamics, we can choose a vector field compatible with the coupled cell structure of figure 24 which has a hyperbolic saddle point on the maximal synchrony subspace, such that the saddle has one-dimensional unstable manifold lying in either $\{\mathbf{A}_1, \mathbf{A}_2\}$ or $\{\mathbf{A}_3, \mathbf{A}_4\}$. It is then straightforward to show that the coupled cell systems associated to this coupled cell network can support simple attracting heteroclinic cycles joining two equilibria in $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ and with connections lying in $\{\mathbf{A}_1, \mathbf{A}_2\} \cup \{\mathbf{A}_3, \mathbf{A}_4\}$.
(3) If we interchange the inputs of cells $\mathbf{A}_3, \mathbf{A}_4$ (so that the output from $\mathbf{A}_1$ goes into the lower input, not the upper), then the coupled cell systems associated to the new network, which is not equivalent to the original network, can support robust simple heteroclinic cycles. (However, for the coupled cell systems associated to this network there do not exist hyperbolic saddle points on the maximal synchrony subspace with one-dimensional unstable manifold contained in either $\{\mathbf{A}_1, \mathbf{A}_2\}$ or $\{\mathbf{A}_3, \mathbf{A}_4\}$.)

(4) If we interchange the inputs of $\mathbf{A}_4$, the resulting network has one non-maximal synchrony class $\{\mathbf{A}_1, \mathbf{A}_2\}$. Another variation, shown in figure 25, again has the single non-maximal synchrony class $\{\mathbf{A}_1, \mathbf{A}_2\}$.
(5) If instead of assuming all four cells are identical, we suppose that the cells $\mathbf{A}_1$ and $\mathbf{A}_2$ are identical and the cells $\mathbf{A}_3$ and $\mathbf{A}_4$ are identical, then the coupled cell systems associated to the network will admit simple attracting heteroclinic cycles connecting two equilibria. The previous analysis continues to apply.

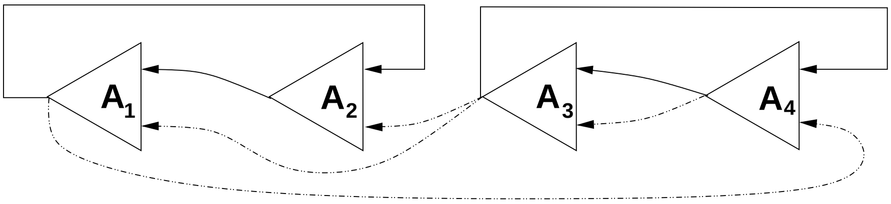


FIGURE 25. A network with one non-maximal synchrony class $\{A_1, A_2\}$ in addition to maximal synchrony.

6.1. **Inflations and quotients of coupled cell networks.** The previous examples show that a more structured approach is needed for the determination and analysis of dynamics in multi-cell networks. What we do is start with a specific identical cell network exhibiting interesting dynamics (for example, one of the three cell architectures $\mathcal{N}_3, \mathcal{N}_4$ that supports robust simple heteroclinic cycles) and then considering it as embedded as a synchrony class in a bigger *transitive* network. The larger network will be an inflation of the smaller network which, in the terminology of Stewart et al. [55, 28], will be a *quotient* of the inflated network (see section 3.4).

Example 6.2. Consider either of the inequivalent identical three cell networks $\mathcal{N}_3, \mathcal{N}_4$ (see figure 16). The coupled cell systems associated to both networks can support heteroclinic cycles and have non-maximal synchrony subspaces $\{\mathbf{A}_1, \mathbf{A}_2\}$ and $\{\mathbf{A}_1, \mathbf{A}_3\}$. The two identical cell network associated to the synchrony classes $\{\mathbf{A}_1, \mathbf{A}_2\}$ and $\{\mathbf{A}_1, \mathbf{A}_3\}$ will be one of the three equivalent networks shown in figure 26. Specifically, the $\{\mathbf{A}_1, \mathbf{A}_2\}$ synchrony class determines the network $\mathcal{N}_1^a$ for both $\mathcal{N}_3$ and $\mathcal{N}_4$. The synchrony class $\{\mathbf{A}_1, \mathbf{A}_3\}$ of $\mathcal{N}_3$ determines $\mathcal{N}_1$ and the synchrony class $\{\mathbf{A}_1, \mathbf{A}_3\}$ of $\mathcal{N}_4$ determines $\mathcal{N}_2$. Conversely, $\mathcal{N}_3$ is the simple inflation of $\mathcal{N}_1^a$ at $\mathbf{A}_1$ or the simple inflation of $\mathcal{N}_1$ at $\mathbf{A}_1$. Similar comments hold for $\mathcal{N}_4$.

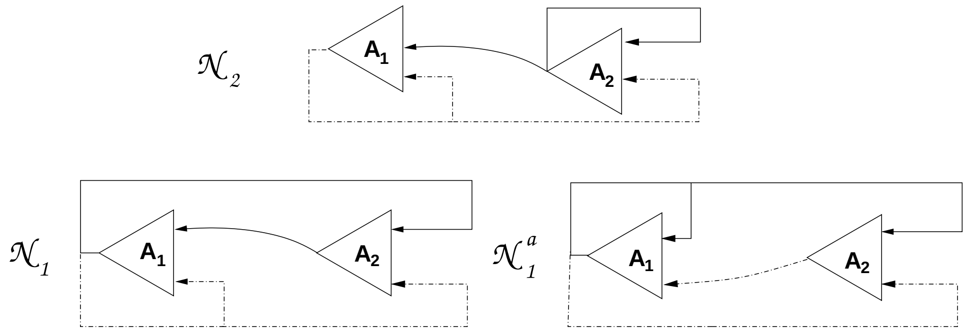


FIGURE 26. Possible quotient networks corresponding to the non-maximal synchrony classes of $\mathcal{N}_3$ and $\mathcal{N}_4$.

Consequently both $\mathcal{N}_3$ and $\mathcal{N}_4$ (the only two input three identical cell network architectures that supports robust heteroclinic cycles) can be obtained by simple inflations of one two-cell networks of figure 26. We remark that five of the network architectures shown in figure 15 arise from a simple inflation of either $\mathcal{N}_1, \mathcal{N}_1^a$ or $\mathcal{N}_2$ and these architectures do not support robust heteroclinic cycles.

Inflating a three-cell network. Next we consider what happens when we inflate $\mathcal{N}_3$ to a four cell network.

Proposition 6.3. *Up to equivalence there are two simple inflations of* $\mathcal{N}_3$ *– see figure 27.*

Proof. Denote the set of cells of $\mathcal{N}_3$ by $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ as in figure 16. Provided we deny slaved subnetworks, there are a total of six four cell networks that can be constructed by simple inflation of a cell in $\mathcal{N}_3$. These determine two equivalence classes of networks. We indicate two representative four cell networks $\mathcal{H}_1, \mathcal{H}_2$ in figure 27. The network $\mathcal{H}_1$ is a simple inflation of $\mathcal{N}_3$ at $\mathbf{A}_1$. The network $\mathcal{H}_2$ is a simple inflation of $\mathcal{N}_3$ at $\mathbf{A}_3$ and is equivalent to either of the two networks we get if we simply inflate $\mathcal{N}_3$ at $\mathbf{A}_2$. □

Following the notation of the above proof, the synchrony classes $\{\mathbf{A}_{i1}, \mathbf{A}_{i2}\}$, $i = 1, 3$, both carry the network architecture $\mathcal{N}_3$ and therefore both synchrony classes support robust simple heteroclinic cycles. Consequently, both network architectures $\mathcal{H}_1, \mathcal{H}_2$ support robust heteroclinic cycles. There remains the issue of whether these cycles can be simple and/or asymptotically stable. It turns out that both network architectures admit a rich synchrony class structure and new synchrony classes are created through the process of inflation. We list all the non-maximal synchrony classes of the networks $\mathcal{H}_1$ and $\mathcal{H}_2$ in table 1. Note

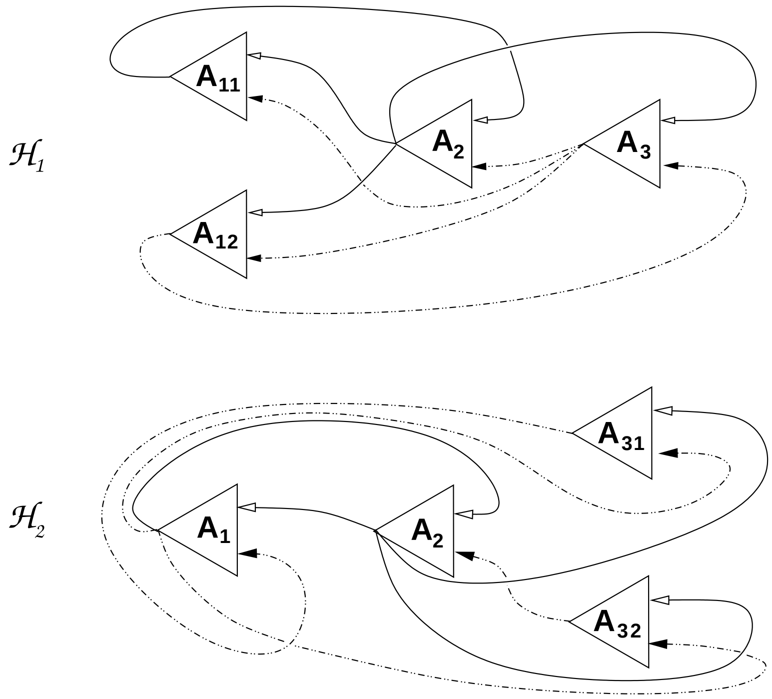


FIGURE 27. Four cell transitive networks obtained by inflating $\mathcal{N}_3$ at $\mathbf{A}_1$ and $\mathbf{A}_3$.

