On the geometry of orientation preserving planar piecewise isometries

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Abstract

Planar piecewise isometries (PWIs) are iterated mappings of subsets of the plane that preserve length (and hence angle and area) on each of a number of disjoint regions. They arise naturally in several applications and are a natural generalization of the well-studied interval exchange transformations.

The aim of this paper is to propose and investigate basic properties of orientation preserving PWIs. We develop a framework with which one can classify PWIs of a polygonal region of the plane with polygonal partition. Basic properties of such maps are discussed and a number of results are proved that relate dynamical properties of the maps to the geometry of the partition. It is shown that the set of such mappings on a given number of polygons splits into a finite number of families; we call these *classes*. These classes may be of varying dimension and may or may not be connected.

The classification of PWIs on n triangles for n up to 3 are discussed in some detail, and several specific cases where n is larger than three are examined. To perform this classification, equivalence under similarity is considered, and an associated *perturbation dimension* is defined as the dimension of a class of maps modulo this equivalence. A class of PWIs is said to be *rigid* if this perturbation dimension is zero.

A variety of rigid and non-rigid classes and several of these rigid classes of PWIs are found. In particular, those with angles that are multiples of π/n for $n = 3$, 4 and 5 give rise to self-similar structures in their dynamical refinements that are considerably simpler than those observed for other angles.

Key words: Piecewise isometry, Discontinuous dynamics, Nonhyperbolic dynamics.

Contents

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1 Introduction

A comprehensive theory for smooth dynamical systems defined by iterated maps or flows has been developed since the 1960s; If the system has a derivative that is nondegenerate (and so expands in some directions and contracts in others) one can apply contraction mapping arguments to show existence of natural ergodic measures, attractors and so on, under suitable assumptions.

For systems devoid of contraction/expansion, these methods do not work and one is often left only with 'bare hands' methods such as geometric constructions. A particular case in point are those of piecewise isometries (PWIs); these are maps that locally have no expansion or contraction. In one dimension, these correspond to the well-studied *interval exchange transformations* [20] (if invertible) or *interval translation maps* if non-invertible [7]. There remain fundamental questions unanswered for such maps; for example, for a typical interval translation map of an interval into a subset of itself, is there a positive Lebesgue measure set that is both forward and backward invariant under the map?

Two-dimensional planar piecewise isometries have been developed as models in several applications and these show rich and beautiful dynamical behaviour under iteration; see for example [18]. We believe the set of PWIs on polygons deserved study for the following reasons.

(a) They are relatively simple to define but nonetheless display a wide variety of dynamics.

(b) PWIs that are non-invertible can reduce to invertible polygonal planar PWIs for the asymptotic dynamics; (see e.g. [24], although the asymptotic dynamics may also occur on a more complicated set, for example the 8-attractors in [13]).

(c) A deeper understanding of the dynamics and bifurcations of PWIs requires an understanding of what mappings may arise in parametrized families of PWIs.

For motivational purposes, we briefly review some examples of maps that can be viewed as planar piecewise isometries.

1.1 Example 1: The overflow oscillation map

The map of $[-1,1)^2$ to itself defined by

$$(x,y) \mapsto (y, g(-x+2y\cos\theta))$$

with $g(x)=x$ for $x \in [-1,1)$ and $g(x+2)=g(x)$ can be viewed as a one-parameter family of PWIs after a simple linear shear of the coordinates parametrized by $\theta \in \mathbf{R}$. This map has been studied by several authors in the electronic engineering and mathematics communities; see for example [1, 3, 4, 9, 10, 19, 21]; there are still many unanswered questions concerning its dynamics; see for example [3, 4, 21]. The sawtooth standard map is also equivalent to this map in a certain parameter region [4].

Figure 1(a) schematically shows the map in sheared coordinates; the phase space is a rhombus composed of two triangles and a hexagon. The dynamics of this defines a partition of the phase space into two sets. The set of points $\mathcal{P}$ that are periodically coded by the regions A, B, C that they visit is a disjoint union of disks foliated with invariant curves whose rotation number is a multiple of θ. These disks have no tangencies for typical θ [6]. The remainder $\mathcal{A}$ (consisting of aperiodically coded points) is a rather mysterious set; it is conjectured that it may have positive Lebesgue measure and that there may be barriers to the diffusion of trajectories throughout this set [3].

1.2 Example 2: A piecewise affine torus map

A closely related map to Example 1 is the piecewise affine map of $[0,1]^2$ studied by Goetz [15]

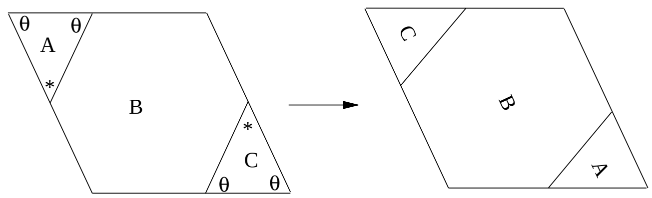
$$(x,y) \mapsto (y, g(-x+2y\cos\theta))$$

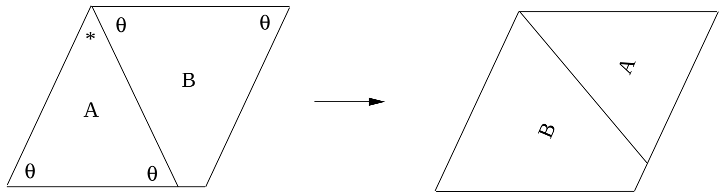
with $g(x)=x$ for $x\in[0,1)$ and $g(x+1)=g(x)$, again with $\theta\in\mathbf{R}$ a parameter. This map is shown in Figure 1(b) after a suitable shear. This phase space is a rhombus composed of a quadrilateral and a triangle. Qualitatively the dynamics of this PWI are very similar to the overflow map with the difference that the invariant sets do not typically possess any symmetry.

1.3 Example 3: Feely's bandpass $\Sigma-\Delta$ modulator

Figure 1(c) shows a PWI that is equivalent to the map

$$\begin{pmatrix} w_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2\cos\theta \end{pmatrix} \begin{pmatrix} w_n \\ t_n \end{pmatrix} + \begin{pmatrix} 0 \\ \mathrm{sgn} w_n - 2\cos\theta \,\mathrm{sgn}\, t_n \end{pmatrix}$$





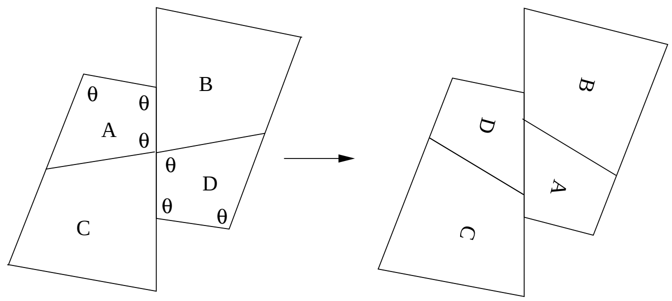


Figure 1: *Three examples of planar piecewise isometries that arise in applications; (a) shows the overflow oscillation map, (b) an piecewise affine map of the torus and (c) a map describing the behaviour of a bandpass $\Sigma - \Delta$ modulator. In each case the letters label the atoms of the partition. On each atom the map acts simply as an isometry, i.e. a rotation and translation. The angle θ in each case is a parameter that can be varied within some range. The $*$ denotes a vertex of symmetry for an isosceles triangle.*

which is the case of quiescent behaviour ($x_n \equiv 0$) with unity gain ($r = 1$) for the iterated mapping

$$w_{n+2} = 2r\cos\theta w_{n+1} - r^2 w_n + 2r\cos\theta(x_{n+1} - \operatorname{sgn} w_{n+1}) - r^2(x_n - \operatorname{sgn} w_n)$$

introduced by Feely and co-workers in [11] for a model of a second order $\Sigma - \Delta$ modulator with a filter in the feedback loop; see [5]. This shows qualitatively similar dynamics to the two other examples.

Other examples of PWIs on other two dimensional manifolds can be found in the recent paper of Scott *et al* [26], and in the work of Tabachnikov and others on dual polygonal billiards [27]. Vivaldi and others have used algebraic dynamics with great success in examining the dynamics of several types of maps that are PWIs with low-order rational rotations [23, 28].

Note that each of three examples above occurs as a one parameter family of piecewise isometries. Clearly there are many ways one could perturb these families while still defining a piecewise isometry. However, the majority of the perturbations will involve one or more of the following:

1. Introducing new atoms to the partition.
2. Introducing addition sides to one or more atom.
3. Destroying the invertibility of the map.

The study here is meant to answer the question of how one can perturb a PWI *without* causing one of the changes 1-3. We will see that, up to equivalence under similarities, Example 1 is part of a two parameter family (see Figure 11) and Example 2 is a one parameter family (see Figure 7 IId) that cannot be perturbed further without one of the changes 1-3 above. Although we do not consider it in any detail here, in [5] we show that Example 3 is part of a family with at least four parameters. However, after identifying symmetrically placed points it reduces to the one parameter family IIa in Figure 9.

1.4 Overview of the paper

This paper aims to give a setting in which one can understand the structure of the set of *orientation preserving invertible piecewise isometries* (PWIs) defined on polygonal partitions. Ultimately the aim is to understand their dynamics, but we claim that a useful classification is a prerequisite for understanding their dynamics.

In particular, we give a setting in which one can ask questions about the behaviour of a 'typical' piecewise isometry. It turns out that this varies considerably, depending on the number of atoms and the number of sides on each polygon.

In Section 2 we define partitions and mappings and discuss the dynamical refinement of the partition induced by the mapping. We give some characterizations of trivial classes of PWIs and discuss reducibility of PWIs to PWIs on subsets of the given set of atoms. We briefly review known results on the complexity and topological entropy of such maps, and raise some open questions.

Section 3 discusses properties of the set of PWIs. In Proposition 4, one of the main results of this paper, we see that the set $\mathrm{PWI}(n,m)$ of piecewise isometries of n atoms that are convex m-gons has the structure of a union of manifolds of differing dimensions; these manifolds we refer to as *classes* of PWIs and their dimensions modulo similarity we refer to as *perturbation dimensions*. We show that the set of possible isometries for a given partition is at most finite, whereas for a "typical" partition it is just the identity map. In other words, the existence of a nontrivial PWI on some partition is a strong constraint on how that partition and map can be deformed while remaining invertible.

In Section 4 we examine the structure of the $\mathrm{PWI}(n,3)$ for small n by exhibiting some classes and discussing their dynamics. We give all piecewise isometries for $n=1$ and 2 and believe we have an exhaustive list for irreducible classes with $n=3$. For $n=4,6$ and 8 we present some examples with interesting dynamical behaviour and embed interval exchange transformations into such classes for larger n. Finally in Section 5 we discuss these results and several open questions concerning the dynamics of PWIs.

2 Partitions and piecewise isometries

We start by introducing the class of partitions we use to define planar piecewise isometries. Throughout, we refer to the plane by its complex parametrization $\mathbf{C}$ and denote two dimensional Lebesgue measure on $\mathbf{C}$ by $\ell(\cdot)$.

2.1 Polygonal partitions and footprints

We define the set $\mathrm{Poly}(m)$ to be the set of bounded open convex polygons in $\mathbf{C}$ with at most m sides ($m \geq 3$). These can be parametrized by ordered sequences of points in the plane that encircle a bounded region in an anticlockwise manner, subject to equivalence under removal of any vertices that lie on straight edges and cyclic permutation of the vertices on the boundary.

The maps we consider are defined on partitions that are a unions of polygons in $\mathrm{Poly}(m)$. More precisely, writing

$$\mathcal{M} = \{M_k\}_{k=1}^{n},$$

with M_k polygons in $\mathbf{C}$ we define the set of partitions that consist of n m-gons to be

$$\mathrm{Part}(n,m) = \{\mathcal{M} \in (\mathrm{Poly}(m))^n \ : \ M_k \cap M_l = \emptyset \ \text{ for all } k \neq l\}.$$

We write the set of all partitions with n polygonal atoms as

$$\mathrm{Part}(n) = \bigcup_{m \geq 3} \mathrm{Part}(n,m).$$

Given a partition $\mathcal{M} \in \mathrm{Part}(n,m)$ we define the *footprint* of this partition on the plane to be the compact subset of the plane

$$\pi(\mathcal{M}) = \bigcup_{k=1}^{n} \overline{M}_k.$$

For any two partitions $\mathcal{M}$ and $\mathcal{N} \in \mathrm{Part}(n,m)$ there is a partial ordering given by *refinement*, i.e. $\mathcal{M} \ll \mathcal{N}$ and we say $\mathcal{M}$ *refines* $\mathcal{N}$ if $M \in \mathcal{M}$ means that $M \subseteq N$ for some $N \in \mathcal{N}$. Clearly, if $\mathcal{M} \ll \mathcal{N}$ then $\pi(\mathcal{M}) \subseteq \pi(\mathcal{N})$.

Moreover, given any two such partitions, we define their *mutual refinement* to be the coarsest refinement that refines both by

$$\mathcal{M} \vee \mathcal{N} = \{M \cap N \ : \ M \in \mathcal{M} \quad \text{and} \quad N \in \mathcal{N}.\}.$$

Observe that $\mathcal{M} \vee \mathcal{N} = \mathcal{N} \vee \mathcal{M}$ and $\mathcal{M} \vee \mathcal{N} \ll \mathcal{M}$ and $\mathcal{N}$.

