Semi-stable laws for intermittent maps

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I certify that all material in this thesis which is not my own work has been identified and that any material that has previously been submitted and approved for the award of a degree by this or any other University has been acknowledged.

Signature D. Coates

Abstract

In this thesis we study probabilistic limit theorems for one-dimensional non-uniformly expanding maps with a single neutral fixed point, commonly known as intermittent maps. In 2004, S. Gouëzel showed that generic Hölder observables satisfy a stable law under the dynamics of the Liverani-Saussol-Vaienti (L.S.V.) family of intermittent maps in the case that an absolutely continuous probability measure is preserved. A key reason for the appearance of stable laws in the setting of Gouëzel's result is the fact that the return time to a particular reference set is regularly varying. We investigate what occurs when this regular variation is not present. In particular, we consider modifications of the L.S.V. map where stable laws fail to hold for generic Hölder observables and show that instead semi-stable laws emerge. We further establish that these semi-stable laws also appear in the context of the usual L.S.V. map for a certain class of oscillatory observables.

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Part I

Introduction and Statement of Results

Introduction

The general problem we are interested in studying here may be described as follows. We take an interval map $T : [0,1] \to [0,1]$ which is *intermittent*:

$$\exists x_0 \in [0,1] \text{ such that } T(x_0) = x_0, \;\; T'(x_0) = 1, \;\; T'(x) > 1 \;\; \forall x \neq x_0,$$

and we seek to understand aspects of the long-term behaviour of the sequence $(T^n(x))_{n\geq 0}$ for points $x \in [0,1]$. The term *intermittent* stems from the fact that a typical orbit $(T^n(x))_{x\geq 0}$ will spend a large period of time close to the neutral fixed point x_0 before briefly behaving chaotically until it again returns close to x_0. The global dynamical picture of an intermittent system thus consists of long laminar phases interrupted by chaotic bursts. Typical examples of such intermittent dynamical systems can be found in [PM80].

Even for simple maps T it can be extremely difficult to gain information on the behaviour of $(T^n(x))_{n\geq 0}$ using either analytic or computational techniques. An approach which is more fruitful, however, is to introduce a probability measure μ and to attempt to understand the behaviour of $(T^n(x))_{n\geq 0}$ probabilistically. If we let $u : [0,1] \to \mathbb{R}$ be a measurable function then we may view $u \circ T^n$ as random variables and thus try and understand the behaviour of averages of the sequence $(u \circ T^n)_{n\geq 1}$ over our space. A very well known, and perhaps most simple result in this direction is Birkoff's ergodic theorem which gives an analogue of the strong law of large numbers. Birkoff's ergodic theorem states that when the measure μ is ergodic for T (i.e. $T^{-1}(E) = E \Rightarrow$ either $(\mu(E) = 0$ or $\mu(E^c) = 0))$ and u is integrable then the average $\frac{1}{n}\sum_{j=0}^{n-1} u \circ T^j(x)$ converges to $\int u d\mu$ for μ-almost every x. The question we are interested in answering is whether there exist sequences $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ such that

$$Z_n := \frac{\sum_{j=0}^{n-1} u \circ T^j - B_n}{A_n}, \tag{1}$$

converges in distribution to something non-degenerate, or more generally, if the sequence $(Z_n)_{n\geq 0}$ has non-degenerate weak limit points. Here, by non-degenerate we mean that limiting random variable is not almost surely constant. For example, one might hope to obtain convergence results for the Z_n analogous to the central limit theorem.

A motivating result for the work we will present here is the following theorem due to Sebastian Gouëzel which gives conditions for the convergence of the Z_n to either Gaussian or (non-Gaussian) stable random variables in the case that u is a Hölder continuous function and T is the following intermittent map

$$T(x) := \begin{cases} x(1 + 2^\beta x^\beta) & \text{for } x \in [0, 1/2) \\ 2x - 1 & \text{for } x \in [1/2, 1]. \end{cases} \tag{2}$$

Theorem ([Gou04a, Theorem 1.3])**.** *Let $\beta \in (1/2, 1)$ and let T be the corresponding map as defined in (2). Let $u : [0, 1] \to \mathbb{R}$ be Hölder continuous with $\int u d\mu = 0$. Then there are two cases.*

1. *If $u(0) \neq 0$, then we have convergence to a stable law*

$$\frac{\sum_{j=0}^{n-1} u \circ T^j}{n^\beta} \xrightarrow[n\to\infty]{d} V_{1/\beta},$$

where $V_{1/\beta}$ is a stable random variable of index $1/\beta$.

2. *If $u(0) = 0$, and the Hölder exponent ν of u such that $\nu > \beta - 1/2$ then we have a central limit theorem*

$$\frac{\sum_{j=0}^{n-1} u \circ T^j}{\sqrt{n}} \xrightarrow[n\to\infty]{d} V \sim \mathcal{N}(0, \sigma^2),$$

for some $\sigma^2 \geq 0$.

A key property of the map defined in (2) that is responsible for the appearance of the stable law in second case of the theorem above is following. If we *induce* on the set $Y := [1/2, 1]$ and and consider the induced observable

$$u_Y(x) := \sum_{j=0}^{\tau(x)-1} u \circ T^j(x),$$

where $\tau : Y \to \mathbb{N}$ is the *first return time* to Y (i.e. $\tau(x) := \min\{n \geq 1 : T^n x \in Y\}$) one may show that when $u(0) \neq 0$ the tail distribution

$$F(y) = \mu(u_Y > y), \tag{3}$$

is *regularly varying*:

$$F(y) = y^{-\frac{1}{\beta}} \ell(y) \text{ where } \lim_{y \to \infty} \frac{\ell(\lambda y)}{\ell(y)} = 1 \ \ \forall \lambda > 0. \tag{4}$$

Indeed, as we will see in Section 1.1.3, having regularly varying tail distributions is essential in the context of independent and identically distributed random variables to obtain convergence to stable random variables (see Theorem 1.1.7). It is the behaviour of the map T near its neutral fixed point which is responsible for the fact that Hölder observables with $u(0) \neq 0$ induce to observables with tail distributions F of the form given in (4).

We are interested in the case that this regular variation is not present in (3), in particular we be interested in the case that an additional oscillatory factor appears in (3) above. We will study three different situations where this oscillatory factor is present in (3), each scenario being a map of the unit interval and class of observables. We will informally introduce our main results here. For precise definitions of the maps we consider and the for the formal statements of our main results we refer the reader to Chapter 2.

In order to introduce the systems we will study we need to introduce the function $M:(0,\infty)\to(0,\infty)$. For some $\varepsilon>0$ we define

$$M(x) \coloneqq 1+\varepsilon\sin\left(\frac{2\pi}{\log c}\log x\right), \tag{5}$$

where $c>1$ is some parameter. A key property of this is that it is *log-periodic*: $M(cx)=M(x)$ for all $x>0$. In what follows, the function M will be our prototypical example of a log-periodic function. In this thesis we will primarily study the following maps.

1. For $\alpha\in(1,2)$ we set $x_0=1$, $x_1=1/2$ and $x_n \coloneqq n^{-\alpha}M(n)$ for $n\geq 2$. We then define the map

$$T_{\exp}(x) \coloneqq \begin{cases} 0, & \text{if } x=0 \\ g_n(x), & x\in[x_{n+1},x_n] \;\; n\geq 2 \\ 2x-1, & \text{if } x\in[1/2,1], \end{cases} \tag{6}$$

where

$$g_n(x) \coloneqq 1+x_n+(1-a_n)\rho_n(x-x_{n+1})-\exp\left\{\log(1-a_n\Delta_{n-1})\frac{x-x_{n+1}}{\Delta_n}\right\}, \tag{7}$$

and

$$\Delta_n \coloneqq x_n-x_{n+1}, \quad \rho_n \coloneqq \frac{\Delta_{n-1}}{\Delta_n},$$

and (a_n) is some strictly decreasing sequence of positive reals converging to 0. The key property of this system is that it maps each interval $[x_{n+1},x_n]$ smoothly and bijectively onto the $[x_n,x_{n-1}]$ with the intervals $([x_{n+1},x_n])_{n\geq 1}$ forming a Markov partition for $T_{\exp}$, this is similar to the piecewise quadratic map studied in [KT18].

2. For $\beta \in (1/2, 1)$ we define

$$T_w(x) \coloneqq \begin{cases} 0, & \text{if } x = 0 \\ x(1 + aM(x)x^{\beta}), & \text{if } x \in (0, 1/2) \\ 2x - 1, & \text{if } x \in [1/2, 1], \end{cases} \tag{8}$$

where a is some constant chosen so that $\frac{1}{2}(1 + aM(\frac{1}{2})(\frac{1}{2})^{\beta}) = 1$.

3. For $\beta \in (1/2, 1)$ we set

$$T_{LSV}(x) \coloneqq \begin{cases} x(1 + 2^{\beta}x^{\beta}), & \text{if } x \in [0, 1/2) \\ 2x - 1, & \text{if } x \in [1/2, 1]. \end{cases} \tag{9}$$

For each of these maps we establish a *semi-stable law*, that is, for certain observables u we show the distributional convergence of the Z_n along subsequences to a non-degenerate *semi-stable* random variable. Semi-stable random variables are generalisation of stable random variables and will be discussed in detail in Section 1.1.4. We show that for $T \in \{T_{\exp}, T_w, T_{LSV}\}$ there exists sequences (k_n) and (A_n) (which may be determined) so that for certain observables u we have that

$$\frac{\sum_{j=0}^{k_n-1} u \circ T^j - k_n \int u d\mu}{A_n} \xrightarrow[n\to\infty]{d} V, \tag{10}$$

where V is a semi-stable random variable and μ is the absolutely continuous invariant probability measure for T. In the case of $T_{\exp}$ and T_w we are able to establish (10) for Hölder observables u which are non-zero at 0, and for T_{LSV} we establish (10) for observables of the form $u(x) = M(x)$. Moreover we can strengthen the distributional convergence which appears in (10) to the following *merging* result. We introduce a function $\gamma : (0, \infty) \to (0, \infty)$, which is defined in terms of the sequence (k_n) that appears in (10) in the following way: for all $s > 0$ small enough we put

$$\gamma(s) = sk_{n(s)} \tag{11}$$

where $k_{n(s)}$ is the unique element of $(k_n)_{n\geq 1}$ such that $\frac{1}{k_n} \leq s < \frac{1}{k_{n-1}}$. One can show that for any all $s > 0$ small enough $\gamma(s)$ is contained in a compact set K. The strengthening of (10) we obtain is

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \mu\left(\frac{\sum_{j=0}^{n-1} u \circ T^j - n \int u d\mu}{A_n} \leq x \right) - \mu(V_{\gamma(1/n)} \leq x) \right| = 0, \tag{12}$$

where $\{V_\lambda : \lambda \in K\}$ is a continuous family of semi-stable random variables, which we introduce in Section 1.1.4, defined in terms of the V which appears (10).

We will now proceed in Chapter 1 to recall relevant background material before giving the formal statements of main findings in Chapter 2. The remainder of this document is then devoted to the proofs of the results given in Chapter 2.

Chapter 1

Background and Terminology

1.1 Limit theorems for sums of i.i.d. random variables

1.1.1 Terminology and Notation

In this section we let $(\Omega, \mathscr{B}, \mu)$ be a probability space, where (Ω, d) a Polish metric space and $\mathscr{B}$ is the Borel σ-algebra. Given a sequence of measures $(\nu_n)_{n\in\mathbb{N}}$ on $(\Omega, \mathscr{B})$ we say that ν_n *converges weakly* to ν and write $\nu_n \xrightarrow[n\to\infty]{w} \nu$ if for every continuous bounded function $u : \Omega \to \mathbb{R}$ we have that $\lim_{n\to\infty} \nu_n(u) = \nu(u)$ where for a measure λ we write $\lambda(u) \coloneqq \int_\Omega u d\lambda$. We say that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$, $X_n : \Omega \to \mathbb{R}$ *converges in distribution to* $X : \Omega \to \mathbb{R}$, and write $X_n \xrightarrow[n\to\infty]{d} X$ if the corresponding distributions converge weakly: $\mu X_n^{-1} \xrightarrow[n\to\infty]{w} \mu X^{-1}$. The topology induced by weak-convergence on the space of probability measures on Ω is metrisable by the Prokhorov metric [Bil99]:

$$d_{\mathcal{P}}(\mu, \nu) \coloneqq \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \ \forall A \in \mathscr{B}\},$$

where we denote by A^ε an ε neighbourhood of a set A. A special case of this metric is the Lévy-Prokhorov $d_{\mathcal{L}}$ metric [Bil99] which captures the notion of weak convergence on the space of distribution functions $\mathscr{F}$ where

$$\mathscr{F} := \{F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0,1] | F \text{ right continuous and decreasing and}$$
$$F(-\infty) = 0,\ F(+\infty) = 1\},$$

and

$$d_{\mathcal{L}}(F, G) := \inf\{\varepsilon > 0 : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon\}.$$

We also recall here the concepts of tightness and stochastic compactness. We say that a collection $\mathcal{F}$ of measures is *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ so that $\mu(K) > 1 - \varepsilon$ for every $\mu \in \mathcal{F}$. A collection of random variables is said to be tight if the corresponding collection of distributions are tight. A collection of measures $\mathcal{F}$ is *stochastically compact* if every sequence of in $\mathcal{F}$ has a sub-sequence that is weakly convergent. Similarly a collection of random variables is called stochastically compact if the corresponding family of distributions is stochastically compact. Thus, both tightness and stochastic compactness are necessary conditions for convergence in distribution, however the converse is in general false: in Section 1.1.4 we will see examples of sequences that are tight but do not converge weakly. The correspondence between stochastic compactness and tightness are given by Prokhorov's theorem which for completeness we state below.

Theorem 1.1.1 (Prokhorov 1956 [Pro56])**.** *Let (Ω, d) be a metric space and let $\mathcal{F}$ be a collection of Borel probability measures on Ω. Then if $\mathcal{F}$ is tight then $\mathcal{F}$ is stochastically compact. Moreover, if (Ω, d) is Polish then $\mathcal{F}$ is tight if and only if it is stochastically compact.*

Remark 1.1.2. As a consequence of the theorem above we have that for real-valued random variables the concepts of tightness and stochastic compactness are equivalent.

1.1.2 Infinitely divisible distributions

For a sequence of independent identically distributed (i.i.d.) random variables $(X_n)_{n\in\mathbb{N}}$ one knows by virtue of the *strong law of large numbers* that the average $\frac{1}{n}\sum_{j=1}^{n} X_j$ will converge to the expectation $\mathbb{E}(X_1)$ almost surely. The *central limit theorem* tells us that if these random variables satisfy $\mathbb{E}(X_n^2) < \infty$ and we replace scaling $\frac{1}{n}$ by $\frac{1}{\sqrt{n}}$ then we no longer have almost sure convergence to a constant, but instead we have convergence in distribution to a normal random variable.

Theorem 1.1.3. *[see for example [Bil12, Theorem 27.1]] Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed random variables with common distribution μ and with finite variance σ^2. Then*

$$\frac{\sum_{j=1}^{n} X_j - n\mathbb{E}(X_1)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} Z \sim \mathcal{N}(0,\sigma^2).$$

The above result has a very simple and elegant proof which utilises Levy's continuity theorem. We present this proof below as it captures some of the key ideas in the arguments that will follow.

Proof of Theorem 1.1.3. We will assume that $\mathbb{E}(X_1) = 0$, the general case will then follow from considering $\tilde{X}_n = X_n - \mathbb{E}(X_n)$. Let $Z_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j$, and consider the *characteristic function* (c.f.) of Z_n,

$$\varphi_n(t) := \mathbb{E}(e^{itZ_n}) = \varphi(t/\sqrt{n})^n, \tag{1.1}$$

where φ is the characteristic function of X_1. By Lévy's continuity theorem (for example see [Kle14, Theorem 15.23]), and the fact that characteristic functions of distributions are unique, it is enough to show that φ_n converges point-wise to the c.f. of a normal distribution. Considering the second order Taylor expansion of $\varphi_n(t)$ about 0 for we see that for fixed t and n large

$$\varphi_n(t) = \left(1 - \frac{\sigma^2 t^2}{2n} + o(1/n)\right)^n \underset{n\to\infty}{\to} e^{\sigma^2 t^2/2},$$

which concludes the proof. □

Later we shall be interested in proving analogous results for identically distributed but non-independent sequences, namely we will replace the i.i.d. random variables $X_1, X_2, \ldots$ with the deterministic sequence $u, u \circ T, u \circ T^2, \ldots$ where $T : \Omega \to \Omega$ is some measure preserving transformation and $u : \Omega \to \mathbb{R}$ is some observable. In this case, if we set $X_n = v \circ T^n$ we note that the sequence $(X_n)_{n\geq 0}$ is identically distributed, as T is measure preserving, but not necessarily independent. Let us note that the key reason why the simple proof above no longer works for this choice of $(X_n)_{n\geq 0}$ is that the equality (1.1) breaks down when the X_n are not independent. In the remainder of this section we will examine the possible distributional limits of appropriately scaled and centred sums of i.i.d. random variables, in particular we are interested in the case that we no longer have finite variance and Theorem 1.1.3 does not apply. We will assume that all random variables appearing in this section are *non-degenerate*, that is to say that they are almost-surely non-constant, or equivalently, that their distribution is not a point mass. Consider a sequence (X_n) of i.i.d. random variables with common distribution μ. Suppose that there exist sequences $(A_n)_{n\geq 1}, (B_n)_{n\geq 1}$ of real numbers so that

$$\frac{\sum_{j=1}^n X_j - A_n}{B_n} \xrightarrow[n\to\infty]{d} Y, \tag{1.2}$$

for some random variable Y with distribution ν. It is well-known that in such a situation the limiting object Y must an *infinitely-divisible distribution*.

Definition 1.1.4. We say that the distribution of a random variable X is *infinitely-*

divisible if for each $n \geq 1$ there exists i.i.d. random variables $X_1, X_2, \dots X_n$ so that

$$X \stackrel{d}{=} X_1 + X_2 + \cdots + X_n.$$

Typical examples of infinitely divisible distributions are the point mass δ_x, the Normal distribution and the Poisson distribution. Moreover, it is clear from the definition that finite sums of infinitely divisible distributions are infinitely-divisible.

Given a random variable Y with distribution ν it is natural to ask which, if any, sequences of random variables (X_n) satisfy (1.2) for some choice of $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 1}$. If the (X_n) are a sequence of i.i.d. random variables with common distribution μ which satisfy (1.2). we say that μ *is in the domain of attraction of* ν and we write $\mu \in \mathbb{D}(\nu)$ (or equivalently $F \in \mathbb{D}(G)$ where $F(x) \coloneqq \mu(-\infty, x])$, $G \coloneqq \nu(-\infty, x])$.

For example, the central limit theorem (Theorem 1.1.3) tells us that $\{\mu : \int x^2 d\mu(x) = \sigma^2\} \subset \mathbb{D}(\nu)$ when ν is the normal distribution with variance σ^2 and mean 0. Let us now give a the representation formula due to Lévy-Khintchine that describes the characteristic function of an infinitely divisible random variable.

A function $\varphi : \mathbb{R} \to \mathbb{C}$ is the characteristic function of an infinitely divisible distribution if and only if there exist:

1. functions $L : (-\infty, 0) \to \mathbb{R}$ and $R : (0, \infty) \to \mathbb{R}$, which we call the *left and right Lévy functions* respectively, that are non-decreasing on their domains and satisfy

   $$L(-\infty) = R(\infty) = 0,$$

   and for every $\varepsilon > 0$

   $$\int_{-\varepsilon}^{0} u^2 \, \mathrm{d}L(u) + \int_{0}^{\varepsilon} u^2 \, \mathrm{d}R(u) < \infty.$$

2. constants $\sigma^2 > 0, \gamma$ such that

$$\varphi(t) = i\gamma t + \frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0}(e^{iut} - 1 - \frac{iut}{1+u^2})\,\mathrm{d}L(u) + \int_{0}^{\infty}(e^{iut} - 1 - \frac{iut}{1+u^2})\,\mathrm{d}R(u).$$

Moreover, the choice of L, R, γ and σ^2 above is unique.

