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Exact propagating wave solutions in reaction cross-diffusion system

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ABSTRACT

Reaction-diffusion systems with cross-diffusion terms in addition to, or instead of, the usual self-diffusion demonstrate interesting features which motivate their further study. The present work is aimed at designing a toy reaction-cross-diffusion model with exact solutions in the form of propagating fronts. We propose a minimal model of this kind which involves two species linked by cross-diffusion, one of which governed by a linear equation and the other having a polynomial kinetic term. We classify the resulting exact propagating front solutions. Some of them have some features of the Fisher-KPP fronts and some features of the ZFK-Nagumo fronts.

1. Introduction

Reaction-diffusion systems are models that are used widely to model physical, chemical, biological and ecological processes. Realistic models of such processes are typically quite complicated and can only be dealt with numerically. However qualitative understanding of the most important features benefits from analytical approaches, even if that requires sacrfices in quantitative accuracy. This may be achieved by using asymptotic methods and/or considering “toy models”.

One of the first and famous “toy” reaction-diffusion systems is the model of propagation of an advantageous gene due to Fisher [1] and Kolmogorov, Petrovsky and Piskunov [2]. We refer to it as Fisher-KPP model. Another early archetypal reaction-diffusion equation was a model of flame propagation considered by Zeldovich and Frank-Kamenetsky [3], which later became known also as Schlögl model [4] and Nagumo equation [5]. We refer to it as ZFK-Nagumo equation. Both models have monotonic propagating wavefront solutions of similar appearances, but each has its own distinct mechanism. The Fisher-KPP model shows the transition from an unstable resting state to a stable resting state, while the ZFK-Nagumo model shows the transition from one stable resting state to another stable resting state. Another qualitative difference between them is that ZFK-Nagumo model exhibits a unique, up to a constant shift in time or space, propagating front solution with a fixed speed and shape, whereas Fisher-KPP model has a family of solutions with a continuous range of possible speeds. The importance of these toy models goes well beyond providing simplest examples. For instance, the ZFK-Nagumo equation can be considered as the fast subsystem in describing pulse waves in the FitzHugh-Nagumo and similar systems using singular perturbation techniques [6,7].

In the last decades, there has been ever increasing attention to reaction-diffusion systems which have cross-diffusion of the dynamic variables in addition or instead of their self-diffusion. These occur in mathematical modeling of various natural phenomena of biological, physical and chemical nature, such as mutual taxis of interacting species, including e.g. spatial segregation phenomena between the competing species [8–10], cell types [11] and human population groups [12], and prey-taxis of predators and evasion of predators by prey [13–22]; interaction of populations of organisms or cells with environment, including e.g. slime mold aggregation [23], tumor angiogenesis [24], amoeboid locomotion [25] and thermoregulation in honey bee colonies [26]; dissipative mechanical processes such as stick-slip motion of geological plates [27,28]; as well as the literal cross-diffusion of reacting chemical species [29–31]. Furthermore, cross-diffusion terms may appear “mathematically”, via adiabatic elimination of fast but diffusing variables [9,10,21,32,33]. Interesting phenomena have been described in such systems, where the cross-diffusion plays an essential role alongside with the self-diffusion and reaction part of the system. This includes e.g. pattern formation via Turing-type instabilities [8–10,14,17,23,30,34,35] and propagation of waves of various kinds [11,19,30,33,36,37]. Overall, the literature on cross-

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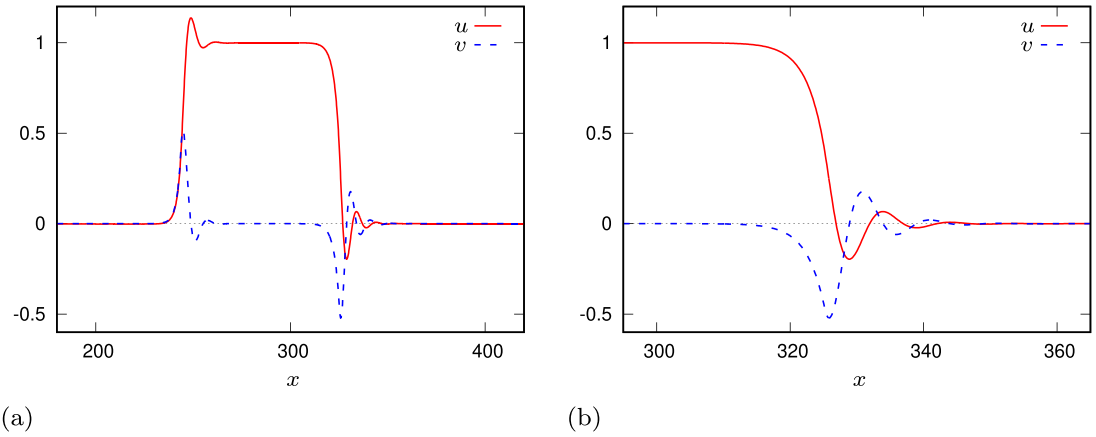

Fig. 1. (a) The direct numerical simulation of (2) the reaction cross-diffusion system with a no-flux boundary exhibits a propagating pulse with $f(u) = u(u-0.3)(1-u)$ and the values of parameters are $\epsilon = 0.001$, $D_u = 5$ and $D_v = 0.5$. (b) At small distance and time, the front of the pulse of reaction cross-diffusion system with a cubic nonlinearity approach to two asymptotic states (u_1, v_1) and (u_2, v_2).

diffusion models is too vast for an exhaustive survey here; some reviews of models and results with further references can be found e.g. in [30,34,38–42].

The focus of this work is on systems with excitable reaction kinetics, motivated by observations that including cross-diffusion in addition or instead of self-diffusion led to new phenomena [15,16,18,20,27,28]. For example, propagating waves in reaction-cross-diffusion systems (RXD) with excitable reaction kinetics could penetrate each other on collision, a behavior that is unusual for excitable systems with self-diffusion only.

The properties of solutions in RXD systems in the above cited motivating works have been mostly studied numerically. An analytical approach has been attempted in [16]. In that work, fast-slow separation between reaction kinetics of two reacting species is assumed. The fast subsystem has piecewise linear kinetics and linear cross-diffusion, and admits exact analytical solutions in the form of propagating fronts. Unlike the Fisher-KPP and ZFK-Nagumo fronts, these front solutions are oscillatory. They can be matched asymptotically with slow pieces to obtain complete asymptotic description of propagating pulses. The fast subsystem in this approach is different from the Fisher-KPP and ZFK-Nagumo equations in two aspects: that it is two-component and it is piecewise linear, as opposed to the two "classical" toy models which are both one-component and with polynomial nonlinearity of the kinetics. At least two components are of course required to have cross-diffusion.

In the present work, we investigate the possibility of having exact front solutions in a cross-diffusion system with polynomial kinetics, unlike piecewise kinetics of Biktashev and Tsyganov [16]. Our leading idea is to postulate the solutions and deduce the governing equations from there. For simplicity and as the first step, we only consider here monotonic fronts, similar to those found in the ZFK-Nagumo equation. Thus it is clear for the outset that as far as are motivating numerical observations are concerned, the present study can only have a methodological value, as the waves observed in excitable cross-diffusion systems typically have oscillatory fronts and backs, as illustrated in Fig. 1.

The paper is organized as follows. The problem formulation is given in Section 2. In Section 3, we consider the possibilities of chosing polynomial nonlinearity for the reaction term. In Section 4, we discuss the simplest aspects of stability of possible solutions. Then we show the correspondent polynomial function suitable for solutions of the wavefront type. These are presented in Section 5. We demonstrate the possibility to have a wavefront solution of the system as generalisation for Fisher-KPP in Section 6 and analyze the choices of the parameters needed to imitate Fisher-KPP model in Section 7. We return to the question of stability, now for the selected wavefront solution, in Section 8. Results of numerical simulation are presented in Section 9. These simulations show that the wavefronts are unstable. These instabilities are investigated in Section 10 and the paper is concluded by discussion in Section 11.

2. Problem formulation

Let us consider the reaction-diffusion system in the form

$$\begin{aligned} u_t &= f(u) - v + D_{uv} v_{xx} + D_{uu} u_{xx}, \\ v_t &= \epsilon(u - v) + D_{vu} u_{xx} + D_{vv} v_{xx}, \end{aligned} \tag{1}$$

where

$$f(u) = u(u-\alpha)(1-u),$$

and the parameters are restricted by $0 \le \epsilon \ll 1$, $\alpha \in (0, 1/2)$.

The system (1) is well studied as a reaction-self-diffusion system, with $D_{uu} > 0$, $D_{vv} \ge 0$ and $D_{uv} = D_{vu} = 0$. If $D_{uv} \ne 0$ and/or $D_{vu} \ne 0$, we have reaction-cross-diffusion system. Regarding the signs of the diffusion coefficients, one common restriction is that their matrix must be positively semi-definite, so in particular, $D_{uu} \ge 0$, $D_{vv} \ge 0$. Regarding the signs of the cross-diffusion coefficients, all sorts of combinations are considered in literature. One of the ways the cross-diffusion terms as in (1) may appear in applications is via linearization of terms describing mutual taxis of dynamic variables, which may describe populations and/or environmental factors affecting populations. For instance, if u represents a population which diffuses and moves towards attractant v, which may be an environmental factor or a prey population and which itself only passively diffuses, then $D_{uv} < 0$ and $D_{vu} = 0$, as e.g. in [14,19,23,35,36]. A similar combination (up to a change of sign of one of the dynamic variables) occurs in description of interaction of geological plates [27,28]. If u and v represent competing species which seek to avoid each other, this leads to $D_{uv} > 0$, $D_{vu} > 0$, as in [8,9]. For predator-prey relationship, on the contrary, one may expect pursuit-evasion behavior, that is, positive prey taxis for predators, i.e. predators seeking prey and prey escaping from predators, so if u component represents prey population and v represents predator population, this means that $D_{uv} > 0$ and $D_{vu} < 0$, as in [13,14,17,19,22]. Well-posedness of an initial or boundary-value problem for this system is not self-evident: examples are known that systems with cross-diffusion are capable of producing solutions blowing up in final time, see e.g. [40]. Some well-posedness results have been established, see e.g. [22,43], however

[43] requires strong ellipticity of the diffusion matrix and [22] requires strong stability properties of the reaction part of the system, neither of which is true in the case we consider. We work under assumption that solutions exist and behave "reasonably"; some evidence for that, even if not rigorous, is provided by the fact that the solutions can be simulated numerically. Clearly the well-posedness for the particular variants of the system of the form (1) we consider here requires separate study. It is beyond the scope of this paper.