	Synchrony classes that lift	New synchrony classes
$\mathcal{H}_1$	$\{\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_2\}$, $\{\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_3\}$.	$\{\mathbf{A}_{11}, \mathbf{A}_{12}\}$, $\{\mathbf{A}_{11}, \mathbf{A}_2\}$,
		$\{\mathbf{A}_{12}, \mathbf{A}_3\}$, $\{\mathbf{A}_{11}, \mathbf{A}_2 \| \mathbf{A}_{12}, \mathbf{A}_3\}$.
$\mathcal{H}_2$	$\{\mathbf{A}_1, \mathbf{A}_2 \| \mathbf{A}_{31}, \mathbf{A}_{32}\}$, $\{\mathbf{A}_1, \mathbf{A}_{31}, \mathbf{A}_{32}\}$.	$\{\mathbf{A}_{31}, \mathbf{A}_{32}\}$, $\{\mathbf{A}_1, \mathbf{A}_{31}\}$.

TABLE 1. Synchrony classes for simple inflations of $\mathcal{N}_3$.

that we say a synchrony class ‘lifts’ if its existence can be directly inferred from the corresponding class in $\mathcal{N}_3$ by identifying the new cells. Otherwise we refer to the synchrony class as a ‘new class’.

The simple heteroclinic cycle that can occur in the coupled cell systems with architecture $\mathcal{N}_3$ lifts to a heteroclinic cycle in $\mathcal{H}_1$ with with connections in $\{\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_2\}$ and $\{\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_3\}$. A straightforward computation shows that this heteroclinic cycle can be simple and asymptotically stable. It is also easy to see that there exist heteroclinic cycles connecting equilibria on $\{\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_2\}$ with connections lying in $\{\mathbf{A}_{11}, \mathbf{A}_{12}\}$ and $\{\mathbf{A}_{11}, \mathbf{A}_2\}$. Both simple and non-simple cycles exist (see also subsequent comments on the possibility of heteroclinic networks in $\mathcal{H}_1$).

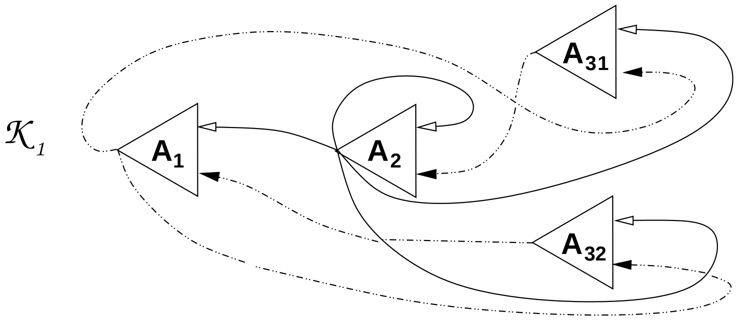


FIGURE 28. A simple inflation of $\mathcal{N}_4$ at $\mathbf{A}_3$.

Proposition 6.4. *(Notation of figure 16.)*

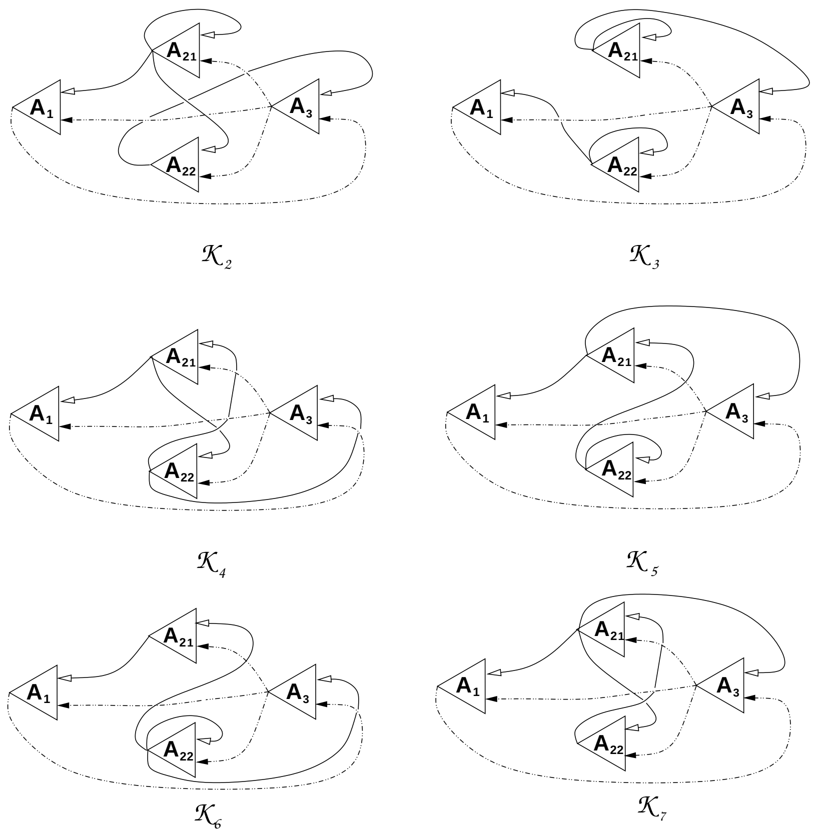
1. *There are no simple transitive inflations of* $\mathcal{N}_4$ *at* $\mathbf{A}_1$.
2. *Up to equivalence, there is one simple transitive inflation of* $\mathcal{N}_4$ *at* $\mathbf{A}_3$ *— see figure 28.*

Proof. Since $\mathbf{A}_1$ has no self-inputs and only one output, there are no simple inflations of $\mathcal{N}_4$ at $\mathbf{A}_1$ (note remarks 3.14(2)).

Turning to the simple inflation of $\mathcal{N}_4$ at $\mathbf{A}_3$, the inputs of $\mathbf{A}_{31}, \mathbf{A}_{32}$ are uniquely determined. Since two inputs are received from the cell $\mathbf{A}_3$, the only choice in the inflation is whether the output of $\mathbf{A}_{31}$ goes to $\mathbf{A}_1$ or $\mathbf{A}_2$. We have shown the case when the output of $\mathbf{A}_{31}$ goes to $\mathbf{A}_2$. It is easy to show that the two networks we obtain are isomorphic. □

Proposition 6.5. *(Notation of figure 16.) Up to equivalence, there are six simple inflations of* $\mathcal{N}_4$ *at* $\mathbf{A}_2$*. These are shown in figure 29.*

Proof. The proof amounts to an enumeration of all the (transitive) inflations of $\mathcal{N}_4$ at $\mathbf{A}_2$ followed by case by case checking of equivalences. As far as checking inequivalence is concerned, it is enough that networks have different synchrony classes and we give in table 2 a complete list of all the synchrony classes of the four cell networks obtained by simple inflation of $\mathcal{N}_4$. Observe that the inequivalence of all seven networks, with the single exception of $\mathcal{K}_3$ and $\mathcal{K}_4$, is immediate from the tabulation of synchrony classes. □



$\mathcal{K}_2$ $\mathcal{K}_3$

$\mathcal{K}_4$ $\mathcal{K}_5$

$\mathcal{K}_6$ $\mathcal{K}_7$

FIGURE 29. Simple inflations of $\mathcal{N}_4$ at $\mathbf{A}_2$.

	Synchrony classes that lift	New synchrony classes
$\mathcal{K}_1$	$\{\mathbf{A}_1, \mathbf{A}_2 \Vert \mathbf{A}_{31}, \mathbf{Y}_{32}\}$, $\{\mathbf{A}_1, \mathbf{X}_{31}, \mathbf{A}_{32}\}$	$\{\mathbf{A}_{31}, \mathbf{A}_{32}\}$, $\{\mathbf{A}_1, \mathbf{A}_{32}\}$.
$\mathcal{K}_2$	$\{\mathbf{A}_1, \mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_3 \Vert \mathbf{A}_{21}, \mathbf{A}_{22}\}$	$\{\mathbf{A}_{21}, \mathbf{Y}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_{21}\}$, $\{\mathbf{A}_1, \mathbf{A}_{22}\}$.
$\mathcal{K}_3$	$\{\mathbf{A}_1, \mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_3 \Vert \mathbf{A}_{21}, \mathbf{A}_{22}\}$	$\{\mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_{22}\}$.
$\mathcal{K}_4$	$\{\mathbf{A}_1, \mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_3 \Vert \mathbf{A}_{21}, \mathbf{A}_{22}\}$	$\{\mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_{22}\}$.
$\mathcal{K}_5$	$\{\mathbf{A}_1, \mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_3 \Vert \mathbf{A}_{21}, \mathbf{A}_{22}\}$	$\{\mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_3\}$.
$\mathcal{K}_6$	$\{\mathbf{A}_1, \mathbf{A}_{21}, \mathbf{A}_{21}\}$, $\{\mathbf{A}_1, \mathbf{A}_3 \Vert \mathbf{A}_{21}, \mathbf{A}_{22}\}$	$\{\mathbf{A}_{21}, \mathbf{A}_{22}\}$.
$\mathcal{K}_7$	$\{\mathbf{A}_1, \mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_3 \Vert \mathbf{A}_{21}, \mathbf{A}_{22}\}$	$\{\mathbf{A}_{21}, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_3\}$, $\{\mathbf{A}_1, \mathbf{A}_{22}\}$, $\{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_{22}\}$.

