The transverse magnetic reflectivity minimum of metals

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Abstract: Metal surfaces, which are generally regarded as excellent reflectors of electromagnetic radiation, may, at high angles of incidence, become strong absorbers for transverse magnetic radiation. This effect, often referred to as the pseudo-Brewster angle, results in a reflectivity minimum, and is most strongly evident in the microwave domain, where metals are often treated as perfect conductors. A detailed analysis of this reflectivity minimum is presented here and it is shown why, in the limit of very long wavelengths, metals close to grazing incidence have a minimum in reflectance given by $\left(\sqrt{2}-1\right)^2$.

OCIS codes: (260.3910) Metal optics; (240.0240) Optics at surfaces

References and links

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1. Introduction

The reflectivity of metals has long been a subject of scientific research. Recently there has been a resurgence of interest in this area. In the visible domain this has arisen primarily as a result of Ebbesen and coworkers observation of strongly enhanced transmission through holey metal films [1]. Further interest has also been stimulated by Pendry's suggestion of using metals as perfect lenses [2] while at much the same time the same author [3] has re-stimulated general interest in the idea of negative index materials and consequentially the possibility of 'cloaking' [4]. Another proposal that structured perfect metals may support surface modes [5] has also stimulated further work at longer wavelength [6]. It is in the context of this interest in metals, and in particular structured metals, as very interesting electromagnetic materials that we wish here to revisit the well-known problem of the simple reflectivity of metals for transverse magnetic or p-polarised radiation.

One of the best treatments of the reflectivity of metals to be found in any textbook is given by Stratton[7], although a more comprehensive coverage of the minimum in reflectivity for p-polarised radiation is given by Humphreys-Owen [8]. The central issues are: (i) What is the minimum p-polarised reflectivity for a metal? and (ii) at what angle of incidence does it occur. In addition, we shall address the question of why the minimum in reflectance at long wavelengths is exactly $\left(\sqrt{2}-1\right)^2$.

The general Fresnel amplitude reflection coefficient for p-polarised light incident on a planar interface between a dielectric, relative permittivity ε_1, and a metal, complex relative permittivity $\varepsilon_m\left(=\varepsilon_r+i\varepsilon_i\right)$, is given by:

$$r_p=\frac{\dfrac{\varepsilon_m}{\varepsilon_1}\cos\theta-\left[\dfrac{\varepsilon_m}{\varepsilon_1}-\sin^2\theta\right]^{1/2}}{\dfrac{\varepsilon_m}{\varepsilon_1}\cos\theta+\left[\dfrac{\varepsilon_m}{\varepsilon_1}-\sin^2\theta\right]^{1/2}} \tag{1}$$

If we define $\alpha=\varepsilon_m/\varepsilon_1$ and $T=\tan^2\theta$ equation (1) can be recast as:

$$r_p=\frac{\alpha-\left[\alpha+(\alpha-1)T\right]^{1/2}}{\alpha+\left[\alpha+(\alpha-1)T\right]^{1/2}}=\frac{\alpha-A}{\alpha+A}$$

where $A=\sqrt{\alpha+(\alpha-1)T}$. From this the reflectivity, $R_p=r_p r_p^*$ may be readily computed. Results for the visible, infra-red and microwave domain are illustrated in Fig. 1.