If $\epsilon = 0$, $v \equiv 0$, the self-diffusion system degenerates to the ZFK-Nagumo equation [3–5] for $u(x, t)$, with an exact propagating front solution. A piecewise linear N-shaped variant of $f(u)$ also leads to exact propagating front solution [5]. Qualitative properties of this equation, including existence of propagating front solutions, persist for a generic N-shape, and for $0 < \epsilon \ll 1$, these solutions can form a basis of asymptotic description, see for instance [6,7].

A similar asymptotic approach for $0 < \epsilon \ll 1$ was considered for the cross-diffusion case of (1) in [16]. To make the problem analytically tractable, the consideration there was restricted to a piecewise linear N-shaped function $f(u)$ and pure cross-diffusion, with self-diffion totally absent, $D_{uu} = D_{vv} = 0$.

In this paper we consider the same system as was dealt with in [16], namely

$$\begin{aligned} u_t &= f(u) - v + D_v v_{xx}, \\ v_t &= \epsilon(u - v) - D_u u_{xx}, \end{aligned} \tag{2}$$

and intend to extend the methodology of Tyson and Keener [6] and Biktashev and Tsyganov [16] for a polynomial function $f(u)$. In absence of self-diffusion terms and in consideration of the chosen signs of the cross-diffusion coefficients, we abbreviate $D_{vu} = D_u$ and $D_{uv} = -D_v$.

We start by recapitulation of the approach of Biktashev and Tsyganov [16] to set the scene and introduce notation and terminology. Direct numerical simulations of (2) with cubic $f(u)$ produces, in particular, solutions in the form of propagating pulses of a fixed shape, as illustrated in Fig. 1. For small ϵ, the width of the pulse grows as $\mathcal{O}(\epsilon^{-1})$. This means that in the limit $\epsilon \to 0$, the wave front and the wave back of the pulse go apart. Our hypothesis is that for very small ϵ, the system we are going to construct, will behave similarly to those discussed in [6] and [16]. Namely, we expect that a typical propagating wave solution will have the form of long stretches where $u(x, t)$ remains near an instant equilibrium of the fast equation, satisfying $f(u) \approx v$, which are interspersed by fast transitions from one such quasi-equibrium to another. Any such transition is approximated by an $\epsilon = 0$ solution in the form of a wave which propagates with constant speed and shape and, far behind and far ahead, approaches constants, corresponding to the above mentioned quasi-equilibria. In particular, a pulse solution such as the one shown in Fig. 1, includes two such fast transitions, a front and a back. Both the front and back represent transitions between two distinct equilibrium points, say (u_1, v_1) and (u_2, v_2).

In the limit $\epsilon \to 0$ the system (2) turns into

$$\begin{aligned} u_t &= f(u) - v + D_v v_{xx}, \\ v_t &= -D_u u_{xx}. \end{aligned} \tag{3}$$

The two equilibria (u_1, v_1) and (u_2, v_2), the asymptotic states of the wave front and the wave back, satisfy $f(u_j) = v_j$. Let $\hat{u}(\xi) = u(x, t)$ and $\hat{v}(\xi) = v(x, t)$ be a propagating wave solution of (3), where $\xi = x - ct$ and $c > 0$. Substituting this into the system (3) yields

$$D_v \frac{d^2\hat{v}}{d\xi^2} + c\frac{d\hat{u}}{d\xi} + f(\hat{u}) - \hat{v} = 0, \tag{4}$$

$$-D_u \frac{d^2\hat{u}}{d\xi^2} + c\frac{d\hat{v}}{d\xi} = 0. \tag{5}$$

As the front asymptotically approaches distinct steady states, we have

$$\hat{u}(\pm\infty) = u_{1,2}, \quad \hat{v}(\pm\infty) = v_{1,2} \tag{6}$$

$$\frac{d\hat{u}}{d\xi}(\pm\infty) = \frac{d\hat{v}}{d\xi}(\pm\infty) = 0. \tag{7}$$

Integrating (5) with respect to ξ gives

$$\hat{v} - \frac{D_u}{c}\hat{u}' = v_\star = \text{const}. \tag{8}$$

When $\xi \to \pm\infty$, we obtain from (8) that $v_\star = \hat{v}_1 = \hat{v}_2$ and then Eq. (4) turns into

$$f(u_{1,2}) = v_\star. \tag{9}$$

We have from Eq. (6) that $c\hat{v}' = D_u\hat{u}''$, hence $\hat{v}'' = D_u\hat{u}'''/c$. Substituting this into (5) yields

$$\begin{aligned} &D_v D_u \hat{u}''' + \left(c^2 - D_u\right)\hat{u}' + c\left(f\left(\hat{u}\right) - v_\star\right) = 0, \\ &\hat{u}(\pm\infty) = u_{1,2}, \end{aligned} \tag{10}$$

where $\hat{u}$ is a wave solution for the reaction cross-diffusion system (3).

This differential equation is deduced by applying the wave variable on the reaction-cross-diffusion system (3). Biktashev and Tsyganov [16] have replaced $f(\hat{u})$ by a piecewise linear function. The fronts that are obtained from the piecewise linear function are oscillatory fronts and are similar to those seen in numerical simulations with cubic $f(\hat{u})$. We seek to consider a polynomial function for $f(\hat{u})$ instead of piecewise linear function, which would still yield explicit analytical solutions for propagating fronts.

3. Selecting the class of the polynomial reaction term

We aim to identify polynomial functions $f(\hat{u})$ which would make the differential Eq. (10) analytically solvable. First we write the Eq. (10) as

$$A\hat{u}''' + B\hat{u}' = \bar{f}(\hat{u}), \tag{11}$$

where

$$A = -\tfrac{D_u D_v}{c}, \quad B = \tfrac{D_u - c^2}{c}, \quad \bar{f}(\hat{u}) = f(\hat{u}) - v_\star.$$

We apply a reduction of order substitution,

$$\frac{d\hat{u}}{d\xi} = y\left(\hat{u}\right). \tag{12}$$

Substituting (12) into (11) gives

$$y\left[A\left(y'^2 + yy''\right) + B\right] = \bar{f}(\hat{u}). \tag{13}$$

We aim that function $\bar{f}(\hat{u})$ is a polynomial. This can be assured if $y(\hat{u})$ is a polynomial.

Let us find the possible degree of the polynomials $y(\hat{u})$ and $\bar{f}(\hat{u})$. Let $\mathcal{P}_n$ be the set of polynomials of degree n. If $y \in \mathcal{P}_n$, then

$$y\left[A\left(y'^2 + yy''\right) + B\right] = \bar{f}(\hat{u}) \in \mathcal{P}_{3n-2}.$$

If $n = 1$ then $\bar{f}(\hat{u})$ is linear, which is not of interest for us, as this cannot produce two distinct solutions for (9). If $n = 2$ then $\bar{f}(\hat{u})$ is quartic. This quartic polynomial is comparable to cubic, in that it can describe bistability, if it has at least three simple roots. Therefore, $y \in P_2$, $\bar{f} \in P_4$ is the simplest suitable choice.

The travelling wave differential equation for ZFK-Nagumo can be solved analytically by a reduction of order [5]. Incidentally, in that solution $y(\hat{u})$ is also quadratic. It is convenient to factorise the quadratic polynomial $y(\hat{u})$,

$$y(\hat{u}) = k\left(\hat{u} - g\right)\left(\hat{u} - h\right), \tag{14}$$

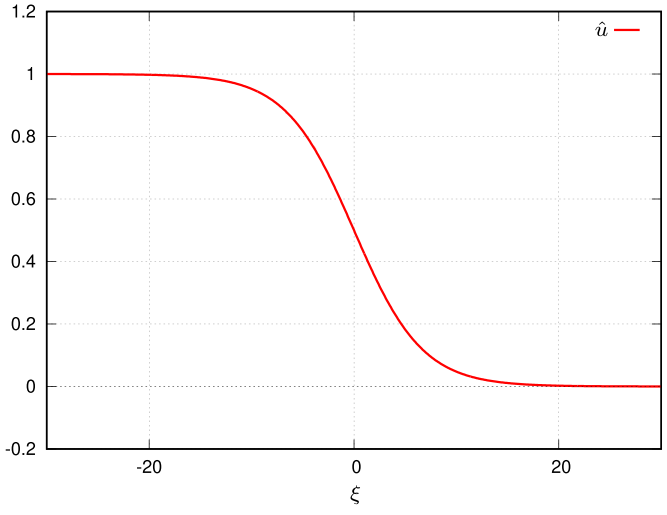

Fig. 2. The solution $\hat{u}(\xi)$ given by (1) for $g = 1$, $h = 0$, $k = 0.3$ and $C = 0$.

for some constants $k \neq 0$, g and h. Note that due to (6), (7) and (12), we have $\{u_1, u_2\} = \{g, h\}$.

From (12) and (14), we obtain

$$\hat{u}(\xi) = \frac{g + h\,e^{\chi}}{1 + e^{\chi}}, \qquad \chi = k(\xi + C)(g - h), \tag{15}$$

where C is an arbitrary constant. The front wave described by (15) is illustrated in Fig. 2.