1.1.3 Stable laws

An important subclass of infinitely divisible distributions are *stable distributions*. The distribution of a random variable X is called *stable* if whenever $X_1, X_2, \ldots, X_n$ are n independent copies of X there exists $A_n, B_n \in \mathbb{R}$ with $A_n > 0$ and

$$X_1 + X_2 + \cdots + X_n \stackrel{d}{=} A_n X + B_n.$$

It is clear from this definition that stable distributions are infinitely divisible and moreover we see that every stable distribution μ is in its own domain of attraction: $\mu \in \mathbb{D}(\mu)$. In fact, we have the following important result:

Theorem 1.1.5 (Gnedenko, Kolmogorov [GK54, Theorem 1, Section 33])**.** *Let μ be a Borel probability measure on $\mathbb{R}$. Then the domain of attraction $\mathbb{D}(\mu)$ of μ is non-empty if and only if μ is stable.*

Let us now refine the definition of a stable distribution by introducing the notion of its index.

Definition 1.1.6 (stable distribution of index α)**.** We will say that a stable random variable is *stable if index* $\alpha \in (0, 2]$ if

$$X_1 + X_2 + \cdots + X_n = n^{1/\alpha} X + B_n$$

for some $B_n \in \mathbb{R}$, whenever $X_1, X_2, \dots, X_n$ are n independent copies of X.

In fact, every stable distribution is stable of some index $\alpha \in (0, 2]$, moreover $\alpha = 2$ corresponds to the distribution being normal [Kle14, Theorem 16.22]. Thus we may unambiguously talk of stable distributions of index α. So far we have that every possible limiting distribution of (1.2). is stable and that every stable distribution appears as a limiting distribution of (1.2). It is also possible to classify completely the domain of attraction $\mathbb{D}(\mu)$ for a stable distribution μ in terms of the tail behaviour of the distribution. We say that $\ell : \mathbb{R} \to \mathbb{R}$ is *slowly varying at* ∞ if for every $t \in \mathbb{R} \setminus 0$ we have that

$$\lim_{x\to\infty} \frac{\ell(tx)}{\ell(x)} = 1;$$

and, we say that $f : \mathbb{R} \to \mathbb{R}$ is *regularly varying* at ∞ with index $p \in \mathbb{R}$ if there exists a slowly varying function ℓ such that

$$f(x) = x^p \ell(x).$$

We may now give the classification of $\mathbb{D}(\mu)$ for non-normal stable distributions μ due to Gnedenko and Kolmogorov.

Theorem 1.1.7 ([GK54]). *Let μ be a stable distribution of index $\alpha \in (0, 2)$. Then $\nu \in \mathbb{D}(\mu)$ if and only if the following two conditions are satisfied.*

tail-balancing

$$\lim_{x\to\infty} \frac{\mu(x, \infty)}{\mu(-\infty, -x) + \mu(x, \infty)} = C \in [0, 1],$$

regularly varying tails *the left and right tail distributions $\mu(-\infty, -x)$ and $\mu(x, \infty)$ for $x > 0$ are regularly varying of index $-\alpha$.*

1.1.4 Semi-stable Laws

A natural way to extend the class of stable distributions is by permitting the convergence in (1.2) to occur along sub-sequences. For a sub-sequence $(k_n)_{n\geq 1}$ of $\mathbb{N}$ we now consider the class of permissible distributions of the limiting random variable Y in the case that

$$Z_{k_n} := \frac{\sum_{j=1}^{k_n} X_j - B_{k_n}}{A_{k_n}} \xrightarrow[n\to\infty]{d} Y. \tag{1.3}$$

In order to obtain a non-trivial subset of infinitely-divisible distributions it makes sense to impose some additional conditions on the sub-sequence $(k_n)_{n\geq 1}$, these conditions are outlined in the following definition of what it means for a distribution to be semi-stable.

Definition 1.1.8 (semi-stable, domain of partial geometric attraction)**.** Let $(k_n)_{n\geq}$ be a sequence of positive integers satisfying one of the following conditions:

$$\liminf_{n\to\infty} \frac{k_{n+1}}{k_n} = c \in (1, \infty) \tag{1.4}$$

$$\limsup_{n\to\infty} \frac{k_{n+1}}{k_n} = c \in [1, \infty) \tag{1.5}$$

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = c \in [1, \infty) \tag{1.6}$$

Non-degenerate distributions which arise as the limit of (1.3) along such sequences (k_n) are called *semi-stable*. We say that a distribution μ is in the *domain of partial geometric attraction* of a semi-stable law ν, written[1] $\mu \in \mathbb{D}_{gp}(\nu)$ along a sequence (k_n) if (1.3) holds and limiting random variable Y has distribution ν.

Remark 1.1.9. We note that trivially any sequence $(k_n)_{n\geq 1}$ satisfying either (1.4) or (1.5) will have a further sub-sequence satisfying (1.6). To simplify the following statements we will only consider semi-stable distributions that arise along sub-sequences satisfying

[1] We will also write $F \in \mathbb{D}_{gp}(G)$ if $\mu \in \mathbb{D}_{gp}(\nu)$, and F is the distribution function of μ and G is the distribution function of ν

(1.6).

We see from the above definition that stable distributions are also semi-stable, in fact we have the following series of strict inclusions

$$\text{stable} \subset \text{semi-stable} \subset \text{infinitely divisible.}$$

Associated to each semi-stable distribution is an *index* $\alpha \in (0, 2]$ and a *period* $c \geq 1$. If the period of semi-stable distribution is equal to 1 then the distribution is stable and if its index is equal to 2 then the distribution is normal. From here on we will only consider semi-stable distributions with index $\alpha \in (0, 2)$ and period $c > 1$. A distribution $\nu_{\alpha,c}$ is semi-stable with index α and period c if and only if its left and right Lévy functions $L : (-\infty, 0) \to \mathbb{R}$ and $R : (0, \infty) \to \mathbb{R}$ are given by

$$L(x) = \frac{M_L(x)}{|x|^\alpha}, \quad R(x) = -\frac{M_R(x)}{x^\alpha}, \tag{1.7}$$

where M_L and M_R are not both identically zero and each non-zero $M \in \{M_R, M_L\}$ satisfies the condition A1 :

A1

- M is right continuous
- M is bounded away from both 0 and ∞
- $M(x)/x^\alpha$ is monotone decreasing
- M is *logarithmically periodic* with period $c^{1/\alpha}$

$$M(c^{1/\alpha}x) = M(x), \quad \forall x > 0. \tag{1.8}$$

Let us fix a non-stable semi-stable distribution of index $\alpha \in (0, 2)$ and period $c > 1$ and let G denote its distribution function. Letting $(k_n)_{n\geq 1}$ be a sequence satisfying

(1.6) and putting $A_{k_n} \coloneqq k_n^{1/\alpha}\ell(k_n)$ for some slowly varying ℓ we may now state the description of $\mathbb{D}_{gp}(G)$ given in [Meg00, Corollary 3]. A distribution function F lies in $\mathbb{D}_{gp}(G)$ along the sequence $(k_n)_{n\geq 1}$ and with normalising coefficients $(A_{k_n})_{n\geq 1}$ (written $F \in \mathbb{D}_{gp}(G, k_n, A_{k_n})$) if and only condition A2 below is satisfied.

A2 For all $x > 0$ sufficiently large we have that

$$\overline{F}(x) \coloneqq 1 - F(x) = x^{-\alpha}\ell^*(x)(M_R(\delta(x)) + h_R(x)), \tag{1.9}$$

and

$$F_-(-x) = x^{-1/\alpha}\ell^*(M_L(-\delta(x)) + h_L(x)) \tag{1.10}$$

where F_- is the left continuous version of F

- $\ell^* : (0, \infty) \to (0, \infty)$ is determined by

$$x^{-\alpha}\ell^*(x) = \sup\{t : t^{-1/\alpha}\ell(1/t) > x\}, \tag{1.11}$$

  so that $x^{1/\alpha}\ell(x)$ and $y^\alpha/\ell^*(y)$ are asymptotic inverses of each other,

- the function δ is defined for all x sufficiently large by $\delta(x) = x/a(x)$ where $a(x)$ is the unique element element of the sequence (A_{k_n}) so that

$$A_{k_n} \leq x < A_{k_{n+1}},$$

- the *error functions* $h_K : (0, \infty) \to (0, \infty)$ with $K \in \{R, L\}$ are any functions for which

$$\lim_{n\to\infty} h_K(A_{k_n}x_0) = 0,$$

  for every continuity point x_0 of M_K.

Remark 1.1.10. We have given above a description of $\mathbb{D}_{gp}(\mu)$ for μ semi-stable for a fixed choice of k_n and A_{k_n}. If one wishes to fix only the sequence k_n then one can take a free choice of the slowly varying function ℓ in the definition of A_{k_n}. On the other hand, if one wishes to fixes only the normalising sequence A_n with the property that $\lim_{n\to\infty} A_{n+1}/A_n = c^{1/\alpha} > 1$ then the convergence in (1.3) may occur along any sequence (k_n) with $\lim_{n\to\infty} k_{n+1}/k_n = c$.

We now state a theorem due to Csörgö and Megyesi which establishes that whenever (1.3) holds along a sequence $(k_n)_{n\geq 1}$ satisfying (1.6) then we also have convergence of $(Z_n)_{n\geq 1}$ to related semi-stable distributions along additional sub-sequences. Let us denote by F_n the distribution function of the scaled and centred sum Z_n:

$$F_n(x) \coloneqq \mathbb{P}\left(Z_n = \frac{\sum_{j=1}^n X_j - B_n}{A_n} \leq x \right), \tag{1.12}$$

and let us suppose that the common distribution μ of the X_n is in $\mathbb{D}_{gp}(\nu)$ for some semi-stable ν. Letting G be distribution function of μ we denote by G_λ for $\lambda > 0$ the distribution function of the semi-stable distribution Lévy functions

$$L_\lambda(x) \coloneqq \frac{M_L(\lambda^{1/\alpha}x)}{|x|^\alpha}, \quad R_\lambda(x) \coloneqq -\frac{M_R(\lambda^{1/\alpha}x)}{x^\alpha}. \tag{1.13}$$

In an analogous way to the definition of the function δ we define γ for all $s > 0$ small enough by putting

$$\gamma(s) = sk_{n(s)} \tag{1.14}$$

where $k_{n(s)}$ is the unique element of $(k_n)_{n\geq 1}$ such that $\frac{1}{k_n} \leq s < \frac{1}{k_{n-1}}$. We note that for any $\varepsilon > 0$ we have that $\gamma(s) \in [1, c+\varepsilon]$ for all $s > 0$ small enough. Given a sequence $(s_n)_{n\geq 1}$ we say that $(\gamma(s_n))_{n\geq 1}$ *converges circularly* to $\lambda \in [1, c)$ and write $\gamma(s_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$ if $\lim_{n\to\infty} \gamma(s_n) = \lambda$ or if $(\gamma(s_n))_{n\geq 1}$ has exactly two limit points 1 and c.

From [CM02, Theorem 1] we know that the sequence $(Z_n)_{n\geq 1}$ is stochastically compact. Moreover $(Z_n)_{n\geq 1}$ is convergent in distribution to a non-degenerate distribution Y' along a sub-sequence $(n_r)_{r\geq 1}$ if and only if $\gamma(1/n_r) \xrightarrow[r\to\infty]{\text{cir}} \lambda$, and if this is the case Y' necessarily has distribution function G_λ. The following result further strengthens the mode of this convergence.

Theorem 1.1.11 (merging of semi-stable distributions [CM02, Theorem 2])**.** *Let G be a semi-stable distribution of index $\alpha \in (0,2)$ and period $c > 1$. Suppose that $X_1, X_2, \ldots$ are i.i.d. with common distribution function $F \in \mathbb{D}_{gp}(G, k_n, A_{k_n})$, where $A_{k_n} := (k_n)^{1/\alpha}\ell(k_n)$ for some slowly varying function ℓ, and where (k_n) is a sequence satisfying (1.6). Then,*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - G_{\gamma(1/n)}(x)| = 0,$$

where F_n is defined in (1.12), and where the family $(G_\lambda)_{\lambda\in[1,c)}$ is defined in terms of G as in (1.13).

Example 1.1.12 (St. Petersburg Paradox)**.** Let us now give a classical example of where semi-stable distributions appear. Consider the following game. A fair coin is tossed until it shows heads, if the coin shows heads on the n^{th} trial the player is rewarded with 2^n units of money. If X is the gain after a single trial of this game we have that $\mathbb{P}(X = 2^n) = 2^{-n}$, and the distribution function of X is given by

$$\mathbb{P}(X \leq x) = 1 - 2^{-\lfloor \log_2 x \rfloor}.$$

Let us note that the expectation of X is infinite. There have been various investigations into the statistical properties of scaled and centred sums of $(X_n)_{n\geq 1}$ where $(X_n)_{n\geq 1}$ is an infinite sequence of independent trials of the game. In 1945 W. Feller [Fel45] proved that

$$\frac{\sum_{j=1}^{n} X_j}{\log_2 n}$$

converges to 1 in probability. In 1985 Martin-Löf [ML85] showed that $Z_n = \frac{\sum_{j=1}^n X_j}{n} - \log_2 n$ has non-degenerate limit points, in particular he showed that Z_{2^n} is convergent in distribution to a non-degenerate random variable. Using the results of Csörgö and Megyesi mentioned above we may classify all the possible weak limit points of the sequence $(Z_n)_{n\geq 1}$.

First let us rewrite the tail distribution of X:

$$\overline{F}(x) = \mathbb{P}(X > x) = 2^{-\lfloor \log_2 x \rfloor} = x^{-1} 2^{-\{\log_2 x\}},$$

where we have denoted by $\{\cdot\}$ the fractional part of a number $\{x\} \coloneqq x - \lfloor x \rfloor$. Setting $c = 2$, $\alpha = 1$ and $M(x) = 2^{-\{\log_2 x\}}$ we see that M satisfies condition A1 and moreover the distribution function F satisfies A2. Thus employing [Meg00, Corollary 3] we retrieve the result of Martin-Lof. Moreover we see from [CM02, Theorem 1] that the set of weak limit points of $(Z_n)_{n\geq 1}$ is the set $\{V_\lambda : \lambda \in (1/2, 1]\}$ where each V_λ has distribution function G_λ as described above.

An observation about mixtures

Consider the following lemma (the proof of which is given in Section 4.6 of the appendix).

Lemma 1.1.13. *Suppose that $M_1, M_2 : \mathbb{R}_{>0} \to \mathbb{R}$ are two right-continuous log-periodic functions with period a and b respectively. Then the function*

$$M = M_1 + M_2$$

is log periodic with period of some period c if and only if $a = b^{p/q}$ for some rational p/q in which case we can take $c = a^q = b^p$.

A *mixture* is a (finite) convex combination of distribution functions:

$$F(x) = \sum_{j=1}^{n} \omega_j F_j(\omega).$$

A natural question to ask is whether mixtures of finite collections $\{F_j\} \subset \bigcup_{G\in\mathscr{G}} \mathbb{D}_{gp}(G)$ remain in $\bigcup_{G\in\mathscr{G}} \mathbb{D}_{gp}(G)$, where here $\mathscr{G}$ denotes the set of all non-stable semi-stable distributions. In fact Lemma 1.1.13 shows that $\bigcup_{G\in\mathscr{G}} \mathbb{D}_{gp}(G)$ is not closed under finite convex combinations and moreover this lemma provides necessary and sufficient conditions on a when a mixture of $\{F_i\} \subset \bigcup_{G\in\mathscr{G}} \mathbb{D}_{gp}(G)$ is an element of $\bigcup_{G\in\mathscr{G}} \mathbb{D}_{gp}(G)$. For example consider

$$F(x) = \frac{1}{2}(F_1(x) + F_2(x)),$$

where $F_1 \in \mathbb{D}_{gp}(G_1)$ and $F_2 \in \mathbb{D}_{gp}(G_2)$ for two distribution functions $G_1, G_2 \in \mathscr{G}$ of period and index c_1, α_1 and c_2, α_2 respectively. If $\alpha_1 \neq \alpha_2$, then we can assume without loss of generality that $\alpha_1 > \alpha_2$. In this case it is clear (from (1.9) and (1.10)) that F will be in $\mathbb{D}_{gp}(G)$ for some G of index α_2 and period c_2. On the other hand, if $\alpha_1 = \alpha_2 = \alpha$ then Lemma 1.1.13 implies that there exists a $G \in \mathscr{G}$ such that $F \in \mathbb{D}_{gp}(G)$ if and only if there exists $p/q \in \mathbb{Q}$ such that

$$c_1^{1/\alpha} = \left(c_2^{1/\alpha}\right)^{p/q},$$

in which case G will have index α and period c_1.

1.2 Limit theorems for intermittent dynamical systems

One of our primary aims is to establish versions of Theorem 1.1.11 where the i.i.d. sequence $(\Omega_n)_{n\geq 1}$ is replaced by the deterministic sequence $(u \circ T^n)_{n\geq 0}$ for certain observables $u : \Omega \to \mathbb{R}$ and certain maps $T : \Omega \to \Omega$. Before stating our main results in this direction we recall some necessary background as well as some existing results in dynamics on which we shall build.

1.2.1 Preliminaries on Gibbs-Markov Maps

Let $T : \Omega \to \Omega$ be a non-singular transformation of a *standard probability space* $(\Omega, \mathscr{B}, m)$, that is we assume that (Ω, d) is a compact Polish metric space, $\mathscr{B}$ is the Borel σ-algebra on Ω, and m is the Lebesgue measure on $(\Omega, \mathscr{B})$. In this context T being non-singular means that $m(T^{-1}(E)) = 0$ if and only if $m(E) = 0$.

We say that an (at most) countable partition $\mathcal{P}$ of Ω is *Markov* for T if

1. $T|_q : q \to Tq$ is a bijection for each partition element $q \in \mathcal{P}$,

2. the σ-algebra generated by the preimages of the partition elements $\mathscr{P} := \sigma(\{T^{-n}q : q \in \mathcal{P}, n \geq 0\})$ coincides with the Borel σ-algebra $\mathscr{B}$ up to a sets of measure zero[2],

3. for each $q \in \mathcal{P}$ we have $Tq \in \sigma(\mathcal{P})$.

A map with Markov partition is called a *Markov map*. Given a Markov map T with partition $\mathcal{P}$ there is natural measure of distance on the space Ω which comes from the notion of the *separation time* $s(x, y)$ of time two points $x, y \in \Omega$ which is the smallest amount of time for two distinct points to lie in different elements of $\mathcal{P}$

$$s(x, y) := \min\{n \geq 0 : T^n x, T^n y \text{ lie in different elements of } \mathcal{P}\}. \tag{1.15}$$

Then for $\theta \in (0, 1)$ we may define the distance d_θ by putting

$$d_\theta(x, y) := \theta^{s(x,y)}. \tag{1.16}$$

We note that the space (Ω, d_θ) is Polish and T is Lipschitz with respect to d_θ[3].

---

[2]This means that for each $A \in \mathscr{B}$ there exists $B \in \mathscr{P}$ such that $m(A \triangle B) = 0$ and vice-versa.

[3]To see that (Ω, d_θ) is Polish one quickly verifies that if $x_n \to x$ in (Ω, d_θ) then $x_n \to x$ in (Ω, d) and

For a function $v : \Omega \to \mathbb{R}$ and a partition element $q \in \mathcal{P}$ we denote by $D_\theta(v)(q)$ the least Lipschitz constant of $v|_q$ with respect to the distance d_θ:

$$D_\theta(v)(q) \coloneqq \sup_{x,y\in q} \frac{|v(x) - v(y)|}{d_\theta(x, y)}.$$

We define the semi-norm

$$|v|_\theta \coloneqq \sup_{q\in\mathcal{P}} D_\theta(v)(q). \tag{1.17}$$

If $|v|_\theta < \infty$ we will say that v is *locally θ-Hölder*. We note that locally θ-Hölder functions may be unbounded. Then, following Section 1 of [AD01] we denote by L_θ the space of bounded locally θ-Hölder functions

$$L_\theta \coloneqq \{v : \Omega \to \mathbb{R} : \|v\|_\theta \coloneqq \|v\|_{L^\infty(m)} + |v|_\theta < \infty\}, \tag{1.18}$$

and remark that $(L_\theta, \|\cdot\|_\theta)$ forms a Banach space. By definition a Markov map is invertible on each partition element. Denoting by $v_q : T^n q \to q$ the inverse of T^n on $q \in \mathcal{P}_n \coloneqq \bigvee_{j=0}^{n-1} T^{-j}\mathcal{P}$ we let v'_q be the Radon-Nikodym derivatives

$$v'_q \coloneqq \frac{dm \circ v_q}{dm}.$$

If T is Markov with partition $\mathcal{P}$ then we say that the tuple $(\Omega, \mathscr{B}, m, T, \mathcal{P})$ is *Gibbs-Markov* if two additional properties are satisfied

1. **big images:**

$$\inf_{q\in\mathcal{P}} m(Tq) > 0,$$

2. **θ-distortion:** there exists a $\theta \in (0, 1)$ and there exists a $C > 0$ so that for all

moreover one can easily check that sequences which are Cauchy in (Ω, d_θ) are also Cauchy in (Ω, d). To see that T is Lipschitz one simple observes that $d_\theta(Tx, Ty) = \theta^{s(Tx,Ty)} \leq \theta^{s(x,y)-1}$.