TABLE 2. Synchrony classes for simple inflations of $\mathcal{N}_4$ at $\mathbf{A}_2$.

Remark 6.6. The network $\mathcal{K}_7$ of figure 29 is isomorphic to the network $\mathcal{H}_1$ of figure 28. That is, $\mathcal{K}_7$ (and so $\mathcal{H}_1$) corresponds to an inflation of

$\mathcal{N}_3$ and an inflation of $\mathcal{N}_4$. This suggests the possibility of the existence of heteroclinic networks for coupled cell systems associated with $\mathcal{K}_7$. At least for one-dimensional cell dynamics, the calculations of sections 5.2 and 5.3 suggest that for the network $\mathcal{K}_7$ there will be two crossing configurations of connections for two of the three flow-invariant planes. Having a connection in one of the planes, one of the vector field components in each of the other two planes is determined. The remaining components of the vector fields for the two other planes are the same, which presents an obstruction to the construction of one of the other two connections. We suspect that this obstruction may always occur when considering networks with only two inputs and one-dimensional dynamics.

Recall that for partitions $\mathcal{X}_j$, $j = 1, 2$, $\mathcal{X}_{12}$ denotes the partition $\{X_1^i \cap X_2^j | X_1^i \in \mathcal{X}_1, X_2^j \in \mathcal{X}_2\}$. Let $r = r(\mathcal{X}_{12})$ denote the number of sets in the partition $\mathcal{X}_{12}$.

Theorem 6.7. *Let $\mathcal{N}$ be a four cell transitive coupled cell network consisting of identical cells with two asymmetric inputs. Suppose that there exist partitions $\mathcal{X}_1 = \{X_1^1, X_1^2\}$ and $\mathcal{X}_2 = \{X_2^1, X_2^2\}$ defining synchrony classes. If the coupled cell networks associated to the synchrony classes defined by $\mathcal{X}_1$ and $\mathcal{X}_2$ correspond to two distinct two cell networks from $\{\mathcal{N}_1, \mathcal{N}_1^a, \mathcal{N}_2\}$ (figure 26), then $r = r(\mathcal{X}_{12}) = 3$ and so the coupled cell network associated to the synchrony class defined by $\mathcal{X}_{12}$ is equivalent to one of the networks $\mathcal{N}_3, \mathcal{N}_4$.*

Proof. By lemma 3.10 we have that $r = r(\mathcal{X}_{12})$ is three or four. Using the transitivity of $\mathcal{N}$, we show that $r \neq 4$. By contradiction, suppose that $r = 4$ and so $|X_i^j| = 2$ for $i, j = 1, 2$. Denote the cells of $\mathcal{N}$ by $\mathbf{C}_1, \ldots, \mathbf{C}_4$. Relabelling cells, we may assume that $\mathcal{X}_1$ defines the synchrony class $\{\mathbf{C}_1, \mathbf{C}_2 \| \mathbf{C}_3, \mathbf{C}_4\}$ and $\mathcal{X}_2$ corresponds to the synchrony class $\{\mathbf{C}_1, \mathbf{C}_3 \| \mathbf{C}_2, \mathbf{C}_4\}$. We consider the case where the coupled cell networks associated to the synchrony classes defined by $\mathcal{X}_1$ and $\mathcal{X}_2$ correspond to the two cell networks $\mathcal{N}_1^a$ and $\mathcal{N}_2$ of figure 26, respectively. (The other cases are addressed in a similar way.) Using the notation of figure 26 for $\mathcal{N}_1^a$, assume that the quotient $\mathcal{N}_1^a$ is determined from by $\mathcal{X}_1$ by taking $\mathbf{A}_1 \in \{\mathbf{C}_1, \mathbf{C}_2\}$ and $\mathbf{A}_2 \in \{\mathbf{C}_3, \mathbf{C}_4\}$. Note that the cell $\mathbf{A}_2$ has only one output going to an input of the second type — shown using a dashed connection in figure 26. Since $\mathcal{N}$ is transitive, $\mathcal{N}$ has one dashed output from the output of the cell $\mathbf{C}_3$ to one of the cells in $\{\mathbf{C}_1, \mathbf{C}_2\}$ and has one dashed output going out from the cell $\mathbf{C}_4$ to the other cell in $\{\mathbf{C}_1, \mathbf{C}_2\}$. Now the coupled cell network defined by the synchrony class $\{\mathbf{C}_1, \mathbf{C}_3 \| \mathbf{C}_2, \mathbf{C}_4\}$ is $\mathcal{N}_2$ and $\mathcal{N}_2$ has no dashed outputs from one of the cells (the second cell in figure 26). Thus either there are no dashed outputs from both of the cells $\mathbf{C}_1, \mathbf{C}_3$ or there are no dashed outputs from both of the cells $\mathbf{C}_2, \mathbf{C}_4$. Either case is a contradiction. $\square$

Theorem 6.8. *Let $\mathcal{N}$ be a four cell transitive coupled cell network consisting of identical cells with two asymmetric inputs. Suppose that there exist synchrony classes $\mathcal{X}_1 = \{X_1^1, X_1^2\}$ and $\mathcal{X}_2 = \{X_2^1, X_2^2\}$. The coupled cell networks associated to the synchrony classes defined by $\mathcal{X}_1$ and $\mathcal{X}_2$ correspond to two distinct two cell networks from $\{\mathcal{N}_1, \mathcal{N}_1^a, \mathcal{N}_2\}$ (figure 26) if and only if the four cell network is equivalent to one of the networks $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \ldots, \mathcal{K}_7$.*

Proof. By propositions 6.3—6.5, the networks $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \ldots, \mathcal{K}_7$ give a complete set (up to equivalence) of four cell transitive networks that are inflations of the three cell networks $\mathcal{N}_3$, $\mathcal{N}_4$. On the other hand, the network $\mathcal{N}_3$ can be obtained by inflating one of the two cell networks $\mathcal{N}_1$ or $\mathcal{N}_1^a$, and $\mathcal{N}_4$ can be obtained by inflating one of the two cell networks $\mathcal{N}_1^a$ or $\mathcal{N}_2$ (recall figure 26).

If the coupled cell networks associated to the synchrony classes defined by $\mathcal{X}_1$ and $\mathcal{X}_2$ correspond to distinct networks in $\{\mathcal{N}_1, \mathcal{N}_1^a, \mathcal{N}_2\}$, then by theorem 6.7 the coupled cell network associated to the synchrony class defined by $\mathcal{X}_{12}$ is a three cell network equivalent to $\mathcal{N}_3$ or to $\mathcal{N}_4$. Thus, $\mathcal{N}$ is an inflation of $\mathcal{N}_3$ or $\mathcal{N}_4$ and so is equivalent to one of the networks $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \ldots, \mathcal{K}_7$. $\square$

Remark 6.9. Coupled cell systems corresponding to the network architectures $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \ldots, \mathcal{K}_7$ all support robust simple heteroclinic cycles.

6.2. More general inflations of a three identical cell network.

If $p \geq 3$, there are no p-fold simple inflations of the three identical cell network $\mathcal{N}_3$ since cells in $\mathcal{N}_3$ have two outputs and no self-inputs. On the other hand, if we simply inflate $\mathcal{N}_3$ at $\mathbf{A}_1$ to get the four cell network $\mathcal{H}_1$ then we can inflate $\mathcal{H}_1$ at $\mathbf{A}_2$ and so obtain a five cell network with synchrony space $\{\mathbf{A}_{11}, \mathbf{A}_{12} \| \mathbf{A}_{21}, \mathbf{A}_{22}\}$. There are various possibilities depending on whether or not $\{\mathbf{A}_{11}, \mathbf{A}_{12}\}$ is synchrony class (of course, if we inflate first at $\mathbf{A}_1$ and then at $\mathbf{A}_2$, $\{\mathbf{A}_{21}, \mathbf{A}_{22}\}$ is always a synchrony class). In figure 30, we show an example where $\{\mathbf{A}_{11}, \mathbf{A}_{12}\}$ and $\{\mathbf{A}_{21}, \mathbf{A}_{22}\}$ are synchrony classes. If instead we had taken the $\mathbf{A}_{22}$-output as input into $\mathbf{A}_{12}$, then $\{\mathbf{A}_{11}, \mathbf{A}_{21}\}$ would not be a synchrony class. Both architectures define the same invariant subspace supporting a heteroclinic cycle.

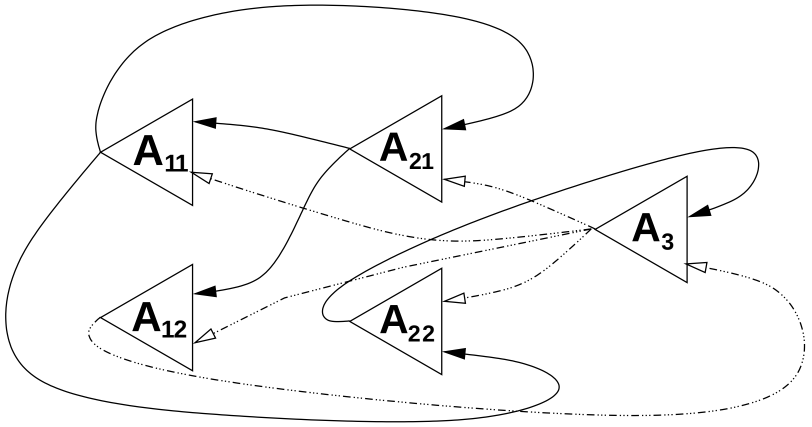


FIGURE 30. Inflating $\mathcal{N}_3$ to a five cell network by a simple inflations at $\mathbf{A}_1$ and $\mathbf{A}_2$.