Once $\hat{u}(\xi)$ is known, we can find $\hat{v}(\xi)$ using (8), as

$$\hat{v}(\xi) = v_\star + \frac{D_u}{c}\hat{u}'(\xi). \tag{16}$$

Obviously, the profile of component $\hat{v}$ represents not a wave front but a pulse. In accordance with (7), we have

$$\hat{v}(\pm\infty) = v_\star.$$

4. On the stability of the front solutions: Continuous spectrum

The stability of any front solution we seek shall depend, in particular, on the stability of its asymptotic spatially uniform steady states, that is, on the continuous spectrum. This, unlike the discrete spectrum, is easily done analytically. The system (3) can be written in the matrix form

$$\mathbf{w}_t = \mathbf{F}(\mathbf{w}) + \mathbf{D}\mathbf{w}_{xx},$$

where

$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}, \mathbf{F}(\mathbf{w}) = \begin{bmatrix} f(u) - v \\ 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 & D_v \\ -D_u & 0 \end{bmatrix}.$$

Suppose $\mathbf{w}^* = [u^*, v^*]^T$ is an equilibrium, i.e. $\mathbf{F}(\mathbf{w}^*) = 0$. We perturb this point,

$$\mathbf{w} = \mathbf{w}^* + \tilde{\mathbf{w}},$$

and in the linear approximation we have

$$\tilde{\mathbf{w}}_t = \mathbf{F}'(\mathbf{w}^*)\tilde{\mathbf{w}} + \mathbf{D}\tilde{\mathbf{w}}_{xx}, \tag{17}$$

where $\mathbf{F}' = \left[\partial\mathbf{F}/\partial\mathbf{w}\right]$ is the Jacobian matrix. By separation of variables, particular solutions of (17) bounded in space can be written as linear combinations of

$$\tilde{\mathbf{w}}(x,t) = e^{i\mu x}e^{\lambda t}\left[C_1 C_2\right], \mu \in \mathbb{R}, \lambda, C_1, C_2 \in \mathbb{C}. \tag{18}$$

Substituting (18) in (17), gives and eigenvalue problem

$$\begin{bmatrix} f'(u^*) & -1 - \mu^2 D_v \\ \mu^2 D_u & 0 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \lambda\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \tag{19}$$

where $f' = \partial f/\partial u$, and the eigenvalues are

$$\lambda_{1,2} = \tfrac{1}{2}\left[f'(u^*) \pm \sqrt{(f'(u^*))^2 - 4\mu^2 D_u - 4\mu^4 D_u D_v}\right], \tag{20}$$

see Fig. 3. Therefore, if $f'(u^*)$ is positive, then $\mathrm{Re}\left(\lambda_{1,2}\right) \geq 0$ and the steady state (u^*, v^*) is unstable, and if $f'(u^*)$ is negative then $\mathrm{Re}\left(\lambda_{1,2}\right) < 0$ for all $\mu \neq 0$, and the state is stable in linear approximation. Of course, even if both asymptotic states are stable, the stability of the whole front solution will still depend on the discrete spectrum; this is outside our scope.

5. Fixing the polynomial reaction term

In this section we will find the particular form of the polynomial function $\bar{f}(\hat{u})$, as well as the parameters A and B that satisfy (13). To achieve this, we substitute (14) into (13), which gives

$$k(\hat{u} - g)(\hat{u} - h)\left\{A\left[k^2\left(2\hat{u} - g - h\right)^2 + 2k^2(\hat{u} - g)(\hat{u} - h)\right] + B\right\} = \bar{f}(\hat{u}). \tag{21}$$

We take our quartic polynomial $\bar{f}(\hat{u})$ in the following form:

$$\bar{f}(\hat{u}) = \sigma(\hat{u} - u_1)(\hat{u} - u_2)(\hat{u} - u_3)(\hat{u} - u_4) \tag{22}$$

where $\{u_1, u_2\} = \{g, h\}$ and without loss of generality $\sigma = \pm 1$; a different scaling of f would just result in a change of the spatial and temporal scale of the solutions.

By substituting (22) into (21), we obtain

$$k(\hat{u} - g)(\hat{u} - h)\left\{A\left[k^2\left(2\hat{u} - g - h\right)^2 + 2k^2(\hat{u} - g)(\hat{u} - h)\right] + B\right\} = \sigma(\hat{u} - u_1)(\hat{u} - u_2)(\hat{u} - u_3)(\hat{u} - u_4). \tag{23}$$

By equating like terms we obtain

$$\begin{aligned}
&[\hat{u}^4]: && \tfrac{6k^3 D_u D_v}{c} = -\sigma;\\
&[\hat{u}^3]: && \tfrac{12k^3 D_u D_v (g+h)}{c} = -\sigma(u_1 + u_2 + u_3 + u_4);\\
&[\hat{u}^2]: && \tfrac{k^3 D_u D_v (7g^2 + 22gh + 7h^2)}{c} + \tfrac{k(c^2 - D_u)}{c}\\
& && = -\sigma(u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_2u_4 + u_3u_4);\\
&[\hat{u}^1]: && \tfrac{k^3 D_u D_v (g+h)(g^2 + 10gh + h^2)}{c} - \tfrac{k(g+h)(-c^2 + D_u)}{c}\\
& && = -\sigma(u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4);\\
&[\hat{u}^0]: && \tfrac{k^3 ghD_u D_v (g^2 + 4gh + h^2)}{c} + \tfrac{gkh(c^2 - D_u)}{c} = -\sigma u_1u_2u_3u_4.
\end{aligned} \tag{24}$$

This imposes five constraints onto a set of 11 parameters k, g, h, σ, D_u, D_v, u_1, u_2, u_3, u_4 and c; hence we can describe all solutions of this system by assigning six of these parameters as free, and then finding the remaining five parameters as dependent on these six free parameters. We restrict consideration to real values of parameters in both groups, except possibly the roots $u_{3,4}$. Moreover, as parameters g and h fix the positions of the pre- and post-front resting states of the solution (15), it convenient to have these two among the free parameters; note also that we have already constrained σ to ± 1.

6. Possible types of solutions

As discussed in the Introduction, this study is not motivated by any real-world applications leading to specific examples of reaction cross-diffusion systems. Rather, we are interested in theoretical possibilities achievable within a certain class of models. With this in mind, we want to see if we can make the reaction cross-diffusion system with quartic polynomial to look like generalizations, in one sense or another, of other well-known models, from the much better studied class of systems with self-diffusion. We shall say that we "imitate" those models. The models that we want

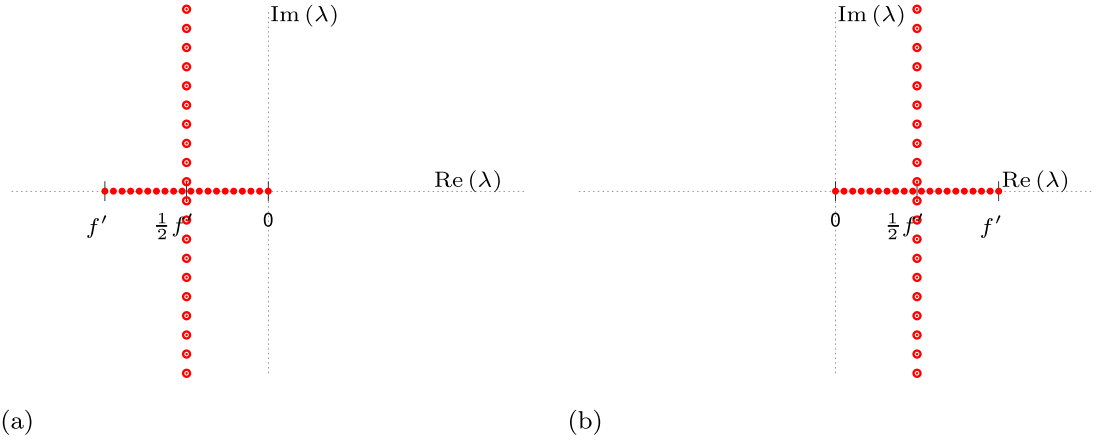

Fig. 3. The continuous spectrum of an equilibrium, for (a) $f' = f'(u^*) < 0$, (b) $f' = f'(u^*) > 0$, according to (20).

to imitate are Fisher-KPP and ZFK-Nagumo. Those models exhibit propagating front solutions with asymptotics

$$u(\xi \to +\infty) = 0, \qquad u(\xi \to -\infty) = 1.$$

If we identify the scalar field u here with the namesake first dynamic variable in our system, then this property can be achieved by letting $g = 0$ and $h = 1$ in (15).

We found in the previous section that the stability of a spatially uniform steady state depends on the sign of the derivative of the quartic polynomial at that state. In terms of stability, to imitate the ZFK-Nagumo wave, we would need a stable pre-front state and a stable post-front state, and consequently an unstable equilibrium in between. To imitate the Fisher-KPP wave front we would need to have an unstable pre-front state and a stable post-front state, with either no or two equilibria in between. In this respect, the possibilities for front waves from the reaction cross-diffusion system with quartic polynomial are constraint by the following proposition.

Proposition 1. *If the boundary-value problem (10) with the nonlinearity defined by (22) and (23) has a travelling wave front solution of the form (15), then the two asymptotic resting states $\{g, h\}$ are either the two outer roots of the quartic polynomial $\bar{f}(\hat{u})$, or its two inner roots.*

Proof. From (23), among the roots of $\bar{f}(\hat{u})$ we have $\{u_1, u_2\} = \{g, h\}$, and the other two roots, $u_{3,4}$, are the roots of the quadratic in the square brackets, which is equivalent to

$$\hat{u}^2 - (g+h)\hat{u} + \frac{g^2 + 4gh + h^2 + B/(Ak^2)}{6} = 0.$$

Hence $\frac{1}{2}(u_3 + u_4) = \frac{1}{2}(g+h)$. If $u_{3,4} \in \mathbb{R}$, $u_3 \neq u_4$, this implies that either g and h are two inner roots while u_3 and u_4 are the two outer roots, or vice versa. If $u_3 = u_4$ the g and h are the two outer roots out of the three, and if $u_{3,4} \notin \mathbb{R}$, then g and h the only two, therefore automatically the outer, roots. □

From Proposition 1, we conclude that of the resting states of the front wave solution, only one can be stable but not both. That means, in the considered reaction cross-diffusion system with the quartic polynomial, it is impossible to imitate ZFK-Nagumo wave in terms of the stability of the resting states, but there is a chance to imitate Fisher-KPP wave. We note, however, that for any given set of parameters of the model, the speed of the front solution is in any case uniquely fixed by (24), see also (28) below, and this feature is characteristic of ZFK-Nagumo fronts rather than Fisher-KPP fronts.

