Exact Harmonic Coefficients for a Magnetic Ring Head

David T. Wilton, Barry K. Middleton, *Member, IEEE*, and Mustafa M. Aziz

***Abstract*—The magnetic field of a ring head has been analyzed by Westmijze, using a conformal mapping, and by Fan, using Fourier techniques. Here these methods are reexamined and combined to give, for the first time, an explicit analytic expression for the harmonic coefficients in the Fan solution.**

***Index Terms*— Heads, frequency response, magnetic fields, mathematics, recording.**

I. Introduction

ONE of the most popular magnetic recording head geometries is the familiar ring head. Two approaches to the analysis of the field of such a head are conformal mapping and Fourier analysis, each of which leads to analytic results although neither is entirely satisfactory in this respect. Westmijze [1] first provided a conformal mapping solution but the mapping requires numerical inversion so that the final result is not fully analytic. Fan's [2] Fourier analysis does provide an explicit result but this is in the form of an infinite series whose coefficients have been determined by solving a large (theoretically infinite) system of linear equations.

Here, each of these methods is examined again and by combining them, two techniques arise which lead to formulas for the coefficients in the Fan solution, thus avoiding the need to solve large systems of equations. The first, motivated by a remark of Mallinson [3], gives an explicit formula for the coefficients, while the second gives a procedure for determining each coefficient in terms of the preceding ones.

The motivation for publication of these results is three-fold. First, the fact that the Fan coefficients, which have previously been computed by a number of authors [2], [4]–[6] and measured experimentally [7] with varying degrees of success, have (after nearly 40 years!) now been derived precisely, is intrinsically interesting and will hopefully make the Fan solution more widely accessible. Second, the first technique described should have wider applicability to other head geometries where a Fourier (Fan-type) solution is known. And third, publicizing successful techniques of combining apparently unrelated methods may encourage similar approaches in other problems.

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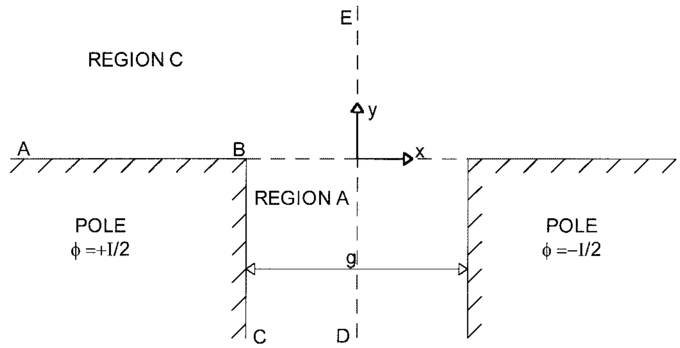

Fig. 1. Ring head model in the z plane.

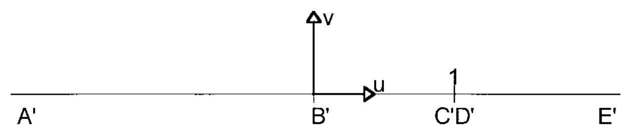

Fig. 2. The complex w plane.

II. Review of Methods

A. Conformal Mapping

First the Westmijze solution is briefly reviewed. The idealized model geometry of a ring head is shown in Fig. 1 where two semi-infinite pole pieces are at magnetic potentials $\pm I/2$ and separated by a gap of total length g. The Schwarz–Christoffel transformation

$$\frac{dz}{dw} = S\,\frac{\sqrt{w}}{w-1} \tag{1}$$

maps the boundary ABCDE (for half a ring head) in the z plane to the real axis in the w plane with the corresponding points $\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{D}', \mathrm{E}'$ shown in Fig. 2. Integration of (1) gives

$$z = S\left[2\sqrt{w} + \ln\left(\frac{\sqrt{w}-1}{\sqrt{w}+1}\right)\right] + z_0 \tag{2}$$

and ensuring that $\mathrm{CD} \to \mathrm{C}'\mathrm{D}'$ and $\mathrm{B} \to \mathrm{B}'$ leads to $S = ig/2\pi$ and $z_0 = 0$.

The magnetostatic potential function in the upper half w plane which takes the value $I/2$ on the real w axis for $-\infty < \mathrm{Re}(w) < 1$ and zero for $\mathrm{Re}(w) \geq 1$ is $\mathrm{Im}[F(w)]$ where

$$F(w) = \frac{I}{2\pi}\,\ln(w-1). \tag{3}$$

By standard theory [8], the components of the magnetic field $\boldsymbol{H} = -\nabla\phi$ in the z plane are given by

$$H_y + iH_x = -\frac{dF(w)}{dw}\,\frac{dw}{dz} \tag{4}$$

producing here

$$H_x - iH_y = \frac{I}{g\sqrt{w}}. \tag{5}$$

To determine the field components at any point $z = x+iy$ in the z plane, the corresponding point w needed in (5) may be found from (2), using, say, the Newton–Raphson numerical iterative method.