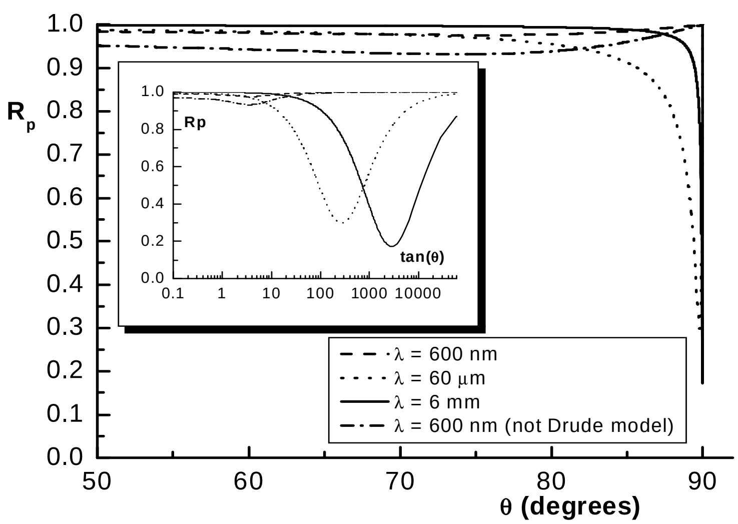

Fig. 1. p-polarised reflectivity from a planar surface as a function of incident angle (θ) for an interface between air and silver described by a Drude model with $\omega_p = 1.32\times10^{16}$ rad/s and $\tau = 1.45\times10^{14}$ s at wavelengths of 600 nm, 60 microns and 6 mm. Also shown are data for a wavelength of 600 nm with permittivity values taken from Palik [9] of $\varepsilon_r = 13.91$, $\varepsilon_i = 0.93$. (Inset: reflectivity as a function of tan(θ) [log scale]).

In each case a clear minimum in the reflectivity is visible, with that minimum progressing to higher angles as the wavelength, and consequently the magnitude of the permittivities, is increased.

Here we are interested in exploring in some detail this minimum. To find the properties of this minimum all that is required is to differentiate R_p with respect to θ or, in this case tan(θ),

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T, and find the real solutions for which this differential is zero. A little mathematical manipulation leads to the solution:

$$A^*\alpha\left(\alpha^* - T\right) + A\alpha^*\left(\alpha - T\right) = 0\,.$$

This can re-expressed after substituting in $\alpha = \alpha_r + i\alpha_i$ and $A = \sqrt{\alpha + (\alpha - 1)T}$, as;

$$\left(1 - \frac{2\alpha_r}{\alpha_r^2 + \alpha_i^2}\right)T^3 + 3T^2 - \left(\alpha_r^2 + \alpha_i^2\right)\left(1 + T\right) = 0$$

Finally substituting in the material parameters leads to the cubic:

$$\left(1 - \frac{2\varepsilon_r\varepsilon_1}{\varepsilon_r^2 + \varepsilon_i^2}\right)\varepsilon_1^2 T^3 + 3\varepsilon_1^2 T^2 - \left(\varepsilon_r^2 + \varepsilon_i^2\right)\left(1 + T\right) = 0 \qquad (2)$$

This may be shown to be identical in form to equation 13 in Humphreys-Owen's paper, with the substitution $\varepsilon_1 = 1$.

Of course equation (2) normally admits three solutions for T but generally only one of these will be purely real while the remaining two constitute a complex conjugate pair. It is the real one which is of primary interest here as we are looking for a real angle solution.

We next examine the reflectivity minimum for limiting cases:

a. Visible domain, for a metal for which $|\varepsilon_r| >> \varepsilon_i, \varepsilon_1$ ***and*** $\varepsilon_r < 0$

Now $\left(1 - \frac{2\varepsilon_1}{\varepsilon_r}\right)T^3 + 3T^2 - \left(\frac{\varepsilon_r}{\varepsilon_1}\right)^2\left(1 + T\right) \approx 0$, in which case one solution is $T = \frac{\varepsilon_r}{\varepsilon_1}$, which is negative and thus tan θ is imaginary. This solution has some interest however as, it gives:

