On local attraction properties and a stability index for heteroclinic connections

Olga Podvigina
Observatoire de la Côte d'Azur, CNRS UMR 6529
BP 4229, 06304 Nice Cedex 4, France, and
International Institute of Earthquake Prediction Theory
and Mathematical Geophysics
84/32 Profsoyuznaya St, 117997 Moscow, Russian Federation

Peter Ashwin
Mathematics Research Institute
Harrison Building
University of Exeter, Exeter EX4 4QF, UK

December 17, 2010

Abstract

Some invariant sets may attract a nearby set of initial conditions but nonetheless repel a complementary nearby set of initial conditions. For a given invariant set $X \subset \mathbb{R}^n$ with a basin of attraction N, we define a *stability index* $\sigma(x)$ of a point $x \in X$ that characterizes the local extent of the basin. Let B_ϵ denote a ball of radius ϵ about x. If $\sigma(x) > 0$, then the measure of $B_\epsilon \setminus N$ relative the measure of the ball is $O(\epsilon^{|\sigma(x)|})$, while if $\sigma(x) < 0$, then the measure of $B_\epsilon \cap N$ relative the measure of the ball is of this order. We show that this index is constant along trajectories, and we relate this orbit invariant to other notions of stability such as Milnor attraction, essential asymptotic stability and asymptotic stability relative to a positive measure set. We adapt the definition to local basins of attraction (i.e. where N is defined as the set of initial conditions that are in the basin and whose trajectories remain local to X).

This stability index is particularly useful for discussing the stability of robust heteroclinic cycles, where several authors have studied the appearance of cusps of instability near cycles that are Milnor attractors. We study simple (robust heteroclinic) cycles in $\mathbb{R}^4$ and show that the local stability indices (and hence local stability properties) can be calculated in terms of the eigenvalues of the linearization of the vector field at steady states on the cycle. In doing this, we extend previous results of Krupa and Melbourne (1995,2004) and give criteria for simple heteroclinic cycles in $\mathbb{R}^4$ to be Milnor attractors.

Key words: Heteroclinic Cycle, Stability, Symmetry, Milnor Attractor

1 Introduction

For many choices of smooth vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, the system on $x \in \mathbb{R}^n$

$$\dot{x} = f(x) \tag{1}$$

has a small subset of $\mathbb{R}^n$ (an attractor) that attracts a large set of initial conditions; these attractors are important for understanding the long term behaviour of trajectories of the system. In this paper we explore the local attraction structure of invariant sets, while in the latter part we focus on a particular class of examples - attracting heteroclinic cycles. More precisely, an invariant set is *asymptotically stable* if it attracts all nearby points; many systems are found to possess invariant sets that are not asymptotically stable, but that are attractors in a weaker sense (e.g. in the sense of Milnor [20]).

Now consider $\xi_1, \ldots, \xi_m$ to be hyperbolic equilibria of (1). A set of connecting trajectories $W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \emptyset$, $j = 1, \ldots, m$, $\xi_{m+1} = \xi_1$, is called a *heteroclinic cycle* between these equilibria. It has been shown that heteroclinic cycles can be robust (persistent to small perturbations) if f is constrained to be symmetric with respect to certain group representations [15, 4, 22], or if f is constrained to preserve certain invariant subspaces [15]. Heteroclinic cycles that are not asymptotically stable may often be observed to be apparently stable in computations. To explain this, weaker notions of stability for heteroclinic cycles sets were introduced in [19, 17, 13] - they do not require attraction in a full neighbourhood of the invariant set; they may even be repelling in a region that is typically cusp-shaped in Poincaré sections to the cycle. The papers [19, 17] define a heteroclinic cycle to be *essentially asymptotically stable* (e.a.s.) if it attracts almost all nearby trajectories, and they define it to be *almost completely unstable* (a.c.u.), if it attracts almost no nearby trajectories. However, as shown in [13] these definitions are not mutually exclusive. (Brannath [9] similarly discusses e.a.s. using the notion of *relative asymptotic stability* from Ura [23].)

The paper is organized as follows: in Section 2 we discuss various definitions of stability, and we relate them to the notion of Milnor attractor and the local geometry of the basin of attraction. We introduce a *stability index* that characterizes the local geometry of the basin of attraction. After proving some basic properties about this invariant of the dynamics, we generalize to a *local stability index* that is the limit of stability indices of local basins of attraction. In Section 3 we discuss the structure of heteroclinic cycles and describe the geometry of local basins of attraction by way of the local stability index and (Poincaré) surfaces of section. We show, under certain assumptions, that the stability index of a connecting trajectory is the stability index on a surface of section.

Section 4 computes the stability indices for robust heteroclinic cycles in $\mathbb{R}^4$; we employ the classification of simple cycles in $\mathbb{R}^4$ into Types A-C by Krupa and Melbourne in a series of papers [16, 17, 18] and calculate the stability indices of the connections in terms of eigenvalues of the linearization at equilibria in the cycle. Finally we discuss some of the limitations and possible further uses of stability indices and related concepts in Section 5.

2 Attractors and the stability index

Various definitions of attraction of invariant sets have been introduced [9, 17, 19, 23] to describe sets that are not asymptotically stable but that are nevertheless attracting in some sense. We review these notions and relate them to Milnor's notion of a measure attractor [20].

2.1 Notions of attraction for invariant sets

In this section we consider a smooth flow $\Phi_t(x)$ on $\mathbb{R}^n$. Two very general notions of attraction are the Milnor and weak attractors discussed in [20] and [6] respectively. For an invariant set $X \subset \mathbb{R}^n$ we define the (global) *basin of attraction* of X to be

$$\mathcal{B}(X) = \{x \in \mathbb{R}^n \;:\; \omega(x) \subset X\}$$

where $\omega(x) = \bigcap_{T>0} \overline{\{\Phi_t(x) \;:\; t > T\}}$ is the ω-limit of x. The following defines attraction properties of X in terms of this basin. We use $\ell(\cdot)$ to denote Lebesgue measure on $\mathbb{R}^n$.

Definition 1 *[6] We say a compact invariant set X is a* weak attractor *if $\ell(\mathcal{B}(X)) > 0$. We say a compact invariant set X is a* Milnor attractor *if it is a weak attractor such that for any proper subset $Y \subset X$ that is compact and invariant we have*

$$\ell(\mathcal{B}(X) \setminus \mathcal{B}(Y)) > 0.$$

We do not assume transitivity of X (a dense orbit); indeed, the main examples we will consider later on are heteroclinic cycles that are not transitive. Note that any weak attractor contains a Milnor attractor [6, Lemma 3.2]. There are various examples of robust heteroclinic cycles (e.g. [19, 9, 13]) that are Milnor attractors, even though they are not asymptotically stable. Let $d(\cdot,\cdot)$ denote the Hausdorff distance between two sets, let

$$B_\epsilon(X) := \{x \in \mathbb{R}^n \;:\; d(x, X) < \epsilon\}$$

denote the ϵ-parallel body of X, and let D^c denote the complement of D in $\mathbb{R}^n$.

Definition 2 *[19] We say a compact invariant set X is* essentially asymptotically stable *(e.a.s.), if there is a set D such that for any open neighbourhood U of X and any $\epsilon > 0$ there exists an open neighbourhood $V \subset U$ of X such that:*

(a) If $x_0 \in V \cap D^c$ then $\Phi_t(x_0) \in U$ for all $t > 0$ and $\lim_{t\to\infty} d(\Phi_t(x_0), X) = 0$,

(b) $\ell(V \cap D^c)/\ell(V) > 1 - \epsilon$.

Intuitively, if X is e.a.s. one might expect that it attracts "almost all" nearby trajectories, while [17] says X is almost completely unstable[1] if it attracts "almost none" of them. However, these definitions do not formalise these intuitive categories very well; as highlighted in

[1]A flow-invariant set X is called *almost completely unstable (a.c.u)*, if there is a set D and an open neighbourhood U of X such that for some $\epsilon > 0$ there exists an open neighbourhood V of X, $V \subset U$, such that (a) for $x_0 \in V \setminus D$ there exists a $t > 0$ with $\Phi_t(x_0) \notin U$; and (b) $\ell(V \cap D^c)/\ell(V) > 1 - \epsilon$.

[13], they are not mutually exclusive and so may yield classifications that are not intuitively helpful. Another useful definition is that of [23] which is used in [9]: for this we consider a set $N \subset \mathbb{R}^n$.

Definition 3 *[23] We say a compact invariant set X with $X \subset \overline{N}$ is* asymptotically stable, relative to *(a.s.r.t.) N if for every neighbourhood U of X there is a neighbourhood V of X such that for all initial $x \in V \cap N$ we have $\Phi_t(x) \in U$ for $t > 0$, and $\omega(x) \subset X$.*

In fact, Brannath [9] interestingly suggests that the authors of [19, 17] had the following definition in mind for e.a.s., but we name it differently to distinguish from the original definition in [19].

Definition 4 *(Adapted from [9]) We say a compact invariant set X is* predominantly asymptotically stable *(p.a.s.) if there is an N such that X is asymptotically stable relative to N and*

$$\lim_{\epsilon \to 0} \frac{\ell(B_\epsilon(X) \cap N)}{\ell(B_\epsilon(X))} = 1.$$

We now give a result that relates these concepts of attraction.

Theorem 2.1 *Suppose that X is a compact invariant set for a continuous flow Φ_t.*

(a) X is p.a.s. $\Rightarrow$ X is e.a.s.

(b) X is e.a.s. $\Rightarrow$ X contains a Milnor attractor.

(c) X is e.a.s. $\Leftrightarrow$ there is an N with $\ell(N \cap A) > 0$ for any neighbourhood A of X, such that X is a.s.r.t. N.

Before proving this theorem, we give a useful lemma that will be used in the proof. For any measurable set N we define the density of N at x to be

$$F(x) = \lim_{\epsilon \to 0} \frac{\ell(B_\epsilon(x) \cap N)}{\ell(B_\epsilon(x))}$$

and recall that the Lebesgue Density Theorem [11] states that for ℓ-almost all $x \in N$ we have $F(x) = 1$. In such a case we say that x is a *point of Lebesgue density* for N.

Lemma 2.1 *Suppose that N has positive measure and Y be any closed and bounded subset of N with zero measure. Then for any $\epsilon > 0$ one can find an open set V containing Y with*

$$\ell(V \cap N)/\ell(V) > 1 - \epsilon.$$

Proof: Although Y need not contain any points of Lebesgue density for N, there is at least one point $x \in N$ of Lebesgue density, and so we choose $\delta > 0$ such that

$$\ell(B_\delta(x) \cap N)/\ell(B_\delta(x)) > 1 - \frac{\epsilon}{2}.$$

Now let $V = B_\delta(x) \cup B_\eta(Y)$. Because of outer regularity of ℓ, η can be chosen small enough to ensure that $\ell(B_\eta(Y))$ is as small as desired, and hence the result holds. **QED**

With a slight modification of the argument, one can assume that V is connected and open in the statement of the above Lemma; however it may be very far from being a ball in terms of the relationship between diameter and volume of the set.

Proof: [of Theorem 2.1] For (a) suppose that X is p.a.s. and let N be a set for which X is a.s.r.t.. By Definition 4, for any $\epsilon > 0$ there exists a $\delta_0 > 0$ such that

$$\frac{\ell(B_\delta(X) \cap N)}{\ell(B_\delta(X))} > 1 - \epsilon$$

for all $\delta < \delta_0$. In Definition 2 we set $N = D^c$ and $V = B_\delta(X)$ (where δ is sufficiently small so that $B_\delta(X) \subset U$), and we prove that X is e.a.s.. For (b), note that this follows because $N = D^c$ is a subset of the basin of attraction of X and has positive measure as $\ell(V \cap N) > 0$ for some set V. Hence it is a weak attractor, and contains a Milnor attractor [6]. Finally, for case (c) suppose firstly that X is e.a.s., then it is stable relative to the set $N = D^c$ and $\ell(N \cap A) > 0$ for any neighbourhood A of X. The converse for (c) follows similarly, on applying Lemma 2.1. **QED**

There are examples that show that, in general, converses of (a,b) do not hold; for a counterexample to the converse of (a) we refer to [13] who present heteroclinic cycles that are, in our terminology, e.a.s. but not p.a.s.. For a counterexample to the converse of (b), there are "unstable attractors" [7], though only for a weaker assumption - that Φ_t is a semiflow. These "unstable attractors" are Milnor attractors that have zero basin measures within a small enough neighbourhood of the attractor. It is not clear whether the converse of (b) is true for flows (possibly subject to some smoothness assumptions). Note that Theorem 2.1 is a generalization of comments already made in [9, p1369] which assume N to be an open set. There are examples of heteroclinic cycles that are not asymptotically stable relative to any open set, but that do seem to be asymptotically stable relative to a positive measure "riddled" set [2].

2.2 Geometry of global basins: the stability index

We suggest that it is useful to distinguish between different local geometries for e.a.s. sets. To this end, consider X an invariant set in $\mathbb{R}^n$ and let $N = \mathcal{B}(X)$ denote its (global) basin of attraction; we assume that the flow Φ_t is smooth. Pick a point $x \in X$, define

$$\Sigma_\epsilon(x) = \frac{\ell(B_\epsilon(x) \cap N)}{\ell(B_\epsilon(x))} \tag{2}$$

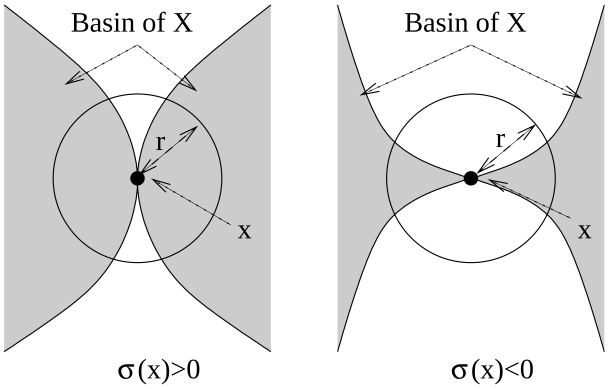

Figure 1: Schematic diagram illustrating how the stability index $\sigma(x)$ of a point $x \in X$ relates to the local geometry of the basin of attraction of X (shaded region). For $\sigma(x) > 0$, the measure of points in a ball of radius r that are in the complement of the basin goes to zero, relative the measure of the ball, as $r^{|\sigma(x)|}$. For $\sigma(x) < 0$, this estimate applies to the basin itself.

and note that $0 \leq \Sigma_\epsilon(x) \leq 1$.

Definition 5 *For a point $x \in X$ we define the* stability index *of X at x to be*

$$\sigma(x) := \sigma_+(x) - \sigma_-(x)$$

which exists when the following converge

$$\sigma_-(x) := \lim_{\epsilon \to 0} \left[\frac{\ln(\Sigma_\epsilon(x))}{\ln(\epsilon)} \right], \qquad \sigma_+(x) := \lim_{\epsilon \to 0} \left[\frac{\ln(1 - \Sigma_\epsilon(x))}{\ln(\epsilon)} \right].$$

We use the convention that $\sigma_-(x) = \infty$ if there is an $\epsilon_0 > 0$ such that $\Sigma_\epsilon(x) = 0$ for all $\epsilon < \epsilon_0$, and $\sigma_+(x) = \infty$ if $\Sigma_\epsilon(x) = 1$ for all $\epsilon < \epsilon_0$. Note that $\sigma_\pm(x) \geq 0$ and so we can assume that $\sigma(x) \in [-\infty, \infty]$.

The stability index may not exist at certain points in X (an example is given in Section 5), and may vary throughout X when it does exist. Figure 1 illustrates how the local geometry of the basin relates to the sign of $\sigma(x)$ for a point $x \in X$. Note that $\sigma(x) = +\infty$ is the "strongest" form of local stability while $-\infty$ is the "weakest". The following Lemma characterizes some basic properties of the index:

Lemma 2.2 *Suppose that $\sigma(x)$ is defined for some $x \in X \subset \mathbb{R}^n$; then the following hold:*

(a) If one of $\sigma_\pm(x)$ converges to a positive value then the other converges to zero (i.e. only one of $\sigma_+(x)$ and $\sigma_-(x)$ can be non-zero).

(b) If $\sigma(x) = c > 0$ then $1 - \Sigma_\epsilon(x) = O(\epsilon^c)$ (and in particular $\Sigma_\epsilon(x) \to 1$ as $\epsilon \to 0$).

(c) If $\sigma(x) = -c < 0$ then $\Sigma_\epsilon(x) = O(\epsilon^c)$ (and in particular $\Sigma_\epsilon(x) \to 0$ as $\epsilon \to 0$).

Proof: For (a) note that if $\sigma_-(x) > 0$ then $\lim_{\epsilon\to 0} \Sigma_\epsilon(x) = 0$; this implies that $1 - \Sigma_\epsilon$ converges to 1 as $\epsilon \to 0$ and so $\sigma_+(x) = 0$. The other case is argued in a similar way. (b) follows on noting by (a) that $c = \sigma(x) = \sigma_+(x) > 0$ and $\sigma_-(x) = 0$. Hence we have from the definition of $\sigma_+(x)$ that $1 - \Sigma_\epsilon(x) = O(\epsilon^c)$. A similar argument gives (c). **QED**

The main result in this section is the following; this can be generalized to cases where $\sigma(x)$ is measured relative to any measurable invariant set N.

Theorem 2.2 *Suppose that $N = \mathcal{B}(X)$ is the basin of X for a measurable invariant set for a C^1-smooth flow $\Phi_t(x)$. Then for any x the index $\sigma(x)$ is constant on trajectories, whenever it is defined.*

Proof: Fix $x \in X$ such that $\sigma(x)$ is defined, and pick any $t > 0$. Let $\phi(x) = \Phi_t(x)$. Because ϕ is a C^1 diffeomorphism, one can find an $\eta > 0$ such that there is an $L > 1$ and an $M > 1$ with

$$\frac{1}{L} < \det(D\phi(y)) < L, \quad \frac{1}{M} < \|D\phi(y)\| < M \tag{3}$$

for all $y \in B_\eta(x)$, where D denotes the derivative (Jacobian) of the map. We assume that L, M are chosen so that the same inequalities are satisfied by $D\phi^{-1}$ for $z \in B_\eta(\phi(x))$. As a consequence of this, one can find η' with $0 < \eta' < \eta$ such that

$$B_{\epsilon/M}(x) \subset \phi^{-1}(B_\epsilon(\phi(x))) \subset B_{M\epsilon}(x) \tag{4}$$

for any $\epsilon < \eta'$. Writing $\chi_N(y)$ as the indicator function for N and $y \in B_\eta(x)$ we have

$$\begin{aligned}
\Lambda = \ell\left(B_\epsilon(\phi(x)) \cap N\right) &= \int_{y\in B_\epsilon(\phi(x))} \chi_N(y)\, d\ell(y) \\
&= \int_{z\in\phi^{-1}(B_\epsilon(\phi(x)))} \chi_N(z) \det(D\phi^{-1}(z))\, d\ell(z)
\end{aligned}$$

where in the last line we have substituted $y = \phi(z)$ and we have used the fact that N is invariant, so that

$$\chi_N(y) = 1 \quad \Leftrightarrow \quad \chi_N(\phi(y)) = 1. \tag{5}$$

Hence, using (3,4),

$$\begin{aligned}
\frac{1}{L}\int_{z\in B_{\epsilon/M}(x)} \chi_N(z)\, d\ell(z) &\le \frac{1}{L}\int_{z\in\phi^{-1}(B_\epsilon(\phi(x)))} \chi_N(z)\, d\ell(z) \\
&< \int_{z\in\phi^{-1}(B_\epsilon(\phi(x)))} \chi_N(z) \det(D\phi^{-1}(z))\, d\ell(z) = \Lambda \\
&< L\int_{z\in\phi^{-1}(B_\epsilon(\phi(x)))} \chi_N(z)\, d\ell(z) \\
&\le L\int_{z\in B_{\epsilon M}(x)} \chi_N(z)\, d\ell(z)
\end{aligned}$$

meaning that for all $\epsilon < \eta'$ we have

$$\frac{1}{L}\ell(B_{\epsilon/M}(x)\cap N) < \ell(B_{\epsilon}(\phi(x))\cap N) < L\ell(B_{\epsilon M}(x)\cap N). \tag{6}$$

This means that from (2), there is a $K = LM^n$ such that for all small enough ϵ we have

$$\frac{1}{K}\Sigma_{\epsilon/M}(x) < \Sigma_{\epsilon}(\phi(x)) < K\Sigma_{\epsilon M}(x)$$

(we have used the property that $\ell(B_{M\epsilon}(x)) = M^n\ell(B_{\epsilon}(x))$ for any $\epsilon > 0$ and x). Hence we have

$$\begin{aligned}
\left[\frac{\ln\epsilon - \ln M}{\ln\epsilon}\right]\frac{\ln(\Sigma_{\epsilon/M}(x))}{\ln(\epsilon/M)} - \frac{\ln K}{\ln\epsilon} &= \frac{\ln(\Sigma_{\epsilon/M}(x))}{\ln\epsilon} - \frac{\ln K}{\ln\epsilon} = \frac{\ln\left(\frac{1}{K}\Sigma_{\epsilon/M}(x)\right)}{\ln\epsilon}\\
&< \frac{\ln(\Sigma_{\epsilon}(\phi(x)))}{\ln\epsilon}\\
&< \frac{\ln\left(K\Sigma_{\epsilon M}(x)\right)}{\ln\epsilon} = \frac{\ln(\Sigma_{\epsilon M}(x))}{\ln\epsilon} + \frac{\ln K}{\ln\epsilon}\\
&= \left[\frac{\ln\epsilon + \ln M}{\ln\epsilon}\right]\frac{\ln(\Sigma_{\epsilon M}(x))}{\ln(\epsilon M)} + \frac{\ln K}{\ln\epsilon}
\end{aligned}$$

and taking the limits as $\epsilon \to 0$ we have

$$\sigma_-(x) \le \sigma_-(\phi(x)) \le \sigma_-(x). \tag{7}$$

A similar argument on substituting N by its complement gives $\sigma_+(x) = \sigma_+(\phi(x))$ and hence the value of $\sigma(x)$ is constant along trajectories of Φ_t. **QED**

Note that this argument works for any C^1-diffeomorphism ϕ for which N is invariant, meaning the result can be used to show that $\sigma(x)$ is invariant under C^1-conjugation - it is an invariant of the dynamics. Note also that although $\sigma(x)$ is constant on a given trajectory, it may depend on which trajectory is chosen.

The stability index can be used to determine e.a.s. and p.a.s. by the following theorem. However, converses of the following theorem are not expected to be true in general as $\sigma(x)$ may be negative on a "lower dimensional" set of trajectories within X, or may not converge.

