Achievable Precision of Close Modes in Operational Modal Analysis: Wide Band Theory

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Abstract

This work takes up the challenge of deriving the 'uncertainty law' for close modes, i.e., closed form analytical expressions for the remaining uncertainty of modal parameters identified using (output-only) ambient vibration data. In principle the uncertainty law can be obtained from the inverse of the Fisher Information matrix of modal parameters. The key mathematical challenges stem from analytical treatment of entangled stochastic dynamics with a large number of modal parameters of different nature and the quest for closed form expressions for identification uncertainty, whose possibility is questionable. Fortunately the problem still admits insightful closed form solution under long data, high signal-to-noise ratio and wide resonance band for identification. Up to modelling assumptions and the use of probability, the uncertainty law dictates the achievable precision of modal properties regardless of the identification algorithm used. A companion paper discusses the insights, verification, scientific implications and recommendation for ambient test planning.

Keywords: ambient modal identification, BAYOMA, close modes, Fisher Information Matrix, operational modal analysis, uncertainty law

1 Introduction

Operational modal analysis (OMA) shows great promise as a feasible and economically viable means for obtaining in-situ modal properties of structures [1][2][3]. This owes primarily to the use of 'output-only' ambient vibration data, which inevitably leads to higher identification (ID) uncertainty in the results compared to properly managed free or forced vibration tests. In addition to algorithms

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for identifying modal properties when data is given, the ability to quantify and manage ID uncertainty is vital to the regular practice of OMA in the near future. ID uncertainty is often quantified in terms of a variance, which can be calculated for a given data set. Method depends on the ID algorithm and philosophy. The main classes are Bayesian and frequentist (non-Bayesian) approach.

Frequentist approach is conventional in OMA, which constructs an estimator for the parameter of interest as a function of data and refers ID uncertainty as the ensemble variability of the estimator over repeated experiments, therefore implying the notion of 'intrinsic variability'. The calculation formulae of uncertainty are always in terms of the 'true' parameter value (assumed to exist) which in application is substituted by the value of the estimator for the given set of data. Perturbation technique is often used to derive the formulae. Examples include [4][5] (frequency domain, maximum likelihood), [6][7] (ARMA model), [8][9][10] (SSI) and [11] (SSI multi-setup) and [12] (ERA).

Bayesian approach [13][14] views modal properties as random variables whose distribution is updated when data is available. ID uncertainty is associated with the spread of their probability density function (PDF) conditional on data and modelling assumptions, i.e., the 'posterior PDF'. Their ID uncertainty is therefore a function of data. The concept of true parameter value is irrelevant and there is no intrinsic uncertainty. Early formulations of Bayesian OMA include [15] (time-domain), [16] (frequency domain, power spectral density, i.e., 'PSD') and [17] (frequency domain, Fast Fourier Transform, i.e., 'FFT'). Fast algorithms that allow practical implementation appears more recently and are focussed on FFT formulation that is found to be most robust in terms of modelling assumptions. See, e.g., [3] (monograph), [18] (frequency domain, general multi-modes), [19] (frequency domain, single-mode, multi-setup).

Managing ID uncertainty requires another level of advance in knowledge beyond the ability to calculate, involving insights on how it depends on various factors that may affect the quality of ID results. Mathematically, it is to discover the fundamental law of how the information about modal properties is extracted from ambient data following stochastic dynamics assumptions. Such discovery need not be possible, but recent research in Bayesian OMA shows that for well-separated modes with long and high signal-to-noise (s/n) ratio data it is possible to obtain closed form analytical expressions for the ID uncertainty in terms of test configuration and environment [20]. Such expressions are collectively referred as 'uncertainty law', motivated from their mathematical origin akin to the Laws of Large Numbers in probability theory. To be useful an uncertainty law needs to capture correctly the influence of governing factors on ID uncertainty but at the same time be simple enough for scientists or engineers to develop intuitions for understanding and test planning.

The identification and calculation of uncertainty of close modes requires much more sophisticated algorithms than well-separated modes. The entangling of modal dynamics with close frequencies obscures intuitions about resonance and poses questions on identifiability and further tightening of OMA precision limit from that of well-separated modes. As part of a recent research campaign to understand and manage ID uncertainty in OMA, this work develops the uncertainty law for close modes. It is reported in two companion papers. This paper develops the mathematical theory. The companion paper [21] discusses the insights, implications, verification and recommendations for planning ambient vibration tests.

As we will see, the mathematics and techniques involved for analysing close modes are much more tedious and complicated than those for well-separated modes. Fortunately, remarkably simple and intuitive expressions for ID uncertainty have been discovered in this work that provide deep insights. The theory also allows us to fundamentally quantify 'how close is close' and how it affects ID uncertainty. In addition to long data and high s/n ratio that were assumed in the uncertainty law for well-separated modes, the theory for close modes requires the additional assumption that the resonance band used for modal ID is wide compared to the separation of modes, i.e., 'wide band' as indicated in the title of this paper. Semi-empirical correction factors have been developed to account for the effect of finite bandwidth in the companion paper.

2 Theory outline

We first outline the mathematical theory for deriving the uncertainty law of close modes. Of fundamental importance is the characteristic nature of two types of ID uncertainty associated with mode shapes discovered previously [23]. Type 1 and Type 2 are respectively associated with the mode shape uncertainty orthogonal to and within the 'mode shape subspace' (MSS), i.e., the linear subspace spanned by the close mode shapes. See Section 2.1 in the companion paper [21] for further qualitative discussion. We will derive the expressions in the companion paper for Type 1 mode shape c.o.v. (coefficient of variation) in (2), Type 2 mode shape c.o.v. in (3), frequency c.o.v. in (8) and damping c.o.v. in (9) (equation numbers here refer to those of the companion paper). Details will follow from Section 3 onwards. The theory assumes that the properties of close modes are identified in the frequency domain using the Fast Fourier Transform (FFT) of ambient vibration data. Following a Bayesian approach (Section 3), the ID results are expressed in terms of a 'posterior' (i.e., given data) PDF of modal parameters, which can be well-approximated by a Gaussian PDF. ID uncertainty is measured by the covariance matrix associated with the PDF. It can be calculated for given data but a 'point-wise' value does not offer any insight on how it depends on test configuration and environment. One way to do that is to introduce a 'frequentist' assumption that the data indeed comes from some 'true' modal properties and study theoretically how the covariance matrix depends on the true parameter values. Asymptotic yet realistic conditions, e.g., long data and high s/n ratio, are applied to simplify analysis and to enable close form expressions. The resulting analytical expressions of ID uncertainty are collectively referred as 'uncertainty law'. For scalar parameters such as frequencies and damping ratios, ID uncertainty is expressed in terms of the coefficient of variation (c.o.v. = standard deviation/mean). For mode shapes that are vector-valued subjected to norm-constraints, a scalar measure of ID uncertainty is the 'mode shape c.o.v.', mathematically equal to the square root sum of eigenvalues of the mode shape covariance matrix; see Section 11.3 of [3].

Deriving the uncertainty law for close modes involves applying the frequentist assumption just mentioned and then asymptotic analysis of the covariance matrix of modal parameters for long data, high s/n ratio and wide band in succession. Previous studies on well-separated modes only involve long data and high s/n ratio. For close modes, the present theory requires 'wide band' as an additional assumption to obtain analytical results of reasonable simplicity for developing intuitions. For long data (Section 4) it can be shown that [22] the covariance matrix is asymptotically equal to the inverse of the Fisher Information Matrix (FIM). In the present OMA context, the FIM can be expressed directly in terms of the theoretical PSD matrix of data; see (1).

The basic target in developing uncertainty law is to obtain analytical expressions for the variance of modal parameters, which are the diagonal entries of the inverse of FIM; or, for mode shapes, the sum of eigenvalues of the corresponding partition in the inverse of FIM. The FIM is a large matrix whose dimension grows linearly with the number of measured degrees of freedom (DOFs) and in a quadratic manner with the number of modes. Taking the inverse of FIM analytically is highly non-trivial, if at all possible. Considering further asymptotic conditions (high s/n and wide band) provides the basic premise for possibility, but the actual pursuit requires a combination of tasks of iterative nature, e.g., applying matrix algebra techniques after observations/guess/intuition about the intrinsic structure of the matrix for simplification.

In addition to long data, when the s/n ratio is high (Section 5) a recent study shows that the FIM can be explicitly expressed in terms of the PSD matrix of modal response and mode shape matrix [23]; see Table 1 later. More importantly, the uncertainty of mode shapes comprises two mutually uncorrelated types: one (Type 1) orthogonal to the mode shape subspace (MSS) they span, and the other (Type 2) within the MSS. The covariance matrix is asymptotically equal to the sum of contributions from the two types. Type 1 was found in well-separated modes, uncorrelated from all other parameters and is often negligible in applications. It is analysed in Section 9 (appendix). Type 2 is unique to close modes. It is correlated with and accounts for the uncertainty of all other parameters. It is the heart of the theory where most effort will be spent on.

By writing Type 2 mode shape uncertainty as a linear combination of basis vectors within the MSS, the FIM and hence Type 2 covariance matrix can be obtained from a FIM $\mathbf{J}'$ of reduced dimension regardless of the number of DOFs. See $\mathbf{J}'$ in (3), which has a dimension of $(m+1)^2 + m(m-1)$ for m close modes. In the remaining theory (Section 6), the scope is narrowed to the case of $m=2$ close modes, where Figure 1 in the companion paper [21] shows a schematic diagram of the singular value spectrum. Direct inverse of $\mathbf{J}'$ is still analytically intractable, but using block matrix inverse formula and matrix inverse lemma it is possible to condense out the auto/cross-PSDs of modal force (parameters in the problem) so that the covariance matrix of frequencies, damping ratios and mode shapes can be expressed in terms of matrices of reduced dimension. See (8) and (9) in Section 6.1, which are proven in Section 11 (appendix). By this time the matrices involved contain sums of frequency response functions and their derivatives; see Table 4. Asymptotic expressions for the sums are then obtained for long data and high s/n ratio. Despite that, it has not been possible to obtain manageable expression for the inverse of $\mathbf{J}'$. Fortunately, when the resonance band is sufficiently wide, the sums are found to simplify and bear special patterns that allow the inverse to be obtained analytically (Section 6.2), eventually delivering the uncertainty law for close modes under wide band situations (Section 6.4 and 6.5). For the sake of discussion, acceleration data is assumed in the theory (Section 3) but the results are applicable for other data types, e.g., velocity. See Section 6.6 for detailed reasoning.

3 Bayesian OMA

Consider modal identification in the frequency domain using 'output-only' ambient vibration data, where the natural frequencies are so close that they must be modelled together in the same resonance band they share [18]. Without loss of generality, the data is assumed to be acceleration $\{\ddot{\mathbf{x}}_j\}_{j=0}^{N-1}$ ($n \times 1$) at n measured DOFs, each with N sample points at time interval Δt (sec). Its scaled FFT at frequency $\mathrm{f}_k = k / N\Delta t$ (Hz) is defined as $\mathcal{F}_k = \sqrt{2\Delta t / N}\Sigma_{j=0}^{N-1}\ddot{\mathbf{x}}_j e^{-2\pi \mathbf{i} jk/N}$, where $\mathbf{i}^2 = -1$; the scaling factor $\sqrt{2\Delta t / N}$ is applied so that $E[\mathcal{F}_k \mathcal{F}_k^*]$ gives the one-sided PSD matrix; a superscripted '*' denotes complex conjugate transpose. In the resonance band the scaled FFT is modelled to comprise m classically damped modes, i.e., $\mathcal{F}_k = \Sigma_{i=1}^{m}\boldsymbol{\varphi}_i \ddot{\eta}_{ik} + \boldsymbol{\xi}_k$, where $\boldsymbol{\varphi}_i$ ($n \times 1$) is the mode shape; $\boldsymbol{\xi}_k$ ($n \times 1$) is data noise assumed to be independent and identically distributed (i.i.d.) among different DOFs with a common PSD S_e within the band (so only band-limited white); $\ddot{\eta}_{ik}$ (scalar) is the scaled FFT of modal acceleration response, whose time-domain counterpart satisfies (omitting dependence on time) $\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = p_i$; $\omega_i = 2\pi f_i$ (rad/sec) and f_i (Hz) is the natural frequency; ζ_i is the damping ratio; p_i is the modal force (per unit modal mass). The modal forces $\{p_i\}_{i=1}^m$ are assumed to be stochastic stationary with a constant PSD matrix $\mathbf{S}$ ($m \times m$ Hermitian and positive definite) within the resonance band (so only band-limited white). The modal properties to be identified comprise $\{f_i\}_{i=1}^m$, $\{\zeta_i\}_{i=1}^m$, $\mathbf{S}$, S_e and $\mathbf{\Phi} = [\boldsymbol{\varphi}_1,...,\boldsymbol{\varphi}_m]$. Accounting for its Herimitian nature, $\mathbf{S}$ has m^2 parameters: m for the real diagonal entries and $m(m-1)$ for the complex-valued lower off-diagonal entries. In total there are $m + m + m^2 + 1 + mn = (m+1)^2 + mn$ parameters, subjected to m unit norm constraints on the mode shapes. ID result is encapsulated in the 'posterior' (i.e., given data) PDF of parameters. For long data, it can be well-approximated by a Gaussian PDF centred at the most probable value (MPV) and with a covariance matrix equal to the Hessian of the negative logarithm of the likelihood function (NLLF) at the MPV.

4 Long data asymptotics and Fisher Information Matrix

For long data and under the frequentist assumption of 'true' parameters, it can be shown that the (posterior) covariance matrix of parameters is asymptotically equal to the inverse of the Fisher Information Matrix (FIM) [22]. The FIM is defined as the expectation of the Hessian of NLLF evaluated at the 'true' parameters, assuming that the data is indeed distributed as the likelihood function. In the OMA problem, the scaled FFT data is complex Gaussian distributed. It follows from a standard result in multivariate statistics [24] that the entry of the FIM with respect to (w.r.t.) generic variables x and y is equal to

$$J_{xy} = tr\Sigma[\mathbf{E}_k^{-1}\mathbf{E}_k^{(x)}\mathbf{E}_k^{-1}\mathbf{E}_k^{(y)}] \tag{1}$$

where $tr(.)$ denotes the trace (i.e., sum of diagonal entries) of the argument matrix; $\mathbf{E}_k = \overline{\mathbf{\Phi}}\mathbf{H}_k\overline{\mathbf{\Phi}}^T + S_e\mathbf{I}_n$ is the PSD matrix of data; $\mathbf{I}_n$ denotes the $n \times n$ identity matrix; the superscripted '(x)' denotes a derivative w.r.t. x ; $\overline{\mathbf{\Phi}}$ is the normalised mode shape matrix whose i th column is $\boldsymbol{\varphi}_i / \|\boldsymbol{\varphi}_i\|$ (so has unit norm); $\mathbf{H}_k$ is the PSD matrix of $\{\ddot{\eta}_{ik}\}_{i=1}^m$ with the (i,j) -entry equal to $S_{ij}h_{ik}h_{jk}^*$; S_{ij} is the (i,j) -entry of $\mathbf{S}$; $h_{ik} = 1/[(1-\beta_{ik}^2) - \mathbf{i}(2\zeta_i\beta_{ik})]$ is the frequency response function (FRF) between p_i and $\ddot{\eta}_{ik}$; and $\beta_{ik} = f_i / \mathrm{f}_k$. The sum in (1) is over all k in the resonance band, where the running index is omitted to simplify notation. Note that the modal properties mentioned above and from this point on refer to the 'true' values under a frequentist assumption.