Table 1
Examining possible choices to imitate Fisher-KPP front. The symbols (↗) and (↘) mean that $\chi(\xi)$ is an increasing or decreasing function, respectively.

Choices				Results		
	g	h	k	χ	$u(+\infty)$	$u(-\infty)$
I	1	0	(+)	↗	0	1
II	1	0	(−)	↘	1	0
III	0	1	(+)	↘	0	1
IV	0	1	(−)	↗	1	0

7. Choice of signs to imitate Fisher-KPP

We have found that there is a possibility to imitate Fisher-KPP front wave, in terms of the stability of the pre-front and post-front equilibria, by reaction cross-diffusion system with quartic polynomial nonlinearity. In this section, we will turn this possibility into reality, by identifying appropriate parameter choices.

Firstly, let us make sure that solution given by (15) satisfies the asymptotic boundary conditions of Fisher-KPP front wave,

$$\hat{u}(+\infty) = 0, \qquad \hat{u}(-\infty) = 1. \tag{25}$$

In Section 5 we found that six parameters in (24) can be arbitrary assigned. We choose k, g and h as three of such free parameters, in order to satisfy (25). We have already committed ourselves to the choice $\{g, h\} = \{0, 1\}$, and we require $k \neq 0$. Table 1 lists the resulting four *a priori* possibilities.

Clearly, choices that comply with (25) are (I) and (III). In both cases, Eq. (14) gives

$$y(\hat{u}) = k\hat{u}(\hat{u}-1), \quad y'(\hat{u}) = 2k(\hat{u}-1), \quad y''(\hat{u}) = 2k, \qquad k > 0. \tag{26}$$

The quartic polynomial $\bar{f}(\hat{u})$ posited in (22) allows $\sigma = 1$ or $\sigma = -1$. Remember that the equation for the coefficients at $\hat{u}^4$ in (24) states

$$6k^3 D_u D_v = -\sigma c. \tag{27}$$

If $\sigma = 1$ then the solution (15) will not satisfy the condition (25): since D_u, D_v and c are positive, equation (27) implies $k < 0$, which is inconsistent with (26).

So, we must choose $\sigma = -1$, which together with $\{g, h\} = \{u_1, u_2\} = \{0, 1\}$ turns the system (24) to

$$\frac{6k^3 D_u D_v}{c} = 1,$$

$$\frac{12k^3 D_u D_v}{c} = 1 + u_3 + u_4,$$

$$\frac{6k^3 D_u D_v}{c} + \frac{k^3 D_u D_v}{c} - k\frac{-c^2 + D_u}{c} = u_3 + u_4 + u_3 u_4,$$

$$\frac{k^3 D_u D_v}{c} - k\frac{-c^2 + D_u}{c} = u_3 u_4,$$
$$0 = 0.$$

Previously, we let variables g, h and k be free parameters. We now add to that list D_u and D_v. The rest of the variables will be dependent on those as follows:

$$c = 6k^3 D_u D_v, \tag{28}$$

$$u_3 = \tfrac{1}{2} - \tfrac{1}{6}\sqrt{3 + 36\rho}, \tag{29}$$

$$u_4 = \tfrac{1}{2} + \tfrac{1}{6}\sqrt{3 + 36\rho}. \tag{30}$$

$$\rho = \frac{k(D_u - c^2)}{c}. \tag{31}$$

where

$$\rho = \frac{k(D_u - c^2)}{c}. \tag{32}$$

The quartic polynomial now has the form

$$\tilde{f}(\hat{u}) = -\hat{u}(\hat{u} - 1)(\hat{u} - u_3)(\hat{u} - u_4), \tag{33}$$

where u_3 and u_4 are given by (29) and (30).

We expect that, in principle, if the quartic polynomial is substituted into the system (3), i.e.

$$u_t = -u(u-1)(u-u_3)(u-u_4) + v_\star - v + D_v v_{xx},$$
$$v_t = -D_u u_{xx}, \tag{34}$$

then the solution of (34) is a front wave which imitates the front wave in Fisher-KPP with respects to the stability of the pre-front and post-front resting states.

The choices of values of the given parameters change the values of the roots u_3 and u_4, which leads to one of the following cases.

- case I: If $\rho \in (\frac{1}{6}, +\infty)$, then $u_{3,4} \in \mathbb{R} \setminus [0, 1]$ and the restings states {0, 1} are inner roots.
- case II: If $\rho = \frac{1}{6}$, then $\{u_3, u_4\} = \{0, 1\}$ and the resting states {0, 1} are the only two roots, both double.
- case III: If $\rho \in (-\frac{1}{12}, \frac{1}{6})$, then $u_{3,4} \in (0, 1)$, $u_3 \neq u_4$, and the resting states {0, 1} are outer of four roots.
- case IV: If $\rho = -\frac{1}{12}$, then $u_3 = u_4 = \frac{1}{2}$, and the resting states {0, 1} are outer of three roots.
- case V: If $\rho \in (-\infty, -\frac{1}{12})$, then $u_{3,4} \in \mathbb{C} \setminus \mathbb{R}$ and the resting states {0, 1} are the only two roots.

Remember that by virtue of (32) and (28), this means that the location of the roots $u_{3,4}$ is determined by the three parameters k, D_u and D_v.

8. Stability of the resting states

Previously, we have linearised the system (3) for general function $f(u)$ about an equilibrium and derived the formula of the eigenvalues (20). Substituting the quartic polynomial function (33) into the function of the eigenvalue yields that, the eigenvalues of the equilibrium $u_1 = 0$ are given by

$$\lambda_{1,2} = \frac{1}{2}\left[u_3 u_4 \pm \sqrt{{u_3}^2 {u_4}^2 - 4\mu^2 D_u - 4\mu^4 D_u D_v}\right], \tag{35}$$

while the eigenvalues of the equilibrium $u_2 = 1$ are given by

$$\lambda_{1,2} = \frac{1}{2}\left[-(1-u_3)(1-u_4) \pm \sqrt{(1-u_3)^2(1-u_4)^2 - 4\mu^2 D_u - 4\mu^4 D_u D_v}\right]. \tag{36}$$

Table 2
The stability of the resting states in the front wave depends on the choice of the roots of the quartic polynomial.

Choice of roots	Pre-front	Post-front	Matching with Fisher-KPP
Case I: Inner	stable	unstable	✗
Case III: Outer	unstable	stable	√
Case IV: Double	unstable	stable	√
Case V: Complex	unstable	stable	√

In the "inner roots" case I, the two roots u_3 and u_4 have different signs, and are to opposite sides of 1. Thus, from (35) and (36) we deduce that the pre-front $u_1 = 0$ is stable and the post-front $u_2 = 1$ is unstable.

The similarity between Fisher-KPP and inner roots case is that both systems have two consecutive roots of $\tilde{f}(u)$ that coincide with the resting states of a wave front. The difference between them is that the pre-front in Fisher-KPP is unstable and the post-front is stable, while in inner roots case it is the other way round, the pre-front is stable and the post-front is unstable.

In the "outer roots" cases III and IV as well as "the only two roots" V, wee see from (35) and (36) that the pre-front $u_1 = 0$ is unstable and the post-front $u_2 = 1$ is stable. This matches the stability of the equilibria in Fisher-KPP model.

The marginal II gives $\mathrm{Re}\left(\lambda_{1,2}\right) = 0$ so the stability of the resting states cannot be established in linear approximation, and requires separate consideration. We leave this outside the scope of this paper.

Table 2 sums up the results of above analysis.

In the next section we will show the result of the numerical simulation for each case.

9. Numerical simulations

9.1. General settings

We simulate numerically the reaction cross-diffusion system

$$u_t = f(u, v) + D_v v_{xx},$$
$$v_t = -D_u u_{xx}, \tag{37}$$

for $-a \le x \le b$ and $t \ge 0$, where the kinetic term $f(u, v)$ is quartic polynomial

$$f(u, v) = -u(u-1)(u-u_3)(u-u_4) - v,$$

and u_3 and u_4 are dependent parameters defined in (29) and (30). We apply no-flux boundary conditions,

$$u_x(-a, t) = u_x(b, t) = v_x(-a, t) = v_x(b, t) = 0,$$

and the initial condition taken from the analytical solution, that is

$$u(x, 0) = \hat{u}(x), \qquad v(x, 0) = \hat{v}(x),$$

where $\hat{u}$ and $\hat{v}$ are defined in (15) and (16).

We will show the results of the simulation for cases I, III, IV and V identified above. For each case, we pick an appropriate set of values of the free parameters to satisfy the correspoinding conditions. Table 3 lists the parameter values used and the corresponding equilibria. Note that the value of D_u for Case IV in the table is given to three decimal places; in fact it was determined from the exact condition that $\rho = -1/12$, which implies

$$D_u = \frac{2 + k^2 D_v}{72 k^6 D_v^2}. \tag{38}$$

The numerical simulations are done using finite differences, fully explicit first order for time and second order central for space. The space discretization interval is $[-a, b] = [-37.5, 150]$ and the

Table 3
Parameters and equilibria in numerical simulations.

Case	I	III	IV	V
Figure(s)	4,10, 11	8	6,9	7
k	1	1	1	1
D_u	1.25	0.2	2.917	0.4
D_v	0.1	0.35	0.1	1.5
u_1	0	0	0	0
$\tilde{f}'(u_1)$	−0.75	0.11	0.25	3.656
u_2	1	1	1	1
$\tilde{f}'(u_2)$	0.75	−0.11	−0.25	−3.656
u_3	1.5	0.874	0.5	0.5 + 1.845i
$\tilde{f}'(u_3)$	−1.5	0.083	0	
u_4	−0.5	0.126	0.5	0.5 − 1.845i
$\tilde{f}'(u_4)$	1.5	−0.083	0	

discretisation steps are $\Delta x = 0.15$ and $\Delta t = 4 \times 10^{-6}$ unless otherwise stated. The choice of the discretization steps is motivated by the numerical stability and accuracy analysis of the scheme, which will be presented later.