Theorem 2.3 *Suppose that for all $x \in X$ the stability index $\sigma(x) \in [-\infty, \infty]$ is defined.*

- *If there is a point $x \in X$ such that $-\infty < \sigma(x)$ then X is essentially asymptotically stable (e.a.s.), and contains a Milnor attractor.*
- *If there is a $c > 0$ such that $c < \sigma(x)$ for all $x \in X$ then X is predominantly asymptotically stable (p.a.s.).*

Proof: (a) The fact that $-\infty < \sigma(x)$ implies in particular that $N = \mathcal{B}(X)$ contains a set of positive measure, and so by Theorem 2.1(c) it is e.a.s.. By the Theorem 2.1(b), X contains a Milnor attractor. (b) Note that the basin of attraction N of X is such that for any δ, $\Sigma_{\epsilon}(x) > 1 - \delta$ for all x, and some ϵ depending on x. By compactness of X one can choose an ϵ small enough that $\ell(B_{\epsilon}(X)\cap N) \ge (1-\delta)\ell(B_{\epsilon}(X))$, implying p.a.s. of X. **QED**

2.3 The local stability index

While Definition 5 considers the global basin of attraction, the stability index $\sigma(x)$ can be adapted to provide a useful concept from purely local properties of the attractor. We define the δ-local basin of attraction to be the basin of attraction of X relative to $B_\delta(X)$; that is,

$$\mathcal{B}_\delta(X) := \{x \ : \ \omega(x) \subset X \text{ and } \Phi_t(x) \in B_\delta(X) \text{ for all } t > 0\}. \tag{8}$$

Note that $\mathcal{B}_\delta(X)$ is forwards, but not necessarily backwards invariant under the flow. The limit of the stability index for points relative to the δ-local basin as $\delta \to 0$ is called the *local stability index* $\sigma_{\mathrm{loc}}(x)$ for X. More precisely, we define

$$\Sigma_{\epsilon,\delta}(x) = \frac{\ell\left(B_\epsilon(x) \cap \mathcal{B}_\delta(X)\right)}{\ell(B_\epsilon(x))} \tag{9}$$

and for a point $x \in X$ we define the local stability index of X at x to be

$$\sigma_{\mathrm{loc}}(x) := \sigma_{\mathrm{loc},+}(x) - \sigma_{\mathrm{loc},-}(x)$$

which exists when the following converge

$$\sigma_{\mathrm{loc},-}(x) := \lim_{\delta\to 0}\lim_{\epsilon\to 0}\left[\frac{\ln(\Sigma_{\epsilon,\delta}(x))}{\ln(\epsilon)}\right], \qquad \sigma_{\mathrm{loc},+}(x) := \lim_{\delta\to 0}\lim_{\epsilon\to 0}\left[\frac{\ln(1-\Sigma_{\epsilon,\delta}(x))}{\ln(\epsilon)}\right]$$

with the same conventions as before. The definition works for discrete time systems as well as continuous time, without further modification. Note that the local stability index is computed for δ small and fixed before taking the limit as $\delta \to 0$.

One can weaken the assumptions in Theorem 2.2 to give the same conclusion; the critical step is that if $\eta < \delta$ and we assume that $\Phi_s(y) \in B_\eta(X)$ for $0 < s < t$ then (5) still holds- and by continuity of ϕ one can choose a small enough ϵ that x and $\phi(x)$ are guaranteed to be in such a $B_\eta(X)$.

2.4 Stability indices for sections to the flow

Suppose that $\Phi_t(x)$ has an attractor $X \subset \mathbb{R}^n$ and pick a point $x \in X$. Let $S \subset \mathbb{R}^n$ be a smooth $n-1$-dimensional subspace containing x that is transverse to the flow at x. One can relate the stability index $\sigma(x)$ or $\sigma_{\mathrm{loc}}(x)$ to the stability index for the dynamics defined by the return map F on S as follows.

Theorem 2.4 *Suppose that X is invariant for a C^1-smooth flow $\Phi_t(x)$ and that N is a (local) basin for X. Suppose that S is a codimension one surface that is transverse to the flow at x; then $\sigma(x)$ can be computed relative to the intersection of N with S on substituting $\Sigma_\epsilon(x)$ by*

$$\Sigma_{\epsilon,S}(x) = \frac{\ell_S(B_\epsilon(x) \cap N \cap S)}{\ell_S(B_\epsilon(x) \cap S)}.$$

Proof: Let $N = \mathcal{B}(X)$; the argument for local basins will be similar. Note that N is invariant implies that it is a union of trajectories. We consider local coordinates in $\mathbb{R}^n$ near x that are the coordinates in S and time. Pick any small $\epsilon > 0$; by simple geometric arguments (i.e. you can always put a cylinder in a larger sphere, and a sphere in a larger cylinder) there is a constant $K > 1$ such that

$$B_{\epsilon/K}(x) \subset (B_\epsilon(x) \cap S) \times [-\epsilon, \epsilon] \subset B_{\epsilon K}(x).$$

Using the product structure of Lebesgue measure we have

$$\ell(B_{\epsilon/K}(x)) < 2\epsilon \times \ell_S(B_\epsilon(x) \cap S) < \ell(B_{\epsilon K}(x))$$

with a similar inequality for $\ell_S(B_\epsilon(x) \cap N \cap S)$. Hence

$$\frac{\ell(B_{\epsilon/K}(x) \cap N)}{\ell(B_{\epsilon K}(x))} < \frac{\ell_S(B_\epsilon(x) \cap N \cap S)}{\ell_S(B_\epsilon(x) \cap S)} < \frac{\ell(B_{\epsilon K}(x) \cap N)}{\ell(B_{\epsilon/K}(x))}$$

meaning that

$$\Sigma_{\epsilon,S}(x) = \frac{\ell_S(B_\epsilon(x) \cap N \cap S)}{\ell_S(B_\epsilon(x) \cap S)}$$

and as before (for the flow) $\Sigma_\epsilon(x)$ satisfies the inequalities

$$\frac{1}{(2K)^n}\Sigma_{\epsilon/K}(x) < \Sigma_{\epsilon,S}(x) < (2K)^n \Sigma_{\epsilon K}(x),$$

where we have used the fact that $\ell(B_{\epsilon K}(x)) = (2K)^n \ell(B_{\epsilon/K}(x))$. In particular, the scalings of these quantities are the same as $\epsilon \to 0$. **QED**

Theorem 2.4 implies, for example, that if there is a return map for the flow on S then the stability index of trajectories for a flow can be computed by examining the stability index for the intersection of the basin with a suitable surface of section.

3 Robust heteroclinic cycles

Suppose that Γ is a finite group acting orthogonally on $\mathbb{R}^n$, and that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant vector field, i.e.

$$f(\gamma x) = \gamma f(x), \quad \text{for all } \gamma \in \Gamma.$$

Let ξ_j, $j = 1, \ldots, m$, be hyperbolic equilibria for

$$\dot{x} = f(x)$$

with stable and unstable manifolds $W^s(\xi_j)$ and $W^u(\xi_j)$ respectively, and let $s_j = W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \emptyset$ be connections between ξ_j and ξ_{j+1}, where $\xi_{m+1} = \xi_1$; then the group orbit X of the equilibria and the connections

$$X = \mathrm{clos}\left(\{\gamma s_j \; : \; j = 1, \ldots, m, \; \gamma \in \Gamma\}\right)$$

is called a *heteroclinic cycle*. Recall that for a group Γ acting on $\mathbb{R}^n$, the isotropy of the point $x \in \mathbb{R}^n$ is the subgroup

$$\Sigma_x = \{\gamma \in \Gamma \quad : \quad \gamma x = x\}$$

while for a subgroup $\Sigma \subset \Gamma$, a *fixed-point subspace* of Σ is the linear subspace

$$\mathrm{Fix}(\Sigma) = \{x \in \mathbb{R}^n \quad : \ \sigma x = x \text{ for all } \sigma \in \Sigma\}.$$

In the absence of symmetry or other constraints, a vector field with a heteroclinic cycle is structurally unstable, i.e. there are arbitrarily small perturbations of f to g, such that the heteroclinic cycle does not exist for the vector field g. For symmetric vector fields, heteroclinic cycles may be robust, as long as each connection is robust within some invariant subspace and only symmetric perturbations are allowed [15].

3.1 Local structure: eigenspaces and simple cycles

A sufficient condition for a cycle X to be *structurally stable* (or *robust*), is that for all j there exists a subspace P_j such that $P_j = \mathrm{Fix}(\Sigma_j)$ for some $\Sigma_j \subset \Gamma$, $s_j \subset P_j$, ξ_{j+1} is a sink in P_j. Denote $L_j = P_j \cap P_{j-1}$. We denote the isotropy subgroup of points in $L_j \setminus \{0\}$ by T_j. Note that $X \ominus Y$, where Y is a linear subspaces of the inner product space X, denotes the orthogonal complement to Y in X.

If X is a structurally stable heteroclinic cycle then the eigenvalues of $(df)_{\xi_j}$ can be divided into four classes:

- Eigenvalues with associated eigenvectors in L_j are called *radial*, the maximal real part of radial eigenvalues being $-r_j$.
- Eigenvalues with associated eigenvectors in $P_{j-1} \ominus L_j$ are called *contracting*, the maximal real part of contracting eigenvalues being $-c_j$.
- Eigenvalues with associated eigenvectors in $P_j \ominus L_j$ are called *expanding*, the maximal real part of expanding eigenvalues being e_j.
- The remaining eigenvalues are called *transverse*, the maximal real part of transverse eigenvalues being t_j.

The heteroclinic cycle $X \in \mathbb{R}^4 \setminus \{0\}$ is called a *simple robust heteroclinic cycle* (in $\mathbb{R}^4$) [18] if for all j:

- All eigenvalues of $(df)_{\xi_j}$ are distinct, $\dim P_j = 2$ and X intersects with each connected component of $L_j \setminus \{0\}$ in at most one point.

For simple cycles, L_j and the three remaining subspaces are one-dimensional, hence there is a unique real eigenvalue of each type. Moreover, for simple cycles either $T_j \cong \mathbb{Z}_2^2$ and $\Sigma_j \cong \mathbb{Z}_2$ for all j or $T_j \cong \mathbb{Z}_2^3$ and $\Sigma_j \cong \mathbb{Z}_2^2$ for all j (see Proposition 3.1 in [18]), and each simple cycle is of one of three types discussed by [18].

Definition 6 *Suppose that X is a simple heteroclinic cycle that is robust for a vector field in $\mathbb{R}^4$ with a finite symmetry group. We say*

- *X is of Type A if $\Sigma_j \cong \mathbb{Z}_2$ for all j.*
- *X is of Type B if there is a subspace Q of $\mathbb{R}^4$ with $\dim(Q) = 3$ such that $Q = \mathrm{fix}(\tilde{\Sigma})$ for some $\tilde{\Sigma} \subset \Gamma$ and $X \subset Q$.*
- *X is of Type C if it is neither of Type A nor of Type B.*

The work of [18] goes on to differentiate between four varieties of Type B cycles (denoted by B_1^+, B_2^+, B_1^- and B_3^-) and three varieties of Type C cycles (denoted by C_1^-, C_2^- and C_4^-), depending on the number of equilibria involved in the cycle and action of the group Γ. In Section 4 we examine the stability of cycles in $\mathbb{R}^4$ using Poincaré maps, where the structure of the maps depend on the type of cycle.

3.2 Local stability for heteroclinic cycles

For a heteroclinic cycle X comprised of one-dimensional connections s_j, $j = 1, \ldots, m$ (s_j is the connection from ξ_{j-1} to ξ_j), its local attraction properties are described by the set of stability indices of the trajectories

$$\sigma = (\sigma_1, \ldots, \sigma_m).$$

where

$$\sigma_j = \sigma(x)$$

for x an arbitrary point on s_j. The following lemma will be useful later on:

Lemma 3.1 *Let a simple heteroclinic cycle X be comprised of one-dimensional connections and suppose that $-\infty < \sigma_j$ for some j. Then X is a Milnor attractor.*

Proof: This follows from the fact that $-\infty < \sigma_j(x)$ implies that $\ell(\mathcal{B}(X)) > 0$ and so X is a weak attractor. Since no invariant subset of the cycle can be a Milnor attractor, X must itself be a Milnor attractor. **QED**

In what follows, we calculate local stability indices for different types of simple robust heteroclinic cycles in $\mathbb{R}^4$. Following [13, 18], to examine stability we construct a Poincaré map in the vicinity of the cycle.

3.3 Stability indices for return maps

Section 3 gave definitions for radial, contracting, expanding and transverse eigenvalues of the linearization $(df)_{\xi_j}$. Simple cycles in $\mathbb{R}^4$ will possess a single eigenvalue of each type. Let

$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$ be local coordinates near ξ_j in the basis of the four associated eigenvectors, and $\tilde{B}_{\tilde{\delta}}(\xi_j)$ be a neighbourhood of ξ_j defined as

$$\tilde{B}_{\tilde{\delta}}(\xi_j) = \{(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z}) \mid \max(|\tilde{u}|, |\tilde{v}|, |\tilde{w}|, |\tilde{z}|) < \tilde{\delta}\}$$

where $\tilde{\delta}$ is small, denote by (u, v, w, z) the scaled coordinates $(u, v, w, z) = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})/\tilde{\delta}$. In the (u, v) plane we will also employ plane polar coordinates (r, θ), $u = r\cos\theta$ and $v = r\sin\theta$. For a small $\tilde{\delta}$, the vector field f can be linearly approximated in $\tilde{B}_{\tilde{\delta}}(\xi_j)$. We assume that δ in (8) is sufficiently small, so that

$$B_\delta(\xi_j) \subset \tilde{B}_{\tilde{\delta}}(\xi_j) \text{ for all } \ j.$$

In $\tilde{B}_{\tilde{\delta}}(\xi_j)$ we consider the linearised system (1)

$$\begin{aligned} \dot{u} &= -r_j u \\ \dot{v} &= -c_j v \\ \dot{w} &= e_j w \\ \dot{z} &= t_j z \end{aligned} \tag{10}$$

This gives an accurate approximations of the nonlinear flow as long as the linear system has no low order resonances.

The connection s_j is tangent to the subspace $u = v = z = 0$. The heteroclinic connection to ξ_j lies in P_{j-1} where local coordinates are u and v. For

$$H_j^{(out)} = \{(u, v, w, z) \ : \ |u|, |v|, |z| \le 1, w = 1\},$$

$$H_j^{(in)} = \{(r, \theta, w, z) \ : \ r = 1, |w|, |z| \le 1\}.$$

the *first return map* $\phi_j : H_j^{(in)} \to H_j^{(out)}$ is defined near each equilibrium in $\tilde{H}_j^{(in)} = Q_j^{(in)} \cap H_j^{(in)}$, where $Q_j^{(in)} = \{(z, w) | \ |z| < |w|^{t_j/e_j}\}$[2]. For each connection we define *connecting diffeomorphisms* $\psi_j : H_j^{(out)} \to H_{j+1}^{(in)}$ and their compositions $\tilde{g}_j = \psi_j \circ \phi_j : \tilde{H}_j^{(in)} \to H_{j+1}^{(in)}$. The Poincaré map is the composition $\tilde{g} = \tilde{g}_m \circ \ldots \circ \tilde{g}_1 : H_1^{(in)} \to H_1^{(in)}$.

As shown in [13, 16], at leading order the maps have the form[3]

$$\phi_j(u, v, w, z) = (u|w|^{r_j/e_j}, vw^{c_j/e_j}, 1, z|w|^{-t_j/e_j}) \tag{11}$$

[2]Strictly speaking, the maps are defined for $|z| < K(1-\delta)|w|^{t_j/e_j}$, where K is a constant and δ is small [13]. For simplicity, we ignore K and δ, because for small z and w they do not enter into asymptotically significant terms.

[3]The maps for negative w are defined as follows. Consider a neighbourhood of a point ξ_j. Two heteroclinic connections enter this neighbourhood, $\xi_{j-1} \to \xi_j$ and $\gamma^{(in)}\xi_{j-1} \to \xi_j$, where $\gamma^{(in)}$ is any symmetry, satisfying $\gamma^{(in)} \in T_j$ and $\gamma^{(in)} \notin \Sigma_{j-1}$. Two heteroclinic connections exit the neighbourhood: $\xi_j \to \xi_{j+1}$ and $\xi_j \to \gamma^{(out)}\xi_{j+1}$, where $\gamma^{(out)}$ is a symmetry, satisfying $\gamma^{(out)} \in T_j$ and $\gamma^{(out)} \notin \Sigma_j$. The local map $\phi_j : H_j^{(in)} \to H_j^{(out)}$ is defined only for w and v of particular signs, say $w > 0$ and $v > 0$. For $w < 0$, the local map acts to $\tilde{H}_j^{(out)}$, where $\tilde{H}_j^{(out)} = \gamma^{(out)} H_j^{(out)}$; for $v < 0$, it is defined in $\gamma^{(in)} H_j^{(in)}$. Due to existence of the symmetries $\gamma^{(in)}$ and $\gamma^{(out)}$, we can consider ϕ_j for arbitrary w and v: by applying these symmetries the local map can be defined for w and v of arbitrary signs.

and

$$\psi_j(u,v,w,z) = (1, \theta_0 + \beta_j u, \alpha_{j_1} v + \alpha_{j_2} z, \alpha_{j_3} v + \alpha_{j_4} z). \tag{12}$$

For these maps only (w,z) coordinates are important [13, 16], and restricting to these coordinates the map $g_j : \mathbb{R}^2 \to \mathbb{R}^2$ we have

$$g_j(w,z) = (\alpha_{j_1} w^{c_j/e_j} + \alpha_{j_2} z|w|^{-t_j/e_j}, \alpha_{j_3} w^{c_j/e_j} + \alpha_{j_4} z|w|^{-t_j/e_j}). \tag{13}$$

For cycles of Type A, generically $\alpha_{jk} \neq 0$ for all $j = 1, \dots, m$, $k = 1,2,3,4$. For cycles of Type B, $\alpha_{j_2} = \alpha_{j_3} = 0$ and $\alpha_{j_1}\alpha_{j_4} \neq 0$. For cycles of Type C, $\alpha_{j_1} = \alpha_{j_4} = 0$ and $\alpha_{j_2}\alpha_{j_3} \neq 0$.

Thus, for a point $x = X \cap H_1^{(in)}$ on the cycle X we have associated the map

$$g = g_m \circ g_{m-1} \circ \dots \circ g_1 : \mathbb{R}^2 \to \mathbb{R}^2. \tag{14}$$

We will also use the notation $g_{l,k} = g_l \circ g_{l-1} \circ \dots \circ g_1 \circ g^k$.

Similarly to the local stability index for a point $x \in X$ (see Section 2.3), we define a stability index σ^g for the map (14). Note that B_ϵ is the ball of radius ϵ in $\mathbb{R}^n$ centered at 0, and we define

$$\mathcal{B}_\delta^g := \{x \ : \ x \in \mathbb{R}^n, \ |g_{l,k} x| < \delta \text{ for all } 0 \le l \le m-1, \ k \ge 0\}, \tag{15}$$

to be the δ-local basin of attraction of 0 in $\mathbb{R}^n$ for the map g (14). The local stability index is defined to be

$$\sigma^g := \sigma_+^g - \sigma_-^g$$

where

$$\sigma_-^g := \lim_{\delta \to 0} \lim_{\epsilon \to 0} \left[\frac{\ln(\Sigma_{\epsilon,\delta})}{\ln(\epsilon)} \right], \qquad \sigma_+^g := \lim_{\delta \to 0} \lim_{\epsilon \to 0} \left[\frac{\ln(1 - \Sigma_{\epsilon,\delta})}{\ln(\epsilon)} \right],$$

with

$$\Sigma_{\epsilon,\delta} = \frac{\ell(B_\epsilon(0) \cap \mathcal{B}_\delta^g)}{\ell(B_\epsilon(0))}. \tag{16}$$

Because of the asymptotic independence of the return map on two of the coordinates, we can effectively reduce the computation of the stability index for heteroclinic cycles in $\mathbb{R}^4$ to a calculation on a section to the cycle in $\mathbb{R}^2$.

Theorem 3.1 *Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ (14) be the map associated with a point $x = X \cap H_1^{(in)}$, where X is a simple heteroclinic cycle in $\mathbb{R}^4$. Then*

$$\sigma_{\rm loc}(x) = \sigma^g.$$

Proof: First, we prove that $\sigma_{\rm loc}(x) \ge \sigma^g$. Denote $Q = [-\epsilon, \epsilon]^2 \times \mathcal{B}_\delta^g$. If $y \equiv (u,v,w,z) \in B_\epsilon(x) \setminus Q$, then there exist j and k, such that $|g_{j,k}(y)| > \delta$. For small δ, a trajectory near the heteroclinic cycle is approximated by the maps ϕ_j (11) and ψ_j (12), where the coordinates (w,z) are independent of (u,v). Hence, for the point $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$, which is the $(k+1)$st intersection of $\Phi_t(y)$ with $H_j^{(in)}$, we have $|(\tilde{w}, \tilde{z})| > \delta/2$. Hence $y \notin \mathcal{B}_{\delta/2}(X)$, and therefore

$$\ell(B_\epsilon(x) \cap \mathcal{B}_{\delta/2}(X)) < 4\epsilon^2 \ell(B_\epsilon \cap \mathcal{B}_\delta^g),$$

implying that $\sigma_{\rm loc}(x) \geq \sigma^g$.