The situation is quite different for $\mathcal{N}_4$ (giving another proof that $\mathcal{N}_3$ and $\mathcal{N}_4$ are inequivalent). In this case there are many ways of inflating to a transitive five cell network by replacing the middle cell by three cells $\mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{23}$. We indicate one possibility in figure 31. Notice that there are nine different ways of connecting the cells $\mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{23}$ to $\mathbf{A}_1$ and $\mathbf{A}_3$. There are a total of 27 ways of interconnecting the cells $\mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{23}$. In all therefore there are 243 ways of inflating $\mathcal{N}_4$ at $\mathbf{A}_2$ to a five cell network. Of course, many of the resulting networks will be equivalent and not all are transitive. For the network shown in figure 31, there are non-maximal synchrony classes: $\{\mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{23}\}$, $\{\mathbf{A}_1, \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{23}\}$, $\{\mathbf{A}_1, \mathbf{A}_3 \| \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{23}\}$, and $\{\mathbf{A}_1, \mathbf{A}_{21}\}$. A simple heteroclinic cycle for $\mathcal{N}_4$ lifts to a heteroclinic cycle with connections in $\{\mathbf{A}_1, \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{23}\}$ and $\{\mathbf{A}_1, \mathbf{A}_3 \| \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{23}\}$. A straightforward computation shows that this cycle can never be a simple (or attracting) heteroclinic cycle.

If we carry out a sequence of simple inflations on $\mathcal{N}_3$ so as to arrive at a transitive network which has r cells defining a synchrony class corresponding to $\mathbf{A}_1$, s cells defining a synchrony class corresponding to $\mathbf{A}_2$ and t cells defining a synchrony class corresponding to $\mathbf{A}_3$, then we must have

$$r + s \geq t, \ s + t \geq r, \ t + r \geq s.$$

This condition is forced by the transitivity requirement (in particular, no slaved subnetworks). If we have equality, say $r + s = t$, then we are blocked from simple inflation of any of the t cells associated to $\mathbf{A}_3$. However, if $r + s = t$, then we must have $s + t > r$ and $t + r > s$ and so we can always simply inflate one of the cells associated to $\mathbf{A}_1$ or $\mathbf{A}_2$.

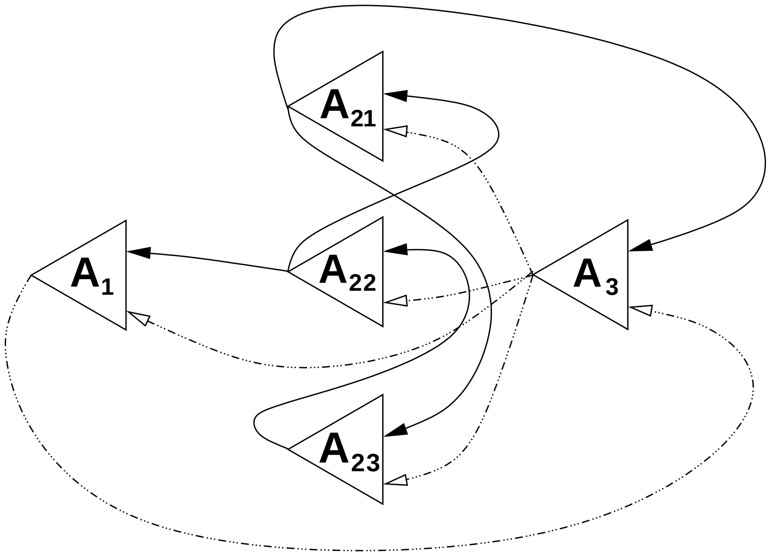


FIGURE 31. A 3-fold simple inflation of $\mathcal{N}_4$ at $\mathbf{A}_2$.

On the other hand, for $\mathcal{N}_4$, we are never blocked from making simple (or p-fold) inflations on cells arising from the cell $\mathbf{A}_2$. Following the previous notation, the constraints on simple inflation are

$$r \leq t, \; t \leq r + s$$

and we get blocked from making simple inflation on cells arising from $\mathbf{A}_1$ if $r = t$ and on simple inflation on cells arising from $\mathbf{A}_3$ if $t = r + s$. The fact that the constraints on the two three cell networks are different provides yet another demonstration that the networks are inequivalent.

Remark 6.10. The necessary conditions given above are special cases of a more general conjecture that has recently been proved by Agarwal. Suppose the network $\mathcal{N}$ has cells $\{\mathbf{C}_1, \cdots, \mathbf{C}_k\}$. Denote the adjacency matrix of $\mathcal{N}$ by A (section 2.2). Then there exists a transitive inflation $\Pi : \mathcal{M} \rightarrow \mathcal{N}$ of $\mathcal{N}$ with $n_j = \#\Pi^{-1}(C_j)$, $1 \leq j \leq k$, if and only if $A(n_1, \cdots, n_k) \geq (n_1, \cdots, n_k)$. We refer to [4] for details of the proof and examples.

7. HETEROCLINIC CYCLES FOR GENERAL NETWORKS OF IDENTICAL CELLS

The coupled cell networks $\mathcal{N}_3$ and $\mathcal{N}_4$ are (up to equivalence) the only three identical cell, two asymmetric input coupled cell networks supporting robust simple heteroclinic cycles. We now give sufficient conditions for the existence of robust heteroclinic cycles in a general n identical cell network with two inputs.

Theorem 7.1. *Let $\mathcal{N}$ be a transitive coupled cell network consisting of identical cells with two (asymmetric) inputs. Suppose that there exist synchrony classes $\mathcal{X}_1 = \{X_1^1, X_1^2\}$ and $\mathcal{X}_2 = \{X_2^1, X_2^2\}$. If $r = r(\mathcal{X}_{12}) = 3$, then the coupled cell network associated to the synchrony class defined by $\mathcal{X}_{12}$ is equivalent to one of the networks $\mathcal{N}_3, \mathcal{N}_4$. In particular, the network architecture $\mathcal{N}$ supports robust heteroclinic cycles.*

Proof. Since $r = 3$, we may relabel the partitions so that $X_1^1 \subset X_2^1$ and $X_1^2 \supset X_2^2$. Set $A = X_2^1 \cap X_1^2$, $B = X_1^1$, and $C = X_2^2$. The network associated to $\mathcal{X}_{12}$ can be identified with a coupled cell network $\mathcal{M}$ with three identical cells $\mathbf{A}, \mathbf{B}, \mathbf{C}$ corresponding respectively to the partition sets A, B, C. Since $A, B \subset X_2^1$, $\{\mathbf{A}, \mathbf{B}\}$ is a synchrony class of $\mathcal{M}$, and since $A, C \subset X_1^2$, $\{\mathbf{A}, \mathbf{C}\}$ is a synchrony class of $\mathcal{M}$. It follows from section 5 that up to equivalence the only three identical cell two asymmetric input coupled cell networks that have two or more non-maximal synchrony classes are $\mathcal{N}_3$ and $\mathcal{N}_4$ (in particular, $\{\mathbf{B}, \mathbf{C}\}$ cannot be a synchrony class). □

Next we give a sufficient condition that denies the existence of robust heteroclinic cycles of a certain type (although it does not guarantee that there are no heteroclinic cycles of other types).

Theorem 7.2. *Let $\mathcal{N}$ be a transitive coupled cell network consisting of identical cells with two (asymmetric) inputs. Suppose that there exist synchrony classes $\mathcal{X}_1 = \{X_1^1, X_1^2\}$ and $\mathcal{X}_2 = \{X_2^1, X_2^2\}$ and $r = r(\mathcal{X}_{12}) = 4$. Then the network architecture $\mathcal{N}$ cannot support robust simple heteroclinic cycles with hyperbolic saddle points on the maximal synchrony subspace with one-dimensional unstable manifold contained in either of the synchrony subspaces defined by $\mathcal{X}_1$ or $\mathcal{X}_2$.*

Proof. Set $A = X_1^1 \cap X_2^1$, $B = X_1^1 \cap X_2^2$, $C = X_1^2 \cap X_1^2$ and $D = X_1^2 \cap X_2^2$. The network associated to $\mathcal{X}_{12}$ can be identified with a coupled cell network $\mathcal{M}$ with four cells $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ corresponding respectively to the partition sets A, B, C, D. Since $A, B \subset X_1^1$, $C, D \subset X_1^2$, $\{\mathbf{A}, \mathbf{B} \| \mathbf{C}, \mathbf{D}\}$ is a synchrony class of $\mathcal{M}$. Denote the corresponding partition by $\mathcal{Y}_1$. Since $A, C \subset X_2^1$, $B, D \subset X_2^2$, $\{\mathbf{A}, \mathbf{C} \| \mathbf{B}, \mathbf{D}\}$ is a synchrony class of $\mathcal{M}$. Denote the corresponding partition by $\mathcal{Y}_2$. We have that $r(\mathcal{Y}_{12}) = 4$. The result now follows from theorem 6.7. □

Remark 7.3. Let $\mathcal{N}$ be a transitive coupled cell network consisting of identical cells with two (asymmetric) inputs. The coupled cell systems corresponding to the network $\mathcal{N}$ can support robust simple heteroclinic cycles with hyperbolic saddle points on the maximal synchrony subspace with one-dimensional unstable manifold contained in two-dimensional synchrony subspaces $\mathbf{S}_1$ and $\mathbf{S}_2$ if and only there exist synchrony classes $\mathcal{X}_1 = \{X_1^1, X_1^2\}$ and $\mathcal{X}_2 = \{X_2^1, X_2^2\}$ defining the synchrony subspaces $\mathbf{S}_1 = \Delta(\mathcal{X}_1)$ and $\mathbf{S}_2 = \Delta(\mathcal{X}_2)$ and such that $r = r(\mathcal{X}_{12}) = 3$.