Two results, derived in [1], are particularly useful here. Westmijze computed fields along the boundary line A′B′C′D′E′ where w is real and hence $\sqrt{w}$ is easily evaluated. In particular along DE, $\sqrt{w} = \sqrt{u} = I/[gH_x(0,y)]$ giving

$$x = 0, \quad y = \frac{g}{\pi}\left[U^{-1} + \frac{1}{2}\,\ln\left(\frac{1-U}{1+U}\right)\right] \tag{6}$$

where $U = H_x(0,y)/H_0$ and $H_0 = I/g$, the deep gap field.

Westmijze also calculated the flux through the coil, using the reciprocity principle, in terms of a "gap loss" function

$$S(\tau) = \mathrm{Im}\left[\frac{1}{2\pi}\int_{-\infty}^{1}\frac{1}{1-w} \cdot e^{(2\tau/\pi)[\sqrt{w}+(1/2)\ln((\sqrt{w}-1)/(\sqrt{w}+1))]}\,dw\right]. \tag{7}$$

If

$$\hat{H}_x(k,y) = \int_{-\infty}^{\infty} H_x(x,y)e^{-ikx}\,dx \tag{8}$$

is the Fourier transform of the horizontal magnetic field, $\hat{H}_x(k,y)$ and $S(\tau)$ are related by

$$\hat{H}_x(k,y) = Ie^{-ky}S(kg/2). \tag{9}$$

B. Fourier Solution

Now consider Fan's Fourier solution. In the gap $(-g/2 \le x \le g/2, -\infty < y \le 0$, region A) the general solution of Laplace's equation taking the correct potentials on the pole pieces is

$$\phi_A(x,y) = -\frac{Ix}{g} - \sum_{n=1}^{\infty} A_n\,\sin\left(\frac{2n\pi x}{g}\right)e^{2n\pi y/g} \tag{10}$$

while above the poles $(-\infty < x < \infty, 0 \le y < \infty$, region C)

$$\phi_C(x,y) = \int_0^{\infty} C(\sigma)\,\sin\left(\frac{2\sigma x}{g}\right)e^{-2\sigma y/g}\,d\sigma. \tag{11}$$

Matching the potential along $y = 0$ gives

$$C(\sigma) = -\frac{I}{\pi}\,\frac{\sin(\sigma)}{\sigma^2} - \sum_{n=1}^{\infty} A_n 2n(-1)^n\,\frac{\sin(\sigma)}{\sigma^2-(n\pi)^2} \tag{12}$$

and matching the vertical field at $y = 0$ in the gap leads to an infinite set of linear equations which, until now, needed to be solved to give the normalized harmonic coefficients $A'_m = A_m/(I/2)$.

III. Exact Harmonic Coefficients

A. Technique 1

The first method presented follows similar work by Mallinson [3] for the "thin" gap head of Westmijze [1] which resulted in an explicit form for the appropriate harmonic coefficients as $2J_0(n\pi)/(n\pi)$. From (11), $C(\sigma)$ is effectively the Fourier transform of the magnetic potential in the head-face plane $(y = 0)$, since for $y \ge 0$

$$\hat{\phi}(k,y) = -i\,\frac{\pi g}{2}\,C(kg/2)e^{-ky} \tag{13}$$

giving

$$\hat{H}_x(k,y) = -\frac{k\pi g}{2}\,C(kg/2)e^{-ky}. \tag{14}$$

Now put $\sigma = m\pi$ in (12), where m is an integer, to give

$$C(m\pi) = -\frac{A_m}{\pi} \tag{15}$$

and then from (14) and (15)

$$A_m = \frac{1}{m\pi}\,\hat{H}_x(2m\pi/g,y)e^{2m\pi y/g}. \tag{16}$$

As observed in [3], the harmonic coefficients are simply multiples of sampled values of the spectral response function $\hat{H}_x(k,y)e^{ky} = -ik\hat{\phi}(k,0)$ at $k = 2m\pi/g$.

This result establishes the link with the conformal mapping solution via (9) giving the normalized coefficients as

$$A'_m = \frac{2}{m\pi}\,S(m\pi). \tag{17}$$

It remains to evaluate the integral (7) when $\tau = m\pi$. If the range of integration is split, the substitution $w = s^2$ for $0 \le w \le 1$ shows that the integrand is totally real there and hence gives no contribution to the result, while the substitution $w = -s^2$ for $-\infty < w \le 0$ leads to

$$S(m\pi) = \mathrm{Im}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{s}{1+s^2}\left(\frac{is-1}{is+1}\right)^m e^{2mis}\,ds\right]. \tag{18}$$

This integral may be evaluated by complex contour integration on closing the contour in the upper half complex s plane and using the method of residues. There is only one pole of the integrand, at $s = i$, and setting $t = s - i$ leads to $S(m\pi) = K_m e^{-2m}$ where K_m is the coefficient of t^m in $(t+i)(t+2i)^{m-1}e^{2mit}$.