Second, we prove that $\sigma_{\rm loc}(x) \leq \sigma^g$. If $y \equiv (u,v,w,z) \in B_\epsilon(x) \cap Q$, then for any intersection $(1, \theta_1, \tilde{w}, \tilde{z})$ (here we are using polar coordinates in the (u,v) plane, θ_1 is the difference between θ and θ_0, the distance of the point from the cycle is denoted $|(\theta_1, \tilde{w}, \tilde{z})|$) of $\Phi_t(y)$ with $H_j^{(in)}$, $|(\tilde{w}, \tilde{z})| < 2\delta$ holds true. Together with (11) and (12), it implies that in any intersection $\theta_1 < \beta\delta^r$, where $\beta = \max(\max_j(\beta_j), 1)$ and $r = \min(\min_j(r_j/e_j), 1)$. Thus, we have proved that for any intersection $\tilde{y}$ of $\Phi_t(y)$ with $H_j^{(in)}$ for any j we have

$$d(\tilde{y}, X) = |(\theta_1, \tilde{w}, \tilde{z})| < 2\beta\delta^r. \tag{17}$$

If a trajectory is close to the heteroclinic cycle, then there exist constants K_j such that $d(s_j, \Phi_t(x)) < K_j d(s_j, \Phi_{T_{j,k}}(x))$ in the interval $\tilde{T}_{j-1,k} < t < T_{j,k}$. (Here $T_{j,k}$ is the time when $\Phi_t(x)$ crosses $H_j^{(in)}$ and $\tilde{T}_{j-1,k}$ is the time when it crosses $H_{j-1}^{(out)}$.) Denote $K = \beta \max_j(K_j)$. In the vicinity of ξ_j we can consider the system (10) using the approximated map f, hence (17) is satisfied at the points of intersection, which implies that

$$d(\Phi_t(y), X) < 2K\delta^r \text{ for all } t > 0.$$

Hence $y \in \mathcal{B}_{2K\delta^r}(X)$, and therefore

$$\ell(B_\epsilon(x) \cap \mathcal{B}_{2K\delta^r}(X)) > \epsilon^2 \ell(B_\epsilon \cap \mathcal{B}^g_\delta).$$

Since $K > 0$ and $r > 0$ do not depend on δ, this implies $\sigma_{\rm loc}(x) \leq \sigma^g$ and therefore $\sigma_{\rm loc}(x) = \sigma^g$. **QED**

4 Stability indices for heteroclinic cycles in $\mathbb{R}^4$

In this section the stability indices for the connections of simple robust heteroclinic cycles in $\mathbb{R}^4$ are calculated in terms of ratios of eigenvalues $a_j = c_j/e_j$ and $b_j = -t_j/e_j$, $1 \leq j \leq m$. We do this relative to the classification of simple heteroclinic cycles in $\mathbb{R}^4$ of Definition 6 and [18]. Here only statements of the main theorems and a sketch of the proof of the main theorem for type A cycles are presented. The complete proofs are given in the Appendices.

Using the maps $\{g_j\ ,\ j = 1, \ldots, m\}$ from the previous section, we define return maps on sections to any of the connections

$$g^{(j)} : H_j^{(in)} \to H_j^{(in)}$$

to be $g^{(j)} = g_{j-1} \circ \ldots \circ g_1 \circ g_m \circ \ldots \circ g_j$, let

$$\sigma_j = \sigma_{j,+} - \sigma_{j,-}$$

and

$$\sigma_{j,\pm} = \sigma_\pm^{g^{(j)}},$$

where $\sigma^g_\pm$ are defined in Section 3.3. Note that in the previous section we used

$$g = g^{(1)}$$

and that σ_j will effectively give the stability index for the connection that intersects $H_j^{(in)}$.

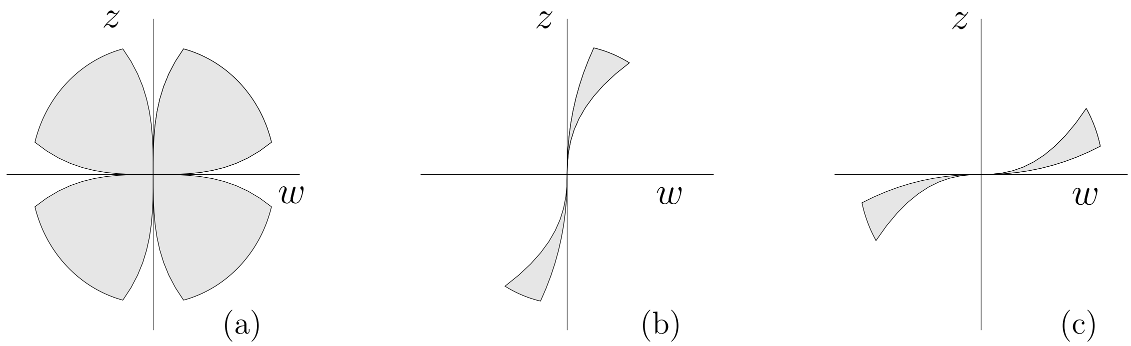

Figure 2: Examples of sets (a) $Q(r,\epsilon)$, (b) $Q_z(f_1(z),f_2(z),\epsilon)$ and (c) and $Q_w(f_1(w),f_2(w),\epsilon)$ used in the discussion of stability of Type A cycles. The sets are shaded.

4.1 Type A cycles

Consider the map $g = g_m \circ \ldots \circ g_1 : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$g_j(w,z) \equiv (g_j^w(w,z), g_j^z(w,z)) = (A_j w^{a_j} + B_j |w|^{b_j} z, C_j w^{a_j} + D_j |w|^{b_j} z). \tag{18}$$

For generic cycles of Type A we can assume $A_jB_jC_jD_j \neq 0$ and $A_jD_j \neq B_jC_j$ for all j. Recall that $g_{l,k} = g_l \circ \ldots \circ g_1 \circ g^k$.

Let us define

$$\begin{aligned} Q(r,\epsilon) &= \{(w,z) \ : \ |(w,z)| < \epsilon, \ |z| > |w|^{1+r}, \ |w| > |z|^{1+r}\}, \\ Q_z(f_1(z),f_2(z),\epsilon) &= \{(w,z) \ : \ |(w,z)| < \epsilon, \ f_1(z) < w < f_2(z)\}, \\ Q_w(f_1(w),f_2(w),\epsilon) &= \{(w,z) \ : \ |(w,z)| < \epsilon, \ f_1(w) < z < f_2(w)\}, \end{aligned}$$

where we assume $f_1(z) = \mathrm{O}(z^{1+r})$, $f_1(w) = \mathrm{O}(w^{1+r})$, $f_1(z) - f_2(z) = \mathrm{O}(z^{1+r_1})$, $f_1(w) - f_2(w) = \mathrm{O}(w^{1+r_1})$ for some $r > 0$ and $r_1 \geq r$. In line with the definitions, the areas of the sets $Q_z(f_1(z),f_2(z),\epsilon)$ and $Q_w(f_1(w),f_2(w),\epsilon)$ are $\mathrm{O}(\epsilon^{r_1+2})$. Examples of the sets $Q(r,\epsilon)$, $Q_z(f_1(z),f_2(z),\epsilon)$ and $Q_w(f_1(w),f_2(w),\epsilon)$ are shown in Figure 2. By Theorem 4.1 below, if the stability index satisfies $\sigma_j > -\infty$, then either the complement to $\mathcal{B}_\delta^g$ in $B_\epsilon(0) \subset H_j^{(in)}$ is empty, or it is the union of the sets $Q_z(f_1(z),f_2(z),\epsilon)$, or the union of $Q_w(f_1(w),f_2(w),\epsilon)$. Whether it is the union of the sets Q_z or Q_w, depends on the difference between the contracting and the transverse eigenvalues.

Examples of the sets $\mathcal{B}_\delta^g$ are shown in Figure 3 for different signs of b_j and $a_j - b_j - 1$. It can be observed that the sets are invariant with respect to the symmetry $(w,z) \to (-w,-z)$. This is so, because the linearised systems (10) evidently possess the symmetry, and the global maps ϕ_j are symmetric (being linear).

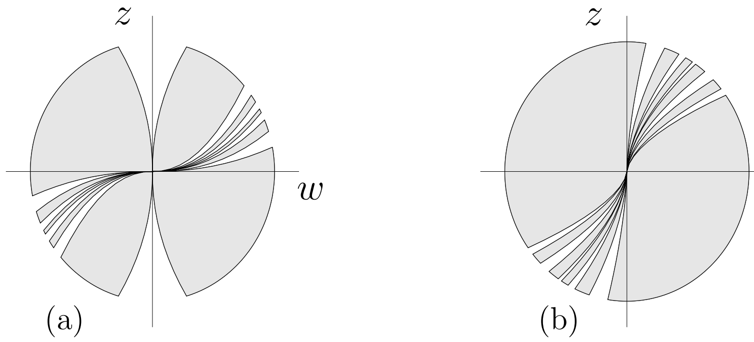

Figure 3: Examples of sets $\mathcal{B}^g_\delta$ (shaded) for Type A cycles for the cases (a) $b_1 < 0$, $a_1 - b_1 < 1$ and (b) $b_1 > 0$, $a_1 - b_1 > 1$.

For calculation of stability indices, we introduce the collection of functions $\{h_{l,j}(y)\}$ for $1 \le j \le m$, $l \le j$, which are defined as follows[4]:

$$
\begin{aligned}
h_{j,j}(y) &= y, \\
h_{l,j}(y) &= \begin{cases} \infty & \text{if } a_l - b_l < 0 \\ \dfrac{a_l h_{l+1,j}(y) - a_l + 1}{a_l - b_l} & \text{if } 0 < a_l - b_l < 1 \\ a_l h_{l+1,j}(y) - b_l & \text{if } a_l - b_l > 1 \end{cases}
\end{aligned}
$$

The next theorem is the main result for Type A cycles, namely it gives the stability indices σ_j for the collection of maps g_j related to Type A cycles. Recall that the coefficients a_j and b_j of the map g_j are related to the eigenvalues of linearisation of (1) near ξ_j as $a_j = c_j/e_j$ and $b_j = -t_j/e_j$. Recall, that $c_j > 0$ and $e_j > 0$ for all j, therefore $a_j > 0$. Following [16], we denote

$$\rho_j = \min(a_j, 1 + b_j),$$

$\rho = \rho_1 \cdots \rho_m$, and note that generically the non-degeneracy conditions

$$a_j \neq 1 + b_j,\ b_j \neq -1,\ \rho \neq 1 \tag{19}$$

are satisfied. The Theorem below is stated and proved more precisely as Theorem A.1.

Theorem 4.1 *For the collection of maps g_j associated with a Type A cycle, the stability indices are:*

(a) If $\rho > 1$ and $b_j > 0$ for all j then $\sigma_{j,+} = \infty$ and $\sigma_{j,-} = 0$ for any j.

[4]If an index takes values $1, \ldots, m$, then the index value modulo m is understood here and below. Note that l in $h_{l,j}$ can take negative values.

(b) If $\rho > 1$, $b_j > -1$ for all j and $b_j < 0$ for $j = J_1, \ldots, J_L$ then $\sigma_{j,-} = 0$ and $\sigma_{j,+}$ are:

$$\sigma_{j,+} = \min_{s=J_1,\ldots,J_L} h_{\tilde{j},s}\left(-\frac{1}{b_s}\right) - 1, \ \textit{where } \tilde{j} = \left\{ \begin{array}{rr} j, & j \le s \\ j-m, & j > s \end{array} \right. .$$

(c) If $\rho < 1$ or there exists j such that $b_j < -1$ then $\sigma_{j,+} = 0$, $\sigma_{j,-} = \infty$ and the cycle is not an attractor.

Proof: (a) Since $\rho > 1$, there exists a $q > 0$ such that

$$\tilde{\rho} = \prod_{j=1}^{m} (\rho_j - q) > 1. \tag{20}$$

By Lemma A.2, for sufficiently small ϵ

$$|g_j(w,z)| < |(w,z)|^{\rho_j - q} \text{ for any } (w,z), \ |(w,z)| < \epsilon. \tag{21}$$

Consider $(w,z) \in H_1^{(in)}$. Inequality (21) implies that for a given δ, we can find an $\epsilon > 0$ such that

$$|g_{j,k}(w,z)| = |g_{j,1}(w,z)||g^k(w,z)| < |g_{j,1}(w,z)||(w,z)|^{\tilde{\rho}^k} < \delta$$
$$\text{for all } 1 \le j \le m, \ k \ge 0, \ \text{and } |(w,z)| < \epsilon.$$

Therefore $\sigma_{1,+} = \infty$ and $\sigma_{1,-} = 0$. The proof for $j > 1$ is similar.

(b) Consider $s = J_l$ for some l, whereby $-1 < b_s < 0$. For any small $q > 0$ and $\delta > 0$ we can find ϵ, such that

$$(w,z) \in Q^{s,s}(\epsilon) \stackrel{\text{def}}{=} Q_z(-|z|^{-1/b_s + q}, |z|^{-1/b_s+q}, \epsilon)$$

implies

$$|g_s(w,z)| > \delta$$

and

$$(w,z) \in H_s^{(in)} \setminus \tilde{Q}^{s,s}(\epsilon), \ \text{where } \tilde{Q}^{s,s}(\epsilon) \stackrel{\text{def}}{=} Q_z(-|z|^{-1/b_s - q}, |z|^{-1/b_s-q}, \epsilon)$$

implies

$$|g_s(w,z)| < \delta.$$

For simplicity, we ignore small q and say that if

$$(w,z) \in \hat{Q}^{s,s}(\epsilon) \stackrel{\text{def}}{=} Q_z(-|z|^{-1/b_s}, |z|^{-1/b_s}, \epsilon),$$

then

$$|g_s(w,z)| > \delta,$$

and if

$$(w,z) \in H_s^{(in)} \setminus \hat{Q}^{s,s}(\epsilon),$$

then

$$|g_s(w,z)| < \delta.$$

We also assume that ϵ is sufficiently small and all estimates for (w,z) required in all the applied lemmas are satisfied.

Denote by $\hat{Q}^{s-1,s}(\epsilon) \subset H_{s-1}^{(in)}$ the preimage of $\hat{Q}^{s,s}(\epsilon)$ under the map g_{s-1}, and by $\hat{Q}^{j,s}(\epsilon) \subset H_j^{(in)}$ the preimage of $\hat{Q}^{s,s}(\epsilon)$ under $g_s \circ g_{s-1} \circ \ldots \circ g_j$. By construction of the sets $\hat{Q}^{j,s}(\epsilon)$, $1 \le j \le s$, for any $(w,z) \in \hat{Q}^{j,s}(\epsilon)$ the inequality

$$|g_s \circ g_{s-1} \circ \ldots \circ g_j(w,z)| > \delta$$

is valid.

The measure (area) of the set $\hat{Q}^{s,s}(\epsilon)$ can be estimated as

$$\ell(\hat{Q}^{s,s}(\epsilon)) = \mathrm{O}(\epsilon^{-1/b_s+1}) = \mathrm{O}(\epsilon^{h_{s,s}(-1/b_s)+1}).$$

By virtue of the definition of functions $h_{l,j}$ and due to Lemmas A.3 and A.4, the measure of the set $\hat{Q}^{1,s}(\epsilon)$ is

$$\ell(\hat{Q}^{1,s}(\epsilon)) = \mathrm{O}(\epsilon^{h_{1,s}(-1/b_s)+1}).$$

(Here and below, the measure of an empty set $\hat{Q}^{j,s}(\epsilon)$ is supposed to be ϵ^∞.)

Denote by $\hat{Q}^{1-m,s}(\epsilon)$ the preimage of the set $\hat{Q}^{1,s}(\epsilon)$ under a complete iteration g along the cycle, and by $\hat{Q}^{1-km,s}(\epsilon)$ the preimage under k iterations. The measure of the set $\hat{Q}^{1-km,s}(\epsilon)$ is

$$\ell(\hat{Q}^{1-km,s}(\epsilon)) = \mathrm{O}(\epsilon^{h_{1-km,s}(-1/b_s)+1}).$$

Since $\rho > 1$, by the same arguments as used in the proof of part (a), if $(w,z) \in H_1^{(in)} \cap B_\epsilon$ does not belong to any $\hat{Q}^{1-km,s}(\epsilon)$ for all $s = J_1, \ldots, J_L$ and $k \ge 0$, then

$$|g_{j,k}(w,z)| < \delta \text{ for all } 1 \le j \le m \text{ and } k \ge 0.$$

By construction of the sets $\hat{Q}^{1-km,s}(\epsilon)$, if $(w,z) \in \hat{Q}^{1-km,s}(\epsilon)$, then

$$|g_{j,k}(w,z)| > \delta \text{ for some } j \text{ and } k.$$

By properties of the functions $h_{l,j}$, the measure of the set $\hat{Q}^{1,s}(\epsilon)$ is larger than that of any other set $\hat{Q}^{1-km,s}(\epsilon)$ for $k > 0$. Since

$$\ell(\cup_{1 \le s \le J_l} \hat{Q}^{1,s}(\epsilon)) = \mathrm{O}(\epsilon^{\min_s h_{1,s}(-1/b_s)+1}),$$

by definition of the stability index, for $j = 1$ the statement of the theorem, part (b), holds true. For other j the proof is similar.

(c) Below we assume that δ and ϵ are sufficiently small, so that for $|(w,z)| < \max(\delta,\epsilon)$ the conditions of all lemmas to be applied hold true.

We consider three following cases, which cover exhaustively all possibilities:

- Suppose $\rho < 1$ and $b_j > -1$ for all j. Since $\rho < 1$, there exists a $q > 0$ such that

$$\tilde{\rho} = \prod_{j=1}^{m} (\rho_j + q) < 1. \tag{22}$$

By Corollary A.1(b), there exists r such that if $(w, z) \in Q(r, \delta)$ then

$$|g_{l,k}(w, z)| > \delta \tag{23}$$

for some l and k. Consequently, if $(w, z) \in (g^K)^{-1}Q(r, \delta)$ for some $K > 0$, then the inequality (23) is satisfied for $k \equiv k + K$. The complement to $Q(r, \delta)$ in B_δ is the union of the sets $\tilde{Q}_z = Q_z(|z|^{1+r}, -|z|^{1+r}, \delta)$ and $\tilde{Q}_w = Q_w(|w|^{1+r}, -|w|^{1+r}, \delta)$. Corollaries A.2 and A.3 imply existence of the limit sets

$$Q_z^{\lim} = \lim_{k\to\infty} (g^k)^{-1}\tilde{Q}_z \neq \emptyset \text{ and } Q_w^{\lim} = \lim_{k\to\infty} (g^k)^{-1}\tilde{Q}_w \neq \emptyset,$$

and by Lemmas A.3 and A.4, (23) is satisfied for $(w, z) \in Q_z^{\lim} \cap Q_w^{\lim}$ for some l and k. Therefore, $\sigma_{1,-} = \infty$.

- Suppose that $b_s < -1$ for some s, and also at least one of the inequalities, $\rho > 1$, or $a_t - b_t < 0$ for some t, is satisfied. Let the set $\hat{Q}^{s,s}(\epsilon)$ be defined as in the proof of part (b). Denote by $R^{s,s}(\epsilon)$ the complement to the set $\hat{Q}^{s,s}(\epsilon)$ in B_ϵ, and by $R^{1-km,s}(\epsilon)$ the preimage of $R^{s,s}(\epsilon)$ under $g_s \circ g_{s-1} \circ \ldots \circ g_1 g^k$. By the arguments of the proof of part (b), the measure of the set $R^{1-km,s}(\epsilon)$ is

$$\ell(R^{1-km,s}(\epsilon)) = \mathrm{O}(\epsilon^{h_{1-km,s}(-b_s)+1}).$$

By properties of the functions $h_{l,j}$, $\lim_{k\to\infty} h_{1-km,s}(-b_s) = \infty$, and therefore $\sigma_{1,-} = \infty$.

- Suppose $\rho < 1$, $a_t - b_t > 0$ for all t, and $b_s < -1$ for some s. Let the sets $R^{1-km,s}(\epsilon)$ be defined as in the previous paragraph. Since $\rho < 1$ and $a_t - b_t > 0$ for all t, the sets do not vanish in the limit $k \to \infty$. By Corollaries A.2 and A.3, the limit set $R^{\lim} = \lim_{k\to\infty} R^{1-km,s}(\epsilon)$ does exist, and by Lemmas A.3 and A.4, inequality (23) is satisfied for all $(w, z) \in R^{\lim}$ for some l and k. Therefore, $\sigma_{1,-} = \infty$.

The proof for $\sigma_{j,-}$, $j > 1$, is similar. **QED**

4.2 Type B and C cycles

Examples of sets $\mathcal{B}_\delta^g$ for Type B and C cycles are shown in Figure 4. The sets are invariant with respect to the symmetries $(w, z) \to (-w, z)$ and $(w, z) \to (w, -z)$, because the linearised systems (10) evidently possess these symmetries, and the global maps ϕ_j are symmetric due to linearity and invariance of the subspaces $(w, 0)$ and $(0, z)$. Therefore, we consider in this

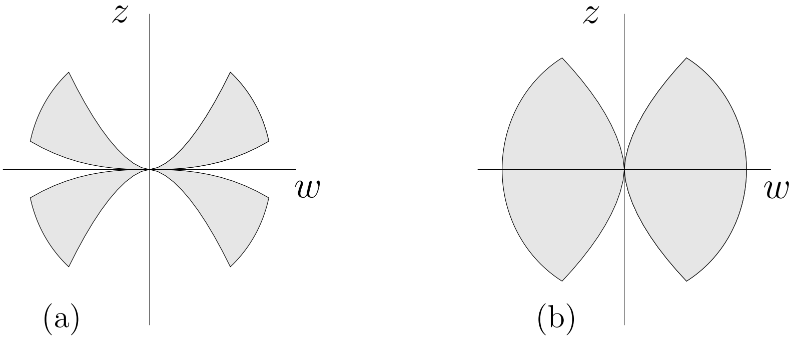

Figure 4: Examples of sets $\mathcal{B}_\delta^g$ (shaded) for Type B or C cycles.

subsection only positive values of w and z; components of $\mathcal{B}_\delta^g$ in other three quadrants are obtained on applying the symmetries.

As noted in [18], the maps $g_j : H_j^{(in)} \to H_{j+1}^{(in)}$ related to the cycles of Types B and C asymptotically to the lowest order have the form $(Ew^{a_j}, Fw^{b_j}z)$ and $(Ew^{b_j}z, Fw^{a_j})$, respectively. In the new coordinates (ζ, η), $\zeta = \ln z$ and $\eta = \ln w$, the maps g_j are linear:

$$g_j(\zeta, \eta) = M_j \begin{pmatrix} \zeta \\ \eta \end{pmatrix} + \begin{pmatrix} \ln E \\ \ln F \end{pmatrix},$$

where the *transition matrices* of the maps are

$$M_j = \begin{pmatrix} a_j & 0 \\ b_j & 1 \end{pmatrix} \text{ and } M_j = \begin{pmatrix} b_j & 1 \\ a_j & 0 \end{pmatrix}$$

for cycles of Types B and C, respectively. In the definition of stability indices, asymptotically small z and w (and therefore asymptotically large negative ζ and η) are assumed. Hence, we ignore finite $\ln E$ and $\ln F$.