$$\sin\theta = \sqrt{\frac{\varepsilon_r}{\varepsilon_r + \varepsilon_1}}\,.$$

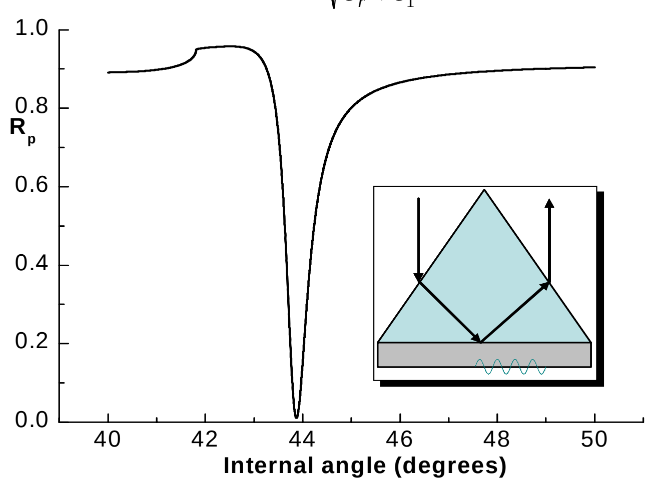

Fig. 2. Reflectivity minimum due to surface plasmon (SP) excitation using the Kretschmann-Raether [11] geometry (inset). Light of 600 nm wavelength is incident upon a 50 nm thick silver film (ε_r =13.91, ε_i = 0.9255) through a glass prism (n = 1.5) with air bounding. The SP is excited at a particular internal angle (measured from the normal to the interface) giving a reflection minimum.

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This is the well known surface plasmon [10] condition (Brewster angle for a metal with pure real permittivity). Though the angle at which the surface plasmon excitation occurs is imaginary it can be excited at real angles if some momentum enhancing method is utilized such as the well known Kretschmann-Raether [11] geometry. In this case a reflection minimum can also occur (Fig. 2), but the physics behind this minimum is very different to that of the pseudo-Brewster angle discussed in this paper.
There are two other solutions which take the form:

$$T = \frac{\varepsilon_r}{2\left(\varepsilon_1\varepsilon_r - 2\varepsilon_1^2\right)}\left[-\left(\varepsilon_r + \varepsilon_1\right) \pm \left(\varepsilon_r^2 - 2\varepsilon_r\varepsilon_1 + 9\varepsilon_1^2\right)^{1/2}\right]$$

Even with $\varepsilon_r < 0$ one of these two solutions is also negative, leaving only one real solution – the one we seek. Substituting in values of –20 for ε_r and 1 for ε_1 gives a θ value of 77°, which agrees with model calculations obtained using Eq. (1).

b. Infra-red region, for a metal for which $\varepsilon_i \approx |\varepsilon_r| >> \varepsilon_1$ ***and*** $\varepsilon_r < 0$

For most metals in the infra-red region of the spectrum there is a wavelength at which the magnitude of the real part of the permittivity becomes equal to the magnitude of the imaginary part of the permittivity. For the Drude model used here to approximately describe the frequency dependent permittivity of silver this occurs at a wavelength of 27.3 μm, at which $|\varepsilon_r| = \varepsilon_i = 18300$. In this situation Eq. (2) can be approximated as:

$$T^3 + 3T^2 - \left(\frac{\varepsilon_r^2 + \varepsilon_i^2}{\varepsilon_1^2}\right)(1+T) \cong 0$$

Whose solution is $T = \sqrt{\left(\varepsilon_r^2 + \varepsilon_i^2\right)/\varepsilon_1^2} = \left(\alpha\alpha^*\right)^{1/2}$. The angle of the reflectivity minimum rapidly approaches 90° and $R_p = \frac{2-\left(2-\sqrt{2}\right)^{1/2}}{2+\left(2-\sqrt{2}\right)^{1/2}} = 44.7\%$ as is shown in Fig. 3.