Example 7.4. Following the proof of theorem 7.2, it is natural to ask whether if $\mathbf{S}_1 = \{\mathbf{A}, \mathbf{B} \| \mathbf{C}, \mathbf{D}\}$ and $\mathbf{S}_2 = \{\mathbf{A}, \mathbf{C} \| \mathbf{B}, \mathbf{D}\}$ are synchrony subspaces it is possible for there to be a network architecture with these synchrony subspaces which supports robust simple heteroclinic cycles with connections in $\mathbf{S}_1, \mathbf{S}_2$ and equilibria in the maximal synchrony subspace? (We assume four cells.) We give a simple construction based on inflation (see [4]). In this case we allow some symmetric inputs.

We start with a 2 cell unidirectional network with cells having two symmetric inputs (see figure 32). We inflate $\mathcal{M}$ at $\mathbf{C}_1$ and $\mathbf{C}_2$; replacing $\mathbf{C}_1$ by $\mathbf{A}, \mathbf{C}$ and $\mathbf{C}_2$ by $\mathbf{B}, \mathbf{D}$. There is more than one way to do this (see [10, 4]); we use the network architecture $\mathcal{M}_1$ shown in figure 33.

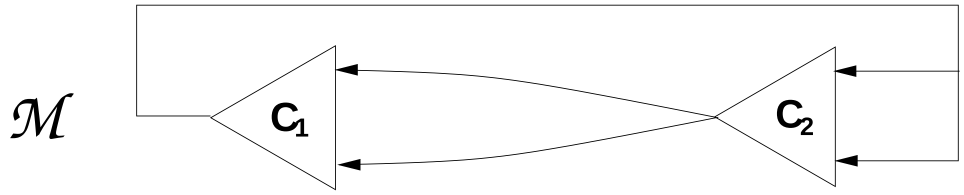


FIGURE 32. Unidirectional network $\mathcal{M}$, symmetric inputs.

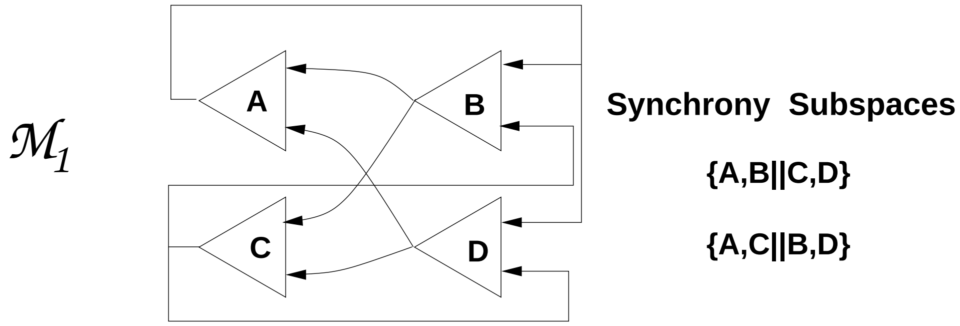


FIGURE 33. Inflating $\mathcal{M}$ to a 4 cell network $\mathcal{M}_1$.

It is easy to check that $\mathcal{M}_1$ has synchrony subspaces $\{\mathbf{A}, \mathbf{B} \| \mathbf{C}, \mathbf{D}\}$ and $\{\mathbf{A}, \mathbf{C} \| \mathbf{B}, \mathbf{D}\}$. For 1-dimensional cell dynamics both synchrony subspaces are of codimension 2. Computing the eigenvalues of the linearization of a model vector field f at an equilibrium p on the maximal synchrony subspace, there is a double eigenvalue and if the unstable manifold of f restricted to $\{\mathbf{A}, \mathbf{B} \| \mathbf{C}, \mathbf{D}\}$ is one-dimensional, then the unstable manifold of p must be at least 2-dimensional. Hence there can be no simple heteroclinic cycles connecting points on the maximal synchrony subspace (it does not help to take asymmetric inputs). However, if we add a new input type and define the architecture $\mathcal{M}_2$ shown in figure 34, then $\mathcal{M}_2$ supports robust attracting simple heteroclinic cycles. On account of the codimension of synchrony subspaces being bigger than one, there is the possibility of switching between connections, even though there are no complex eigenvalues.

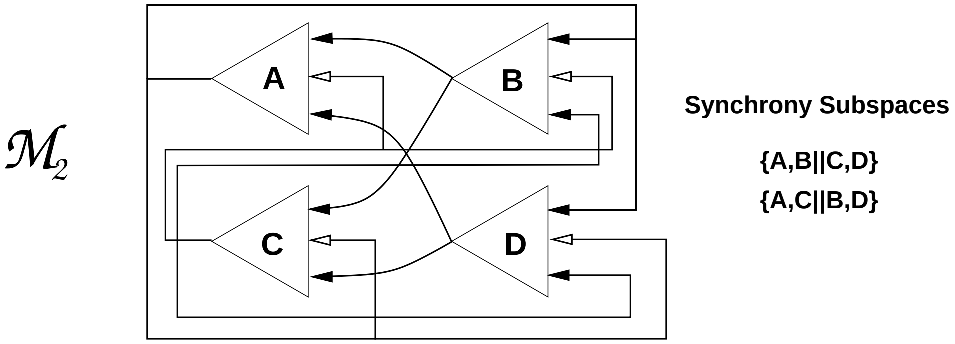


FIGURE 34. The network architecture $\mathcal{M}_2$ supporting robust heteroclinic cycles.

7.1. **A network with six identical cells.** As an illustration and application of some of our results, we investigate the possibility of robust heteroclinic cycles for coupled cell systems modelling the coupled cell network $\mathcal{L}$ shown in figure 35 and considered in [21]. We keep the same notation that was used in [21]: we emphasize that the cells are identical with asymmetric inputs. Later we exploit the A, B, C notation to describe what happens in this example when we allow for more than one class of cell.

One may verify that $\mathcal{L}$ has seven non-maximal synchrony classes

$$\begin{aligned}
&\mathbf{S}_1 = \{\mathbf{A_1}, \mathbf{A_2} \| \mathbf{B_1}, \mathbf{B_2}, \mathbf{C_1}, \mathbf{C_2}\},\ \mathbf{S}_2 = \{\mathbf{B_1}, \mathbf{B_2} \| \mathbf{A_1}, \mathbf{A_2}, \mathbf{C_1}, \mathbf{C_2}\},\\
&\mathbf{S}_3 = \{\mathbf{A_1}, \mathbf{A_2} \| \mathbf{B_1}, \mathbf{B_2} \| \mathbf{C_1}, \mathbf{C_2}\},\ \mathbf{S}_4 = \{\mathbf{B_1}, \mathbf{B_2} \| \mathbf{A_2}, \mathbf{C_1}, \mathbf{C_2}\},\\
&\mathbf{S}_5 = \{\mathbf{B_1}, \mathbf{B_2} \| \mathbf{C_1}, \mathbf{C_2}\},\ \mathbf{S}_6 = \{\mathbf{B_1}, \mathbf{B_2}\},\ \mathbf{S}_7 = \{\mathbf{C_1}, \mathbf{C_2}\}.
\end{aligned}$$

(If we assume symmetric inputs, then $\mathbf{S}_8 = \{\mathbf{A}_1, \mathbf{A}_2 \| \mathbf{B_1}, \mathbf{C_1} \| \mathbf{B_2}, \mathbf{C_2}\}$ is also a synchrony class and $\mathcal{L}$ has a $\mathbb{Z}_2$-symmetry generated by the cell permutation $\mathbf{A_1} \leftrightarrow \mathbf{A_2}$, $\mathbf{B_1} \leftrightarrow \mathbf{C_1}$, $\mathbf{B_2} \leftrightarrow \mathbf{C_2}$, see [21] for more details. Note that this shows that theorem 3.8 fails if we allow symmetric

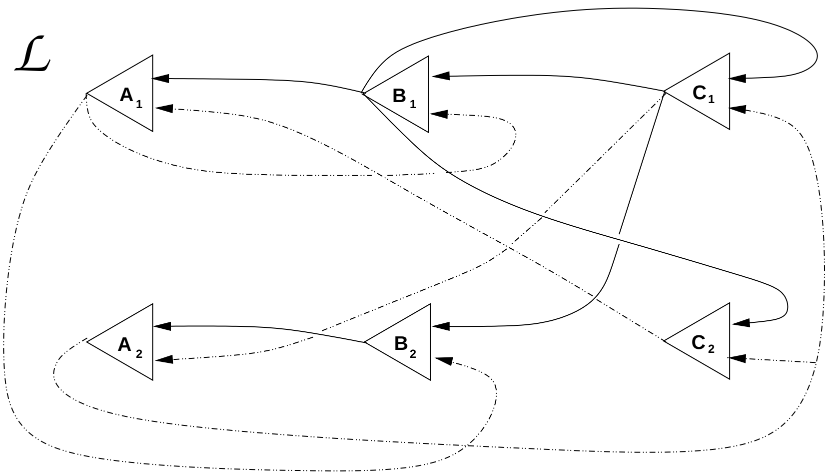


FIGURE 35. The six identical cell network $\mathcal{L}$ with asymmetric inputs.

inputs: take $\mathcal{X}_1$ to be the partition defined by $\mathbf{S}_8$ and $\mathcal{X}_2$ the partition defined by $\mathbf{S}_1$.)