$n \geq 0$, all $q \in \mathcal{P}_n$ and almost every $x, y \in q$ we have that

$$\left| \frac{v_q'(x)}{v_q'(y)} - 1 \right| \leq C d_\theta(x, y).$$

Uniformly expanding C^2 interval maps

Example 1.2.1. A important example of Gibbs-Markov maps are *uniformly expanding C^2 Markov interval maps.* A non-singular map $T : \Omega \to \Omega$ of a compact interval $\Omega \subset \mathbb{R}$ is a C^2 *Markov interval map* if there is a Markov partition $\mathcal{P}$ of Ω into sub-intervals for which $T|_q$ is strictly monotone and admits a C^2 extension on a neighbourhood of the closure $\overline{q}$ of each $q \in \mathcal{P}$, and *Adler's condition* is satisfied:

$$\sup_{x \in \Omega} \frac{|T''(x)|}{T'(x)^2} < \infty. \tag{1.19}$$

If T is a C^2 Markov interval map that is *uniformly expanding*:

$$\inf_{x \in \Omega} |T'(x)| = \lambda > 1$$

then it is shown in [Aar97, Proposition 4.3.3] that there exists some $\theta \in (0, 1)$ so that T is Gibbs-Markov with θ-distortion.

1.2.2 Existence of the a.c.i.p. and properties of the transfer operator

Throughout this section we let $(\Omega, \mathscr{B}, m, T, \mathcal{I})$ be a topologically mixing Gibbs-Markov map with θ-distortion. We consider the *Frobenius-Perron-Ruelle* transfer operator $\mathcal{L} : L^1(m) \to L^1(m)$ defined by the relation

$$\int \mathcal{L}(f) \cdot g \, \mathrm{d}m = \int f \cdot g \circ T \, \mathrm{d}m, \quad \forall f \in L^1(m), g \in L^\infty(m), \tag{1.20}$$

one checks that $\mathcal{L}(f)$ is given by the equation

$$\mathcal{L}(f) = \sum_{q\in\mathcal{P}}^{\infty} 1_q v_q' f \circ v_q.$$

It is straightforward to check from the definition that the operator $\mathcal{L}$ is bounded positive linear operator $L^1 \to L^1$ with $\|\mathcal{L}\|_{L^1} = 1$. From [AD01] we know by the Corollary to Renyi's property that T preserves an exact probability measure $\mu = \rho\, dm$ with $h \in L^\infty$ which is bounded from below away from 0. Moreover, from [AD01, Corollary 1.5] and [AD01, Theorem 1.6] we have that $h \in L_\theta$, $\mathcal{L} \in \mathrm{Hom}(L_\theta, L_\theta)$ and that $\mathcal{L}|_{L_\theta}$ has a simple isolated eigenvalue at 1 and *a spectral gap*.

Definition 1.2.2 (spectral gap)**.** We say that a bounded linear operator $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ acting on a Banach space $(\mathcal{B}, \|\cdot\|)$ has a *spectral gap* if

$$\mathcal{T} = \lambda P + N, \tag{1.21}$$

where P is a projection onto a 1-dimensional subspace of $\mathcal{B}$, N is a bounded linear operator with spectral radius $\rho(N) < |\lambda|$ and $NP = PN = 0$.

Remark 1.2.3. Let us briefly remark on some consequences of an operator $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ having a spectral gap. Writing $\mathcal{T}$ as in equation (1.21) above we note that the fact that $\rho(N) < |\lambda|$ and the fact that $NP = PN = 0$ imply that $\lambda^{-n}\mathcal{T}^n$ converges to the projection P exponentially fast. Indeed, we have that

$$\mathcal{T}^n = \lambda^n P + N^n, \tag{1.22}$$

and moreover, employing the spectral radius formula, we have for $\varepsilon > 0$ small enough and n large enough that $\|N^n\| = (\rho(N) + \varepsilon)^n < \lambda^n$.

We also note that as the name suggests $\mathcal{T}$ having a spectral gap implies that the spectrum $\sigma(\mathcal{T})$ of $\mathcal{T}$ consists of simple isolated eigenvalue at λ with remaining eigenvalues lying within a disc of radius strictly smaller that $|\lambda|$:

$$\sigma(\mathcal{T}) = \{\lambda\} \cup A, \text{ where } \exists\gamma > 0 \text{ such that } A \subset \{z \in \mathbb{C} : |z| \leq e^{-\gamma}|\lambda|\}.$$

Moreover, one can check that the projection P is the projection onto the eigenspace associated with λ, in particular $\mathcal{T}v = \lambda v$ if and only if $v \in \operatorname{Im} P$.

1.2.3 Inducing and invariant measures

Later we shall exploit the properties of Gibbs-Markov maps described above in order to establish statistical limit theorems. However, the systems for which we wish to establish these limit theorems are intermittent interval maps that are not Gibbs-Markov. A common approach to overcome this obstacle is to *induce*. The main idea is to choose a reference set, say Y, and for points $x \in Y$ define a new system $T_Y : Y \to Y$, by letting $T_Y(x) = x'$ where x' is the first point in the orbit of x under our original system that lies in Y. By inducing one hopes that the new system is easier to study than the original and that we may gain information about the original system by examining the induced one. In this section we will introduce this notion formally and briefly discuss some of the consequences when the induced system preserves an ergodic absolutely continuous probability measure.

Let $T : \Omega \to \Omega$ be a non-singular transformation of a standard probability space $(\Omega, \mathscr{B}, m)$ and let $Y \subset \Omega$ be of positive measure. Let us suppose that the orbit under T of almost every $x \in \Omega$ hits the set Y in the sense that

$$\Omega = \bigcup_{n=1}^{\infty} T^{-n}Y \mod m.$$

In this case we may define the *first return time* $\tau : Y \to \mathbb{N}$ to Y by setting

$$\tau(x) := \min\{n \geq 1 : T^n(x) \in Y\}.$$

We may then define the *induced map* $T_Y : Y \to Y$ on Y by setting $T_Y(x) = T^{\tau(x)}(x)$. We denote by $m_Y := \frac{m|_Y}{m(Y)}$ the normalised Lebesgue measure restricted to Y and we assume that T_Y preserves an ergodic absolutely continuous probability measure which we denote by μ_Y. Let us now define a new measure μ on Ω by setting

$$\mu(E) := \sum_{n=0}^{\infty} \mu_Y(T^{-n}(E) \cap \{\tau > n\}). \tag{1.23}$$

As T is non-singular by assumption it is clear that μ is absolutely continuous with respect to m. Since $\mu(\Omega) = \sum_{n\geq 0} \mu_Y(\tau > n) = \int_Y \tau d\mu_Y$ that the measure μ is finite if and only if τ is integrable with respect to μ_Y. Regardless of whether or not μ is finite we see that μ is an invariant measure for T. To see this we calculate that

$$\begin{aligned}
\mu(T^{-1}(E)) &= \sum_{n\geq 0} \mu_Y(T^{-(n+1)}E \cap \{\tau > n\}) \\
&= \sum_{n\geq 1} \mu_Y(T^{-n}E \cap \{\tau = n\}) + \sum_{n\geq 1} \mu_Y(T^{-n}E \cap \{\tau > n)).
\end{aligned}$$

Examining the first series on the right hand side above and using the fact that $(\{\tau = n\})_{n\geq 1}$ forms a disjoint partition of Y we see that

$$\begin{aligned}
\sum_{n\geq 1} \mu_Y(T^{-n}(E) \cap \{\tau = n\}) &= \sum_{n\geq 1} \mu_Y(T_Y^{-1}(E \cap Y) \cap \{\tau = n\}) \\
&= \mu_Y\left(\bigcup_{n\geq 1} T_Y^{-1}(E \cap Y) \cap \{\tau = n\}\right) \\
&= \mu_Y(T_Y^{-1}(E \cap Y)) \\
&= \mu_Y(E \cap Y) = \mu_Y(T^{-0}(E) \cap \{\tau > 0\}),
\end{aligned}$$

and so we can conclude that

$$\mu(T^{-1}(E)) = \mu(E),$$

as required. Proceeding in a similar way to the above one can also show μ is ergodic[4].

In summary, we have seen that if the induced systems posses an ergodic absolutely continuous invariant probability measure and the return time is integrable with respect to this measure then the measure defined in (1.23) forms an ergodic absolutely continuous probability measure for the original system.

1.2.4 Intermittent interval maps

In this thesis we will study semi-stable laws for certain intermittent interval maps. The intermittent maps we consider are all derived from the *Pomeau-Manneville* (*P.M.*) family of maps. The P.M. maps are a one-parameter family of piecewise expanding maps of the unit interval of the form $T_{PM}(x) \coloneqq x + x^{1+\beta} \mod 1$, where $\beta > 0$ is a positive parameter. These maps are named after Pomeau and Manneville who in the late 1970s first studied numerical approximations of these maps to investigate phenomena of intermittency in certain physical systems, namely the intermittent transition to turbulence in convective fluids [PM80]. Such maps have also seen applications to modelling various intermittent phenomena outside of physics, for example one can see [BHK07, BH07] where P.M. maps are used in the statistical analysis of long memory processes in financial markets.

We will focus on the *Liverani-Saussol-Vaienti* (*L.S.V.*) family of maps and modifications thereof. The L.S.V. map, first introduced in [LSV99], with parameter $\beta > 0$ is the map

[4] In the sense that invariant sets have zero measure or their complement have zero measure, in this way ergodicity makes sense regardless of whether the measure is finite or not.

$T_{LSV} : [0, 1] \to [0, 1]$ given by

$$T_{LSV}(x) := \begin{cases} x(1 + 2^\beta x^\beta) & \text{for } x \in [0, 1/2) \\ 2x - 1 & \text{for } x \in [1/2, 1]. \end{cases} \tag{1.24}$$

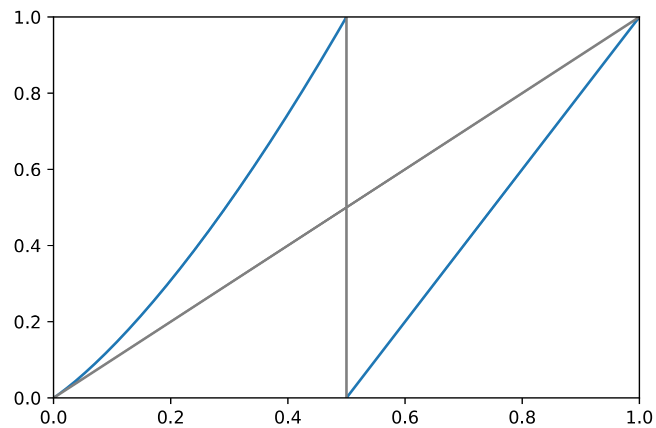

Figure 1.1: The L.S.V. map with parameter $\beta = \frac{2}{3}$.

We note that the L.S.V. maps are a simplification of the P.M. maps where the second non-linear branch of T_{PM} is replaced by the the linear branch that appears in (1.24). This map has been a large subject of interest in dynamical systems in the last two decades. The map T_{LSV} provides one the simplest examples of a non-uniformly hyperbolic dynamical system: the uniform hyperbolicity is violated at a single neutral fixed point $T'_{LSV}(0) = 1$, and away from this fixed point the map is uniformly expanding. A significant challenge in studying limit theorems for this map lies precisely in the fact that it is not uniformly expanding.

The L.S.V. map preserves an absolutely continuous invariant probability measure when $\beta \in (0, 1)$, and when $\beta > 1$ there still exists an absolutely continuous invariant measure, though in this case the measure is infinite. Both the finite and the infinite measure cases have been of interest in recent years. In the infinite measure case one can see for example [TZ06, MT12]. In the finite measure case several developments have been made in the study of the statistical properties of T_{LSV} (see for example [LSV99, You99, Sar02, Gou04b, BT16] and the references therein), here we highlight some recent results that are of particular interest to us as they relate to the long-term distributional behaviour of scaled and centred Birkoff sums of regular observables.

In [You99], amongst many other things, estimates on the decay of correlations are used to give a central limit theorem for Hölder observables u under the dynamics of the L.S.V. map in the case that $\beta < 1/2$, as in the case that $\beta < 1/2$ the correlations are summable. In [Gou04a] Gouëzel gave a complete picture of the limiting behaviour appropriately scaled and centred Birkoff sums $Z_n := \frac{1}{A_n}(\sum_{k=1}^{n-1} u \circ T^j - B_n)$ for Hölder observables u. If $\beta = 1/2$ then a central limit theorem still holds, but with a non-standard normalisation $(A_n = \sqrt{n\log(n)})$. If $\beta \in (1/2, 1)$ Gouëzel showed that Z_n will converge[5] to a Gaussian random variable if $u(0) = 0$, and will converge to a stable random variable if $u(0) \neq 0$. This result, and the techniques used to establish it, form the principal foundation for the work presented in this thesis; in the following sub-section we will discuss result in further detail. Stronger results in the same direction are established in [DM09, MZ15] where a *weak invariance principle* is given. Roughly speaking the weak invariance principle ensures that whenever Z_n converges to a Gaussian random variable the process $\tilde{Z}_n(t) := Z_{\lfloor nt \rfloor}$ will converge weakly to a Brownian motion, and whenever Z_n converges to a stable random variable the process $\tilde{Z}_n(t)$ will converge weakly to a stable Lévy process. More recently in [CDKM20] in the situations where $\tilde{Z}_n(t)$ converges weakly to a Brownian motion an *almost sure invariance principle* has been established: $\tilde{Z}_n = W_n + r_n$ almost surely where W_n is a Brownian motion, and r_n is an error or rate of convergence (this rate is further quantified in [CDKM20]).

Stable laws for the L.S.V. map

In this section we examine in further detail some of the results in [Gou04a] on stable laws for the L.S.V. map. As we shall see later, the L.S.V. map preserves a measure μ

[5] Under some mild additional assumptions on the Hölder exponent of u

that is absolutely continuous with respect to Lebesgue. If $\beta \in (0,1)$ then μ is finite, and if $\beta > 1$ then μ is only σ-finite with $\mu([0,1]) = +\infty$. In [Gou04a, Theorem 1.3] Gouëzel establishes the following result:

Let $\beta \in (1/2, 1)$ and let T be the corresponding L.S.V. map. Suppose that $u : [0,1] \to \mathbb{R}$ is Hölder with $\int udm = 0$. Then we have two cases

- If $u(0) \neq 0$, then we have convergence to a stable law

  $$\frac{\sum_{j=0}^{n-1} u \circ T^j}{n^\beta} \xrightarrow[n\to\infty]{d} V_{1/\beta},$$

  where $V_{1/\beta}$ is a stable random variable of index $1/\beta$.

- If $u(0) = 0$, and the Hölder exponent ν of u such that $\nu > \beta - 1/2$ then we have a central limit theorem

  $$\frac{\sum_{j=0}^{n-1} u \circ T^j}{\sqrt{n}} \xrightarrow[n\to\infty]{d} V \sim \mathcal{N}(0, \sigma^2),$$

  for some $\sigma^2 \geq 0$.

To understand the dichotomy present in the above result let us describe very briefly some aspects of its proof.

As the map T is not uniformly expanding it is somewhat difficult to study the statistical properties of T directly. Inducing on the set $Y := [1/2, 1]$ one can check that the resulting map T_Y is a uniformly expanding C^2 Markov interval map with respect to the partition $(J_n := \{\tau = n\})$ and thus (see Example 1.2.1) is Gibbs-Markov.

One can calculate the tail distribution of the return time using the following procedure. Consider the sequence $x_0 = 1$, and $x_{n+1} = T^{-1}(x_n) \cap [0, 1/2]$. If we then set $I_0 := [1/2, 1]$, $I_n := [x_{n+1}, x_n)$ then T maps I_{n+1} bijectively onto I_n for each $n \geq 1$ and $T(I_1) = Y \setminus \{1\}$ and we note that the map T is Markov with respect to the partition $(I_n)_{n\geq 0}$. Letting

$J_1 \coloneqq \left[\frac{1}{2}x_1 + \frac{1}{2}, \frac{1}{2}x_0 + \frac{1}{2}\right]$ and $J_n \coloneqq \left[\frac{1}{2}x_n + \frac{1}{2}, \frac{1}{2}x_{n-1} + \frac{1}{2}\right)$ we see that $T(J_n) = I_{n-1}$, and that $\{\tau = n\} = J_n$. We may then find the tail distribution of the return time in terms of the sequence x_n by calculating

$$m_Y(\tau > n) = \sum_{k=n+1} m_Y(J_k) = x_n.$$

The asymptotics of the sequence x_n may be estimated by using the definition of the map

$$\begin{aligned} \frac{1}{x_n^\beta} &= \frac{1}{x_{n+1}^\beta}\left(1 + 2^\beta x_{n+1}^\beta\right)^{-\beta} \\ &= \frac{1}{x_{n+1}^\beta}\left(1 - \beta 2^\beta x_{n+1}^\beta + O(x_{n+1}^{2\beta})\right) \\ &= \frac{1}{x_{n+1}^\beta} - \beta 2^\beta + o(1), \end{aligned}$$

Summing we then find that

$$\frac{1}{x_n^\beta} = 1 + \beta 2^\beta n + o(n),$$

and so

$$\implies x_n = \frac{1}{2}(\beta n)^{-\frac{1}{\beta}}(1 + o(1)).$$

In particular one finds that x_n is regularly varying with index $\alpha \coloneqq \frac{1}{\beta}$. By considering higher order terms in the expansion above one finds that

$$x_n = \frac{1}{2}\alpha^\alpha n^{-\alpha}(1 + O((\log n)/n), \tag{1.25}$$

see for example [Hol05], or [Ter16] for even higher order terms. As the induced map is Gibbs-Markov we know that the density h lies in L_θ, and so

$$\mu_Y(\tau > n) = \int_{\frac{1}{2}}^{\frac{1}{2}x_n + \frac{1}{2}} h dm_Y = h\left(\frac{1}{2}\right) x_n(1 + o(1)). \tag{1.26}$$

From (1.26) and our comments in Section 1.2.3 that the measure μ defined by (1.23) gives and invariant measure for T. Moreover, we note that as τ is only integrable with respect to μ_Y if $\alpha > 1$ we see that μ is an a.c.i.p. for T if and only if $\beta \in (0, 1)$.

We can begin to see where the two cases in the Gouëzel's result above emerge when consider the tail distribution of the induced observable $u_Y := \sum_{j=0}^{\tau-1} u \circ T^k$ for a Hölder function $u : [0, 1] \to \mathbb{R}$. Gouëzel shows that if $u(0) = 0$ and the Hölder exponent ν of u is such that $\nu > \beta - 1/2$ then $u \in L^2(\mu_Y)$, which is why a central limit theorem appears in the second case in [Gou04a, Theorem 1.3] (see [Gou04a, Theorem 1.1] for why central limit theorems appear when the induced observable is in L^2 in more general settings). On the other hand if $u(0) \neq 0$ Gouëzel shows that

$$\mu_Y(u_Y \geq x) = C\mu_Y(\tau \geq x)(1 + o(1)). \tag{1.27}$$

Note that the sequence $(u_Y \circ T^n)_{n\geq 1}$ is identically distributed but not independent. However, if we imagined that the sequence $(u_Y \circ T_Y^n)_{n\geq 0}$ was i.i.d. we would know from the previous sections that we may obtain a stable law as from (1.25) and (1.27) we know that the tail distribution of u_Y is regularly varying with index $-\alpha$.

In what follows we are interested in systems similar to the L.S.V. map where this regular variation is not present in the tail distribution of the observable. We will either work with a different intermittent map whose return time no longer has a regularly varying tail, or we will consider observables for the L.S.V. that do not inherit the distribution of the return time. In either scenario we will see that the induced observable will have tail distribution of the form $f(x)M(x)$ where f is regularly varying and M is oscillatory.

Let us summarise briefly the key steps involved in obtaining a stable law for the L.S.V. maps, one can find a detailed survey of the approach outlined below in [Gou15].

1. Induce on the set Y and show that the induced map is Gibbs-Markov, and whence preserves an a.c.i.p. μ_Y and the transfer operator of the induced map will have

good functional analytic properties on a Banach space L_θ

2. For the observable which we want to prove the limit theorem for we study the behaviour of the tail distribution of induced observable, in particular we show that tail distribution is in the domain of attraction of a stable law.