5 High s/n asymptotics and principal uncertainties

The FIM in (1) in terms of $\mathbf{E}_k$ still gives no insight on ID uncertainty. Motivated from the possibility of analytical results in well-separated modes, the asymptotic behaviour of the FIM for high s/n ratio ($S_e \to 0$) has been recently investigated [23]. Asymptotic expressions for the FIM explicitly in terms of $\mathbf{H}_k$ and $\overline{\mathbf{\Phi}}$ have been obtained. The results are summarised in Table 1. The entries related to S_e are omitted as it can be deduced that S_e is asymptotically uncorrelated from all other parameters and therefore can be ignored in the calculation of their uncertainty.

Table 1 Asymptotic FIM for high s/n ratio (entries for S_e omitted); $\mathbf{\Phi}$: denotes the 'vectorisation' of $\mathbf{\Phi} = [\varphi_1, \ldots, \varphi_m]$, i.e., a $nm \times 1$ vector obtained by stacking $\{\varphi_i\}_{i=1}^m$ column-wise; $\mathbf{Q} = \mathbf{I}_n - \overline{\mathbf{\Phi}}(\overline{\mathbf{\Phi}}^T \overline{\mathbf{\Phi}})^{-1} \overline{\mathbf{\Phi}}^T$, $\mathbf{Q}_i = (\overline{\mathbf{\Phi}}^T \overline{\mathbf{\Phi}})^{-1} \overline{\mathbf{\Phi}}^T (\mathbf{I}_n - \overline{\varphi}_i \overline{\varphi}_i^T)$ and $[\mathbf{Q}_i]$ denotes a block diagonal matrix containing the $\mathbf{Q}_i$ s; $[\boldsymbol{e}_j \boldsymbol{e}_i^T]$ denotes a matrix whose $(\boldsymbol{i}, \boldsymbol{j})$-partition is $\boldsymbol{e}_j \boldsymbol{e}_i^T$ ($\boldsymbol{i}, \boldsymbol{j} = \mathbf{1}, \ldots, \boldsymbol{m}$); $\boldsymbol{e}_i$ is a $\boldsymbol{m} \times \mathbf{1}$ vector whose $\boldsymbol{i}$th entry is 1 and other entries are zero ($\boldsymbol{e}_j$ is similarly defined)

	$x, y = f, \zeta, \mathbf{S}$	$\mathbf{\Phi} := [\varphi_1; \ldots; \varphi_m]$
$x, y = f, \zeta, \mathbf{S}$	$tr \Sigma \mathbf{H}_k^{-1} \mathbf{H}_k^{(x)} \mathbf{H}_k^{-1} \mathbf{H}_k^{(y)}$	$2\mathbf{e}_i^T (\mathrm{Re} \Sigma \mathbf{H}_k^{(x)} \mathbf{H}_k^{-1}) \mathbf{Q}_i$
$\mathbf{\Phi}:$	Symmetric	$J_{\mathbf{\Phi}:\mathbf{\Phi}:} \sim J_{\mathbf{\Phi}:\mathbf{\Phi}:}^{(1)} + J_{\mathbf{\Phi}:\mathbf{\Phi}:}^{(2)}$
		$J_{\mathbf{\Phi}:\mathbf{\Phi}:}^{(1)} = 2S_e^{-1} (\mathrm{Re} \Sigma \mathbf{H}_k) \otimes \mathbf{Q}$
		$J_{\mathbf{\Phi}:\mathbf{\Phi}:}^{(2)} = 2[\mathbf{Q}_i]^T \left\{ (\mathrm{Re} \Sigma \mathbf{H}_k \otimes \mathbf{H}_k^{-T}) + N_f [\mathbf{e}_j \mathbf{e}_i^T] \right\} [\mathbf{Q}_i]$

Based on an analytical eigenvalue analysis of the asymptotic FIM, it was found that the ID uncertainty comprises two characteristic types: one (Type 1) associated with the uncertainty of mode shapes orthogonal to the 'mode shape subspace' (MSS) they span, and the other (Type 2) within the MSS. The eigenvectors of the asymptotic FIM of Type 1 and Type 2 lie in two orthogonally complementary subspaces. This implies that the two types of uncertainties are uncorrelated and so the covariance matrix is simply equal to the sum of their contributions. Type 1 uncertainty was previously found in well-separated modes [20], where mode shapes are uncorrelated from all other parameters (e.g., frequency, damping). It diminishes with increased data quality (smaller S_e), vanishes for noiseless data and is often negligible in applications. As Type 1 mode shape uncertainty is uncoupled from that of the remaining parameters and it is orthogonal to Type 2 uncertainty, its contribution to the mode shape covariance matrix can be obtained by simply taking the inverse of $J^{(1)}_{\mathbf{\Phi}:\mathbf{\Phi}:}$ in Table 1. Details can be found in Section 9 (appendix). Type 1 induces no uncertainty in other parameters, i.e., their uncertainty comes only from Type 2, and is generally correlated with mode shapes. Because of the correlation, Type 2 covariance matrix of all parameters must be determined together. It can be found from the inverse of a FIM of reduced dimension. Specifically, for Type 2 one can write the uncertain mode shape deviation $\Delta\mathbf{\Phi}:$ as

$$\Delta\mathbf{\Phi}:=\begin{bmatrix}\Delta\boldsymbol{\varphi}_1\\ \vdots\\ \Delta\boldsymbol{\varphi}_m\end{bmatrix}=\begin{bmatrix}\mathbf{U}_1\boldsymbol{\alpha}_1\\ \vdots\\ \mathbf{U}_m\boldsymbol{\alpha}_m\end{bmatrix}=\underbrace{\begin{bmatrix}\mathbf{U}_1 & & \\ & \ddots & \\ & & \mathbf{U}_m\end{bmatrix}}_{\mathbf{U}}\underbrace{\begin{bmatrix}\boldsymbol{\alpha}_1\\ \vdots\\ \boldsymbol{\alpha}_m\end{bmatrix}}_{\boldsymbol{\alpha}}=\mathbf{U}\boldsymbol{\alpha} \tag{2}$$

where $\mathbf{U}_i$ ($n\times(m-1)$) contains in its columns the $(m-1)$ basis vectors in the MSS and $\boldsymbol{\alpha}_i$ ($(m-1)\times 1$) is a vector of coordinates for $\Delta\boldsymbol{\varphi}_i$ w.r.t. the basis; $\mathbf{U}$ is a block diagonal matrix comprising the $\mathbf{U}_i$s. Type 2 uncertainty of $\boldsymbol{\varphi}_i$ is associated with that of $\boldsymbol{\alpha}_i$. Let the full set of modal parameters be partitioned as $\mathbf{a}=[\varpi;\mathbf{\Phi}:]$ where ϖ comprises all parameters except the mode shapes. By choosing an orthonormal basis for $\mathbf{U}_i$, i.e., $\mathbf{U}_i^T\mathbf{U}_i=\mathbf{I}_{m-1}$, the eigenvalue problem $\mathbf{J}\mathbf{a}=\lambda\mathbf{a}$ can be reduced to $\mathbf{J}'\mathbf{v}=\lambda\mathbf{v}$ where $\mathbf{v}=[\varpi;\boldsymbol{\alpha}]$ is a reduced vector. The original FIM $\mathbf{J}$ and reduced FIM $\mathbf{J}'$ are related by

$$\mathbf{J}=\begin{bmatrix}J_{\varpi\varpi} & J_{\varpi\mathbf{\Phi}:}\\ \text{sym.} & J_{\mathbf{\Phi}:\mathbf{\Phi}:}\end{bmatrix}\qquad\qquad \mathbf{J}'=\begin{bmatrix}J_{\varpi\varpi} & J_{\varpi\mathbf{\Phi}:}\mathbf{U}\\ \text{sym.} & \mathbf{U}^T J^{(2)}_{\mathbf{\Phi}:\mathbf{\Phi}:}\mathbf{U}\end{bmatrix} \tag{3}$$

where the subscripts indicate the parameters the partitions correspond to; in the (2,2)-block entry of $\mathbf{J}'$, we have used $\mathbf{U}^T J_{\mathbf{\Phi}:\mathbf{\Phi}:}\mathbf{U}=\mathbf{U}^T J^{(2)}_{\mathbf{\Phi}:\mathbf{\Phi}:}\mathbf{U}$ because $J^{(1)}_{\mathbf{\Phi}:\mathbf{\Phi}:}\mathbf{U}=\mathbf{0}$ ($\mathbf{U}_i$ is orthogonal to Type 1 uncertainty). Generally, the diagonal entries of $\mathbf{J}'^{-1}$ give the variance of parameters. The sum of diagonal entries of $\mathbf{J}'^{-1}$ corresponding to $\boldsymbol{\alpha}_i$ gives the square of Type 2 mode shape c.o.v. of Mode i. To see this, the covariance matrix of $\boldsymbol{\varphi}_i$ with Type 2 uncertain deviation $\Delta\boldsymbol{\varphi}_i=\mathbf{U}_i\boldsymbol{\alpha}_i$ is

$\mathbf{C}_{\boldsymbol{\varphi}_i}^{(2)} = \mathbf{U}_i \mathbf{C}_{\boldsymbol{\alpha}_i} \mathbf{U}_i^T$ where $\mathbf{C}_{\boldsymbol{\alpha}_i} = E[\boldsymbol{\alpha}_i \boldsymbol{\alpha}_i^T]$ is the covariance matrix of $\boldsymbol{\alpha}_i$. By noting that $\mathbf{U}_i^T \mathbf{U}_i = \mathbf{I}_{m-1}$,

$$\delta_{\boldsymbol{\varphi}_i}^{(2)^2} = tr[\mathbf{C}_{\boldsymbol{\varphi}_i}^{(2)}] = tr(\mathbf{U}_i \mathbf{C}_{\boldsymbol{\alpha}_i} \mathbf{U}_i^T) = tr(\mathbf{U}_i^T \mathbf{U}_i \mathbf{C}_{\boldsymbol{\alpha}_i}) = tr(\mathbf{C}_{\boldsymbol{\alpha}_i}) \tag{4}$$

where in the third equality we have used the cyclic property of trace, i.e., $tr(AB) = tr(BA)$.

6 Theory for two close modes

Excluding S_e from the parameter set, $\mathbf{J}'$ has a dimension of $2m^2 + m$, i.e., 10 for $m = 2$. Inverting $\mathbf{J}'$ analytically from first principles is still intractable. In the remaining development of the theory presented in this section, we narrow the scope to two close modes and explore further simplifications that can be attained. In this case the full set of parameters is shown in the first row of Table 2. A typical picture of the singular value spectrum is illustrated in Figure 1 of the companion paper [21]. The complex-valued modal force cross-PSD is parameterised in terms of its real and imaginary parts, i.e., $S_{21} = U_{21} + \mathbf{i}V_{21}$. For $m = 2$ modes, each mode shape has $(m-1) = 1$ direction of Type 2 uncertainty represented by

$$\Delta\boldsymbol{\varphi}_i = \mathbf{U}_i \alpha_i \qquad \mathbf{U}_1 = \frac{\boldsymbol{\varphi}_2 - \rho\boldsymbol{\varphi}_1}{\sqrt{1-\rho^2}} \qquad \mathbf{U}_2 = \frac{\boldsymbol{\varphi}_1 - \rho\boldsymbol{\varphi}_2}{\sqrt{1-\rho^2}} \tag{5}$$

where $\rho = \boldsymbol{\varphi}_1^T \boldsymbol{\varphi}_2$ is the cosine of angle between mode shapes (assuming $\| \boldsymbol{\varphi}_i \| = 1$), also called the 'modal assurance criterion' (MAC) in modal ID literature; $\mathbf{U}_i$ is the basis vector, essentially a unit vector along the tangential direction of rotation from $\boldsymbol{\varphi}_i$ to the other mode shape. Clearly, $\mathbf{U}_i$ lies in the subspace spanned by the two mode shapes. Check also that $\boldsymbol{\varphi}_i^T \mathbf{U}_i = 0$ and $\mathbf{U}_i^T \mathbf{U}_i = 1$. When α_1 is small it can be interpreted as the uncertain angle of rotation from $\boldsymbol{\varphi}_1$ to $\boldsymbol{\varphi}_2$; α_2 can be interpreted similarly. The reduced FIM $\mathbf{J}'$ for $m = 2$ modes can be obtained from (3). The results are shown in Table 3 and derivations can be found in Section 10 (appendix). It is written in a normalised form as

$$\mathbf{J}' = N_f \, diag(\mathbf{a}) \bar{\mathbf{J}}' diag(\mathbf{a}) \tag{6}$$

where N_f is the number of FFT points in the resonance band; $\mathbf{a}$ is a vector of scaling factors given in the second row of Table 2, defined after inspection of the entries in $\mathbf{J}'$; $diag(\mathbf{a})$ denotes a diagonal matrix comprising the entries in $\mathbf{a}$. Clearly,

$$\mathbf{J}'^{-1} = N_f^{-1} diag(\mathbf{a})^{-1} \overline{\mathbf{J}}'^{-1} diag(\mathbf{a})^{-1} \quad (7)$$

The target is now to obtain analytically the inverse $\overline{\mathbf{J}}'^{-1}$.

Table 2 Parameters of reduced FIM $\mathbf{J}'$ and normalising factors in $\mathbf{a}$ of (6); $|\chi|$ is the modulus of coherence between the two modal forces

Parameter	$f_1, f_2, \zeta_1, \zeta_2$	S_{11}	S_{22}	U_{21}, V_{21}	α_1, α_2
Factor	1	$\dfrac{1}{S_{11}(1-\|\chi\|^2)}$	$\dfrac{1}{S_{22}(1-\|\chi\|^2)}$	$\dfrac{1}{\sqrt{S_{11}S_{22}}(1-\|\chi\|^2)}$	$\dfrac{1}{\sqrt{1-\rho^2}}$

Table 3 High s/n asymptotic expressions for condensed and normalised FIM $\bar{\mathbf{J}}'$ (2 modes). See (6) for relation to reduced FIM $\mathbf{J}'$ and Section 10 for derivation; $r_1=\sqrt{S_{11}/S_{22}}$, $r_2=\sqrt{S_{22}/S_{11}}$, $\chi=S_{21}/\sqrt{S_{11}S_{22}}$ (modal force coherence), $\rho=(\boldsymbol{\varphi}_1^T\boldsymbol{\varphi}_2)/\|\boldsymbol{\varphi}_1\|\|\boldsymbol{\varphi}_2\|$ (MAC); see Table 4 for definitions of $A_{x_iy_i}$, $a_{x_iy_j}$, a_{xi} , a_{xij} , a_i'

