On computing the H_2 norm using the polynomial Diophantine equation

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Abstract: An explicit algorithm will be presented for computing the H_2 norm of a single input single output system from the coefficients in its transfer function. The algorithm follows directly from Cauchy's residue theorem, and the most computationally intensive step involves solving a polynomial Diophantine equation. This can be efficiently solved using subresultant sequences in a fraction-free variant of the extended Euclidean algorithm. The coefficients in these subresultant sequences correspond to the Hurwitz determinants, whereby a stability test can be obtained alongside computing the H_2 norm with little additional computational effort. Implementations of the algorithm symbolically, in exact arithmetic, and in floating-point arithmetic will be presented. These will be applied to examples of passive train suspension systems that optimise passenger comfort. The examples will demonstrate the algorithm's greater robustness and computational efficiency relative to H_2 norm algorithms requiring the computation of the controllability or observability Gramians. Finally, applications to the realisation of optimal lumped-parameter systems will be discussed.

Keywords: Linear systems; H_2 optimal control; Mechanical systems; Symbolic computation; Real algebraic geometry; Subresultant sequences.

1. INTRODUCTION

The H_2 norm is a widely used metric for characterising system performance. It corresponds to the square root of the power spectral density of the system's output in response to zero mean white noise input of unit power spectral density. The design of the famous Linear Quadratic Gaussian controller corresponds to a H_2 norm minimisation problem. The H_2 norm is also a natural measure of system performance for the design of mechanical networks, such as vehicle suspension systems. For example, it is commonly used to characterise passenger comfort as a vehicle traverses a rough surface (see, e.g., Wang et al., 2009). There is therefore a need for efficient algorithms for the computation of the H_2 norm of a given system. In safety critical applications, or when numerical robustness is a consideration, it can be desirable to compute the H_2 norm using exact arithmetic. Moreover, in the design of lumped-parameter systems, it can be desirable to obtain symbolic expressions for the H_2 norm in terms of the system's parameters. For example, this is useful in the design of optimal mechanical networks, or in other structured H_2 norm optimisation problems, where it is necessary to choose one or more system parameters to optimise a H_2 norm performance measure.

In this paper, an algorithm will be presented for the computation of the H_2 norm of a single input single output system from the coefficients in its transfer function. The most computationally demanding step in the algorithm corresponds to solving a polynomial Diophantine equation. This equation arises from the application of Cauchy's residue theorem to evaluate the frequency domain integral formula for the H_2 norm. It will be shown how this equation can be solved efficiently by a fraction-free variant of the extended Euclidean algorithm, which corresponds to the computation of subresultant polynomials generated from the even and odd part of the denominator polynomial in the system's transfer function. Moreover, the coefficients in these subresultant polynomials correspond to Hurwitz determinants, whereby the stability of the system can be determined alongside the H_2 norm computation. Some examples from the design of mechanical networks will be presented and used to demonstrate the robustness and computational efficiency of the algorithm as compared to H_2 norm algorithms that involve computation of the controllability or observability Gramians. The algorithm is also applicable to the design of optimal lumped-parameter systems and more general structured H_2 norm optimisation problems.

The notation employed is as follows. $\mathbb{R}$ denotes the real numbers; $\mathbb{R}^{m\times n}$ the real matrices with m rows and n columns; and $\mathbb{R}[s]$ and $\mathbb{R}(s)$ the univariate polynomials and rational variables in the indeterminate s, respectively. If $p \in \mathbb{R}[s]$, then $\deg(p(s))$ denotes its degree, and $\mathrm{LC}(p(s))$ its leading coefficient. If $G \in \mathbb{R}(s)$, then $\|G\|_2$ denotes its H_2 norm. For a complex number z, its conjugate is denoted z^*, and j denotes the imaginary unit $\sqrt{-1}$. Finally if $x \in \mathbb{R}$, then $\lceil x \rceil$ rounds $x \in \mathbb{R}$ up to the next integer, and $\lfloor x \rfloor$ rounds $x \in \mathbb{R}$ down to the previous integer.

