Hopf-Galois Module Structure Of Some Tamely Ramified Extensions

Submitted by

Paul James Truman

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I certify that all material in this thesis which is not my own work has been identified and that no material is included for which a degree has previously been conferred upon me.

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Paul James Truman

Abstract

We study the Hopf-Galois module structure of algebraic integers in some finite extensions of p-adic fields and number fields which are at most tamely ramified. We show that if L/K is a finite unramified extension of p-adic fields which is Hopf-Galois for some Hopf algebra H then the ring of algebraic integers $\mathfrak{O}_L$ is a free module of rank one over the associated order $\mathfrak{A}_H$. If H is a commutative Hopf algebra, we show that this conclusion remains valid in finite ramified extensions of p-adic fields if p does not divide the degree of the extension. We prove analogous results for finite abelian Galois extensions of number fields, in particular showing that if L/K is a finite abelian domestic extension which is Hopf-Galois for some commutative Hopf algebra H then $\mathfrak{O}_L$ is locally free over $\mathfrak{A}_H$. We study in greater detail tamely ramified Galois extensions of number fields with Galois group isomorphic to $C_p \times C_p$, where p is a prime number. Byott has enumerated and described all the Hopf-Galois structures admitted by such an extension. We apply the results above to show that $\mathfrak{O}_L$ is locally free over $\mathfrak{A}_H$ in all of the Hopf-Galois structures, and derive necessary and sufficient conditions for $\mathfrak{O}_L$ to be globally free over $\mathfrak{A}_H$ in each of the Hopf-Galois structures. In the case $p = 2$ we consider the implications of taking $K = \mathbb{Q}$. In the case that p is an odd prime we compare the structure of $\mathfrak{O}_L$ as a module over $\mathfrak{A}_H$ in the various Hopf-Galois structures.

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