	$x,y=f_1,\zeta_1$	$x,y=f_2,\zeta_2$	S_{11}	S_{22}	U_{21}	V_{21}	α_1	α_2
$x,y=f_1,\zeta_1$	$4A_{x_1y_1}+\frac{2\lvert\chi\rvert^2}{1-\lvert\chi\rvert^2}\mathrm{Re}\,a_{x_1y_1}$	$\frac{-2\lvert\chi\rvert^2}{1-\lvert\chi\rvert^2}\mathrm{Re}\,a_{x_1y_2}$	$2\,\mathrm{Re}\,a_{x1}$	0	$-2\,\mathrm{Re}\,\chi a_{x1}$	$-2\,\mathrm{Im}\,\chi a_{x1}$	$-2\rho\,\mathrm{Re}\,a_{x1}-\frac{2}{1-\lvert\chi\rvert^2}\times(\rho\,\mathrm{Re}\,a_{x1}+r_1\,\mathrm{Re}\,\chi a_{x11})$	$\frac{2}{1-\lvert\chi\rvert^2}\times(\rho\lvert\chi\rvert^2\,\mathrm{Re}\,a_{x1}+r_2\,\mathrm{Re}\,\bar{\chi}a_{x12})$
$x,y=f_2,\zeta_2$		$4A_{x_2y_2}+\frac{2\lvert\chi\rvert^2}{1-\lvert\chi\rvert^2}\mathrm{Re}\,a_{x_2y_2}$	0	$2\,\mathrm{Re}\,a_{x2}$	$-2\,\mathrm{Re}\,\bar{\chi}a_{x2}$	$2\,\mathrm{Im}\,\bar{\chi}a_{x2}$	$\frac{2}{1-\lvert\chi\rvert^2}\times(\rho\lvert\chi\rvert^2\,\mathrm{Re}\,a_{x2}+r_1\,\mathrm{Re}\,\chi a_{x21})$	$-2\rho\,\mathrm{Re}\,a_{x2}-\frac{2}{1-\lvert\chi\rvert^2}\times(\rho\,\mathrm{Re}\,a_{x2}+r_2\,\mathrm{Re}\,\bar{\chi}a_{x22})$
S_{11}			1	$\lvert\chi\rvert^2$	$-2\,\mathrm{Re}\,\chi$	$-2\,\mathrm{Im}\,\chi$	$-2(\rho+r_1\,\mathrm{Re}\,\chi\overline{a_1'})$	0
S_{22}				1	$-2\,\mathrm{Re}\,\chi$	$-2\,\mathrm{Im}\,\chi$	0	$-2(\rho+r_2\,\mathrm{Re}\,\chi a_2')$
U_{21}					$2(1+\mathrm{Re}\,\chi^2)$	$2\,\mathrm{Im}\,\chi^2$	$2(\rho\,\mathrm{Re}\,\chi+r_1\,\mathrm{Re}\,a_1')$	$2(\rho\,\mathrm{Re}\,\chi+r_2\,\mathrm{Re}\,a_2')$
V_{21}						$2(1-\mathrm{Re}\,\chi^2)$	$2(\rho\,\mathrm{Im}\,\chi+r_1\,\mathrm{Im}\,a_1')$	$2(\rho\,\mathrm{Im}\,\chi-r_2\,\mathrm{Im}\,a_2')$
α_1		Symmetric					$\frac{2}{1-\lvert\chi\rvert^2}[\rho^2(2-\lvert\chi\rvert^2)+r_1^2a_1+2r_1\rho\,\mathrm{Re}(\chi\overline{a_1'})]$	$\frac{2}{1-\lvert\chi\rvert^2}[1-(1+\rho^2)\lvert\chi\rvert^2-\rho(r_1\,\mathrm{Re}\,\chi\overline{a_1'}+r_2\,\mathrm{Re}\,\chi a_2')-\mathrm{Re}(\chi^2 d)]$
α_2								$\frac{2}{1-\lvert\chi\rvert^2}[\rho^2(2-\lvert\chi\rvert^2)+r_2^2a_2+2r_2\rho\,\mathrm{Re}(\chi a_2')]$

6.1 Parameter condensation by block matrix inverse

Taking analytically the inverse of $\bar{\mathbf{J}}'$ (a 10×10 matrix) is intractable unless there is some special structure in the matrix. Upon exploration, it was found that $\{S_{11}, S_{22}, U_{21}, V_{21}\}$ can be analytically 'condensed out' of the inverse. Let $(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}$ and $(\bar{\mathbf{J}}'^{-1})_{\varpi\varpi}$ denote the partitions in the inverse $\bar{\mathbf{J}}'^{-1}$ w.r.t. $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]$ and $\varpi = \{f_1, f_2, \zeta_1, \zeta_2\}$, respectively. Using block matrix inverse formula and matrix inverse lemma, it is shown in Section 11 that

$$\underset{2\times 2}{(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}} = diag\begin{Bmatrix} 1/r_1 \\ 1/r_2 \end{Bmatrix} \left[\bar{\mathbf{A}}_\alpha - \frac{4|\chi|^2}{(1-|\chi|^2)^2} \mathbf{B}_\alpha \right]^{-1} diag\begin{Bmatrix} 1/r_1 \\ 1/r_2 \end{Bmatrix} \tag{8}$$

$$\underset{4\times 4}{(\bar{\mathbf{J}}'^{-1})_{\varpi\varpi}} = \underset{4\times 4}{\mathbf{A}_\varpi^{-1}} + \frac{4|\chi|^2}{(1-|\chi|^2)^2} \underset{4\times 2}{(\mathbf{A}_\varpi^{-1}\mathbf{y})} \underbrace{diag\{r_1, r_2\}(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}\, diag\{r_1, r_2\}}_{2\times 2} \underset{2\times 4}{(\mathbf{A}_\varpi^{-1}\mathbf{y})^T} \tag{9}$$

where $r_1 = \sqrt{S_{11}/S_{22}}$ and $r_2 = \sqrt{S_{22}/S_{11}}$; $\chi = S_{21}/\sqrt{S_{11}S_{22}} = |\chi| e^{\mathbf{i}\phi}$;

$$\bar{\mathbf{A}}_\alpha = 2\underbrace{\begin{bmatrix} a_1 - |a_1'|^2 & \mathrm{Re}(1 - a_1' a_2') \\ \text{sym.} & a_2 - |a_2'|^2 \end{bmatrix}}_{\bar{\mathbf{A}}_{\alpha 0}} + \frac{2|\chi|^2}{1-|\chi|^2} \underbrace{\begin{bmatrix} a_1 - |a_1'|^2 & -\mathrm{Re}[(d - \bar{a}_1' a_2') e^{2\mathbf{i}\phi}] \\ \text{sym.} & a_2 - |a_2'|^2 \end{bmatrix}}_{\bar{\mathbf{A}}_{\alpha 1}} \tag{10}$$

$$\mathbf{A}_\varpi = 2\left(\mathbf{A}_{\varpi 0} + \frac{|\chi|^2}{1-|\chi|^2} \mathbf{A}_{\varpi 1} \right) \tag{11}$$

$$\mathbf{A}_{\varpi 0} = 2\begin{bmatrix} C_{f_1 f_1} & 0 & C_{f_1\zeta_1} & 0 \\ & C_{f_2 f_2} & 0 & C_{f_2\zeta_2} \\ & & C_{\zeta_1\zeta_1} & 0 \\ \text{sym.} & & & C_{\zeta_2\zeta_2} \end{bmatrix} \tag{12}$$

$$\mathbf{A}_{\varpi 1} = \begin{bmatrix} c_{f_1 f_1} & -c_{f_1 f_2} & c_{f_1\zeta_1} & -c_{f_1\zeta_2} \\ & c_{f_2 f_2} & -c_{f_2\zeta_1} & c_{f_2\zeta_2} \\ & & c_{\zeta_1\zeta_1} & -c_{\zeta_1\zeta_2} \\ \text{sym.} & & & c_{\zeta_2\zeta_2} \end{bmatrix} \tag{13}$$

$$\mathbf{B}_\alpha = \mathbf{y}^T \mathbf{A}_\varpi^{-1} \mathbf{y} \tag{14}$$

$$\mathbf{y}=\mathrm{Re}\begin{bmatrix} (a_{f11}-a_{f1}\overline{a_1'})e^{\mathbf{i}\phi} & -(a_{f12}-a_{f1}\overline{a_2'})e^{-\mathbf{i}\phi} \\ -(a_{f21}-a_{f2}\overline{a_1'})e^{\mathbf{i}\phi} & (a_{f22}-a_{f2}\overline{a_2'})e^{-\mathbf{i}\phi} \\ (a_{\zeta 11}-a_{\zeta 1}\overline{a_1'})e^{\mathbf{i}\phi} & -(a_{\zeta 12}-a_{\zeta 1}\overline{a_2'})e^{-\mathbf{i}\phi} \\ -(a_{\zeta 21}-a_{\zeta 2}\overline{a_1'})e^{\mathbf{i}\phi} & (a_{\zeta 22}-a_{\zeta 2}\overline{a_2'})e^{-\mathbf{i}\phi} \end{bmatrix} \tag{15}$$

The entry for the x_i-row ($x=f,\zeta$; $i=1,2$) and j th column ($j=1,2$) of $\mathbf{y}$ has the pattern

$$(-1)^{i-j}\,\mathrm{Re}[(a_{xij}-a_{x_i}\overline{a_j'})e^{s_j\mathbf{i}\phi}] \qquad s_1=1,\ s_2=-1 \tag{16}$$

Equations (10), (12), (13) and (15) contain coefficients (e.g., a_1, a_1') that are sums involving the FRFs and their derivatives. Their definitions are given in Table 4.

Table 4 Coefficients involving sums of FRFs; superscripted '(x)' denotes a derivative w.r.t. x

Relevance	Definition ($x,y=f,\zeta$)
$\overline{\mathbf{A}}_\alpha$ in (10)	$a_1=N_f^{-1}\Sigma\lvert\frac{h_{1k}}{h_{2k}}\rvert^2$, $a_2=N_f^{-1}\Sigma\lvert\frac{h_{2k}}{h_{1k}}\rvert^2$ $a_1'=N_f^{-1}\Sigma\frac{h_{1k}}{h_{2k}}$, $a_2'=N_f^{-1}\Sigma\frac{h_{2k}}{h_{1k}}$, $d=N_f^{-1}\Sigma\frac{h_{1k}^*}{h_{1k}}\frac{h_{2k}}{h_{2k}^*}$ $a_i-\lvert a_i'\rvert^2\ge 0$ $(a_1-\lvert a_1'\rvert^2)(a_2-\lvert a_2'\rvert^2)\ge\lvert 1-a_1'a_2'\rvert^2\ge[\mathrm{Re}(1-a_1'a_2')]^2$
$\mathbf{A}_{\varpi 0}$ in (12)	$A_{x_i}=N_f^{-1}\,\mathrm{Re}\Sigma\frac{h_{ik}^{(x_i)}}{h_{ik}}$, $A_{x_i y_j}=N_f^{-1}\Sigma\,\mathrm{Re}\frac{h_{ik}^{(x_i)}}{h_{ik}}\mathrm{Re}\frac{h_{jk}^{(y_j)}}{h_{jk}}$ $C_{x_i y_j}=A_{x_i y_i}-A_{x_i}A_{y_j}$
$\mathbf{A}_{\varpi 1}$ in (13)	$a_{x_i}=N_f^{-1}\Sigma\frac{h_{ik}^{(x_i)}}{h_{ik}}$, $a_{x_i y_j}=N_f^{-1}\Sigma\frac{h_{ik}^{(x_i)}}{h_{ik}}(\frac{h_{jk}^{(y_j)}}{h_{jk}})^*$ $c_{x_i y_j}=\mathrm{Re}(a_{x_i y_j}-a_{xi}\overline{a_{yj}})$
$\mathbf{y}$ in (15)	$a_{xi1}=N_f^{-1}\Sigma\frac{h_{ik}^{(x_i)}}{h_{ik}}(\frac{h_{1k}}{h_{2k}})^*$, $a_{xi2}=N_f^{-1}\Sigma\frac{h_{ik}^{(x_i)}}{h_{ik}}(\frac{h_{2k}}{h_{1k}})^*$

6.2 Wide band asymptotics

The partitions $(\overline{\mathbf{J}}'^{-1})_{\alpha\alpha}$ and $(\overline{\mathbf{J}}'^{-1})_{\varpi\varpi}$ in (8) and (9) involve matrices and inverses that depend on the coefficients in Table 4. The next natural step is to obtain asymptotic expressions for the coefficients, analogous to what was done for well-separated modes [20]. Generally, the coefficient is written as a Riemann sum and then approximated by an integral, which is asymptotically correct for long data. The resulting integral is then determined analytically using complex integration. Involving two FRFs, the resulting expressions are much more complicated than their single mode counterparts. They generally depend on the frequencies, damping ratios and the bandwidth factor κ in the specification of resonance band $[f_1(1-\kappa\zeta_1), f_2(1+\kappa\zeta_2)]$; see also a schematic diagram in Figure 1 of the companion paper [21]. Despite the asymptotic expressions for the coefficients, it has not been possible to obtain manageable expressions for $(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}$ and $(\bar{\mathbf{J}}'^{-1})_{\varpi\varpi}$. Nevertheless, it has been discovered that some coefficients simplify and bear some pattern when the resonance band is sufficiently wide; see mathematical definition shortly. Table 5 and Table 6 summarise the wide band asymptotic expressions for those coefficients. They allow $\mathbf{A}_{\varpi}$ and its inverse, the diagonal entries of $\overline{\mathbf{A}}_{\alpha}$, and the (1,1), (2,2), (3,1) and (4,2)-entries of $\mathbf{y}$ to be expressed in a manageable manner.

In Table 5,

$$c_i = \frac{f_j\zeta_j}{f_i\zeta_i} - 1 \qquad e_i = \frac{f_j - f_i}{f_i\zeta_i} \qquad d_i = \sqrt{c_i^2 + e_i^2} \tag{17}$$

are dimensionless quantities that are intuitively related to the difference in damping ratios, frequencies and in an overall sense, respectively; the index j refers to the other mode. These quantities appear frequently in the asymptotic expressions of coefficients and their definitions are motivated accordingly. The parameter d_i, referred as 'modal disparity', is an overall measure that makes its way into the final expressions of uncertainty law. Wide band refers to $\kappa >> e_i$ and $\kappa >> 1$. The parameter $b_i = \pi\zeta_i N_{ci} / N_f$ appears frequently as a scaling factor, where $N_{ci} = T_d f_i$ is a dimensionless data duration as a multiple of natural period. Since $N_f \propto \kappa\zeta_i N_{ci}$, it can be reasoned that $b_i = O(1/\kappa) << 1$ for wide band. In Table 6, the quantities

$$g_1 = \frac{2\sqrt{(2+c_1)(2+c_2)}}{|e_1 e_2| + (2+c_1)(2+c_2)} \qquad g_2 = \frac{2\sqrt{|e_1 e_2|}}{|e_1 e_2| + (2+c_1)(2+c_2)} \tag{18}$$

are less intuitive but they make their way into the final expression of uncertainty law. They, and related quantities such as q_1 and q_2 (see later), are perhaps manifestation of modal entangling.