2. COMPUTING THE H_2 NORM USING THE POLYNOMIAL DIOPHANTINE EQUATION

The H_2 norm of a linear system has a number of well known equivalent characterisations. From the perspective of system performance, it is most naturally characterised as the power spectral density of the system's output when the input is zero-mean white noise whose power spectral density is equal to the identity matrix. For the purposes of computation, there are three alternative characterisations, corresponding to the system's impulse response, frequency response, and the controllability or observability Gramian. For the case of a (rational and strictly proper) single input single output system with impulse response $g(t)$, frequency response $G(s) \in \mathbb{R}(s)$ (the Laplace transform of $g(t)$), and state-space realization (A, B, C) (i.e., $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$ satisfy $G(s) = C(sI - A)^{-1}B$), these three alternative characterisations are as described next.

Firstly, the H_2 norm is the 2-norm of the impulse response, i.e., $\|G\|_2^2 = \int_{-\infty}^{\infty} g(t)^2 \mathrm{d}t$. Secondly, using Parseval's theorem, this can be evaluated using the frequency response $G(j\omega)$ as follows:

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)^* G(j\omega)\mathrm{d}\omega. \tag{1}$$

Thirdly, $\|G\|_2^2 = B^T L_o B = C L_c C^T$, where L_o and L_c are the observability and controllability Gramians, which are the solutions to the Lyapunov equations $A^T L_o + L_o A + C^T C = 0$ and $A L_c + L_c A^T + BB^T = 0$, respectively.

Similar characterisations to the above also hold for multi-input multi-output systems (see, e.g., Zhou et al., 1996, pp. 112–113). It should be noted that the H_2 norm as defined above requires the system to be stable, and accordingly this will be assumed to be the case throughout.

Owing to the abundance of efficient computational methods for solving Lyapunov equations, such as those characterising the observability and controllability Gramians, then algorithms for computing the H_2 norm typically employ the third of the aforementioned characterisations. While the Lyapunov equations themselves are linear in the entries in the observability and controllability Gramians, the size of such equations are considerably greater than the state dimension, and the solutions depend in a complicated manner on the entries in the matrices A, B and C. In contrast, in this paper, an algorithm will be presented based on the second of the aforementioned characterisations for the H_2 norm. The most computationally demanding step in the calculation corresponds to solving a structured linear equation of dimension equal to the state dimension. Moreover, the structural properties of this equation can be handled in a computationally efficient manner via a fraction-free variant of the extended Euclidean algorithm.

The algorithm for the computation of the H_2 norm will be stated in terms of the coefficients in the numerator and denominator polynomials of the transfer function:

$$G(s) = \frac{c(s)}{a(s)} = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \ldots + c_1 s + c_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}, \tag{2}$$

where, without loss of generality, we let $a_n > 0$, and we assume that $c(s)$ and $a(s)$ have no common roots in the closed right half plane (whereupon G is stable if and only if the roots of $a(s)$ are all in the open left half plane). Following Ablowitz (2003, pp. 220–221), it follows that the integral in equation (1) is equal to $\oint_D G(-s)G(s)\mathrm{d}s/(2\pi j)$, where D is a contour that traverses the imaginary axis from a point $s = -jR$ to the point $s = jR$, then follows a semicircular arc of radius R into the left half plane, for any given $R > 0$ such that this contour encloses all of the poles of $G(s)$ (see also Zhou et al., 1996).

Since the roots of $a(s)$ are all in the open left half plane, then $a(s)$ and $a(-s)$ have no roots in common, whereupon there exists a unique solution to the polynomial Diophantine equation $c(-s)c(s) = a(s)x(s) + a(-s)y(s)$ for which the degrees of $x(s)$ and $y(s)$ are strictly less than n. Moreover, it can be noted that if the pair $(x(s), y(s))$ satisfies the aforementioned equation, then so too does the pair $(y(-s), x(-s))$, and it follows that $x(s) = y(-s)$. In other words, $y(s)$ is the unique polynomial whose degree is strictly less than n that solves the equation

$$c(-s)c(s) = a(s)y(-s) + a(-s)y(s). \tag{3}$$

It follows that

$$2\pi j\|G\|_2^2 = \oint_D \frac{c(s)c(-s)}{a(s)a(-s)}\mathrm{d}s = \oint_D \frac{y(-s)}{a(-s)}\mathrm{d}s + \oint_D \frac{y(s)}{a(s)}\mathrm{d}s.$$