Recall that the coefficients a_j and b_j of the matrices are related to the eigenvalues of linearisation of (1) near ξ_j as $a_j = c_j/e_j$ and $b_j = -t_j/e_j$. For the map $g = g_m \circ \ldots \circ g_1$ the transition matrix is $M = M(g) = M_m \cdots M_1$. We introduce the notation: $M_{j,k}$ and $M^{(j)}$ denote transition matrices for the maps $g_{j,k}$ and $g^{(j)}$, respectively; $M^{(l,j)} = M_l \cdots M_j$; λ_1^j, λ_2^j, $\mathbf{v}_1^j = (v_{11}^j, v_{12}^j)$ and $\mathbf{v}_2^j = (v_{21}^j, v_{22}^j)$ denote eigenvalues and associated eigenvectors of the matrix $M^{(j)}$, respectively. If the eigenvalues are real, $\lambda_1^j \geq \lambda_2^j$ is assumed. (Generically the eigenvalues are different.)

A necessary condition for (w, z) to belong to $\mathcal{B}_\delta^g$ (see Subsection 3.3) is that $g^k(w, z)$ is bounded for all k. To leading order, the map $g : (\zeta, \eta) \to (\zeta, \eta)$ is described by the transition matrix $M(g)$. Due to linearity of the map, in the new coordinates the condition that the iterates $(\zeta_k, \eta_k)^t = M^k(\zeta, \eta)^t$ are bounded by an $S < 0$ (i.e., $\zeta_n < S$ and $\eta_n < S$, or, in the original coordinates, $w < \mathrm{e}^S$ and $z < \mathrm{e}^S$) generically is equivalent to $\lim_{k\to\infty} M^k(\zeta,\eta)^t = (-\infty,-\infty)^t$.

We denote

$$U^{-\infty}(M) = \{(x,y):\ x \le 0,\ y \le 0,\ \lim_{n\to\infty} M^n(x,y)^t = (-\infty,-\infty)^t\}.$$

Evidently, $U^{-\infty}(M) = \emptyset$ implies $\mathcal{B}^g_\delta = \emptyset$. The conditions for $U^{-\infty}(M) \neq \emptyset$ are given in Lemma B.1 in terms of eigenvalues and eigenvectors of matrix M. They are:

(i) the eigenvalues are real;

(ii) $\lambda_1 > 1$;

(iii) $\lambda_1 > |\lambda_2|$;

(iv) $v_{11}v_{12} > 0$.

In terms of entries of a 2×2 matrix $M = (a_{ij})$ (where we assume $a_{11} > a_{22}$) the conditions are (Lemma B.2):

$$\text{(i)} \qquad \frac{(a_{11}-a_{22})^2}{4} + a_{12}a_{21} \ge 0 \tag{24}$$

$$\text{(ii)} \qquad \max\left(\frac{a_{11}+a_{22}}{2},\ a_{11}+a_{22}-a_{11}a_{22}+a_{12}a_{21}\right) > 1 \tag{25}$$

$$\text{(iii)} \qquad \frac{a_{11}+a_{22}}{2} > 0 \tag{26}$$

$$\text{(iv)} \qquad a_{21} > 0. \tag{27}$$

For calculation of stability indices we introduce the sets $U_R = \{(\zeta,\eta) \mid \max(\zeta,\eta) < R\}$ and

$$U_R(\alpha_1,\beta_1,q_1;\alpha_2,\beta_2,q_2) = \\ \{(\zeta,\eta) \in U_R\ :\ (\alpha_1+q_1)\zeta + (\beta_1+q_1)\eta < 0,\ (\alpha_2+q_2)\zeta + (\beta_2+q_2)\eta < 0\},$$

where $R < 0$. If all entries of the matrices M_j, $1 \le j \le m$, are non-negative, the stability indices of the related map can be calculated using the following Theorem which is stated and proved as Theorem B.1.

Theorem 4.2 *Let g be a map related to simple heteroclinic cycle of Types B or C and M_j, $1 \le j \le m$, its transition matrices. Suppose that for all j, $1 \le j \le m$, all entries of the matrices are non-negative. Then:*

(a) If the transition matrix $M = M_m \cdots M_1$ satisfies condition (ii) (see (25)), then $\sigma_{j,+} = \infty$ and $\sigma_{j,-} = 0$ for all j and moreover the cycle is asymptotically stable.

(b) Otherwise, $\sigma_{j,+} = 0$ and $\sigma_{j,-} = \infty$ for all j and the cycle is not an attractor.

For calculation of stability indices of matrices with negative entries we introduce the following functions

$$f^+(\alpha,\beta)=\begin{cases}\infty, & \alpha\ge 0,\ \beta\ge 0,\\ 0, & \alpha\le 0,\ \beta\le 0,\\ -\dfrac{\beta}{\alpha}-1, & \alpha<0,\ \beta>0,\ \dfrac{\beta}{\alpha}<-1,\\ 0, & \alpha<0,\ \beta>0,\ \dfrac{\beta}{\alpha}>-1,\\ -\dfrac{\alpha}{\beta}-1, & \alpha>0,\ \beta<0,\ \dfrac{\alpha}{\beta}<-1,\\ 0, & \alpha>0,\ \beta<0,\ \dfrac{\alpha}{\beta}>-1\end{cases},$$

$f^-(\alpha,\beta)=f^+(-\alpha,-\beta)$ and

$$f^{\text{index}}(\alpha,\beta)=f^+(\alpha,\beta)-f^-(\alpha,\beta)$$

and prove in the following theorem that the set $\mathcal{B}_\delta(g)$ in $\hat{H}_1^{(in)}$ (and similarly for $\hat{H}_j^{(in)}$ with $j>1$) in the (ζ,η) coordinates is $U_R(\alpha_1,\beta_1,0;\alpha_2,\beta_2,0)$. We denote the latter set in original coordinates (w,z) as

$$\tilde{U}_R(\alpha_1,\beta_1,0;\alpha_2,\beta_2,0)$$

and then by Definition 5

$$\begin{aligned}\sigma &= \lim_{R\to-\infty}\frac{\ln(\ell(\tilde{U}_R\setminus\tilde{U}_R(\alpha_1,\beta_1,0;\alpha_2,\beta_2,0)))-\ln(\ell(\tilde{U}_R(\alpha_1,\beta_1,0;\alpha_2,\beta_2,0)))}{\ln(\ell(\tilde{U}_R))}\\ &= \min(f^{\text{index}}(\alpha_1,\beta_1),f^{\text{index}}(\alpha_2,\beta_2)).\end{aligned}$$

The Theorem below is stated and proved as Theorem B.2.

Theorem 4.3 *Let X be a simple heteroclinic cycle of Type B or C and M_j, $1\le j\le m$ the associated transition matrices. We denote by $j=j_1,\dots j_L$ the indices, for which some of the entries of M_j are negative; they are all non-negative for all remaining j.*

(a) If for at least one of $j=j_l+1$ the matrix $M^{(j)}$ does not satisfy conditions (i)-(iv) of Lemma B.2, then the cycle is repelling and $\sigma_j=-\infty$ for all j.

(b) If the matrices $M^{(j)}$ satisfy conditions (i)-(iv) of Lemma B.2 for all $j=j_l+1$, then there exist numbers $(\alpha_1^j,\beta_1^j,\alpha_2^j,\beta_2^j)$, $1\le j\le m$, such that

(i) $U_0(\alpha_1^j,\beta_1^j,0;\alpha_2^j,\beta_2^j,0)\ne\emptyset$, $1\le j\le m$.

(ii) For any $S<0$ and $q>0$ there exists $R<0$ such that

$$M^{(l,j)}(M^{(j)})^k(U_R(\alpha_1^j,\beta_1^j,-q;\alpha_2^j,\beta_2^j,-q))\subset U_S \text{ for all } l,\quad 1\le l\le m,\ k\ge 0,$$

(iii)

$$\lim_{k\to\infty}(M^{(l,j)})(M^{(j)})^k(\zeta,\eta)^t=(-\infty,-\infty) \text{ for all } (\zeta,\eta)\in U_0(\alpha_1^j,\beta_1^j,0;\alpha_2^j,\beta_2^j,0).$$

(iv)

$$U_0(\alpha_1^j,\beta_1^j,0;\alpha_2^j,\beta_2^j,0)=U^{-\infty}(M^{(j)})\cap\bigcap_{1\le l\le L}(M^{(j_l,j)})^{-1}U_0\cap\bigcap_{1\le l\le L}(M^{(j_l+m,j)})^{-1}U_0.$$

(v) If $\lambda_2\ge 0$ then

$$U_0(\alpha_1^j,\beta_1^j,0;\alpha_2^j,\beta_2^j,0)=U^{-\infty}(M^{(j)})\cap\bigcap_{1\le l\le L}(M^{(j_l,j)})^{-1}U_0.$$

and the cycle is a Milnor attractor.

Note that

$$(M^{(j,s)})^{-1}U_0\cap U_0=U_0(\alpha_{11}^{(j,s)},\alpha_{12}^{(j,s)},0;\alpha_{21}^{(j,s)},\alpha_{22}^{(j,s)},0),$$

where $\alpha^{(j,s)}$ are entries of the matrix $M^{(j,s)}$.

Theorems 4.1-4.3 imply the following Corollary:

Corollary 4.1 *For simple heteroclinic cycles in $\mathbb{R}^4$, $\sigma_j=-\infty$ for some j if and only if $\sigma_j=-\infty$ for all j.*

4.2.1 Calculation of stability indices for Type B cycles

Types B_1^+ and B_1^- The cycles of Types B_1^+ and B_1^- have transition matrices

$$M=\begin{pmatrix} a & 0\\ b & 1\end{pmatrix}.$$

Corollary B.1 implies that if $a<1$ or $b<0$, then the cycles are not attractors and the stability index is $-\infty$, otherwise they are attracting and the stability index is ∞.

Types B_2^+ For cycles of Type B_2^+ the product of transition matrices is

$$M_1M_2=\begin{pmatrix} a_1a_2 & 0\\ b_1a_2+b_2 & 1\end{pmatrix}$$

with eigenvalues a_1a_2 and 1, and the associated eigenvectors $(a_1a_2-1,b_1a_2+b_2)$ and $(0,1)$, respectively (for M_2M_1, simply swap the indices 1 and 2 in the expressions to obtain the corresponding eigenvectors).

Theorems 4.2 and 4.3 imply to obtain the following classification:

- If $b_1 < 0$ and $b_2 < 0$, then the cycle is not an attractor and all stability indices are $-\infty$.
- Suppose $b_1 > 0$ and $b_2 > 0$.
  - If $a_1a_2 < 1$, then the cycle is not an attractor and the stability indices are $-\infty$.
  - If $a_1a_2 > 1$, then the cycle is locally attracting and the stability indices are ∞.
- Suppose $b_1 < 0$ and $b_2 > 0$.
  - If $a_1a_2 < 1$ or $b_1a_2 + b_2 < 0$, then the cycle is not an attractor and the stability indices are $-\infty$.
  - If $a_1a_2 > 1$ and $b_1a_2 + b_2 > 0$, then the stability indices are $\sigma_1 = f^{\text{index}}(b_1, 1)$ and $\sigma_2 = \infty$.

Type B_3^- For cycles of Type B_3^- the product of transition matrices is

$$M_3M_2M_1 = \begin{pmatrix} a_1a_2a_3 & 0 \\ b_3a_1a_2 + b_2a_1 + b_1 & 1 \end{pmatrix}.$$

Its eigenvalues are $a_1a_2a_3$ and 1 with associated eigenvectors

$$(a_1a_2a_3 - 1, b_3a_1a_2 + b_2a_1 + b_1), \quad (0, 1).$$

(For $M_2M_1M_3$ and $M_1M_3M_2$, the quantities are obtained by cyclic permutation of the indices.) Theorems 4.2 and 4.3 imply obtain the following classification:

- If $b_1 < 0$, $b_2 < 0$ and $b_3 < 0$, then the cycle is not an attractor and the stability indices are all $-\infty$.
- Suppose $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$.
  - If $a_1a_2a_3 < 1$, then the cycle is not an attractor and the stability indices are $-\infty$.
  - If $a_1a_2a_3 > 1$, then the cycle is locally attracting and the stability indices are ∞.
- Suppose $b_1 < 0$, $b_2 > 0$ and $b_3 > 0$.
  - If $a_1a_2a_3 < 1$ or $b_1a_2a_3 + b_3a_2 + b_2 < 0$, then the cycle not an attractor and the stability indices are $-\infty$.
  - If $a_1a_2a_3 > 1$ and $b_1a_2a_3 + b_3a_2 + b_2 > 0$, then the stability indices are $\sigma_1 = f^{\text{index}}(b_1, 1)$, $\sigma_2 = \infty$ and $\sigma_3 = f^{\text{index}}(b_3 + b_1a_3, 1)$.
- Suppose $b_1 < 0$, $b_2 < 0$ and $b_3 > 0$.
  - If $a_1a_2a_3 < 1$ or $b_2a_1a_3 + b_1a_3 + b_3 < 0$ or $b_1a_2a_3 + a_2b_3 + b_2 < 0$, then the cycle is not an attractor and the stability indices are $-\infty$.
  - If $a_1a_2a_3 > 1$, $b_2a_1a_3 + b_1a_3 + b_3 > 0$ and $b_1a_2a_3 + a_2b_3 + b_2 > 0$, then the stability indices are $\sigma_1 = \min(f^{\text{index}}(b_1, 1), f^{\text{index}}(b_1 + b_2a_1, 1))$, $\sigma_2 = f^{\text{index}}(b_2, 1)$ and $\sigma_3 = f^{\text{index}}(b_3 + b_1a_3, 1)$.

4.2.2 Calculation of stability indices for Type C cycles

Type C_1^- Cycles of Type C_1^- have a transition matrix of the form

$$M = \begin{pmatrix} b & 1 \\ a & 0 \end{pmatrix}.$$

Corollary B.1 implies that the cycle is attracting and the stability index is ∞ whenever $b \geq 0$ and $a + b > 1$; otherwise it is not an attractor and the stability index is $-\infty$.

Type C_2^- The product of transition matrices for cycles of Type C_2^- is

$$M_1 M_2 = \begin{pmatrix} b_1 b_2 + a_2 & b_1 \\ a_1 b_2 & a_1 \end{pmatrix}.$$

Denote by λ_1 and λ_2 eigenvalues of the matrix $M_1 M_2$ (which will be the same as those for $M_2 M_1$).

Theorems 4.2 and 4.3 imply the following:

- If $b_1 < 0$ and $b_2 < 0$, then the cycle is not an attractor and the stability indices are $-\infty$.
- Suppose that $b_1 > 0$ and $b_2 > 0$.
  - If
  $$\max(b_1 b_2 + a_2 + a_1, 2(b_1 b_2 + a_2 + a_1 - a_1 a_2)) < 2$$
  then the cycle is not an attractor and the stability indices are $-\infty$.
  - Otherwise the cycle is locally attracting and the stability indices are ∞.
- Suppose $b_1 < 0$ and $b_2 > 0$.
  - If
  $$(b_1 b_2 + a_2 - a_1)^2 - 4 a_1 b_1 b_2 < 0,$$
  or
  $$\max(b_1 b_2 + a_2 + a_1, 2(b_1 b_2 + a_2 + a_1 - a_1 a_2)) < 2,$$
  or
  $$b_1 b_2 - a_1 + a_2 < 0,$$
  then the cycle is not an attractor and the stability indices are $-\infty$.
  - If none of the listed conditions are satisfied, the stability indices are
  $$\sigma_1 = f^{\text{index}}\left(\frac{b_1 b_2 + a_1 - \lambda_2}{b_2}, 1\right) \text{ and } \sigma_2 = f^{\text{index}}\left(\frac{\lambda_2 - b_1 b_2 - a_2}{b_1}, -1\right). \tag{28}$$

Type C_4^- The transition matrix for cycles of Type C_4^- is

$$M \equiv M^{4,1} \equiv M_4 M_3 M_2 M_1 =$$
$$\begin{pmatrix} (b_1b_2+a_1)(b_3b_4+a_3)+b_1a_2b_4 & (b_3b_4+a_3)b_2+b_4a_2 \\ a_4b_3(b_1b_2+a_1)+a_2a_4b_1 & a_4b_2b_3+a_2a_4 \end{pmatrix}$$

Denote by λ_1 and λ_2 eigenvalues and by $\mathbf{v}_1^{4,1}$ and $\mathbf{v}_2^{4,1}$ the associated eigenvectors of the matrix. The trace and determinant are:

$$\mathrm{tr}(M) = b_1b_2b_3b_4 + b_1b_2a_3 + b_3b_4a_1 + b_1b_4a_2 + a_4b_2b_3 + a_1a_3 + a_2a_4, \ \det(M) = a_1a_2a_3a_4.$$

Theorems 4.2 and 4.3 imply the following:

- If $b_j < 0$ for all j, then the cycle is not an attractor and the stability indices are $-\infty$.
- Suppose that $b_j > 0$ for all j.
  - If
  $$\max(\mathrm{tr}M, 2\,\mathrm{tr}M - 2\det M) < 2, \tag{29}$$
  then the cycle is not an attractor and the stability indices are $-\infty$.
  - Otherwise the cycle is locally attracting and the stability indices are ∞.
- Suppose that $b_1 < 0$ and $b_j > 0$ for $2 \le j \le 4$.
  - If (29) holds, or
  $$(\mathrm{tr}M)^2 - 4\det M < 0 \tag{30}$$
  or
  $$v_{11}^{1,2} v_{12}^{1,2} < 0 \tag{31}$$
  then the cycle is not an attractor and the stability indices are $-\infty$.
  - If none of the listed conditions are satisfied, the stability indices are
  $$\sigma_j = f^{\mathrm{index}}(v_{22}^{j+3,j}/h^{j+3,j}, -v_{21}^{j+3,j}/h^{j+3,j}) \ \text{ where}$$
  $$h^{j+3,j} = v_{11}^{j+3,j}v_{22}^{j+3,j} - v_{12}^{j+3,j}v_{21}^{j+3,j} \ \text{ and } \ v_{11}^{j+3,j} > 0, \ v_{12}^{j+3,j} > 0.$$
  ($v_{11}^{j+3,j} > 0$ and $v_{12}^{j+3,j} > 0$ can be assumed, because (31) implies that $v_{11}^{j+3,j}v_{12}^{j+3,j} > 0$ for all j).
- Suppose that $b_1 < 0$, $b_2 < 0$, $b_3 > 0$ and $b_4 > 0$.
  - If (29), or (30), or (31), or
  $$v_{11}^{2,3}v_{12}^{2,3} < 0, \tag{32}$$
  holds then the cycle is not an attractor and the stability indices are $-\infty$.

– If none of the listed conditions are satisfied then the stability indices are

$$\sigma_1 = \min(f^{\rm index}(v_{22}^{4,1}/h^{4,1}, -v_{21}^{4,1}/h^{4,1}), f^{\rm index}(b_1b_2+a_1, b_2), f^{\rm index}(b_1, 1)),$$

$$\sigma_2 = \min(f^{\rm index}(v_{22}^{1,2}/h^{1,2}, -v_{21}^{1,2}/h^{1,2}), f^{\rm index}(b_2, 1)),$$

$$\sigma_3 = \min(f^{\rm index}(v_{22}^{2,3}/h^{2,3}, -v_{21}^{2,3}/h^{2,3}), f^{\rm index}(b_1b_3b_4+b_1a_3+b_3a_4, b_1b_4+a_4)),$$

$$\sigma_4 = \min(f^{\rm index}(v_{22}^{3,4}/h^{3,4}, -v_{21}^{3,4}/h^{3,4}),$$
$$f^{\rm index}(b_1b_4+a_4, b_1), f^{\rm index}(b_1b_2b_4+b_2a_4+a_1b_4, b_1b_2+a_1)).$$

- Suppose that $b_1 < 0$, $b_2 > 0$, $b_3 < 0$ and $b_4 > 0$.

  – If (29), or (30), or (31), or

  $$v_{11}^{3,4}v_{12}^{3,4} < 0, \tag{33}$$

  holds then the cycle is not an attractor and the stability indices are $-\infty$.

  – If none of the listed conditions are satisfied then the stability indices are

  $$\sigma_j = f^{\rm index}(v_{22}^{j+3,j}/h^{j+3,j}, -v_{21}^{j+3,j}/h^{j+3,j}).$$

- Suppose that $b_1 < 0$, $b_2 < 0$, $b_3 < 0$ and $b_4 > 0$.

  – If at least one of (29)-(33) is satisfied, then the cycle is not an attractor and the stability indices are $-\infty$.

  – If none of the listed conditions are satisfied, the stability indices are

$$\sigma_1 = \min(f^{\rm index}(v_{22}^{4,1}/h^{4,1}, -v_{21}^{4,1}/h^{4,1}), f^{\rm index}(b_1, 1), f^{\rm index}(b_1b_2+a_1, b_2),$$
$$f^{\rm index}(b_1b_2b_3+a_1b_3+b_1a_2, b_2b_3+a_2)),$$

$$\sigma_2 = \min(f^{\rm index}(v_{22}^{1,2}/h^{1,2}, -v_{21}^{1,2}/h^{1,2}), f^{\rm index}(b_2b_3+a_2, b_3), f^{\rm index}(b_2, 1)),$$

$$\sigma_3 = \min(f^{\rm index}(v_{22}^{2,3}/h^{2,3}, -v_{21}^{2,3}/h^{2,3}), f^{\rm index}(b_1b_3b_4+b_1a_3+b_3a_4, b_1b_4+a_4), f^{\rm index}(b_3, 1)),$$

$$\sigma_4 = \min(f^{\rm index}(v_{22}^{3,4}/h^{3,4}, -v_{21}^{3,4}/h^{3,4}), f^{\rm index}(b_1b_4+a_4, b_1), f^{\rm index}(b_1b_4a_2+a_2a_4, b_1a_2)).$$

4.3 Comparison with earlier results

Asymptotic stability of heteroclinic cycles has been previously examined in a number of papers. Type A heteroclinic cycles were considered by Krupa and Melbourne [16, 17]. In the first paper [16], cycles with negative transverse eigenvalues were investigated and the condition $\rho > 1$ was found to be necessary and sufficient for asymptotic stability of cycles. In the second paper [17] it was shown that cycles with some positive transverse eigenvalues are essentially asymptotically stable, if and only if $\rho > 1$ and the condition $t_j > -1$ is satisfied for all j. This result is a special cases of our Theorem 4.1. The stability of Types B and C cycles with positive transverse eigenvalues was studied in [18]. The conditions for asymptotic stability presented in this paper are equivalent to ours as given in subsections 4.2.1 and 4.2.2 for the special cases $b_j > 0$.