We refer to elements of the partition $\mathcal{M}$ as the *atoms* of that partition. If $\overline{M}_k \cap \overline{M}_l$ is non-empty for $k \neq l$ we say that these atoms *abut*.

There are a number of ways in which two polygonal atoms A and B can abut, depending on whether the intersection is a point or a line and whether it contains vertices of A and/or B. Note that by convexity at most one edge and two vertices from each atom can be involved in the abuttal, if all vertex angles are not π.

We consider two features to characterize the partitions, firstly the footprint type (based on the number of sides on the footprint) and secondly the graph of abuttals, coloured by type of abuttal. Using these properties we get a set of possible graphs to give a guide as to which partitions are possible.

For $\mathrm{Part}(n,m)$ we distinguish possible types of abuttal between any two of the n atoms, A and B into the five illustrated in Figure 2 up to reflection of the diagram an interchange of the atoms (giving a total of 10 abuttal types). Since there are mn edges and at most $\frac{mn}{2}(m(n-1))$ possible pairs of edges, there are at most $\frac{10}{2}mn(m(n-1))$ different types of partition up to equivalence by linear deformation of the elements while preserving the type of abuttal between each pair of sides. Many of these graphs are not planar and hence not realizable by a partition, hence this is far from being an optimal upper bound on the number of possible abuttal types.

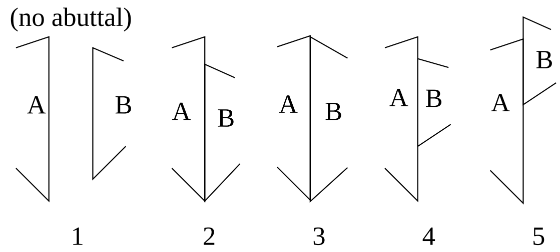


Figure 2: *Five possible ways in which two sides of polygonal atoms A and B in a partition can abut. We disregard abuttals that occur at only one point.*

2.2 Isometries, similarities and piecewise isometries

Consider the *Special Euclidean* group SE(2) of isometries of the plane $\mathbf{C} = \mathbf{R}^2$ that preserve the metric induced by the Euclidean norm $|x+iy| = \sqrt{x^2+y^2}$, and such that the Jacobian of the mapping is unity (preserving orientation). Note that any $g \in \mathrm{SE}(2)$ can be represented as a pair $(g_\theta, g_t) \in S^1 \times \mathbf{C}$ acting on the plane by

$$g(z) = e^{ig_\theta} z + g_t.$$

The group SE(2) (which we will write as $\mathrm{ISO}(\mathbf{R}^2)$ from here on) is a normal subgroup of the group of orientation preserving *similarities* $\mathrm{SIM}(\mathbf{R}^2)$ given by the set of affine mappings f that scale the Euclidean metric by a constant amount $\lambda \in \mathbf{R}^+$ everywhere; namely $\mathrm{ISO}(\mathbf{R}^2) = \ker(\lambda)$ where the λ is the scaling that is a homomorphism $\lambda : \mathrm{SIM}(\mathbf{R}^2) \to (\mathbf{R}^+, \times)$.

For any $\mathcal{M} \in \mathrm{Part}(n,m)$ we define

$$\mathrm{ISO}(\mathcal{M})$$

the set of *invertible piecewise isometries* on $\mathcal{M}$ to be the set of maps of $\pi(\mathcal{M})$ to itself that satisfy PWI_1, PWI_2 and PWI_3 where:

PWI_1: f is an isometry on each $M_k \in \mathcal{M}$.

PWI_2: $f(\mathcal{M})$ is a partition.[1]

PWI_3: The footprint is invariant under f in the sense that $\pi(\mathcal{M}) = \pi(f(\mathcal{M}))$.

Remark 1 *The reason for choosing conditions PWI_2 and PWI_3 rather than any others is that PWI_3 ensures that arbitrarily many iterates of f are defined on a full measure subset of the footprint, and PWI_2 ensures that the map is invertible on a full measure subset.*

We ignore any differences between maps that on the boundary of the partition $\pi(\mathcal{M}) \setminus \cup M_k$. More precisely, we consider the natural metric topology on $\mathrm{PWI}(n,m)$ which says that $(f,\mathcal{M})$ and $(g,\mathcal{N})$ are separated by a distance $d(f,g) = \int |\tilde{f}(x) - \tilde{g}(x)|\, dx$ where $\tilde{f} : \mathbf{C} \to \mathbf{C}$ and $\tilde{g}$ are extensions of the maps to maps of the plane that are the identity outside $\pi(\mathcal{M})$ and $\pi(\mathcal{N})$ respectively.

[1] Note that if $f : \mathbf{C} \to \mathbf{C}$, by $f(\mathcal{M})$ we mean $\{f(M) \ : \ M \in \mathcal{M}\}$.

A consequence is that we identify maps that are are equal on full measure subsets of $\pi(\mathcal{N})$. Restricting to $\mathrm{PWI}(n,m)$ this is the same topology as that induced by the topology of the embedding into $\mathbf{R}^{2nm+3m}$ after identifying parametrizations that give identical maps.

If M is a single polygon then we denote by $\mathrm{ISO}(M)$ the set of isometries that map the single element partition $\mathcal{M} = \{M\}$ to itself. The set of all piecewise isometries can be decomposed into

$$\mathrm{PWI}(n,m) = \{(f,\mathcal{M}) \ : \ \mathcal{M} \in \mathrm{Part}(n,m) \ \text{ and } f \in \mathrm{ISO}(\mathcal{M})\},$$

$$\mathrm{PWI}(n) = \bigcup_{m\geq 3} \mathrm{PWI}(n,m) \quad \text{and} \quad \mathrm{PWI} = \bigcup_{n\geq 1} \mathrm{PWI}(n).$$

Similarly, for any $\mathcal{M} \in \mathrm{Part}(n,m)$ we define $\mathrm{SIM}(\mathcal{M})$ to be the set of *injective* maps of $\cup M_k$ to $\mathbf{C}$ that are similarities on each M_k.

We say a map $(f,\mathcal{M})$ is *non-degenerate* in $\mathrm{PWI}(n,m)$ if it is not equivalent to any $(g,\mathcal{N}) \in \mathrm{PWI}(n-1,m)$ or $\mathrm{PWI}(n,m-1)$. A simple consequence of this is that (a) all atoms are non-empty and (b) at least one atom is a proper m-gon (although not all need to be).

2.3 Invertibility

The assumption that the mapping is injective puts subtle constraints not only on the mapping parameters but also on the geometry of the partition. The following lemma gives three equivalent conditions that PWI_2 and PWI_3 hold.

Lemma 1 *Suppose that* $f : \pi(\mathcal{M}) \to \pi(\mathcal{M})$ *is an* invertible piecewise isometry *with partition* $\mathcal{M} = \{M_k\}_{k=1}^n$*, then following are true:*

(i) f *is injective when restricted to* $\bigcup_{k=1}^n M_k$.

(ii) For any $l \neq k$ *we have*

$$f(M_k) \cap f(M_l) = \emptyset.$$

(iii) The inverse map f^{-1} *can be defined as a PWI with partition* $f(\mathcal{M})$.

Proof Note that (ii) is a direct consequence of the stipulation that $f(\mathcal{M})$ is a partition. (i) Suppose that $x \in f(\mathcal{M})$, then by (ii) there is at most one k such that $x \in f(M_k)$ and we choose $f|_{M_k}^{-1}(x)$. Note that the inverse is uniquely defined on $\cup_k f(M_k)$ (it need not be so on $\pi(\mathcal{M}) \setminus \cup f(M_k)$). (iii) follows from (i) by assigning the inverse map to take arbitrary values on $\pi(\mathcal{M}) \setminus \cup_k f(M_k)$. ∎

2.4 Refinements of piecewise isometries and dynamics

A piecewise isometry $(f,\mathcal{M})$ will restrict to any refinement $\mathcal{N} \ll \mathcal{M}$ with $\pi(\mathcal{M}) = \pi(\mathcal{N})$, i.e. if $\mathcal{N} \ll \mathcal{M}$ with $\pi(\mathcal{M}) = \pi(\mathcal{N})$ then $f \in \mathrm{ISO}(\mathcal{N})$. Even if $\pi(\mathcal{N}) \subsetneq \pi(\mathcal{M})$, f will in certain circumstances induce a first return map on a full measure refinement of $\mathcal{N}$.

Iteration of f generates a sequence of *dynamical refinements*

$$\mathcal{M}^{(n)} = \bigvee_{k=0}^{n} f^{-k}(\mathcal{M})$$

for $n = 0, 1, 2, \cdots$ where $\mathcal{M}^{(n)} \ll \mathcal{M}^{(n-1)}$. We define the *asymptotic refinement*

$$\mathcal{M}^{(\infty)} = \bigvee_{n=-\infty}^{\infty} f^k(\mathcal{M})$$

but this infinite refinement may no longer be understood as a partition in the above sense. More precisely, $\mathcal{M}^{(\infty)}$ is a possibly uncountable collection of subsets of $\mathbf{R}^2$ where $V \in \mathcal{M}^{(\infty)}$ if and only if there is a sequence $\{n_k\}_{k \in \mathbf{Z}}$ with $1 \leq n_k \leq n$ such that

$$V = \bigcap_{k=-\infty}^{\infty} f^{-k}(M_{n_k}). \tag{1}$$

Equivalently, V is the set of points x such that $f^k(x) \in M_{n_k}$.

Each such V we refer to as a *cell* for the mapping, following the notation of Goetz *et al.*. Each cell is convex and can have dimension 0, 1 or 2. Examples 1 and 2 will show that all possibilities for the dimension of cells can occur for planar PWIs.

We partition the asymptotic refinement into a disjoint union of two sets of cells

$$\mathcal{M}^{(\infty)} = \mathcal{P}^{(\infty)} \cup \mathcal{A}^{(\infty)} \tag{2}$$

where a cell V is said to be in $\mathcal{P}$ if and only if it has non-empty interior. Note that following results in Goetz [15] and others, the cells in $\mathcal{P}$ are periodically coded and hence contain only periodic and quasiperiodic points. Also, even though cells in $\mathcal{P}$ have interior they will not typically be open due to a subset of points on the boundary following the same coding[2]. The footprints of these asymptotic refinements correspond to the sets $\mathcal{P}$ and $\mathcal{A}$ mentioned in 1.1.

The union $\bigcup_{V \in \mathcal{M}^{(\infty)}} V$ is a *full measure* partition of $\pi(\mathcal{M})$ into cells. Examples 1 and 2, detailed later on, show that $\mathcal{P}^{(\infty)}$ may be empty.

The asymptotic refinement $\mathcal{M}^{(\infty)}$ gives a characterization of all forward orbits by their itinerary through the atoms of $\mathcal{M}$. Work on specific examples of piecewise isometries indicates that this asymptotic refinement may show a surprising degree of complexity, for example [14, 1]. On the other hand, if $\mathcal{M}^{(\infty)}$ is finite then it is equal to a refinement of the form $\mathcal{M}^{(n)}$ for some n.

Characterization of simple piecewise isometries We now characterize some simple types of piecewise isometries. If $f(\mathcal{M}) = \mathcal{M}$ we say that f *acts by permutation of the atoms* and we see immediately that $\mathcal{M}^{(n)} = \mathcal{M}$ for all n.

If there is an $p > 1$ such that $\mathcal{M}^{(p+1)} = \mathcal{M}^{(p)} \neq \mathcal{M}^{(p-1)}$, then we say f is *preperiodic* with preperiod p and note that it acts by permutation of atoms on the refinement $\mathcal{M}^{(p)}$. Note that $\mathcal{M}^{(\infty)} = \mathcal{M}^{(p)}$.

If $(f, \mathcal{M})$ is such that for some minimal $p \geq 1$ with $f^p(x) = x$ on a full measure of $\pi(\mathcal{M})$ then we say f is *periodic* with period p. If f is not periodic, we say it is *aperiodic* (note that an aperiodic PWI may still be such that all orbits are periodic, as long as the set of minimal periods is unbounded).

[2] We are grateful to M. Mendes for pointing this out

2.5 Permutation of atoms

Suppose that $f \in \mathrm{PWI}(n,m)$ acts by permutation of its atoms $\{M_k\}_{k=1}^{n}$. Then there is a $\sigma \in S_n$ such that $f(M_k) = M_{\sigma(k)}$ for all k, where S_n is the group of permutations of $\{1, \cdots, n\}$. Note that $\sigma \in S_n$ induces a partition of $\{1, \cdots, n\}$ into *orbit classes* given by the equivalence $i \sim j$ if $\sigma^k(i) = j$ for some k. We write I to be a minimal set of representatives from the orbit classes. For any i we denote by $k(i)$ the minimum $k > 0$ such that $f^{k(i)}(M_i) = M_i$. Note that $\sum_{i \in I} k(i) = n$.

The dynamics of PWIs acting as permutations can then be characterized by the following Proposition.