9.2. The inner roots case

As shown above, in this case the pre-front equilibrium $u_1 = 0$ is stable, while the post-front equilibrium $u_2 = 1$ is unstable. Hence we expect in simulations that the post-front state evolves to another, stable equilibrium. This is indeed what happens in simulations, see Fig. 4.

For the parameters used in this simulations, the unstable equilibrium $u_2 = 1$ is surrounded by the pre-front equilibrium $u_1 = 0$ and the upper stable equilibrium $u_4 = 1.5$. Thus in this case we expect the post-front state attracted to either of these two stable equilibria.

In fact, the solution curiously does both, i.e. is first attracted to the upper stable equilibrium, $u_4 = 1.5$, but does not stay there for long and departs for the lower stable equilibrium, $u_1 = 0$. As a result, a pulse-shaped solution develops, with the pre-front and post-front states at $u_1 = 0$, and the plateau state near $u_4 = 1.5$. This phenomenology is similar to that observed in [16] for excitable (i.e. one stable equilibrium) cross-diffusion systems, including oscillatory front and oscillatory back, both trigger waves from one stable equilibrium to another — and is of course very far from the initial condition which is a monotonic front from a stable equilibrium to an unstable one.

9.3. The result of simulation of distinct real roots, double roots and complex roots

The behavior of the propagating wave front for the distinct real roots case and double roots case is quite similar. The simulation shows that the numerical propagating wave remains close to the analytical wave for a period of time. Then an oscillation appears near the onset of the front. After that the oscillation grows as the time evolves, which causes the numerical solution to break up. The results of the simulation of distinct real roots case is shown in Fig. 5 while the results of double roots case is shown in Fig. 6.

For complex roots case, we observe that the instability occurs earlier than all previous cases (inner roots case, outer roots case and double roots case). Moreover, the numerical front does not last as long as those front waves in the other cases, see Fig. 7.

10. The instability of the solution

In the previous sections we have shown the results of direct numerical simulation on reaction cross-diffusion system (37) where the initial condition is an exact analytical wave solution. This analytical solution presents a monotonic wave front.

Table 4
Comparison between the theoretical instability time T_{inst}, and time T_{break} to break-up in numerics, in the four selected simulations.

Case	T_{inst}	T_{break}
Inner roots	4371.3	30
Outer roots	7805.9	112
Double roots	1873.2	52
Complex roots	910.7	7

We have considered four cases, corresponding to different positions of the roots of the quartic polynomial. In all four cases considered, there are oscillations which appear near the onset of the wave front. These oscillations grow as time evolves, which obviously means that the propagating wave front is not stable. We now would like to address the question whether this was due to dynamical instability in the underlying partial differential equations, or numerical instability, i.e. artefact of the numerical scheme used.

Our plan on how to distinguish numerical instability from the numerical is as follows. If the instability is numerical, then its features shall significantly depend on details of the numerical scheme. For instance, the oscillations could be reduced by changing the discretisation steps. Conversely, the dynamical instability the behavior of the solution may be affected by refining the discretisation steps only slightly, if the simulation is "resolved".

A crude theoretical analysis of numerical stability of the scheme we use can be achieved by removing the kinetic terms from system (37). In this way, we obtain the following

$$u_t = D_v v_{xx},$$
$$v_t = -D_u u_{xx}.$$

For the forward-time, central-space discretization on the grid $x \in \Delta x\,\mathbb{Z}$, $t \in \Delta t\,\mathbb{Z}$, using the standard von Neumann stability analysis, for the Fourier component $(u,\, v) \propto e^{iqx}$ we find the amplification factor ν, such that

$$|\nu(q)|^2 = 1 + 16 D_u D_v \Delta t^2 \Delta x^{-4} \sin^4(q\Delta x/2), \tag{39}$$

which means that the numerical scheme is unstable as the condition $|\nu| \le 1$ will not be satisfied, in principle, for any choice of discretization steps.

However, let us look at the quantitative aspect of the numerical instability. Namely, let us estimate the time it takes for the numerical instability to grow to macroscopic value. Supposing, for a crude estimate, that the seed of the instability comes from round-off errors, so is of the order of machine epsilon ε, and it will become significant when it grows to an order of 1. Then, with the amplification factor $\nu(q)$, the number of time steps required for that will be at least $\ln|1/\varepsilon| / \max_q(\ln|\nu(q)|)$. Taking the leading order approximation for the $\ln|\nu(q)|$ in (39), we get the time interval required for the instability to grow to macroscopic size as

$$T_{\text{inst}} \approx \frac{\ln\left|\varepsilon^{-1}\right| \Delta x^4}{8 \Delta t D_u D_v}.$$

By substituting the values of parameters we used in our simulation, we see that in all cases T_{inst} is much bigger than the time T_{break} taken for the numerical waves to break up. Table 4 clarifies more by numbers. We took $\varepsilon = 10^{-15}$.

This comparison suggests that even though the numerical scheme is formally unstable, this instability cannot affect the numerical solutions on the time intervals involved. This means that there is no need to look for more sophisticated, stable methods to simulate the solutions presented. This also means that the numerical instability cannot explain the behavior observed in our numerics, and we must consider the possibility of a dynamical instability.

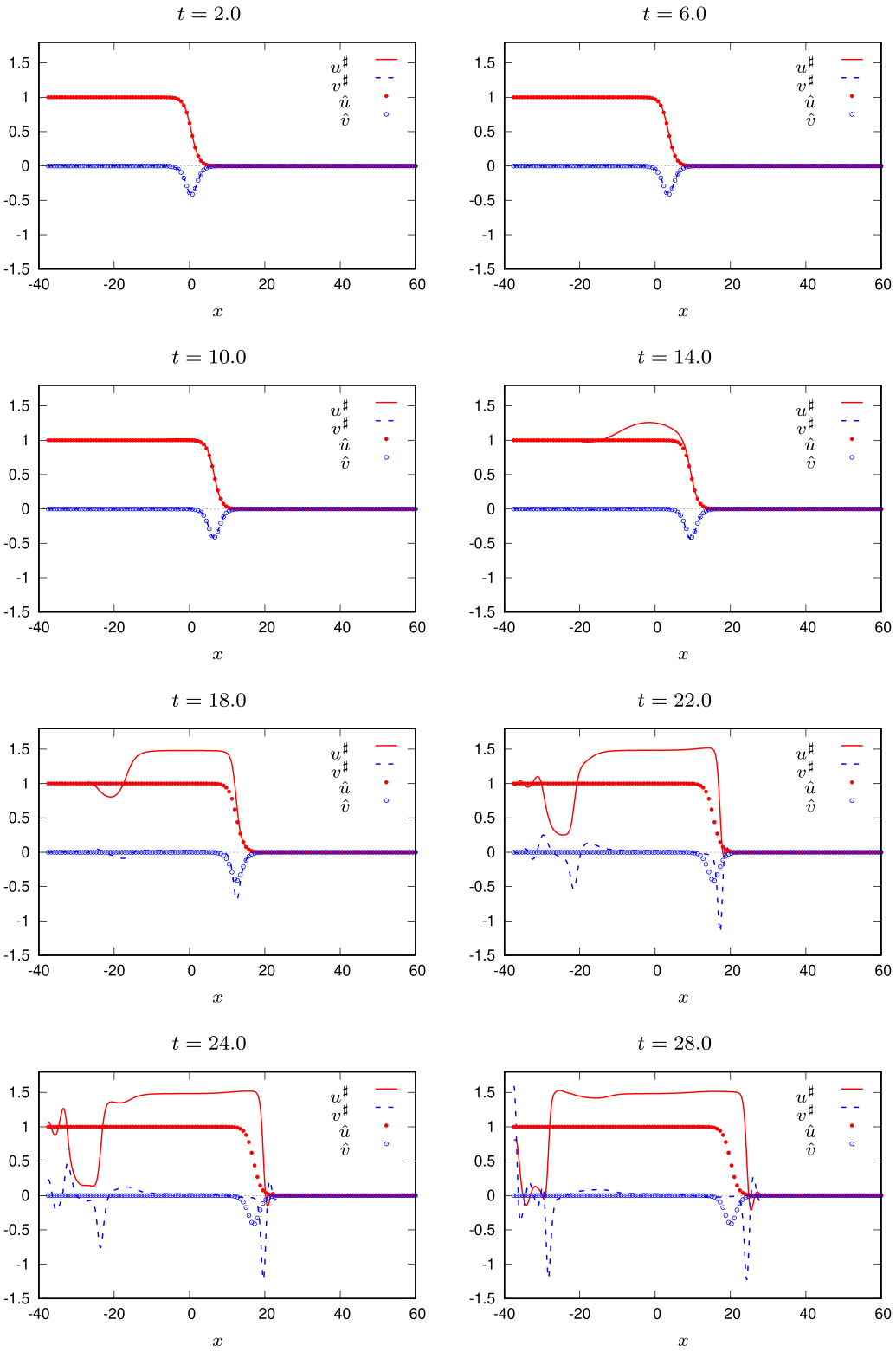

Fig. 4. The numerical simulation of reaction cross-diffusion system with quartic polynomial where the resting states of the front coincides with the **inner roots** of the quartic. The values of parameters in this simulations are $D_u = 1.25$, $D_v = 0.1$ and $k = 1$. Here and in the subsequent figures, $u^\sharp = u^\sharp(x,t)$, $v^\sharp = v^\sharp(x,t)$ is the numerical solution, whereas $\hat{u} = \hat{u}(x - ct)$, $\hat{v} = \hat{v}(x - ct)$ is the analytical solution used as the initial condition for the numerics.

So, according to our plan, we have verified the plausibility of a dynamical instability by repeating the simultions at different discretization steps. We have repeated each of the simulations, once with bigger discretization steps and once with smaller discretization steps. We have found that the behavior of the solution does not significantly change even after we refine the discretisation. More precisely, once the oscillations appear, we have found the growth rate of the oscillation is the same in all different discretisation steps. Fig. 8 illustrates that for the "outer roots" case: even though the moment of onset of the instability depends on the discretization, its growth rate is not affected by it.