3. We then obtain a limit theorem for the induced system using the *spectral method* (also referred to as the Nagaev-Guivarc'h, or Aaronson-Denker method) see [AD01] or the aforementioned review [Gou15].

4. Pull back the limit theorem using techniques due to Melbourne and Török [MT04] (see also [Gou08] for a simple application of this method)

In what follows we will adapt this general regime in order to prove the limit theorems which we will now present.

Chapter 2

Results

We will now give precise statements of our main results. Throughout this chapter we let $p : \mathbb{R} \to \mathbb{R}$ be a non-constant periodic Lipschitz function with period $\frac{1}{\alpha} \log c$ where $\alpha \in (1, 2)$ and $c > 1$, and we assume further that the second derivative of p is bounded. We let $a > 0$ be constant and let $\varepsilon > 0$ be a small parameter. We define the function $M : (0, \infty) \to (0, \infty)$ by setting

$$M(x) := a(1 + \varepsilon p(\log(x))), \tag{2.1}$$

and note that M is log-periodic with period $c^{1/\alpha}$. We let $\varepsilon > 0$ be small enough so that M is bounded away from 0. As p is Lipschitz we know that p and p' are bounded, thus, decreasing ε if required, we know that the function $x \mapsto M(x)/x^\alpha$ is strictly decreasing. In particular we note that the function M satisfies A1 . For example we could take

$$M(x) = a \left[1 + \varepsilon \sin \left(\frac{2\pi}{\alpha} \frac{\log x}{\log c} \right) \right]. \tag{2.2}$$

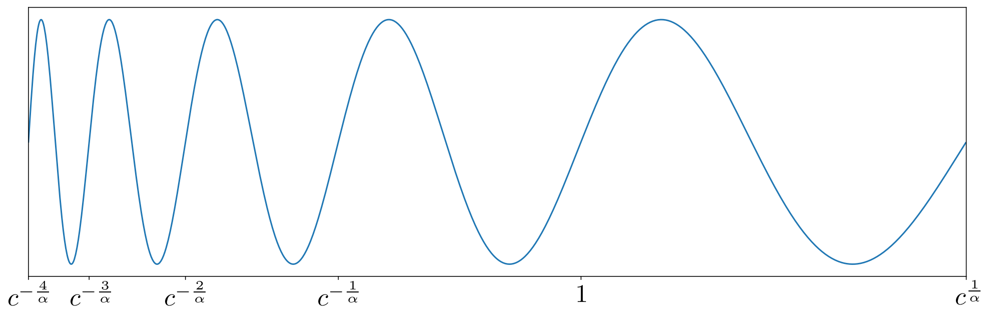

Figure 2.1: Plot of M as given in (2.2) with $c = 2$ and $\alpha = 3/2$.

2.1 Semi-stable limit theorems for a non-i.i.d. Markov Chain

We now turn to our first main result: a semi-stable law for a non-i.i.d. Markov chain. The Markov chain that we study is formed by taking an intermittent interval interval map $T : [0,1] \to [0,1]$ together with a "generic" Hölder observable $u : [0,1] \to \mathbb{R}$ and then considering the sequence $(u \circ T^n)_{n\geq 0}$, here by "generic" we mean $u(0) \neq 0$. The map T is to be defined piecewise on intervals $(I_n)_{n\geq 0}$ forming a partition of $[0,1]$. In particular we will stipulate that $T = \sum_{n\geq 0} 1_{I_n} g_n$ where for $n \geq 1$, g_n maps I_n smoothly and bijectively onto I_{n-1} and T_0 maps I_0 smoothly and bijectively onto $[0,1]$.

We define the sequence $(x_n)_{n\geq 0}$ by setting

$$x_0 \coloneqq 1, \quad x_n \coloneqq n^{-\alpha} M(n) \text{ for } n \geq 1 \tag{2.3}$$

and choose the constant a in the definition of M so that $x_1 = \frac{1}{2}$, i.e. we set $a \coloneqq (2(1 + \varepsilon p(0)))^{-1}$. We then define the intervals $(I_n)_{n\geq 0}$ by setting

$$I_0 \coloneqq [1/2, 1], \quad I_n \coloneqq [x_{n+1}, x_n) \text{ for } n \geq 1,$$

and define

$$\Delta_n \coloneqq x_n - x_{n+1}, \quad \rho_n \coloneqq \frac{\Delta_{n-1}}{\Delta_n}.$$

We will take $\varepsilon > 0$ small enough in the definition of M so that Δ_n is strictly decreasing[1] and whence

$$\rho_n > 1. \tag{2.4}$$

We define the map $T_{exp} : [0,1] \to [0,1]$ by setting

$$T_{exp}(x) := \begin{cases} 0, & \text{if } x = 0 \\ g_n(x), & x \in [x_{n+1}, x_n] \;\; n \geq 2 \\ 2x - 1, & \text{if } x \in [1/2, 1], \end{cases} \tag{2.5}$$

where $g_n : I_n \to I_{n-1}$ is given by

$$g_n(x) := 1 + x_n + (1 - a_n)\rho_n(x - x_{n+1}) - \exp\left\{\log(1 - a_n\Delta_{n-1})\frac{x - x_{n+1}}{\Delta_n}\right\}, \tag{2.6}$$

and where $(a_n) \subset (0,1)$ is any strictly decreasing sequence converging to zero so that for each $n \in \mathbb{N}$ we have

$$a_n^2 < \left(1 - \frac{1}{\rho_n}\right)\Delta_{n-1}. \tag{2.7}$$

The map $T_{\exp}|_{[0,1/2)}$ is continuous and piecewise C^∞. One can readily verify by our choice of (a_n) that $T'(x) > 1$ for each $x \neq 0$ (see (4.1)). Moreover we have that T is differentiable from the right at 0 with $T'(0) = 1$. To see this, suppose that $y_n \to 0$ with $y_n > 0$ and let the sequence j_n be such that $y_n \in I_{j_n}$ for each n. We then obtain the bounds

$$1 = \frac{x_{j_n}}{x_{j_n}} \leq \frac{T_{\exp}(y_n) - T(0)}{y_n - 0} \leq \frac{x_{j_n - 1}}{x_{j_n + 1}} = \frac{x_{j_n - 1}}{x_{j_n}}\frac{x_{j_n}}{x_{j_n + 1}}.$$

As M is assumed to be continuous and as $n^{-\alpha}M(n)$ is strictly decreasing we can use Proposition 4.4.1 in the Appendix in order to conclude that the product on the right of the above converges to 1.

[1] One can check this is possible by taking the derivative of the map $x \mapsto x^{-\alpha}aM(x) - (x-1)^{-\alpha}aM(x-1)$ and checking the derivative is negative for $\varepsilon > 0$ sufficiently small.

We later show that $T_{\exp}$ has an a.c.i.p. which here we denote by μ. We fix a Hölder continuous observable with $u(0) \neq 0$ and set

$$F_n(x) \coloneqq \mu\left(\frac{\sum_{j=0}^{n-1} u \circ T_{\exp}^j - \int u d\mu}{n^{1/\alpha}} \leq x\right). \tag{2.8}$$

We let G be the distribution function of a semi-stable random variable whose left and right Lévy functions, L and R are given by

$$L(x) \equiv 0, \quad R(x) = -\frac{M(x)}{x^\alpha},$$

in the case that $u(0) > 0$, and

$$L(x) = \frac{M(x)}{|x|^\alpha}, \quad R(x) \equiv 0,$$

if $u(0) < 0$. We let $k_n = \lfloor c^n \rfloor$, $A_n = n^{1/\alpha}$ where c is as in the definition of the function M. For $\lambda \in [1, c)$ we define the distribution function G_λ in terms of G as in (1.13) and define the function γ in terms of $(k_n)_{n\geq 1}$ as in (1.14). The following theorem states that the observable u will satisfy a semi-stable law under the dynamics of T_{exp}, with $F_{k_n}(x) \to G(x)$, and that the distribution functions $(F_n)_{n\geq 1}$ will merge to the family $(G_\lambda)_{\lambda\in[1,\tilde{c})}$.

Theorem A. Let $\alpha \in (1, 2)$ and let $c > 1$. Let (a_n) be any strictly sequence in $(0, 1)$ that satisfies (2.7) and let $\varepsilon > 0$ in the definition of M be small enough so that (2.4) holds. Then, for any Hölder observable $u : [0, 1] \to \mathbb{R}$ with $u(0) \neq 0$ the distribution functions $(F_n)_{n\geq 1}$ given in (2.8) merge to the family $(G_\lambda)_{\lambda\in[1,c)}$ in the following sense:

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - G_{\gamma(1/n)}(x)| = 0. \tag{2.9}$$

In particular, whenever (k'_n) is a strictly increasing sequence of positive integers with

$\gamma(a_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda \in [1, c)$, we know

$$\frac{\sum_{j=0}^{k'_n-1} v \circ T_w^j - k'_n \int v d\mu}{A_{k'_n}} \xrightarrow[n\to\infty]{d} V_\lambda, \tag{2.10}$$

where V_λ is a semi-stable random variable with distribution function G_λ.

We will see when we turn to the proof of Theorem A that the return time τ has tail distribution

$$\mu_Y(\tau > n) = Cx_n(1 + o(1)),$$

indeed, this is precisely how the map is constructed. The fact that in this case we know the exact tail behaviour of the return time means that we are able to identify precisely the limiting distributions G_λ in Theorem A. As we explain at the beginning of Part II, knowing precisely the tail behaviour of the return will greatly simplify the proof of this theorem. In this situation we have mostly just to check that the scheme presented in [Gou15] and outlined in Section 1.2.4 may be applied in this setting. The main obstacle in applying the general scheme presented at the end of Section 1.2.4 is that the argument of Melbourne and Török for pulling-back the limit law from the induced setting (see [MT04] and [Gou08]) does not hold when there is only subsequential convergence in the induced system. In Section 3.1 we rectify this issue and present modification of pull-back method of Melbourne and Török which allows for subsequential limit theorems to be pulled back from the induced system. This somewhat artificial construction of an intermittent interval map provides our first example of semi-stable laws existing for a non-i.i.d. system and provides a relatively simple setting in which to check that the general method of establishing limit theorems outlined Section 1.2.4 can be indeed modified to prove semi-stable limit theorems in this context. Our following theorem however will be a somewhat more “natural” perturbation of the L.S.V. map where a semi-stable law holds and the proof of this fact is significantly more involved. Before presenting our next result let us briefly make some comments on why the map presented here is non-i.i.d..

If, in the place of the definition given in (2.6), we had defined $g_n : I_n \to I_{n-1}$ to be the affine map that takes I_n bijectively to I_{n-1} we have obtained the same limit theorem for the resulting map $T_{\exp}$. However, the Markov chain we would be studying would be asymptotically i.i.d.. Let us briefly explain why this is the case in order to demonstrate why the map we have described above leads to a non-i.i.d. Markov chain.

Let us assume that $T : [0,1] \to [0,1]$ is an interval map that maps each interval I_{n+1} bijectively and smoothly onto I_{n-1}. Then T is Markov with respect to I_n. Let us suppose for sake of simplicity that $T|_{I_0}(x) = 2x - 1$ as we have with $T_{\exp}$ above. Then, inducing on the interval $Y \coloneqq I_0$ we have that the induced map $T_Y \coloneqq T^\tau$, is Markov with respect to the partition formed by the intervals $J_n \coloneqq [\frac{1}{2}x_n + \frac{1}{2}], \frac{1}{2}x_{n-1} + \frac{1}{2})$, and consists of countably many full branches $T_Y(J_n) = [0,1]$.

If we suppose further that $T|_{I_n}$ is linear then so is $T_Y|_{J_n}$ and one readily verifies that T_Y preserves the Lebesgue measure m_Y. For $y_0, \ldots, y_{n-1} \in \mathbb{N}$ let us denote by $[y_0, \ldots, y_{n-1}]$ an n-cylinder so that $x \in [y_0, \ldots, y_{n-1}]$ if and only if $T_Y^k(x) \in J_{y_k}$ for each $k = 0, 1, \ldots, n-1$. We note that as T_Y is linear on each J_n the Lebesgue measure acts in the following way on cylinders

$$m_Y[y_0, \ldots, y_{n-1}] = m_Y\left(\bigcap_{j=0}^{n-1} T_Y^{-j} J_{y_j}\right) = \prod_{j=0}^{n-1} m_Y(J_{y_j}).$$

Now we can see that the sequence $(T_Y^j)_{j\geq 1}$ is asymptotically independent in the following sense. First let us take two cylinders $C_1 \coloneqq [y_0, y_2, \ldots, y_{j-1}], C_2 \coloneqq [z_0, z_1, \ldots, z_{k-1}] \subset Y$ of length $j \geq 1$ and $k \geq 1$ respectively. Now letting n be arbitrary we see that if m is large enough we have that

$$
\begin{aligned}
& m_Y(\{T_Y^n \in C_1\} \cap \{T_Y^{n+m} \in C_2\}) \\
&= m_Y\left(\left(\bigcup_{\xi_0,\dots,\xi_{n-1}} [\xi_0,\dots,\xi_{n-1},y_0,\dots,y_{j-1}]\right) \cap \left(\bigcup_{\xi_0,\dots,\xi_{n+m-1}} [z_0,\dots,z_{k-1}]\right)\right) \\
&= m_Y\left(\bigcup_{\xi_0,\dots,\xi_{n+m}} [\xi_0,\dots,\xi_{n-1},y_0,\dots,y_{j-1},\xi_{n+j+1},\dots,\xi_{n+m-1},z_0,\dots,z_{k-1}]\right) \\
&= m_Y(C_1)m_Y(C_2).
\end{aligned}
$$

The key property that allows the above to occur is that

$$m_Y(J_n \cap T_Y^{-1}J_k) = m_Y(J_n)m_Y(J_k),$$

for each n and k. If the branches of the induced map T_Y are non-linear then the above phenomenon does not usually occur. Imagine that for a given n and k the interval J_k is contained in the in the left half $[0,1]$ and suppose that the slope of $T_Y|_{J_n}$ is steeper on the left hand side of J_n than the right. Then

$$m_Y(J_n \cap T_Y^{-1}J_k) < m_Y(J_n)m_Y(J_k).$$

In general when $T_Y|_{J_n}$ is non-linear the Lebesgue measure is not invariant but, under sufficient distortion conditions (as discussed in Section 1.2.2), there is an invariant measure μ equivalent to Lebesgue and there is a *distortion constant* $C \geq 1$ such that $1/C \leq \frac{d\mu}{dm_Y} \leq C$, and $1/C \leq \frac{\mu(J_n)\mu(J_k)}{\mu(J_n \cap T_Y^{-1}(J_k))} \leq C$ (see [Aar97, 4.3.1]). The condition $\mu(J_n \cap T_Y^{-1}(J_k)) = \mu(J_n)\mu(J_k)$ is then only verified if $C = 1$. If the sequence $(T_Y^j)_{j\geq 1}$ is asymptotically i.i.d., then we do not need the machinery introduced in the following chapters in order to establish a semi-stable law. Though the asymptotically i.i.d. case is slightly more involved than the i.i.d. scenario it is possible to apply the results of Csörgö and Megyesi in [CM02] almost directly.

2.2 Semi-stable laws for a wobbly intermittent map

In this section we present our second main result which is joint work with M. Holland and D. Terhesiu (see [CHT19]). The map we consider in this section is alteration of the L.S.V. where the constant 2^β is replaced with the oscillatory function M.

Let $\alpha \in (1, 2)$ and define the map $T_w : [0,1] \to [0,1]$ by

$$T_w(x) \coloneqq \begin{cases} 0 & \text{for } x = 0 \\ x(1 + M(x)x^{1/\alpha}) & \text{for } x \in [0, 1/2), \\ 2x - 1 & \text{for } x \in [1/2, 1], \end{cases} \tag{2.11}$$

where M is the logarithmically periodic function with period $c^{1/\alpha}$ defined in (2.1) and the constant a appearing in (2.1) is chosen to be such that $\frac{1}{2}(1 + M(\frac{1}{2})\frac{1}{2}^{1/\alpha}) = 1$.

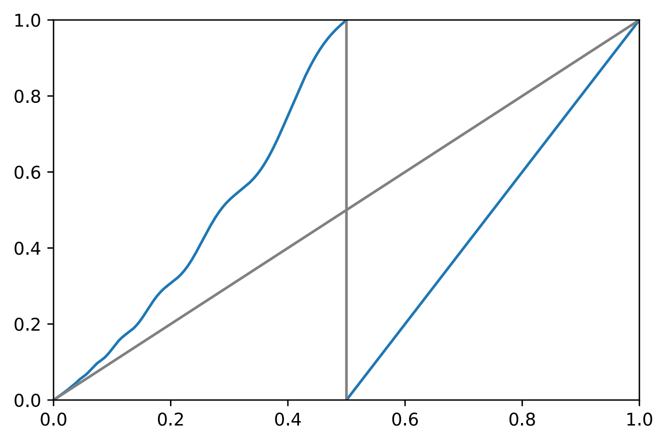

Figure 2.2: Plot of the map T_w as defined in (2.11) with M given by (2.2) and $\alpha = 3/2$, $c = 2$ and $\varepsilon = 1/10$.

Let us now fix $\tilde{c} \coloneqq c^{1/\alpha}$, $k_n = \tilde{c}^n$, $\ell(y) \coloneqq \frac{\tilde{c}^{y/\alpha}}{\lfloor \tilde{c}^{y/\alpha} \rfloor}$, $A_n = n^{1/\alpha}\ell(n)$.

We fix a Hölder observable $u : [0,1] \to \mathbb{R}$ with $u(0) \neq 0$ and define

$$F_n(x) \coloneqq \mu\left(\frac{\sum_{j=0}^{n-1} u \circ T_w^j - n\int u d\mu}{A_n} \leq x\right). \tag{2.12}$$

We then define function γ in terms of the sequence $(k_n)_{n\geq 1}$ as in (1.14). The following theorem then states that the observable u will satisfy a semi-stable law under the dynamics of T_w along the sequence (k_n) and that, as in Theorem 1.1.11, the distribution functions (F_n) will merge to a family of semi-stable distribution functions $(G_\lambda)_{\lambda\in[1,\tilde{c})}$.

Theorem B. Let $\alpha \in (1,2)$ and $c > 1$. Then for all $\varepsilon > 0$ sufficiently small there exists a semi-stable distribution G of index α and period $\tilde{c}$ so that for any Hölder observable $u : [0,1] \to \mathbb{R}$ with $u(0) \neq 0$ the distribution functions (F_n), defined in (2.12), merge to the family $(G_\lambda)_{\lambda\in[1,\tilde{c})}$, defined in terms of G as in (1.13), in the sense that:

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - G_{\gamma(1/n)}(x)| = 0. \tag{2.13}$$

In particular, whenever $(k'_n)_{n\geq 1}$ is a strictly increasing sequence of positive integers with $\gamma(k'_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$, we have

$$\frac{\sum_{j=0}^{k'_n-1} u \circ T_w^j - k'_n \int u d\mu}{A_{k'_n}} \xrightarrow[n\to\infty]{d} V_\lambda, \tag{2.14}$$

where V_λ is a semi-stable random variable with distribution function G_λ.

The map T_w is in many senses a more natural modification of the L.S.V. map than the piecewise map T_{exp} defined previously. Here we are simply replacing the constant coefficient of $x^{1+\beta}$ in the L.S.V. map with an oscillatory function. Unlike in the situation of Theorem A here we are unable to determine precisely the tail behaviour of the return time. We can show that there exists a semi-stable distribution G (of index α and period $\tilde{c}$) and a $F \in \mathbb{D}_{gp}(G)$ so that $\mu(\tau > n) = (1 - F(n))(1 + o(1))$, but we cannot determine precisely what this F is. This is why we only have an existence result in Theorem B above and is a consequence of the fact that in this situation it is far more involved to study the behaviour of $\mu(\tau > n)$ (see Section 4.2 for details), in particular we can no longer just apply methods like those given in [Hol05], or [Ter16].