Proposition 1 *Suppose that $f \in PWI(n,m)$ acts as a permutation on its atoms with $\sigma \in S_n$. Then the following are true:*

(i) Within each orbit class the atoms M_i are isometrically related.

(ii) The map $f^{k(i)}|_{M_i}$ is an $r(i)$-fold rotation for some $0 \le r(i) \le m$.

(iii) f is periodic with period that divides $p = lcm\{k(i)r(i) \ : \ i \in I\}$.

Proof (i) follows from the fact that $f|_{M_i} : M_i \to M_{\sigma(i)}$ is an isometry; by induction, there is an isometry between any two atoms within the same orbit class. (ii) is because the m-gon M_i is invariant under the orientation preserving isometry $g = f^{k(i)}|_{M_i}$. Hence g must be a rotation with order $r(i)$ that is at most m. (iii) follows by noting that for any i, the action of $f^{k(i)}$ is an $r(i)$-fold rotation and so all points in M_i have period $k(i)r(i)$ or $k(i)$ (the centre). If we take the lowest common multiple of all of these quantities, this iterate will send all atoms back to themselves by the identity transformation. Note that the period of the map can be seen to be bounded above by $m\phi(n)$ where $\phi(n)$ is the lowest common multiple of all integers less than or equal to n. ∎

In other words, the dynamics of such an f is determined by the cycle type of the permutation σ and the sets of integers $k(i)$ and $r(i)$, up to scaling within the orbit classes and arbitrary non-intersecting placement of the atoms.

In particular, all permutation PWIs are periodic; the following Proposition implies that the converse is also true if one chooses a suitable refinement. It also demonstrates that periodicity on a refinement and preperiodicity are essentially the same property.

Proposition 2 *Suppose that $(f, \mathcal{M})$ is a piecewise isometry; f is periodic if and only if there is a refinement $\mathcal{N} \ll \mathcal{M}$ such that f is a permutation on $\mathcal{N}$. Moreover, a piecewise isometry $(f, \mathcal{M})$ is p-periodic if and only if it is q-preperiodic for some $q \le p$.*

Proof If there is such a refinement, we apply Proposition 1. Conversely, suppose that f is periodic with period p. Then $\mathcal{M}^{(p)} = \mathcal{M}^{(p+1)}$ and so $f(\mathcal{M}^{(p)}) = \mathcal{M}^{(p)}$; thus $\mathcal{M}^{(p)}$ is a refinement on which f acts as a permutation.

To demonstrate the second statement, suppose that f is p-periodic. Then $f^{p+n}(\mathcal{M}) = f^n(\mathcal{M})$ and so $\mathcal{M}^{(p)} = \mathcal{M}^{(p+1)}$, hence it is q-preperiodic for some $q \le p$. Conversely, suppose it is q-preperiodic. Then f^q acts as an permutation on $\mathcal{M}^{(q)}$. Hence it is periodic for some $p \ge q$ by Proposition 1. ∎

We remark that we are unsure as to whether q in Proposition 2 may in fact be taken to be a factor of p.

2.6 Reducibility of PWIs

Suppose that a piecewise isometry $(f, \mathcal{M}) \in \mathrm{PWI}(n, m)$ is such that for some proper subset $\mathcal{N} \subsetneq \mathcal{M}$ there is *first return map*

$$f_{\mathcal{N}}(x) = \left\{ f^{k(x)}(x) \;:\; k(x) = \min_{k \geq 1}\{k \;:\; f^k(x) \in \pi(\mathcal{N})\} \right\}$$

with the return time $k(x)$ bounded, such that $f_{\mathcal{N}}$ is in $\mathrm{PWI}(\mathcal{N})$, i.e. it is a piecewise isometry on the partition $\mathcal{N}$. Then we say that the map is *reducible* to a piecewise isometry on $\mathcal{N}$ and note that $\mathcal{N} \in \mathrm{Part}(k, m)$ for some $k < n$. If it is not reducible on any proper subset of the set of atoms, we say it is *irreducible*. Clearly by induction, successive reduction of a reducible map will terminate with an irreducible first return map (on a sub-partition of the original that contains at least one atom).

Reducibility is defined relative to a particular $\mathrm{PWI}(n, m)$. In fact, a map that is be irreducible with respect to some $\mathrm{PWI}(n, m)$ will not be irreducible in $\mathrm{PWI}(n', m)$ for large enough $n' > n$.

Remark 2 *If $(f, \mathcal{M}) \in PWI(n)$ for $n \geq 2$ is reducible on some proper subset $\mathcal{N}$ of $\mathcal{M}$ and if the first return time to $\mathcal{N}$ is constant, then the map may be reducible onto $\mathcal{N}^c = \mathcal{M} \setminus \mathcal{N}$. However, more generally the first return map on $\mathcal{N}^c$ is not a map in $PWI(\mathcal{N}^c)$. All permutation PWIs on more than one atom are reducible; however a PWI that is not a permutation may or may not be reducible.*

Remark 3 *Given any irreducible map $(f, \mathcal{M})$ it is possible to construct many maps that reduce to this map as follows. Take one atom, say M_1 and make an isometric copy, M. Define $\tilde{\mathcal{M}} = \mathcal{M} \cup \{M\}$ and a map by $\tilde{f} = f$ on $\mathcal{M} \setminus M_1$, $\tilde{f}(M_1) = M$, $\tilde{f}(M) = f(M_1)$. Then $\tilde{f}$ reduces to f on $\mathcal{M}$.*

Remark 4 *We conjecture that if $\mathcal{M}^{(\infty)}$ has cells with arbitrary high period then it must necessarily contain aperiodic cells and f is aperiodic. For example, [1, 19] have studied an example of a PWI where* almost all *orbits are periodic, but there is still a set of aperiodic cells that has Hausdorff dimension approximately* 1.24.

2.7 Topological entropy and complexity

The topological entropy of a PWI is the growth rate of the symbolic language generated by the dynamics, i.e. the exponential rate of growth of the size of partition $\mathcal{M}^{(n)}$ as n increases

$$h_{top} = \lim_{n \to \infty} \frac{1}{n} \log |\mathcal{M}^{(n)}|.$$

Buzzi [8] has recently shown that $h_{top} = 0$ for PWIs in arbitrary dimensions. This general result supports the conjecture that the growth of $|\mathcal{M}^{(n)}|$ is polynomial for PWIs. There is also numerical evidence [25] that the growth is polynomial for typical piecewise isometries, namely it seems that the *complexity*, a constant $p \in \mathbf{R}$ such that

$$p(f) = \lim_{n \to \infty} \frac{\log |\mathcal{M}^{(n)}|}{\log n}$$

exists and is finite. Note that any PWI that is periodic will have $p = 0$; Examples 1 and 2 show that one can find $p = 1$ and 2 respectively.

Example 1 *A PWI of* $[0,1]^2$ *of the form* $(x,y) \mapsto (x+\alpha(\bmod 1), y)$ *has complexity* $p=1$ *for any irrational* α. *Note that in this example* $\mathcal{P}^{(\infty)}$ *is empty and* $\mathcal{A}^{(\infty)}$ *is an uncountable number of line segments with endpoints not included.*

Example 2 *A PWI of* $[0,1]^2$ *with* $(x,y) \mapsto (x+\alpha \ (\bmod 1), y+\beta \ (\bmod 1))$ *has complexity* $p=2$ *if* α *and* β *are independent irrationals. This example has* $\mathcal{P}^{(\infty)}$ *empty and* $\mathcal{A}^{(\infty)}$ *is an uncountable number of points.*

Moreover, we have the following lemma which is applicable to general PWIs. Although it may be well known in certain circles, we have not seen this result in the literature and so include it here.

Lemma 2 *For any piecewise isometry* $f: X \to X$ *on finite partition* $\mathcal{M}$ *of* X *of any dimension,* $p(f) \notin (0,1)$.

Proof Consider the dynamical refinements $\mathcal{M}^{(n)}$ of f. If there exists some p such that $|\mathcal{M}^{(p+1)}| = |\mathcal{M}^{(p)}|$, then $\mathcal{M}^{(p+1)} = \mathcal{M}^{(p)}$, and f is preperiodic with preperiod less than or equal p. Therefore $p(f)=0$. If for all $n \geq 0$, $|\mathcal{M}^{(n+1)}| \neq |\mathcal{M}^{(n)}|$, then

$$|\mathcal{M}^{(n+1)}| \geq |\mathcal{M}^{(n)}| + 1, \ \forall n \geq 0,$$

therefore $|\mathcal{M}^{(n)}| \geq n + |\mathcal{M}|$, this implies $p(f) \geq 1$ and so there is no PWI such that $0 < p(f) < 1$. ∎

An interval exchange transformation (IET) is 1-dimensional piecewise isometry, and it can be embedded into a planar piecewise isometry (see Section 4.7 later). For these 1-dimensional piecewise isometries, we have

Proposition 3 *The complexity of an IET is either 0 or 1. More precisely, for an IET* $f, p(f) = 0$ *if and only if* f *is preperiodic; therefore* $p(f) = 1$ *if and only if* f *is non-preperiodic.*

Proof To see this, consider the dynamical refinements $\mathcal{M}^{(n)}$ of an IET f of N intervals. From [20] (Page 472), $\mathcal{M}^{(n)}$ consists of at most $(n+1)(N-1)+1$ intervals and exactly this many if there are no saddle connections, as it is generic, i.e., it has no rigid intervals. Therefore we have

$$|\mathcal{M}^{(n)}| \leq (n+1)(N-1)+1,$$

which implies $p(f) \leq 1$. By Lemma 2, we have $p(f)=0$ or 1. By the arguments in the proof of Lemma 2, it is clear that $p(f)=0$ if and only if f is preperiodic; or in other words, $p(f)=1$ if and only if f is aperiodic. ∎

Numerical experiments of Poggiaspala and others [25] seem to suggest that the complexity is in fact always less than or equal to 2 for planar piecewise isometries. For the more general case of PWIs in $\mathbf{R}^d$, one might conjecture that the complexity is always less than or equal to d. However a clear understanding of $p(f)$ is, to the knowledge of the authors, still lacking for piecewise isometries.

The following result relates reducibility and complexity of PWIs.

Lemma 3 *Suppose that* $(f, \mathcal{M})$ *is reducible to some* $(g, \mathcal{N})$ *with* $\mathcal{N} \ll \mathcal{M}$ *and* g *is the first return map of* f *to* $\mathcal{N}$. *Then* $p(f) \geq p(g)$.

Proof Consider any A and B elements in the partition $\mathcal{N}^{(n)}$ induced by g. Suppose that the first return to $\mathcal{N}$ occurs after k_i iterates of f, $i = 1, 2, \ldots, n, \ldots$, and $k_i \leq K$. Then any point $a \in A$ and $b \in B$ must have different itineraries under f after considering at most nK iterates of f. i.e. we have $|\mathcal{M}^{(N)}| \geq |\mathcal{N}^{(n)}|$, where $N = \sum_{i=1}^{n} k_i$. Therefore we have

$$\begin{aligned} p(f) &= \lim_{n\to\infty} \frac{\log|\mathcal{M}^{(N)}|}{\log N} \geq \lim_{n\to\infty} \frac{\log|\mathcal{N}^{(n)}|}{\log N} \\ &\geq \lim_{n\to\infty} \frac{\log|\mathcal{N}^{(n)}|}{\log n}\left(1 + \frac{\log K}{\log n}\right)^{-1} \\ &= \lim_{n\to\infty} \frac{\log|\mathcal{N}^{(n)}|}{\log n} = p(g). \end{aligned}$$

∎

We conjecture that the following is true, but have as yet not been able to prove this. If this is true, then there is an irreducible 'core' for any PWI that generates all the complexity of the shift associated with it.

Conjecture 1 *Suppose that $(f, \mathcal{M}) \in PWI(n, m)$. Then there is an irreducible $(g, \mathcal{N})$ reduction of $(f, \mathcal{M})$ such that*

$$p(f) = p(g).$$

3 Classes of PWIs and similarity

We state some results about the basic structure of the set of PWIs. Because the inverse of a PWI is also a PWI, they form a group under composition. Clearly PWI(n) $\subset$ PWI(k) and also PWI(n, m) $\subset$ PWI(k, l) whenever $n \leq k$ and $m \leq l$. Moreover, composition of PWIs gives an increase in the number of atoms characterized by the following Lemma.

Lemma 4 *Consider $(n, m) \in \mathbf{N}^2$*

(i) If $(f, \mathcal{M}) \in PWI(n)$ and $(g, \mathcal{N}) \in PWI(m)$ with $\pi(\mathcal{M}) = \pi(\mathcal{N})$, then $(f \circ g, \mathcal{M}) \in PWI(mn)$.

(ii) If $(f, \mathcal{M}) \in PWI(n)$ then for any $\alpha > 0$ there is a $C(\alpha)$ such that $(f^k, \mathcal{M}) \in PWI([C(\alpha)n^{\alpha|k|}])$ for all $k \in \mathbf{Z}$. In other words, $\mathcal{M}^{(k)} \in Part([Cn^{\alpha|k|}])$ for all k.

(iii) There is an embedding $PWI(n, k) \subseteq PWI(n(k-2), 3)$.