Table 5 Wide band asymptotic expressions for coefficients with simple pattern; $b_i = \pi\zeta_i N_{ci} / N_f$

Relevance	Expression
$\overline{\mathbf{A}}_\alpha$ in (10)	$a_i \sim 1 + b_i c_i (c_i + 2) + b_i e_i^2$ $a_i' \sim 1 + b_i c_i - b_i e_i \mathbf{i}$ $a_i - \lvert a_i' \rvert^2 \sim b_i d_i^2$ where $d_i^2 = c_i^2 + e_i^2$ $\mathrm{Re}(1 - a_1' a_2') = b_1 b_2 (e_1 e_2 - c_1 c_2) << a_i - \lvert a_i' \rvert^2$
$\mathbf{A}_{\varpi 0}$ in (12)	$C_{f_i f_i} \sim \dfrac{b_i}{2 f_i^2 \zeta_i^2}$, $C_{\zeta_i \zeta_i} \sim \dfrac{b_i}{2\zeta_i^2}$, $C_{f_i \zeta_i} \sim \dfrac{b_i}{2 f_i \zeta_i}$
$\mathbf{y}$ in (15)	$a_{fii} - a_{fi} \overline{a_i'} \sim \dfrac{b_i}{f_i \zeta_i}(-e_i + c_i \mathbf{i})$ $a_{\zeta ii} - a_{\zeta i} \overline{a_i'} \sim \dfrac{b_i}{\zeta_i}(-c_i - e_i \mathbf{i})$

Table 6 Wide band asymptotic expressions for $c_{x_i y_j}$ **(** $x, y = f, \zeta$ **) in** $\mathbf{A}_{\varpi 1}$ **of (13);** $b_i = \pi\zeta_i N_{ci} / N_f$ **. To obtain the expression, take the entry and multiply by the factors indicated, e.g.,** $c_{f_1 \zeta_2} = (-g_2)(\sqrt{b_1} / f_1\zeta_1)(\sqrt{b_2} / \zeta_2)$

	f_1 ($\times \sqrt{b_1} / f_1\zeta_1$)	f_2 ($\times \sqrt{b_2} / f_2\zeta_2$)	ζ_1 ($\times \sqrt{b_1} / \zeta_1$)	ζ_2 ($\times \sqrt{b_2} / \zeta_2$)
f_1 ($\times \sqrt{b_1} / f_1\zeta_1$)	1	g_1	ζ_1	$-g_2$
f_2 ($\times \sqrt{b_2} / f_2\zeta_2$)		1	g_2	ζ_2
ζ_1 ($\times \sqrt{b_1} / \zeta_1$)			1	g_1
ζ_2 ($\times \sqrt{b_2} / \zeta_2$)	Symmetric			1

6.3 Limit identity and consistent coefficients

Issues remain for the (1,2)-entry of $\overline{\mathbf{A}}_{\alpha 1}$ in (10) that involves d ; and the (1,2), (2,1), (3,2) and (4,1)-entries of $\mathbf{y}$ in (15) that involve terms of the form $(a_{xij} - a_{x_i} \overline{a_j'})$ ($i \neq j$). It was not possible to find patterns of similar simplicity as in Table 5 and Table 6. An alternative, more fundamental, means is used to determine the coefficients. In particular, it was discovered that the coefficients must satisfy the identity

$$\mathbf{y}^T \mathbf{A}_{\varpi 1}^{-1} \mathbf{y} = \overline{\mathbf{A}}_{\alpha 1} \tag{19}$$

This follows from the observation that when $|\chi| \to 1$, i.e., perfectly coherent modal forces, $\overline{\mathbf{A}}_{\alpha}$ in (10) is dominated by $2|\chi|^2 \overline{\mathbf{A}}_{\alpha 1}/(1-|\chi|^2)$, $\mathbf{A}_{\varpi}$ in (11) is dominated by $2|\chi|^2 \mathbf{A}_{\varpi 1}/(1-|\chi|^2)$ and so from (14) $\mathbf{B}_{\alpha} \approx \mathbf{y}^T \mathbf{A}_{\varpi 1}^{-1} \mathbf{y}(1-|\chi|^2)/2|\chi|^2$. Substituting these into (8) suggests that the bracket is dominated by $2|\chi|^2 (\overline{\mathbf{A}}_{\alpha 1} - \mathbf{y}^T \mathbf{A}_{\varpi 1}^{-1} \mathbf{y})/(1-|\chi|^2)$. This would imply $(\overline{\mathbf{J}}'^{-1})_{\alpha\alpha} \to \mathbf{0}$ as $|\chi| \to 1$, i.e., mode shape uncertainty vanishes when modal forces are perfectly coherent, which is generally not true. See Section 12.1 for a formal analysis of this issue, which shows that (19) must hold. This 'limit identity' provides an effective means for obtaining the remaining coefficients in a consistent manner, which is otherwise a tedious and difficult task. Details can be found in Section 12.2.

As a result, the following expressions are obtained that allow $(\overline{\mathbf{J}}'^{-1})_{\alpha\alpha}$ in (8) and $(\overline{\mathbf{J}}'^{-1})_{\varpi\varpi}$ in (9) to be determined later in Sections 6.4 and 6.5, respectively. The matrix $\overline{\mathbf{A}}_{\alpha}$ in (10) is given by

$$\overline{\mathbf{A}}_{\alpha} \sim 2\underbrace{\begin{bmatrix} b_1 d_1^2 & 0 \\ \text{sym.} & b_2 d_2^2 \end{bmatrix}}_{\overline{\mathbf{A}}_{\alpha 0}} + \frac{2|\chi|^2}{1-|\chi|^2}\underbrace{\begin{bmatrix} b_1 d_1^2 & -q_1\sqrt{b_1 b_2}\, d_1 d_2 \\ \text{sym.} & b_2 d_2^2 \end{bmatrix}}_{\overline{\mathbf{A}}_{\alpha 1}} \tag{20}$$

where $q_1 = -g_1 \cos 2\phi + g_2 \sin 2\phi$. It follows from using $a_i - |a_i'|^2$ from Table 5, noting that $\mathrm{Re}(1 - a_1' a_2')$ is negligible and obtaining $\mathrm{Re}[(d - \overline{a_1'} a_2')e^{2\mathbf{i}\phi}] = q_1\sqrt{b_1 b_2}\, d_1 d_2$ from the limit identity (see (84) and (104)). For $\mathbf{A}_{\varpi}$ in (11), using Table 5 and Table 6 to obtain $\mathbf{A}_{\varpi 0}$ and $\mathbf{A}_{\varpi 1}$ in (85), we have

$$\mathbf{A}_{\varpi} \sim diag(\mathbf{a})\overline{\mathbf{A}}_{\varpi} diag(\mathbf{a}) \qquad \mathbf{a} = \begin{bmatrix} \dfrac{\sqrt{b_1}}{f_1 \zeta_1} & \dfrac{\sqrt{b_2}}{f_2 \zeta_2} & \dfrac{\sqrt{b_1}}{\zeta_1} & \dfrac{\sqrt{b_2}}{\zeta_2} \end{bmatrix} \tag{21}$$

$$\overline{\mathbf{A}}_{\varpi} \sim \frac{2}{1-|\chi|^2}\begin{bmatrix} 1 & -|\chi|^2 g_1 & 0 & |\chi|^2 g_2 \\ & 1 & -|\chi|^2 g_2 & 0 \\ & & 1 & -|\chi|^2 g_1 \\ \text{sym.} & & & 1 \end{bmatrix} \tag{22}$$

The matrix $\overline{\mathbf{A}}_{\varpi}$ has a special structure that allows its inverse to be obtained analytically as

$$\overline{\mathbf{A}}_{\varpi}^{-1} \sim \frac{1-|\chi|^2}{2(1-q_2^2|\chi|^4)}\begin{bmatrix} 1 & |\chi|^2 g_1 & 0 & -|\chi|^2 g_2 \\ & 1 & |\chi|^2 g_2 & 0 \\ & & 1 & |\chi|^2 g_1 \\ \text{sym.} & & & 1 \end{bmatrix} \tag{23}$$

The vector $\mathbf{y}$ in (15) is derived in Section 12.2 using the limit identity; see (79) and (91). The matrix $\mathbf{B}_\alpha$ in (14) is given by (see Section 12.4 for derivation)

$$\begin{aligned} \mathbf{B}_\alpha &= \frac{1-|\chi|^2}{2(1-q_2^2|\chi|^4)}\times \\ & diag\begin{Bmatrix} \sqrt{b_1}d_1 \\ \sqrt{b_2}d_2 \end{Bmatrix}\begin{bmatrix} 1+q_2^2(1-2|\chi|^2) & -[2-(1+q_2^2)|\chi|^2 q_1] \\ \text{sym.} & 1+q_2^2(1-2|\chi|^2) \end{bmatrix} diag\begin{Bmatrix} \sqrt{b_1}d_1 \\ \sqrt{b_2}d_2 \end{Bmatrix} \end{aligned} \tag{24}$$

where $q_2^2 = g_1^2 + g_2^2$. The vector $\mathbf{A}_{\varpi}^{-1}\mathbf{y}$ in (9) is given by (see Section 12.6 for derivation)

$$\mathbf{A}_{\varpi}^{-1}\mathbf{y} = \frac{1-|\chi|^2}{2} diag(\mathbf{a})^{-1}\mathbf{x}\ \ diag\{d_1\sqrt{b_1}, d_2\sqrt{b_2}\} \tag{25}$$

$$\mathbf{x} = \frac{-1}{1-q_2^2|\chi|^4}\begin{bmatrix} (1-q_2^2|\chi|^2)\cos(\phi - s_1\phi_1) & -(1-|\chi|^2)q_2\sin(\phi + s_1\phi_1 - \psi) \\ -(1-|\chi|^2)q_2\sin(\phi + s_2\phi_2 - \psi) & (1-q_2^2|\chi|^2)\cos(\phi - s_2\phi_2) \\ -(1-q_2^2|\chi|^2)\sin(\phi - s_1\phi_1) & (1-|\chi|^2)q_2\cos(\phi + s_1\phi_1 - \psi) \\ -(1-|\chi|^2)q_2\cos(\phi + s_2\phi_2 - \psi) & (1-q_2^2|\chi|^2)\sin(\phi - s_2\phi_2) \end{bmatrix} \tag{26}$$

where $s_1 = 1$ and $s_2 = -1$; $\tan\phi_i = c_i / e_i$ and $\tan\psi = g_1 / g_2$. The parameters involved in the wide band theory are summarised in Table 7. Some properties are also included. The identity $(g_1 - r)^2 + g_2^2 = r^2$ with $r = 1/\sqrt{(c_1+2)(c_2+2)}$ can be shown algebraically. A geometric interpretation is illustrated in Figure 6 of the companion paper. For modes with well-separated frequencies, $|e_i| \gg 1$ for both modes. Correspondingly, $|d_i| = O(|e_i|) \gg 1$ is large; $g_1 = O(1/|e_1 e_2|)$, $g_2 = O\left(1/\sqrt{|e_1 e_2|}\right)$ and $q_i = O\left(1/\sqrt{|e_1 e_2|}\right)$ are small compared to 1. As the frequency separation increases, i.e., $|e_i| \to \infty$, the point (g_1, g_2) on the circle in Figure 6 of the companion paper converges to the origin.

Table 7 Summary of parameters in the wide band theory

Relevance	Definition
Modal properties	f_1, f_2 (natural frequencies), ζ_1, ζ_2 (damping ratios), S_{11}, S_{22} (modal force auto-PSDs), $S_{21} = U_{21} + \mathbf{i}V_{21}$ (modal force cross-PSD), S_e (noise

	PSD), α_1, α_2 (angles of mode shape uncertainty) $\chi = S_{21} / \sqrt{S_{11} S_{22}} = \lvert \chi \rvert e^{\mathbf{i}\phi}$ (modal force coherence) $\rho = \boldsymbol{\varphi}_1^T \boldsymbol{\varphi}_2$ (cosine of angle between mode shapes)
Disparity	$c_i = \dfrac{f_j \zeta_j}{f_i \zeta_i} - 1$, $e_i = \dfrac{f_j - f_i}{f_i \zeta_i}$, $d_i = \sqrt{c_i^2 + e_i^2}$, $r_i = \sqrt{\dfrac{S_{ii}}{S_{jj}}}$
Modal entangling	$g_1 = \dfrac{2\sqrt{(2+c_1)(2+c_2)}}{\lvert e_1 e_2 \rvert + (2+c_1)(2+c_2)}$, $g_2 = \dfrac{2\sqrt{\lvert e_1 e_2 \rvert}}{\lvert e_1 e_2 \rvert + (2+c_1)(2+c_2)}$ $q_1 = -g_1 \cos 2\phi + g_2 \sin 2\phi = q_2 \sin(2\phi - \psi)$ $q_2^2 = g_1^2 + g_2^2$ $\tan \phi_i = c_i / e_i$, $\phi_2 = \phi_1 \pm \pi$, $\tan \psi = g_1 / g_2$
Miscellaneous	$N_{ci} = T_d f_i$ (dimensionless data duration) $b_i = \dfrac{N_{ci}}{N_f} \pi \zeta_i = O(1/\kappa) << 1$
Identities/properties	$b_1 c_1 + b_2 c_2 = 0$, $b_1 e_1 + b_2 e_2 = 0$, $c_1 / e_1 = c_2 / e_2$ c_1 and c_2 of opposite sign; e_1 and e_2 of opposite sign; $c_1 + c_2 = -c_1 c_2$, $e_1 + e_2 = -c_1 e_2 = -c_2 e_1$ $c_2 = -c_1 / (c_1 + 1)$, $e_2 = -e_1 / (c_1 + 1)$ $g_1 = \dfrac{2(c_1+2)\sqrt{c_1+1}}{(c_1+2)^2 + e_1^2}$, $g_2 = \dfrac{2e_1\sqrt{c_1+1}}{(c_1+2)^2 + e_1^2}$, $q_2 = 2\sqrt{\dfrac{c_1+1}{(c_1+2)^2 + e_1^2}}$ $(g_1 - r)^2 + g_2^2 = r^2$ and $q_2 = \sqrt{2 r g_1}$ where $r = \dfrac{1}{\sqrt{(c_1+2)(c_2+2)}} \le \dfrac{1}{2}$

6.4 Type 2 covariance matrix of mode shapes

Substituting (20) and (24) into (8) and taking inverse gives

$$(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha} = diag\{\frac{1}{r_1 d_1 \sqrt{b_1}}, \frac{1}{r_2 d_2 \sqrt{b_2}}\} \; \mathbf{D} \; diag\{\frac{1}{r_1 d_1 \sqrt{b_1}}, \frac{1}{r_2 d_2 \sqrt{b_2}}\} \tag{27}$$

$$\mathbf{D} = \frac{1 - q_2^2 \lvert \chi \rvert^4}{2(1 - q_2^2 \lvert \chi \rvert^2)(1 - q_1^2 \lvert \chi \rvert^4)} \begin{bmatrix} 1 & -q_1 \lvert \chi \rvert^2 \\ -q_1 \lvert \chi \rvert^2 & 1 \end{bmatrix} \tag{28}$$

Applying the factor $(1 - \rho^2) / N_f$ implied by (7) and Table 2 gives the covariance matrix of $\{\alpha_1, \alpha_2\}$:

$$\mathbf{C}_\alpha = \frac{1 - \rho^2}{N_f} (\bar{\mathbf{J}}'^{-1})_{\alpha\alpha} \tag{29}$$

Reading off the ith diagonal entry of $\mathbf{C}_\alpha$ gives the square of the Type 2 mode shape c.o.v.:

$$\delta''^2_{\boldsymbol{\varphi}_i} = \delta^2_{\alpha_i} = \mathbf{C}_\alpha(i,i) = \frac{1-\rho^2}{N_f r_i^2 d_i^2 b_i}\mathbf{D}(i,i) \tag{30}$$

Substituting $\mathbf{D}(i,i)$ from (28) and simplifying gives the final expression in (3) of the companion paper:

$$\delta''^2_{\boldsymbol{\varphi}_i} = \frac{1-\rho^2}{2\pi\zeta_i N_{ci} r_i^2 d_i^2} \times \frac{1-q_2^2 |\chi|^4}{(1-q_1^2|\chi|^4)(1-q_2^2|\chi|^2)} \tag{31}$$

The $(1,2)$-entry of $\mathbf{C}_\alpha$ is given by

$$\mathbf{C}_\alpha(1,2) = \frac{1-\rho^2}{N_f r_1 r_2 d_1 d_2 \sqrt{b_1 b_2}}\mathbf{D}(1,2) = -q_1|\chi|^2\,\delta_{\alpha_1}\delta_{\alpha_2} \tag{32}$$

This implies that the correlation between α_1 and α_2 is $-q_1|\chi|^2$. Note that mode shape uncertainty comprises a contribution from Type 1 (orthogonal to mode shape subspace) as well, although that one is asymptotically small and often negligible; see (44) in Section 9 (appendix).