Since the roots of $a(s)$ are all in the open left half plane, then D contains all of the poles of $y(s)/a(s)$ and none of the poles of $y(-s)/a(-s)$, whereupon by Cauchy's residue theorem it follows that

$$\|G\|_2^2 = \frac{1}{2\pi j}\oint_D \frac{y(s)}{a(s)}\mathrm{d}s = \sum_{j=1}^{n} \mathrm{Res}\left(\frac{y(s)}{a(s)}, s_j\right),$$

where $s_1, \ldots, s_n$ denote the roots of the polynomial $a(s)$, and $\mathrm{Res}(y(s)/a(s), s_j)$ denotes the residue of $y(s)/a(s)$ at s_j ($j = 1, \ldots, n$). This is most conveniently evaluated using the concept of the *residue at infinity* (see Ablowitz, 2003, pp. 211–212). Specifically, from (Ablowitz, 2003, equations (4.1.13) and (4.1.14)), it follows that

$$\|G\|_2^2 = \sum_{j=1}^{n} \mathrm{Res}\left(\frac{y(s)}{a(s)}, s_j\right) = \lim_{s\to\infty}\left(\frac{sy(s)}{a(s)}\right) = \frac{y_{n-1}}{a_n},$$

where y_{n-1} denotes the coefficient of s^{n-1} in $y(s)$.

Now, let

$$\begin{aligned} c^e(s) &= c_0 + c_2 s + \ldots + c_{2\lfloor\frac{n-1}{2}\rfloor} s^{\lfloor\frac{n-1}{2}\rfloor}, \\ c^o(s) &= c_1 + c_3 s + \ldots + c_{2\lceil\frac{n-3}{2}\rceil+1} s^{\lceil\frac{n-3}{2}\rceil}, \\ \text{and } z(s) &= (c^e(s))^2 - s(c^o(s))^2, \end{aligned} \tag{4}$$

whereupon $c(s) = c^e(s^2) + sc^o(s^2)$ and

$$c(-s)c(s) = (c^e(s^2) - sc^o(s^2))(c^e(s^2) + sc^o(s^2)) = z(s^2).$$

Further, let

$$\begin{aligned} a^e(s) &= a_0 + a_2 s + \ldots + a_{2\lfloor\frac{n}{2}\rfloor} s^{\lfloor\frac{n}{2}\rfloor}, \\ \text{and } a^o(s) &= a_1 s + a_3 s^2 + \ldots + a_{2\lceil\frac{n}{2}\rceil-1} s^{\lceil\frac{n}{2}\rceil}, \end{aligned} \tag{5}$$

whereupon $a(s) = a^e(s^2) + \frac{1}{s}a^o(s^2)$. It can be shown that $a^e(s)$ and $a^o(s)$ do not share any common roots since the roots of $a(s)$ are all in the open left half plane, and it follows that there exist unique polynomials $f(s)$ and $g(s)$ such that the degree of $f(s)$ (resp. $g(s)$) is strictly less than the degree of $a^o(s)$ (resp. $a_e(s)$) and

$$a^e(s)f(s) + a^o(s)g(s) = z(s). \tag{6}$$

Then, with the notation

$$y(s) = \tfrac{1}{2}(f(s^2) - sg(s^2)), \tag{7}$$

it is easily shown that the degree of y is strictly less than n, and that $a(s)y(-s) + a(-s)y(s) = c(-s)c(s)$. In other words, $y(s)$ in equation (7) is the unique solution to equation (3) for which the degree of $y(s)$ is strictly less than n, whereupon the coefficient y_{n-1} of s^{n-1} in the polynomial $y(s)$ is determined from the solutions $f(s)$ and $g(s)$ to the polynomial Diophantine equation (6).

3. EFFICIENT COMPUTATION OF THE SOLUTION TO THE POLYNOMIAL DIOPHANTINE EQUATION

It has been shown that the H_2 norm of the system whose transfer function is as in (2) is equal to y_{n-1}/a_n, where y_{n-1} is the coefficient of s^{n-1} in the polynomial $y(s)$ in (7), where $f(s)$ and $g(s)$ are the solutions to the polynomial Diophantine equation (6). Here, $a_e(s)$, $a_o(s)$ and $z(s)$ are directly determined from the coefficients $c_0, c_1, \ldots c_{n-1}$ and $a_0, a_1, \ldots, a_n$ in the transfer function $G(s)$ using equations (4) and (5). In this and the next section, efficient algorithms for the computation of the solution to this Diophantine equation will be presented.