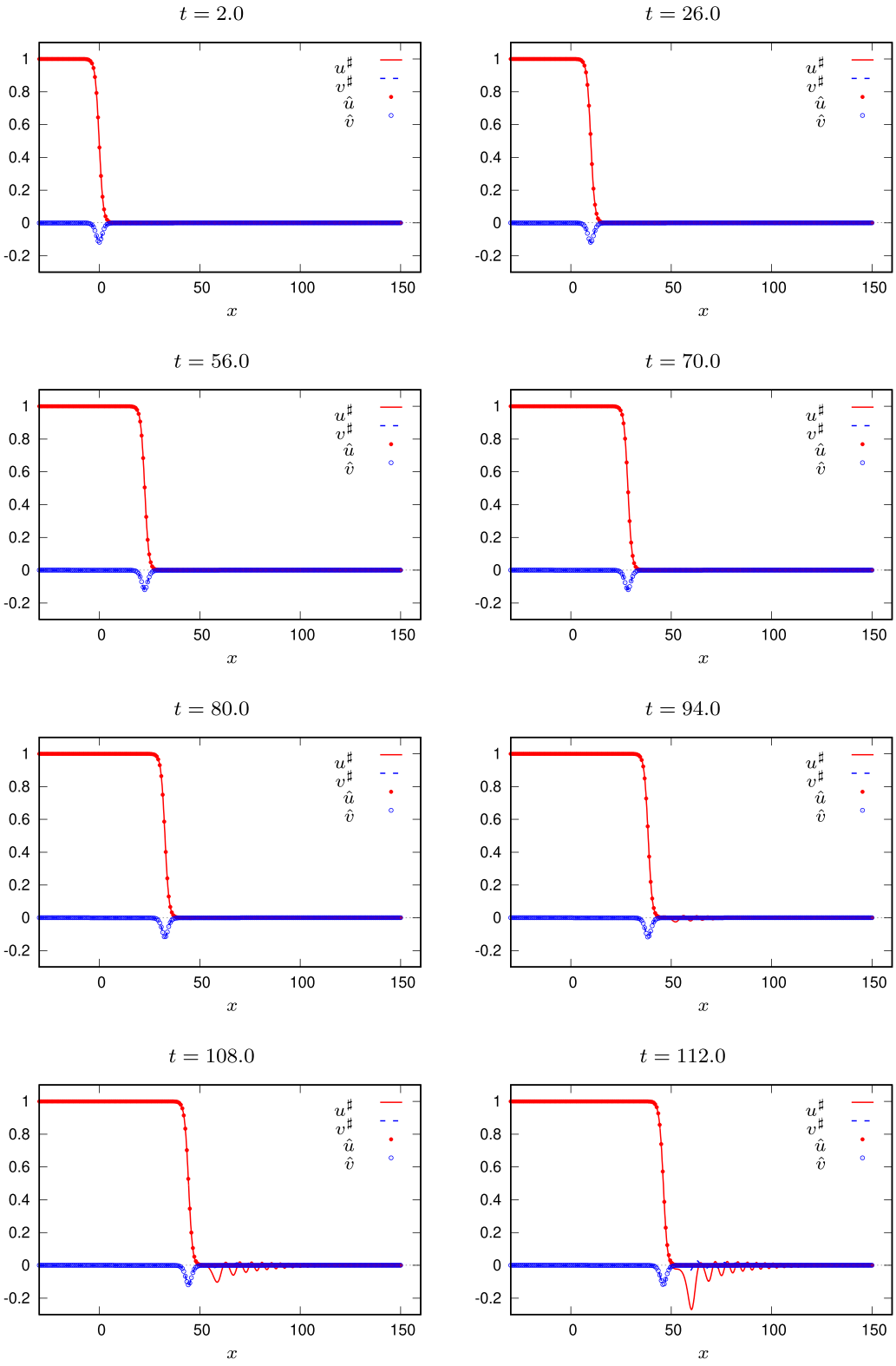

Fig. 5. The numerical simulation of reaction cross-diffusion system with quartic polynomial where the resting states of the front coincides with the **outer roots** of the quartic. The values of parameters in this simulations are $D_u = 0.2$, $D_v = 0.35$ and $k = 1$.

The same thing happened in double roots case and complex roots case. Change of discretisation steps changes the time of the onset of the instability, but not the growth rate of the instability, as can be seen in Figs. 9 and 10.

For the "inner roots" case, the initial condition is a front of invasion of an unstable state into a stable state, and the numerical simulation show behavior different from other cases: now the instability appears, at first, as the elevation of the u-field right behind the front. So we observe how this instability changes with different discretization steps. The result is shown in Fig. 11. We see, again, that the time of the onset of the instability does depend on the discretization steps, but the growth rate remains the

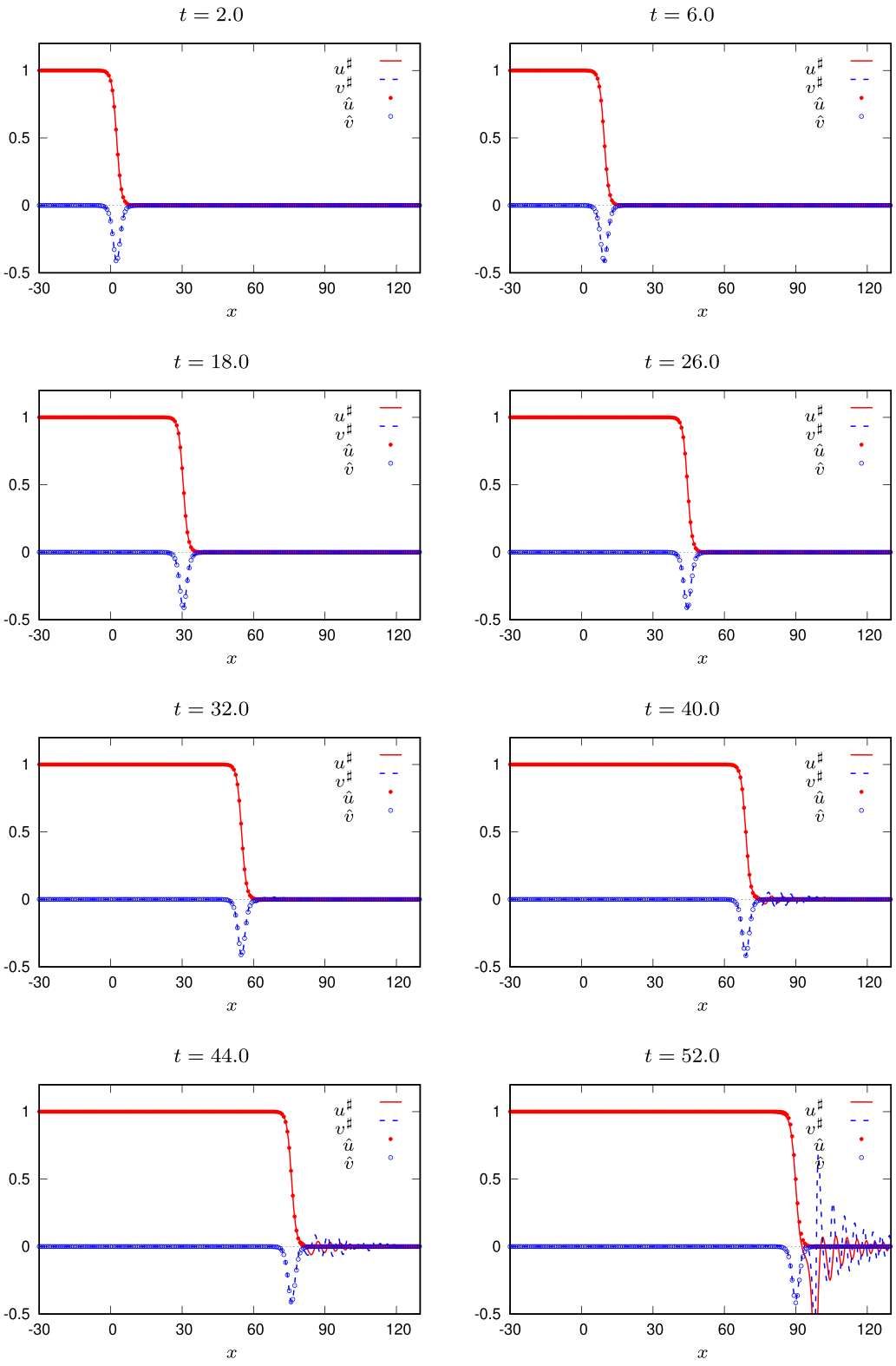

Fig. 6. The numerical simulation of reaction cross-diffusion system with quartic polynomial where there are **double roots** and the resting states are simple roots. The values of parameters in these simulations are $D_v = 0.1$ and $k = 1$ where D_u is given in the formula (38).

same. The subsequent behavior of the solution also remains qualitatively similar, involving formation of a propagating pulse with a plateau and a back — even though shifted in time and differing in detail, which is of course only expectable for a solution affected by a dynamical instability.

We can conclude that insofar as it may be established by numerical simulations, the analytical front solutions are dynamically unstable: they yield to solutions with oscillatory fronts, which are beyond the main scope of the current paper and requires separate study.