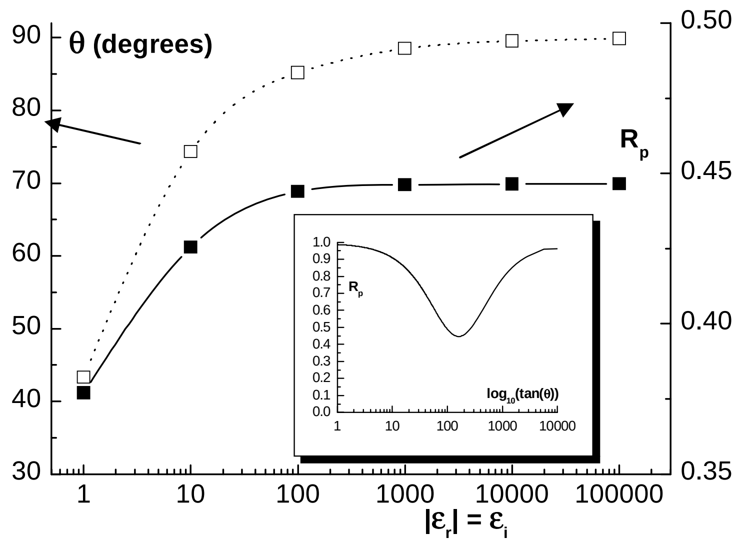

Fig. 3. The angle (open squares) and reflectivity (solid squares) at the reflectivity minimum as a function of $|\varepsilon_r| = \varepsilon_i$. The angle of the minimum is observed to rapidly approach grazing incidence, whilst the reflectivity of the minimum asymptotically approaches 44.7%. Inset: The reflection minimum for the case when $|\varepsilon_r| = \varepsilon_i = 18300$ which occurs at a wavelength of 27.3 μm for the Drude model with the parameters used here.

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c. Microwave domain, for a metal for which $\varepsilon_i >> |\varepsilon_r| >> \varepsilon_1$ or $n \approx k >> n_1$

Finally we consider the microwave region of the spectrum, a domain in which metals are often considered as perfect conductors, and as such as perfect reflectors. In this case equation 2 becomes:

$$T^3 + 3T^2 - \left(\frac{\varepsilon_i^2}{\varepsilon_1^2}\right)(1+T) \cong 0,$$

whereupon $T = \varepsilon_i / \varepsilon_1$.

Substituting this back into Eq. 1 gives

$$r_p = \frac{i\alpha_i - [i\alpha_i + (i\alpha_i - 1)\alpha_i]^{1/2}}{i\alpha_i + [i\alpha_i + (i\alpha_i - 1)\alpha_i]^{1/2}} = \frac{1-(-i)^{1/2}}{1+(-i)^{1/2}},$$

which leads directly to $R_p = (\sqrt{2}-1)^2 = 17.16\%$.

For both cases b and c above the reflectivity minimum tends to a very high angle and the reflectivity toward a limit value. In the case of the microwave domain this limit value is given by $(\sqrt{2}-1)^2$, but what is the physics of this limit value? This is the question we now address.

Consider p-polarised radiation of wavevector k_0 incident onto a surface at an angle θ. The incident medium has index n_1 (purely real) and the metal has permittivity $\varepsilon_m (= \varepsilon_r + i\varepsilon_i)$. Define normalized k vectors ($k' = k / k_0$) such that $k'_z = k'_{zr} + k'_{zi} = k_{zr} / k_0 + k_{zi} / k_0$ with normalized (conserved) in plane wavevector $k'_x = n_1 \sin\theta$.

This gives ${k'_{zr}}^2 - {k'_{zi}}^2 = \varepsilon_r - \varepsilon_1 \sin^2\theta$ and $2k'_{zr}k'_{zi} = \varepsilon_i$. Combining these gives a quadratic equation:

$$\left[{k'_{zr}}^2\right]^2 + \left[-\varepsilon_r + \varepsilon_1 \sin^2\theta\right]{k'_{zr}}^2 - \frac{\varepsilon_i^2}{4} = 0,$$

the real solutions of which are

$${k'_{zr}}^2 = \frac{\varepsilon_r - \varepsilon_1 \sin^2\theta + \left[\varepsilon_r^2 + \varepsilon_i^2 + \varepsilon_1^2 \sin^4\theta - 2\varepsilon_1\varepsilon_r \sin^2\theta\right]^{1/2}}{2}$$

and

$${k'_{zi}}^2 = \frac{-\varepsilon_r + \varepsilon_1 \sin^2\theta + \left[\varepsilon_r^2 + \varepsilon_i^2 + \varepsilon_1^2 \sin^4\theta - 2\varepsilon_1\varepsilon_r \sin^2\theta\right]^{1/2}}{2}.$$