Proof To see (i), suppose that $\mathcal{M} = \{M_i\}$ and $\mathcal{N} = \{N_i\}$. The set $\{N_i \cap g^{-1}(M_j)\}$ is a partition for $f \circ g$ where $i = 1..m$ and $j = 1..n$. Hence there are at most nm atoms for the composition (some of these may be empty). The proof of (ii) follows directly from the result of Buzzi [8] that the topological entropy is zero (it is easy to observe that for $\alpha = 1$ one can set $C = 1$). Given any partition into n atoms each of which is a polygon with at most k sides, we can triangulate all the polygons by at most $k - 2$ triangles each and so (iii) follows. ∎

Remark 5 *If a PWI has finite complexity, $C(\alpha)$ can be chosen to be arbitrarily small for large enough k. If it is possible to have a PWI with infinite complexity the scaling $C(\alpha)$ (for large k) will contain interesting information about the asymptotic growth in size of the partition.*

3.1 A parametrization of PWI(n, m)

For fixed m and n, the set of possible $(f, \mathcal{M}) \in \mathrm{PWI}(n, m)$ is a finite dimensional space. The set of possible partitions for such a map can be parametrized by the coordinates of the vertices of the polygons and so $\mathcal{M} \in \mathrm{Part}(n, m)$ can be parametrized by a vector in $\mathbf{R}^{2nm}$. The map itself can be expressed using an angle θ and a translation (x, y) for each atom meaning that $\mathrm{PWI}(n, m)$ can be embedded into $\mathbf{R}^{2nm+3n}$. The requirement that M is non-overlapping and that the isometry is invertible on M provides many constraints, meaning that this embedding of $\mathrm{PWI}(n, m)$ will typically have very high codimension in $\mathbf{R}^{2nm+3n}$.

We say two mappings $(f, \mathcal{M})$ and $(g, \mathcal{N})$ in PWI are *similar* if there is a $\sigma \in \mathrm{SIM}(\mathbf{R}^2)$ that conjugates them, i.e. such that $f(\sigma(x)) = \sigma(g(x))$ for all $x \in \cup_k M_k$; see the following Lemma.

Lemma 5 *The group of similarities SIM$(\mathbf{R}^2)$ acts on PWI(n, m) by conjugation. This action has finite kernel, i.e. if $(f, \mathcal{M}) \in$ PWI(n, m) then*

$$\{\sigma \in \mathit{SIM}(\mathbf{R}^2) \ : \ f\sigma = \sigma f\}$$

is a finite group (it is a cyclic or dihedral group).

Proof First, we note that if $\sigma \in \mathrm{SIM}(\mathbf{R}^2)$ then $g = \sigma^{-1} f \sigma$ maps $\sigma^{-1}(\mathcal{M})$ to itself. Since σ preserves polygons, $\sigma^{-1}(\mathcal{M}) \in \mathrm{Part}(n, m)$. For any $M_i \in \mathcal{M}$ we have $g|_{\sigma^{-1}(M_i)}$ is an isometry and hence $(g, \sigma^{-1}(\mathcal{M})) \in \mathrm{PWI}(n, m)$.

Now suppose that $\sigma^{-1} f \sigma = f$ with $(f, \mathcal{M}) \in \mathrm{PWI}(n, m)$. This means that $\sigma(\mathcal{M}) = \mathcal{M}$ and so σ is a similarity that can only permute the atoms in $\mathcal{M}$. It must therefore be an isometry and has finite order as $\pi(\mathcal{M})$ is a bounded polygon. Hence only a finite number of σ will commute with f. This group is the group of symmetries of a bounded polygon and hence it is cyclic or dihedral. ∎

As a consequence of this lemma, if we consider $\mathrm{PWI}(n, m)$ under conjugation by similarities, passing through each $(f, \mathcal{M})$ there are four-dimensional group orbits under $\mathrm{SIM}(\mathbf{R}^2)$ corresponding to the action of the group $\mathrm{SIM}(\mathbf{R}^2)$. These correspond to translation, rotation and scaling of the partition as a whole.

Proposition 4 *The set PWI$(n, 3)$ has the structure of a stratified space, i.e. it is the disjoint union of a finite number of manifolds.*

Proof We first show that the set $\mathrm{PWI}(n, 3)$ can be described as a semialgebraic set embedded in some $\mathbf{R}^N$ and then use results in [22] to obtain the conclusion. Recall that a subset of $\mathbf{R}^n$ is *semialgebraic* if it can be defined by finite intersection and union of a number of sets that are defined by algebraic inequalities of the form $p(x) \leq 0$ where $p(x)$ is a polynomial function of its variables.

For $(f, \mathcal{M}) \in \mathrm{PWI}(n, 3)$ we consider the locations of the kth vertex of the ith atom by $\{x_k^i\}_{k=1..n}^{i=1..3}$ with each $x_k^i \in \mathbf{R}^2$ and embed these in $\mathbf{R}^3$ by taking the last component as zero.

Without loss of generality we assume that $[x_k^2 - x_k^1, x_k^3 - x_k^1, (0,0,1)] > 0$ meaning that the triangles are oriented in the same direction[3]. The triangles k and l overlap precisely if there exist i and j in $\{1,2,3\}$ and $0 < \alpha_{ij}, \beta_{ij} < 1$ such that

$$\alpha_{ij} x_k^i + (1-\alpha_{ij}) x_k^{i+1} = \beta_{ij} x_l^j + (1-\beta_{ij}) x_l^{j+1}$$

where the superscripts are taken modulo three. We now parameterize the transformations $z \mapsto e^{i\theta} z + w$ by setting $e^{i\theta} = R + iS$, $w = U + iV$ and requiring that $R^2 + S^2 = 1$. The transformed atoms have vertices that are affine combinations of the old ones with the transformations. In particular they are algebraic functions of the parameters x_k^i and the transformations as expressed as above. These are not permitted to intersect each other (a semi-algebraic condition), and moreover they are not permitted to intersect the polygonal complement of M (also a semi-algebraic condition). Hence the set PWI$(n,3)$ can be viewed as a semi-algebraic subset of

$$\mathbf{R}^{10n} = \{x_k^1, x_k^2, x_k^3, R_k, S_k, U_k, V_k\}_{k=1}^n$$

where we identify maps that are same up to permutation of vertices and atoms. Results on semialgebraic sets [22], see eg [12, p128] then imply that this embedding of PWI$(n,3)$ is a Whitney stratified subset of $\mathbf{R}^{10n}$. By taking the canonical stratification of PWI$(n,3)$ (i.e. the stratification with the largest possible strata of given dimension) this stratification is unique. ∎

Note that Lemma 4(iii) implies that we can embed PWI(n,m) inside some PWI$(n',3)$ and hence get a similar result for PWI(n,m). Each connected stratum of this canonical stratification we refer to as a *class* within PWI(n,m).

Remark 6 *Observe that the parametrization of some $(f, \mathcal{M}) \in PWI(n,3)$ by a vector in $\mathbf{R}^{10n}$ is unique up to (a) cyclic permutation of the coordinates on each triangle and (b) permutation of the atoms. Hence, maps that are equivalent to each other by this finite group of such symmetries are parametrizations of the same map. Forming the quotient by this finite group will not destroy the stratified structure of $PWI(n,3)$ (see Lemma 6 for a more precise statement).*

The remainder of the paper is concerned with understanding the behaviour of these classes of maps. In particular we see that

(a) PWI(n,m) has a number connected components and classes within the connected components. The number of these grows with n and m.

(b) The classes can meet in non-trivial ways.

(c) The geometry of the class determines the range of possible dynamics within that class.

3.2 Conjugacy up to similarity and perturbation dimension

If f and g are similar then their domains and atoms are similar by the same similarity transformation. Moreover, their dynamics are the same in that they have the same order

[3]Where $[a,b,c]$ is the usual triple scalar product in $\mathbf{R}^3$.

periodic orbits and there is a one-to-one map between the possible symbol sequences (which track the itineraries through the partitions) for these orbits, except those orbits that intersect a boundary of an atom.

Given some $(f, \mathcal{M}) \in \mathrm{PWI}(n, m)$, we define the *perturbation dimension* of f within $\mathrm{PWI}(n, m)$ to be

$$\dim_p(f, \mathcal{M}) = \dim(S/\mathrm{SIM}(\mathbf{R}^2)) \tag{3}$$

where S is the stratum in the canonical stratification of $\mathrm{PWI}(n, m)$ containing f. This corresponds to the dimension of this stratum modulo similarity. Since every $(f, \mathcal{M})$ is nondegenerate in some $\mathrm{PWI}(n', m')$, one can clearly define a perturbation dimension for every such map. Note also that an equivalent map can have different perturbation dimensions for different m, n.

Lemma 6 *The perturbation dimension is well-defined for the set of all nondegenerate maps in $PWI(n, m)$.*

Proof Consider any nondegenerate $(f, \mathcal{M}) \in \mathrm{PWI}(n, m)$. We choose any atom $M_1 \in \mathcal{M}$ that is a non-trivial m-gon. The action by $\mathrm{SIM}(\mathbf{R}^2)$ on this atom, and hence on the whole $\mathcal{M}$ is fixed-point free (i.e. $\sigma M_1 = M_1$ for some similarity if any only if σ is the identity). Hence, if S is the stratum in the canonical stratification of $\mathrm{PWI}(n, m)$ containing f, we have

$$\dim_p(f) = \dim(S/\mathrm{SIM}(\mathbf{R}^2)) = \dim(S) - \dim(\mathrm{SIM}(\mathbf{R}^2)) = \dim(S) - 4.$$

∎

Remark 7 *One could similarly define a weaker notion of similarity where f and g are 'weakly similar' if there is a* piecewise *similarity that conjugates them. Under the notion of weak similarity, we could identify piecewise isometries where different components of $\pi(\mathcal{M})$ are moved around using different similarities. However, we do not investigate this in detail here as the two notions are equivalent for the irreducible PWIs considered here.*

Rigid PWIs We say $(f, \mathcal{M})$ is *rigid in* $\mathrm{PWI}(n, m)$ if

$$\dim_p(f, \mathcal{M}) = 0.$$

Clearly, a rigid class is closed and hence disconnected from all other classes in $\mathrm{PWI}(n, m)$. Note that rigidity in $\mathrm{PWI}(n, m)$ does not imply rigidity for $\mathrm{PWI}(n', m')$ for any $n' > n$, $m' \geq m$. We can define an arbitrary $f \in \mathrm{PWI}$ as *rigid* if it is rigid within some $\mathrm{PWI}(n, m)$.

One might at first think that rigidity is such a strong constraint that only very simple PWIs (for example rotations of regular $n-1$-gons in $\mathrm{PWI}(n, 3)$) are rigid. However this is not the case and we will see there are many different rigid classes in $\mathrm{PWI}(2, 3)$ and $\mathrm{PWI}(3, 3)$.

3.3 Possible PWIs on given partition

The next result shows that the set of isometries with a given partition is finite.

Proposition 5 *For any n and m and $\mathcal{M} \in Part(n, m)$ the set $ISO(\mathcal{M})$ is finite. Hence the dimension of the set $PWI(n, m)$ is at most $2nm$. Moreover, there is an open dense set of $\mathcal{M} \in Part(n, m)$ such that the only PWI in $ISO(\mathcal{M})$ is the identity map.*

Proof Given any $\mathcal{M} \in \mathrm{Part}(n,m)$, consider the graph with vertices M_k and an edge from M_k to M_l if $\dim(\overline{f(M_k)} \cap \overline{f(M_l)}) = 1$, i.e. if they have a common edge. Note that firstly, there are only finitely many possible distinct graphs on the n vertices. Secondly, if any image is perturbed either a tangency will be destroyed, an overlap will be created or images will lie outside $\pi(\mathcal{M})$ (note that this requires that $\pi(\mathcal{M})$ is compact and therefore bounded by a polygonal curve). Hence one cannot perturb the map while retaining the same graph of abuttals. Finally, any $\cup_k f(M_k)$ is a polygon that can be fitted into $\pi(\mathcal{M})$ in only a finite number of ways given by $\mathrm{ISO}(\pi(\mathcal{M}))$.

To obtain the dimension bound, note we can parameterize $\mathrm{Part}(n,m)$ using the $2nm$ coordinates of the vertices and we have shown that there are only finitely many possible PWIs with that partition.

For the last part, suppose we have a partition $\mathcal{M}$ with $M = \pi(\mathcal{M})$; an arbitrarily small scaling perturbation can be applied to all elements to disconnect all elements in the partition. A second arbitrarily small perturbation can be applied to ensure that no two atoms have a side with the same length and hence the only invertible map must fix every atom. Clearly being disjoint and asymmetric is an open condition and so the proof is complete. ∎

Remark 8 *If the partition is mapped to itself under an element in $ISO(M)$ then this symmetry is also a PWI. The previous result relies on the fact that we only allow bounded polygonal atoms; simple examples show that $ISO(M)$ need not be finite if atoms have curved boundaries and/or if they are unbounded.*

Proposition 5 shows that the geometry of the PWI is essentially given by the structure of the partition, up to a finite number of cases. It also shows that for most partitions with a given footprint, $M = \pi(\mathcal{M})$ the only PWIs will be $\mathrm{ISO}(M)$ and so for most M the only PWIs are the identity.