Theorem B is directly comparable to [Gou04a, Theorem 1.3]. We see that by altering the L.S.V. map in this way, that the stable laws of Gouëzel fail to hold and that we only have subsequential convergence in distribution for generic Hölder observables. Though not explicitly mentioned in the result above, in the situations of both Theorem A and B a central limit theorem will hold in the case that $u(0) = 0$, provided that the Hölder exponent ν of u is such that $\nu > \beta - 1/2$. This follows from the fact that when we induce the induced observables will be square integrable (see Lemma 3.0.3) when $\nu > \beta - 1/2$. We also have that, like in the case of the L.S.V. map, a central limit for Hölder observables will hold for $\beta \in (0, 1/2)$, regardless of the Hölder exponent (again this is because the induced observable will always be square integrable). So for the maps T_{exp} and T_w we can give a complete picture of the distributional convergence of scaled and centred Birkoff sums of Hölder observables for $\beta \in (0, 1/2) \cup (1/2, 1)$. We cannot however draw any conclusion when $\beta = \frac{1}{2}$, nor can we draw any conclusion when $\beta \geq 1$. However, for the case $\beta > 1$, where the invariant measure is infinite, piecewise linear and piecewise quadratic maps similar to T_{exp} have been studied in [KT18] where subsequential limit theorems similar to a Darling-Kac theorem have been established.

2.3 Wobbly observables for the L.S.V. map

Our next result establishes a semi-stable law for certain logarithmically periodic observables under the dynamics of the L.S.V. map. Throughout this section we let $T_{LSV} : [0, 1] \to [0, 1]$ be the L.S.V. map as defined (1.24) with parameter $\beta \coloneqq \frac{1}{\alpha} \in (1/2, 1)$ and denote by μ the a.c.i.p. for T (cf. section 1.2.4). We saw in section 1.2.4 that Hölder observables that are non-zero at the neutral fixed point satisfy a stable law. The reason that stable laws appear for such observables is the fact when we induce the induced observable will inherit the distribution of the return time, which we know is regularly varying (see Lemma 3.0.3 for a version of Gouëzel's argument). In order to stop this phenomenon from occurring we will consider observables that oscillate faster the closer we get the neutral fixed point.

Let $u : (0, 1] \to \mathbb{R}$ be given by $u(x) = M(x) = a(1 + \varepsilon p(\log x))$ with $a > 0$ an arbitrary constant, $\varepsilon > 0$ a small parameter and p a periodic non-constant Lipschitz function with period $\frac{1}{\alpha}\log(c)$. This observable, as we show in Section 4.3.1, is *not* Hölder continuous.

Define

$$F_n(x) := \mu\left(\frac{\sum_{j=0}^{n-1} u \circ T_{LSV}^j - \int u d\mu}{n^{1/\alpha}} \leq x\right). \tag{2.15}$$

We let $\tilde{c} = c^{1/\alpha}$ $k_n = \lfloor \tilde{c}^n \rfloor$, $A_n = n^{1/\alpha}$ and define the function γ in terms of $(k_n)_{n\geq 1}$ as in (1.14).

Our next result states that any such log-periodic observable u will satisfy a semi-stable law along the subsequence (k_n) under the dynamics of T_{LSV} and that the distribution functions $(F_n)_{n\geq 1}$ will merge a family of distribution functions $(G_\lambda)_{\lambda\in[1,\tilde{c})}$

Theorem C. Let $\alpha \in (1, 2)$ and let $c > 1$. Let $u : (0, 1] \to \mathbb{R}$ be a log-periodic observable with period $c^{1/\alpha}$ of the form $u(x) = M(x)$, where M is as in (2.1). Then there exists a distribution function G of index α and period $\tilde{c}$ such that the distribution functions $(F_n)_{n\geq 1}$, defined in (2.15) merge to the family $(G_\lambda)_{\lambda\in[1,\tilde{c})}$, defined in terms of G as in (1.13), in the sense that:

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - G_{\gamma(1/n)}(x)| = 0. \tag{2.16}$$

In particular, if $(k'_n)_{n\geq 1}$ is any strictly increasing of integers with $\gamma(k'_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$, then

$$\frac{\sum_{j=0}^{k'_n-1} v \circ T_{LSV}^j - k'_n \int v d\mu}{A_{k'_n}} \xrightarrow[n\to\infty]{d} V_\lambda, \tag{2.17}$$

where V_λ is a semi-stable random variable with distribution function G_λ.

A remark on semi-stable laws for the doubling map

Theorem C shows tells us that if take a log-periodic observable of the form (2.1) then we can obtain a semi-stable under the dynamics of the L.S.V. map. Let us now remark on a family of observables for which one can obtain a semi-stable law for a uniformly expanding map. Let $T : [0, 1] \to [0, 1]$ be the doubling map

$$T(x) = 2x \mod 1,$$

and let us recall that T is a Gibbs-Markov map with a.c.i.p. given by the Lebesgue measure m.

In [Gou08] it is shown that observables of the form $u(x) = x^{-1/\alpha}$ for $\alpha \in (1, 2)$ we have that

$$\frac{\sum_{j=0}^{n-1} u \circ T^j(x) - n \int u dm}{n^{1/\alpha}},$$

converges in distribution to an α stable law. We now comment on how this result changes in the case that an additional oscillatory factor is introduced to the observable u.

Let us fix $\alpha \in (1, 2)$ and let $M : (0, \infty) \to \mathbb{R}$ be

- bounded away from 0 and ∞,
- continuous,
- log-periodic of period c: $M(cx) = M(x)$, note the difference with condition A2
- so that $x^{-1/\alpha} M(x)$ is strictly decreasing.

We then consider observables of the form

$$u(x) = x^{-1/\alpha} M(x).$$

We define $k_n \coloneqq \lfloor c^n \rfloor$ and put $A_n = n^{1/\alpha}$.

In this setting we can check directly that the tail distribution of u satisfies A2 with respect to the (k_n) and (A_n), indeed

$$m(u > x) = \inf\{y : u(y) > x\} = u^{-1}(x).$$

By Theorem 3 and Corollary 3 in [Meg00] we know that if we put $Q(1-x) = u(x)$ the Q is the inverse of a distribution function F which lies in domain of partial geometric attraction of a semi-stable distribution of index α and period c (i.e. F must satisfies A2). Then one can use a slight modification of the techniques we present in Chapter 3 in order to establish a semi-stable for u under the dynamics of the doubling map. In the case of the doubling map however it is not necessary to use the methods that are to be presented in Chapter 3, as when we induce we form an asymptotically i.i.d. Markov chain which can be studied using i.i.d. techniques alone.

Part II

Proofs

Introduction and outline of the proofs

Here we give the proofs to the results presented in Chapter 2. As mentioned before we will adapt the regime presented at the end of Section 1.2.4 to establish the limit theorems given in Chapter 2. The main steps are as follows

1. We induce on the set $Y := [1/2, 1]$ and show that the induced map T_Y is a C^2 Markov interval map. Whence the induced map will have an a.c.i.p. μ_Y and the Ruelle-Perron-Frobenius operator of T_Y will have a spectral gap on the space L_θ for some $\theta \in (0, 1)$. In the case of Theorem C this is already known (see [Gou04a]), and for Theorem A and Theorem B we show that Adler's distortion condition holds for the maps T_{exp} and T_w.

2. For the observable for which we want to prove the limit theorem we study the behaviour of the tail distribution of induced observable, in particular we show that there is a semi-stable law G and a distribution function $F \in \mathbb{D}_{gp}(G)$ so that $m_Y(u_Y > x) = 1 - F(x)$. For Theorem A and Theorem B this amounts to carefully studying the behaviour of the return time τ for the maps T_{exp} and T_w. As we mentioned before, for T_{exp} the tail behaviour of τ follows almost immediately from the construction of T_{exp}. On the other hand, when studying T_w we need to work much harder to understand the behaviour of τ. In the case of T_w we first employ arguments similar to those presented in [Hol05] to understand the leading asymptotics of the sequence (x_n) satisfying the relation $T(x_{n+1}) = x_n$ (see Proposition 4.2.1 and [CHT19, Proposition 4.1] for details), then we have to develop new methods to show that $n^\alpha x_n$ is asymptotically log-periodic (see Proposition 4.2.2 and [CHT19, Proposition 4.3 and Lemma 4.4]). Once the behaviour of the return time is established we can employ a version of the argument given in [Gou04a, Proof of Theorem 1.1] (see Lemma 3.0.3) to conclude the desired behaviour of u_Y. For Theorem C we have to examine the tail behaviour of u_Y directly. We will show that there is a lot of similarity between

$\mu(u_Y > x)$ and the tail behaviour of the return time of T_w and we are then able to use techniques similar to those developed in Proposition 4.2.2 in order to examine $\mu(u_Y > x)$.

3. We then obtain a limit theorem for the induced system using the *spectral method* of Aaronson-Denker (see Sections 3.0.1 and 3.0.2 and [AD01]).

4. Pull back the limit theorem from the induced system using an adaptation of the arguments given in [Gou08], namely we will adapt the proof of [Gou08, Theorem 4.6] so that distributional convergence along subsequences may be pulled back.

5. Finally we strengthen the distributional convergence in the previous step to a *merging* using techniques developed from [CM02]

In Chapter 3 we will collect the common aspects of the proofs of each of the Theorems given in Chapter 2. In particular, we will give a set of assumptions (A4 and A5) under which we can establish merging to a semi-stable law using steps (3)-(5) of the outline above. The remaining sections are then devoted to establishing that the systems described in Chapter 2 satisfy A4 and A5 which we will do using step (1)-(2) of the outline above.

Chapter 3

Semi-stable laws for intermittent interval maps

Here we present the common elements of the proofs of the results presented in Chapter 2. In this chapter we will consider an interval map $T : [0,1] \to [0,1]$ with a neutral fixed point that satisfies the following assumptions:

A4 1. $T : [0,1] \to [0,1]$ is given by

$$T(x) := \begin{cases} g(x) & \text{for } x \in [0,1/2), \\ 2x-1 & \text{for } x \in [1/2,1], \end{cases}$$

where

(a) g is continuous[1], strictly increasing, injective and piecewise $C^{1+\varepsilon}$

(b) $g(0) = 0$

[1] This assumption is not strictly necessary , see [CHT19] for a case when g has countably many points of discontinuity. Although the approach in essence remains the same if g is discontinuous the arguments are much more clear if we make this simplifying assumption. The key difference for a discontinuous g as studied in [CHT19] is that one has to induce more than once to form a Gibbs-Markov map.

2. Letting $Y = [1/2, 1]$ and letting $\tau : Y \to \mathbb{N}$ be given by $\tau(x) := \min\{n \geq 1 : T^n x \in Y\}$ we assume:

   (a) The induced map $T_Y := T^\tau$ on Y is a uniformly expanding Gibbs-Markov map with θ-distortion with respect to the partition $(J_n := \{x \in Y : \tau(x) = n\})_{n\geq 1}$, and whence preserves an absolutely continuous invariant probability measure which we shall denote by μ_Y.

   (b) There exists a semi-stable distribution $\tilde{G}$ of index $\alpha \in (1, 2)$ and period $\tilde{c} \geq 1$, a sequence $(\tilde{k}_n)_{n\geq 1}$ with $\lim_{n\to\infty} \frac{\tilde{k}_{n+1}}{\tilde{k}_n} = \tilde{c}$, a slowly varying function $\tilde{\ell} : \mathbb{R} \to \mathbb{R}$, and distribution function $\tilde{F} \in \mathbb{D}_{gp}(\tilde{G}, \tilde{k}_n, \tilde{A}_n)$ such that $m_Y(\tau > n) = 1 - \tilde{F}(n)$, where $\tilde{A}_n := n^{1/\alpha}\tilde{\ell}(n)$, and $m_Y := \frac{m|_Y}{m(Y)}$.

From A4 and Proposition 4.7.1 we know that τ is integrable, and so, following our remarks in Section 1.2.3, we know that T has an ergodic a.c.i.p. μ which as defined in equation (1.23). We define the sequence $(x_n)_{\geq 1}$ by setting

$$x_0 = 1, \quad x_1 = 1/2, \quad x_n = g^{-n}(1/2) \text{ for } n \geq 2. \tag{3.1}$$

We note that as we have assumed g to be continuous and strictly increasing the sequence $(x_n)_{n\geq 0}$ is well-defined, strictly decreasing and $\lim_{n\to\infty} x_n = 0$. We define the intervals I_n and J_n by setting:

$$I_0 := \left[\frac{1}{2}, 1\right], \quad I_n := [x_{n+1}, x_n) \text{ for } n \geq 1, \tag{3.2}$$

$$J_1 := \left[\frac{3}{4}, 1\right], \quad J_n := \left[\frac{1}{2}x_n + \frac{1}{2}, \frac{1}{2}x_{n-1} + \frac{1}{2}\right) \text{ for } n \geq 2, \tag{3.3}$$

and we note that $T(I_n) = I_{n-1}$, and $T(J_n) = I_{n-1}$. Moreover the map T is Markov for the partition $\{I_n\}_{n\geq 0}$, and the dynamics of T can be represented by the diagram in Figure 3.1.

We see that the sets J_n are precisely the subsets of Y which will first return to Y under

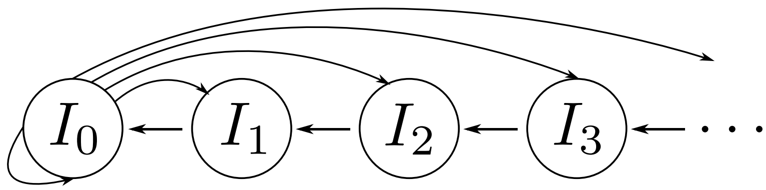

Figure 3.1: Diagram showing the symbolic dynamics of the the map T under the assumption A4.

n iterations of T:

$$\{\tau = n\} = J_n.$$

Moreover, as we saw in Section 1.2.4, the tail distribution of the τ is given by the x_n:

$$m_Y(\tau > n) = \sum_{k=n+1}^{\infty} m_Y(J_k) = x_n, \tag{3.4}$$

and there is a constant C_τ such that

$$\mu_Y(\tau > n) = C_\tau x_n(1 + o(1)). \tag{3.5}$$

In what follows we let $s : Y \times Y \to \mathbb{N}$ denote the separation time under T_Y (see equation (1.15)). We extend s to all of $[0,1] \times [0,1]$: if x, y lie in the same element of the partition $\{I_n\}_{n\geq 1}$ we set

$$s(x, y) := s(x', y') + 1,$$

where $x', y' \in [0,1]$ are the first returns of x and y to Y respectively, otherwise we set $s(x,y) = 0$.

Remark 3.0.1. We note that if T is the L.S.V. map (see (1.24)) with parameter $\beta \in (1/2, 1)$ then T satisfies our assumption A4. As mentioned in Section 1.2.4 we know that in the case of the L.S.V. map the map $n \mapsto m_Y(\tau > n)) = x_n$ is regularly varying with index $-\alpha = -\frac{1}{\beta}$, and so we know (see Theorem 1.1.7) that τ will satisfy A4 with $\tilde{G}$ being a stable distribution , $\tilde{c} = 1$ and $\tilde{k}_n = n$. In Sections 4.1 and 4.2 we will show that the maps defined in Sections 2.1 and 4.2 respectively will also satisfy A4 by studying the asymptotic behaviour of the sequence x_n.

We will show that observables $v : Y \to \mathbb{R}$ will satisfy a semi-stable law under the dynamics of the induced map if v has the following properties:

A5 1. There exists $\delta \in (0,1)$ such that

$$\sum_{n=1}^{\infty} D_\theta v(J_n)^\delta m_Y(J_n) < \infty, \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} \|v|_{J_n}\|_\infty^\delta m_Y(J_n) < \infty. \tag{3.7}$$

2. There exists a semi-stable distribution G of index α and period $c \geq 1$, a sequence $(k_n)_{n\geq 1}$ with $\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = c$, a slowly varying function $\ell : \mathbb{R} \to \mathbb{R}$, and distribution function $F \in \mathbb{D}_{gp}(G, k_n, A_n)$ such that $\mu_Y(v > x) = 1 - F(x)$, where $A_n := n^{1/\alpha}\ell(n)$.

Remark 3.0.2. We note that the return time $\tau : Y \to \mathbb{N}$ will satisfy A5. It is clear from A4 and (3.5) that τ will satisfy A5.2. As τ is constant on each J_n we have that $D_\theta \tau(J_n) = 0$, which yields (3.6). Also, as choosing $\delta \in (0,1)$ such that $\delta - \alpha < -1$ we see that $\sum_{n=1}^{\infty} \|\tau|_{J_n}\|_\infty^\delta m_Y(\tau = n) = \sum_{n=1}^{\infty} n^\delta (m_Y(\tau > n-1) - m_Y(\tau > n)) = \sum_{n=1}^{\infty} O(n^{-\alpha+\delta}\ell(n))$ is finite[2], and so (3.7) holds. It is also clear from (3.4) and (3.5) that A5.2 holds.

In the next lemma we will show that Hölder continuous observables $u : [0,1] \to \mathbb{R}$ that

[2]This comes from the general fact that if a function $f : \mathbb{R} \to \mathbb{R}$ is regularly varying index $p < -1$, so that $f(x) = x^p L(x)$ for some slowly varying function $L : \mathbb{R} \to \mathbb{R}$, we have that

$$\int_x^\infty f(t)dt = O(x^{p+1}L(x)),$$

for each $x > 0$ (see Proposition 4.7.1).

are non-zero at the neutral fixed point will induce to give to observables u_Y on Y which satisfy A5.

Lemma 3.0.3. *Assume the set up of A4. Let $u : [0,1] \to \mathbb{R}$ be Hölder continuous. Then there exists a $\delta \in (0,1)$ so that Equations (3.6) and (3.7) hold for u_Y, and we have the following two cases.*

1. *If $u(0) = 0$ and the Hölder exponent ν of u is such that $\nu > \frac{1}{\alpha} - \frac{1}{2}$ then $u_Y \in L^2(\mu_Y)$.*

2. *If $u(0) \neq 0$ then u_Y satisfies A5. Moreover, if $u(0) > 0$ there exists a $C > 0$ such that*

$$\mu_Y(u_Y > x) = C_u \mu_Y(\tau > x)(1 + o(1)),$$
$$\mu_Y(u_Y < -x) = o(x^{-\alpha}),$$

or, if $u(0) < 0$

$$\mu_Y(u_Y < -x) = C_u \mu_Y(\tau > x)(1 + o(1)),$$
$$\mu_Y(u_Y > x) = o(x^{-\alpha}).$$

The below proof is taken from [Gou04a] with only small modifications to generalise to the present setting.

Proof. As the induced map T_Y is uniformly expanding we then have that for each $x, y \in [0,1]$, $|x - y| < C\lambda^{-s(x,y)}$ where $\lambda = \inf T_Y'(x) > 1$. Then as u is Hölder with some exponent $\nu \in (0,1]$ we obtain $|u(x) - u(y)| \leq C\theta_0^{s(x,y)}$ where $\theta_0 = \lambda^{-\nu}$. For

$x, y \in J_n$ we know that $s(T^j x, T^j y) = s(x, y)$ and thus the induced observable satisfies

$$|u_Y(x) - u_Y(y)| = \left| \sum_{k=0}^{n-1} u \circ T^k(x) - u \circ T^k(y) \right| \leq Cnd_{\theta_0}(x, y),$$

yielding $D_{\theta_0} u_Y(J_n) \leq Cn$. Also, as u is Hölder, u is bounded and so

$$|u|_{J_n}(x)| \leq Cn.$$

Thus, in order to show that u satisfies (3.6) and (3.7) it is enough to find a $\delta \in (0, 1)$ so that

$$\sum_{n=1}^{\infty} n^{\delta} m_Y(J_n) < \infty.$$

Letting $\delta \in (0, 1)$ be such that $\delta\alpha > 1$ we see that $\sum_{n=1}^{\infty} n^{\delta} m_Y(J_n) = \mathbb{E}_{m_Y}(\tau^{\delta})$ is finite by Proposition 4.7.1 as $m_Y(\tau > n) = O(n^{-\alpha}\ell(n))$.

Let us first suppose that $u(0) = 0$. We note that for a point $x \in J_n$ its orbit under T will satisfy

$$T^j x \in I_{n-j}, \ j = 1, 2, \ldots, n,$$

and so we may calculate the following bound

$$|u_Y(x)| \leq C \sum_{j=0}^{n-1} |T^j x|^{\nu} \leq C' \left(1 + \sum_{j=1}^{n-1} x_j^{\nu}\right) \leq C'' \left(1 + \sum_{j=1}^{n-1} j^{-\alpha\nu}\right) \leq C'' n^{1-\alpha\nu}.$$

Thus, letting $q > 0$ we see that

$$\begin{aligned}
\int_Y |u_Y(x)|^q dm_Y &\leq C'' \sum_{n=1}^{\infty} m_Y(J_n) n^{q(1-\alpha\nu)} \\
&= C'' \left[x_0 + \sum_{n=1} x_n \left((n+1)^{q(1-\alpha\nu)} - n^{q(1-\alpha\nu)} \right) \right] \\
&\leq C''' \left[x_0 + \sum_{n=1} n^{-\alpha} n^{q(1-\alpha\nu)-1} \right].
\end{aligned}$$

If $\nu > \frac{1}{\alpha} - \frac{1}{2}$ then we see from the above that $u_Y \in L^2(m_Y)$. Since m_Y and μ_Y are equivalent we have that $u_Y \in L^2(\mu_Y)$, which concludes the first part of our claim which concludes the first part of our claim.

