

# Hopf-Galois Module Structure Of Some Tamely Ramified Extensions

Submitted by

**Paul James Truman**

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Paul James Truman

# Abstract

We study the Hopf-Galois module structure of algebraic integers in some finite extensions of  $p$ -adic fields and number fields which are at most tamely ramified. We show that if  $L/K$  is a finite unramified extension of  $p$ -adic fields which is Hopf-Galois for some Hopf algebra  $H$  then the ring of algebraic integers  $\mathfrak{D}_L$  is a free module of rank one over the associated order  $\mathfrak{A}_H$ . If  $H$  is a commutative Hopf algebra, we show that this conclusion remains valid in finite ramified extensions of  $p$ -adic fields if  $p$  does not divide the degree of the extension. We prove analogous results for finite abelian Galois extensions of number fields, in particular showing that if  $L/K$  is a finite abelian domestic extension which is Hopf-Galois for some commutative Hopf algebra  $H$  then  $\mathfrak{D}_L$  is locally free over  $\mathfrak{A}_H$ . We study in greater detail tamely ramified Galois extensions of number fields with Galois group isomorphic to  $C_p \times C_p$ , where  $p$  is a prime number. Byott has enumerated and described all the Hopf-Galois structures admitted by such an extension. We apply the results above to show that  $\mathfrak{D}_L$  is locally free over  $\mathfrak{A}_H$  in all of the Hopf-Galois structures, and derive necessary and sufficient conditions for  $\mathfrak{D}_L$  to be globally free over  $\mathfrak{A}_H$  in each of the Hopf-Galois structures. In the case  $p = 2$  we consider the implications of taking  $K = \mathbb{Q}$ . In the case that  $p$  is an odd prime we compare the structure of  $\mathfrak{D}_L$  as a module over  $\mathfrak{A}_H$  in the various Hopf-Galois structures.

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