CHAPTER V.

ON THE THEORY OF STEREOSCOPIC VISION.

Having, in the preceding chapter, described the ocular, the reflecting, and the lenticular stereoscopes, and explained the manner in which the two binocular pictures are combined or laid upon one another in the last of these instruments, we shall now proceed to consider the theory of stereoscopic vision.

In order to understand how the two pictures, when placed the one above the other, rise into relief, we must first explain the manner in which a solid object itself is, in ordinary vision, seen in relief, and we shall then shew how this process takes place in the two forms of the ocular stereoscope, and in the lenticular stereoscope. For this purpose, let $ABCD$, Fig. 19, be a section of the frustum of a cone, that is, a cone with its top cut off by a plane $CEDG$, and having $AEBG$ for its base. In order that the figure may not be complicated, it will be sufficient to consider how we see, with two eyes, $l$ and $r$, the cone as projected upon a plane passing through its summit $CEDG$. The points $l$, $r$ being the points of sight, draw the lines $RA$, $RB$, which will cut the plane on which the projection is to be made in the points $a$, $b$, so that $ab$ will represent the line $AB$, and a circle, whose diameter is $ab$, will represent the base of the cone, as seen by the right
eye r. In like manner, by drawing LA, LB, we shall find that A'B' will represent the line AB, and a circle, whose diameter is A'B', the base AEBG, as seen by the left eye. The summit, CEDG, of the frustum being in the plane of projection, will be represented by the circle CEDG. The representation of the frustum ABOD, therefore, upon a plane
surface, as seen by the left eye \( L \), consists of two circles, whose diameters are \( AB, CD \); and, as seen by the right eye, of other two circles, whose diameters are \( ab, cd \), which, in Fig. 20, are represented by \( AB, CD \), and \( ab, cd \). These

![Diagram of circles representing stereo images.](image)

plane figures being also the representation of the solid on the retina of the two eyes, how comes it that we see the solid and not the plane pictures? When we look at the point \( B \), Fig. 19, with both eyes, we converge upon it the optic axes \( LB, RB \), and we therefore see the point single, and at the distances \( LB, RB \) from each eye. When we look at the point \( D \), we withdraw the optic axes from \( B \), and converge them upon \( D \). We therefore see the point \( D \) single, and at the distances \( LD, RD \) from each eye; and in like manner the eyes run over the whole solid, seeing every point single and distinct upon which they converge their axes, and at the distance of the point of convergence from the observer. During this rapid survey of the object, the whole of it is seen distinctly as a solid, although every point of it is seen double and indistinct, excepting the point upon which the axes are for the instant converged.

From these observations it is obvious, that when we look with both eyes at any solid or body in relief, we see more of the right side of it by the right eye, and more of the left side
of it by the left eye. The right side of the frustum $ABCD$, Fig. 19, is represented by the line $DB$, as seen by the right eye, and by the shorter line $DB'$, as seen by the left eye. In like manner, the left side $AC$ is represented by $CA'$, as seen by the left eye, and by the shorter line $CA'$, as seen by the right eye.

When the body is hollow, like a wine glass, we see more of the right side with the left eye, and more of the left side with the right eye.

If we now separate, as in Fig. 20, the two projections shown together on Fig. 19, we shall see that the two summits, $CD, CD'$, of the frustum are farther from one another than the more distant bases, $AB, AB'$, and it is true generally that in the two pictures of any solid in relief, the similar parts that are near the observer are more distant in the two pictures than the remoter parts, when the plane of perspective is beyond the object. In the binocular picture of the human face the distance between the two noses is greater than the distance between the two right or left eyes, and the distance between the two right or left eyes greater than the distance between the two remoter ears.

We are now in a condition to explain the process by which, with the eyes alone, we can see a solid in relief by uniting the right and left eye pictures of it,—or the theory ocular stereoscope. In order to obtain the proper relief we must place the right eye picture on the left side, and the left eye picture on the right side, as shown in Fig. 21, by the pictures $ABCD, ABD$, of the frustum of a cone, as obtained from Fig. 19.