A cycle of Type C_2^- with one positive and one negative transverse eigenvalues was considered in [21]. Conditions (12)-(14) in [21], under which the set of points satisfying $\lim_{k\to\infty} g^k(w,z) = (0,0)$ is non-empty, are equivalent to our conditions that $\sigma_1 > -\infty$ for $b_1 < 0$ and $b_2 > 0$. Existence of the set $U_0(\alpha_1^j, \beta_1^j, 0; \alpha_2^j, \beta_2^j, 0)$ was also noted in [21]. It was termed $\hat{\Sigma}_{\beta_+}$ and was defined by the condition $z > w^{\beta_+}$, which we express as $\beta_+\zeta - \eta < 0$. This inequality implies that the stability index σ_1 is equal to $f^{\mathrm{index}}(\beta_+, -1)$. Substituting in it β_+ defined in the beginning of [21, Theorem 3.2], we obtain the value σ_2 given in (28) (subject to the appropriate change of indices of t_j and c_j).

5 Discussion

Although it is natural to investigate cusp-like basins of attraction for heteroclinic cycles, to our knowledge this is the first paper to identify the algebraic order of the cusp as being an invariant of the dynamics — we use the stability index $\sigma(x)$ to characterize the local geometry of basins of attraction near invariant sets X in general, and heteroclinic cycles in particular. This quantity might be especially useful in describing the local structure of a range of invariant sets, for example, for riddled and intermingled basins of attraction [1, 6].

In the latter part of the paper we have calculated how the stability indices depend on the cycle structure and eigenvalues for simple (robust heteroclinic) cycles in $\mathbb{R}^4$. Clearly, transition matrices can be used to study the stability of simple cycles in higher-dimensional systems or for more complex cycles; however, we expect such a classification to be so complex that the results can hardly be enlightening — and so we have not attempted this.

Our approach should give some insight into the structure of heteroclinic networks [3, 8] and heteroclinic switching [5, 10, 12, 13, 14]. For some cycles (of Type A where at least one difference between transverse and contracting eigenvalues is negative, or of Type B where at most one transverse eigenvalue is positive) we find one of the stability indices to be ∞. Consequently, if a heteroclinic network involves such a cycle, no switching is possible for perturbations on that particular connecting trajectory (almost all trajectories that are near a connection where $\sigma = \infty$ will stay near the cycle for all $t > 0$). Some of the results in this paper may be extended to general robust heteroclinic cycles; it seems plausible that Corollary 4.1 holds for simple heteroclinic cycles in $\mathbb{R}^n$ where all connections are one-dimensional manifolds, because for such cycles the Poincaré map along the cycle becomes linear after the same change of coordinates. It should also be possible to extend to cases of compact but not finite symmetry, though again this is likely to be quite involved owing to the complexity of heteroclinic cycles between relative equilibria.

Finally, we emphasise that there is no *a priori* reason why the limits in the definition of the stability index exist. We give below an example where the stability index can be shown not to converge. This is probably not a generic situation though; the example below is highly degenerate, and the generic conditions for the cycles in $\mathbb{R}^4$, as detailed in the previous section, all result in computable stability indices.

An example where σ_+ and σ_- do not exist. We consider a (non-invertible) map $M : [0, \infty) \to [0, \infty)$ with $X = \{0\}$ and consider its basin of attraction $N = \mathcal{B}(X)$. Define a sequence $\epsilon_k = \exp(-2^k)$ and

$$M(y) = \begin{cases} (\epsilon_{2k-1}(y - \epsilon_{2k+2}) - \epsilon_{2k}(y - \epsilon_{2k+1}))/(\epsilon_{2k+1} - \epsilon_{2k+2}), & \text{if } \epsilon_{2k+2} < |y| \le \epsilon_{2k+1} \\ 0, & \text{if } \epsilon_{2k+1} < |y| \le \epsilon_{2k} \\ 0, & \text{if } y = 0 \end{cases}.$$

Then

$$\Sigma_{\epsilon_{2k}} > \frac{\epsilon_{2k} - \epsilon_{2k+1}}{\epsilon_{2k}} = 1 - \epsilon_{2k},$$

whereas

$$\Sigma_{\epsilon_{2k+1}} < \frac{\epsilon_{2k+2}}{\epsilon_{2k+1}} = \epsilon_{2k+1}.$$

Hence, $\ln(\Sigma_{\epsilon_{2k}})/\ln(\epsilon_{2k}) < \ln(1-\epsilon_{2k})/\ln(\epsilon_{2k})$ and $\ln(\Sigma_{\epsilon_{2k+1}})/\ln(\epsilon_{2k+1}) > 1$, and therefore the limit defining $\sigma_-(0)$ does not exist; it can similarly be shown that $\sigma_+(0)$ does not exist. This example can clearly be extended to a continuous map M. Although it is not easy to see how to extend to a map that is differentiable at $y = 0$ with the same properties, it may well be possible to produce a smooth example in dimension two or more.

Other global measures of stability The stability index $\sigma(x)$ describes the local geometry of the basin of attraction of an invariant set X. One can define global and local *stability numbers* of a flow invariant set X as follows:

$$n_{\mathrm{glob}}(X) = \lim_{\epsilon \to 0} \frac{\ell(B_\epsilon(X) \cap \mathcal{B}(X))}{\ell(B_\epsilon(X))},$$

$$n_{\mathrm{loc}}(X) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\ell(B_\epsilon(X) \cap \mathcal{B}_\delta(X))}{\ell(B_\epsilon(X))}$$

where $\mathcal{B}_\delta$ is defined as in (8). Clearly, from the definition one can verify that $0 \le n_{\mathrm{glob}}(X) \le 1$, $0 \le n_{\mathrm{loc}}(X) \le 1$, and $n_{\mathrm{glob}}(X) = 1$ if and only if X is p.a.s. for a local basin of attraction. Note however that stability number is not an invariant of the dynamics; we believe that only the classification into whether $n(X) = 0$, $0 < n(X) < 1$ or $n(X) = 1$ and scaling properties will be invariant under smooth conjugation. For a heteroclinic cycle comprised of one-dimensional connections, the stability number can be related to its stability indices by the following:

$$n_{\mathrm{glob}}(X) = \frac{\sum_{\sigma_j > 0} \ell^1(s_j)}{\sum \ell^1(s_j)}.$$

where $\ell^1(s_j)$ is the length of the connection s_j.

Acknowledgements

Part of the research of OP was carried out during visit to the University of Exeter during January to April 2008. We are grateful to the Royal Society for the support of the visit. OP was also financed by the grants ANR-07-BLAN-0235 OTARIE from Agence Nationale de la Recherche, France, and 07-01-92217-CNRSL_a from the Russian foundation for basic research.

References

[1] J. Alexander, J.A. Yorke, Z. You, and I. Kan. Riddled Basins. *Intl. J. Bif. Chaos*, **2**, 795 (1992).

[2] P. Ashwin and P. Chossat. Attractors for robust heteroclinic cycles with continua of connections. *J. Nonlin. Sci.*, **8**, 103–129 (1998).

[3] P. Ashwin and M. Field. Heteroclinic networks in coupled cell systems. *Arch. Rational Mech. Anal.*, **148** 107–143 (1999).

[4] P. Ashwin and J. Montaldi. Group theoretic conditions for existence of robust relative homoclinic trajectories. *Math. Proc. Camb. Phil. Soc.* **133**, 125–141 (2002).

[5] P. Ashwin, G. Orosz, J. Wordsworth and S. Townley. Reliable switching between cluster states for globally coupled phase oscillators. *SIAM J. Appl. Dynamical Systems* **6**, (2007), 728–758.

[6] P. Ashwin and J.R. Terry. On riddling and weak attractors. *Physica D* **142**, 87–100 (2000).

[7] P. Ashwin and M. Timme. Unstable attractors: existence and robustness in networks of oscillators with delayed pulse coupling. *Nonlinearity* **18**, 2035–2060 (2005).

[8] P. Ashwin and O.M. Podvigina. Noise-induced switching near a depth two heteroclinic network arising in Boussinesq convection. *Chaos* **20**, 023133 (2010).

[9] W. Brannath. Heteroclinic networks on the tetrahedron. *Nonlinearity* **7**, 1367–1384 (1994).

[10] R. Driesse and A.-J. Homburg. Essentially asymptotically stable homoclinic networks. *Dynamical Systems* **24**, 459 – 471 (2009).

[11] K.J. Falconer. *Fractal geometry.* John Wiley, Chichester (1990).

[12] A.J. Homburg and J. Knobloch. Switching homoclinic networks. *Dynamical Systems* **25**, 361–368 (2010).

[13] V. Kirk and M. Silber. A competition between heteroclinic cycles. *Nonlinearity* **7**, 1605–1621 (1994).

[14] V. Kirk, E. Lane, A.M. Rucklidge and M. Silber. A mechanism for switching near a heteroclinic network *Dynamical Systems* **25**, 323?-349 (2010).

[15] M. Krupa. Robust heteroclinic cycles. *Journal of Nonlinear Science*, **7**, 129–176 (1997).

[16] M. Krupa and I. Melbourne. Asymptotic stability of heteroclinic cycles in systems with symmetry. *Ergodic Theory Dyn. Syst.*, **15**, 121–148 (1995).

[17] M. Krupa and I. Melbourne. Nonasymptotically stable attractors in $\mathbf{O}(2)$ mode interaction. *Normal Forms and Homoclinic Chaos* (W.F. Langford and W. Nagata, eds.) Fields Institute Communications **4**, Amer. Math. Soc., 1995, 219–232.

[18] M. Krupa and I. Melbourne. Asymptotic stability of heteroclinic cycles in systems with symmetry, II. *Proc. Roy. Soc. Edinburgh* **134A**, 1177–1197 (2004).

[19] I. Melbourne. An example of a non-asymptotically stable attractor. *Nonlinearity* **4**, 835–844 (1991).

[20] J. Milnor. On the concept of attractor. *Commun. Math. Phys.* **99**, 177–95 (1985), *and* On the concept of attractor: correction and remarks *Commun. Math. Phys.* **102**, 517–19 (1985).

[21] C.M. Postlethwaite. A new mechanism for stability loss from a heteroclinic cycle. *Dynamical Systems* **25**, 305–22 (2010).

[22] N. Sottocornola. Robust homoclinic cycles in $\mathbb{R}^4$. *Nonlinearity* **16**, 1–24 (2003).

[23] T. Ura. On the flow outside a closed invariant set; stability, relative stability and saddle sets. *Contributions to Differential Equations* **3**, 249–294 (1964).

A Type A cycles

In this Appendix we present a proof of the main theorem for calculation of stability indices for Type A cycles. Consider the map $g = g_m \circ \ldots \circ g_1 : \mathbb{R}^2 \to \mathbb{R}^2$, where[5]

$$g_j(w,z) \equiv (g_j^w(w,z), g_j^z(w,z)) = (A_j w^{a_{1j}}|z|^{a_{2j}} + B_j|w|^{b_{1j}}z^{b_{2j}}, C_j w^{a_{1j}}|z|^{a_{2j}} + D_j|w|^{b_{1j}}z^{b_{2j}}). \tag{34}$$

For generic cycles of Type A, $A_jB_jC_jD_j \neq 0$ and $A_jD_j \neq B_jC_j$ holds true for all j. In the first two lemmas in this appendix we assume that $a_{1j} + a_{2j} < b_{1j} + b_{2j}$. No generality

[5]By virtue of (13), $a_{2j} = 0$ and $b_{2j} = 1$. However, the first two lemmas in this Appendix are proved for arbitrary a_{ij} and b_{ij}.

is lost, because the expression (18) for the map $g_j(w,z)$ is invariant under the transformation $(A_j, B_j, C_j, D_j; a_{1j}, a_{2j}, b_{1j}, b_{2j}; w, z) \to (B_j, A_j, D_j, C_j; b_{2j}, b_{1j}, a_{2j}, a_{1j}; z, w)$. Recall that $g_{l,k} = g_l \circ \ldots \circ g_1 \circ g^k$.

Let us define

$$\begin{array}{rcl} Q(r,\epsilon) & = & \{(w,z) : |(w,z)| < \epsilon,\ |z| > |w|^{1+r},\ |w| > |z|^{1+r}\}, \\ Q_z(f_1(z), f_2(z), \epsilon) & = & \{(w,z) : |(w,z)| < \epsilon,\ f_1(z) < w < f_2(z)\}, \\ Q_w(f_1(w), f_2(w), \epsilon) & = & \{(w,z) : |(w,z)| < \epsilon,\ f_1(w) < z < f_2(w)\}, \end{array}$$

where we assume $f_1(z) = \mathrm{O}(z^{1+r})$, $f_1(w) = \mathrm{O}(w^{1+r})$, $f_1(z) - f_2(z) = \mathrm{O}(z^{1+r_1})$, $f_1(w) - f_2(w) = \mathrm{O}(w^{1+r_1})$ for some $r > 0$ and $r_1 \geq r$.

We start the study of stability by proving a few lemmas about properties of the maps g_j.

Lemma A.1 *Suppose $a_{1j} + a_{2j} > 0$. For any $q > 0$ and $r > 0$ satisfying*

$$(1-r)(b_{1j} - a_{1j}) + b_{2j} - a_{2j} > 0,\ b_{1j} - a_{1j} + (1-r)(b_{2j} - a_{2j}) > 0,\ a_{1j} + a_{2j} - q > 0$$

and

$$r < \min\left(\left|\frac{q}{2a_{1j}}\right|, \left|\frac{q}{2a_{2j}}\right|\right) \tag{35}$$

there exists an $\epsilon_j > 0$, such that

$$g_j(Q(r,\epsilon_j)) \subset Q(r, \epsilon_j^{a_{1j}+a_{2j}-q}), \tag{36}$$

$$|g_j(w,z)| < |(w,z)|^{a_{1j}+a_{2j}-q} \text{ for any } (w,z) \in Q(r,\epsilon_j) \tag{37}$$

and

$$|g_j(w,z)| > |(w,z)|^{a_{1j}+a_{2j}+q} \text{ for any } (w,z) \in Q(r,\epsilon_j). \tag{38}$$

Proof: At least one of the coefficients, a_{1j} or a_{2j}, is positive. Assume $a_{1j} > 0$. The inequality $a_{1j} + a_{2j} < b_{1j} - b_{2j}$ implies $b_{1j} - a_{1j} + b_{2j} - a_{2j} > 0$ and hence at least one of two differences, $b_{1j} - a_{1j}$ or $b_{2j} - a_{2j}$, is positive. For the sake of definiteness we assume without loss of generality that $b_{1j} - a_{1j} > 0$. Set $\epsilon_j < 1$ satisfying the following inequalities:

$$\epsilon_j^{(1-r)(b_{1j}-a_{1j})+b_{2j}-a_{2j}} < \min\left(\left|\frac{A_j}{2B_j}\right|, \left|\frac{C_j}{2D_j}\right|\right) \tag{39}$$

$$\epsilon_j^{q/2} |(3A_j/2, 3C_j/2)| < 1, \tag{40}$$

$$\epsilon_j^{r(a_{1j}+a_{2j}-q)} < \left|\frac{A_j}{3C_j}\right| \quad \text{and} \tag{41}$$

$$\epsilon_j^{q/2} K < |(A_j/2, C_j/2)|, \tag{42}$$

where[6] $K = 2^{a+b+q+1}$.

[6]Here, and below, we use the norm $|(w,z)| = (w^2 + z^2)^{1/2}$. If a different norm is employed, the proofs remain similar but some constants will be modified.

Assume that

$$(w,z) \in Q(r,\epsilon_j) \tag{43}$$

and re-write (18) as

$$g_j(w,z) = w^{a_{1j}}|z|^{a_{2j}}(A_j + B_j w^{b_{1j}-a_{1j}} z^{b_{2j}-a_{2j}}, C_j + D_j w^{b_{1j}-a_{1j}} z^{b_{2j}-a_{2j}}).$$

Due to (39),

$$|w^{b_{1j}-a_{1j}} z^{b_{2j}-a_{2j}}| < |z|^{(b_{1j}-a_{1j})/(1+r)+b_{2j}-a_{2j}} < \epsilon_j^{(1-r)(b_{1j}-a_{1j})+b_{2j}-a_{2j}} < \min\left(\left|\frac{A_j}{2B_j}\right|, \left|\frac{C_j}{2D_j}\right|\right). \tag{44}$$

Therefore, due to (35), (40) and (43)

$$\begin{aligned} |g_j(w,z)| &< |w^{a_{1j}} z^{a_{2j}}(3A_j/2, 3C_j/2)| < |z|^{a_{1j}/(1+r)+a_{2j}}|(3A_j/2, 3C_j/2)| \\ &< |(w,z)|^{a_{1j}(1-r)+a_{2j}-q/2}\epsilon_j^{q/2}|(3A_j/2, 3C_j/2)| < |(w,z)|^{a_{1j}+a_{2j}-q}. \end{aligned} \tag{45}$$

Thus, (37) is proved.

The inequalities (44), (41) and (45) imply that for any $q > 0$ and $r > 0$ the ϵ_j can be chosen so that

$$\left|\frac{g_j^w(w,z)}{g_j^z(w,z)}\right| > \left|\frac{A_j}{3C_j}\right| > \epsilon_j^{r(a_{1j}+a_{2j}-q)} > |g_j^z(w,z)|^r.$$

Hence

$$|g_j^w(w,z)| > |g_j^z(w,z)|^{1+r},$$

and similarly

$$|g_j^z(w,z)| > |g_j^w(w,z)|^{1+r}.$$

Therefore (36) holds because of (45).

Combining (44), (43), (35) and (42) we obtain

$$\begin{aligned} |g_j(w,z)| &> |w^{a_{1j}} z^{a_{2j}}(A_j/2, C_j/2)| > |z|^{a_{1j}(1+r)+a_{2j}}|(A_j/2, C_j/2)| \\ &> |z|^{a_{1j}(1+r)+a_{2j}+q/2}\epsilon_j^{-q/2}|(A_j/2, C_j/2)| > K|z|^{a_{1j}+a_{2j}+q}. \end{aligned} \tag{46}$$

Assume that $a_{2j} < 0$. Estimates (44), (43), (35) and (42) imply that

$$\begin{aligned} |g_j(w,z)| &> |w^{a_{1j}} z^{a_{2j}}(A_j/2, C_j/2)| > |w|^{a_{1j}+a_{2j}/(1+r)}|(A_j/2, C_j/2)| \\ > |w|^{a_{1j}+a_{2j}(1-r)}|(A_j/2, C_j/2)| &> |w|^{a_{1j}+a_{2j}(1-r)+q/2}\epsilon_j^{-q/2}|(A_j/2, C_j/2)| \\ &> K|w|^{a_{1j}+a_{2j}+q}. \end{aligned} \tag{47}$$

If $a_{2j} > 0$ then

$$|g_j(w,z)| > |w|^{a_{1j}+a_{2j}(1+r)}|(A_j/2, C_j/2)| > K|w|^{a_{1j}+a_{2j}+q}.$$

Thus,

$$|g_j(w,z)| > \frac{1}{2}K(|z|^{a_{1j}+a_{2j}+q} + |w|^{a_{1j}+a_{2j}+q}) > |(w,z)|^{a_{1j}+a_{2j}+q}.$$

QED

Corollary A.1 *Suppose that the conditions of Lemma A.1 are satisfied for $r > 0$, $q > 0$ and for all $1 \leq j \leq m$*

(a) If $\prod_{1\leq j\leq m}(a_{1j} + a_{2j} - q) > 1$ then for any $\delta > 0$ there exists an $\epsilon > 0$ such that $|g_{l,k}(w, z)| < \delta$ for all $(w, z) \in Q(r, \epsilon)$ and for any $1 \leq l \leq m$ and $k \geq 0$.

(b) Denote $\delta_0 = \min_{1\leq j\leq m} \epsilon_j$. If $\prod_{1\leq j\leq m}(a_{1j} + a_{2j} + q) < 1$ then for any $\delta < \delta_0$ and any $(w, z) \in Q(r, \delta)$ there exist l and k such that $|g_{l,k}(w, z)| > \delta$.

Lemma A.2 *Suppose $a_{1j}, a_{2j}, b_{1j}, b_{2j} \geq 0$. For any $q > 0$ there exists an $\epsilon > 0$ such that*

$$|g_j(w, z)| < |(w, z)|^{a_{1j}+a_{2j}-q} \text{ for any } (w, z) \text{ with } |(w, z)| < \epsilon. \tag{48}$$

Proof: Let $\epsilon < \min((|A_j| + |C_j| + |B_j| + |D_j|)^{-1/q}, 1)$. Thus, for such (w, z) we have

$$\begin{array}{rcl} |g_j(w, z)| & < & (|A_j| + |C_j|)|w^{a_{1j}} z^{a_{2j}}| + (|B_j| + |D_j|)|w^{b_{1j}} z^{b_{2j}}| \\ & < & (|A_j| + |C_j| + |B_j| + |D_j|)|(w, z)|^{a_{1j}+a_{2j}} < |(w, z)|^{a_{1j}+a_{2j}-q}. \end{array}$$

QED

The following lemmas and the main theorem of this subsection consider the special case relevant to Type A cycles, where the maps $g_j(w, z)$ have $a_{2j} = 0$ and $b_{2j} = 1$, i.e. they simplify to

$$g_j(w, z) = (A_j w^{a_j} + B_j|w|^{b_j} z, C_j w^{a_j} + D_j|w|^{b_j} z) \tag{49}$$

(we set $a_{1j} \equiv a_j$, $b_{1j} \equiv b_j$ and do not assume necessarily that $a_j < b_j + 1$).

Lemma A.3 *Suppose $a_j - b_j > 1$.*

(a) For any $q > 0$, $f_1(z)$, $f_2(z)$, $\tilde{p}$ and p, such that $\tilde{p} > p > 1$ and

$$f_1'(z) = \mathrm{O}(z^{p-1}),\ f_1(z) - f_2(z) = \mathrm{O}(z^{\tilde{p}}),\ \text{as } z \to 0 \tag{50}$$

there exist $\tilde{f}_1(w)$, $\tilde{f}_2(w)$ and $\epsilon_j > 0$ such that

$$\tilde{f}_1'(w) = \mathrm{O}(w^{a_j-b_j-1}),\ \tilde{f}_1(w) - \tilde{f}_2(w) = \mathrm{O}(w^{a_j\tilde{p}-b_j}) \text{ for } w \to 0,$$

and

$$\begin{array}{c} g_j(Q_w(\tilde{f}_1(w), \tilde{f}_2(w), \epsilon_j)) \subset Q_z(f_1(z), f_2(z), \epsilon_j^{a_j-q}), \\ g_j(Q_w(\tilde{f}_1(w), \tilde{f}_2(w), \epsilon_j)) \supset Q_z(f_1(z), f_2(z), \epsilon_j^{a_j+q}), \\ |g_j(w, z)| < |(w, z)|^{a_j-q},\ |g_j(w, z)| > |(w, z)|^{a_j+q} \end{array}$$

for all $(w, z) \in Q_w(\tilde{f}_1(w), \tilde{f}_2(w), \epsilon_j)$.