Inside the metal the field is described by $\boldsymbol{E} = E_x(\hat{\boldsymbol{x}}, (A+iB)\hat{\boldsymbol{z}})e^{ik_x x}e^{ik_{zr}z}e^{-k_{zi}z}$, and consequently $\nabla \cdot \boldsymbol{E} = ik_x E_x + (A+iB)(ik_{zr} - k_{zi})E_x = 0$. Resulting in $E_x = 0$ or $ik_x + (A+iB)(ik_{zr} - k_{zi}) = 0$.

The second of these solutions results in the following equations:

$$A^2 + B^2 = \left[\frac{E_z}{E_x}\right]^2 = \frac{k_x^2}{k_{zr}^2 + k_{zi}^2} = \frac{{k'_x}^2}{{k'_{zr}}^2 + {k'_{zi}}^2}, \text{ and } \tan\phi = \frac{B}{A} = -\frac{k_{zi}}{k_{zr}} = -\frac{k'_{zi}}{k'_{zr}}.$$

We now require solutions for the field components when $|\varepsilon_i| >> |\varepsilon_r|$ and $|\varepsilon_i| >> \varepsilon_1$.

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With $k_{zr}'^{\,2} = \dfrac{\varepsilon_r - \varepsilon_1 \sin^2\theta + \varepsilon_i}{2}$ and $k_{zi}'^{\,2} = \dfrac{-\varepsilon_r + \varepsilon_1 \sin^2\theta + \varepsilon_i}{2}$ we obtain

$$\left[\frac{E_z}{E_x}\right]^2 = A^2 + B^2 = \frac{\varepsilon_1 \sin^2\theta}{\varepsilon_i} \tag{3}$$

$$\tan\phi = -\frac{k_{zi}}{k_{zr}} = -\frac{\varepsilon_i^{1/2}\left(1 - \varepsilon_r/2\varepsilon_i\right)}{\varepsilon_i^{1/2}\left(1 + \varepsilon_r/2\varepsilon_i\right)} \approx -1. \tag{4}$$

Thus the phase difference between E_x and E_z in the metal is 45°, the key result, as it means that the incident field in which E_x and E_z are in phase cannot match the fields inside the metal. Stratton points this out on p523 although it appears to have been largely overlooked. A reflected field is now essential.

From Eqs. (3) and (4) inside the metal we have $E_z/E_x = \sqrt{\varepsilon_1/\varepsilon_i}\sin\theta\ e^{i\pi/4}$, whilst in the incident dielectric medium: $E_x^{in} = \cos\theta$, $E_z^{in} = \sin\theta$, $E_x^{ref} = -E_o\cos\theta$, $E_z^{ref} = E_o\sin\theta$, where we assume an incident field of 1 and E_0 is the electric field reflected from the interface. The superscripts 'in' and 'ref' refer to the incident and reflected fields respectively. Since tangential E is conserved we have $\cos\theta - E_o\cos\theta = E_x$ or

$$1 - E_o = E_x/\cos\theta. \tag{5}$$

Normal D is also conserved, and since $\mathbf{D} = \varepsilon\,\mathbf{E}$ and, in the wavelength range of interest, $\varepsilon_m \cong i\varepsilon_i$, we can write $\varepsilon_1 \sin\theta + \varepsilon_1 E_o \sin\theta = \varepsilon_m E_z \cong i\varepsilon_i E_z = i\varepsilon_i \varepsilon_1 in\theta\ E_x e^{i\pi/4}$ which, upon rearranging, gives

$$1 + E_o \cong i\left(\frac{\varepsilon_i}{\varepsilon_1}\right)^{1/2} E_x e^{i\pi/4}. \tag{6}$$