4 Classification of PWI$(n, 3)$

We now consider the classification of $\mathrm{PWI}(n,3)$, concentrating on understanding the classes for small n. We are especially interested in those classes that include maps that are irreducible and aperiodic, i.e. such that no part reduces to a permutation on a finer partition (see Proposition 2).

We focus on $(f,\mathcal{M}) \in \mathrm{PWI}(n,3)$ that are irreducible within $\mathrm{PWI}(n,3)$. Clearly, those that are reducible can be understood in terms of elements in $\mathrm{PWI}(n-1,3)$ etc. For larger n we see that a classification of the classes in $\mathrm{PWI}(n,3)$ very quickly becomes a daunting problem in combinatorial geometry. Recall also that the classification for arbitrary $\mathrm{PWI}(n,m)$ can be studied by embedding it into $\mathrm{PWI}(n',3)$ using Lemma 4(iii).

We start by giving a result concerning the perturbation dimensions of the permutation classes.

Lemma 7 *Suppose that $f \in PWI(n,3)$ acts as a permutation σ on its atoms, and that there are k orbit equivalence classes for σ. Then*

$$3n + k - 4 \leq \dim_p \leq 3n + 3k - 4.$$

There are classes that realize each of these bounds.

Proof We have n triangular atoms that can be divided into $k \leq n$ groups such that f acts transitively within each group. Since each atom can be rotated and translated as long as intersections are avoided, and each group can be scaled arbitrarily, the class containing f must have at least $3n-3$ degrees of freedom from placement of the atoms and $k-1$ degrees of freedom from scaling of the groups; hence the lower bound. The only degrees of freedom left come from two angles that can be used to specify each triangular atom modulo similarity. Hence the upper bound has $2k$ more dimensions than the lower bound.

To obtain a class that realizes the lower bound, we map each atom so that the first return to each triangle is a rotation by $2\pi/3$; this means that all triangles are equilateral. To obtain a class that realizes the upper bound, we map each atom so that the first return to each triangle is the identity map. ∎

4.1 Construction of PWIs with isosceles atoms

As a slight diversion, we give a Lemma that we use to construct many classes of nontrivial PWIs that are not permutations of the atoms.

Lemma 8 *Suppose that a partition $\mathcal{M} \in Part(n,3)$ is such that*

(a) all triangles in $\mathcal{M}$ are isosceles, and

(b) there is a reflection κ such that $\kappa(\pi(\mathcal{M})) = \pi(\mathcal{M})$.

Then there is a piecewise isometry $(f, \mathcal{M})$ such that $f(\mathcal{M}) = \kappa(\mathcal{M})$. Moreover, for any continuously parametrized family of such partitions where all triangles are isosceles, one can find a continuous family of f that are piecewise isometries on this family of partitions.

Proof We define f on each $M_k \in \mathcal{M}$ by requiring that $f(M_k) = \kappa M_k$. Since M_k is isosceles, it can be mapped onto $\kappa(M_k)$ by a uniquely defined isometry and we are done (if M_k is equilateral, note that there are three possible choices for this isometry). For any family of such partitions, if the atoms change continuously (in the Hausdorff metric) then the isometries connecting them can be chosen to vary continuously. ∎

Remark 9 *Suppose that a partition $\mathcal{M}$ satisfies the hypotheses of Lemma 8.*

(a) If there is more than just one reflection symmetry and/or rotational symmetries of $\pi(\mathcal{M})$, these can be composed with the PWI to obtain several more nontrivial PWIs with the same partition.

(b) If there are atoms in $\mathcal{M}$ that are isometric then these can be permuted to give further nontrivial PWIs. (c) If a family of such partitions has a triangle that is equilateral it is possible that the isometries cannot be chosen continuously; this is because the vertex of symmetry of the isosceles triangle may change discontinuously at that point.

Lemma 8 and (a) and (b) above allow one to construct a large number of classes of piecewise isometries, but not all. For example, in $PWI(3,3)$ one can find non-trivial isometries that contain scalene triangles. Moreover, it is not necessary to have footprints that have a line of reflectional symmetry, as we see in $PWI(3,3)$.

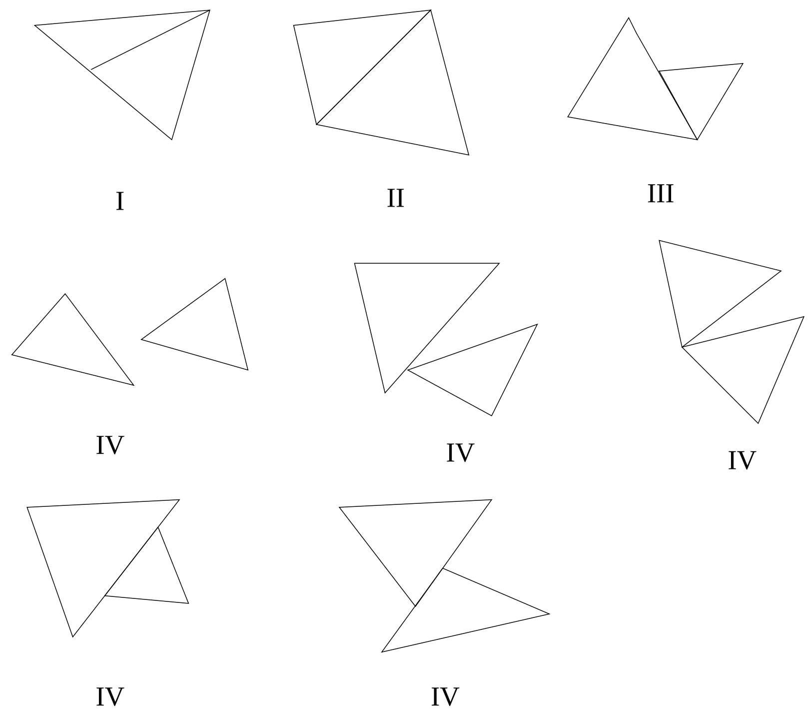


Figure 3: *Seven possible footprints for two triangular atoms shown schematically. The type labelled as IV all have a footprint that is a hexagon and they support only piecewise isometries that are permutations, shown in Figure 4. Types I-III support, in addition to the permutations, a number of nontrivial PWIs that are shown in Figure 5.*

4.2 PWIs on a single triangle.

The maps in PWI$(1,3)$ are simply the isometries that map a triangle to itself. Since any triangle is parametrized by two angles α and β (modulo similarity) one can see that if $\alpha=\beta=\pi/3$ then there are two possible PWIs given by rotation by 0, and $2\pi/3$ (or its inverse) whereas for any other triangle there is only the identity. Hence $\dim_p(f)=2$ for the identity and the nontrivial class is rigid. PWI$(1,3)$ is the disconnected union of these two classes.

Similarly, consider PWI$(1,n)$ parametrized by $n-1$ angles modulo a similarity. For all n-gons, the identity map has $\dim_p = n-1$ whereas all other classes are rigid and periodic.

4.3 Two triangular atoms

The next most simple case is PWI$(2,3)$; we give a full classification for this case. We classify the transformations in PWI$(2,3)$ firstly by properties of footprint and then examine the possible transformations on those atoms. Figure 3 schematically shows all possible abuttal properties of two triangles; there are seven possibilities in all up to interchange of the atoms. We classify these into *footprint types* I-IV depending on the number of sides on the footprint (starting with type I for a triangular footprint, type II for quadrilaterals etc).

The hexagonal footprint types IV will only support piecewise isometries that are per-

Case	Triangle A	Triangle B	mapping	$\dim_p$
IVa	$(\alpha, \beta, 1)$	(γ, δ, a)	identity on A and B	8
IVb	$(\alpha, \beta, 1)$	$(\alpha, \beta, 1)$	exchange A and B, identity on return	5
IVc	$(\pi/3, \pi/3, 1)$	(α, β, a)	rotate A by $\pm\pi/3$, identity on B	6
IVd	$(\pi/3, \pi/3, 1)$	$(\pi/3, \pi/3, a)$	rotate both by $\pm\pi/3$	4
IVe	$(\pi/3, \pi/3, 1)$	$(\pi/3, \pi/3, 1)$	exchange such that rotated by $\pm\pi/3$ on return	3

Table 1: *The six permutation classes in PWI*$(2, 3)$*, illustrated in Figure 4. In all cases, three of the perturbation dimensions come from the arbitrary placing of triangle B relative to triangle A. Observe that the perturbations dimensions verify the bounds given in Lemma 7.*

mutations of the atoms; this is because the footprint has six non-trivial vertices and we must map each atom such that a vertex is in one of the vertices of the footprint. These permutations are detailed in Table 1 and illustrated in Figure 4. In the table we parameterize a triangle by the triple (α, β, a) where α and β are the internal angles on a side of length $a > 0$. Note that for these permutations

(i) All atoms of the partition are mapped to themselves either by f or the second iterate of the map.

(ii) On return of an atom to itself, the only rotations allowed are 0, $\pi/3$ and $2\pi/3$.

(iii) The classes are uniquely determined by the minimum number of iterates to return both elements to themselves and the rotation each element experiences on its first return. All of these permutation classes are disconnected in PWI$(2, 3)$.

Classes in PWI$(2, 3)$ that are not permutations The set PWI$(2, 3)$ contains several classes with footprint types I-III that are not permutations. All of these are rigid, several are aperiodic and all can be constructed using Lemma 8. The partitions of these are shown in Figure 5 corresponding to PWIs with footprint types illustrated in Figure 3. We believe that this covers all cases for two triangles; regarding a proof, we note that this would be possible by considering general triangles and an exhaustive consideration of the possibilities of how they map to each other. Given the very large number of these possibilities we have not attempted this as yet. The reader may wish to satisfy themselves that this is so by considering the case for PWI$(1, 3)$!

Figure 6 sketches the structure of the asymptotic refinements for these classes. Note that Ia, IIIb and IIIc have very similar structure; specifically, Ia has been examined by Goetz in [17] and Tabachnikov [27], and the self-similar structure of the periodically coded cells is shown in detail in the same figure. The case IIc is the case investigated by Adler et al in [1] which is also self-similar. Table 2 summarizes the information about all classes in PWI$(2, 3)$ that are not permutations. Observe that classes Ia, IIIb and IIIc can all be renormalized to 3, 4 and 4 copies (respectively) of first return maps equivalent to Ia.

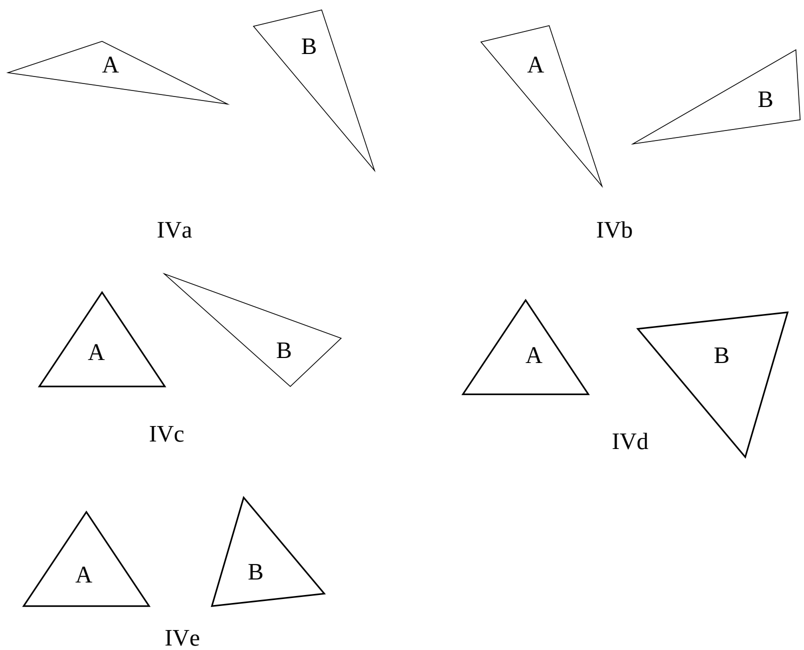


Figure 4: *The partitions for the classes in $PWI(2,3)$ corresponding to permutations of atoms as described in Table 1. Equilateral triangles are denoted by bold outlines.*

Case	ρ	$\dim_p$	Period
IIb	$\pi/5$	0	60
IId	$\pi/5$	0	15
IIa	$\pi/4$	0	4
IIIa	$\pi/4$	0	3
Ia,IIIb,IIIc	$\pi/5$	0	∞
Ib	$\pi/7$	0	∞
IIc	$\pi/4$	0	∞

Table 2: *Summary of the dynamics of the non-permutation classes in $PWI(2,3)$ illustrated in Figure 5. ρ indicates that all rotation angles for this class are integer multiples of ρ. All these classes are rigid (i.e. $\dim_p = 0$) and five of them are aperiodic (denoted by period=∞).*