(b) For any $q>0$, $f_1(w)$, $f_2(w)$, $\tilde p$ and p, such that $\tilde p > p > 1$ and

$$f_1'(w) = \mathrm{O}(w^{p-1}),\ f_1(w) - f_2(w) = \mathrm{O}(w^{\tilde p}),\ \text{as } w \to 0, \tag{51}$$

there exist $\tilde f_1(w)$, $\tilde f_2(w)$ and $\epsilon_j > 0$ such that

$$\tilde f_1'(w) = \mathrm{O}(w^{a_j - b_j - 1}),\ \tilde f_1(w) - \tilde f_2(w) = \mathrm{O}(w^{a_j \tilde p - b_j})\ \text{as } w \to 0,$$

and

$$\begin{gathered} g_j(Q_w(\tilde f_1(w), \tilde f_2(w), \epsilon_j)) \subset Q_w(f_1(w), f_2(w), \epsilon_j^{a_j - q}), \\ g_j(Q_w(\tilde f_1(w), \tilde f_2(w), \epsilon_j)) \supset Q_w(f_1(w), f_2(w), \epsilon_j^{a_j + q}), \\ |g_j(w,z)| < |(w,z)|^{a_j - q},\ |g_j(w,z)| > |(w,z)|^{a_j + q} \end{gathered}$$

for all $(w,z) \in Q_w(\tilde f_1(w), \tilde f_2(w), \epsilon_j)$.

Proof: (a) Let $\tilde f_{1,2}(w)$ be respectively the solutions of

$$A_j w^{a_j} + B_j |w|^{b_j} \tilde f_1(w) = f_1(C_j w^{a_j} + D_j |w|^{b_j} \tilde f_1(w)) \tag{52}$$

and

$$A_j w^{a_j} + B_j |w|^{b_j} \tilde f_2(w) = f_2(C_j w^{a_j} + D_j |w|^{b_j} \tilde f_2(w)). \tag{53}$$

The functions $\tilde f_{1,2}(w)$ are defined for any small w and due to (50)

$$\tilde f_{1,2}(w) = -\frac{A_j}{B_j} w^{a_j - b_j} + \mathrm{O}(w^{a_j p - b_j}). \tag{54}$$

Substitution of $w \equiv w + \delta w$ into (52) implies

$$A_j (w+\delta w)^{a_j} + B_j |(w+\delta w)|^{b_j} \tilde f_1(w+\delta w) = f_1(C_j (w+\delta w)^{a_j} + D_j |(w+\delta w)|^{b_j} \tilde f_1(w+\delta w)). \tag{55}$$

Subtracting now (52) from (55), dividing the result by δw and taking the limit $\delta w \to 0$, we obtain that

$$\tilde f_1'(w) = -\frac{A_j(a_j - b_j)}{B_j} w^{a_j - b_j - 1} + \mathrm{O}(w^{a_j p - b_j - 1}).$$

Subtraction of (53) from (52) yields

$$\begin{aligned} B_j |w|^{b_j} (\tilde f_1(w) - \tilde f_2(w)) = f_1(C_j w^{a_j} + D_j |w|^{b_j} \tilde f_1(w)) - f_1(C_j w^{a_j} + D_j |w|^{b_j} \tilde f_2(w)) + \\ f_1(C_j w^{a_j} + D_j |w|^{b_j} \tilde f_2(w)) - f_2(C_j w^{a_j} + D_j |w|^{b_j} \tilde f_2(w)). \end{aligned} \tag{56}$$

Because of (50) and (54)

$$\begin{aligned} & f_1(C_j w^{a_j} + D_j |w|^{b_j} \tilde f_1(w)) - f_1(C_j w^{a_j} + D_j |w|^{b_j} \tilde f_2(w)) \\ \approx f_1'((C_j - \frac{D_j A_j}{B_j}) w^{a_j}) & D_j |w|^{b_j} (\tilde f_1(w) - \tilde f_2(w)) = \mathrm{O}(w^{a_j(p-1) + b_j})(\tilde f_1(w) - \tilde f_2(w)) \end{aligned} \tag{57}$$

and

$$f_1(C_jw^{a_j}+D_j|w|^{b_j}\tilde{f}_2(w))-f_2(C_jw^{a_j}+D_j|w|^{b_j}\tilde{f}_2(w))=\mathrm{O}(w^{a_j\tilde{p}}).$$

Hence (56) takes the form

$$B_jw^{b_j}(\tilde{f}_1(w)-\tilde{f}_2(w))=\mathrm{O}(w^{a_j(p-1)+b_j})(\tilde{f}_1(w)-\tilde{f}_2(w))+\mathrm{O}(w^{a_j\tilde{p}}).$$

Since $p>1$ and $a_j>1$, the first term in the r.h.s. of this expression is asymptotically smaller than the l.h.s. and it can be ignored. Thus,

$$\tilde{f}_1(w)-\tilde{f}_2(w)=\mathrm{O}(w^{a_j\tilde{p}-b_j}).$$

The asymptotic relation (54) implies that there exist $\tilde{\epsilon}$ and K such that

$$\left|z+\frac{A_j}{B_j}w^{a_j-b_j}\right|<K|w^{a_jp-b_j}|$$

for $(w,z)\in Q_w(\tilde{f}_1(w),\tilde{f}_2(w),\tilde{\epsilon})$. Suppose that ϵ_j satisfies

$$\epsilon_j<\min(\tilde{\epsilon},1),\ \epsilon_j^q\left(|C_j|+\left|\frac{A_jD_j}{B_j}\right|+(|B_j|+|D_j|)K\right)<1,$$

$$K\epsilon_j^{a_j(p-1)}<\min\left(\frac{1}{2}\left|\frac{C_j}{D_j}-\frac{A_j}{B_j}\right|,\left|\frac{A_j}{B_j}\right|\right),$$

$$\epsilon_j^{-q}\left|C_j-\frac{D_jA_j}{B_j}\right|>4,\quad \epsilon_j^{a_j-b_j-1}<\left|\frac{B_j}{A_j}\right|.$$

Then

$$\begin{array}{rcl}|g_j(w,z)| & = & |w^{a_j}(A_j+B_jw^{b_j-a_j}z,C_j+D_jw^{b_j-a_j}z)| \\ & < & |w|^{a_j}(|C_j|+|\frac{A_jD_j}{B_j}|+(|B_j|+|D_j|)K)<|(w,z)|^{a_j-q}\end{array}$$

and

$$|g_j(w,z)|>|w^{a_j}(C_j+D_jw^{b_j-a_j}z)|>|w|^{a_j}\frac{1}{2}\left|C_j-\frac{D_jA_j}{B_j}\right|>2|w|^{a_j+q}>|(w,z)|^{a_j+q}.$$

From (52) and (53), the proof of part (a) is complete.

(b) Set $\tilde{f}_{1,2}(w)$ to be respectively the solutions of

$$C_jw^{a_j}+D_j|w|^{b_j}\tilde{f}_1(w)=f_1(A_jw^{a_j}+B_j|w|^{b_j}\tilde{f}_1(w))$$

and

$$C_jw^{a_j}+D_j|w|^{b_j}\tilde{f}_2(w)=f_2(A_jw^{a_j}+B_j|w|^{b_j}\tilde{f}_2(w)).$$

The remainder of the proof is similar to case (a). **QED**

Corollary A.2 *Lemma A.3 implies that the maps $f_l \to \tilde{f}_l$ are monotonic in the following sense:*

(a) Let f_0 be such that $f_1(z) < f_0(z) < f_2(z)$ for all $|z| < \epsilon_j^{a_j - q}$. Define $\tilde{f}_0(w)$ by analogy with $\tilde{f}_{1,2}(w)$ in part (a) of that lemma.

$$\text{If } \tilde{f}_1(w) > \tilde{f}_2(w), \text{ then } \tilde{f}_1(w) > \tilde{f}_0(w) > \tilde{f}_2(w),$$
$$\text{If } \tilde{f}_1(w) < \tilde{f}_2(w), \text{ then } \tilde{f}_1(w) < \tilde{f}_0(w) < \tilde{f}_2(w)$$

for all $|w| < \epsilon_j$.

(b) Let f_0 be such that $f_1(w) < f_0(w) < f_2(w)$ for all $|w| < \epsilon_j^{a_j - q}$. Define $\tilde{f}_0(w)$ by analogy with $\tilde{f}_{1,2}(w)$ in part (b) of that lemma.

$$\text{If } \tilde{f}_1(w) > \tilde{f}_2(w), \text{ then } \tilde{f}_1(w) > \tilde{f}_0(w) > \tilde{f}_2(w),$$
$$\text{If } \tilde{f}_1(w) < \tilde{f}_2(w), \text{ then } \tilde{f}_1(w) < \tilde{f}_0(w) < \tilde{f}_2(w)$$

for all $|w| < \epsilon_j$.

Lemma A.4 *Suppose that $0 < a_j - b_j < 1$.*

(a) For any $q > 0$, $f_1(z)$, $f_2(z)$, $\tilde{p}$ and p, such that $\tilde{p} > p > 1$ and

$$f_1'(z) = \mathrm{O}(z^{p-1}), \;\; f_1(z) - f_2(z) = \mathrm{O}(z^{\tilde{p}}), \;\; \text{as } z \to 0, \tag{58}$$

there exist $\tilde{f}_1(z)$, $\tilde{f}_2(z)$ and $\epsilon_j > 0$ such that

$$\tilde{f}_1'(z) = \mathrm{O}(z^{1/(a_j-b_j)-1}), \;\; \tilde{f}_1(z) - \tilde{f}_2(z) = \mathrm{O}(w^{(a_j\tilde{p}-a_j+1)/(a_j-b_j)}) \text{ for } z \to 0,$$

and

$$g_j(Q_z(\tilde{f}_1(z), \tilde{f}_2(z), \epsilon_j)) \subset Q_z(f_1(z), f_2(z), \epsilon_j^{a_j/(a_j-b_j)-q}),$$
$$g_j(Q_z(\tilde{f}_1(z), \tilde{f}_2(z), \epsilon_j)) \supset Q_z(f_1(z), f_2(z), \epsilon_j^{a_j/(a_j-b_j)+q}),$$
$$|g_j(w,z)| < |(w,z)|^{a_j/(a_j-b_j)-q}, \;\; |g_j(w,z)| > |(w,z)|^{a_j/(a_j-b_j)+q},$$

for all $(w,z) \in Q_z(\tilde{f}_1(z), \tilde{f}_2(z), \epsilon_j)$.

(b) For any $q > 0$, $f_1(w)$, $f_2(w)$, $\tilde{p}$ and p, such that $\tilde{p} > p > 1$ and

$$f_1'(w) = \mathrm{O}(w^{p-1}), \;\; f_1(w) - f_2(w) = \mathrm{O}(w^{\tilde{p}}), \;\; \text{as } w \to 0, \tag{59}$$

there exist $\tilde{f}_1(z)$, $\tilde{f}_2(z)$ and $\epsilon_j > 0$ such that

$$\tilde{f}_1'(z) = \mathrm{O}(z^{1/(a_j-b_j)-1}), \;\; \tilde{f}_1(z) - \tilde{f}_2(z) = \mathrm{O}(z^{(a_j\tilde{p}-a_j+1)/(a_j-b_j)}) \text{ for } z \to 0,$$

and

$$g_j(Q_z(\tilde{f}_1(z), \tilde{f}_2(z), \epsilon_j) \subset Q_w(f_1(w), f_2(w), \epsilon_j^{a_j/(a_j-b_j)-q}),$$
$$g_j(Q_z(\tilde{f}_1(z), \tilde{f}_2(z), \epsilon_j) \supset Q_w(f_1(w), f_2(w), \epsilon_j^{a_j/(a_j-b_j)+q}),$$
$$|g_j(w,z)| < |(w,z)|^{a_j/(a_j-b_j)-q}, \;\; |g_j(w,z)| > |(w,z)|^{a_j/(a_j-b_j)+q},$$
$$\text{for all } (w,z) \in Q_z(\tilde{f}_1(z), \tilde{f}_2(z), \epsilon_j).$$

Proof: (a) Let $\tilde{f}_{1,2}(z)$ be solutions of

$$A_j\tilde{f}_1^{a_j}(z)+B_j|\tilde{f}_1|^{b_j}(z)z=f_1(C_j\tilde{f}_1^{a_j}(z)+D_j|\tilde{f}_1|^{b_j}(z)z) \tag{60}$$

and

$$A_j\tilde{f}_2^{a_j}(z)+B_j|\tilde{f}_2|^{b_j}(z)z=f_2(C_j\tilde{f}_2^{a_j}(z)+D_j|\tilde{f}_2|^{b_j}(z)z). \tag{61}$$

The functions $\tilde{f}_{1,2}(z)$ are defined for any small z. Note that

$$\tilde{f}_{1,2}(z)=-\left(\frac{B_j}{A_j}\right)^{1/(a_j-b_j)}z^{1/(a_j-b_j)}+\mathrm{O}(z^{(a_jp-a_j+1)/(a_j-b_j)}). \tag{62}$$

We substitute $z\to z+\delta z$ into (60), subtract (60) from the obtained equation, divide the result by δz and take the limit $\delta z\to 0$. Since

$$f_1^{a_j}(z+\delta z)-f_1^{a_j}(z)=f_1^{a_j}(z)\left(1+\frac{f_1(z+\delta z)-f_1(z)}{f_1(z)}\right)^{a_j}-f_1^{a_j}(z)\approx a_jf_1^{a_j-1}(f_1(z+\delta z)-f_1(z)),$$

and similar estimate holds true for b_j, we obtain that

$$\tilde{f}_1'(z)=-\frac{1}{(a_j-b_j)}\left(\frac{B_j}{A_j}\right)^{1/(a_j-b_j)}z^{1/(a_j-b_j)-1}+\mathrm{O}(z^{(a_jp-a_j+1)/(a_j-b_j)-1}). \tag{63}$$

Subtracting (61) from (60) we obtain

$$A_j(\tilde{f}_1^{a_j}-\tilde{f}_2^{a_j})+B_jz(|\tilde{f}_1|^{b_j}-|\tilde{f}_2|^{b_j})= \tag{64}$$

$$f_1(C_j\tilde{f}_1^{a_j}+D_j|\tilde{f}_1|^{b_j}z)-f_2(C_j\tilde{f}_1^{a_j}+D_j|\tilde{f}_1|^{b_j}z)+f_2(C_j\tilde{f}_1^{a_j}+D_j|\tilde{f}_1|^{b_j}z)-f_2(C_j\tilde{f}_2^{a_j}+D_j|\tilde{f}_2|^{b_j}z).$$

Since

$$\tilde{f}_1^{a_j}-\tilde{f}_2^{a_j}=(\tilde{f}_1-\tilde{f}_2)\mathrm{O}(z^{(a_j-1)/(a_j-b_j)}),$$
$$z(|\tilde{f}_1|^{b_j}-|\tilde{f}_2|^{b_j})=(\tilde{f}_1-\tilde{f}_2)\mathrm{O}(z^{(a_j-1)/(a_j-b_j)}),$$
$$f_1(C_j\tilde{f}_1^{a_j}+D_j|\tilde{f}_1|^{b_j}z)-f_2(C_j\tilde{f}_1^{a_j}+D_j|\tilde{f}_1|^{b_j}z)=\mathrm{O}(z^{\tilde{p}a_j/(a_j-b_j)})$$

and

$$f_2(C_j\tilde{f}_1^{a_j}+D_j|\tilde{f}_1|^{b_j}z)-f_2(C_j\tilde{f}_2^{a_j}+D_j|\tilde{f}_2|^{b_j}z)=(\tilde{f}_1-\tilde{f}_2)\mathrm{O}(z^{(pa_j-1)/(a_j-b_j)}),$$

(64) implies that

$$\tilde{f}_1(z)-\tilde{f}_2(z)=\mathrm{O}(z^{(\tilde{p}a_j-a_j+1)/(a_j-b_j)}).$$

Due to (62) there exist $\tilde{\epsilon}$ and K such that

$$\left|z+\frac{A_j}{B_j}w^{a_j-b_j}\right|<K|z|^{(a_jp-b_j)/(a_j-b_j)}\text{ for }(w,z)\in Q_w(\tilde{f}_1(z),\tilde{f}_2(z),\tilde{\epsilon}).$$

Suppose that ϵ_j satisfies

$$\epsilon_j<\min(\tilde{\epsilon},1),\quad \epsilon_j^q\left|\frac{2B_j}{A_j}\right|^{a_j/(a_j-b_j)}\left(|C_j|+\left|\frac{A_jD_j}{B_j}\right|+(|B_j|+|D_j|)K\right)<1,$$

$$K\epsilon_j^{a_j(p-1)/(a_j-b_j)} < \min\left(\frac{1}{2}\left|\frac{C_j}{D_j}-\frac{A_j}{B_j}\right|,\left|\frac{A_j}{B_j}\right|\right),$$

$$\epsilon_j^{-q}\frac{1}{4}\left|\frac{B_j}{2A_j}\right|^{a_j/(a_j-b_j)}\left|C_j-\frac{D_jA_j}{B_j}\right|>1 \quad \text{and} \quad \epsilon_j^{1/(a_j-b_j)-1}<\left(\frac{B_j}{A_j}\right)^{1/(a_j-b_j)}.$$

Then

$$\begin{aligned}|g_j(w,z)| &= |w^{a_j}(A_j+B_jw^{b_j-a_j}z, C_j+D_jw^{b_j-a_j}z)|\\ &< \left|\frac{2B_j}{A_j}\right|^{a_j/(a_j-b_j)}|z|^{a_j/(a_j-b_j)}\left(|C_j|+\left|\frac{A_jD_j}{B_j}\right|+(|B_j|+|D_j|)K\right)\\ &< |(w,z)|^{a_j/(a_j-b_j)-q}\end{aligned} \tag{65}$$

and

$$\begin{aligned}|g_j(w,z)| &> |w^{a_j}(C_j+D_jw^{b_j-a_j}z)|\\ &> |w|^{a_j}\frac{1}{2}\left|C_j-\frac{D_jA_j}{B_j}\right|\\ &> 2|z|^{a_j/(a_j-b_j)+q} > |(w,z)|^{a_j/(a_j-b_j)+q}.\end{aligned} \tag{66}$$

Hence part (a) is proved. The proof for the part (b) is similar. **QED**

Corollary A.3 *Lemma A.4 implies that the maps $f_l \to \tilde{f}_l$ are monotonic in the following sense:*

(a) Let f_0 be such that $f_1(z) < f_0(z) < f_2(z)$ for all $|z| < \epsilon_j^{a_j/(a_j-b_j)-q}$. Define $\tilde{f}_0(z)$ by analogy with $\tilde{f}_{1,2}(z)$ in part (a) of that lemma.

$$\text{If } \tilde{f}_1(z) > \tilde{f}_2(z), \text{ then } \tilde{f}_1(z) > \tilde{f}_0(z) > \tilde{f}_2(z),$$
$$\text{If } \tilde{f}_1(z) < \tilde{f}_2(z), \text{ then } \tilde{f}_1(z) < \tilde{f}_0(z) < \tilde{f}_2(z).$$

for all $|z| < \epsilon_j$.

(b) Let f_0 be such that $f_1(w) < f_0(w) < f_2(w)$ for all $|w| < \epsilon_j^{a_j/(a_j-b_j)-q}$. Define $\tilde{f}_0(z)$ by analogy with $\tilde{f}_{1,2}(z)$ in part (b) of that lemma.

$$\text{If } \tilde{f}_1(z) > \tilde{f}_2(z), \text{ then } \tilde{f}_1(z) > \tilde{f}_0(z) > \tilde{f}_2(z),$$
$$\text{If } \tilde{f}_1(z) < \tilde{f}_2(z), \text{ then } \tilde{f}_1(z) < \tilde{f}_0(z) < \tilde{f}_2(z).$$

for all $|z| < \epsilon_j$.