6.5 Covariance matrix of frequencies and damping ratios

Substituting $(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}$ from (27) and $\mathbf{A}_\varpi^{-1}\mathbf{y}$ from (25) into (9), the covariance matrix of $\varpi = \{f_1, f_2, \zeta_1, \zeta_2\}$ is given by

$$\mathbf{C}_\varpi = N_f^{-1}(\bar{\mathbf{J}}'^{-1})_{\varpi\varpi} = N_f^{-1} diag(\mathbf{a})^{-1}[\bar{\mathbf{A}}_\varpi^{-1} + |\chi|^2\,\mathbf{x}\mathbf{D}\mathbf{x}^T]diag(\mathbf{a})^{-1} \tag{33}$$

where $\mathbf{D}$ is given by (28), $\bar{\mathbf{A}}_\varpi^{-1}$ is given by (23) and $\boldsymbol{a}$ is given by (21). The entries of $\mathbf{C}_\varpi$ can be obtained by noting that the (i,j)-entry of $\mathbf{x}\mathbf{D}\mathbf{x}^T$ is $\sum_{r,s} x_{ir} D_{rs} x_{js}$. Performing the multiplication for the first two diagonal entries and simplifying gives the variance of frequencies. Dividing by f_i^2 gives the c.o.v.s (squared):

$$\delta^2_{f_i} \sim \frac{\zeta_i}{2\pi N_{ci}}\left[\frac{1-|\chi|^2}{1-q_2^2|\chi|^4} + \frac{|\chi|^2 R_{f_i}}{(1-q_2^2|\chi|^2)(1-q_1^2|\chi|^4)(1-q_2^2|\chi|^4)}\right] \tag{34}$$

where the first term $\zeta / 2\pi N_{ci}$ is the familiar squared c.o.v. of frequency for well-separated modes [20]; the bracketed term carries the influence of coherence with

$$\begin{aligned} R_{f_i} &= (1-q_2^2 |\chi|^2)^2 \cos^2(\phi - s_i\phi_i) + (1-|\chi|^2)^2 q_2^2 \sin^2(\phi + s_i\phi_i - \psi) \\ &+ 2q_1q_2 |\chi|^2 (1-q_2^2|\chi|^2)(1-|\chi|^2)\cos(\phi - s_i\phi_i)\sin(\phi + s_i\phi_i - \psi) \end{aligned} \tag{35}$$

The posterior c.o.v. δ_{ζ_i} of damping ratio is of the same form as in (34) except that the cosines and sines are swapped in R_{ζ_i} :

$$\delta_{\zeta_i}^2 \sim \frac{1}{2\pi\zeta_i N_{ci}} \left[\frac{1-|\chi|^2}{1-q_2^2|\chi|^4} + \frac{|\chi|^2 R_{\zeta_i}}{(1-q_2^2|\chi|^2)(1-q_1^2|\chi|^4)(1-q_2^2|\chi|^4)} \right] \tag{36}$$

$$\begin{aligned} R_{\zeta_i} &= (1-q_2^2 |\chi|^2)^2 \sin^2(\phi - s_i\phi_i) + (1-|\chi|^2)^2 q_2^2 \cos^2(\phi + s_i\phi_i - \psi) \\ &+ 2q_1q_2 |\chi|^2 (1-q_2^2|\chi|^2)(1-|\chi|^2)\sin(\phi - s_i\phi_i)\cos(\phi + s_i\phi_i - \psi) \end{aligned} \tag{37}$$

Again, the first term $1/2\pi\zeta_i N_{ci}$ in (36) is the familiar squared c.o.v. of damping for well-separated modes.

6.6 Other data types

For the sake of discussion, data was assumed to be acceleration in Section 3 but the results in this work are generally applicable to other data types, e.g., velocity. The reasoning is as follow. The asymptotic FIM in Table 1 is expressed in terms of the modal response PSD matrix $\mathbf{H}_k$ and mode shapes through $\mathbf{Q}$ and $\mathbf{Q}_i$. Data type only affects $\mathbf{H}_k$ through the FRF h_{ik}, which in turn only affects the coefficients in Table 4. When data other than acceleration is used the FRF will be scaled by a conversion factor. For example, the FRF of velocity data is equal to the FRF of acceleration data divided by $2\pi \mathrm{f}_k \mathbf{i}$ where f_k is the FFT frequency (Hz). From Table 4, the coefficients all involve ratios of h_{ik} or its derivatives and so they are not affected by the data type. The only exception is the s/n ratio γ_i in (44), which should be revised to be the PSD ratio of the corresponding data type, e.g., divided by $(2\pi \mathrm{f}_k)^2$ for velocity instead of acceleration data.

7 Conclusions

Closed form analytical asymptotic expressions have been derived for the posterior c.o.v.s of natural frequencies, damping ratios and mode shapes for two close modes following classically damped stochastic linear dynamics identified with long ambient data, high s/n ratio and wide resonance band; see (31), (34), (36) and (44). Compared to well-separated modes, developing uncertainty law for close modes is much more challenging, owing to the correlation of mode shapes with all parameters and hence high dimension (ten for two modes) of the Fisher Information Matrix involved; asymptotic sums involving two frequency response functions and their derivatives; and highly non-trivial hidden algebraic pattern. Fortunately under wide band situation the problem still admits remarkably simple expressions and they have been discovered in this work. Insights, verification, scientific implications and recommendation for ambient test planning are discussed in the companion paper. Up to modelling assumptions and the use of probability for uncertainty quantification, the uncertainty law dictates the achievable precision regardless of the modal identification method used. This is because the Bayesian approach processes information (no more, no less) from data in strict accordance with probability and modelling assumptions. Although this work assumes acceleration data in its development, it is also applicable to other data types (e.g., velocity, displacement) provided that the s/n ratio is defined in a consistent manner with the data type (see Section 6.6).

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9 Appendix. Type 1 mode shape covariance matrix

It was shown in [23] that for long data and high s/n ratio ($S_e \to 0$) the covariance matrix of the vectorisation of mode shapes, $\mathbf{\Phi} := [\boldsymbol{\varphi}_1; \ldots; \boldsymbol{\varphi}_m]$ ($mn \times 1$), comprises the contributions from two types, i.e., $\mathbf{C}_{\mathbf{\Phi}:} = \mathbf{C}_{\mathbf{\Phi}:}^{(1)} + \mathbf{C}_{\mathbf{\Phi}:}^{(2)}$. Type 2 uncertainty is correlated among all parameters and so $\mathbf{C}_{\mathbf{\Phi}:}^{(2)}$ can only be obtained by determining the inverse of the reduced FIM $\mathbf{J}'$ in (3), which is highly non-trivial and discussed elsewhere in the main body of the paper. This section discusses $\mathbf{C}_{\mathbf{\Phi}:}^{(1)}$, which can be determined much more easily because Type 1 mode shape uncertainty is uncorrelated from all other parameters. It is equal to the inverse of $J^{(1)}_{\mathbf{\Phi}:\mathbf{\Phi}:} = 2N_f S_e^{-1}\mathbf{H} \otimes \mathbf{Q}$ in Table 1, where $\mathbf{H} = \Sigma \mathbf{H}_k / N_f$ is the average PSD matrix of data and $\mathbf{Q} = \mathbf{I}_n - \overline{\mathbf{\Phi}}(\overline{\mathbf{\Phi}}^T \overline{\mathbf{\Phi}})^{-1}\overline{\mathbf{\Phi}}^T$. By noting that $\mathbf{Q}\overline{\mathbf{\Phi}} = \mathbf{0}$ and $\mathbf{Qx} = \mathbf{x}$ for any $\mathbf{x}$ orthogonal to the mode shape subspace (MSS), it can be reasoned that $\mathbf{Q}$ is a zero mapping in the MSS and an identity mapping in the orthogonal complement of MSS. Using $(A \otimes B)^+ = A^+ \otimes B^+$ where the superscripted '+' denotes a pseudo-inverse, one obtains

$$\mathbf{C}^{(1)}_{\mathbf{\Phi}:} = \frac{S_e}{2N_f}\mathbf{H}^{-1} \otimes \mathbf{Q} \tag{38}$$

since $\mathbf{H}$ has full rank (assuming imperfect modal force coherence) and $\mathbf{Q}^+ = \mathbf{Q}$ (identity mapping in the orthogonal complement of MSS). Similar to $\mathbf{C}_{\mathbf{\Phi}:}$, the covariance matrix of each mode shape $\boldsymbol{\varphi}_i$ comprises contributions from the two types. The square of mode shape c.o.v. is equal to the trace of the covariance matrix and so has the same property. Thus,

$$\mathbf{C}_{\boldsymbol{\varphi}_i} = \mathbf{C}^{(1)}_{\boldsymbol{\varphi}_i} + \mathbf{C}^{(2)}_{\boldsymbol{\varphi}_i} \qquad \delta^2_{\boldsymbol{\varphi}_i} = \delta'^2_{\boldsymbol{\varphi}_i} + \delta''^2_{\boldsymbol{\varphi}_i} \tag{39}$$

Extracting the (i,i)-partition of $\mathbf{C}^{(1)}_{\mathbf{\Phi}:}$ in (38) gives the Type 1 covariance matrix of $\boldsymbol{\varphi}_i$:

$$\mathbf{C}^{(1)}_{\boldsymbol{\varphi}_i} = \frac{S_e}{2N_f}\mathbf{H}^{-1}(i,i)\mathbf{Q} \tag{40}$$

where $\mathbf{H}^{-1}(i,i)$ denotes the (i,i)-entry of $\boldsymbol{H}^{-1}$. Summing the eigenvalues of $\mathbf{C}^{(1)}_{\boldsymbol{\varphi}_i}$ gives the square of Type 1 mode shape c.o.v.:

$$\delta'^2_{\boldsymbol{\varphi}_i} = \frac{(n-m)S_e}{2N_f}\mathbf{H}^{-1}(i,i) \tag{41}$$

where we have used the fact that $\mathbf{Q}$ has m zero eigenvalues (in MSS) and $(n-m)$ eigenvalues equal to 1 (in the orthogonal complement of MSS). Since $\delta'^2_{\boldsymbol{\varphi}_i} \propto S_e$ from (44) and $\delta''^2_{\boldsymbol{\varphi}_i}$ in (31) does not depend on S_e, Type 1 mode shape uncertainty is asymptotically small for high s/n ratio and is often negligible in applications. When there is only one mode ($m = 1$) it can be verified that (44) reduces to the formula for the square of mode shape c.o.v. in [20]. This is a consistent result because in this case the orthogonal complement of MSS is null and $\delta''_{\varphi_i} = 0$.

Note that $\mathbf{H}(i,j) = S_{ij}\Sigma h_{ik} h_{jk}^* / N_f$. For high s/n ratio and wide band,

$$\frac{1}{N_f}\Sigma h_{ik} h_{ik}^* \sim \frac{b_i}{4\zeta_i^2} \qquad \frac{1}{N_f}\Sigma h_{1k} h_{2k}^* \sim \frac{\sqrt{b_1 b_2}}{4\zeta_1\zeta_2}(g_1 - \mathbf{i}g_2) = \frac{\sqrt{b_1 b_2}}{4\zeta_1\zeta_2} q_2 e^{\mathbf{i}(\psi - \pi/2)} \tag{42}$$

where $b_i = \pi\zeta_i N_{ci} / N_f$, $q_2 = \sqrt{g_1^2 + g_2^2}$, g_1 and g_2 are given in Table 7. Using these results for $m = 2$ modes gives

$$\mathbf{H} = diag\{\frac{\sqrt{b_1 S_{11}}}{2\zeta_1}, \frac{\sqrt{b_2 S_{22}}}{2\zeta_2}\}\begin{bmatrix} 1 & \text{sym.} \\ \overline{\chi} q_2 e^{-\mathbf{i}(\psi - \pi/2)} & 1 \end{bmatrix} diag\{\frac{\sqrt{b_1 S_{11}}}{2\zeta_1}, \frac{\sqrt{b_2 S_{22}}}{2\zeta_2}\} \tag{43}$$

and hence $\mathbf{H}^{-1}(i,i) = 4\zeta_i^2 / b_i S_{ii}(1 - |\chi|^2 q_2^2)$. Substituting this and $b_i = \pi\zeta_i N_{ci} / N_f$ into (41) and simplifying gives

$$\delta'^2_{\varphi_i} = \underbrace{\overbrace{\frac{1}{2\pi\zeta_i N_{ci}}}^{\substack{\text{same as for} \\ \text{damping}}} \times \overbrace{\frac{1}{\gamma_i}}^{\text{s/n effect}} \times \overbrace{(n-2)}^{\substack{\text{Dim.}\perp \\ \text{to MSS}}}}_{\substack{\text{analogous to well-} \\ \text{separated modes}}} \times \underbrace{\frac{1}{1 - q_2^2 |\chi|^2}}_{\substack{\text{coherence} \\ \text{effect}}} \qquad \gamma_i = \frac{S_{ii}}{4 S_e \zeta_i^2} \tag{44}$$

where γ_i is the modal s/n ratio considering Mode i only.

10 Appendix. Derivation of reduced FIM in Table 3

In this section we derive the expressions for the reduced FIM $\mathbf{J}'$ in (6) and hence its normalised form $\overline{\mathbf{J}}'$ in Table 3. The results are derived separately for three groups of parameters, i.e., $\{f_i, \zeta_i\}$, $\{S_{ij}\}$ and $\boldsymbol{\alpha}$, as they have different mathematical roles in $\mathbf{E}_k$ of (1). We use the short-cut notation '$x, y = \{f, \zeta\}$' to denote that the variables $\{x, y\}$ are frequencies or damping ratios; and '$x, y = \mathbf{S}$' to denote that $\{x, y\}$ are auto/cross PSDs of modal force.