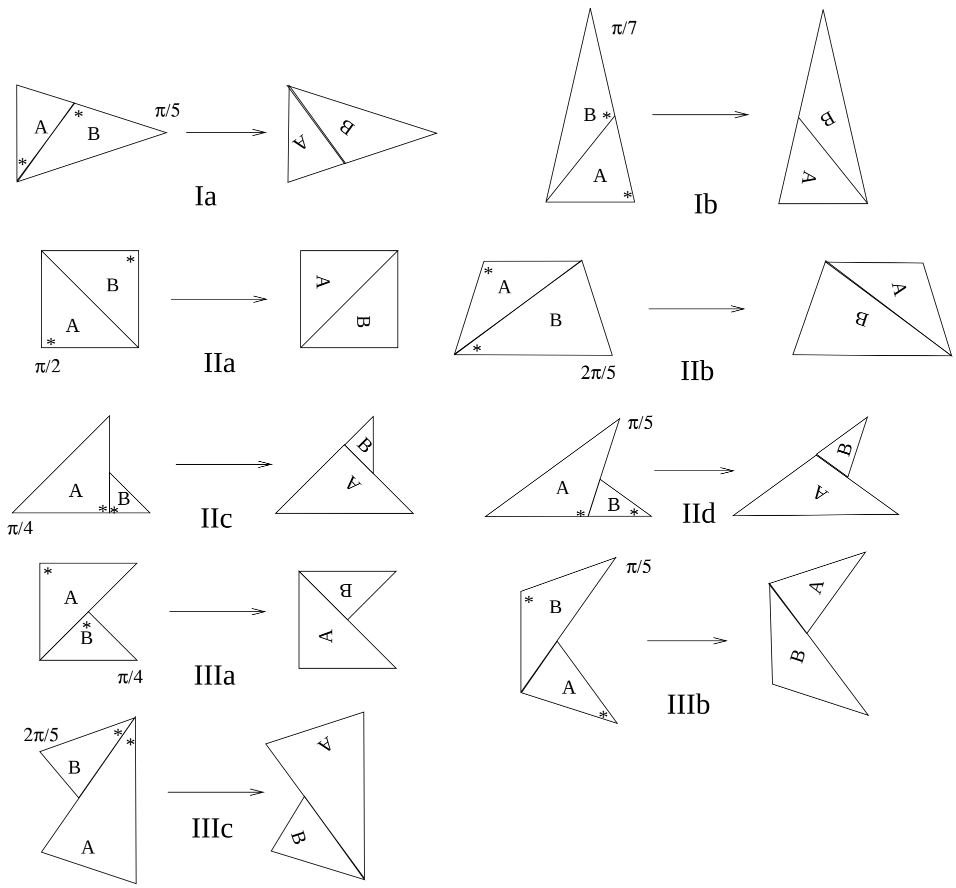
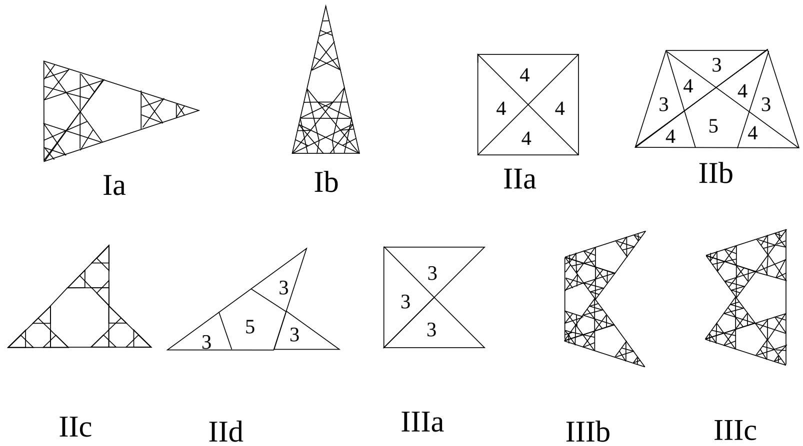
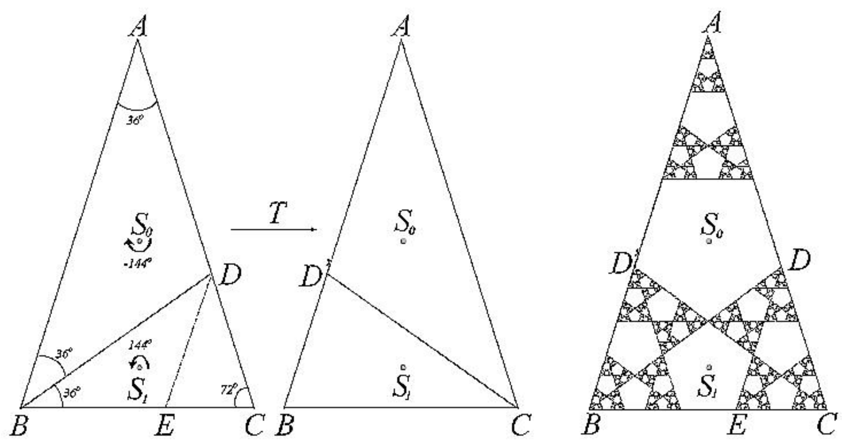


Figure 5: *The footprint types I-III in Figure 3 support the above illustrated piecewise isometries that are not permutations of the atoms. Triangles that are isosceles are indicated by a* $*$ *at the vertex of symmetry of the triangle. All of these classes are rigid (i.e. have perturbation dimension zero); all angles in IIa, IIc and IIIa are multiples of* $\pi/4$*, those in Ia, IIb, IId, IIIb and IIIc are multiples of* $\pi/5$ *and those in Ib has angles that are multiples of* $\pi/7$*.*



(a)



(b)

Figure 6: *(a) A sketch of the dynamically defined cells for the classes in Figure 5. Class IIa is periodic with period 4, IIb is periodic with period 60, IId is periodic with period 15 and IIIa is periodic with period 3. The other PWIs (Ia, Ib, IIc and IIIb and IIIc) are aperiodic and have an infinite number of cells. (b) shows the detail of the cell structure we refer to as class Ia, computed by Goetz in [17] (reproduced from that paper with permission of the author). The two centres of rotation are at S_0 and S_1. The class Ib has no obvious exact self-similar structure in its cells.*

4.4 Three triangular atoms

We now consider PWI$(3,3)$. If the map is reducible in PWI$(3,3)$, we can understand it in terms of reduced maps in PWI$(n,3)$ with $n=1,2$. Similarly, maps in PWI$(3,3)$ where two atoms form a triangle that is mapped around without discontinuity can be viewed as a map in PWI$(2,3)$ where one of the atoms has been subdivided. We do not separately list the partitions that appear for the inverses of maps.

The remaining irreducible classes in PWI$(3,3)$ are classified by footprint type, with I being triangular, II quadrilateral etc. The largest footprint type, VII, will only support permutations. These irreducible classes are shown in Figure 7 and tabulated in Table 3. We believe that we have found most classes; again, we have not rigorously proven this. The irreducible classes shown in Figure 7 have perturbation dimension that is 0 or 1.

Remark 10 *We make a number of observations on Table 3:*

1. *The class piecewise affine torus maps considered by Goetz [15] and shown in Figure 1(b) correspond to class IId in $PWI(3,3)$.*
2. *Several classes have isometries with rotations that can be irrational multiples of π (marked as rotation order ∞).*
3. *Several classes (for example Ie-f) are not rigid, but nonetheless the angles of rotation fixed as rational.*
4. *Many classes are periodic with a variety of periods.*
5. *Class Ie-f, IIc, IId are interesting in that they contains scalene triangles in their partitions.*
6. *Several classes (eg IIIa, IIIg) are such that two adjacent atoms move under the same isometry.*
7. *For three atoms, we have found no rigid PWIs that have angles other than multiples of π/n, where $n = 3,4,5,6,7,11$ or 13, at their vertices.*

Example of junctions of classes The set PWI$(3,3)$ provides the first examples of distinct classes of maps (with differing perturbation dimensions) that accumulate on others. Figure 8 illustrates such an example. In the class of permutations of the form (c) with $\dim_p = 6$ one can find as a special case (b) which is a limit of the irreducible class with $\dim_p = 1$ shown in (a); this is in the class illustrated in Figure 7(e).

4.5 Four triangular atoms

Four triangular atoms will of course support more varied classes than three. Rather than attempt to list the irreducible classes, we present a few examples that give a glimpse of new phenomena. Some irreducible classes of maps in PWI$(4,3)$ are illustrated in Figure 9 and listed in Table 4.

Remark 11 *Class IIa is a mapping of a trapezium to itself such that the isometry on each atom is a rotation by amounts α and $\alpha - \pi$ where α is one of the internal angles of the trapezium. This family is determined uniquely (up to similarity) by this angle and so*

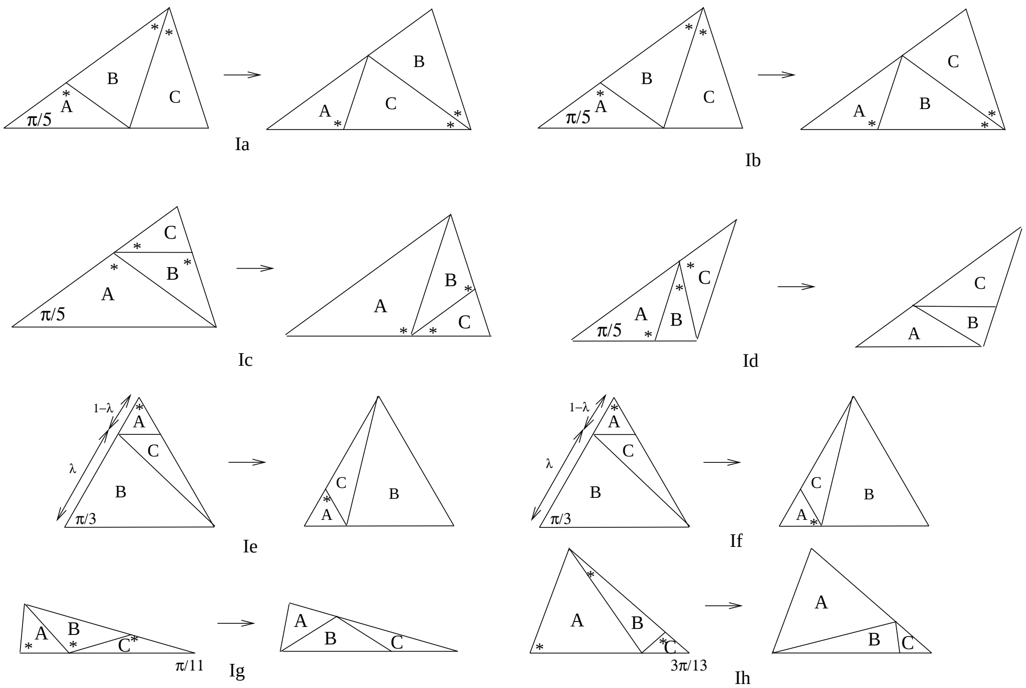


Figure 7: *Partitions for different irreducible classes in $PWI(3,3)$ with footprint type I. In subsequent footprint types, angles marked α indicate that there is a one-dimensional set of angles for this class; fixed angles indicate that these cannot change in the construction. Sides marked λ and μ indicate there is a one-dimensional set where this side changes in proportion to the others (NB for clarity here we do not rotate the indexing letters on the atoms).*