Lemma A.5 *Suppose $a_j > 0$ and $a_j - b_j < 0$. Then for any $\epsilon_0 > 0$ and $r > 0$ there exists an $\epsilon > 0$ such that*

$$g_j(w,z) \in Q(r,\epsilon_0) \text{ for any } (w,z) \text{ with } |(w,z)| < \epsilon.$$

Proof: Let ϵ satisfy

$$\epsilon^{b_j-a_j+1} < \min\left(\left|\frac{A_j}{2B_j}\right|, \left|\frac{C_j}{2D_j}\right|\right),$$

$$\epsilon^{a_j}|(3A_j/2, 3C_j/2)| < \epsilon_0$$

and

$$\epsilon^{ra_j} < \min\left(\left|\frac{C_j}{2}\left(\frac{3A_j}{2}\right)^{-1-r}\right|, \left|\frac{A_j}{2}\left(\frac{3C_j}{2}\right)^{-1-r}\right|\right).$$

Therefore, if $|(w,z)| < \epsilon$ then

$$|g_j(w,z)| = |w^{a_j}||(A_j + B_j w^{b_j-a_j}z, C_j + D_j w^{b_j-a_j}z)| < \epsilon^{a_j}|(3A_j/2, 3C_j/2)| < \epsilon_0,$$

$$|g_j^w(w,z)|^{1+r}/|g_j^z(w,z)| < |w^{ra_j}|\frac{|3A_j/2|^{1+r}}{|C_j/2|} < 1$$

and similarly

$$|g_j^z(w,z)|^{1+r}/|g_j^w(w,z)| < 1.$$

QED

Lemma A.6 *For any $\tilde{q} > 0$ there exists an $\epsilon_j > 0$ such that*

$$|g_j(w,z)| < |(w,z)|^{\beta_j} \text{ where } \beta_j = \min(a_j/2, |b_j|\tilde{q}/2, |1+b_j|/2),$$

for all $(w,z) \in B_{\epsilon_j} \setminus \tilde{Q}_1$, where

$$\widetilde{Q}_1 = \begin{cases} \emptyset & \text{if } b_j > 0 \\ Q_z(-|z|^{-1/b_j-\tilde{q}}, |z|^{-1/b_j-\tilde{q}}, \epsilon_j) & \text{if } b_j < 0. \end{cases}$$

Proof: If

$$(|A_j| + |C_j|)\epsilon_j^{a_j/2} < 1/2, \;\; (|B_j| + |D_j|)\epsilon_j^{|b_j|\tilde{q}/2} < 1/2, \;\; (|B_j| + |D_j|)\epsilon_j^{|1+b_j|/2} < 1/2$$

then one can verify that

$$|g_j(w,z)| < (|A_j| + |C_j|)|w|^{a_j} + (|B_j| + |D_j|)|w^{b_j}z| < |(w,z)|^{\beta_j}.$$

QED

Lemma A.7 *For any $\tilde{q} > 0$, $r > 0$ and $\epsilon > 0$ there exists an $\epsilon_j > 0$ such that*

$$g_j(w,z) \in Q(r,\epsilon) \text{ for all } (w,z) \in B_{\epsilon_j} \setminus (\widetilde{Q}_1 \cup \widetilde{Q}_2 \cup \widetilde{Q}_3),$$

where $\widetilde{Q}_1$ is defined in Lemma A.6,

$$\widetilde{Q}_2 = \begin{cases} \emptyset & \text{if } a_j - b_j < 0 \\ Q_z(-(\frac{B_j}{A_j})^{\alpha_1} z^{\alpha_1} - |z|^{\alpha_2}, -(\frac{B_j}{A_j})^{\alpha_1} z^{\alpha_1} + |z|^{\alpha_2}, \epsilon_j) & \text{if } 0 < a_j - b_j < 1 \\ Q_w(-\frac{A_j}{B_j} w^{\alpha_3} - |w|^{\alpha_4}, -\frac{A_j}{B_j} w^{\alpha_3} + |w|^{\alpha_4}, \epsilon_j) & \text{if } a_j - b_j > 1 \end{cases},$$

$$\widetilde{Q}_3 = \begin{cases} \emptyset & \text{if } a_j - b_j < 0 \\ Q_z(-(\frac{D_j}{C_j})^{\alpha^1} z^{\alpha_1} - |z|^{\alpha_2}, -(\frac{D_j}{C_j})^{\alpha_1} z^{\alpha_1} + |z|^{\alpha_2}, \epsilon_j) & \text{if } 0 < a_j - b_j < 1 \\ Q_w(\frac{C_j}{D_j} w^{\alpha_3} - |w|^{\alpha_4}, \frac{C_j}{D_j} w^{\alpha_3} + |w|^{\alpha_4}, \epsilon_j) & \text{if } a_j - b_j > 1 \end{cases},$$

and

$$\alpha_1 = \frac{1}{a_j - b_j},\ \alpha_2 = \frac{a_j(r+1) - 2a_j + 2}{2(a_j - b_j)},\ \alpha_3 = a_j - b_j,\ \alpha_4 = \frac{a_j(r+1)}{2} - b_j.$$

Proof: We start with the proof of existence of ϵ_w such that in the case $a_j - b_j > 1$ the condition

$$(w, z) \in B_{\epsilon_w} \setminus (\widetilde{Q}_1 \cup \widetilde{Q}_2 \cup \widetilde{Q}_3)$$

implies that

$$|g_j^w(w, z)| > |g_j^z(w, z)|^{1+r}. \tag{67}$$

If $|A_j w^{a_j} + B_j |w|^{b_j} z| < |A_j w^{a_j}/2|$ then

$$|g_j^w(w, z)| > |B_j w^{a_j(r+1)/2}| \text{ and } |g_j^z(w, z)| < (|C_j| + |3A_j D_j / B_j|)|w|^{a_j}.$$

For $|w^{a_j(r+1)/2}| < |B_j|(|C_j| + |3A_j D_j/B_j|)^{-r-1}$ the inequality (67) holds true.

If $|A_j w^{a_j} + B_j |w|^{b_j} z| > |A_j w^{a_j}/2|$ then

$$|g_j^w(w, z)| > \min(|A_j/2|, |B_j/3|) \max(|w^{a_j}|, |w^{b_j} z|)$$

and

$$|g_j^z(w, z)| < 2 \max(|C_j|, |D_j|) \max(|w^{a_j}|, |w^{b_j} z|).$$

Hence if

$$|(w, z)|^{r\beta_j} < \min(|A_j/2|, |B_j/3|)(2\max(|C_j|, |D_j|))^{-r-1},$$

where β_j is defined in Lemma A.6, then (67) holds.

The proof of existence of ϵ_z such that $|g_j^z(w, z)| > |g_j^w(w, z)|^{1+r}$ is similar. Denote by $\tilde{\epsilon}_j$ the ϵ_j from Lemma A.6. Then $\epsilon_j = \min(\epsilon_w, \epsilon_z, \tilde{\epsilon}_j)$, by the definition of the set $Q(r, \epsilon)$, satisfies the the condition of the lemma. For the case $0 < a_j - b_j < 1$ the proof is similar and is not presented, for the case $a_j - b_j < 0$ the statement of the Lemma follows from Lemma A.5. **QED**

Let the collection of functions $\{h_{l,j}(y)\}$ for $1 \le j \le m$, $l \le j$, be defined as follows[7]:

$$\begin{aligned} h_{j,j}(y) &= y, \\ h_{l,j}(y) &= \begin{cases} \infty & \text{if } a_l - b_l < 0 \\ \dfrac{a_l h_{l+1,j}(y) - a_l + 1}{a_l - b_l} & \text{if } 0 < a_l - b_l < 1 \\ a_l h_{l+1,j}(y) - b_l & \text{if } a_l - b_l > 1 \end{cases} \end{aligned}$$

This collection has the following properties:

(a)
$$h_{l,j}(y_0 + y_1) = h_{l,j}(y_0) + a_{l,j} y_1,$$
where $a_{l,j} = \prod_{l \le s < j} \max(a_s, a_s/(a_s - b_s))$ is defined for $l \le j$.

(b) If there exist J such that $a_J - b_J < 0$, then $h_{l,j}(y) = \infty$ for $l \le J$.

(c) If $a_{j-m,j} > 1$, then $h_{l-m,j}(y) > h_{l,j}(y)$ for any $y \ge 1$.

(d) If $a_{j-m,j} > 1$, then $\lim_{l \to -\infty} h_{l,j}(y) = \infty$ for any $y \ge 1$.

The next theorem gives the main result for Type A cycles, namely it gives the stability indices σ_j for the collection of maps g_j related to Type A cycles. The coefficients a_j and b_j of the map g_j are related to the eigenvalues of linearisation of (1) near ξ_j as $a_j = c_j/e_j$ and $b_j = -t_j/e_j$. Recall, that $c_j > 0$ and $e_j > 0$ for all j and therefore $a_j > 0$. Following [16], we denote
$$\rho_j = \min(a_j, 1 + b_j),$$
$\rho = \rho_1 \cdots \rho_m$, and note that generically the non-degeneracy conditions (19) apply.

Theorem A.1 *(reproduces Theorem 4.1) For the collection of maps g_j associated with a Type A cycle, the stability indices are:*

(a) If $\rho > 1$ and $b_j > 0$ for all j then $\sigma_{j,+} = \infty$ and $\sigma_{j,-} = 0$ for any j.

(b) If $\rho > 1$, $b_j > -1$ for all j and $b_j < 0$ for $j = J_1, \ldots, J_L$ then $\sigma_{j,-} = 0$ and $\sigma_{j,+}$ are:

$$\sigma_{j,+} = \min_{s=J_1,\ldots,J_L} h_{j,s}\left(-\frac{1}{b_s}\right) - 1.$$

(c) If $\rho < 1$ or there exists j such that $b_j < -1$ then $\sigma_{j,+} = 0$, $\sigma_{j,-} = \infty$ and the cycle is not an attractor.

[7]If an index takes values $1, \ldots, m$, then the index value modulo m is understood here and below.

Proof: (a) Since $\rho > 1$, there exists a $q > 0$ such that

$$\prod_{j=1}^{m} (\rho_j - q) > 1. \tag{68}$$

By Lemma A.2, for any j there exist ϵ_j such that

$$|g_j(w,z)| < |(w,z)|^{\rho_j - q} \text{ for any } (w,z) \text{ with } |(w,z)| < \epsilon_j. \tag{69}$$

For a given δ, choose an $\epsilon > 0$ satisfying

$$\epsilon^{\rho_{1,j}} < \min(\delta, \epsilon_j, 1) \text{ and } \rho_{1,j} = \prod_{s=1}^{j} (\rho_s - q), \text{ for all } 1 \le j \le m. \tag{70}$$

Consider $(w,z) \in H_1^{(in)}$. If $|(w,z)| < \epsilon$, then (69) and (70) imply that $|g_{j,0}(w,z)| < \delta$. Due to (68), $|g(w,z)| < \epsilon$ and hence $|g_{j,k}(w,z)| < \delta$ for all $0 \le j \le m-1$, $k \ge 0$. Therefore $\sigma_{1,+} = \infty$ and $\sigma_{1,-} = 0$. The proof for $j > 1$ is similar.

(b) First, let us prove that

$$\sigma_{1,+} < h_{1,s}\left(-\frac{1}{b_s}\right) - 1 + \tilde{q}_1, \tag{71}$$

where $\tilde{q}_1$ is any small number and $s = J_l$ for some l. Denote $q_1 = \tilde{q}_1 / a_{1,s}$. Assume that q satisfies (68). Define the sets $Q^{j,s}(\epsilon_j)$ by the following rule:

$$Q^{s,s}(\epsilon_s) = Q_z(-|z|^{-1/b_s + q_1}, |z|^{-1/b_s + q_1}, \epsilon_s).$$

For a given $Q^{j+1,s}(\epsilon_{j+1})$ the set $Q^{j,s}(\epsilon_j)$ is

$$Q^{j,s}(\epsilon_j) = \begin{cases} \emptyset & \text{if } a_j - b_j < 0 \\ Q_z(\tilde{f}_1(z), \tilde{f}_2(z), \epsilon_j),\ \epsilon_j = \epsilon_{j+1}^{(a_j - b_j)/(a_j - (a_j - b_j)q)} & \text{if } 0 < a_j - b_j < 1 \\ Q_w(\tilde{f}_1(w), \tilde{f}_2(w), \epsilon_j),\ \epsilon_j = \epsilon_{j+1}^{1/(a_j - q)} & \text{if } a_j - b_j > 1, \end{cases} \tag{72}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are the functions defined in Lemmas A.3 and A.4. Denote by $\tilde{\epsilon}_j$ the ϵ_j from Lemmas A.3 and A.4 and set $\epsilon_0 = \min_j \tilde{\epsilon}_j^{1/\hat{a}_{1,j}}$, where $\hat{a}_{l,j} = \prod_{l \le s < j} (\max(a_s, a_s/(a_s - b_s)) - q)$. Examples of the sets $Q^{j,s}(\epsilon_s)$ are shown in Figure 5.

For any $\delta > 0$ we can find an $\hat{\epsilon} > 0$ such that

$$|g_s(w,z)| > \delta \text{ for all } (w,z) \in Q_z(-|z|^{-1/b_s + q_1}, |z|^{-1/b_s + q_1}, \hat{\epsilon}).$$

Hence, if $\epsilon < \min(\epsilon_0, \hat{\epsilon}^{1/\hat{a}(1,s)})$ then

$$|g_s \circ \ldots \circ g_1(w,z)| > \delta \text{ for all } (w,z) \in Q^{1,s}(\epsilon).$$

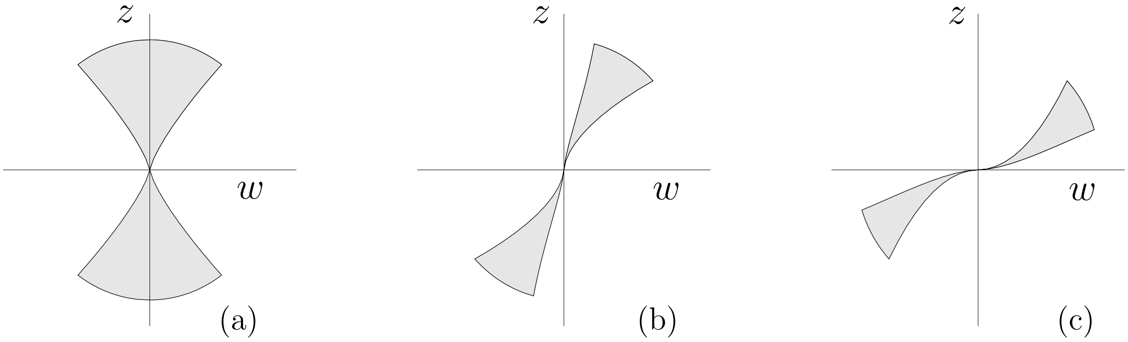

Figure 5: Examples of the sets (a) $Q^{s,s}(\epsilon_s)$, (b) $Q^{s-1,s}(\epsilon_{s-1})$ for $C_{s-1}D_{s-1} < 0$ and $0 < a_{s-1} - b_{s-1} < 1$ and (c) $Q^{s-2,s}(\epsilon_{s-2})$, for $A_{s-2}B_{s-2} < 0$ and $a_{s-2} - b_{s-2} > 1$.

Since

$$\ell(Q^{1,s}(\epsilon)) = \mathrm{O}(\epsilon^{h_{1,s}(-1/b_s+q_1)+1}),$$

and $h_{1,s}(1/b_s + q_1) - h_{1,s}(1/b_s) = \tilde{q}_1$ the inequality (71) is proved.

Second, we prove that

$$\sigma_{1,+} > h_{1,\tilde{J}}(-1/b_{\tilde{J}}) - 1 - \tilde{q}_1, \tag{73}$$

where $\tilde{q}_1$ is any small number, $q_1 = \tilde{q}_1/a_{1,\tilde{J}}$, and $\tilde{J}$ is the value of J_l where

$$\min_{1\le l\le L}(h_{1,J_l}(-1/b_{J_l}))$$

is achieved. Assume that q satisfies (68) and r and q satisfy the conditions of Lemma A.1 for all j. Set

$$Q^{s,s}(\epsilon_s) = Q_z(-|z|^{-1/b_s-q_1}, |z|^{-1/b_s-q_1}, \epsilon_s) \text{ for } s = 1, \dots, J_l,$$

$Q_l^{1,1}(\epsilon_1) = \tilde{Q}_l$ where $\tilde{Q}$, $l = 2, 3$, and ϵ_1 are defined in Lemma A.7 for $j = 1$. The sets $Q^{j,s}$ and $\widetilde{Q}_l^{j,1}$, $l = 2, 3$, are defined by (72). Denote

$$y_0 = \begin{cases} \infty & \text{if } a_1 - b_1 < 0 \\ (a_1(r+1)/2 - a_1 + 1)/(a_1 - b_1) & \text{if } 0 < a_1 - b_1 < 1 \\ a_1(r+1)/2 - b_1 & \text{if } a_1 - b_1 > 1. \end{cases}$$

Since $y_0 > 1$, by property (d) of the functions $h_{l,j}(y)$, there exists $k > 0$ such that $h_{1-mk,1}(y_0) > h_{1,\tilde{J}}(-1/b_{\tilde{J}})$. Hence by Lemmas A.6 and A.7,

$$\text{if } (w, z) \in B_\epsilon \setminus ((\cup_{0\le s\le k,\ 1\le l\le L} Q^{1-sm,J_l}) \cup \widetilde{Q}_2^{1-km,1} \cup \widetilde{Q}_3^{1-km,1}) \text{ then } g^k(w, z) \in Q(r, \tilde{\epsilon}),$$

where $\epsilon^{k\beta} < \tilde{\epsilon}$, $\beta = \prod_{1\le j\le m} \beta_j$. Since

$$\ell(Q^{1,\tilde{J}}(\epsilon)) = \mathrm{O}(\epsilon^{h_{1,\tilde{J}}(-1/b_{\tilde{J}}-q_1)+1}),$$

$$\ell(\widetilde{Q}_{2,3}^{1-km,1}(\epsilon)) = \mathrm{O}(\epsilon^{h_{1-mk,1}(y_0)+1}) = \mathrm{o}(\epsilon^{h_{1,\tilde{J}}(-1/b_{\tilde{J}}-q_1)+1})$$

and

$$\ell(Q^{1-sm,K}(\epsilon)) = \mathrm{O}(\epsilon^{h_{1-sm,K}(-1/b_K-q_1)+1}) = \mathrm{o}(\epsilon^{h_{1,\tilde{J}}(-1/b_{\tilde{J}}-q_1)+1}) \text{ if } s \neq 0,\ K \neq \tilde{J}$$

by Corollary A.1, part (b) is proved.

(c) Let r and q_3 be such that the conditions of Lemma A.1 are satisfied for all j where $b_j > -1$. For all j where $b_j < -1$ assume that r also satisfies

$$b_j + r < -1. \tag{74}$$

Let $\widetilde{Q}_2$ and $\widetilde{Q}_3$ be defined as in Lemma A.7 for $j = 1$. By the same arguments employed in that lemma, for sufficiently small δ, if $(w, z) \in B_\delta \setminus (\widetilde{Q}_2 \cup \widetilde{Q}_3)$ then either $g_1(w, z) > \delta$ or $g_1(w, z) \in Q(r, \delta)$. In the latter case, if all $b_j > -1$ then $g_{l,k}(w, z) > \delta$ (for small enough δ) for some l and k due to Corollary A.1(b). If there exist $b_j < -1$ then $g_{j,1}(w, z) > \delta$ for small δ due to (74). Therefore, $g_{l,k}(w, z) > \delta$ for some l and k for all (w, z), such that $(w, z) \in B_\delta \setminus (\widetilde{Q}_2 \cup \widetilde{Q}_3)$.

We now consider $(w, z) \notin B_\delta \setminus (\widetilde{Q}_2 \cup \widetilde{Q}_3)$. There are two (generically) mutually exclusive cases:

- Suppose that at least one of the inequalities, $\prod_{1\le j\le m} a_j > 1$ or $a_t - b_t < 0$ for some t, is satisfied. Denote $Q_{2,3}^{1,1} = \widetilde{Q}_{2,3}$ and define the sets $Q_{2,3}^{1-l,1}$ and y_0 in the same way as in the proof of the part (b). Since $\lim_{k\to\infty} h_{1-km,1}(y_0) = \infty$, $\sigma_{1,-}$ is arbitrary large.

- Suppose that $\prod_{1\le j\le m} a_j < 1$ and $a_t - b_t > 0$ for all t. There exist $q_2 > 0$ such that $\prod_{1\le j\le m}(a_j + q_2) < 1$. By Corollaries A.2 and A.3 there exist limits as $k \to \infty$ of f_l bounding $Q_{2,3}^{1-km+j,1}(\epsilon)$ for some $\epsilon < \tilde{\epsilon}$. Hence, finite values of ϵ_j can be found such that Lemmas A.3 and A.4 hold true for $Q_{2,3}^{1-km+j,1}$ for any $k > 0$. For any $\delta < \min_j(\epsilon_j)$, by Lemmas A.3 and A.4 the following is satisfied

$$|g_j(w, z)| > (w, z)^{a_j+q_2} \text{ for all } (w, z) \in Q_{2,3}^{j-km,1}(\delta). \tag{75}$$

Take any (w_0, z_0), $|(w_0, z_0)| = \alpha < \delta$. There exists $k > 0$ such that $\alpha^{k\hat{a}_{1,m}} > \delta$, where $\hat{a}_{l,j} = \prod_{l\le s<j}(a_s + q_2)$. Hence, due to (75), if $|g_{l,s}(w_0, z_0)| < \delta$ for all $1 \le l \le m$ and $s < k$, and $|g^k(w_0, z_0)| < \delta$ then $g^k(w_0, z_0) \in B_\delta \setminus (\widetilde{Q}_2 \cup \widetilde{Q}_3)$.

QED

B Types B and C cycles

In this section we present proofs of Theorems and Lemmas employed for calculation of stability indices of Types B and C cycles. To leading order, the maps $g_j : H_j^{(in)} \to H_{j+1}^{(in)}$

associated with the cycles of Types B and C reduce to $g_j(w,z) = (Ew^{a_j}, Fw^{b_j}z)$ and $g_j(w,z) = (Ew^{b_j}z, Fw^{a_j})$, respectively. As noted in Section 4.2, it suffices to consider only positive values of w and z. In the coordinates (ζ,η), $\zeta = \ln z$ and $\eta = \ln w$, the maps g_j take the form:

$$g_j(\zeta,\eta) = M_j \begin{pmatrix} \zeta \\ \eta \end{pmatrix}.$$

(In what follows, the constants E and F are ignored, see discussion in Section 4.2.) The *transition matrices* of the maps are

$$M_j = \begin{pmatrix} a_j & 0 \\ b_j & 1 \end{pmatrix} \text{ and } M_j = \begin{pmatrix} b_j & 1 \\ a_j & 0 \end{pmatrix}$$

for cycles of Types B and C, respectively. Recall that the coefficients a_j and b_j of the map g_j are related to the eigenvalues of linearisation of (1) near ξ_j as $a_j = c_j/e_j$ and $b_j = -t_j/e_j$. As in Appendix A, the stability indices are calculated in terms of exponents of the maps g_j, a_j and b_j. For the map $g = g_m \circ \ldots \circ g_1$ the transition matrix is $M = M(g) = M_m \cdots M_1$. We introduce the notation: $M_{j,k}$ and $M^{(j)}$ denote transition matrices for the maps $g_{j,k}$ and $g^{(j)}$, respectively; $M^{(l,j)} = M_l \cdots M_j$; λ_1^j, λ_2^j, $\mathbf{v}_1^j = (v_{11}^j, v_{12}^j)$ and $\mathbf{v}_2^j = (v_{21}^j, v_{22}^j)$ denote eigenvalues and associated eigenvectors of the matrix $M^{(j)}$, respectively. If the eigenvalues are real, $\lambda_1^j \geq \lambda_2^j$ is assumed.

B.1 The set $U^{-\infty}(M)$

A necessary condition for (w,z) to belong to $\mathcal{B}_\delta^g$ (see Subsection 3.3) is that $g^k(w,z)$ is bounded for all k by a small $\delta > 0$. Since $g^k(w,z)$ is bounded, in the new coordinates $(\zeta,\eta) = (\ln w, \ln z)$ the iterates $(\zeta_k,\eta_k) = g^k(\zeta,\eta)$ are bounded from above: $\zeta_k < S$ and $\eta_k < S$ for some large in absolute value negative S. Due to linearity of g, this generically implies that $\lim_{k\to\infty} g^k(\zeta,\eta) = (-\infty,-\infty)$.

We denote

$$U^{-\infty}(M) = \{(x,y):\ x \leq 0,\ y \leq 0,\ \lim_{n\to\infty} M^n(x,y)^t = (-\infty,-\infty)^t\}.$$

Lemma B.1 *The dependence of $U^{-\infty}(M)$ on eigenvalues and eigenvectors is as follows:*

(i) If the λ_i are complex, then $U^{-\infty}(M) = \emptyset$.