If we take the two set partitions $\mathcal{X}_i$ defined by $\mathbf{S}_i$, $i = 1, 2$, then $r(\mathcal{X}_{12}) = 3$ and theorem 7.1 applies. The synchrony class defined by $\mathcal{X}_{12}$ is $\mathbf{S}_3$ and it is easy to check that the coupled cell network $\mathcal{P}$ obtained by restricting to $\mathbf{S}_3$ is equivalent to $\mathcal{N}_3$ — see figures 36, 16 and note the equivalence with $\mathcal{N}_3$ is defined by a permutation of cells $\mathbf{A}$ and $\mathbf{C}$. The network $\mathcal{P}$ has synchrony classes $\{\mathbf{A}, \mathbf{C}\}$ and $\{\mathbf{B}, \mathbf{C}\}$ and

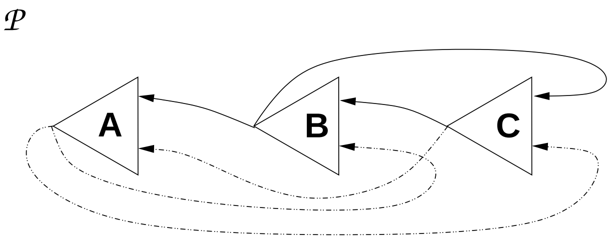


FIGURE 36. The network $\mathcal{P}$ on $\mathbf{S}_3$.

so coupled cell systems modelling $\mathcal{L}$ can support heteroclinic cycles between pairs of equilibria lying in the maximal synchrony class $\mathbf{S} = \mathbf{S}(\mathcal{F})$ and with connections in $\mathbf{S}_2$ and $\mathbf{S}_1$. Assume 1-dimensional cell dynamics modelled by an ordinary differential equation defined by $f =$

$f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$. Take coordinates $(a_1, a_2, b_1, b_2, c_1, c_2)$ on the phase space $\mathbb{R}^6$. The linearization $J_f(\mathbf{p})$ of the differential equation at an equilibrium $\mathbf{p} \in \mathbf{S} = \Delta(\mathbb{R}^6)$ is given by

$$J(\mathbf{p}) = \begin{pmatrix} \alpha & 0 & \beta & 0 & 0 & \gamma \\ 0 & \alpha & 0 & \beta & \gamma & 0 \\ \gamma & 0 & \alpha & 0 & \beta & 0 \\ \gamma & 0 & 0 & \alpha & \beta & 0 \\ 0 & \gamma & \beta & 0 & \alpha & 0 \\ 0 & \gamma & \beta & 0 & 0 & \alpha \end{pmatrix}, \tag{7.12}$$

where $\alpha = \frac{\partial f}{\partial x}(\mathbf{p})$, $\beta = \frac{\partial f}{\partial y}(\mathbf{p})$, $\gamma = \frac{\partial f}{\partial z}(\mathbf{p})$. Assume $\beta\gamma \neq 0$ and set $\rho = -\frac{\beta}{\gamma}$. In table 3 we list the eigenvalues of $J(\mathbf{p})$ and indicate which synchrony classes(s) contain the corresponding eigenspaces.

Eigenvalue	Multiplicity	Simplicity	Eigenspace
$\alpha+\beta+\gamma$	1	Simple	$\mathbf{S}(\mathcal{L})$
α	3	Not simple	$\{(\rho t, u, \rho^{-1}u, \rho^{-1}t, t, u) \mid (t,u) \in \mathbb{R}^2\}$
$\alpha-\beta$	1	Simple	$\subset \{\mathbf{B_1}, \mathbf{B_2} \Vert \mathbf{A_1}, \mathbf{A_2}, \mathbf{C_1}, \mathbf{C_2}\}$
$\alpha-\gamma$	1	Simple	$\subset \{\mathbf{A_1}, \mathbf{A_2} \Vert \mathbf{B_1}, \mathbf{B_2}, \mathbf{C_1}, \mathbf{C_2}\}$

TABLE 3. Eigenspace data for $J(\mathbf{p})$.

Since we can choose α, β, γ so that $\alpha, \alpha+\beta+\gamma < 0$ and $\gamma < \alpha < \beta$ or $\beta < \alpha < \gamma$ there are no local obstructions to choosing f so that the unstable manifold of $\mathbf{p}$ is 1-dimensional and lies in either $\mathbf{S}_1 = \{\mathbf{A_1}, \mathbf{A_2} \Vert \mathbf{B_1}, \mathbf{B_2}, \mathbf{C_1}, \mathbf{C_2}\}$ or $\mathbf{S}_2 = \{\mathbf{B_1}, \mathbf{B_2} \Vert \mathbf{A_1}, \mathbf{A_2}, \mathbf{C_1}, \mathbf{C_2}\}$. Since we have previously shown there are no global obstructions to constructing a simple heteroclinic cycle for the coupled cell systems associated to the network $\mathcal{N}_3$, the network architecture $\mathcal{L}$ can support simple heteroclinic cycles with connections in $\mathbf{S}_1, \mathbf{S}_2$.

This is not the only type of simple heteroclinic cycle supported by the network architecture $\mathcal{L}$. We claim there exist coupled systems that support a simple attracting heteroclinic cycle between two equilibria on the synchrony subspace $\mathbf{S}_5 = \{\mathbf{B_1}, \mathbf{B_2} \Vert \mathbf{C_1}, \mathbf{C_2}\}$ with connections in $\mathbf{S}_6 = \{\mathbf{B_1}, \mathbf{B_2}\}$ and $\mathbf{S}_7 = \{\mathbf{C_1}, \mathbf{C_2}\}$. Unlike the previous cycle, the existence of this cycle cannot be deduced from the existence of a cycle on a synchrony subspace as $\mathbf{S}_6$ and $\mathbf{S}_7$ are not contained in a synchrony subspace. We give a few details of the computations that show it is possible to choose 1-dimensional cell dynamics for which there exists a simple heteroclinic cycle. As usual this breaks into a local computation and a verification that there are no global obstructions for the existence of 1-dimensional connections. We sketch below the local computations needed for this.

Assume one-dimensional cell dynamics and an equilibrium at $\mathbf{p} = (a_1, a_2, b, b, c, c) \in \mathbf{S}_5$. Let $f = f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ model the dynamics and suppose

$$\begin{aligned}
\frac{\partial f}{\partial x}(a_1; b, c) &= \alpha_1,\ \frac{\partial f}{\partial y}(a_1; b, c) = \beta_1,\ \frac{\partial f}{\partial z}(a_1; b, c) = \gamma_1 \\
\frac{\partial f}{\partial x}(a_2; b, c) &= \alpha_2,\ \frac{\partial f}{\partial y}(a_2; b, c) = \beta_2,\ \frac{\partial f}{\partial z}(a_2; b, c) = \gamma_2 \\
\frac{\partial f}{\partial x}(b; c, a_1) &= \alpha_3,\ \frac{\partial f}{\partial y}(b; c, a_1) = \beta_3,\ \frac{\partial f}{\partial z}(b; c, a_1) = \gamma_3 \\
\frac{\partial f}{\partial x}(c; b, a_2) &= \alpha_4,\ \frac{\partial f}{\partial y}(c; b, a_2) = \beta_4,\ \frac{\partial f}{\partial z}(c; b, a_2) = \gamma_4.
\end{aligned}$$

The linearization of the vector field determined by f at $\mathbf{p}$ is

$$J(\mathbf{p}) = \begin{pmatrix} \alpha_1 & 0 & \beta_1 & 0 & 0 & \gamma_1 \\ 0 & \alpha_2 & 0 & \beta_2 & \gamma_2 & 0 \\ \gamma_3 & 0 & \alpha_3 & 0 & \beta_3 & 0 \\ \gamma_3 & 0 & 0 & \alpha_3 & \beta_3 & 0 \\ 0 & \gamma_4 & \beta_4 & 0 & \alpha_4 & 0 \\ 0 & \gamma_4 & \beta_4 & 0 & 0 & \alpha_4 \end{pmatrix}. \tag{7.13}$$

Noting that $\mathbf{S}_5$ is $J(\mathbf{p})$-invariant, the matrix of $J(\mathbf{p})|\mathbf{S}_5 = J_{BC}$ is

$$J_{BC} = \begin{pmatrix} \alpha_1 & 0 & \beta_1 & \gamma_1 \\ 0 & \alpha_2 & \beta_2 & \gamma_2 \\ \gamma_3 & 0 & \alpha_3 & \beta_3 \\ 0 & \gamma_4 & \beta_4 & \alpha_4 \end{pmatrix}. \tag{7.14}$$

We have similar expressions for $J_B = J(\mathbf{p})|\mathbf{S}_5$ and $J_C = J(\mathbf{p})|\mathbf{S}_6$. For a simple cycle, we require that all eigenvalues of $J_{BC}(\mathbf{p})$ are strictly negative. Since $\mathbf{S}_6 \supset \mathbf{S}_5$ and both spaces are $J(\mathbf{p})$-invariant, we see that the eigenvalue of J_B with eigenline transverse to $\mathbf{S}_5$ must be α_4. This is because the sum of the eigenvalues of $J_{BC}(\mathbf{p})$ is $\alpha_1+\alpha_2+\alpha_3+\alpha_4$, while the sum of the eigenvalues of J_B is $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$. Similarly the eigenvalue of J_C with eigenline transverse to $\mathbf{S}_5$ must be α_3. Hence for a simple cycle with saddle at $\mathbf{p}$ we must have

$$\alpha_3\alpha_4 < 0,$$

and all eigenvalues of J_{BC} strictly negative. It is not hard to choose the $\alpha_i, \beta_i, \gamma_i$ to achieve this. For example, if we take $\alpha_1 = \alpha_2 = \alpha_4 = -1$, $\alpha_3 = 1$, and $\beta_1, \gamma_1, \gamma_3 = 0$, then the characteristic equation of J_{BC} is of the form $(1 + \lambda)(\lambda^3 + \lambda^2 + A\lambda + B)$ where we can choose A, B freely (in terms of $\beta_2, \beta_3, \beta_4, \gamma_2, \gamma_4$). It is easy to check that we can choose A, B so that the equation has all four roots negative.