10.1 Entries J_{xy} for $x, y = \{f, \zeta\}$

Recall from Table 1 that for any $x, y = \{f, \zeta\}$, the entry of FIM is

$$J_{xy} = tr\Sigma\mathbf{H}_k^{-1}\mathbf{H}_k^{(x)}\mathbf{H}_k^{-1}\mathbf{H}_k^{(y)} \tag{45}$$

where $\mathbf{H}_k$ is the PSD matrix of modal response. For analysis purpose we write $\mathbf{H}_k = [s_r h_{rk}]\boldsymbol{\chi}[s_r h_{rk}]^*$ where $s_r = \sqrt{S_{rr}}$, $\boldsymbol{\chi}$ ($m \times m$) is a coherence matrix with (i, j)-entry $\boldsymbol{\chi}(i, j) = S_{ij} / \sqrt{S_{ii}S_{jj}}$, and $[s_r h_{rk}]$ denotes a diagonal matrix containing $\{s_1 h_{1k}, \ldots, s_m h_{mk}\}$. We first show the general formula

$$J_{xy} = 2\,Re\,tr\Sigma[h_{rk}^{-1}h_{rk}^{(x)}h_{rk}^{-1}h_{rk}^{(y)}] + 2\,Re\,tr\Sigma\boldsymbol{\chi}^{-1}[h_{rk}^{-1}h_{rk}^{(x)}]\boldsymbol{\chi}[h_{rk}^{-1}h_{rk}^{(y)}]^* \tag{46}$$

First note that

$$\mathbf{H}_k^{(x)} = [s_r h_{rk}^{(x)}]\boldsymbol{\chi}[s_r h_{rk}]^* + [s_r h_{rk}]\boldsymbol{\chi}[s_r h_{rk}^{(x)}]^* \tag{47}$$

Substituting into (45) and using $tr[X(A + A^*)X(B + B^*)] = 2\,\mathrm{Re}\,tr[XAX(B + B^*)]$ for Hermitian X (see Appendix of [23]) gives

$$J_{xy} = 2\,\mathrm{Re}\,tr\Sigma\mathbf{H}_k^{-1}[s_r h_{rk}^{(x)}]\boldsymbol{\chi}[s_r h_{rk}]^*\mathbf{H}_k^{-1}([s_r h_{rk}^{(y)}]\boldsymbol{\chi}[s_r h_{rk}]^* + [s_r h_{rk}]\boldsymbol{\chi}[s_r h_{rk}^{(y)}]^*) \tag{48}$$

Substituting $\mathbf{H}_k^{-1} = [s_r^{-1}h_{rk}^{-1}]^*\boldsymbol{\chi}^{-1}[s_r^{-1}h_{rk}^{-1}]$, the product before the parenthesis can be simplified as

$$\begin{aligned}
&\mathbf{H}_k^{-1}[s_r h_{rk}^{(x)}]\boldsymbol{\chi}[s_r h_{rk}]^*\mathbf{H}_k^{-1} \\
&= [s_r^{-1}h_{rk}^{-1}]^*\boldsymbol{\chi}^{-1}[s_r^{-1}h_{rk}^{-1}] \times [s_r h_{rk}^{(x)}]\boldsymbol{\chi}[s_r h_{rk}]^* \times [s_r^{-1}h_{rk}^{-1}]^*\boldsymbol{\chi}^{-1}[s_r^{-1}h_{rk}^{-1}] \\
&= [s_r^{-1}h_{rk}^{-1}]^*\boldsymbol{\chi}^{-1}[h_{rk}^{-1}h_{rk}^{(x)}][s_r^{-1}h_{rk}^{-1}]
\end{aligned} \tag{49}$$

Substituting this into (48) and expanding the parenthesis there, the first term is given by

$$\begin{aligned}
&2\,\mathrm{Re}\,tr\Sigma[s_r^{-1}h_{rk}^{-1}]^*\boldsymbol{\chi}^{-1}[h_{rk}^{-1}h_{rk}^{(x)}][s_r^{-1}h_{rk}^{-1}][s_r h_{rk}^{(y)}]\boldsymbol{\chi}[s_r h_{rk}]^* \\
&= 2\,\mathrm{Re}\,tr\Sigma\boldsymbol{\chi}[s_r h_{rk}]^*[s_r^{-1}h_{rk}^{-1}]^*\boldsymbol{\chi}^{-1}[h_{rk}^{-1}h_{rk}^{(x)}][h_{rk}^{-1}h_{rk}^{(y)}] \\
&= 2\,\mathrm{Re}\,tr\Sigma[h_{rk}^{-1}h_{rk}^{(x)}h_{rk}^{-1}h_{rk}^{(y)}]
\end{aligned} \tag{50}$$

where in arriving at the first equality we have used the cyclic property of trace, i.e., $tr(AB) = tr(BA)$, to move $\boldsymbol{\chi}[s_r h_{rk}]^*$ at the right end to the left end. Similarly, the second term in (48) after expanding the parenthesis is given by

$$
\begin{aligned}
&2\operatorname{Re} tr\Sigma[s_r^{-1}h_{rk}^{-1}]^* \boldsymbol{\chi}^{-1}[h_{rk}^{-1}h_{rk}^{(x)}][s_r^{-1}h_{rk}^{-1}][s_r h_{rk}]\boldsymbol{\chi}[s_r h_{rk}^{(y)}]^* \\
&= 2\operatorname{Re} tr\Sigma\boldsymbol{\chi}^{-1}[h_{rk}^{-1}h_{rk}^{(x)}]\boldsymbol{\chi}[s_r h_{rk}^{(y)}]^*[s_r^{-1}h_{rk}^{-1}]^* \\
&= 2\operatorname{Re} tr\Sigma\boldsymbol{\chi}^{-1}[h_{rk}^{-1}h_{rk}^{(x)}]\boldsymbol{\chi}[h_{rk}^{-1}h_{rk}^{(y)}]^*
\end{aligned}
\tag{51}
$$

where in arriving at the first equality we have used the cyclic property of trace to move $[s_r^{-1}h_{rk}^{-1}]^*$ at the left end to the right end. Combining (50) and (51) gives (46).

The first term in (46) is zero if x and y do not belong to the same mode. Otherwise it is equal to $2\operatorname{Re}\Sigma h_{ik}^{-1}h_{ik}^{(x_i)}h_{ik}^{-1}h_{ik}^{(y_i)}$ when both belong to Mode i. Using $\operatorname{Re}(ab) = 2(\operatorname{Re} a)(\operatorname{Re} b) - \operatorname{Re}(ab^*)$, it can be written as

$$
2\operatorname{Re}\Sigma h_{ik}^{-1}h_{ik}^{(x_i)}h_{ik}^{-1}h_{ik}^{(y_i)} = N_f(4A_{x_i y_i} - 2\operatorname{Re} a_{x_i y_i})
\tag{52}
$$

where

$$
A_{x_i y_j} = N_f^{-1}\Sigma \operatorname{Re} h_{ik}^{-1}h_{ik}^{(x_i)} \operatorname{Re} h_{jk}^{-1}h_{jk}^{(y_j)} \qquad a_{x_i y_j} = N_f^{-1}\Sigma h_{ik}^{-1}h_{ik}^{(x_i)}(h_{jk}^{-1}h_{jk}^{(y_j)})^*
\tag{53}
$$

On the other hand, by noting that $[h_{rk}^{-1}h_{rk}^{(x_i)}] = h_{ik}^{-1}h_{ik}^{(x_i)}\mathbf{e}_i\mathbf{e}_i^T$ where $\mathbf{e}_i$ is a $m\times 1$ zero vector except at the i th entry equal to 1, the second term in (46) can be simplified as

$$
\begin{aligned}
&2\operatorname{Re} tr\Sigma\boldsymbol{\chi}^{-1}h_{ik}^{-1}h_{ik}^{(x_i)}\mathbf{e}_i\mathbf{e}_i^T\boldsymbol{\chi}(h_{jk}^{-1}h_{jk}^{(y_j)})^*\mathbf{e}_j\mathbf{e}_j^T \\
&= 2\operatorname{Re}\Sigma h_{ik}^{-1}h_{ik}^{(x_i)}(h_{jk}^{-1}h_{jk}^{(y_j)})^*(\mathbf{e}_j^T\boldsymbol{\chi}^{-1}\mathbf{e}_i)(\mathbf{e}_i^T\boldsymbol{\chi}\mathbf{e}_j) \\
&= 2N_f \operatorname{Re}[a_{x_i y_j}\boldsymbol{\chi}^{-1}(j,i)\boldsymbol{\chi}(i,j)]
\end{aligned}
\tag{54}
$$

where in arriving at the first equality we have used the cyclic property of trace to move $\mathbf{e}_j^T$ from the right end to the left end. In summary, combining the two terms gives

$$
J_{x_i y_j} = N_f \times \begin{cases} 4A_{x_i y_i} + 2\operatorname{Re}\{a_{x_i y_i}[\boldsymbol{\chi}^{-1}(i,i) - 1]\} & i = j \\ 2\operatorname{Re}[a_{x_i y_j}\boldsymbol{\chi}^{-1}(j,i)\boldsymbol{\chi}(i,j)] & i \neq j \end{cases}
\tag{55}
$$

For the case of two modes, substituting the values of $\boldsymbol{\chi}^{-1}(j,i)$ and $\boldsymbol{\chi}(i,j)$ from the following gives the expressions for $\{f_1, f_2, \zeta_1, \zeta_2\}$ in Table 3:

$$\boldsymbol{\chi} = \begin{bmatrix} 1 & \bar{\chi} \\ \chi & 1 \end{bmatrix} \qquad \boldsymbol{\chi}^{-1} = \frac{1}{1-|\chi|^2}\begin{bmatrix} 1 & -\bar{\chi} \\ -\chi & 1 \end{bmatrix} \tag{56}$$

10.2 Entries J_{xy} for $x, y = \mathbf{S}$

For $\{x, y\}$ from $\mathbf{S}$, we write $\mathbf{H}_k = [h_{rk}]\mathbf{S}[h_{rk}]^*$. Substituting $\mathbf{H}_k^{-1} = [h_{rk}^{-1}]^*\mathbf{S}^{-1}[h_{rk}^{-1}]$ into (45) and using $\mathbf{H}_k^{(x)} = [h_{rk}]\mathbf{S}^{(x)}[h_{rk}]^*$ gives

$$\begin{aligned} J_{xy} &= tr\Sigma[h_{rk}^{-1}]^*\mathbf{S}^{-1}[h_{rk}^{-1}][h_{rk}]\mathbf{S}^{(x)}[h_{rk}]^*[h_{rk}^{-1}]^*\mathbf{S}^{-1}[h_{rk}^{-1}][h_{rk}]\mathbf{S}^{(y)}[h_{rk}]^* \\ &= N_f tr(\mathbf{S}^{-1}\mathbf{S}^{(x)}\mathbf{S}^{-1}\mathbf{S}^{(y)}) \end{aligned} \tag{57}$$

where we have used the cyclic property of trace to move $[h_{rk}]^*$ from the right end to the left end. For the case of two modes, parameterising $S_{21} = U_{21} + \mathbf{i}V_{21}$ and using the following gives the expressions for $\{S_{11}, S_{22}, U_{21}, V_{21}\}$ in Table 3:

$$\mathbf{S}^{(S_{11})} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{S}^{(S_{22})} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{S}^{(U_{21})} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{S}^{(V_{21})} = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} \tag{58}$$

10.3 Entries J_{xy} for $x = \{f, \zeta\}$ and $y = \mathbf{S}$

For x from $\{f, \zeta\}$ and y from $\mathbf{S}$, substituting $\mathbf{H}_k^{(x)}$ from (47) into (45) gives

$$\begin{aligned} J_{xy} &= tr\Sigma\mathbf{H}_k^{-1}\left\{[s_r h_{rk}^{(x)}]\boldsymbol{\chi}[s_r h_{rk}]^* + [s_r h_{rk}]\boldsymbol{\chi}[s_r h_{rk}^{(x)}]^*\right\}\mathbf{H}_k^{-1}\mathbf{H}_k^{(y)} \\ &= 2\,\mathrm{Re}\,tr\Sigma\mathbf{H}_k^{-1}[s_r h_{rk}^{(x)}]\boldsymbol{\chi}[s_r h_{rk}]^*\mathbf{H}_k^{-1}\mathbf{H}_k^{(y)} \end{aligned} \tag{59}$$

where we have used the identity $tr[X(A + A^*)XY] = 2\,\mathrm{Re}\,tr[XAXY]$ for Hermitian X and Y. Substituting $\mathbf{H}_k^{-1} = [s_r^{-1}h_{rk}^{-1}]^*\boldsymbol{\chi}^{-1}[s_r^{-1}h_{rk}^{-1}]$ and $\mathbf{H}_k^{(y)} = [h_{rk}]\mathbf{S}^{(y)}[h_{rk}]^*$, and simplifying using the cyclic property of trace gives the general expression:

$$J_{xy} = 2\,\mathrm{Re}\,tr\Sigma[s_r^{-1}]\boldsymbol{\chi}^{-1}[s_r^{-1}h_{rk}^{-1}h_{rk}^{(x)}]\mathbf{S}^{(y)} \tag{60}$$

Evaluating it for the case of two modes using (56) and (58) gives the expressions in the cross terms for $\{f_1, f_2, \zeta_1, \zeta_2\}$ and $\{S_{11}, S_{22}, U_{21}, V_{21}\}$ in Table 3.

10.4 Entries involving $\{\alpha_1, \alpha_2\}$

When there are two modes, each mode shape has $(m-1)=1$ direction of Type 2 uncertainty, which is within the mode shape subspace but orthogonal to the mode shape (norm-constrained). For Mode 1, the basis vector along this direction is simply $\boldsymbol{\varphi}_2 - \rho\boldsymbol{\varphi}_1$ where $\rho = \boldsymbol{\varphi}_1^T \boldsymbol{\varphi}_2$ is the modal assurance criterion (MAC) between $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ (assume $\|\boldsymbol{\varphi}_i\|=1$). Similarly, for Mode 2 the basis vector is $\boldsymbol{\varphi}_1 - \rho\boldsymbol{\varphi}_2$. The basis matrix $\mathbf{U}$ ($2n \times 2$) for Type 2 uncertain mode shape deviation $\Delta\boldsymbol{\Phi} := \mathbf{U}\boldsymbol{\alpha}$ in (2) is taken as

$$\mathbf{U} = \frac{1}{\sqrt{1-\rho^2}} \begin{bmatrix} \boldsymbol{\varphi}_2 - \rho\boldsymbol{\varphi}_1 & \\ & \boldsymbol{\varphi}_1 - \rho\boldsymbol{\varphi}_2 \end{bmatrix} \tag{61}$$

Check that the two columns of $\mathbf{U}$ are orthogonal and they have unit norm, i.e., $\mathbf{U}^T\mathbf{U} = \mathbf{I}_2$. Substituting $\mathbf{U}$ into the partitions of $\mathbf{J}'$ in (3) related to $\{\alpha_1, \alpha_2\}$ and simplifying gives the expressions in Table 3.

11 Appendix. Derivation of $(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}$ in (8) and $(\bar{\mathbf{J}}'^{-1})_{\varpi\varpi}$ in (9)

The partitions of inverses $(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}$ and $(\bar{\mathbf{J}}'^{-1})_{\varpi\varpi}$ are derived by condensing out the partitions of modal force auto/cross PSDs using block matrix inverse formula and matrix inverse lemma [25][26]. Let the reduced and normalised FIM $\bar{\mathbf{J}}'$ in (6) be partitioned as

$$\bar{\mathbf{J}}' = \begin{bmatrix} \bar{\mathbf{J}}'_{\varpi\varpi} & \bar{\mathbf{J}}'_{\varpi s} & \bar{\mathbf{J}}'_{\varpi\alpha} \\ & \bar{\mathbf{J}}'_{ss} & \bar{\mathbf{J}}'_{s\alpha} \\ \text{sym.} & & \bar{\mathbf{J}}'_{\alpha\alpha} \end{bmatrix} \tag{62}$$

where ϖ ($2m \times 1$) comprises the frequencies and damping ratios, s ($m^2 \times 1$) comprises the modal force auto/cross PSDs and α ($m(m-1)\times 1$) comprises the coordinates of Type 2 mode shape deviations w.r.t. the basis $\mathbf{U}$ in (2). According to the block matrix inverse formula, if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{63}$$

then

$$B = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22} \\ -A_{22}^{-1}A_{21}B_{11} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix} \tag{64}$$

The essence is that a particular partition (say) B_{22} in the full inverse can be expressed in terms of inverses of a lower dimension. Once B_{22} has been obtained, B_{11} and B_{12} can be expressed in terms of B_{22}.