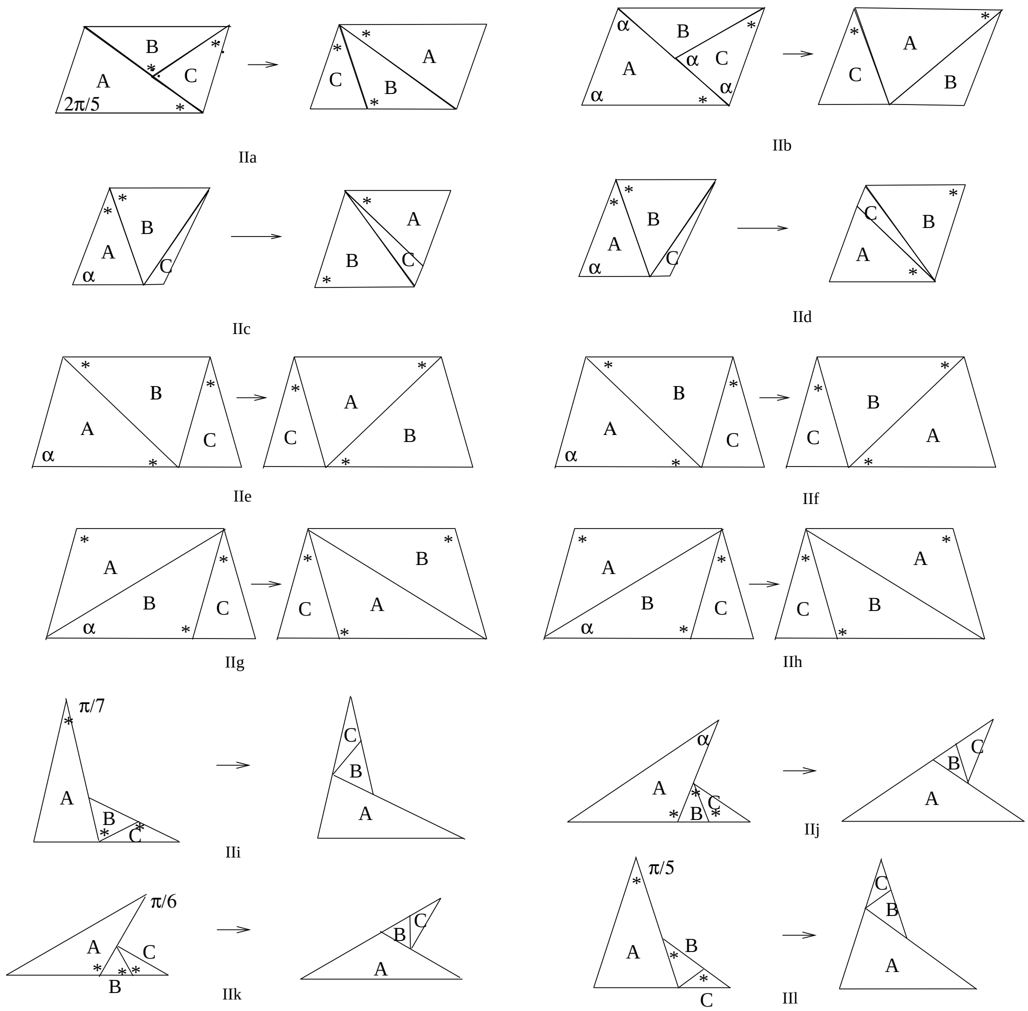


Figure 7 (continued) $PWI(3,3)$ *type II.*

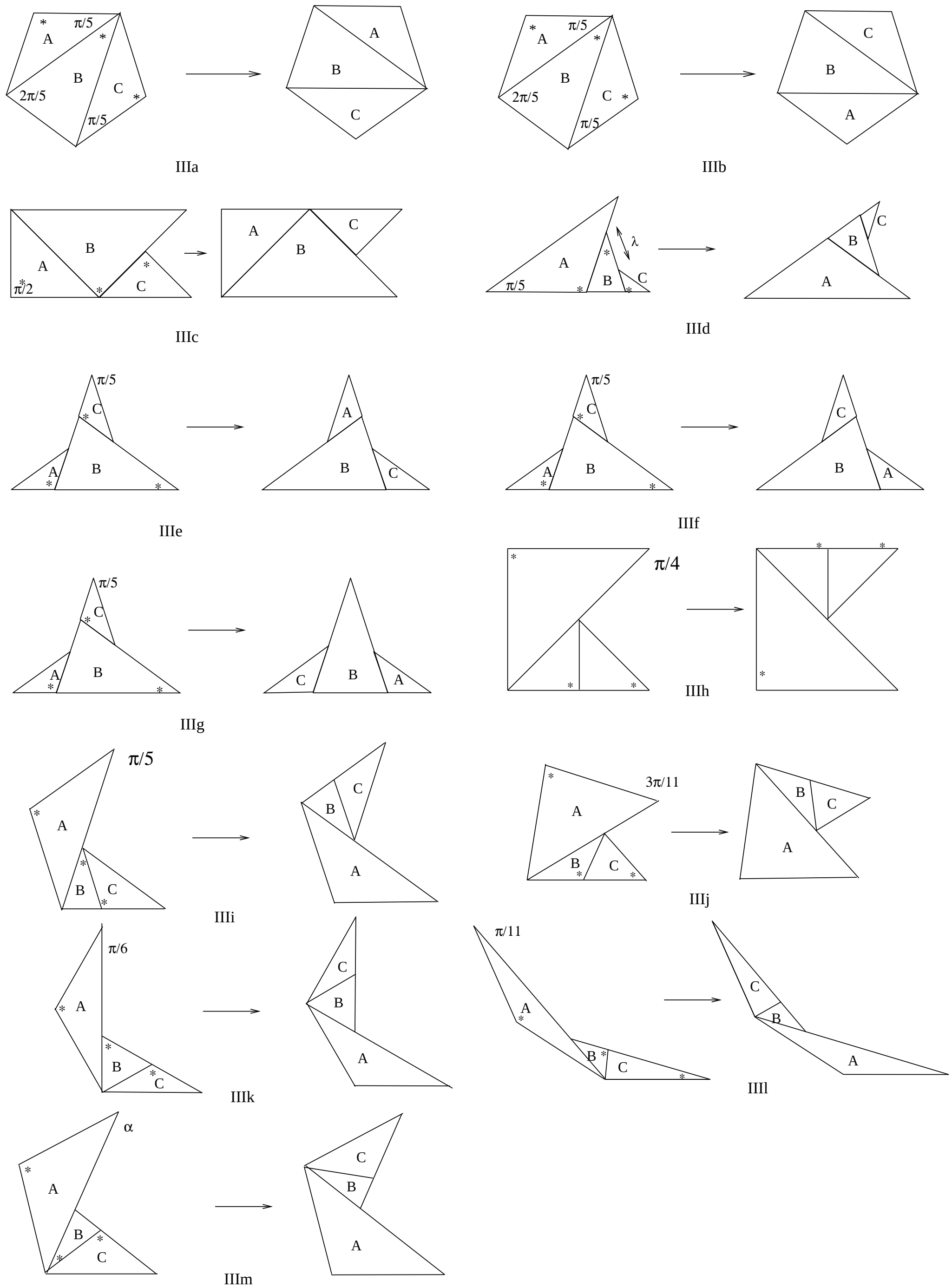


Figure 7 (continued) $PWI(3,3)$ *type III.*

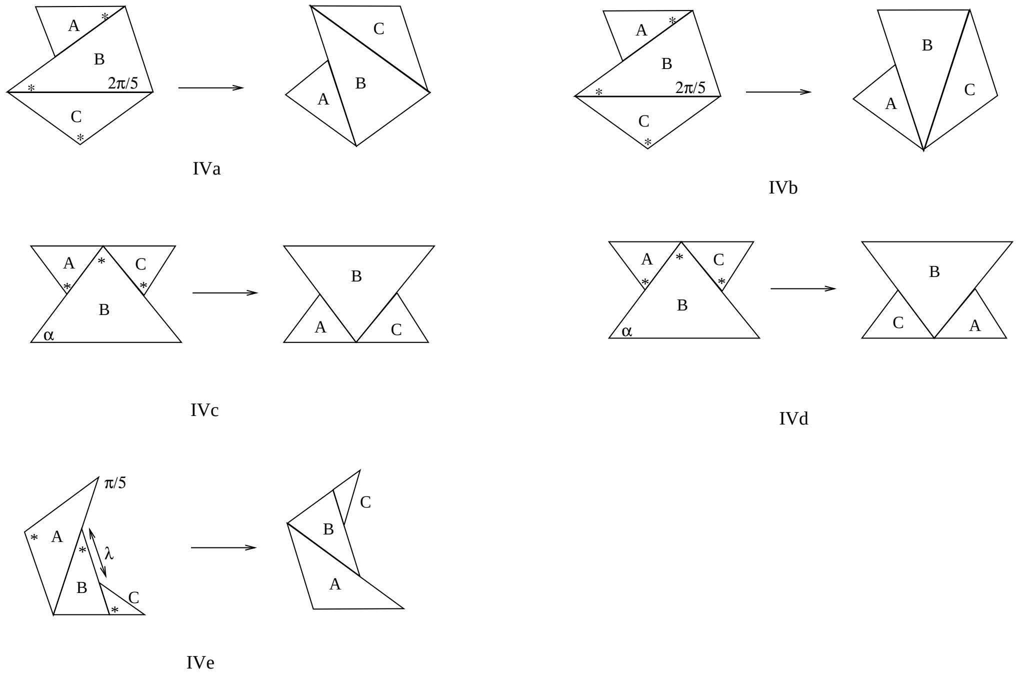


Figure 7 (continued) $PWI(3,3)$ *type IV.*

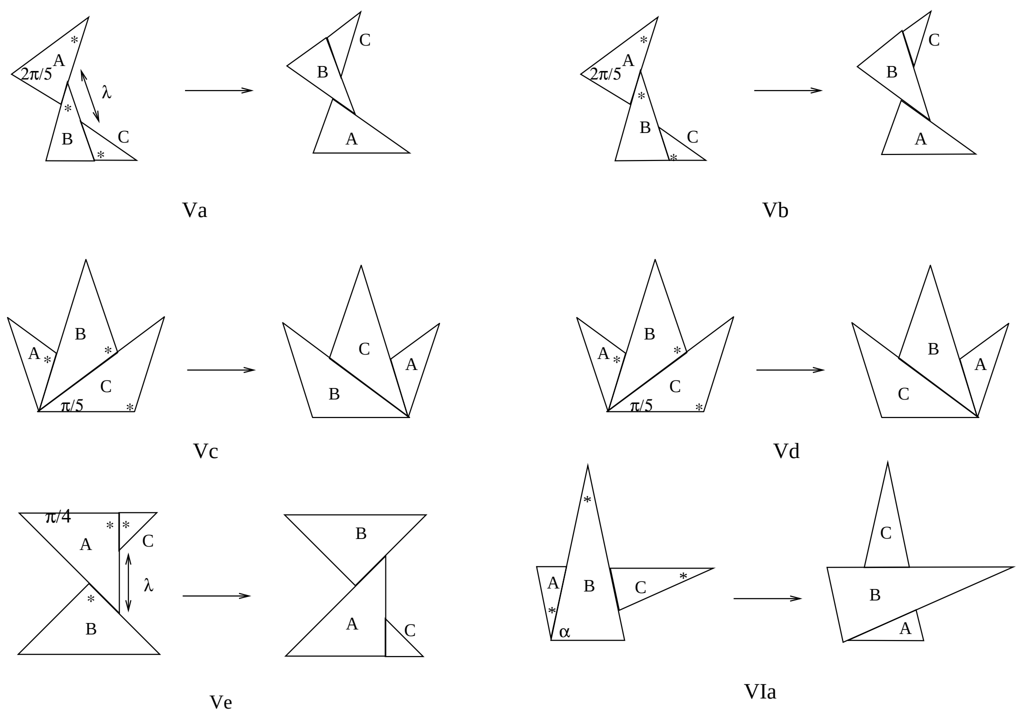


Figure 7 (continued) $PWI(3,3)$ *types V and VI.*

Case	$\dim_p$	Period	Rotation Order
IIIb	0	30	5
IIIc	0	10	4
IIIh	0	6	4
IIIk	0	6	6
IIIa	0	5	5
IVc-d	1	4	∞
Ie-f	1	3	3
Ih	0	∞	13
Ig,IIIj	0	∞	11
IIi	0	∞	7
IIk	0	∞	6
Ia-d, IIa,IIl	0	∞	5
IIId-i, IVa-b, Vb-d	0	∞	5
IVe, Va	1	∞	5
Ve	1	∞	4
IIb-IIh, IIj, VIa	1	∞	∞
IIIl	1	∞	∞

Table 3: *Summary of the dynamics of the non-permutation classes in $PWI(3,3)$ illustrated in Figure 7. The period corresponds to the global periodicity. It is assigned period ∞ if there is an aperiodic member of that class. A class is marked as having rotation order n if all atoms are rotated by integer multiples of π/n; it is assigned to ∞ if can be rotations that are irrational in the class.*

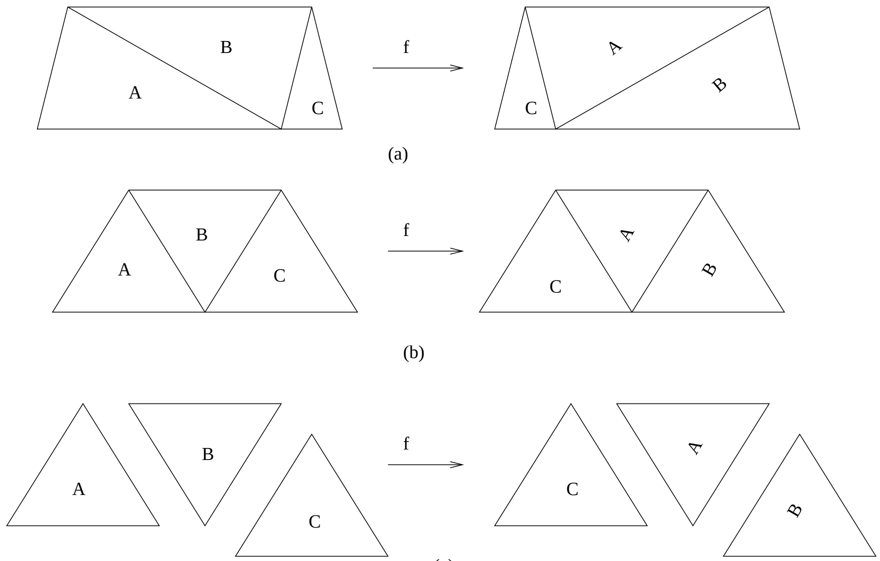


Figure 8: *PWI*$(3,3)$ *yields the simplest examples of classes that are not closed. The class IIe of maps of the form (a) includes a map (b) that is a permutation, reducible and has much larger perturbation dimension, namely (c). Hence (b) is contained in the closure of both (a) and (c).*

the perturbation dimension of the family is 1. On identifying points with their symmetric image, this map corresponds to Feely's bandpass $\Sigma - \Delta$ *modulator, the PWI on 4 atoms mentioned in the introduction and shown in Figure 1(c).*

Class IIb has perturbation dimension one and is periodic for all λ. *However, the period depends on* λ. *Figure 10(a) shows the coding of the regions under iteration for the case* $1/3 < \lambda < 1/2$. *If* $\frac{1}{n-1} \leq \lambda < \frac{1}{n}$ *then the map can be seen to have period* n.