(ii) If the λ_i are real and $\lambda_1 \leq 1$ or $|\lambda_2| > \lambda_1$, then $U^{-\infty}(M) = \emptyset$;

(iii) If the λ_i are real and $v_{11}v_{12} < 0$, then $U^{-\infty}(M) = \emptyset$;

(iv) If the λ_i are real, $\lambda_1 > 1$, $v_{11}v_{12} > 0$ and $v_{21}v_{22} \leq 0$, then

$$U^{-\infty}(M) = \{(x,y):\ x \leq 0,\ y \leq 0\};$$

(v) *If the* λ_i *are real,* $\lambda_1 > 1$, $v_{11}v_{12} > 0$ *(whereby we assume* $v_{11} > 0$ *and* $v_{12} > 0$*), and* $v_{21}v_{22} > 0$*, then*

$$U^{-\infty}(M) = \{(x,y):\ x \le 0,\ y \le 0,\ (v_{11}v_{22} - v_{12}v_{21})^{-1}(v_{22}x - v_{21}y) < 0\};$$

(vi) *If the* λ_i *are real,* $\lambda_1 > 1$, $v_{11}v_{12} = 0$ *and* $\lambda_2 \le 1$*, then* $U^{-\infty}(M) = \emptyset$*;*

(vii) *If the* λ_i *are real,* $\lambda_1 > 1$, $v_{11}v_{12} = 0$, $\lambda_2 > 1$ *and* $v_{21}v_{22} \le 0$*, then* $U^{-\infty}(M) = \{(x,y):\ x \le 0,\ y \le 0\}$*;*

(viii) *If the* λ_i *are real,* $\lambda_1 > 1$, $v_{11}v_{12} = 0$ *(whereby we assume* $v_{11} \ge 0$ *and* $v_{12} \ge 0$*),* $\lambda_2 > 1$ *and* $v_{21}v_{22} > 0$*, then*

$$U^{-\infty}(M) = \{(x,y):\ x \le 0,\ y \le 0,\ (v_{11}v_{22} - v_{12}v_{11})^{-1}(v_{22}x - v_{21}y) < 0\}.$$

Proof: Let the eigenvalues be complex conjugate, $\lambda_{1,2} = s\mathrm{e}^{\pm i\phi}$. Then

$$M^n(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = s^n(\mathbf{v}_1(\alpha\cos n\phi - \beta\sin n\phi) + \mathbf{v}_2(\alpha\sin n\phi + \beta\cos n\phi)) =$$

$$s^n(\cos n\phi(\alpha v_{11} + \beta v_{21}) + \sin n\phi(-\beta v_{11} + \alpha v_{21}), \cos n\phi(\alpha v_{12} + \beta v_{22}) + \sin n\phi(-\beta v_{12} + \alpha v_{22})).$$

Because the eigenvalues are complex, $\phi \ne k\pi$. Hence for any $N_0 > 0$ there exists $N > N_0$ such that

$$x_n \equiv s^n(\cos n\phi(\alpha v_{11} + \beta v_{21}) + \sin n\phi(-\beta v_{11} + \alpha v_{21})) > 0.$$

This proves part (i).

If the eigenvalues are real and distinct, the map M^n in the basis comprised of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, $(x,y) = h_1\mathbf{v}_1 + h_2\mathbf{v}_2$, takes the form

$$M^n(h_1,h_2) \equiv \begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix} = \lambda_1^n h_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} + \lambda_2^n h_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}. \tag{76}$$

If $|\lambda_1| \le 1$ and $|\lambda_2| \le 1$ (recall that $\lambda_2 < \lambda_1$), then for any (h_1,h_2), (76) has a finite limit as $n \to \infty$. If $|\lambda_2| > \lambda_1$, then $\lambda_2 < 0$ and hence in (76) the sign of h_j^n $(j = 1,2)$ (76) alternates for odd and even n, if n is large enough. Part (ii) is proved.

Assume that $\lambda_1 > 1$ and $|\lambda_2| < \lambda_1$. If $v_{11}v_{12} \ne 0$, to leading order $h_1^{(n)} = \lambda_1^n h_1 v_{11}$ and $h_2^{(n)} = \lambda_1^n h_1 v_{12}$ for $n \to \infty$. Thus, if the signs of v_{11} and v_{12} are different, then the limits of $h_1^{(n)}$ and $h_2^{(n)}$ have different signs, and (iii) is proved. If $v_{11}v_{12} > 0$ (and assuming without any loss of generality that they are positive), the limits of $h_j^{(n)}$, $j = 1,2$, are $-\infty$ in the points (x,y) such that $h_1 < 0$. h_1 is negative for any $x < 0$ and $y < 0$, if $v_{21}v_{22} \le 0$, which proves part (iv). If $v_{21}v_{22} > 0$, then the set of (x,y) for which $h_1 < 0$ satisfies the inequality $(v_{11}v_{22} - v_{12}v_{11})^{-1}(v_{22}x - v_{21}y) < 0$, and so part (v) is proved.

Assume that $v_{12} = 0$; if $\lambda_2 \le 1$, in (76) the limit of $h_2^{(n)}$ for $n \to \infty$ is either $+\infty$ or it does not exist. Thus, part (vi) is proved. The proofs of statements (vii) and (viii) are similar to the proofs of (iv) and (v) and are omitted. **QED**

Lemma B.1 can be used to give the dependence of $U^{-\infty}(M)$ on the matrix entries, $M = (a_{ij})$, $i,j = 1,2$.

Lemma B.2 *Let λ_1 and λ_2 ($\lambda_1 > \lambda_2$, if they are real; generically $\lambda_1 \neq \lambda_2$) be the eigenvalues of the matrix $M = (a_{ij})$, $a_{11} > a_{22}$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ be the associated eigenvectors. Then*

(a) *(i) the eigenvalues are real if and only if*

$$\frac{(a_{11}-a_{22})^2}{4} + a_{12}a_{21} \geq 0 \tag{77}$$

(ii) $\lambda_1 > 1$ if and only if

$$\max\left(\frac{a_{11}+a_{22}}{2},\ a_{11}+a_{22}-a_{11}a_{22}+a_{12}a_{21}\right) > 1 \tag{78}$$

(iii) $\lambda_1 > |\lambda_2|$ if and only if

$$\frac{a_{11}+a_{22}}{2} > 0 \tag{79}$$

(iv) $v_{11}v_{12} > 0$ if and only if

$$a_{21} > 0. \tag{80}$$

(b) If λ_j, $j = 1, 2$, are real, then $a_{12}v_{22} = (\lambda_2 - a_{11})v_{21}$ and $\lambda_2 - a_{11} < 0$.

Proof: The eigenvalues of the matrix M can be expressed as

$$\lambda_{1,2} = \frac{a_{11}+a_{22}}{2} \pm \sqrt{\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}a_{21}}.$$

Statements (a)(i)-(iv) follow from examination of this formula and on noting that the eigenvector $\mathbf{v}_1$ satisfies $a_{21}v_{11} + a_{22}v_{12} = \lambda_1 v_{12}$. If the eigenvalues are real, then $\lambda_2 - a_{11} < 0$ and $\mathbf{v}_2$ satisfies $a_{11}v_{21} + a_{12}v_{22} = \lambda_2 v_{11}$, which proves statement (b). **QED**

Corollary B.1 *Assume that entries of a matrix M satisfy the conditions (i)-(iv) of Lemma B.2. Then $U^{-\infty}(M) = \{(x, y) :\ x \leq 0,\ y \leq 0,\ (\lambda_2 - a_{11})x - a_{12}y > 0\}$ for $a_{12} < 0$, and $U^{-\infty}(M) = \{(x, y) :\ x \leq 0,\ y \leq 0\}$ for $a_{12} \geq 0$.*

B.2 Maps and neighbourhoods

The condition $w^2 + z^2 < \epsilon$ in (ζ, η) coordinates is equivalent to the condition[8] $\max(\zeta, \eta) < R$, where $R < 0$; small ϵ corresponds to large $|R|$. In this subsection we examine how R-neigbourhoods of $(-\infty, -\infty)$ are transformed by linear maps.

Let M be an invertible linear map $M : \mathbb{R}^2 \to \mathbb{R}^2$. Define

$$U_R = \{(\zeta, \eta) \mid \max(\zeta, \eta) < R\}$$

[8]In other words, instead of $|(w, z)| = (w^2 + z^2)^{1/2}$, an equivalent norm $|(w, z)| = \max(|w|, |z|)$ can be employed.

and

$$U_R(\alpha_1,\beta_1,q_1;\alpha_2,\beta_2,q_2) = \\ \{(\zeta,\eta)\in U_R \ : \ (\alpha_1+q_1)\zeta+(\beta_1+q_1)\eta<0,\ (\alpha_2+q_2)\zeta+(\beta_2+q_2)\eta<0\},$$

where $R<0$.

Lemma B.3 *For any $S<0$ and $q>0$ there exists $R<0$ such that*

$$M(U_R\cap M^{-1}U_0(1,0,-q;0,1,-q))\subset U_S. \tag{81}$$

Proof: We split the neighbourhood in two parts

$$U_0(1,0,-q;0,1,-q)=V_1\cup V_2,$$

where $V_1=U_0(1,0,-q;0,1,-q)\cap U_S$ and $V_2=U_0(1,0,-q;0,1,-q)\setminus U_S$ and denote

$$\tilde{R}=\min\{\zeta,\eta \ : \ (\zeta,\eta)\in M^{-1}V_2\}.$$

$\tilde{R}$ is finite, since V_2 is bounded and M invertible. The inclusion (81) takes place for any $R<\tilde{R}$, because

$$M(U_R\cap M^{-1}U_0(1,0,-q;0,1,-q))\cap V_2=\emptyset$$

due to $R<\tilde{R}$. **QED**

Note, that if $(\zeta,\eta)\in U_R\setminus M^{-1}U_0$, then $\max(\tilde{\zeta},\tilde{\eta})\geq 0$, where $(\tilde{\zeta},\tilde{\eta})=M(\zeta,\eta)$.

Lemma B.4 *Denote $\widetilde{U}=U_0(\alpha_1,\beta_1,-q_1;\alpha_2,\beta_2,-q_2)$. Suppose $\widetilde{U}\subseteq U^{-\infty}(M)\neq\emptyset$ and $(v_{11},v_{12})\in\widetilde{U}$.*

(i) If $\lambda_2\geq 0$, then $M^k(\widetilde{U})\subset\widetilde{U}$ for any $k>0$;

(ii) $M^{2k}(\widetilde{U})\subset\widetilde{U}$ for any $k>0$.

Proof: (i) If $(x,y)\in\widetilde{U}$, then $\alpha(x,y)\in\widetilde{U}$ for any $\alpha>0$. If $(x,y)\in\widetilde{U}$ is represented as $(x,y)=\alpha\mathbf{v}_1+\beta\mathbf{v}_2$, then, due to convexity of $\widetilde{U}$, $\alpha\mathbf{v}_1+\tilde{\beta}\mathbf{v}_2\in\widetilde{U}$ for any $|\tilde{\beta}|$, such that $|\tilde{\beta}|\leq|\beta|$ and $\tilde{\beta}\beta\geq 0$.

For any $(x,y)=\alpha\mathbf{v}_1+\beta\mathbf{v}_2\in\widetilde{U}$ the identity

$$M^k(x,y)=\lambda_1^k(\alpha\mathbf{v}_1+\beta\lambda_2^k/\lambda_1^k\mathbf{v}_2) \tag{82}$$

holds. Since $\lambda_1>\lambda_2>0$, due to the arguments above, $M^k(x,y)\in\widetilde{U}$.

(ii) If k is even, the identity (82) implies the statement for negative λ_2 as well. **QED**

Lemma B.5 *Consider the set*

$$U^{-\infty}(M) = U_0(\alpha_1, \beta_1, 0; \alpha_2, \beta_2, 0).$$

For any $q_1 > 0$, $q_2 > 0$ and $S < 0$ there exists an $R < 0$ such that

(i) If $\lambda_2 \geq 0$, then $M^k U_R(\alpha_1, \beta_1, -q_1; \alpha_2, \beta_2, -q_2) \subset U_S$ for any $k > 0$;

(ii) $M^{2k} U_R(\alpha_1, \beta_1, -q_1; \alpha_2, \beta_2, -q_2) \subset U_S$ for any $k > 0$.

Proof: (i) Consider a set $\widetilde{W}$ comprised of two line segments, one segment being $x = -1$, $y = [-1, 0]$ and the other one $y = -1$, $x = [-1, 0]$. Denote $W = \widetilde{W} \cap \bar{U}_0(\alpha_1, \beta_1, -q_1; \alpha_2, \beta_2, -q_2)$. The iterates $(x_k, y_k) = M^k (x_0, y_0)^t$ are bounded from above for any $(x_0, y_0) \in W$ by some $\widetilde{Q} < 0$, because $W \subset U^{-\infty}(M)$. Due to compactness of W, the bounds for the iterates are uniform for all $(x_0, y_0) \in W$ by some $Q < 0$. Since the maps M^k are linear, $\max(x_k, y_k) < Q$ for a given (x_0, y_0) implies $(\tilde{x}_k, \tilde{y}_k) < \alpha Q$ for $(\tilde{x}_0, \tilde{y}_0) = \alpha(x_0, y_0)$. Part (i) is thus proved for $R = -S/Q$.

Statement (ii) is a consequence of Lemma B.4 (ii). **QED**

B.3 Main theorems

In this subsection we prove two theorems on stability indices for maps related to heteroclinic cycles of Types B or C, employing the Lemmas proved in Sections B.1 and B.2.

Theorem B.1 *(reproduces Theorem 4.2) Let g be a map related to simple heteroclinic cycle of Types B or C and M_j, $1 \leq j \leq m$, its transition matrices. Suppose that for all j, $1 \leq j \leq m$, all entries of the matrices are non-negative. Then:*

(a) If the transition matrix $M = M_m \cdots M_1$ satisfies condition (a)(ii) of Lemma B.2, then $\sigma_{j,+} = \infty$ and $\sigma_{j,-} = 0$ for all j and therefore the cycle is asymptotically stable.

(b) Otherwise, $\sigma_{j,+} = 0$ and $\sigma_{j,-} = \infty$ for all j and the cycle is not an attractor.

Proof: (a) Suppose that the matrix $M \equiv M^{(1)}$ satisfies condition (a)(ii) of Lemma B.2. For a map $M^{(j)}: H_j^{(in)} \to H_j^{(in)}$, the condition can be expressed as

$$\max\left(\operatorname{tr}\left(M^{(j)}\right), 2\operatorname{tr}\left(M^{(j)}\right) - 2\det\left(M^{(j)}\right)\right) > 2. \tag{83}$$

Hence if the condition is satisfied by $M^{(j)}$ for any one value of j, it is satisfied for all $1 \leq j \leq m$. Any $M^{(j)}$ have non-negative entries, as it is a product of matrices with non-negative entries. Therefore, $M^{(j)}$ satisfies conditions (i) and (iii) of part (a) the Lemma. Due to the assumptions $\lambda_1 > \lambda_2$ and $a_{11} > a_{22}$, the condition (iv) is satisfied. Hence $U^{-\infty}(M^{(j)}) = \mathbb{R}^2_-$ for all j.

Consider the images of the lines $\alpha(-1+q,-q)$ and $\beta(-q,-1+q)$, where $0<q<1$, $\alpha\in\mathbb{R}_+$ and $\beta\in\mathbb{R}_+$, under the mappings $M^{j,1}$ or $M^{j,1}M$. Since all entries of the matrices $M^{j,1}$ and $M^{j,1}M$ are non-negative, the images take the form $\alpha(-r_1,-r_2)$ for $r_i>0$ and $\alpha\in\mathbb{R}_+$. Thus, for any positive $q<1$ there exists a $q_j>0$, such that

$$M^{j,1}U_0(1,0,-q;0,1,-q)\subset U_0(1,0,-q_j;0,1,-q_j)$$

and

$$M^{j,1}MU_0(1,0,-q;0,1,-q)\subset U_0(1,0,-q_j;0,1,-q_j).$$

Apply Lemma B.3 to mappings $M^{j,1}$ and $M^{j,1}M$, setting any $S<0$ and $q=q_j$. According to the Lemma, we can find S_j, such that

$$M^{j,1}U_{S_j}(1,0,-q;0,1,-q)\subset U_S \text{ and } M^{j,1}MU_{S_j}(1,0,-q;0,1,-q)\subset U_S. \qquad (84)$$

Denote

$$\widetilde{S}=\min_j S_j.$$

By Lemmas B.4 and B.5 (where Lemma B.5 is applied for $S=\widetilde{S}$), there exists R such that

$$M^{2k}U_R(1,0,-q;0,1,-q)\subset U_{\widetilde{S}}(1,0,-q;0,1,-q) \text{ for all } k\geq 0.$$

Thus, (84) implies

$$M_{j,k}U_R(1,0,-q;0,1,-q)\subset U_S \text{ for all } k\geq 0.$$

Hence, $\sigma_{1,+}>1/q-1$ for any q, which implies $\sigma_{1,+}=\infty$ and $\sigma_{1,-}=0$. The proof holds true for $j>1$ as well, and therefore part (a) is proved.

For part (b), if the matrix $M\equiv M^{(1)}$ does not satisfy the condition (ii) of the Lemma, by Lemma B.1 the set $U^{-\infty}(M)$ is empty, $\sigma_{1,+}=0$ and $\sigma_{1,-}=\infty$. Since condition (83) is satisfied or not satisfied by all $M^{(j)}$ simultaneously, $\sigma_{j,+}=0$ and $\sigma_{j,-}=\infty$ for all $1\geq j\geq m$. **QED**

Theorem B.2 *(reproduces Theorem 4.3). Let X be a simple heteroclinic cycle of Types B or C and M_j, $1\leq j\leq m$ the associated transition matrices. We denote by $j=j_1,\dots j_L$ the indices, for which some of the entries of M_j are negative; they are all non-negative for all remaining j.*

- *(a) If at least for one of $j=j_l+1$ the matrix $M^{(j)}$ does not satisfy conditions (i)-(iv) of Lemma B.2, then the cycle is repelling and $\sigma_j=-\infty$ for all j.*
- *(b) If the matrices $M^{(j)}$ satisfy conditions (i)-(iv) of Lemma B.2 for all $j=j_l+1$, then there exist numbers $(\alpha_1^j,\beta_1^j,\alpha_2^j,\beta_2^j)$, $1\leq j\leq m$, such that*
  - *(i) $U_0(\alpha_1^j,\beta_1^j,0;\alpha_2^j,\beta_2^j,0)\neq\emptyset$, $1\leq j\leq m$.*

(ii) For any $S < 0$ and $q > 0$ there exists $R < 0$ such that

$$M^{(l,j)}(M^{(j)})^k(U_R(\alpha_1^j, \beta_1^j, -q; \alpha_2^j, \beta_2^j, -q)) \subset U_S \quad \text{for all } l, \quad 1 \le l < m, \ k \ge 0.$$

(iii)

$$\lim_{k\to\infty}(M^{(l,j)}(M^{(j)})^k(\zeta, \eta)) = (-\infty, -\infty), \quad \text{for all } \ (\zeta, \eta) \in U_0(\alpha_1^j, \beta_1^j, 0; \alpha_2^j, \beta_2^j, 0).$$

(iv)

$$U_0(\alpha_1^j, \beta_1^j, 0; \alpha_2^j, \beta_2^j, 0) = U^{-\infty}(M^{(j)}) \cap \left(\bigcap_{1\le l\le L} (M^{(j_l,j)})^{-1} U_0 \right) \cap \left(\bigcap_{1\le l\le L} (M^{(j_l+m,j)})^{-1} U_0 \right).$$

(v) If $\lambda_2 \ge 0$ then

$$U_0(\alpha_1^j, \beta_1^j, 0; \alpha_2^j, \beta_2^j, 0) = U^{-\infty}(M^{(j)}) \cap \left(\bigcap_{1\le l\le L} (M^{(j_l,j)})^{-1} U_0 \right).$$

The cycle is a Milnor attractor.

Proof: For (a), as noted in the proof of Theorem 4.2, the matrices $M^{(j)}$ will simultaneously satisfy, or not satisfy, conditions (i)-(iii) of Lemma B.2 for all j. Suppose the condition (iv) is not satisfied for some $j = J$. For any s, the iterates $(M^{(J)})^k M^{(J,s)}(x, y)$ on increasing k become aligned with (v_{11}^J, v_{12}^J), see (76). Since $v_{11}^J v_{12}^J \le 0$, the iterates escape from $U_S \subset \hat{H}_J^{(in)}$ for any $S < 0$ for a sufficiently large k. Hence part (a) is proved.

For (b) suppose that, for j_l+1 the matrix $M^{(j_l+1)}$ satisfies conditions (iv) of Lemma B.2. The matrices M_j, $j_l+1 \le j \le j_{l+1}-1$, have positive entries, $\mathbf{v}^j = M_{j-1} \ldots M_{j_l+2} M_{j_l+1} \mathbf{v}^{j_l+1}$, therefore $v_{11}^{(j_l+1)} v_{12}^{(j_l+1)} > 0$ implies $v_{11}^{(j)} v_{12}^{(j)} > 0$ for any j, $j_l+2 \le j \le j_{l+1}$. Hence, it suffices to check condition (iv) for $j = j_l + 1$, $1 \le l \le L$.

Denote

$$\widetilde{U}_j = U^{-\infty}(M^{(j)}) \cap \left(\bigcap_{1\le l\le L} (M^{(j_l,j)})^{-1} U_0 \right) \cap \left(\bigcap_{1\le l\le L} (M^{(j_l+m,j)})^{-1} U_0 \right)$$

The set is non-empty, because it includes a neighbourhood of the point (v_{11}^j, v_{12}^j) on the plane (since this point belongs to all sets in the intersection). Since all the sets are of the type $U_0(\alpha_1, \beta_1, 0; \alpha_2, \beta_2, 0)$, the intersection is also of the required type $U_0(\alpha_1^j, \beta_1^j, 0; \alpha_2^j, \beta_2^j, 0)$. Consider $\widetilde{U} = \widetilde{U}_1$.

Due to Lemma B.4 and definition of the set $\widetilde{U}$,

$$M_{j,k}\widetilde{U} \subset U^{-\infty}(M^{(j)}) \text{ for all } 1 \le j \le m, \ k \ge 0.$$

By the same arguments as employed in the proof of Theorem 4.2, this inclusion implies that for any $q > 0$ and $S < 0$ there exists $R < 0$, such that

$$M_{j,k}U_R(\alpha_1^1, \beta_1^1, -q; \alpha_2^1, \beta_2^1, -q) \subset U_S \text{ for all } 1 \leq j \leq m,\ k \geq 0,$$

and therefore $-\infty < \sigma_1$. The proof for σ_j with $j > 1$ is similar. Finally, by Theorem 2.3, X is a Milnor attractor, since the inequality $-\infty < \sigma_j$ is satisfied for all j. **QED**