7.2. **Coexisting cycles and attracting heteroclinic networks.** It is clear that both heteroclinic cycles described above can coexist in a coupled cell systems modelling the network $\mathcal{L}$. Moreover, we can also arrange that both cycles are asymptotically attracting (without going into any detail, this amounts to the contracting eigenvalues dominating the expanding eigenvalues at the saddle points). It is likely that there are robust heteroclinic networks supported by the network architecture $\mathcal{L}$. We describe one possible scenario (we have not checked all of the eigenvalue conditions on linearizations at equilibria). It is consistent with the invariant subspace structure that there are connections contained in $\mathbf{S}_6$ from a saddle on $\mathbf{S}_5$ to a saddle on $\mathbf{S}_2$ which in turn connects to saddle on $\mathbf{S}_1$. The latter saddle can connect via a robust 1-dimensional connection in $\mathbf{S}_3$ to a second saddle on $\mathbf{S}_1$ which can then connect to a second saddle on $\mathbf{S}_2$ which then connects within $\mathbf{S}_6$ to a second saddle on $\mathbf{S}_5$ which in turn connects by a robust 1-dimensional connection in $\mathbf{S}_7$ back to the original saddle on $\mathbf{S}_5$. In Appendix A we briefly discuss the case of symmetric inputs for this network.

7.3. **Multiple cell classes.** Instead of assuming identical cells in $\mathcal{L}$, suppose that there are three cell classes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. Denote the corresponding network by $\mathcal{L}_1$. The maximal synchrony subspace for $\mathcal{L}_1$ will be $\mathbf{S}_3 = \{\mathbf{A}_1, \mathbf{A}_2 \| \mathbf{B}_1, \mathbf{B}_2 \| \mathbf{C}_1, \mathbf{C}_2\}$ and the non-maximal synchrony classes will be $\mathbf{S}_5$, $\mathbf{S}_6$, $\mathbf{S}_7$. Exactly as for $\mathcal{L}$, there exist coupled cell systems associated to $\mathcal{L}_1$ which support robust simple heteroclinic cycles with equilibria on $\mathbf{S}_5$ and connections in $\mathbf{S}_6$ and $\mathbf{S}_7$. In particular, it is not necessary to have three identical cells for there to be robust simple heteroclinic cycles. If we define $\mathcal{L}_2$ by taking two cell classes $\mathbf{A}$ and $\mathbf{B} = \mathbf{C}$, then the maximal synchrony subspace will be $\mathbf{S}_1$ and $\mathbf{S}_3$ will be a non-maximal synchrony class. As before $\mathcal{L}_2$ supports robust simple heteroclinic cycles with equilibria on $\mathbf{S}_5$ and connections in $\mathbf{S}_6$ and $\mathbf{S}_7$. Finally, if we take two cell classes $\mathbf{A} = \mathbf{C}$ and $\mathbf{B}$ to define the network $\mathcal{L}_3$, then $\mathbf{S}_2$ will be the maximal synchrony class and $\mathbf{S}_3, \ldots, \mathbf{S}_7$ will all be synchrony classes. As before, the network $\mathcal{L}_3$ supports robust simple heteroclinic cycles. The networks $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ do not support a robust simple cycle with connections lying in $\mathbf{S}_1$ and $\mathbf{S}_2$. This is simply because if we break the identical cell structure we change the maximal synchrony class and one or other of $\mathbf{S}_1$ and $\mathbf{S}_2$ ceases to be a synchrony class. There remains the problem of describing bifurcations of this heteroclinic cycle when we break the identical cell structure.

8. Discussion

A crucial aspect in the area of coupled cell networks is to characterize how dynamics of the underlying coupled cell systems depends on the network architecture. One fundamental property of coupled cells is that the network structure forces the existence of subspaces that are flow-invariant under the associated coupled cell systems – the synchrony subspaces. These should have an important role in the kinds of dynamics that can occur, and a significant step in understanding the dynamics forced by the network architecture is the classification of these patterns of synchrony. In [9] there is work in progress towards this classification.

An important question that we address in this paper is the existence of (robust) heteroclinic cycles in non-symmetric coupled cell systems forced by these synchrony spaces. We mention the setting of symmetric dynamical systems, where the natural flow invariant subspaces correspond to fixed-point subspaces and heteroclinic behaviour occurs in a robust manner. One of the novelties of our work is that small and simple asymmetric networks with one- or two-dimensional cell dynamics can support robust heteroclinic phenomena, and we give three ODE examples of such cycles in quite different dynamical settings. We have emphasized the case of relatively small but nontrivial networks of identical cells, usually with asymmetric inputs and modelled by ODEs.

The second novelty of our results is the contribution to the design theory of coupled cell networks with prescribed dynamical behaviour. Using the technique of inflation (an inverse process to that of forming a quotient network) we show how it is possible to construct network architectures supporting robust simple heteroclinic cycles and heteroclinic networks in a systematic way. Although our focus is on cells with asymmetric inputs, it is interesting to consider cells whose inputs are all of the same type. The latter should include types of network examined by other authors, for example [8, 27].

We feel that very natural problems are to identify natural families of networks that possess common dynamical features (such as robust heteroclinic cycles and heteroclinic networks), and to explore ways in which one can combine small networks with "known" dynamics ("motifs") so as to construct larger networks with specific dynamical properties.

One particular result from our investigation is that dynamics does not depend on the precise network architecture; it only depends on the structure up to dynamical equivalence. In many cases, it is possible to work with either input or output equivalence and we refer to [5, 6] where the relations between input and output equivalence and dynamical equivalence are investigated and shown to be equivalent in many contexts. However, it still remains to find effective ways of determining equivalence for large networks of coupled oscillators or networks with a more hierarchical and heterogenous structure. We would expect that various forms of equivalence (dynamical, input and output) are likely to play an important role in clarifying and organizing the design of larger networks.

Finally, in this work we have restricted to cases where the network is transitive and one cannot split off autonomous systems that may force other parts of the system (that is, we assume there are no slaved networks). There are some results on the decomposition of non-transitive networks, for example [21], and the question of two forcing subsystems with robust heteroclinic attractors is likely to be of particular interest and could be approached with [15]. Other directions of particular interest to investigate using our framework are symbolic and measure dynamics [3] as well as bifurcation from synchrony [24, 11] and 'forced symmetry breaking' from symmetric to asymmetric inputs. There are also many outstanding questions about bifurcation from heteroclinic cycles and dynamics near heteroclinic networks, in particular switching [7, 35] and we believe the methods in the paper should be useful in approaching these problems.

Appendix A. Bifurcation from symmetric inputs

Suppose the network $\mathcal{L}$ of section 7.1 has *symmetric* inputs. The network defined by restriction to the synchrony class $\mathbf{S}_3$ is $\mathbf{D}_3$-equivariant, where $\mathbf{D}_3$ is the dihedral group of order 6 (isomorphic to the symmetric group S_3). If we assume dynamics given by ordinary differential equations then dynamics on $\mathbf{S}_3$ are given by

$$\mathbf{x}'_A = f(\mathbf{x}_A;\mathbf{x}_B,\mathbf{x}_C), \tag{A.15}$$

$$\mathbf{x}'_B = f(\mathbf{x}_B;\mathbf{x}_A,\mathbf{x}_C), \tag{A.16}$$

$$\mathbf{x}'_C = f(\mathbf{x}_C;\mathbf{x}_A,\mathbf{x}_B). \tag{A.17}$$

where $f(\mathbf{x};\mathbf{u},\mathbf{v})$ is symmetric in $\mathbf{u},\mathbf{v}$. Any $\mathbf{D}_3$-symmetric solution to these equations determines a (synchronous) solution lying in the maximal synchrony class. In general, an *asymmetric* solution of the equations determines a solution lying in $\mathbf{S}_3$. One way of obtaining such a solution is via a $\mathbf{D}_3$-equivariant Hopf bifurcation from a $\mathbf{D}_3$-symmetric equilibrium of the equations (A.15,A.16,A.17). This solution will determine three pairs of hyperbolic periodic solutions for the coupled cell systems corresponding to the network $\mathcal{L}$, each differing by a phase which is a multiple of $2\pi/3$ [25]. That is, each of the cell pairs $\{\mathbf{A_1}, \mathbf{A_2}\}$, $\{\mathbf{B_1}, \mathbf{B_2}\}$, $\{\mathbf{C_1}, \mathbf{C_2}\}$ will oscillate synchronously with the same frequency but each pair will be $\pm 2\pi/3$ out of phase with the other pairs (see [28, §5] for another example of this type that depends on an interesting variation of the Hopf bifurcation argument). Now suppose we allow inputs to be asymmetric. As we break symmetry of inputs, the triple of hyperbolic periodic solutions for the coupled cell systems corresponding to the network $\mathcal{L}$ will persist except that the phase differences will now only be approximately $\pm 2\pi/3$.

References

[1] M Abeles, H Bergman, I Gat, I Meilijson, E. Seidmann and M. Tishby, 'Cortical activity flips among quasi-stationary states', *PNAS* **92** (1995), 8616–8620.