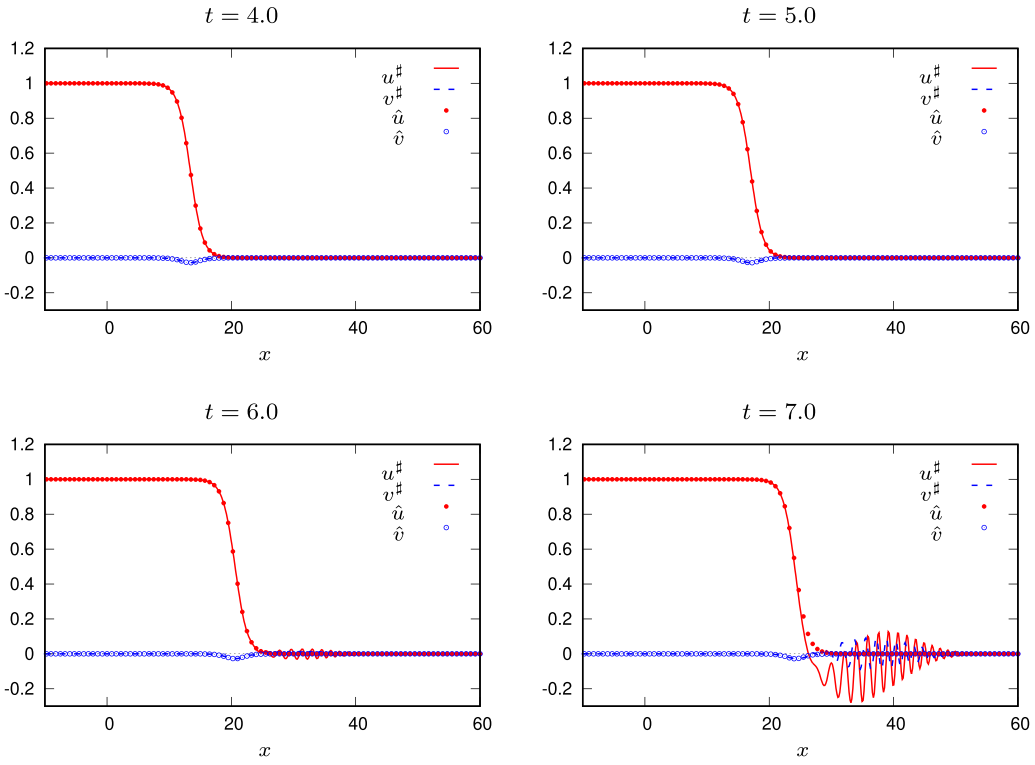

Fig. 7. The numerical simulation of reaction cross-diffusion system with quartic polynomial where there are two **complex conjugate roots**. The values of parameters in these simulations are $D_u = 0.4, D_v = 1.5$ and $k = 1$. The instability make the numerical solution run away at $t = 8$.

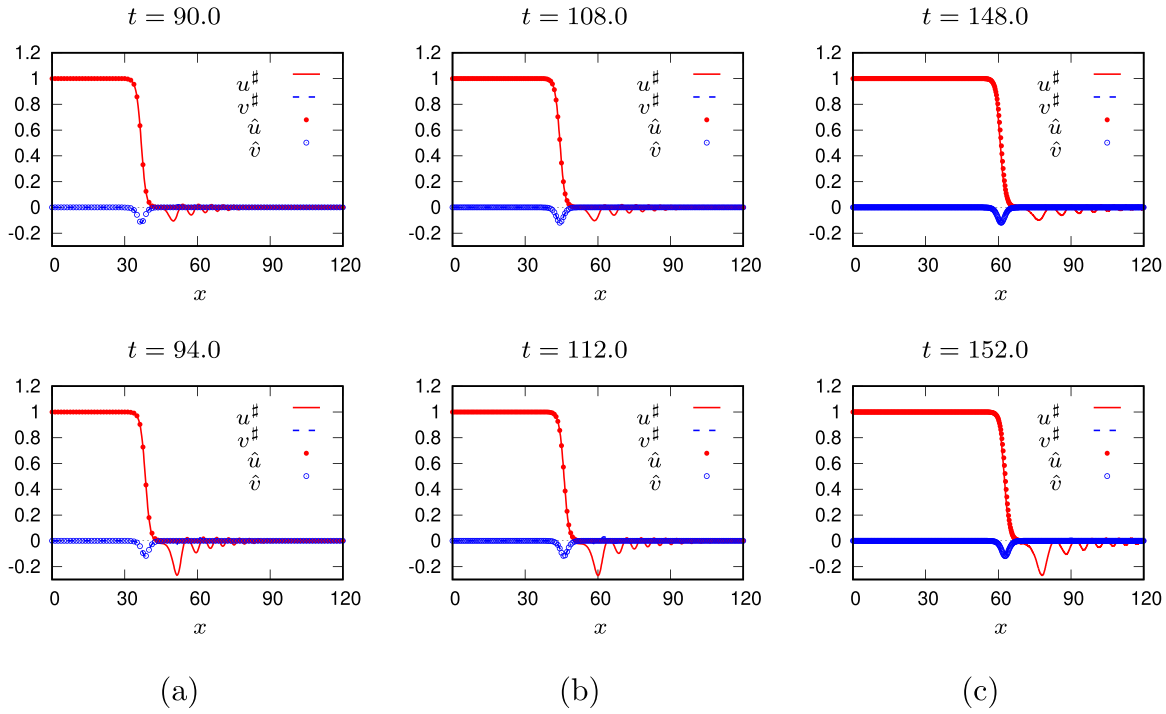

Fig. 8. The dynamical instability appears for **outer roots case**. The behavior of the solution does not change even after the steps are refined. The values of parameters are $k = 1$, $D_u = 0.2$ and $D_v = 0.35$. The discretisation is: (a) $\Delta x = 0.25,\ \Delta t = 4 \times 10^{-5}$; (b) $\Delta x = 0.15,\ \Delta t = 4 \times 10^{-6}$; (c) $\Delta x = 0.05,\ \Delta t = 1 \times 10^{-7}$.

11. Discussion

The main purpose of the paper, which has been successfully achieved, was to demonstrate the feasibility, and provide an example, of constructing a PDE model of a certain class which has desirable analytical solutions. Regardless of the utility of the particular example we have considered, we hope that the technique we used may be helpful in other problems similarly formulated.

More specifically, our aim has been a reaction-cross-diffusion system with a polynomial nonlinearity, which would have solutions in the form of a propagating front. We have found that to achieve that, the nonlinearity must be at least quartic, in which

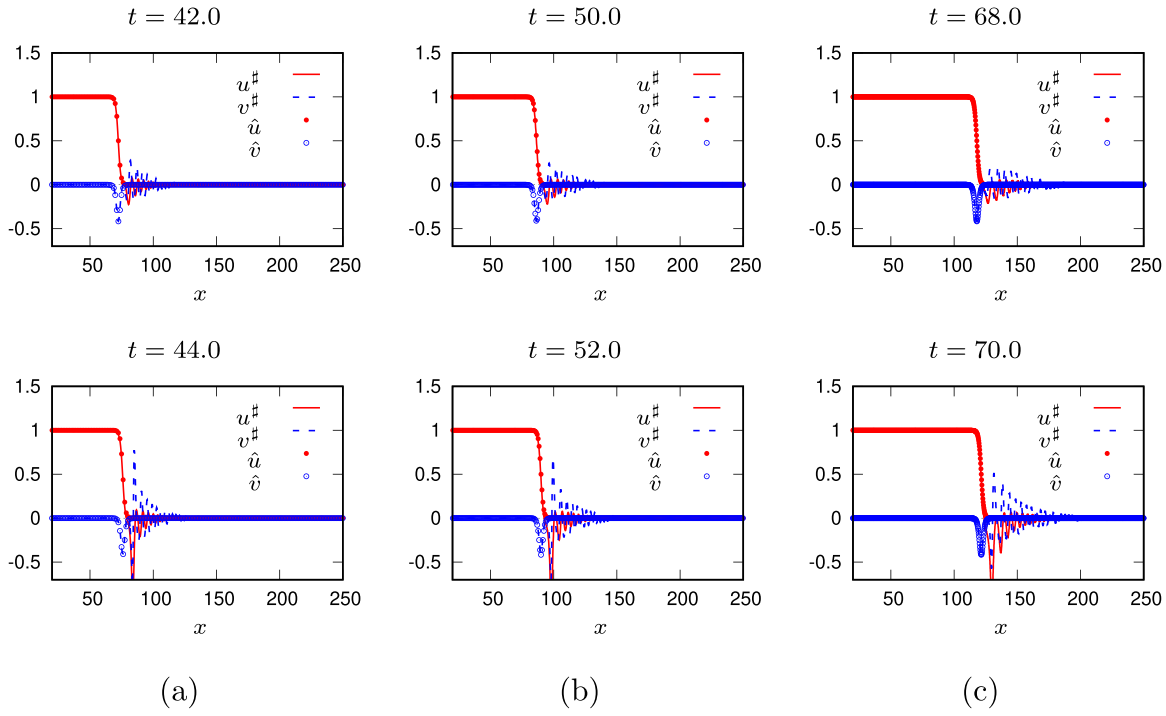

Fig. 9. The dynamical instability appears for **double roots case**. Each column represents the front wave for different discretisation steps. The behavior of the solution does not change even if the steps are refined. The values of parameters are $k = 1$ and $D_v = 0.1$. The discretisation is: (a) $\Delta x = 0.25,\ \Delta t = 4 \times 10^{-5}$; (b) $\Delta x = 0.15,\ \Delta t = 4 \times 10^{-6}$; (c) $\Delta x = 0.05,\ \Delta t = 1 \times 10^{-7}$.