First, by equating coefficients of s in equation (6), it follows that the coefficients in the polynomials $f(s)$ and $g(s)$ can be obtained by finding the solution x to the linear equation

$$H^T x = b, \tag{8}$$

where H is the $n \times n$ Hurwitz matrix

$$H = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ a_n & a_{n-2} & a_{n-4} & \cdots \\ 0 & a_{n-1} & a_{n-3} & \cdots \\ 0 & a_n & a_{n-2} & \cdots \\ & & & \ddots \end{bmatrix}, \tag{9}$$

and $b^T = [z_{n-1} \; z_{n-2} \; \cdots \; z_1 \; z_0]$ where $z(s) = z_{n-1}s^{n-1} + z_{n-2}s^{n-2} + \ldots + z_1 s + z_0$. Here, if n is odd (resp., even), then the odd (resp., even) entries in the solution x to equation (8) correspond to the coefficients in $f(s)$ in equation (6), and the even (resp., odd) entries correspond to the coefficients in $g(s)$, in descending degree. In either case, it follows that $y_{n-1} = (-1)^{n+1} x_1/2$, where x_1 denotes the first entry in the solution x to equation (8).

The preceding characterisation of the H_2 norm is similar to the approach taken by Betser et al. (1995) to characterise the solution P to the Lyapunov equation $-PA - A^T P = Q$ in the case that A is a a companion matrix. In contrast to the preceding derivation, the result of Betser et al. (1995) used the theory of matrix polynomials. A similar approach was followed by Hughes (2016), where an alternative characterisation was also obtained in terms of the Bezoutian of the polynomials $a^o(s)$ and $a^e(s)$, which allows one to exploit the algorithms of Bini and Gemignani (1998) for efficient triangularisation of Bezoutians.

The benefits of the aforementioned formula will now be illustrated by considering a train suspension design problem. The authors Wang et al. (2009) considered the suspension system is shown in Fig. 1, where m_s, m_b and m_w represent the masses of the train body, bogie and wheel; and k_w and c_w represent the stiffness and damping coefficient for the train wheel. One problem considered by Wang et al. (2009) was to obtain suspension admittances $Q_1(s) = k_s/s + K_1(s)$ and $Q_2(s) = k_b/s + K_2(s)$ in order to optimise passenger comfort, which corresponds to minimising the H_2 norm of the transfer function from z_r to $\frac{dz_s}{dt}$ (hereafter denoted by J_1). In one example, the authors considered the constant admittance $K_1(s) = 8870$,

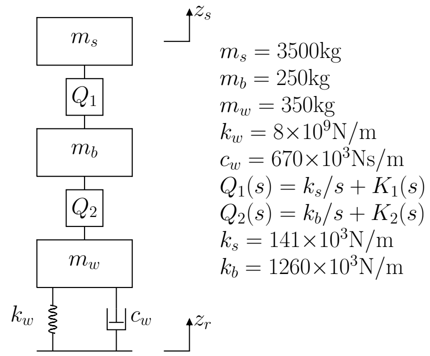

Fig. 1. One wheel train model (see (Wang et al., 2009)).

and $K_2(s)$ was optimised over the class of bicubic positive-real functions. This resulted in the admittance denoted $K_2^{\text{3rd}}(s)$ by Wang et al. (2009, p., 814). However, when attempting to evaluate the H_2 norm J_1 for the resulting network using the `NormH2` function in Maple 2018, an error is returned claiming the system is unstable. In contrast, the `LinearSolve` command in Maple 2018 is capable of successfully solving equation (8) in order to determine J_1.

On the other hand, the `norm` command in MATLAB 2019a can compute J_1 for this example. However, MATLAB 2019a returns a similar error claiming the system is unstable when attempting to calculate the H_2 norm J_3 defined by Wang et al. (2009), when $K_1(s) = K_1^{\text{3rd}}(s)$ and $K_2(s) = K_2^{\text{3rd}}(s)$ as defined on p. 818 of that paper (here, J_3 characterises the dynamic wheel load). In contrast, the backslash operator in MATLAB 2019a can solve equation (8) to determine J_3 in this case.

This solution method via the polynomial Diophantine equation (6) also lends itself to both exact computation over the integers and to symbolic computation. This is of particular interest in safety critical applications, and in structured H_2 norm optimisation problems. In these cases, the solution can be efficiently obtained via the algorithm to be described in the next section.