Now let us suppose that $u(0) \neq 0$ and let us write

$$u(x) = u(0) + \tilde{u}(x),$$

where $\tilde{u}(0) = 0$. Then we have that

$$u_Y(x) = \tau(x)u(0) + \tilde{u}_Y(x).$$

Let us first consider the case that $u(0) > 0$. By what we have just seen we know that we may take $q > \alpha$ so that $\tilde{u}_Y \in L^q(\mu_Y)$ and so by Markov's inequality

$$\mu_Y(\tilde{u}_Y > x) \leq x^{-q} \int |\tilde{u}_Y|^q d\mu_Y = o(x^{-\alpha}).$$

Thus we obtain

$$\mu_Y(u_Y > x) = C_u \mu_Y(\tau > x)(1 + o(1)),$$

for some positive constant C_u, and

$$\mu_Y(u_Y < -x) = o(x^{-\alpha}).$$

To conclude the case that $u(0) > 0$ we note that as τ is integer valued we have from (3.5) that

$$\mu_Y(\tau > x) = \mu(\tau > \lfloor x \rfloor) = C_\tau m_Y(\tau > \lfloor x \rfloor)(1 + o(1)).$$

Letting $\tilde{F}$, $\tilde{G}$, $\tilde{k}_n$ and $\tilde{A}_n$ be as in A4 we know that $\tilde{F} \in \mathbb{D}_{gp}(\tilde{G}, \tilde{k}_n, \tilde{A}_n)$ is such that

$$1 - \tilde{F}(n) = m_Y(\tau > n).$$

We can then use Lemma 4.5.1 in the appendix to see that the distribution function

$x \mapsto m_Y(\tau \leq \lfloor x \rfloor)$ is also in $\mathbb{D}_{gp}(\tilde{G}, \tilde{k}_n, \tilde{A}_n)$, and thus $F(x) := \mu_Y(u_Y \leq x)$ is in $\mathbb{D}_{gp}(\tilde{G}, \tilde{k}_n, \tilde{A}_n)$. This concludes that u_Y satisfies A5 with $G \equiv \tilde{G}$, $k_n \equiv \tilde{k}_n$ and $A_n \equiv \tilde{A}_n$.

The case that $u(0) < 0$ follows in the same way. □

We may now state the main result of this chapter. We define the function γ and the distribution function G_λ for $\lambda \in [1, c)$ in terms of the G and the $(k_n)_{n\geq 1}$ that appears in A5 as in A1 . We also define the sequence $(k'_n)_{n\geq 1}$ by putting $k'_n := \lfloor k_n \mu(Y) \rfloor$ and define the function $\tilde{\gamma}$ in terms of the sequence $(k'_n)_{n\geq 1}$ as in A1 .

Theorem 3.0.4. *Assume the set up of A4. If $u : [0,1] \to \mathbb{R}$ is such that the induced observable $u_Y : Y \to \mathbb{R}$, given by $u_Y(x) := \sum_{j=0}^{\tau(x)-1} u \circ T^j(x)$, satisfies A5 then*

$$\lim_{n\to\infty} |F_n(x) - G_{\tilde{\gamma}(1/n)}| = 0,$$

where

$$F_n(x) := \mu\left(\frac{1}{A_n}\left(\sum_{j=0}^{n-1} u \circ T^j - n \int u d\mu\right) \leq x\right).$$

In order to prove Theorem 3.0.4 we first will first prove that a semi-stable law holds for the induced system.

Proposition 3.0.5. *Assume the set up of A4. If $v : Y \to \mathbb{R}$ satisfies A5 then for every strictly increasing sequence of positive integers $(a_n)_{n\geq 1}$ such that $\gamma(1/a_n) \xrightarrow[n\to\infty]{cir} \lambda$ we have that*

$$\frac{\sum_{j=0}^{a_n-1} v \circ T_Y^j - a_n \int v d\mu_Y}{A_{a_n}} \xrightarrow[n\to\infty]{d} V_\lambda,$$

where V_λ has distribution function G_λ.

Remark 3.0.6. Proposition 3.0.5 implies that the sequence (Z_n) given by

$$Z_n := \frac{\sum_{j=0}^{n-1} v \circ T_Y^j - n \int v d\mu_Y}{A_n},$$

is stochastically compact (and whence tight by Remark 1.1.2). To see this we recall that for any $\varepsilon > 0$ we have that $\gamma(s) \in [1, c+\varepsilon]$ for all $s > 0$ sufficiently small. Thus, for any sequence $(a_n)_{n\geq 1}$ we know by Bolzano-Weierstrass that there exists a subsequence $(a'_n)_{n\geq 1}$ so that $\gamma(1/a_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda \in [1, c)$ which yields

$$Z_{a'_n} \xrightarrow[n\to\infty]{d} V_\lambda,$$

by the Lemma above. Although we could establish a merging for the induced system this is not required in order to obtain Theorem 3.0.4. As we will see later, the stochastic compactness of (Z_n) is enough to allow us to conclude Theorem 3.0.4.

3.0.1 Preliminaries

Let us assume the set up of of assumption A4, and let us suppose that $v : Y \to \mathbb{R}$ is an observable on Y satisfying assumption A5.

We denote by $\mathcal{L} : L_\theta \to L_\theta$ the Perron-Frobenius-Ruelle transfer operator associated with the induced map T_Y (cf. equation (1.20) and Section 1.2.1). For $t \in \mathbb{R}$ we define the *characteristic function operator* $\mathcal{L}_t$ by

$$\mathcal{L}_t(f) = \mathcal{L}(e^{it\tilde{v}} f),$$

where for convenience we write $\tilde{v} := v - \int v d\mu$. In this section we will establish some key properties of the operators $(\mathcal{L}_t)_{t\in\mathbb{R}}$. Our first claim is that the mapping $t \mapsto \mathcal{L}_t$ is continuous at 0 with respect to the operator norm inherited from the Banach space $(L_\theta, \|\cdot\|_\theta)$ in the following sense.

Proposition 3.0.7. *There exists a* $C > 0$ *such that*

$$\|\mathcal{L}_t - \mathcal{L}_0\|_\theta \leq C|t|^\delta.$$

Before proving the above proposition we introduce the operator $R_n(f) \coloneqq L(1_{\{\tau=n\}}f)$ and establish the following Proposition.

Proposition 3.0.8. *There exists a positive constant* C *such that for each* $n \geq 1$ *we have*

$$\|R_n\|_\theta \leq C m_Y(J_n).$$

Proof. Let $\xi_n : Y \to J_n$ denote the inverse of $T_Y|_{J_n}$ so that we may write

$$\mathcal{L}f(x) = \sum_{n=1}^{\infty} 1_{J_n} \xi_n'(x) \cdot f \circ \xi_n(x).$$

Let us fix $n \geq 1$ and let us estimate $\|R_n f\|_\theta$ for $f \in L_\theta$.

Since $T_Y|_{J_n}$ is C^1 there exists by the mean value theorem an x_0 such that

$$T_Y'(x_0) = \frac{m(Y)}{m(J_n)} = \frac{1}{m(J_n)}.$$

Thus, by the inverse function theorem, we have that

$$\xi_n'(x_0) = m_Y(J_n).$$

As we have assumed that T_Y has θ-distortion we know that for every $x, y \in J_n$,

$$\left| \frac{\xi_n'(x)}{\xi_n'(y)} - 1 \right| \leq C d_\theta(x, y).$$

We may then calculate that for any $x, y \in J_n$

$$\begin{aligned}
|\xi_n'(x) f \circ \xi_n(x) - \xi_n'(y) f \circ \xi_n(y)| &\leq |\xi_n'(y)| \left| \frac{\xi_n'(x)}{\xi_n'(y)} f \circ \xi_n(x) - f \circ \xi_n(y) \right| \\
&\leq |\xi_n'(y)| \left| f \circ \xi_n(x) \left(\frac{\xi_n'(x)}{\xi_n'(y)} - 1 \right) + f \circ \xi_n(x) - f \circ \xi_n(y) \right| \\
&\leq |\xi_n'(y)| \left(\|f\|_\infty C d_\theta(x, y) + |f|_\theta d_\theta(\xi_n(x), \xi_n(y)) \right) \\
&\leq \tilde{C} |\xi_n'(y)| \|f\|_\theta d_\theta(x, y), \qquad (3.8)
\end{aligned}$$

where in the final line we have used that $d_\theta(\xi_n(x), \xi_n(y)) = \theta d_\theta(x, y)$. Then as

$$|\xi_n'(y)| \leq |\xi_n'(x_0)| \left(1 + \left| \frac{\xi_n'(y)}{\xi_n'(x_0)} - 1 \right| \right) \leq m_Y(J_n)(1 + C d_\theta(y, x_0)),$$

and as $d_\theta(y, x_0)$ is bounded we obtain that

$$|\xi_n'(x) f \circ \xi_n(x) - \xi_n'(y) f \circ \xi_n(y)| \leq B m_Y(J_n) \|f\|_\theta d_\theta(x, y),$$

for some constant $B > 0$ which yields $|R_n f|_\theta \leq B m_Y(J_n) \|f\|_\theta$.

Let us now estimate $\|R_n f\|_\infty$. Since $R_n f(x) = 1_{J_n}(x) \xi_n'(x) f \circ \xi_n(x)$, we have from (3.8) that

$$\begin{aligned}
|R_n f(x)| &\leq |\xi_n'(x_0) f \circ \xi_n(x_0)| + |\xi_n'(x) f \circ \xi_n(x) - \xi_n'(x_0) f \circ \xi_n(x_0)| \\
&\leq |\xi_n'(x_0)| \|f\|_\infty + \tilde{C} m_Y(J_n) \|f\|_\theta \\
&\leq C' m(J_n) \|f\|_\theta.
\end{aligned}$$

Putting these estimates together we obtain $\|R_n\|_\theta \leq D m(J_n)$, for some constant $D > 0$ independent of n. $\square$

Now we may give a proof of Proposition 3.0.7.

Proof of Proposition 3.0.7. Let us fix $\delta \in (0,1)$ so that v satisfies (3.6) and (3.7).

Since $\mathcal{L}(f) = \sum_{n=1}^{\infty} R_n f$, we have that

$$\|(\mathcal{L}_t - \mathcal{L})f\|_\theta = \|\mathcal{L}(e^{it\tilde{v}} - 1)f\|_\theta \leq \sum_{n=1}^{\infty} \|R_n\|_\theta \|1_{J_n}(e^{it\tilde{v}} - 1)f\|_\theta.$$

Let us fix $C > 1$ so that $|e^{ia} - 1| \leq C|a|^\delta$, for all $a \in \mathbb{R}$ (see Proposition 4.8.1 in the appendix).

Let us first estimate $|1_{J_n}(e^{it\tilde{v}} - 1)f|_\theta$. Letting $x, y \in J_n$ we have that,

$$\begin{aligned}
|(e^{it\tilde{v}(x)} - 1)f(x) - (e^{it\tilde{v}(y)} - 1)f(y)| &\leq |(e^{it\tilde{v}(x)} - 1)(f(x) - f(y))| + |(e^{it\tilde{v}(x)} - e^{it\tilde{v}(x)})f(y)| \\
&\leq C|t|^\delta |\tilde{v}(x)|^\delta |f|_\theta d_\theta(x,y) + |e^{it\tilde{v}(x)}||f(y)||e^{it(\tilde{v}(x)-\tilde{v}(y))} - 1| \\
&\leq C|t|^\delta \|\tilde{v}|_{J_n}\|_\infty^\delta |f|_\theta d_\theta(x,y) + |t|^\delta D_\theta \tilde{v}(J_n)^\delta d_\theta(x,y)^\delta \|f\|_\infty \\
&\leq C|t|^\delta \|f\|_\theta (\|\tilde{v}|_{J_n}\|_\infty^\delta + D_\theta u_Y(J_n)^\delta) d_\theta(x,y)
\end{aligned}$$

Similarly we have that for $x \in J_n$,

$$|(e^{it\tilde{v}(x)} - 1)f(x)| \leq C\|f\|_\infty |t|^\delta |\tilde{v}(x)|^\delta \leq C|t|^\delta \|f\|_\theta \|\tilde{v}|_{J_n}\|_\infty.$$

Combining these estimates and using Proposition 3.0.8 together with (3.6) and (3.7) we obtain

$$\|(\mathcal{L}_t - \mathcal{L})f\|_\theta \leq C|t|^\delta \|f\|_\theta \left(2\sum_{n=1}^{\infty} \|\tilde{v}|_{J_n}\|_\infty^\delta m_Y(J_n) + \sum_{n=1}^{\infty} D_\theta \tilde{v}(J_n)^\delta m_Y(J_n) \right) \leq C'|t|^\delta \|f\|_\theta,$$

which concludes the proof. □

We know that the operator $\mathcal{L} = \mathcal{L}_0$ acting on the space L_θ has a simple isolated eigenvalue at 1 and a spectral gap: $\mathcal{L} = P + N$ (see Definition 1.2.2 and the discussion preceding it).

As, from Proposition 3.0.7, we know that $\|\mathcal{L}_t - \mathcal{L}_0\| \leq C|t|^\delta$ we also know that the families of maximal eigenvalues (κ_t) and the corresponding eigenprojections (P_t) of the operators $\mathcal{L}_t$ will depend continuously on t in a neighbourhood of $t = 0$ in the same as the family $(\mathcal{L}_t)$. To conclude this fact we can employ [Gou15, Proposition 2.3] which itself is a reformulation of more general statements given in [Kat66, IV.3.6 and Theorem VII.1.8]. In the Lemma below we state formally the key consequences of [Gou15, Proposition 2.3] in our present setting.

Lemma 3.0.9 ([Gou15, Proposition 2.3])**.** *For $|t|$ sufficiently small we have that $\mathcal{L}_t$ has a spectral gap with*

$$\mathcal{L}_t = \kappa_t P_t + N_t,$$

and moreover

$$|\kappa_t - \kappa| = O(|t|^\delta), \text{ and } \|P_t - P\| = O(|t|^\delta),$$

as $|t|$ approaches zero.

3.0.2 Proof of Lemma 3.0.5

With the preliminaries in place we now turn to concluding the proof of Lemma 3.0.5. In the proof below we will first employ the *spectral method* in order to establish convergence to semi-stable random variable along subsequences $(a_n)_{n\geq 1}$ for which $\gamma(1/a_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$. We will then to proceed to strengthen this convergence to that claimed in Lemma 3.0.5 by using techniques derived from [CM02].

Proof of Lemma 3.0.5. We let (X_n) be a sequence of independent identically distributed random variables on $(Y, \mathscr{B}_Y, \mu_Y)$ with common distribution equal to that of v.

Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of positive integers such that $\gamma(1/a_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$. We know from our assumptions on the distribution of v (namely A5.2) and Theorem

1.1.11 that

$$\frac{\sum_{j=1}^{a_n-1} X_j - a_n \int_Y X_1 d\mu_Y}{A_{a_n}} \xrightarrow[n\to\infty]{d} V_\lambda, \tag{3.9}$$

where V_λ is a semi-stable random variable with distribution function G_λ. We claim that

$$\frac{\sum_{j=1}^{a_n-1} u_j - a_n \int_Y u_1 d\mu_Y}{A_{a_n}} \xrightarrow[n\to\infty]{d} G_\lambda. \tag{3.10}$$

Let us denote by $\psi_n(t) := \int e^{it\sum_{j=0}^{n-1}\tilde{v}\circ T^j} d\mu$ the characteristic function of $\sum_{j=0}^{n-1} \tilde{v}\circ T^j$ and let us denote by $\phi_n(t) = \int e^{it\left(\sum_{j=0}^{n-1} X_j - n\int X d\mu\right)} d\mu$ the characteristic function of the corresponding i.i.d. sequence $\sum_{j=0}^{n-1} X_j - n\int X_j d\mu$.

We will now relate the maximal eigenvalue κ_t of $\mathcal{L}_t$ with the characteristic function ϕ_1. Let $f_t := \frac{P_t 1}{\int P_t dm}$ and note that f_t is an eigenvector of $\mathcal{L}_t$ corresponding to the eigenvalue κ_t, in particular note that $f_0 \equiv 1$. For $|t| > 0$ sufficiently small we know from Proposition 3.0.7 and Lemma 3.0.9 that $\|\mathcal{L}_t - \mathcal{L}_0\| = O(|t|^\delta)$ and $\|f_t - f_0\| = O(|t|^\delta)$. We may then calculate that

$$\kappa_t = \int \mathcal{L}_t f_t dm = \int \mathcal{L}_t f_0 dm + \int (\mathcal{L}_t - \mathcal{L})(f_t - f_0) dm = \phi_1(t) + O(|t|^{2\delta}).$$

Following Remark 1.2.3 we know that

$$\|N_t^n\| = \|\mathcal{L}_t^n - P_t^n\| = o(\kappa_t^n).$$

So, for any strictly increasing sequence $(a_n)_{n\geq 1}$ we may calculate that for large n

$$\begin{aligned}\psi_{a_n}(t/A_{a_n}) &= \int \mathcal{L}_{t/A_{a_n}} h dm_Y \\ &= \kappa_{t/A_{a_n}}^{a_n} \int P_{t/A_{a_n}} h dm_Y + \int N_{t/A_{a_n}}^{a_n} h dm_Y \\ &= \kappa_{t/A_{a_n}}^{a_n} \left[\int (P_{t/A_{a_n}} - P)h + Ph dm_Y + o(1) \right] \\ &= \kappa_{t/A_{a_n}}^{a_n} (1 + o(1)), \\ &= \phi_1(t/A_{a_n})^{a_n} (1 + o(1)).\end{aligned} \tag{3.11}$$

We know from (3.9) that

$$\lim_{n\to\infty} \phi_{a_n}(t/A_{a_n}) = \lim_{n\to\infty} \phi_1(t/A_{a_n})^{a_n} = \psi_{V_\lambda}(t),$$

where ψ_{V_λ} is the characteristic function of V_λ. By equation (3.11) we then obtain

$$\lim_{n\to\infty} \psi_{a_n}(t/A_{a_n}) = \psi_{V_\lambda}(t),$$

which, by Lévy's continuity theorem concludes (3.10) and the Lemma. □

3.1 Inducing semi-stable laws

Having established Lemma 3.0.5 we may now move to the proof of Theorem 3.0.4. In the proof below we will pull back the semi-stable law established in Lemma 3.0.5 using a modification of the argument given in the [Gou08, Theorem 4.3]. We will then argue that the distributional convergence established can be strengthened to convergence claimed in Theorem 3.0.4.

Proof of Theorem 3.0.4. We fix an observable $u : [0,1] \to \mathbb{R}$ and assume that the induced observable satisfies A5. Let us suppose further that $\int u d\mu = 0$. The general case then follows by applying the conclusion of the theorem to $\tilde{u} = u - \int u d\mu$.

We begin by relating the ergodic sum $u_n := \sum_{j=0}^{n-1} u \circ T^j$ with the induced ergodic sum $U_n := \sum_{j=0}^{n-1} u_Y \circ T_Y^j$. We note that by the definition of U and T_Y we have that $(u_n)_{n\geq 1}$ is a sub-sequence of $(U_n)_{n\geq 1}$. By passing to the natural extension if required we assume without loss of generality that the map T is invertible. Then we may write

$$\sum_{j=0}^{n-1} u \circ T^j(x) = \sum_{j=0}^{N(x,n)-1} u_Y \circ T_Y^j(x) + Hu \circ T^{-k}(x),\ \forall x \in Y,$$

where $N(x,n)$ denotes the *lap number*

$$N(x,n) := \sum_{j=1}^{n-1} 1_Y \circ T^j(x),$$

and

$$Hu(x) := \sum_{j=1}^{\psi(x)} u \circ T^{-k}(x),$$

where we have put $\psi(x) := \inf\{j \geq 1 : T^{-j}(x) \in Y\}$. Since $A_n \to \infty$ and as μ is T invariant we have that the quantity $Hu \circ T^n$ converges to 0 in measure.