11.1 Mode shapes

Taking '1' for $\{\varpi, s\}$ and '2' for α, the partition of inverse for α can be obtained from the (2,2)-partition in (64):

$$(\overline{\mathbf{J}}'^{-1})_{\alpha\alpha} = \left\{ \overline{\mathbf{J}}'_{\alpha\alpha} - \begin{bmatrix} \overline{\mathbf{J}}'_{\varpi\alpha} \\ \overline{\mathbf{J}}'_{s\alpha} \end{bmatrix}^T \begin{bmatrix} \overline{\mathbf{J}}'_{\varpi\varpi} & \overline{\mathbf{J}}'_{\varpi s} \\ \text{sym.} & \overline{\mathbf{J}}'_{ss} \end{bmatrix}^{-1} \begin{bmatrix} \overline{\mathbf{J}}'_{\varpi\alpha} \\ \overline{\mathbf{J}}'_{s\alpha} \end{bmatrix} \right\}^{-1} \tag{65}$$

Using (64) again but now with '1' for s and '2' for ϖ, the inverse inside the brace in (65) is given by

$$\begin{bmatrix} \overline{\mathbf{J}}'_{\varpi\varpi} & \overline{\mathbf{J}}'_{\varpi s} \\ \text{sym.} & \overline{\mathbf{J}}'_{ss} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}_{\varpi}^{-1} & \text{sym.} \\ -\overline{\mathbf{J}}'^{-1}_{ss}\overline{\mathbf{J}}'_{s\varpi}\mathbf{A}_{\varpi}^{-1} & \overline{\mathbf{J}}'^{-1}_{ss} + \overline{\mathbf{J}}'^{-1}_{ss}\overline{\mathbf{J}}'_{s\varpi}\mathbf{A}_{\varpi}^{-1}\overline{\mathbf{J}}'_{\varpi s}\overline{\mathbf{J}}'^{-1}_{ss} \end{bmatrix} \tag{66}$$

$$\mathbf{A}_{\varpi} = \overline{\mathbf{J}}'_{\varpi\varpi} - \overline{\mathbf{J}}'_{\varpi s}\overline{\mathbf{J}}'^{-1}_{ss}\overline{\mathbf{J}}'_{s\varpi} \tag{67}$$

Using (66), the second term in the brace in (65) can be written as, after algebra,

$$\begin{aligned} &\begin{bmatrix} \overline{\mathbf{J}}'_{\varpi\alpha} \\ \overline{\mathbf{J}}'_{s\alpha} \end{bmatrix}^T \begin{bmatrix} \overline{\mathbf{J}}'_{\varpi\varpi} & \overline{\mathbf{J}}'_{\varpi s} \\ \text{sym.} & \overline{\mathbf{J}}'_{ss} \end{bmatrix}^{-1} \begin{bmatrix} \overline{\mathbf{J}}'_{\varpi\alpha} \\ \overline{\mathbf{J}}'_{s\alpha} \end{bmatrix} \\ &= \overline{\mathbf{J}}'^T_{s\alpha}\overline{\mathbf{J}}'^{-1}_{ss}\overline{\mathbf{J}}'_{s\alpha} + (\overline{\mathbf{J}}'_{\varpi s}\overline{\mathbf{J}}'^{-1}_{ss}\overline{\mathbf{J}}'_{s\alpha} - \overline{\mathbf{J}}'_{\varpi\alpha})^T \mathbf{A}_{\varpi}^{-1} (\overline{\mathbf{J}}'_{\varpi s}\overline{\mathbf{J}}'^{-1}_{ss}\overline{\mathbf{J}}'_{s\alpha} - \overline{\mathbf{J}}'_{\varpi\alpha}) \end{aligned} \tag{68}$$

Substituting into (65) gives

$$(\overline{\mathbf{J}}'^{-1})_{\alpha\alpha} = [\mathbf{A}_{\alpha} - \mathbf{z}^T\mathbf{A}_{\varpi}^{-1}\mathbf{z}]^{-1} \tag{69}$$

$$\mathbf{A}_{\alpha} = \overline{\mathbf{J}}'_{\alpha\alpha} - \overline{\mathbf{J}}'^T_{s\alpha}\overline{\mathbf{J}}'^{-1}_{ss}\overline{\mathbf{J}}'_{s\alpha} \qquad \mathbf{z} = \overline{\mathbf{J}}'_{\varpi s}\overline{\mathbf{J}}'^{-1}_{ss}\overline{\mathbf{J}}'_{s\alpha} - \overline{\mathbf{J}}'_{\varpi\alpha} \tag{70}$$

The significance of (69) is that the matrices are of reduced dimension and it turns out that the inverses can be obtained in an analytically manageable form. For the case of two modes, substituting the partitions of $\overline{\mathbf{J}}'$ from Table 1 and simplifying gives

$$\mathbf{z} = \frac{2|\chi|}{1-|\chi|^2}\mathbf{y} \; diag\{r_1, r_2\} \tag{71}$$

where $\mathbf{y}$ is given by (15). Substituting into (69) gives (8) with the term $\mathbf{B}_\alpha = \mathbf{y}^T \mathbf{A}_\varpi^{-1}\mathbf{y}$ in (14). The inverse $\bar{\mathbf{J}}_{ss}'^{-1}$ can be obtained analytically. Substituting it together with the other partitions $\bar{\mathbf{J}}'_{\alpha\alpha}$ and $\bar{\mathbf{J}}'_{s\alpha}$ gives $\mathbf{A}_\alpha = diag\{r_1, r_2\}\bar{\mathbf{A}}_\alpha diag\{r_1, r_2\}$ where $\bar{\mathbf{A}}_\alpha$ is given in (10).

11.2 Natural frequencies and damping ratios

The partition of inverse for ϖ can be obtained by simply swapping the roles of ϖ and α in (69):

$$(\bar{\mathbf{J}}'^{-1})_{\varpi\varpi} = [\mathbf{A}_\varpi - \mathbf{z}\mathbf{A}_\alpha^{-1}\mathbf{z}^T]^{-1} \tag{72}$$

where $\mathbf{A}_\varpi$, $\mathbf{A}_\alpha$ and $\mathbf{z}$ are the same as before. The bracketed matrix is $2m \times 2m$ and it is almost singular when $\mathbf{z}\mathbf{A}_\alpha^{-1}\mathbf{z}^T$ (rank $m(m-1)$) dominates. This singularity can be bypassed by using the matrix inverse lemma:

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1} \tag{73}$$

Taking $A = \mathbf{A}_\varpi$, $U = \mathbf{z}$, $V = \mathbf{z}^T$ and $C = -\mathbf{A}_\alpha^{-1}$, and noting that $(C^{-1}+VA^{-1}U)^{-1}$ is just $-(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}$, we obtain

$$(\bar{\mathbf{J}}'^{-1})_{\varpi\varpi} = \mathbf{A}_\varpi^{-1} + (\mathbf{A}_\varpi^{-1}\mathbf{z})(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}(\mathbf{A}_\varpi^{-1}\mathbf{z})^T \tag{74}$$

Thus, once $(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}$ has been obtained, it can be substituted into (74) to give $(\bar{\mathbf{J}}'^{-1})_{\varpi\varpi}$ without further taking inverse. For the case of two modes, substituting (71) into (74) gives (9). For wide band, $\mathbf{A}_\varpi^{-1}$ and $(\bar{\mathbf{J}}'^{-1})_{\alpha\alpha}$ can be obtained analytically in a manageable form; see (21), (23) and (27).

12 Appendix. Limit identity and consistent coefficients

In the development of uncertainty law for two close modes, asymptotic expressions were first obtained using complex integration for all the coefficients in Table 4. For wide band situation some coefficients simplify and bear a mathematical pattern (Table 5 and Table 6) but others do not. Moreover, substituting the coefficients into the FIM was found to give unstable numerical behaviour when taking inverse to give the covariance matrix, setting aside accuracy issues. Such problem was later found to be associated with the violation of a condition that generated an artificial singular term in the calculations when $|\chi|$ is large. The condition turns out to provide an effective and fundamental means for obtaining a full set of consistent coefficients. This issue was discussed qualitatively in Section 6.3 and is now addressed formally in this section. We first derive the condition and then use it to determine the coefficients.

12.1 Limit identity

The artificial singular term appears in the bracket of $(\overline{\mathbf{J}}'^{-1})_{\alpha\alpha}$ in (8) when $|\chi|\to 1$. To study the behaviour of $(\overline{\mathbf{J}}'^{-1})_{\alpha\alpha}$ as $|\chi|\to 1$, let $z=(1-|\chi|^2)/|\chi|^2$ and write $\mathbf{A}_{\varpi}$ in (11) as

$$\mathbf{A}_{\varpi}=\frac{2}{z}\left[\mathbf{A}_{\varpi 1}+z\mathbf{A}_{\varpi 0}\right] \qquad\qquad z=\frac{1-|\chi|^2}{|\chi|^2} \tag{75}$$

Clearly, $z\to 0$ when $|\chi|\to 1$. Using a second order Taylor expansion for the inverse of the bracketed term in (75) w.r.t. z, $\mathbf{A}_{\varpi}^{-1}$ and hence $\mathbf{B}_{\alpha}$ are given by, as $z\to 0$,

$$\mathbf{A}_{\varpi}^{-1}\sim\frac{z}{2}\left[\mathbf{A}_{\varpi 1}^{-1}-z\mathbf{A}_{\varpi 1}^{-1}\mathbf{A}_{\varpi 0}\mathbf{A}_{\varpi 1}^{-1}+z^2\mathbf{A}_{\varpi 1}^{-1}\mathbf{A}_{\varpi 0}\mathbf{A}_{\varpi 1}^{-1}\mathbf{A}_{\varpi 0}\mathbf{A}_{\varpi 1}^{-1}\right] \tag{76}$$

$$\mathbf{B}_{\alpha}=\mathbf{y}^T\mathbf{A}_{\varpi}^{-1}\mathbf{y}\sim\frac{z}{2}\left[\mathbf{y}^T\mathbf{A}_{\varpi 1}^{-1}\mathbf{y}-z\mathbf{C}_1+z^2\mathbf{C}_2\right] \tag{77}$$

where $\mathbf{C}_1=\mathbf{y}^T\mathbf{A}_{\varpi 1}^{-1}\mathbf{A}_{\varpi 0}\mathbf{A}_{\varpi 1}^{-1}\mathbf{y}$; and $\mathbf{C}_2=\mathbf{y}^T\mathbf{A}_{\varpi 1}^{-1}\mathbf{A}_{\varpi 0}\mathbf{A}_{\varpi 1}^{-1}\mathbf{A}_{\varpi 0}\mathbf{A}_{\varpi 1}^{-1}\mathbf{y}$. Substituting $\mathbf{B}_{\alpha}$ in (77) and $\overline{\mathbf{A}}_{\alpha}=2\overline{\mathbf{A}}_{\alpha 0}+2|\chi|^2\,\overline{\mathbf{A}}_{\alpha 1}/(1-|\chi|^2)$ from (10) into (8) and simplifying gives, for $|\chi|\to 1$,

$$(\overline{\mathbf{J}}'^{-1})_{\alpha\alpha}\sim\frac{1}{2}diag\begin{Bmatrix}1/r_1\\1/r_2\end{Bmatrix}\left[\frac{\overline{\mathbf{A}}_{\alpha 1}-\mathbf{y}^T\mathbf{A}_{\varpi 1}^{-1}\mathbf{y}}{1-|\chi|^2}+\left(\overline{\mathbf{A}}_{\alpha 0}-\overline{\mathbf{A}}_{\alpha 1}+\frac{\mathbf{C}_1}{|\chi|^2}\right)-\frac{1-|\chi|^2}{|\chi|^4}\mathbf{C}_2\right]^{-1}diag\begin{Bmatrix}1/r_1\\1/r_2\end{Bmatrix} \tag{78}$$

If $\overline{\mathbf{A}}_{\alpha 1}\neq\mathbf{y}^T\mathbf{A}_{\varpi 1}^{-1}\mathbf{y}$ then the term inside the bracket involving $(1-|\chi|^2)^{-1}$ will grow unbounded as $|\chi|\to 1$. This would imply $(\overline{\mathbf{J}}'^{-1})_{\alpha\alpha}\to\mathbf{0}$, i.e., mode shape uncertainty vanishes in the limit, which is generally not true. Thus we must have $\mathbf{y}^T\mathbf{A}_{\varpi 1}^{-1}\mathbf{y}=\overline{\mathbf{A}}_{\alpha 1}$ as in (19). This is a general condition that must hold for any bandwidth.

12.2 Wide band coefficients from limit identity

The 'limit identity' in (19) provides an effective means for obtaining the asymptotic expressions of the entries in $\mathbf{y}$ in (15). For this purpose, write

$$\mathbf{y} = diag(\mathbf{a})\begin{bmatrix} F_{11} & -F_{12} \\ -F_{21} & F_{22} \\ Z_{11} & -Z_{12} \\ -Z_{21} & Z_{22} \end{bmatrix} diag\begin{Bmatrix} \sqrt{b_1} \\ \sqrt{b_2} \end{Bmatrix} \tag{79}$$

where, by construction,

$$F_{ij} = \frac{f_i \zeta_i}{b_i} \mathrm{Re}\left[(a_{fij} - a_{fi}\overline{a'_j}) e^{s_j \mathbf{i}\phi} \right] \qquad Z_{ij} = \frac{\zeta_i}{b_i} \mathrm{Re}\left[(a_{\zeta ij} - a_{\zeta i}\overline{a'_j}) e^{s_j \mathbf{i}\phi} \right] \tag{80}$$

$s_1 = 1$ and $s_2 = -1$; $b_i = \pi \zeta_i N_{ci} / N_f$ and $N_{ci} = T_d f_i$ is a dimensionless data duration as a multiple of natural period. In (79), the matrices on the left and right are introduced so that $diag(\mathbf{a})$ will cancel out with $diag(\mathbf{a})^{-1}$ of $\mathbf{A}_{\varpi 1}^{-1}$ (see (85) later); and $diag\{\sqrt{b_1}, \sqrt{b_2}\}$ follows the same scaling of $\overline{\mathbf{A}}_{\alpha 1}$ (see (83) later). The limit identity does not provide enough equations to determine all $\{F_{ij}, Z_{ij}\}$, but we can use the wide band expressions in Table 5 for $\{F_{ii}, Z_{ii}\}$ to determine the remaining ones $\{F_{ij}, Z_{ij} : i \neq j\}$. Specifically, using the expressions of $a_{fii} - a_{fi}\overline{a'_i}$ and $a_{\zeta ii} - a_{\zeta i}\overline{a'_i}$ in Table 5, we can write

$$F_{ii} = -e'_i \qquad Z_{ii} = -c'_i \tag{81}$$

where

$$e'_i = \mathrm{Re}[(e_i - c_i \mathbf{i}) e^{\mathbf{i} s_i \phi}] \qquad c'_i = \mathrm{Re}[(c_i + e_i \mathbf{i}) e^{\mathbf{i} s_i \phi}] \qquad (s_1 = 1,\ s_2 = 1) \tag{82}$$

Then we can determine F_{21} and Z_{21} from the (1,1)-entry of the limit identity; and similarly F_{12} and Z_{12} from the (2,2)-entry. After that, we can substitute them to evaluate the (1,2)-entry of the LHS of the identity, equating which with the RHS will give the (1,2)-entry of $\overline{\mathbf{A}}_{\alpha 1}$ in a consistent manner. The details are presented as follows.