[2] V S Afraimovich, V P Zhigulin & M I Rabinovich. 'On the origin of reproducible sequential activity in neural circuits', *Chaos* **14**(4) (2004), 1123–1129.

[3] V S Afraimovich and L A Bunimovich. 'Dynamical Networks: interplay of topology, interactions and local dynamics', *Nonlinearity* **20** (2007), 1761–1771.

[4] N Agarwal. 'Inflation of strongly connected networks', *Math. Proc. Camb. Phil. Soc.*, accepted for publication.

[5] N Agarwal and M Field. 'Dynamical equivalence of networks of coupled dynamical systems I: asymmetric inputs', *Nonlinearity*, to appear.

[6] N Agarwal and M Field. 'Dynamical equivalence of networks of coupled dynamical systems II: general case', *Nonlinearity*, to appear.

[7] M A D Aguiar, S B S D Castro and I S Labouriau. 'Dynamics near a heteroclinic network', *Nonlinearity* **18**(1) (2005), 391–414.

[8] M A D Aguiar and A P S Dias. 'Minimal coupled cell networks', *Nonlinearity* **20** (2007), 193–219.

[9] M A D Aguiar and A P S Dias. 'Synchrony subspaces of coupled cell networks', (2010), in preparation.

[10] M A D Aguiar, A P S Dias, M Golubitsky and M C A Leite. 'Bifurcations from regular quotient networks: A first insight', *Physica D* **238** (2009), 137-155.

[11] F Antoneli, A P S Dias and R C Paiva. 'Hopf Bifurcation in Coupled Cell Networks with Interior Symmetries', *SIAM J. Appl. Math.* **7** (2008), 220-248.

[12] P Ashwin and J Borresen, 'Discrete computation using a perturbed heteroclinic network', *Phys. Rev. E* **70** (2004), 026203.

[13] P Ashwin, G Orosz, J Wordsworth and S Townley, 'Reliable switching between cluster states for globally coupled phase oscillators', *SIAM Journal on Applied Dynamical Systems* **6**, (2007), 728–758.

[14] P Ashwin and M Field. 'Heteroclinic networks in coupled cell systems', *Arch. Rational Mech. Anal.* **148** (1999), 107–143.

[15] P Ashwin and M Field. 'Product dynamics for homoclinic attractors', *Proc. Royal Soc., ser. A* **461** (2005), 155–177.

[16] P Ashwin and J Swift. 'The dynamics of n weakly coupled identical oscillators', *Journal of Nonlinear Science* **2** (1992), 69–108.

[17] A P S Dias and J Lamb. 'Local bifurcation in symmetric coupled cell networks: linear theory', *Physica D* **223** (2006), 93–108.

[18] A P S Dias and I Stewart. 'Linear equivalence and ODE-equivalence for coupled cell networks', *Nonlinearity* **18** (2005), 1003–1020.

[19] B Dionne, M Golubitsky and I Stewart. 'Coupled cells with internal symmetry: I. Wreath products', *Nonlinearity* **9** (1996), 559–574.

[20] M J Field. *Dynamics, Bifurcation and Symmetry*, Pitman Research Notes in Mathematics, **356**, 1996.

[21] M J Field. 'Combinatorial Dynamics', *Dynamical Systems* **19** (2004), 217–243.

[22] M J Field *Dynamics and Symmetry* (Imperial College Press Advanced Texts in Mathematics — Vol. 3, 2007.)

[23] M J Field and R W Richardson. 'Symmetry breaking and branching patterns in equivariant bifurcation theory II', *Arch. Rational Mech. and Anal.* **120** (1992), 147–190.

[24] M Golubitsky, M Pivato and I Stewart. 'Interior symmetry and local bifurcation in coupled cell networks'. *Dynamical Systems* **19** (2004), 389–407.

[25] M Golubitsky, D G Schaeffer and I N Stewart. *Singularities and Groups in Bifurcation Theory, Vol. II* (Appl. Math. Sci. **69**, Springer-Verlag, New York, 1988).

[26] M Golubitsky, I Stewart, P L Buono and J J Collins. 'Symmetry in locomotor central pattern generators and animal gaits', *Nature* **401** (1999), 693–695.

[27] M Golubitsky and I Stewart. 'Nonlinear dynamics of networks: the groupoid formalism'. *Bull. Amer. Math. Soc.* **43** (2006), 305–364.

[28] M Golubitsky, I Stewart and A Török. 'Patterns of Synchrony in Coupled Cell Networks with Multiple Arrows', *SIAM J. Appl. Dynam. Sys.* **4**(1) (2005), 78–100.

[29] J Guckenheimer and P Holmes. 'Structurally stable heteroclinic cycles', *Math. Proc. Camb. Phil. Soc.* **103** (1988), 189–192.

[30] D Hansel, G Mato, C Meunier. 'Phase dynamics for weakly coupled Hodgkin-Huxley neurons', *Europhys Letts* **23** (1993), 367–372.

[31] J Hofbauer. 'Heteroclinic cycles on the simplex', *Proc. Int. Conf. Nonlinear Oscillations, Janos Bolyai Math. Soc. Budapest*, 1987.

[32] J Hofbauer. 'Heteroclinic cycles in ecological differential equations', *Tatra Mountains Math. Publ.* **4** (1994), 105–116.

[33] J Hofbauer and K Sigmund. *The Theory of Evolution and Dynamical Systems*, Cambridge University Press, Cambridge, 1988.

[34] J Hofbauer and K Sigmund. *Evolutionary Games and Replicator Dynamics*, Cambridge University Press, Cambridge, 1998.

[35] A J Homburg and J Knobloch. 'Switching homoclinic networks', to appear in *Dynamical Systems* (2010).

[36] O Karabacak and P Ashwin. 'Heteroclinic Ratchets in a System of Four Coupled Oscillators', *J. Nonlinear Sci.* **20** (2010), 105–129.

[37] N Kashtan & U Alon. Spontaneous evolution of modularity and network motifs, *PNAS* **102** (39) (2005), 13773–13778.

[38] V Kirk and M Silber. 'A competition between heteroclinic cycles', *Nonlinearity* **7**(6) (1994), 1605–1622.

[39] I Kiss, C Rusin, H Kori and J Hudson. 'Engineering complex dynamical structures: sequential patterns and desynchronization', *Science* **316** (2007), 1886-9.

[40] B P Kitchens. *Symbolic Dynamics* (Univeritext, Spring-Verlag, Berlin, Heidelberg, New York, 1998).

[41] M Krupa. 'Robust heteroclinic cycles', *Journal of Nonlinear Science* **7** (1997), 129–176.

[42] M Krupa and I Melbourne. 'Asymptotic stability of heteroclinic cycles in systems with symmetry', *Ergod. Th. & Dynam. Sys.* **15** (1995), 121–147.

[43] M Krupa and I Melbourne. 'Asymptotic stability of heteroclinic cycles in systems with symmetry, II', *Proc. Roy. Soc. Edinburgh A* **134A** (2004), 1177-1197.

[44] M C A Leite and M Golubitsky. 'Homogeneous three-cell networks', *Nonlinearity* **19** (2006), 2313–2363.

[45] R M May and W J Leonard. 'Nonlinear aspects of competition between three species', *SIAM J. Appl. Math.*, **29** (1975), 243–253.

[46] I Melbourne, P Chossat and M Golubitsky. 'Heteroclinic cycles involving periodic solutions in mode interactions with $O(2)$ symmetry', *Proc. Roy. Soc. Edinburgh* **113A** (1989), 315–345.

[47] R-M Memmesheimer and M Timme. 'Designing the Dynamics of Spiking Neural Networks', *Phys. Rev. Letts* **97** (2006), 188101.

[48] T Nowotny and M Rabinovich. 'Dynamical Origin of Independent Spiking and Bursting Activity in Neural Microcircuits', *Phys. Rev. Letts* **98** (2007), 128106.

[49] A Pikovsky, M Rosenblum, and J Kurths. 'Synchronization; A Universal Concept in Nonlinear Sciences', Cambridge University Press, 2001.

[50] M Rabinovich, R Huerta and G Laurent. 'Transient Dynamics for Neural Processing', *Science* **321** (2008), 48-50.

[51] G L dos Reis. 'Structural stability of equivariant vector fields on two manifolds', *Trans. Amer. Math. Soc.*, **283** (1984), 633–643.

[52] J G Restrepo, E Ott and B R Hunt. 'Emergence of synchronization in complex networks of interacting dynamical systems', *Physica D* **224** (2006), 114–122.

[53] A Scheel and P Chossat. 'Bifurcation d'orbites périodiques à partit d'un cycle homoclinic symétrique', *C. R. Acad. Sci. Paris* **314**, Sér. I (1992), 49–54.

[54] I Stewart. 'The lattice of balanced equivalence relations of a coupled cell network', *Math. Proc. Cambridge Philos. Soc.* **143** (1) (2007), 165–183.

[55] I Stewart, M Golubitsky and M Pivato. 'Symmetry groupoids and patterns of synchrony in coupled cell networks', *SIAM J. Appl. Dynam. Sys.* **2** (4) (2003), 606–646.

[56] S Strogatz. 'From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators', *Physica D* **143** (2000), 1–20.

[57] G.B. Ermentrout. *xppaut* (dynamical systems software). Available from `http://www.math.pitt.edu/~bard/bardware/` (2000).

[58] V Zhigulin. 'Dynamical Motifs: Building Blocks of Complex Dynamics in Sparsely Connected Random Networks' *Phys. Rev. Letts* **92** (2004), 238701.

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