Let us fix a strictly increasing sequence $(a_n)_{n\geq 1} \subset \mathbb{N}$ such that $\gamma(1/a_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$. Then by Lemma 3.0.5

$$\frac{U_{a_n}}{A_{a_n}} \xrightarrow[n\to\infty]{d} V_\lambda,$$

where V_λ has distribution function G_λ. Following Remark 3.0.2 we also know that τ satisfies Lemma 3.0.5. Setting $\tau_n := \sum_{j=0}^{n-1} \tau \circ T_Y^j$ we then know from Remark 3.0.6 that the sequence

$$\frac{\tau_n - n\mu(Y)}{\tilde{A}_n},$$

is tight where $\tilde{A}_n$ is as in A4.

We set $a'_n := \lfloor a_n \mu(Y) \rfloor$ and claim that

$$\frac{u_{a'_n}}{A_{a'_n}} \xrightarrow[n\to\infty]{d} V_\lambda. \tag{3.12}$$

From our discussion above we know that it is sufficient to show that

$$\frac{\sum_{j=0}^{N(\cdot,a_n')-1} U \circ T_Y^j}{A_{a_n'}} \xrightarrow[n\to\infty]{d} V_\lambda. \tag{3.13}$$

We will in fact show that (3.13) converges in distribution with respect to μ_Y on Y, then by Theorem 4.1 [Gou08] the result will follow.

As $\alpha \in (1,2)$ we may choose an $\eta > 0$ small enough so that $\frac{1}{\alpha} + \eta - 1 < 0$. We set $\beta(n) := n^{\frac{1}{\alpha}+\eta}$ and aim to show that

$$\frac{N(\cdot, a_n') - a_n}{\beta(a_n)} \xrightarrow[n\to\infty]{\mu_Y} 0. \tag{3.14}$$

From the definition of the lap number we have that $N(x,k) \geq n$ if and only if $\tau_n < k$. Thus we have that for any $\varepsilon > 0$

$$\begin{aligned} \mu_Y\left\{x \in Y : \frac{N(x,a_n') - a_n}{\beta(a_n)} \geq \varepsilon\right\} &= \mu_Y\{x : N(x,a_n') \geq \beta(a_n)\varepsilon + a_n\} \\ &= \mu_Y\{x : \tau_{a_n+\beta(a_n)\varepsilon}(x) < a_n'\}. \end{aligned} \tag{3.15}$$

As $\tau \geq 1$ we have that the trivial inequality

$$\tau_{a_n+\beta(a_n)\varepsilon} \geq \tau_{a_n} + \beta(a_n)\varepsilon,$$

which yields

$$\mu_Y\{x : \tau_{a_n+\beta(a_n)\varepsilon}(x) < a_n'\} \leq \mu\left(\frac{\tau_{a_n} - a_n'}{\beta(a_n)} < -\varepsilon\right).$$

Proceeding as above we obtain

$$\mu_Y\left\{x \in Y : \frac{N(x,a_n') - a_n}{\beta(a_n)} \leq -\varepsilon\right\} \leq \mu\left(\frac{\tau_{a_n} - a_n'}{\beta(a_n)} \geq \varepsilon\right).$$

In particular we have that

$$\mu_Y \left\{ x \in Y : \frac{|N(x, a_n') - a_n|}{\beta(a_n)} \geq \varepsilon \right\} \leq \mu \left(\frac{|\tau_{a_n} - a_n'|}{\beta(a_n)} \geq \varepsilon \right)$$

Writing

$$\frac{\tau_{a_n} - a_n'}{\beta(a_n)} = a_n^{-\eta} \tilde{\ell}(a_n) \frac{\tau_{a_n} - a_n \mu(Y)}{\tilde{A}_{a_n}} + \frac{a_n \mu(Y) - a_n'}{\beta(a_n)},$$

we know that the last term on the right converges to 0 as $\beta(n) \to \infty$ and the first term on the right converges to 0 in measure as the sequence

$$\frac{\tau_{a_n} - a_n \mu(Y)}{\tilde{A}_{a_n}},$$

is tight and, since $\tilde{\ell}$ is slowly varying, $a_n^{-\eta} \tilde{\ell}(a_n) \to 0$. This concludes (3.14).

We now show that for each $\varepsilon > 0$

$$\lim_{n \to \infty} \mu_Y \left(x : \frac{\left| \sum_{j=0}^{N(x, a_n')-1} u_Y \circ T^j(x) - \sum_{j=0}^{a_n - 1} u_Y \circ T^j(x) \right|}{A_{a_n}} \geq \varepsilon \right) = 0. \qquad (3.16)$$

Letting $\varepsilon > 0$ we calculate

$$\begin{aligned} \mu_Y \left(\frac{|U_{N(\cdot, a_n')} - U_{a_n}|}{A_{a_n}} \geq \varepsilon \right) \leq \mu_Y \left(|N(\cdot, a_n') - a_n| \geq \beta(a_n) \right) \\ + \mu_Y(x : \exists j \in [-\beta(a_n), \beta(a_n)] \text{ such that} \\ N(x, a_n') = a_n + j \text{ and} \\ |U_{a_n + j} - U_{a_n}| \geq \varepsilon A_{n_r}). \end{aligned}$$

The first term on the right of above vanishes by what we have just shown so we only need to deal with second term. Let us suppose that

$$x \in \{x : \exists j \in [-\beta(a_n), \beta(a_n)] \text{ such that } N(x, a_n') = a_n + j \text{ and } |U_{a_n + j} - U_{a_n}| \geq \varepsilon A_{a_n}\},$$

and to begin with let us suppose further that $j \geq 0$. Then we see that

$$|U_{a_n+j} - U_{a_n}| = |U_j| \circ T_Y^{a_n}(x),$$

and

$$\mu_Y\left(\frac{|U_j| \circ T_Y^{a_n}}{A_{a_n}} \geq a\right) = \mu_Y\left(\frac{|U_j|}{A_j}\frac{A_j}{A_{a_n}} \geq a\right).$$

Moreover

$$\frac{|U_j|}{A_j}\frac{A_j}{A_{a'_n}} \leq \frac{|U_j|}{A_j}\frac{A_{\beta(a'_n)}}{A_{a'_n}}. \tag{3.17}$$

Using the definition of $\beta(n)$ we have that

$$\frac{A_{\beta(n)}}{A_n} = n^{\frac{1}{\alpha}\left(\frac{1}{\alpha}+\eta-1\right)}\frac{\ell(\beta(n))}{\ell(n)}.$$

Letting $0 < \delta < \frac{1}{\alpha}$ and $C > 1$ we may choose an n_0 by Potter's bounds so that whenever $n, \beta(n) > n_0$ we have

$$\frac{\ell(\beta(n))}{\ell(n)} \leq C\left(\frac{\beta(n)}{n}\right)^{-\delta} = Cn^{n^{-\delta(\frac{1}{\alpha}+\eta-1)}}.$$

Thus, for n large enough we have that

$$\frac{A_{\beta(n)}}{A_n} \leq Cn^{\left(\frac{1}{\alpha}-\delta\right)\left(\frac{1}{\alpha}+\eta-1\right)},$$

which converges to 0 by our choice of η and δ.

We know from Remark 3.0.6 that the sequence $|U_j|/A_j$ is tight which gives that (3.17) converges to zero in measure as claimed. Proceeding in the same way as above one may show that if$j \in [-\beta(n), 0)$ we also have that

$$\frac{|U_{a_n+j} - U_{a_n}|}{A_{a_n}},$$

converges to 0 in measure, which concludes our claim.

We have shown that whenever (a_n) is such that $\gamma(1/a_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$ we obtain (3.12) as claimed.

We now claim that whenever (b_n) is such that $\tilde{\gamma}(1/b_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$ we have that

$$\frac{u_{b_n}}{A_{b_n}} \xrightarrow[n\to\infty]{d} V_\lambda,$$

where we recall the function $\tilde{\gamma}$ is defined in terms of the sequence $k'_n := \lfloor k_n \mu(Y) \rfloor$.

First, let us note that whenever $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences with $b_n = a_n(1 + o(1))$ then

$$\frac{u_{b_n}}{A_{b_n}} \xrightarrow[n\to\infty]{d} V \Leftrightarrow \frac{u_{a_n}}{A_{a_n}} \xrightarrow[n\to\infty]{d} V_\lambda.$$

Now let us suppose that $(b_n)_{n\geq 1}$ is such that $\tilde{\gamma}(1/b_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$. Let $a_n = \lfloor b_n/\mu(Y) \rfloor$ and note that $a'_n = b_n(1 + o(1)$. Thus, to conclude our current claim it is enough to show that $\gamma(1/a_n) \to \lambda$. Letting $k_{p(n)}$ and $k'_{\tilde{p}(n)}$ be the elements of $(k_n)_{n\geq 1}$ and $(k'_n)_{n\geq 1}$ respectively so that

$$\gamma(1/a_n) = \frac{k_{p(n)}}{a_n}, \text{ and } \tilde{\gamma}(1/b_n) = \frac{k'_{\tilde{p}(n)}}{b_n}.$$

Let us note that

$$\begin{aligned} &k'_{\tilde{p}(n)-1} < b_n \leq k'_{\tilde{p}(n)} \\ &\Leftrightarrow k_{\tilde{p}(n)-1} \left(1 - \frac{\{k'_{\tilde{p}(n)-1}\mu(Y)\}}{k'_{\tilde{p}(n)-1}}\right) < \frac{b_n}{\mu(Y)} \leq k_{\tilde{p}(n)} \left(1 - \frac{\{k'_{\tilde{p}(n)}\mu(Y)\}}{k'_{\tilde{p}(n)}}\right). \end{aligned}$$

Taking integer parts of the final inequality we may conclude that for all n large enough $k_{p(n)} \in \{k_{\tilde{p}(n)-2}, k_{\tilde{p}(n)-1}, k_{\tilde{p}(n)}\}$. Thus

$$\gamma(1/a_n) = \frac{k_{p(n)}}{a_n} = \frac{k_{\tilde{p}(n)}\mu(Y)}{b_n}\frac{k_{p(n)}}{k_{\tilde{p}(n)}}(1 + o(1)) = \frac{k'_{\tilde{p}(n)}}{b_n}\frac{k_{p(n)}}{k_{\tilde{p}(n)}}(1 + o(1)) \xrightarrow[n\to\infty]{\text{cir}} \lambda.$$

This establishes that whenever $(b_n)_{n\geq 1}$ is strictly increasing sequence of integers with

$\tilde{\gamma}(1/b_n) \xrightarrow[n\to\infty]{\text{cir}} \lambda$ we have that

$$\frac{u_{b_n}}{A_{b_n}} \xrightarrow[n\to\infty]{d} V_\lambda.$$

Now, following the first part of the proof of [CM02, Theorem 2], we move to concluding the merging aspect of the result. We will argue by the fact that the distribution function G_λ is uniformly continuous for each $\lambda \in [1, c)$ (see [CM02, Lemma 2, Theorem 2]).

Let $(a_n)_{n\geq 1}$ be an arbitrary subsequence of $\mathbb{N}$. For every $\varepsilon > 0$ we know that $\gamma(1/a_n) \in [1, c+\varepsilon]$ for all n sufficiently large. So, Bolzano-Weierstrass we have that there exists a further subsequence, say $(a_{n_j})_{j\geq 1}$, such that $\gamma(1/a_{n_j}) \xrightarrow[j\to\infty]{\text{cir}} \lambda \in [1, c)$ and so by what we have just shown that

$$\lim_{j\to\infty} d_{\mathcal{L}}(F_{a_{n_j}}, G_\lambda) = 0. \tag{3.18}$$

Moreover, one can check that the family $(G_\lambda)_{\lambda\in[1,c)}$ is continuous with respect to the topology of weak convergence (cf. proof of [CM02, Theorem 2]). In particular we have that

$$\lim_{j\to\infty} d_{\mathcal{L}}(G_{\gamma(1/a_{n_j})}, G_\lambda) = 0. \tag{3.19}$$

We claim that the weak convergence of the distributions function in equations (3.18) and (3.19) can be strengthened to uniform convergence.

Let $\varepsilon > 0$ be arbitrary. By the continuity of G_λ we may choose a $0 < \delta < \varepsilon$ so that

$$|x_1 - x_2| \leq 2\delta \Leftrightarrow |G_\lambda(x_1) - G_\lambda(x_2)| \leq \varepsilon.$$

Then by equations (3.18) and (3.19) we may take $N \in \mathbb{N}$ large enough so that for every $j \geq N$ we have

$$d_{\mathcal{L}}(F_{a_{n_j}}, G_\lambda) < \delta, \;\; d_{\mathcal{L}}(G_{\gamma(1/a_{n_j})}, G_\lambda) < \delta$$

Letting $x \in \mathbb{R}$ be arbitrary we know by the definition of the Lévy distance that

$$G_\lambda(x-\delta) - G_\lambda(x+\delta) - \delta \leq F_{a_{n_j}}(x) - G_\lambda(x+\delta) \leq \delta,$$

and since $0 \geq G_\lambda(x-\delta) - G_\lambda(x+\delta) \geq -\varepsilon$ we have that

$$|F_{a_{n_j}}(x) - G_\lambda(x+\delta)| \leq \delta + \varepsilon \leq 2\varepsilon$$

Thus we obtain

$$|F_{a_{n_j}}(x) - G_\lambda(x)| \leq |F_{a_{n_j}}(x) - G_\lambda(x+\delta)| + |G_\lambda(x) - G_\lambda(x+\delta)| \leq 3\varepsilon.$$

Repeating the above calculation with $G_{\gamma(1/a_{n_j})}$ in the place of F we obtain

$$|G_{\gamma(1/a_{n_j})}(x) - G_\lambda(x)| \leq 3\varepsilon.$$

Thus, our claim that

$$\lim_{j\to\infty} |F_{a_{n_j}} - G_\lambda|_\infty = 0 \text{ and } \lim_{j\to\infty} |G_{\gamma(1/a_{n_j})} - G_\lambda|_\infty = 0$$

is established. By a further application of the triangle inequality, we see that

$$\lim_{j\to\infty} |F_{a_{n_j}} - G_{\gamma(1/a_{n_j})}|_\infty = 0.$$

Finally, as the sequence $(a_n)_{n\geq 1}$ with which we started was arbitrary we obtain

$$\lim_{j\to\infty} |F_n - G_{\gamma(1/n)}|_\infty = 0,$$

which concludes the proof. □

Chapter 4

Proofs of main results

In this chapter we give the proofs of our main results: Theorems A, B and C. In each case we will verify that the system in question satisfies the assumptions A4 and the induced observable satisfies A5 so that we may apply Theorem A to conclude. In the case of Theorems A and B it is enough to verify A4 as we can then use Lemma 3.0.3 to show that the induced observable will satisfy A5. In the case of Theorem C we know already that T_{LSV} satisfies A4 (see Remark 3.0.1), however our observable in this case is chosen so that 3.0.3 does not apply. Thus for Theorem C we will need to verify A5 directly.

4.1 Proof of Theorem A

Throughout this section we let $T \equiv T_{exp}$ be as defined in Section 2.1 and let $u : [0,1] \to \mathbb{R}$ be Hölder continuous observables which is non-zero at 0. It is clear from the definition of T that (1) of A4 holds. We also know by the definition of T and the discussion following of the statement of the assumptions A4 that

$$m_Y(\tau > n) = x_n,$$

where x_n is as given in (2.3). In particular, we recall that with $m_Y(\tau > n) = n^{-\alpha} M(n)$ and M satisfies A1 . So, if F is such that $1 - F(x) = x^{-\alpha} M(x)$ for all x large enough we know that $F \in \mathbb{D}_{gp}(G, k_n, A_n)$ where $k_n = \lfloor c^n \rfloor$, $A_n = n^{1/\alpha}$ and G is the distribution function of a semi-stable distribution with right Lévy function

$$R(x) = -\frac{M(x)}{x^{\alpha}},$$

and left Lévy function $L \equiv 0$. This establishes 2(b) of A4.

In order to see that T satisfies A4 it is sufficient to show that T_Y is a uniformly expanding C^2 Markov interval map (see Example 1.2.1).

Proof of Theorem A. Let g_n be as defined in (2.6). Considering the first derivative

$$g_n'(x) = (1 - a_n)\rho_n - \frac{1}{\Delta_n} \log(1 - a_n \Delta_{n-1}) \exp\left\{ \log(1 - a_n \Delta_{n-1}) \frac{x - x_{n+1}}{\Delta_n} \right\},$$

we note that g_n' is positive and decreasing and so obtains its infimum at x_n which yields

$$\begin{aligned} \inf_{x \in I_n} g_n'(x) &= (1 - a_n)\rho_n - \frac{1}{\Delta_n} \log(1 - a_n \Delta_{n-1})(1 - a_n \Delta_{n-1}) \\ &\geq \rho_n (1 - a_n^2 \Delta_{n-1}) > 1 \end{aligned} \tag{4.1}$$

by our choice of the sequence a_n. This readily implies that $T_Y := T^{\tau}$ will be uniformly expanding. Now we argue that T has the Adler property. First we calculate

$$g_n''(x) = -\frac{1}{\Delta_n^2} \log(1 - a_n \Delta_{n-1})^2 \exp\left\{ \log(1 - a_n \Delta_{n-1}) \frac{x - x_{n+1}}{\Delta_n} \right\},$$

which is negative and strictly increasing. Whence, $|g_n''|$ will obtain its supremum at x_{n+1}, and so

$$\sup_{x \in I_n} |T''(x)| = \frac{1}{\Delta_n^2} \log(1 - a_n \Delta_{n-1})^2 \leq a_n^2 \rho_n^2.$$

Thus we have that

$$\sup_{x\in I_n} \frac{|T''(x)|}{|T'(x)|^2} \leq \frac{a_n^2}{(1-a_n^2\Delta_n)^2} \to 0,$$

and so

$$\sup_n \sup_{x\in I_n} \frac{|T''(x)|}{|T'(x)|^2} = C < \infty.$$

Now we argue that the induced map also satisfies the Adler condition. Letting

$$\mathrm{Adl}(f) = \frac{|f''|}{|f'|^2},$$

a simple calculation shows that

$$\mathrm{Adl}(T^n)(x) \leq \mathrm{Adl}(T^{n-1}) \circ T(x) + \frac{\mathrm{Adl}(T)(x)}{(T^{n-1})' \circ T(x)},$$

for each $x \in Y$. Iterating the above relation we obtain that for all $x \in Y$

$$\mathrm{Adl}(T^n)(x) \leq \mathrm{Adl}(T) \circ T^{n-1}(x) + \sum_{k=1}^{n-1} \frac{\mathrm{Adl}(T) \circ T^{k-1}(x)}{(T^{n-k})' \circ T^k(x)} \leq \sum_{k=0}^{n-2} \mathrm{Adl}(T) \circ T^k(x).$$

Fixing $x \in J_n$ we know that $T^k(x) \in I_{n-k}$, and

$$\mathrm{Adl}(T) \circ T^k(x) \leq \sup_{y\in I_{n-k}} \frac{|T''(y)|}{(T'(y))^2} \leq a_{n-k}^2.$$

Thus, using our assumption that the a_n satisfy (2.7), we find

$$\mathrm{Adl}(T^n)(x) \leq C + \sum_{k=1}^{n-1} a_k^2 \leq C + \sum_{k=1}^{n-1} \Delta_{k-1} \leq C + 1,$$

yielding the Adler property for the induced map. This yields that T is uniformly expanding C^2 Markov interval map and concludes that A4 hold for T.

Now we may apply Lemma 3.0.3 to the observable u to see that u_Y satisfies A5 with G, k_n, and A_n as defined in Section 2.1. Applying Theorem 3.0.4 then concludes the proof. □

4.2 Proof of Theorem B

Throughout this section we will let $T \equiv T_w$ as defined in 2.2 and let $u : [0,1] \to \mathbb{R}$ be Hölder observable with $u(0) \neq 0$. In order to prove B we will verify the assumption A4. Then, as in the previous section, we will use Lemma 3.0.3 to show that u_Y satisfies A5 and employ Theorem 3.0.4 in order to conclude. We begin by examining the tail behaviour of the return time.