4. EXACT OR SYMBOLIC COMPUTATION USING SUBRESULTANT SEQUENCES

In this section, an algorithm for solving the polynomial Diophantine equation (6) will be provided that is particularly suitable for exact or symbolic computation. The algorithm amounts to the computation of subresultant and remainder sequences in a generalised version of the extended Euclidean algorithm (see, e.g., Basu et al., 2006, Section 8.3). As will be shown in a follow on paper, these results can be derived by consideration of the LU decomposition of the Hurwitz matrix H in (8), in a manner similar to that adopted by Hughes (2014). In this follow on paper, it will also be shown that the coefficients correspond to Hurwitz determinants, thus facilitating a stability test alongside the H_2 norm computation at little additional computational cost. It will also be shown that the algorithm is fraction free (i.e., the outputs at each stage in the computation are integers whenever the inputs are integers), which allows for efficient exact or symbolic computation.

In the case that n is odd, it can be shown that the stability of the system guarantees the existence of coefficients $\beta_i > 0$

and polynomials $q_i(s)$ $(i = 1, \ldots, (n+1)/2)$ that satisfy the recursive formulae:

$$q_1(s){=}a^e(s), \;\; \beta_1{=}\mathrm{LC}(q_1(s)) \;\&\; q_2(s){=}h_1(s)q_1(s){-}\beta_1^2 a^o(s),$$

where $\deg(q_2(s)) = \deg(q_1(s)) - 1$; and

$$\beta_i = \mathrm{LC}(q_i(s)) \text{ and } q_{i+1}(s) = h_i(s)q_i(s) - \tfrac{\beta_i^2}{\beta_{i-1}^2} q_{i-1}(s),$$

for $i = 2, \ldots, (n-1)/2$, where $\deg(q_{i+1}(s)) = \deg(q_i(s)) - 1$ for $i = 1, 2, \ldots, (n-1)/2$; and $\beta_{(n+1)/2} = \mathrm{LC}(q_{(n+1)/2}(s))$. In other words, $q_2(s)$ (resp., $q_{i+1}(s)$) is the negative of the remainder of $\beta_1^2 a^o(s)$ (resp., $(\beta_i^2/\beta_{i-1}^2)q_{i-1}(s)$) upon division by $q_1(s)$ (resp., $q_i(s)$). Next, define polynomials $v_i(s)$ $(i = 1, 3, \ldots, n)$ by the recursive formulae:

$$v_1(s) = 1, \quad v_3(s) = h_1(s), \text{ and}$$

$$v_{2i+1}(s) = h_i(s)v_{2i-1}(s) - \tfrac{\beta_i^2}{\beta_{i-1}^2} v_{2i-3}(s), \;\; (i = 2, \ldots, \tfrac{n-1}{2}).$$

Finally, let $h(s)$ and $r(s)$ be the quotient and remainder in the division of $v_n(s)z(s)$ by $p_1(s)$, i.e.,

$$v_n(s)z(s){=}h(s)p_1(s){+}r(s) \;\&\; \deg(r(s)){<}\deg(p_1(s)). \quad (10)$$

Then it can be shown that $y_{n-1} = (-1)^{n+1} r_{(n-1)/2}/\beta_{(n+1)/2}$ where $r_{(n-1)/2}$ denotes the coefficient of $s^{(n-1)/2}$ in $r(s)$. Thus, $\|G\|_2^2 = (-1)^{n+1} r_{(n-1)/2}/(2\beta_{(n+1)/2}a_n)$.

On the other hand, when n is even, it can be shown that the stability of the system guarantees the existence of coefficients $\alpha_i > 0$ and polynomials $p_i(s)$ $(i = 2, \ldots, (n/2)+1)$ that satisfy the recursive formulae:

$$p_1 = a^e(s), \;\; \alpha_1 = \mathrm{LC}(p_1(s)) \;\&\; p_2(s) = g_1(s)p_1(s){-}\alpha_1 a^o(s),$$

$$\alpha_2 = \mathrm{LC}(p_2(s)) \;\&\; p_3(s) = g_2(s)p_2(s){-}\tfrac{\alpha_2^2}{\alpha_1} p_1(s),$$

where $\deg(p_3(s)) = \deg(p_2(s)) - 1 = \deg(p_1(s)) - 2$; and

$$\alpha_i = \mathrm{LC}(p_i(s)) \text{ and } p_{i+1}(s) = g_i(s)p_i(s) - \tfrac{\alpha_i^2}{\alpha_{i-1}^2} p_{i-1}(s),$$