In order to unite these two dissimilar projections, we must converge the optical axes to a point nearer the ob-
server, or look at some point about M. Both pictures will immediately be doubled. An image of the figure ab will advance towards P, and an image of AB will likewise advance towards R; and the instant these images are united, the frustum of a cone, which they represent, will appear in
relief at MN, the place where the optic axes meet or cross each other. At first the solid figure will appear in the middle, between the two pictures from which it is formed and of the same size, but after some practice it will appear smaller and nearer the eye. Its smallness is an optical illusion, as it has the same angle of apparent magnitude as the plane figures, namely, \( \triangle MNL = \triangle ABL \); but its position at MN is a reality, for if we look at the point of our finger held beyond M the solid figure will be seen nearer the eye. The difficulty which we experience in seeing it of the size and in the position shewn in Fig. 21, arises from its being seen along with its two plane representations, as we shall prove experimentally when we treat in a future chapter of the union of similar figures by the eye.

The two images being thus superimposed, or united, we shall now see that the combined images are seen in relief in the very same way that in ordinary vision we saw the real solid, ABCD, Fig. 19, in relief, by the union of the two pictures of it on the retina. From the points A, B, C, D, a, b, c, d, draw lines to L and R, the centres of visible direction of each eye, and it will be seen that the circles AB, a b, representing the base of the cone, can be united by converging the optical axes to points in the line MN, and that the circles CD, c d, which are more distant, can be united only by converging the optic axes to points in the line OP. The points A, a, for example, united by converging the axes to m, are seen at that point single; the points B, b at n single, the points C, c at o single, the points D, d at p single, the centres s, s of the base at M single, and the centres s', s' of the summit plane at N single. Hence the eyes L and R see the combined pictures at
MN in relief, exactly in the same manner as they saw in relief the original solid MN in Fig. 19.

In order to find the height MN of the conical frustum thus seen, let d = distance of F; d = s's, the distance of the two points united at M; d' = s's', the distance of the two points united at N; and c = LR = 2½ inches, the distance of the eyes. Then we have—

\[
MP = \frac{d}{c+d} \\
NP = \frac{d'}{c+d'} \\
MN = \frac{d}{c+d} - \frac{d'}{c+d'}
\]

If \( d = 9.24 \) inches,
\( c = 2.50 \), then
\( d = 2.14 \),
\( d' = 2.42 \), and
\( MN = 0.283 \), the height of the cone.

When \( c = d \), \( MP = \frac{d}{2c} \).

As the summit plane op rises above the base mn by the successive convergency of the optic axes to different points in the line np, it may be asked how it happens that the conical frustum still appears a solid, and the plane op where it is, when the optic axes are converged to points in the line mn, so as to see the base distinctly? The reason of this is that the rays emanate from op exactly in the same manner, and form exactly the same image of it, on the two retinas as if it were the summit cd, Fig. 19, of the real solid when seen with both eyes. The only effect of the advance of the point of convergence from n to m is to throw the image of n a little to the right side of the
optical axis of the left eye, and a little to the left of the optical axis of the right eye. The summit plane \(ap\) will therefore retain its place, and will be seen slightly doubled and indistinct till the point of convergence again returns to it.

It has been already stated that the two dissimilar pictures may be united by converging the optical axes to a point beyond them. In order to do this, the distance \(ss'\) of the pictures, Fig. 21, must be greatly less than the distance of the eyes \(l, n\), in order that the optical axes, in passing through similar points of the two plane pictures, may meet at a moderate distance beyond them. In order to explain how the relief is produced in this case, let \(AB, cd, ab, cd\), Fig. 22, be the dissimilar pictures of the frustum of a cone whose summit is \(cd\), as seen by the right eye, and \(cd\) as seen by the left eye. From \(l\) and \(n\), as before, draw lines through all the leading points of the pictures, and we shall have the points \(a, a\) united at \(m\), the points \(b, b\) at \(n\), the points \(c, c\) at \(o\), and the points \(d, d\) at \(p\), the points \(s, s\) at \(m\), and the points \(s', s'\) at \(n\), forming the cone \(mnop\), with its base \(mn\) towards the observer, and its summit \(op\) more remote. If the cone had been formed of lines drawn from the outline of the summit to the outline of the base, it would now appear hollow, the inside of it being seen in place of the outside as before. If the pictures \(AB, ab\) are made to change places the combined picture would be in relief, while in the case shown in Fig. 21 it would have been hollow. Hence the right-eye view of any solid must be placed on the left hand, and the left-eye view of it on the right hand, when we wish to obtain it in relief by converging the optical axes to a point between the pictures and the eye, and vice versa when we wish to obtain
it in relief by converging the optic axes to a point beyond the pictures. In every case when we wish the combined pictures to represent a hollow, or the converse of relief, their places must be exchanged.
In order to find the height \( MN \), or rather the depth of the cone in Fig. 22, let \( d, \ d', \ c, \ c' \), represent the same quantities as before, and we shall have

\[
\begin{align*}
MP &= \frac{d}{c-d} \\
NP &= \frac{d'd}{c-d'}, \text{ and} \\
OP &= \frac{d'd'}{c-d'} - \frac{d}{c-d}
\end{align*}
\]

When \( d, \ c, \ d', \ c' \) have the same values as before, we shall have

\[ MN = 18.7 \text{ feet} \]

When \( c = d \), \( MP \) will be infinite.