Now

$$K_m = \sum_{n=0}^{m-1}\left[\binom{m-1}{n}(2i)^{m-1-n}\right] \cdot \left[\frac{-(2mi)^{m-n-1}(m+n)}{(m-n)!}\right] \tag{19}$$

where the term in the first bracket is the coefficient of t^n in the binomial expansion of $(t+2i)^{m-1}$ and the term in the

TABLE I
EXACT HARMONIC COEFFICIENTS FOR A RING HEAD

m	A'_m	A'_m (correct to 8 d.p.)
1	$-\frac{2e^{-2}}{\pi}$	−0.08615712
2	$\frac{5e^{-4}}{\pi}$	0.02915024
3	$-\frac{58e^{-6}}{3\pi}$	−0.01525422
4	$\frac{539e^{-8}}{6\pi}$	0.00959250
5	$-\frac{6934e^{-10}}{15\pi}$	−0.00668033
6	$\frac{38081e^{-12}}{15\pi}$	0.00496516
7	$-\frac{918970e^{-14}}{63\pi}$	−0.00386090
8	$\frac{109167851e^{-16}}{1260\pi}$	0.00310358
9	$-\frac{166282598e^{-18}}{315\pi}$	−0.00255909
10	$\frac{9303339907e^{-20}}{2835\pi}$	0.00215301

second bracket is the coefficient of t^{m-n} in the Taylor series expansion of $(t+i)e^{2mit}$. This simplifies to

$$K_m = \sum_{n=0}^{m-1} \binom{m-1}{n} \frac{m+n}{(m-n)!} (-1)^{m-n}(4m)^{m-n-1}. \tag{20}$$

Hence the normalized harmonic coefficients are

$$A'_m = \frac{2e^{-2m}}{m\pi} K_m. \tag{21}$$

The first ten values are shown in Table I. The values given in [6] are seen to be correct to the 6 d.p. quoted there.

B. Technique 2

The second technique which enables exact evaluation of the harmonic coefficients uses the Westmijze result (6) which may be rearranged as

$$U = \tanh(U^{-1} - y\pi/g). \tag{22}$$

Using an exponential series for tanh [9] gives

$$U = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-2nU^{-1}} e^{2n\pi y/g}. \tag{23}$$

Another expression for U is found from the Fourier representation (10) as

$$U = 1 + \sum_{n=1}^{\infty} A'_n n\pi e^{2n\pi y/g}. \tag{24}$$

The idea now is to match the coefficients of powers of $\varepsilon = e^{2\pi y/g}$ in each of (23) and (24) but first (24) must be substituted into the exponential term on the right-hand side of (23) since (23) is implicit in U. If $\alpha_n = A'_n n\pi$, (24) becomes

$$U = 1 + \sum_{n=1}^{\infty} \alpha_n \varepsilon^n \tag{25}$$

and then (23) is

$$U = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-2n(1+\Sigma_{n=1}^{\infty} \alpha_n\varepsilon^n)^{-1}} \varepsilon^n. \tag{26}$$

Expanding U^{-1} in (26) as a power series in ε valid for $|\Sigma_{n=1}^{\infty} \alpha_n\varepsilon^n| < 1$, which is satisfied for y sufficiently negative, expressing $e^{-2nU^{-1}}$ as a Taylor series and then equating coefficients of ε^n in (25) and (26) leads to

$$n=1: \quad \alpha_1 = -2e^{-2}$$
$$n=2: \quad \alpha_2 = -4e^{-2}\alpha_1 + 2e^{-4} = 10e^{-4}$$
$$n=3: \quad \alpha_3 = -4e^{-2}\alpha_2 + 8e^{-4}\alpha_1 - 2e^{-6} = -58e^{-6}$$

etc. Each α_n is given in terms of previous ones but it has not proved possible to establish a general formula.

The computer algebra package Mathematica [10] has been used to produce further values which are entirely consistent with the coefficients A'_n in Table I and provides a valuable independent check on their correctness.

IV. CONCLUSIONS

A fresh look at two long-established techniques for determining the magnetic field of a ring head has given further insight into this problem. In particular an explicit formula for the harmonic coefficients of Fan's solution has been derived. Also the relationship between such coefficients and the spectral response/gap loss function, observed in [3], has been reinforced. Such a relationship holds for any head geometry where a Fourier representation of the form (11) is valid.

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