For $\overline{\mathbf{A}}_{\alpha 1}$ in (10), using $a_i - |a'_i|^2 \sim b_i d_i^2$ from Table 5 for the diagonal entries, we can write

$$\overline{\mathbf{A}}_{\alpha 1} \sim diag\begin{Bmatrix} \sqrt{b_1} \\ \sqrt{b_2} \end{Bmatrix} \begin{bmatrix} d_1^2 & -d_{12} \\ \text{sym.} & d_2^2 \end{bmatrix} diag\begin{Bmatrix} \sqrt{b_1} \\ \sqrt{b_2} \end{Bmatrix} \tag{83}$$

where the (1,2)-entry has been written in terms of

$$d_{12} = \frac{\mathrm{Re}[(d - \overline{a_1'} a_2')e^{2\mathbf{i}\phi}]}{\sqrt{b_1 b_2}} \tag{84}$$

On the other hand, substituting the coefficients in Table 6 into (12) and (13) gives

$$\mathbf{A}_{\varpi 0} \sim diag(\mathbf{a}) diag(\mathbf{a}) \qquad \mathbf{A}_{\varpi 1} \sim diag(\mathbf{a}) \begin{bmatrix} 1 & -g_1 & 0 & g_2 \\ & 1 & -g_2 & 0 \\ & & 1 & -g_1 \\ \text{sym.} & & & 1 \end{bmatrix} diag(\mathbf{a}) \tag{85}$$

where $\mathbf{a}$ is given by (21); the (1,3) and (2,4)-entry are asymptotically zero for small damping. The matrix $\mathbf{A}_{\varpi 1}$ has a special structure that allows its inverse to be obtained analytically as

$$\mathbf{A}_{\varpi 1}^{-1} = \frac{1}{1-q_2^2} diag(\mathbf{a})^{-1} \begin{bmatrix} 1 & g_1 & 0 & -g_2 \\ & 1 & g_2 & 0 \\ & & 1 & g_1 \\ \text{sym.} & & & 1 \end{bmatrix} diag(\mathbf{a})^{-1} \tag{86}$$

where $q_2^2 = g_1^2 + g_2^2$. Substituting $F_{ii} = -e_i'$ and $Z_{ii} = -c_i'$ from (81) into $\mathbf{y}$ in (79), $\mathbf{A}_{\varpi 1}^{-1}$ from (86) and $\overline{\mathbf{A}}_{\alpha 1}$ from (83), the limit identity reads, after cancelling the factors $diag\{\sqrt{b_1}, \sqrt{b_2}\}$ on the RHS and LHS,

$$\frac{1}{1-q_2^2} \begin{bmatrix} e_1' & F_{12} \\ F_{21} & e_2' \\ c_1' & Z_{12} \\ Z_{21} & c_2' \end{bmatrix}^T \begin{bmatrix} 1 & g_1 & 0 & -g_2 \\ & 1 & g_2 & 0 \\ & & 1 & g_1 \\ \text{sym.} & & & 1 \end{bmatrix} \begin{bmatrix} e_1' & F_{12} \\ F_{21} & e_2' \\ c_1' & Z_{12} \\ Z_{21} & c_2' \end{bmatrix} = \begin{bmatrix} d_1^2 & -d_{12} \\ \text{sym.} & d_2^2 \end{bmatrix} \tag{87}$$

Performing matrix multiplication, the (1,1)-entry of the LHS of (87) reads

$$(1-q_2^2)^{-1}[F_{21}^2 + 2(c_1' g_2 + e_1' g_1)F_{21} + Z_{21}^2 + 2(c_1' g_1 - e_1' g_2)Z_{21} + c_1'^2 + e_1'^2] \tag{88}$$

Completing the squares, it can be rewritten as

$$(1-q_2^2)^{-1}[(F_{21}+c_1'g_2+e_1'g_1)^2+(Z_{21}+c_1'g_1-e_1'g_2)^2+C] \tag{89}$$

where $C=c_1'^2+e_1'^2-(c_1'g_2+e_1'g_1)^2-(c_1'g_1-e_1'g_2)^2$. Expanding the squares and simplifying gives $C=d_1^2(1-q_2^2)$. Substituting into (89), the (1,1)-entry on the LHS of (87) now reads

$$(1-q_2^2)^{-1}[(F_{21}+c_1'g_2+e_1'g_1)^2+(Z_{21}+c_1'g_1-e_1'g_2)^2]+d_1^2 \tag{90}$$

This must be equal to the (1,1)-entry on the RHS of (87), i.e., d_1^2. The two squared terms in the bracket of (90) must then be identically zero, which gives the expressions of F_{21} and Z_{21}. Applying the same procedure for the (2,2)-entry on the RHS of (87) gives F_{12} and Z_{12}. The overall results are

$$\begin{aligned} F_{12}&=c_2'g_2-e_2'g_1 \qquad & Z_{12}&=-c_2'g_1-e_2'g_2 \\ F_{21}&=-c_1'g_2-e_1'g_1 \qquad & Z_{21}&=-c_1'g_1+e_1'g_2 \end{aligned} \tag{91}$$

12.3 Coefficient d_{12} in (83)

Substituting (91) into the LHS of (87) to obtain the (1,2)-entry and equating with the (1,2)-entry of the RHS gives

$$(1-q_2^2)^{-1}(T_1+T_2+T_3)=-d_{12} \tag{92}$$

where

$$\begin{aligned} T_1&=g_1(c_1'c_2'+e_1'e_2')+g_2(c_1'e_2'-c_2'e_1') \\ T_2&=g_1(F_{12}F_{21}+Z_{12}Z_{21})+g_2(-F_{12}Z_{21}+F_{21}Z_{12}) \\ T_3&=e_1'F_{12}+e_2'F_{21}+c_1'Z_{12}+c_2'Z_{21} \end{aligned} \tag{93}$$

Substituting (91) and simplifying, it turns out that T_2 and T_3 can be expressed in terms of T_1:

$$T_2=q_2^2T_1 \qquad\qquad T_3=-2T_1 \tag{94}$$

Substituting into (92) and simplifying then gives $d_{12}=T_1$, i.e.,

$$d_{12}=g_1(c_1'c_2'+e_1'e_2')+g_2(c_1'e_2'-c_2'e_1') \tag{95}$$

See Section 12.5 later that obtains the final form $d_{12}=q_1d_1d_2$ in (104) by using compound angle formula.

12.4 Derivation of $\mathbf{B}_\alpha$ in (24)

The entries of $\mathbf{B}_\alpha$ can be evaluated by leveraging on the algebra already performed in Sections 12.2 and 12.3. The key observation is that replacing g_i in $\mathbf{A}_{\varpi 1}$ in (85) by $|\chi|^2 g_i$ and multiplying the resulting matrix by $2/(1-|\chi|^2)$ gives $\mathbf{A}_\varpi$ in (21). This implies that replacing g_i by $|\chi|^2 g_i$ and q_2^2 by $q_2^2|\chi|^4$ in $\mathbf{A}_{\varpi 1}^{-1}$ in (86) and multiplying the resulting expression by $(1-|\chi|^2)/2$ will give $\mathbf{A}_\varpi^{-1}$. Doing the same for $\mathbf{y}^T\mathbf{A}_{\varpi 1}^{-1}\mathbf{y}$ will give $\mathbf{B}_\alpha = \mathbf{y}^T\mathbf{A}_\varpi^{-1}\mathbf{y}$, without the need to perform the tedious matrix multiplications again. For the (1,1)-entry of $\mathbf{B}_\alpha$, we can make use of (90):

$$\begin{aligned}\mathbf{B}_\alpha(1,1) &= \left(\mathbf{y}^T\mathbf{A}_\varpi^{-1}\mathbf{y}\right)_{(1,1)} \\ &= \frac{b_1(1-|\chi|^2)}{2}\times\left\{\frac{[F_{21}+|\chi|^2(c_1'g_2+e_1'g_1)]^2+[Z_{21}+|\chi|^2(c_1'g_1-e_1'g_2)]^2}{1-q_2^2|\chi|^4}+d_1^2\right\} \\ &= \frac{b_1(1-|\chi|^2)}{2(1-q_2^2|\chi|^4)}\left[(1-|\chi|^2)^2(F_{21}^2+Z_{21}^2)+(1-q_2^2|\chi|^4)d_1^2\right]\end{aligned} \tag{96}$$

after recognising $c_1'g_2+e_1'g_1=-F_{21}$ and $c_1'g_1-e_1'g_2=-Z_{21}$ from (91). Using (91), one obtains

$$F_{21}^2+Z_{21}^2=(g_1^2+g_2^2)(c_1'^2+e_1'^2)=q_2^2d_1^2 \tag{97}$$

since $c_1'^2+e_1'^2=c_1^2+e_1^2=d_1^2$. Substituting (97) into (96) and simplifying gives the final expression for $\mathbf{B}_\alpha(1,1)$, which can be generalised to read

$$\mathbf{B}_\alpha(i,i)=b_id_i^2\frac{(1-|\chi|^2)}{2(1-q_2^2|\chi|^4)}\left[1+q_2^2(1-2|\chi|^2)\right] \qquad (i=1,2) \tag{98}$$

Similarly, using the LHS of (92),

$$\mathbf{B}_\alpha(1,2)=\left(\mathbf{y}^T\mathbf{A}_\varpi^{-1}\mathbf{y}\right)_{(1,2)}=\frac{\sqrt{b_1b_2}(1-|\chi|^2)}{2}\times\frac{|\chi|^2T_1+|\chi|^2T_2+T_3}{1-q_2^2|\chi|^4} \tag{99}$$

Note that $T_1=d_{12}$ and (94) implies $T_2=q_2^2d_{12}$ and $T_3=-2d_{12}$. Substituting into (99) gives

$$\mathbf{B}_\alpha(1,2)=-\frac{\sqrt{b_1b_2}\,d_{12}(1-|\chi|^2)}{2(1-q_2^2|\chi|^4)}\left[2-(1+q_2^2)|\chi|^2\right] \tag{100}$$

Assembling (98) and (100) in matrix form and using $d_{12} = qd_1d_2$ from (104) (see later) gives (24).

12.5 Expressions in terms of phase angles

Some quantities such as d_{12} in (95) can be written in a compact manner in terms of phase angles. Write $e_i - c_i\mathbf{i} = d_i e^{-\mathbf{i}\phi_i}$ where $d_i^2 = c_i^2 + e_i^2$ and $\tan\phi_i = c_i / e_i$. Then the terms in (82) can be written as

$$\begin{aligned}(e_i - c_i\mathbf{i})e^{\mathbf{i}s_i\phi} &= d_i e^{-\mathbf{i}(\phi_i - s_i\phi)} \\ (c_i + e_i\mathbf{i})e^{\mathbf{i}s_i\phi} &= (e_i - c_i\mathbf{i})\mathbf{i}e^{\mathbf{i}s_i\phi} = d_i e^{-\mathbf{i}(\phi_i - s_i\phi - \pi/2)}\end{aligned} \tag{101}$$

where $s_1 = 1$ and $s_2 = -1$. Taking the real part gives $e_i' = d_i\cos(\phi_i - s_i\phi)$ and $c_i' = d_i\sin(\phi_i - s_i\phi)$, or equivalently,

$$e_i' = d_i\cos(\phi - s_i\phi_i) \qquad\qquad c_i' = -s_i d_i\sin(\phi - s_i\phi_i) \tag{102}$$

Substituting into (95) gives

$$\begin{aligned}d_{12} &= g_1d_1d_2\left[-\sin(\phi-\phi_1)\sin(\phi+\phi_2)+\cos(\phi-\phi_1)\cos(\phi+\phi_2)\right] \\ &\quad + g_2d_1d_2\left[-\sin(\phi-\phi_1)\cos(\phi+\phi_2)-\sin(\phi+\phi_2)\cos(\phi-\phi_1)\right] \\ &= d_1d_2\left[g_1\cos(2\phi-\phi_1+\phi_2)-g_2\sin(2\phi-\phi_1+\phi_2)\right]\end{aligned} \tag{103}$$

Note that c_1 and c_2 are of opposite sign; so are e_1 and e_2. Also $c_1/e_1 = c_2/e_2$. These imply that $\phi_2 = \phi_1 \pm \pi$. Applying this to (103) and noting that a phase shift of π changes the sign of cosine and sine gives

$$d_{12} = q_1d_1d_2 \tag{104}$$

where $q_1 = -g_1\cos 2\phi + g_2\sin 2\phi$. If we write $g_1 = q_2\sin\psi$ and $g_2 = q_2\cos\psi$ where $\tan\psi = g_1/g_2$ and $q_2^2 = g_1^2 + g_2^2$, then using compound angle formula we can write $q_1 = q_2\sin(2\phi - \psi)$. Together with (102) it can be shown that

$$\begin{aligned}c_i'g_2 + s_ie_i'g_1 &= -s_id_iq_2\sin(\phi - s_i\phi_i - \psi) \\ c_i'g_1 - s_ie_i'g_2 &= -s_id_iq_2\cos(\phi - s_i\phi_i - \psi)\end{aligned} \qquad s_1 = 1, s_2 = -1 \tag{105}$$

These identities are used for obtaining the final expression of $\mathbf{A}_{\varpi}^{-1}\mathbf{y}$ in (25). See Section 12.6 for details.

12.6 Derivation of $\mathbf{A}_{\varpi}^{-1}\mathbf{y}$ in (25)

To derive the expression of $\mathbf{A}_{\varpi}^{-1}\mathbf{y}$ in (25), use $\mathbf{y}$ in (79) and $\mathbf{A}_{\varpi}^{-1} = diag(\mathbf{a})^{-1}\overline{\mathbf{A}}_{\varpi}^{-1}diag(\mathbf{a})^{-1}$ where $\overline{\mathbf{A}}_{\varpi}^{-1}$ is given by (23). When multiplying, use the following to simplify the (1,1), (2,2), (3,1) and (4,2)-entries:

$$\begin{aligned} g_1F_{21} - g_2Z_{21} &= -e_1'q_2^2 \qquad & g_1Z_{21} + g_2F_{21} &= -c_1'q_2^2 \\ g_1F_{12} + g_2Z_{12} &= -e_2'q_2^2 \qquad & g_1Z_{12} - g_2F_{12} &= -c_2'q_2^2 \end{aligned} \tag{106}$$

These identities can be verified by direct substitution of (91). One then obtains

$$\mathbf{A}_{\varpi}^{-1}\mathbf{y} = \frac{1-|\chi|^2}{2}diag(\mathbf{a})^{-1}\mathbf{x}'diag\{\sqrt{b_1},\sqrt{b_2}\} \tag{107}$$

$$\mathbf{x}' = \frac{-1}{1-q_2^2|\chi|^4}\begin{bmatrix} (1-q_2^2|\chi|^2)e_1' & (1-|\chi|^2)(c_2'g_2 - e_2'g_1) \\ -(1-|\chi|^2)(c_1'g_2 + e_1'g_1) & (1-q_2^2|\chi|^2)e_2' \\ (1-q_2^2|\chi|^2)c_1' & -(1-|\chi|^2)(c_2'g_1 + e_2'g_2) \\ -(1-|\chi|^2)(c_1'g_1 - e_1'g_2) & (1-q_2^2|\chi|^2)c_2' \end{bmatrix} \tag{108}$$

Use (102) and (105) to write the entries of $\mathbf{x}'$ in terms of phase angles. Then use $\phi_2 = \phi_1 \pm \pi$ to write the (1,2) and (3,2)-entries in terms of ϕ_1; and the (2,1) and (4,1) in terms of ϕ_2. These effectively involve a sign change only. After that, taking out the factors d_1 and d_2 will give the vector $\mathbf{x}$ in (26) and hence the final expression of $\mathbf{A}_{\varpi}^{-1}\mathbf{y}$ in (25). The factors d_1 and d_2 are absorbed into the rightmost diagonal matrix in (25).

13 References

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