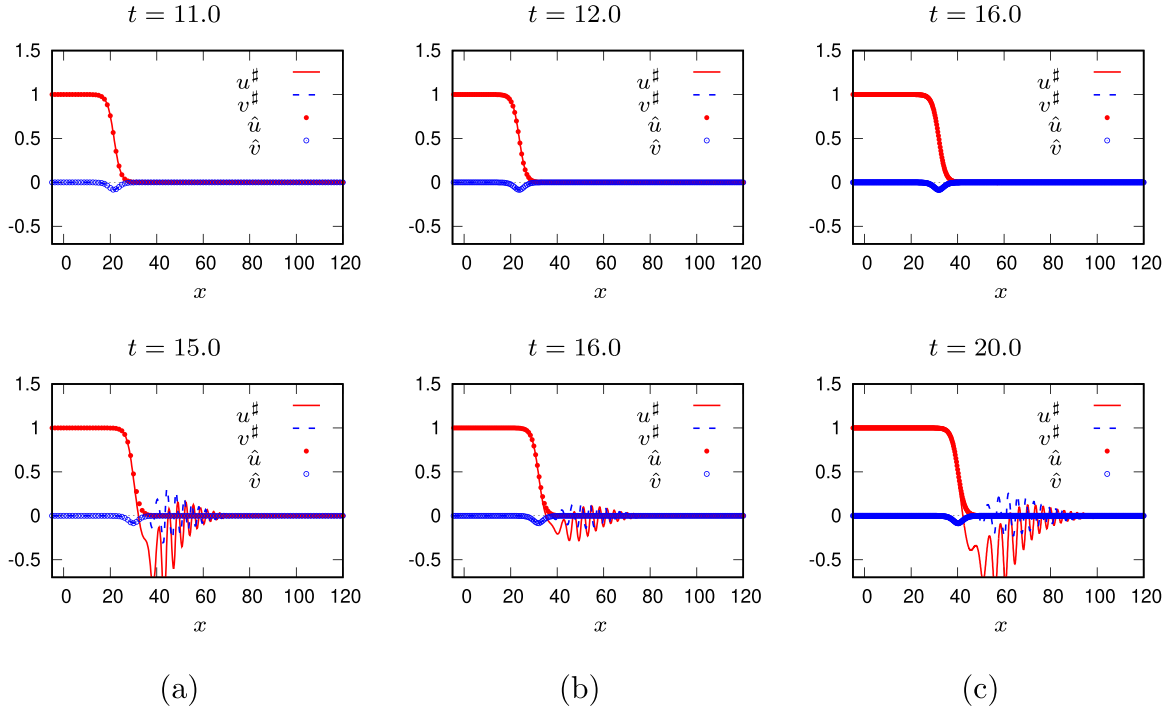

Fig. 10. The dynamical instability appears for **complex roots case**. Each column represents the front wave for different discretisation steps. The behavior of the solution does not change even if the steps are refined. The values of parameters are $k = 1$, $D_u = 1.25$ and $D_v = 0.1$. The discretisation is: (a) $\Delta x = 0.25,\ \Delta t = 4 \times 10^{-5}$; (b) $\Delta x = 0.15,\ \Delta t = 4 \times 10^{-6}$; (c) $\Delta x = 0.05,\ \Delta t = 1 \times 10^{-7}$.

case the system may indeed have solutions in the form of monotonic propagating fronts. The situation is similar to ZFK-Nagumo model rather than Fisher-KPP model in that for given parameters of the system, the speed and shape of the front solution are uniquely defined.

We have further established that in terms of stability of pre-front and post-front equilibria, the proposed model may be likened to the Fisher-KPP system (one of the equilibria is stable and the other unstable) but not ZFK-Nagumo (with both equilibria stable).

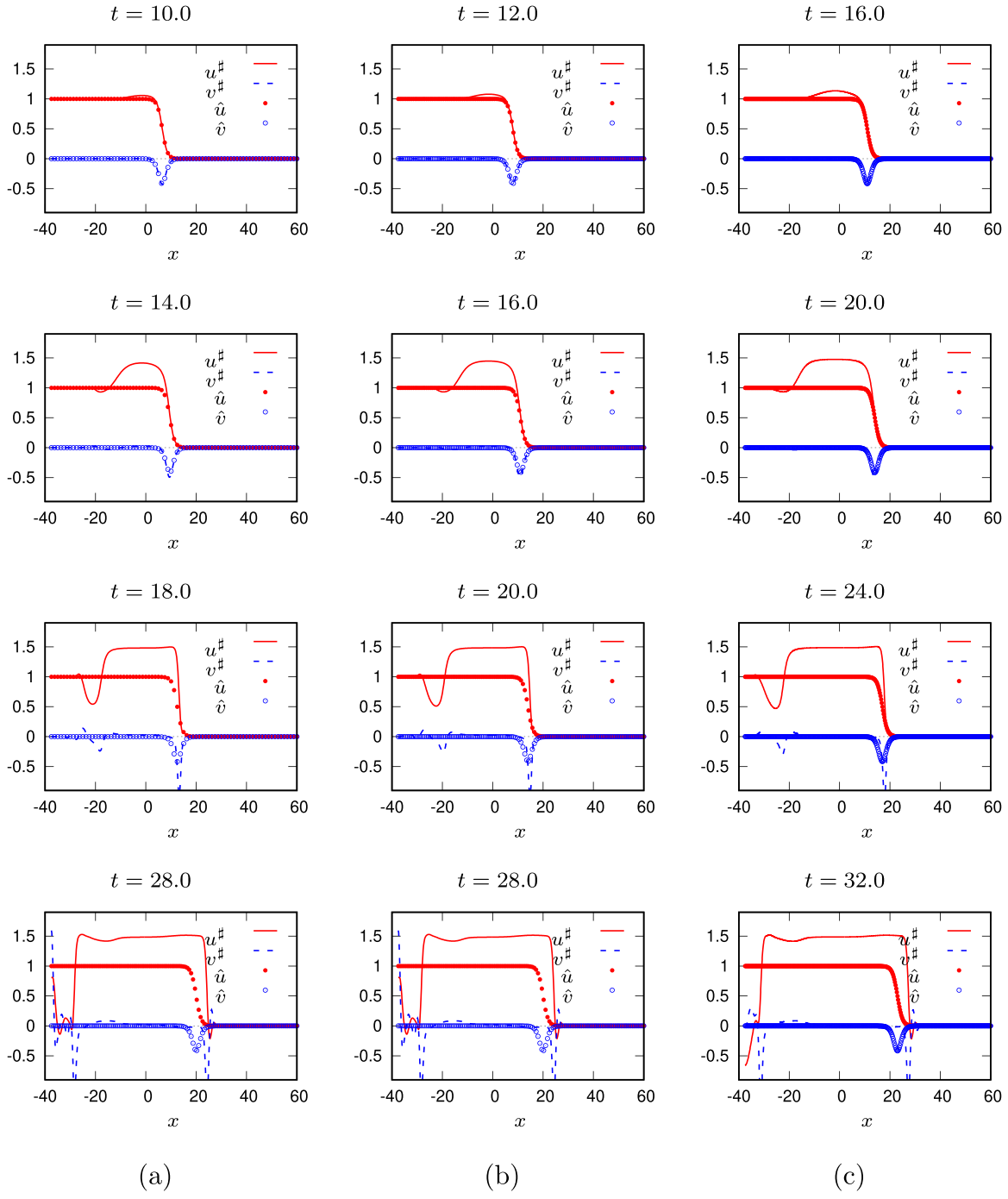

Fig. 11. The dynamical instability appears for **inner roots case**. Each column represents the front wave for different discretisation steps. The behavior of the solution does not change even the steps are refined. The values of parameters are $k = 1$, $D_u = 1.25$ and $D_v = 0.1$. The discretisation is: (a) $\Delta x = 0.25$, $\Delta t = 4 \times 10^{-5}$; (b) $\Delta x = 0.15$, $\Delta t = 4 \times 10^{-6}$; (c) $\Delta x = 0.05$, $\Delta t = 1 \times 10^{-7}$.

The quartic nonlinearity can be of various different classes depending on behavior of its four roots: when the asymptotic equilibria are two inner roots, two outer roots out of four, two outer roots out of three, the only two simple roots (with the other two being complex) and two double roots.

We have made simulations of selected examples of the proposed model belonging to different algebraic classes, and in all of these examples it happened that the analytical solutions are dynamically unstable, with some of the instabilities distinct from those related to the unstable pre-front equilibrium. Since the conclusion about instability of the solutions is based only on direct numerical simulations of arbitrarily selected examples, it requires further investigation, both theoretically and numerically, perhaps including continuation of propagating wave solutions rather than just direct numerical simulations, and wider parametric searches. A good survey of the relevant theory can be found in [44], and examples of numerical tools suitable for this task are AUTO [45] and WAVETRAIN [46].

Returning to feasibility of proposed PDE system as a model of real processes, we recall that KPP-Fisher is a viable model despite the unstable pre-front state. As it is well known, there are two inter-related reasons for that. One reason is the positivity of the

equation: non-negative initial conditions guarantee that the solution will remain non-negative at all times. Since the linearly unstable pre-front state is 0, i.e. at the border of the domain invariant under the system, this motivates restriction on the class of perturbations considered to those that would respect the positivity. The other reason is also related to the fact that the pre-front state is 0, but is of physical rather than mathematical nature: it motivates applications in which the dynamic field represent an essentially non-negative quantity with the meaning of a concentration of some kind; specifically, in the seminal papers [1,2] it was population density. With that physical sense of the dynamic field, the magnitude of physically feasible perturbations related to fluctuations must decay as the system gets closer to the pre-front state, and exactly vanish at that state. This motivates consideration of solutions in specially constructed functional spaces that take this issue into account, in which the solution may be stable — despite the formal instability of the pre-front state in the sense of generic dynamical systems theory. In this context, the possibility of, and, as numerics show, preference for, the non-monotonic fronts is only possible because the class of model we consider does not possess the positivity property. Here we note that the models with linear cross-diffusion cannot have that property in principle, see e.g. [31].

The above consideration motivates possible continuation of the present work:

- ZFK-Nagumo type fronts, i.e. monotonic fronts with stable pre-front and stable post-front states, may be sought for in models with polynomial nonlinearity of degrees higher than four;
- Reasonably stable monotonic fronts switching from a zero pre-front state may be observed in models with nonlinear cross-diffusion, e.g. “pursuit-evasion” type mutual taxis of the components;
- As the fronts actually observed in numerical simulations of cross-diffusion models so far are typically oscillatory, search of exact solutions of that kind would involve “inventing” an ansatz more sophisticated than that given by (14) and (15).

All that should be considered in the context that the problem addressed in this paper is about the “fast subsystem” in (2), and encompasses just the first step in the singular perturbation theory in the limit $\epsilon \to 0$.

Data access statement

All the research data supporting this publication are provided within this paper.

Declaration of Competing Interest

Authors are not aware of any conflicts of interests related to the submitted manuscript.

CRediT authorship contribution statement

Abdullah Aldurayhim: Investigation, Methodology, Visualization, Writing - original draft. **Vadim N. Biktashev:** Conceptualization, Methodology, Supervision, Visualization, Writing - review & editing.

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