Combining Eq. (5) and (6) leads to

$$E_o \cong \frac{i\left(\frac{\varepsilon_i}{\varepsilon_1}\right)^{1/2}\cos\theta\ e^{i\pi/4} - 1}{i\left(\frac{\varepsilon_i}{\varepsilon_1}\right)^{1/2}\cos\theta\ e^{i\pi/4} + 1} = \frac{iCe^{i\pi/4} - 1}{iCe^{i\pi/4} + 1}.$$

Then the reflected intensity is

$$I_{ref} = \left(\frac{iCe^{i\pi/4} - 1}{iCe^{i\pi/4} + 1}\right)\left(\frac{-iCe^{-i\pi/4} - 1}{-iCe^{-i\pi/4} + 1}\right) = \frac{C^2 + 1 - \sqrt{2}C}{C^2 + 1 + \sqrt{2}C}.$$

It is now a simple matter to find the minimum with respect to C, which occurs when $C^2 = 1$. Taking this solution gives: $I_{ref} = \left(\sqrt{2} - 1\right)^2 = 17.16\%$ as predicted. (Note this also gives $\left|r_{p\,\min}\right| = \sqrt{2} - 1$, and $\sqrt{\varepsilon_i/\varepsilon_1}\cos\theta = 1$, which can be rewritten as $\tan\theta \cong \sqrt{\varepsilon_i/\varepsilon_1}$ and is the same as was obtained when solving the cubic.) The minimum value of 17.16% and the dependence of the angle of

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the minimum on $\sqrt{\varepsilon_i}$ is demonstrated in Fig. 4 in which the reflectivity curves for various values of ε_i are plotted as functions of tan θ (log scale).

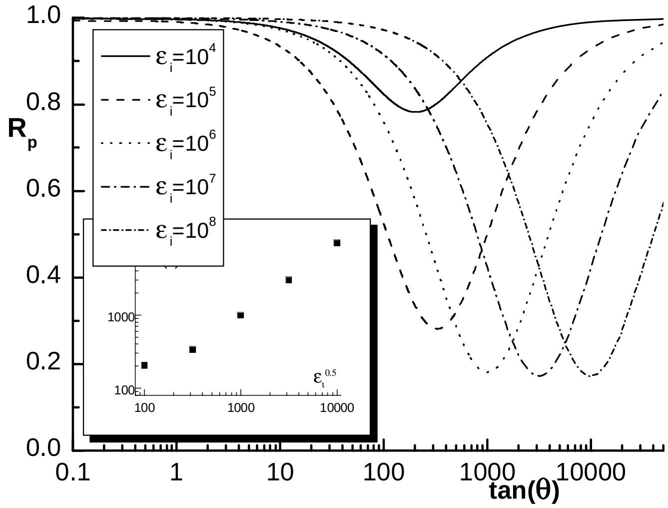

Fig. 4. Reflectivity curves for various values of ε_i as functions of tanθ (log scale) with $\varepsilon_1 = 1$ and $\varepsilon_r = -40000$ (such that for the curve with $\varepsilon_i = 1\times10^7$ the system approximates that for silver at a wavelength of 1cm). The asymptotic limit of the minimum reflectance of 17.16% when $\varepsilon_i >> |\varepsilon_r| >> \varepsilon_1$ is clearly evident. Inset: tan(θ) (log scale) as a function of $\sqrt{\varepsilon_i}$ (log scale) demonstrating the validity of the equation obtained for the angle of the reflectance minimum $\tan\theta \cong \sqrt{\varepsilon_i/\varepsilon_1}$ in this limit.

In summary we have drawn attention to the curious minimum in the p-polarised reflectivity of metals which, at long wavelengths, gives an elegant solution for the reflectivity of $\left(\sqrt{2}-1\right)^2$. It has been shown that, because of the 45° phase difference between the normal and tangential components of ***E*** within a metal at these wavelengths, the reflected field can not be zero since the incident field has no out of phase components to match the fields at the boundary. The minimum reflectivity of $\left(\sqrt{2}-1\right)^2$ follows from this 45° phase difference. It should be noted, however, that this minimum limit on the reflectivity is only true for a single interface planar system. If the interface is structured in some manner this minimum value can be lowered.

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