

# **Compressibility Approximations in Jovian Regimes: A Normal Mode Analysis**

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## Abstract

The atmospheres and interiors of planetary and stellar systems has been studied in various forms, though the complexity of these systems currently makes full replication of their dynamics impossible. In order to make these complex systems tractable it is necessary to make significant simplifications. For many years models have concentrated on making use of the Boussinesq approximation, often with a constant density reference profile. More recently, anelastic approximations have been developed to allow for the analysis of some degree of reference state density variation. The validity of these approximations is well understood for modelling the adiabatic, inviscid, terrestrial atmosphere, however their use in modelling other regimes remains equivocal. We consider the fully compressible, Boussinesq, anelastic, quasi-hydrostatic and pseudo-incompressible equation sets governing fluid flow within a rotating, differentially heated system. We consider both tangent plane and spherical shell geometries and conduct a normal mode analysis in order to examine the validity of these approximations outside of terrestrial parameters. We find the compressibility approximations can cause spurious distortion of the normal mode solutions including misrepresentation of the frequencies, growth/decay rates, and modal structure. This in turn can have knock on effects including energy redistribution. The level of distortion is found to be dependent on mode type, reference profile and geometry and varies according to approximated equation set. Selected eigenmodes and frequencies are presented and discussed.

We conclude that the most suitable approximated equation sets for use in modelling the various regions of the Jovian atmosphere depends primarily on the type of wave it is necessary to reflect most accurately; and that the scale analyses, upon which the approximated equation sets are based, provide a good indication of the regimes in which their use is appropriate.

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