Classes IIc and IId have three parameters (up to similarity) and have the dynamics of interval exchanges. In case IIc we have an orientation reversing section whereas IId is orientation preserving. The general case of these examples is considered in more detail in Section 4.7. In IIc the map is periodic (see Figure 10) whereas in IId all points are effectively subject to a circle map with rotation number λ *and hence if* $\lambda \notin \mathbf{Q}$ *the map is aperiodic.*

4.6 PWIs with larger numbers of triangular atoms

We consider a few specific cases with larger than four atoms; these include some previously studied examples of planar piecewise isometries.

Six triangular atoms In this case we find a class of maps with $\dim_p = 2$ that include several very interesting examples, notably the overflow oscillation map discussed in the introduction and illustrated in Figure 1(a). In fact the one-parameter family is a one parameter restriction (namely the symmetric PWIs) of the two parameter family shown in Figure 11; this is parametrized by the angle α and the length ratio μ; the ratio λ is determined by these two quantities.

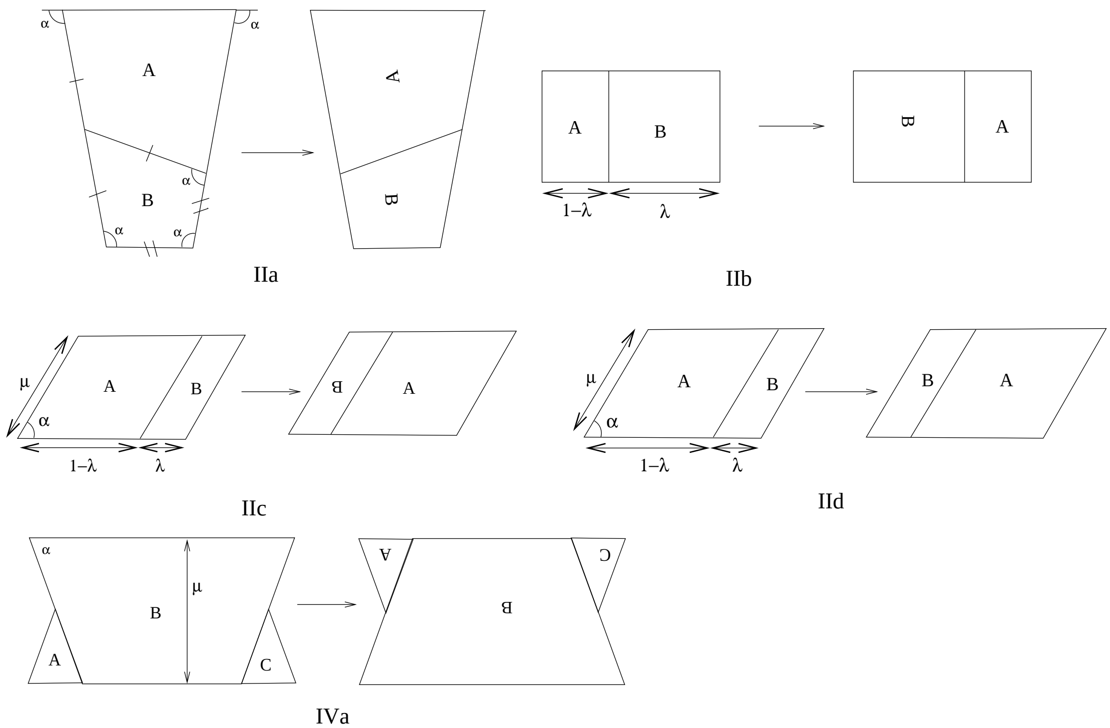


Figure 9: *Six irreducible classes in $PWI(4,3)$ parametrized (up to similarity) by α, λ, μ subject to certain constraints; one can easily find many other classes. For clarity, quadrilaterals are not shown divided into their triangular atoms. As in previous diagrams, the left diagrams show the partitions and the right diagrams show the images of the partition after one iteration. For clarity, IIa-d are shown as maps in $PWI(2,4)$ with two quadrilateral atoms; these are divided into two triangles each to give the map in $PWI(4,3)$. Classes IIa and IIb have perturbation dimension one, IVa has perturbation dimension 2 and IIc, IId have perturbation dimension three.*

Class	$\dim_p$	Periodic?	Rotations
IIa	1	No	$(\alpha, \alpha - \pi)$
IIb	1	Yes	$(\pi/4, 0)$
IIc	3	Yes	$(0, 0)$
IId	3	No	$(0, \pi)$
IVa	2	Yes	(π, π, π)

Table 4: *Six irreducible classes of maps in $PWI(4,3)$ as illustrated in Figure 9. The angles are the rotations of the atoms under one iteration. The classes are signified periodic if all maps in the class are periodic.*

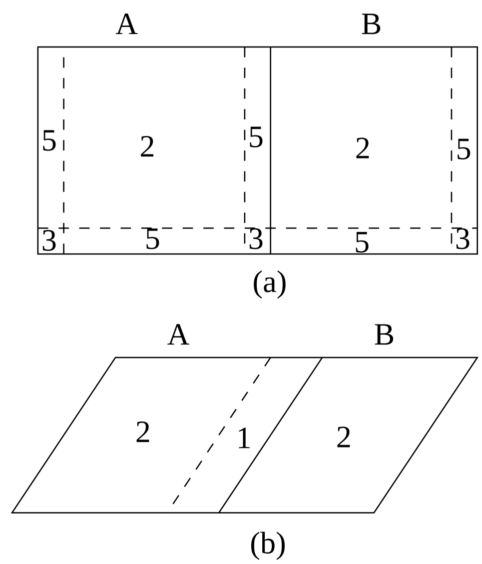


Figure 10: *Periodicities of the codings for the partition for two classes in* $PWI(4,3)$*. (a) corresponds to the partition in Figure 9 IIb whereas (b) corresponds to the partition in Figure 9 IIc. These classes are illustrated for* $\frac{1}{3} < \lambda < \frac{1}{2}$*; for different* λ *there will exist regions with different itineraries.*

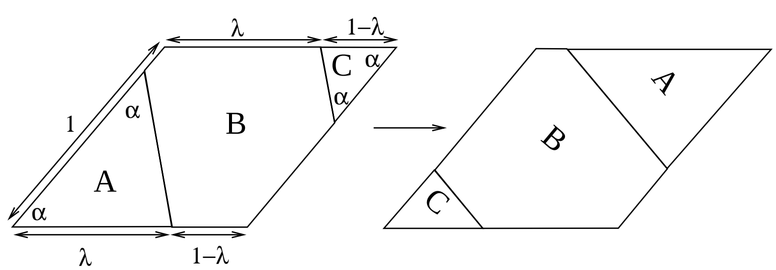


Figure 11: *The family of aperiodic PWIs in* $PWI(6,3)$ *parametrized by the angle* $0 < \alpha < \pi/2$ *and the length ratio* $0 < \lambda < 1$*, subject to the constraint that* $0 < 2\lambda\cos\alpha < 1$*. For clarity this is shown as a map in* $PWI(3,6)$ *with one hexagonal atom and two triangles; the hexagon can be divided up into four triangles to give the map in* $PWI(6,3)$*.*

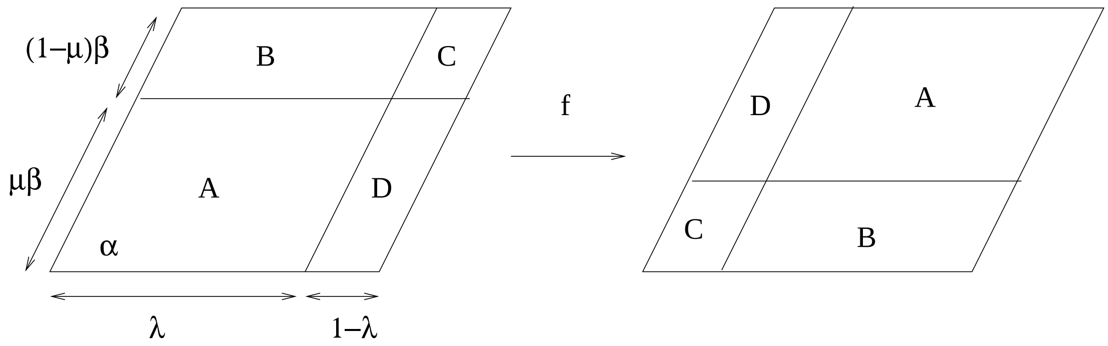


Figure 12: *A class of maps in* $PWI(8,3)$ *with perturbation dimension four such that almost all points have dense orbits whenever* λ, μ *and* 1 *are rationally independent. The map acts purely by translations, i.e. without any rotation.*

Eight triangular atoms The simplest example we are aware of where almost all initial points have orbits that are dense in the footprint is in PWI$(8,3)$. We view an irrational translation on a torus as a piecewise isometry, as shown in Figure 12. In the case that $(1,\lambda,\mu)$ are rationally independent of each other, almost all orbits will have closure that is the whole footprint.

4.7 Interval exchanges and PWIs

As noted in the case for PWI$(4,3)$ the dynamics of interval exchange transformation (IET) can be embedded into planar PWIs. In fact one can embed any IET in a PWI as follows. Let IET(n) be the set of interval exchange transformations on the interval $[0,1]$ continuous on n subintervals (these may be orientation preserving or reversing on each interval).

Proposition 6 *Given any* $f \in IET(n)$ *there is a family of* $(g,\mathcal{M}) \in PWI(2n,3)$ *and a map* $\pi(M) \to [0,1]$ *such that* g *is semiconjugate to* f*.*

Proof Consider $f : [0,1] \to [0,1]$ an IET with discontinuities at $0 = r_0 < r_1 < \cdots < r_n = 1$. We define n polygons M_i, $i = 1..n$ by translating the intervals (r_{i-1}, r_i) in the x-axis a distance $\mu > 0$ in a direction $\alpha \neq 0$ relative to the x-axis. This defines a two parameter family of parallelograms with a partition into n smaller parallelograms. The map g can then be defined to act on these parallelograms by translation and π rotation if necessary so that $f\sigma = \sigma g$, where σ is projection onto the x axis along a vector in direction α. Each quadrilateral can be decomposed as two triangles to show that $(g,\mathcal{M}) \in$ PWI$(2n,3)$. ∎

5 Discussion

By considering invertible orientation preserving piecewise isometries on a finite number of convex m-gons we have demonstrated that the set of such maps has a highly ordered and nontrivial geometric structure. Even for two or three triangles a large range of different dynamical behaviours are possible for such families of piecewise isometries, and these raise a number of interesting questions.

5.1 Geometric and Dynamical problems

Some problems concerning the geometry of PWIs:

1. Complete a classification of classes in $\mathrm{PWI}(n, 3)$. Since the number of possible abuttal types grows very quickly with n, we expect this will require the use of sophisticated computer-aided classification tools.

2. Understand the ways in which classes of PWIs can meet.

3. Understand the asymptotic refinement of PWIs in terms of a renormalization theory.

There is a range of open conjectures regarding the dynamics of particular examples of piecewise isometries; see for example [1, 3, 4, 15, 19] etc. Using the setting here, we can interpret questions about generic classes families of piecewise isometries as questions about typical members of classes within $\mathrm{PWI}(n, m)$. In particular, we have highlighted how the geometry of the partition tightly constrains, in a rather non-trivial way, the possible ways of perturbing a particular PWI without introducing more atoms or increasing the number of sides on the atoms.

Note also that the dynamically very interesting behaviours can be seen already in cases that have small numbers of atoms and low perturbation dimension. However seeing that, for example, to find a map that has all orbits dense one needs to look in $\mathrm{PWI}(8, 3)$ or $\mathrm{PWI}(4, 4)$, it does raise the question of whether typical behaviour for larger numbers of atoms is substantially different from that for small numbers.

There are many open questions are related to the set of aperiodically coded points $\mathcal{A}^{(\infty)}$ for a map $(f, \mathcal{M})$.

1. Is $\mathcal{A}^{(\infty)}$ non-empty if f is aperiodic?

2. Does $\mathcal{A}^{(\infty)}$ have positive measure footprint?

3. What dynamics is possible in $\mathcal{A}^{(\infty)}$?

4. How are cells created/destroyed within classes of PWIs?

In another paper [6] we have examined the invariant disk packings induced by PWIs for a particular family and show that these disk packings (see also [26] for disk packings in similar systems), although computer simulation seems to indicate that they are dense, are in fact loosely packed in the sense that they typically have no tangencies. The framework discussed is paper justifies considering the single perturbation parameter in [6]; this is the only degree of freedom we have in perturbing this map unless we are content to (a) break the symmetry of the map, (b) add more atoms to the partition or (c) increase the number of sides to at least one of the polygonal atoms.

In recent work, Aharonov, Devaney and Elias [2] have shown how smooth approximations of a piecewise linear map may be used to give information about invariant curves in the dynamics of the approximated map. However, the maps we consider are not just piecewise linear but also discontinuous in nature, and moreover the properties discussed are properties of the dynamics of the discontinuity itself.

Acknowledgement

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