We have already explained how the two binocular pictures are combined or laid upon one another in the lenticular stereoscope. Let us now see how the relief is obtained. The two plane pictures \( a\  b\  c\  d\ ), \( A\  B\  C\  D\ ), in Fig. 18, are, as we have already explained, combined or simply laid upon one another by the lenses \( LL, L'L' \), and in this state are shown by the middle circles at \( A\  A'\  b\  b' \), \( C\  C'\  d\  d' \). The images of the bases \( AB, \ ab \) of the cone are accurately united in the double base \( AB, \ ab \), but the summits of the conical frustrum remain separate, as seen at \( CD, \ c'd' \). It is now the business of the eyes to unite these, or rather to make them appear as united. We have already seen how they are brought into relief when the summits are refracted so as to pass one another, as in Fig. 18. Let us therefore take the case shown in Fig. 20, where the summits \( CD, \ cd \) are more distant than the bases \( AB, \ ab \). The union of these figures is instantly effected, as shown in Fig. 23, by converging the optic axes to points \( m \) and \( n \) successively, and thus uniting \( o \) and \( c \) and \( d \) and \( d' \), and making these points of the summit plane appear at \( m \) and \( n \), the
points of convergence of the axes LM, RM, and LN, RN.

In like manner, every pair of points in the summit

plane, and in the sides AM, BN of the frustum, are converged to points corresponding to their distance from the base AB of the original solid frustum, from which the plane
pictures $A B C D$, $a b c d$, were taken. We shall, therefore, see in relief the frustum of a cone whose section is $A M N B$.

The theory of the stereoscope may be expressed and illustrated in the following manner, without any reference to binocular vision:—

1. When a drawing of any object or series of objects is executed on a plane surface from one point of sight, according to the principles of geometrical perspective, every point of its surface that is visible from the point of sight will be represented on the plane.

2. If another drawing of the same object or series of objects is similarly executed on the same plane from a second point of sight, sufficiently distant from the first to make the two drawings separate without overlapping, every point of its surface visible from this second point of sight will also be represented on the plane, so that we shall have two different drawings of the object placed, at a short distance from each other, on the same plane.

3. Calling these different points of the object 1, 2, 3, 4, &c., it will be seen from Fig. 24, in which $L, R$ are the
two points of sight, that the distances 1, 1, on the plane $M\overline{N}$, of any pair of points in the two pictures representing the point 1 of the object, will be to the distance of any other pair 2, 2, representing the point 2, as the distances $1'P, 2'P$ of the points of the object from the plane $M\overline{N}$, multiplied inversely by the distances of these points from the points of sight $L, E$, or the middle point $O$ between them.

4. If the sculptor, therefore, or the architect, or the mechanist, or the surveyor, possesses two such pictures, either as drawn by a skilful artist or taken photographically, he can, by measuring the distances of every pair of points, obtain the relief or prominence of the original point, or its distance from the plane $M\overline{N}$ or $A\overline{E}$; and without the use of the stereoscope, the sculptor may model the object from its plain picture, and the distances of every point from a given plane. In like manner, the other artists may determine distances in buildings, in machinery, and in the field.

5. If the distance of the points of sight is equal to the distance of the eyes $L, E$, the two plane pictures may be united and raised into relief by the stereoscope, and thus give the sculptor and other artists an accurate model, from which they will derive additional aid in the execution of their work.

6. In stereoscopic vision, therefore, when we join the points 1, 1 by converging the optic axes to $1'$ in the line $P\overline{O}$, and the points 2, 2 by converging them to $2'$ in the same line, we place these points at the distances $o1, o2$, and see the relief, or the various differences of distance which the sculptor and others obtained by the method which we have described.
7. Hence we infer, that if the stereoscopic vision of relief had never been thought of, the principles of the instrument are involved in the geometrical relief which is embodied in the two pictures of an object taken from two points of sight, and in the prominence of every part of it obtained geometrically.