for $i = 3, \ldots, n/2$, where $\deg(p_{i+1}(s)) = \deg(p_i(s)) - 1$ for $i = 1, 2, \ldots, n/2$; and $\alpha_{(n/2)+1} = \mathrm{LC}(p_{(n/2)+1}(s))$. In other words, $p_2(s)$ (resp., $p_3(s)$, $p_{i+1}(s)$) is the negative of the remainder of $\alpha_1 a^o(s)$ (resp., $(\alpha_2^2/\alpha_1)p_1(s)$, $(\alpha_i^2/\alpha_{i-1}^2)p_{i-1}(s)$) upon division by $p_1(s)$ (resp., $p_2(s)$, $p_i(s)$). Next, define polynomials $v_i(s)$ $(i = 0, 2, \ldots, n)$ by the recursive formulae:

$$v_0(s) = 0, \quad v_2(s) = -\alpha_1, \quad v_4(s) = -\alpha_1 g_2(s), \text{ and}$$

$$v_{2i}(s) = g_i(s)v_{2(i-1)}(s) - \tfrac{\alpha_i^2}{\alpha_{i-1}^2} v_{2(i-2)}(s), \;\; (i = 3, \ldots, \tfrac{n}{2}),$$

and let $h(s)$ and $r(s)$ be the quotient and remainder in the division of $v_n(s)z(s)$ by $p_1(s)$, as in (10). Then it can be shown that $y_{n-1} = (-1)^{n+1} r_{(n/2)-1}/\alpha_{(n/2)+1}$ where $r_{(n/2)-1}$ denotes the coefficient of $s^{(n/2)-1}$ in $r(s)$. Thus, $\|G\|_2^2 = (-1)^{n+1} r_{(n/2)-1}/(2\alpha_{(n/2)+1}a_n)$.

It can be shown that the coefficients in the aforementioned polynomials $q_i(s), p_i(s)$ and $v_i(s)$ are all minors of the Hurwitz matrix H in (9), whereupon these coefficients are integers whenever the coefficients in $a(s)$ are integers. Moreover, β_i (resp., α_i) is equal to the $2i{-}1$th (resp., $2(i{-}1)$th) leading principal minor of H. Then, by the Liénard Chipart criterion (Gantmacher, 1980, pp. 221), it follows that if the coefficients of $q_1(s)$ (resp., $p_1(s)$) are all strictly positive, and β_i (resp., α_i) is strictly positive for $i = 1, 2, \ldots$, then the system is guaranteed to be stable.

To finish, consider again the train suspension in Fig. 1. A second example of Wang et al. (2009) considered the case

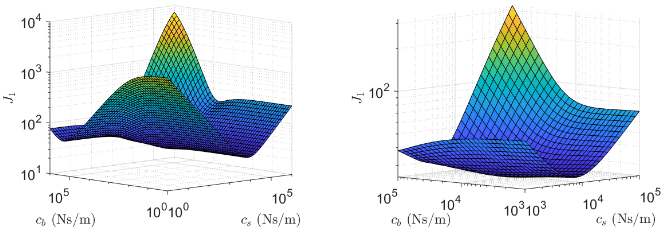

Fig. 2. J_1 as a function of c_s and c_b.

in which $K_1(s) = c_s$ and $K_2(s) = c_b$, where $c_s \geq 0$ and $c_b \geq 0$ are chosen to minimise J_1. Here, the aforementioned algorithm obtains an analytical expression for J_1^2 as the ratio of two polynomials in c_s and c_b. This leads to the visualisation in Fig. 2, suggesting that J_1 has a unique minimum in the region of non-negative c_b and c_s. This can be confirmed analytically by computing the partial derivatives of J_1^2 with respect to c_s and c_b and solving the resulting pair of bivariate polynomial equations that determine the local stationary points. It is thus found that there is a unique local minimum for $c_s \geq 0$ and $c_b \geq 0$, which occurs when $c_s = 7.616$Ns/mm and $c_b = 11.844$Ns/mm, resulting in $J_1 = 24.92$. Note that these damping values differ slightly from the values obtained by Wang et al. (2009) ($c_s = 7.607$Ns/mm and $c_b = 11.851$Ns/mm), but the resulting H_2 norm agrees up to the seventh significant figure. The method presented here also guarantees that the global minimum has been found, in contrast to the optimisation approach employed by Wang et al. (2009) and other commonly used nonlinear optimisation methods. Furthermore, the method is generally applicable to H_2 norm optimisation problems for any system whose transfer function coefficients are specified parametrically.

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