4.2.1 Tail behaviour of the return time

Here we aim to show that the return time τ has tail distribution

$$m_Y(\tau > x) = x^{-\alpha}(\tilde{M}(x) + H(x)),$$

which satisfies (1.9), in particular we will show that the $\tilde{M}$ that appears in the expression above is log-periodic with period c^{1/α^2} and satisfies A1 . The proofs presented here are based on the arguments given in [CHT19, Section 4]

Proposition 4.2.1 (see [CHT19, Proposition 4.1]). *Let the sequence x_n be defined by relation $x_n = T|_{[0,1/2]}(x_{n+1})$ with $x_0 = 1$. Then*

$$x_n = n^{-\alpha}(M_0(n) + E_0(n)),$$

where

$$M_0(n) = \left(\frac{1}{\alpha n}\sum_{j=1}^{n} M(x_j)\right)^{-\alpha}, \quad E_0(n) = O\left(\frac{\log n}{n}\right).$$

Proof. Put $\beta = \frac{1}{\alpha}$. Using the fact that $T(x_{n+1}) = x_n$, we calculate

$$\begin{aligned}
\frac{1}{x_n^\beta} &= \frac{1}{x_{n+1}^\beta}\left(1 + x_{n+1}(M(x_{n+1}))\right)^{-\beta} \\
&= \frac{1}{x_{n+1}^\beta}\left(1 - \beta x_{n+1}^\beta M(x_{n+1}) + \frac{\beta(\beta+1)}{2} x_{n+1}^{2\beta}(x_{n+1})^2(1+o(1))\right) \\
&= \frac{1}{x_{n+1}^\beta} - \beta M(x_{n+1}) + \frac{\beta(\beta+1)}{2} x_{n+1}^\beta M(x_{n+1})^2(1+o(1))
\end{aligned} \tag{4.2}$$

Since M bounded and $x_n \to 0$ we find that by summing that,

$$\frac{1}{x_n^\beta} = 1 + \beta \sum_{j=1}^{n} M(x_j) - e_j, \tag{4.3}$$

where $e_j := \frac{\beta(\beta+1)}{2} x_j^\beta M(x_{n+1})(1+o(1))$. Since $e_n = o(1)$ in the limit as $n \to \infty$, we see by substituting (4.3) into the right hand side of (4.2) that

$$\begin{aligned}
\frac{1}{x_n^\beta} &= 1 + \beta \sum_{j=1}^{n} M(x_j) - \frac{\beta(\beta+1)}{2} \sum_{j=1}^{n} \frac{M(x_j)^2}{1 + \sum_{k=1}^{j}(M(x_k) - e_j)}(1+o(1)) \\
&= 1 + \beta \sum_{j=1}^{n} M(x_j) - \frac{\beta(\beta+1)}{2} \sum_{j=1}^{n} \frac{M(x_j)^2}{1 + \sum_{k=1}^{j} M(x_k)}(1+o(1))
\end{aligned}$$

Putting $\tilde{E}_0(n) := \frac{1}{n} - \frac{\beta(\beta+1)}{2n} \sum_{j=1}^{n} \frac{M(x_j)^2}{1+\sum_{k=1}^{j} M(x_k)}(1+o(1))$ the above yields: $x_n^{-\beta} = n\left(\frac{\beta}{n}\sum_{j=1}^{n} M(x_j) + \tilde{E}_0(n)\right)$, and thus

$$\begin{aligned}
x_n &= \frac{1}{n^\alpha}\left(M_0(n)^{-1/\alpha} + \tilde{E}_0(n)\right)^{-\alpha} \\
&= \frac{1}{n^\alpha}\left(M_0(n) - \alpha M_0(n)^{1+1/\alpha}\tilde{E}_0(n)(1+o(1))\right)
\end{aligned}$$

Let us now study the error term $E_0(n) := -\alpha M_0(n)^{1+1/\alpha}\tilde{E}_0(n)(1+o(1))$. As M is bounded above and below by positive constants we know that $M_0(n) = O(1)$ and so $\tilde{E}_0(n) \sim \frac{1}{n}\left(1 + \sum_{j=1}^{n} \frac{1}{1+j}\right) = O\left(\frac{\log n}{n}\right)$, which concludes our claim. □

Proposition 4.2.2 (see [CHT19, Proposition 4.3 and Lemma 4.4])**.** *The function M_0 is asymptotically log-periodic with period c^{1/α^2} in the sense that*

$$\frac{M_0(c^{1/\alpha^2}n)}{M_0(n)} = 1 + O\left(\frac{(\log n)^2}{n}\right),$$

in the limit as $n \to \infty$.

Let us extend the functions M_0 and E_0 to the positive reals by setting $M_0(x) = M_0(\lfloor x \rfloor)$ and $E_0(x) = E_0(\lfloor x \rfloor)$.

Proof. Let us define $m_0(x) \coloneqq \frac{1}{\alpha x}\sum_{j=1}^{\lfloor x \rfloor} M(x_j)$ so that $x_n = (nm_0(n))^{-\alpha}(1 + o(1))$. Let us note that as $x_n \sim (nm_0(n))^{-\alpha}$ we have that $m_0(n) \sim \frac{1}{n\alpha}\sum_{j=1}^{n} M((nm_0(n))^{-\alpha})$. Consider a continuous analogue $\overline{m}$ of m_0 defined implicitly by the relation

$$\overline{m}(x) \coloneqq \frac{1}{\alpha x}\int_1^x M((u\overline{m}(u))^{-\alpha})du.$$

We will now show that with $\hat{c} \coloneqq \frac{\log c}{\alpha^2}$

$$\frac{\overline{m}(e^{\hat{c}}x)}{\overline{m}(x)} = 1 + O(1/x)$$

in the limit as $x \to \infty$.

Let $v(x) \coloneqq x\overline{m}(x)$, then from the definition of $\overline{m}$ we find that

$$v(x) = \frac{1}{\alpha}\int_1^x M(v(u)^{-\alpha})du.$$

Differentiating we then obtain

$$v'(x) = \frac{1}{\alpha}M(v(x)^{-\alpha}).$$

Using the definition of M we find that

$$v'(x) = a(1 + \varepsilon p(-\alpha \log v(x))).$$

We now make the substitution $e^{u(x)} = v(x)$ in the above. Making the substitution we obtain

$$u'(x)e^{u(x)} = a(1 + \varepsilon p(-\alpha u(x))) \Rightarrow x = \frac{1}{a}\int_0^u \frac{e^z}{1 + \varepsilon p(-\alpha z)} dz - C,$$

for some integration constant C. Let us define

$$g(u) := \frac{1}{a}\int_0^u \frac{e^z}{1 - \varepsilon p(-\alpha z)} dz - C,$$

so that $g(u(x)) = x$. The integral $g(u)$ does not admit a closed form, however we can study some of its properties by exploiting the log-periodicity of M.

Let us note that if $u = \hat{c}\ell + z_0$ and with $\ell \in \mathbb{N}$ and $z_0 \in [0, \hat{c})$ we may divide up the integral in $g(u)$ in the following way

$$g(u) = \frac{1}{a}\left(\sum_{j=0}^{\ell-1}\int_{\hat{c}j}^{\hat{c}(j+1)} \frac{e^z}{1 + \varepsilon p(-\alpha z)} dz + \int_{\hat{c}\ell}^{\hat{c}\ell+z_0} \frac{e^z}{1 + \varepsilon p(-\alpha z)} dz.\right) - C.$$

Now, making the substitution $w + \hat{c}j = z$ in each of the integrals above, and using the fact that p is periodic with period $\frac{\log c}{\alpha} = \alpha\hat{c}$, we obtain

$$\begin{aligned} g(u) &= \frac{1}{a}\left(\sum_{j=0}^{\ell-1} e^{\hat{c}} g(\hat{c}) + e^{\hat{c}\ell} g(z_0)\right) - C, \\ &= \frac{1}{a}\left(\frac{e^{\hat{c}\ell} - 1}{e^{\hat{c}} - 1} g(\hat{c}) + e^{\hat{c}\ell} g(z_0)\right) - C. \end{aligned}$$

Now for $x > 0$ arbitrary let us write $x = e^{\hat{c}k + w_0}$ for some $k \in \mathbb{N}$ and some $w_0 \in [0, \hat{c})$ and let x' be such that $u(x') = u(x) + \hat{c}$. Then letting $\ell \in \mathbb{N}$ and $x_0 \in [0, \hat{c})$ be such that $u(x) = \hat{c}\ell + x_0$, we see that

$$\begin{aligned} x' = g(u(x')) &= \frac{1}{a}\left(\frac{e^{\hat{c}(\ell+1)}-1}{e^{\hat{c}}-1}g(\hat{c}) + e^{\hat{c}(\ell+1)}g(z_0)\right) - C \\ &= \frac{e^{\hat{c}}}{a}\left(\frac{e^{\hat{c}\ell}-1+(1-e^{-\hat{c}})}{e^{\hat{c}}-1}g(\hat{c}) + e^{\hat{c}\ell}g(z_0)\right) - C \\ &= e^{\hat{c}}\left(g(u(x)) + \frac{1-e^{-\hat{c}}}{e^{\hat{c}-1}} + C\right) - C. \end{aligned}$$

In particular, we find that for some constant C_1 independent of x, x', we have that

$$\frac{x'}{x} = e^{\hat{c}} + \frac{\tilde{C}}{x}.$$

Thus in the limit as $x \to \infty$ we see that $x' = e^{\hat{c}}x + O(1)$. Now we may calculate that

$$u(e^{\hat{c}x}) - u(x) - \hat{c} = u(e^{\hat{c}}x) - u(x'),$$

and by the mean value theorem we may choose ξ_x between $e^{\hat{c}x}$ and x' so that

$$u(\hat{c}x) - u(x) = u'(\xi_x)(e^{\hat{c}x} - x') = O(u'(\xi_x)).$$

Since $u'(x) = \frac{v'(x)}{v(x)} = O(1/x)$ and since $\xi_x \sim e^{\hat{c}}x$ we find that

$$u(e^{\hat{c}x}) - u(x) = \hat{c} + O(1/x).$$

Thus

$$\frac{\overline{m}(e^{\hat{c}}x)}{\overline{m}(x)} = \frac{1}{e^{\hat{c}}}e^{\hat{c}+O(1/x)} = 1 + O(1/x).$$

Now we relate $\overline{m}$ with m_0. Let us note that the function

$$y \mapsto M(y^{-\alpha}) = a\left(1 + \varepsilon p(-\alpha \log(y))\right),$$

is log-periodic with period c^{1/α^2} and has a finite number of intervals of monotonicity in $\left[c^{1/\alpha^2}, c^{2/\alpha^2}\right]$. Then since $C_1 x \leq m_0(x) \leq C_2 x$ we see that function $h(x) := M((xm_0(x))^{-\alpha})$ has $O(\log x)$ intervals of monotonicity within the interval $[1, x]$. Then for each interval I on which h is monotone we see that

$$\left| \int_I h(u)du - \sum_{j \in I \cap \mathbb{N}} h(j) \right| \leq \sup_I h - \inf_I h \leq \tilde{C},$$

for some constant $\tilde{C}$ independent of I. Thus, as there $O(\log(x))$ intervals of monotonicity, we obtain

$$\int_1^x h(u)du = \sum_{j=1}^{\lfloor x \rfloor} h(j) + O(\log x). \tag{4.4}$$

We now wish to compare $m_0(x)$ and $\frac{1}{\alpha x} \sum_{j=1}^{\lfloor x \rfloor} h(j)$. First we calculate

$$\begin{aligned} \log x_n - \log((nm_0(n))^{-\alpha}) &= \log((nm_0(n))^{-\alpha} + n^{-\alpha} E_0(n)) - \log((nm_0(n))^{-\alpha}) \\ &= \log\left(1 + \frac{E_0(n)}{m_0(n)^\alpha}\right) \\ &= O(E_0(n)) = O\left(\frac{\log n}{n}\right). \end{aligned}$$

Then since p is Lipschitz by assumption we see that

$$M(x_n) - M((nm_0(n))^{-\alpha}) = O\left(\frac{\log n}{n}\right),$$

and thus

$$m_0(n) - \frac{1}{\alpha j} \sum_{j=1}^{n} h(j) = O\left(\frac{1}{n} \sum_{j=1}^{n} \frac{\log j}{j}\right) = O\left(\frac{(\log n)^2}{n}\right).$$

Thus, we can relate m_0 and $\overline{m}_0$ as (4.4) yields

$$\begin{aligned} m_0(x) &= \frac{1}{\alpha x} \sum_{j=1}^{\lfloor x \rfloor} M((jm_0(j))^{-\alpha}) + O\left(\frac{(\log x)^2}{x}\right) \\ &= \frac{1}{\alpha x} \int_1^x M((um_0(u))^{-\alpha})du + O\left(\frac{(\log x)^2}{x}\right) \end{aligned}$$

and so

$$m_0(x) = \overline{m}(x)\left(1 + O\left(\frac{(\log(x))^2}{x}\right)\right) \tag{4.5}$$

Now we may compute

$$\begin{aligned}\frac{m_0(e^{\hat{c}}x)}{m_0(x)} &= \frac{\overline{m}(e^{\hat{c}}x)\left(1 + O\left(\frac{(\log e^{\hat{c}}x)^2}{e^{\hat{c}}x}\right)\right)}{\overline{m}(x)\left(1 + O\left(\frac{(\log x)^2}{x}\right)\right)} \\ &= \frac{\overline{m}(e^{\hat{c}}x)}{\overline{m}(x)}\left(1 + O\left(\frac{(\log x)^2}{x}\right)\right) \\ &= 1 + O\left(\frac{(\log x)^2}{x}\right).\end{aligned}$$

The proposition then follows as $M_0 = m_0^{-\alpha}$. □

Set $\tilde{c} := c^{1/\alpha}$ and set $k_n = \lfloor \tilde{c}^n \rfloor$, $\ell(y) := \frac{c^{y/\alpha^2}}{\lfloor c^{y/\alpha^2} \rfloor}$ and $A_n = n^{1/\alpha}\ell(n)$. Then, by definition, we have that

$$\begin{aligned}&\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = \tilde{c}, \\ &\lim_{y\to\infty} \ell(y) = 1, \\ &A_{k_n} = \tilde{c}^{n/\alpha}, \quad \lim_{n\to\infty} \frac{A_{k_{n+1}}}{A_{k_n}} = \tilde{c}^{1/\alpha}.\end{aligned}$$

Proposition 4.2.3. *For each $x > 0$ the limit*

$$\lim_{n\to\infty} M_0(A_{k_n}x)$$

exists and moreover the function

$$\tilde{M}(x) = \lim_{n\to\infty} M_0(A_{k_n}x)$$

is right continuous and log-periodic with period $\tilde{c}^{1/\alpha}$.

Proof. Let us fix some $x > 0$. We claim that the sequence $M(k_n x)$ is Cauchy. Indeed, as $A_{k_{n+1}} = c^{1/\alpha^2} A_{k_n}$ we may use the previous proposition to see that

$$|M_0(A_{k_{n+1}}x) - M_0(A_{k_n}x)| \leq M_0(A_{k_n}x) \left| \frac{M_0(c^{1/\alpha^2} A_{k_n} x)}{M_0(A_{k_n}x)} - 1 \right| = O\left(\frac{\log(A_{k_n}x)^2}{A_{k_n}x} \right).$$

From this we obtain that

$$\begin{aligned}
|M_0(A_{k_{n+m}}x) - M_0(A_{k_n}x)| &= O\left(\sum_{j=0}^{m-1} \frac{(\log(A_{k_{n+j}}x))^2}{A_{k_{n+j}}x} \right) \\
&= O\left(\sum_{j=0}^{m-1} \frac{(\log(c^{j/\alpha^2} A_{k_n} x))^2}{c^{j/\alpha^2} A_{k_n} x} \right) \\
&= O\Bigg(\frac{(\log A_{k_n}x)^2}{A_{k_n}x} \sum_{j=0}^{m-1} \frac{1}{c^{j/\alpha^2}} + \frac{2 \log A_{k_n}x \log c^{1/\alpha^2}}{A_{k_n}x} \sum_{j=0}^{m-1} \frac{j}{c^{j/\alpha^2}} \\
&\qquad + \frac{(\log c^{1/\alpha^2})^2}{A_{k_n}x} \sum_{j=0}^{m-1} \frac{j^2}{c^{j/\alpha^2}} \Bigg) \\
&= O\left(\frac{(\log A_{k_n}x)^2}{A_{k_n}x} \right)
\end{aligned}$$

It is then clear that $\tilde{M}$ is log-periodic with period $\tilde{c}^{1/\alpha} = c^{1/\alpha^2}$ as

$$\tilde{M}(c^{1/\alpha^2}x) = \lim_{n\to\infty} M_0(A_{k_{n+1}}x) = \tilde{M}(x).$$

In order to see that $\tilde{M}$ is continuous we let $x > 0$ be arbitrary and let $\delta > 0$. Then we see that

$$\begin{aligned}
&\frac{1}{\lfloor c^{n/\alpha^2}(x+\delta) \rfloor} \sum_{j=1}^{\lfloor c^{n/\alpha^2}(x+\delta) \rfloor} M(x_j) - \frac{1}{\lfloor c^{n/\alpha^2}x \rfloor} \sum_{j=1}^{\lfloor c^{n/\alpha^2}x \rfloor} M(x_j) \\
&\quad = \left(\frac{\lfloor c^{n/\alpha^2}x \rfloor}{\lfloor c^{n/\alpha^2}(x+\delta) \rfloor} - 1 \right) \frac{1}{\lfloor c^{n/\alpha^2}x \rfloor} \sum_{j=1}^{\lfloor c^{n/\alpha^2}x \rfloor} M(x_j) + \frac{1}{\lfloor c^{n/\alpha^2}(x+\delta) \rfloor} \sum_{j=\lfloor c^{n/\alpha^2}x \rfloor}^{\lfloor c^{n/\alpha^2}(x+\delta) \rfloor} M(x_j),
\end{aligned}$$

and so

$$\tilde{M}(x+\delta)^{-1/\alpha} - \tilde{M}(x)^{-1/\alpha} = -\tilde{M}(x)\frac{\delta}{x+\delta} + O\left(\delta/x+\delta\right),$$

which yields the right continuity of $\tilde{M}$. □

We know that $F(x) \coloneqq m_Y(\tau > x) = x^{-\alpha}(M_0(x) + E_0(x))$. We may rewrite this as

$$F(x) = x^{-\alpha}(\tilde{M}(x) + H(x)),$$

where $H(x) \coloneqq M_0(x) - \tilde{M}(x) + E_0(x)$. We aim to show that F satisfies (1.9). We have already established that $\tilde{M}$ is log-periodic with period $\tilde{c}^{1/\alpha}$, right continuous, and bounded away from both 0 and ∞. It remains to show that the error function H satisfies $\lim_{n\to\infty} H(A_{k_n}x) = 0$ for each $x > 0$. Fixing $x > 0$ we see that

$$\lim_{n\to\infty} H(A_{k_n}x) = \lim_{n\to\infty} M_0(A_{k_n}x) - \lim_{m\to\infty} M_0(A_{k_n}A_{k_m}x) + E_0(A_{k_n}x) = \tilde{M}(x) - \tilde{M}(x) = 0,$$

as $A_{k_m}A_{k_n} = A_{k_{n+m}}$. This concludes that (2,b) of A4 holds.

4.2.2 Distortion properties and concluding the theorem

We now turn to establishing 2(a) of A4 before concluding the proof.

Proposition 4.2.4. *The induced map $T_Y : Y \to Y$ satisfies Adler's distortion condition*

$$\sup_{x\in Y} \frac{|T''(x)|}{T_Y^2(x)^2} < \infty.$$

Proof. Calculating the first and second derivatives of T we find

$$T'(x) = 1 + (\beta+1)x^{\beta}M(x) + x^{\beta+1}M'(x),$$

$$T''(x) = \beta(\beta+1)x^{\beta-1}M(x) + 2(\beta+1)x^{\beta}M'(x) + x^{\beta+1}M''(x).$$

We recall that $I_n = [x_{n+1}, x_n)$ and $J_n = \{\tau = n\}$, and $T(J_n) = I_{n-1}$. Note that $T^j(I_n) = I_{n-j+1}$ for $1 \le j \le n-1$. Since $M(x) = a(1 + \varepsilon p(\log x))$ and p is Lipschitz we know that $M(x) = O(1)$, $M'(x) = O(x^{-1})$. Moreover as we have assumed that the second derivative of p is bounded, we see that $M''(x) = O(x^{-2})$ and we obtain the following bounds for each $n \in \mathbb{N}$

$$T''|_{I_n} \le Bn^{\alpha-1}.$$

for some positive constant B independent of n. Let us put $C \coloneqq \max\{\sup|p(x)|, \sup|p'(x)|\}$, we then obtain that

$$\begin{aligned} T'|_{I_n}(x) &\ge 1 + a((\beta+1)(1-\varepsilon C) - \varepsilon C)x^{\beta} \\ &\ge 1 + a((\beta+1))(1-\varepsilon C) - \varepsilon C)x_{n+1}^{\beta} \end{aligned}$$

From Proposition 4.2.1 we know that

$$x_{n+1}^{\beta} = \frac{M_0(n+1)^{\beta}}{n+1}\left(1 + \frac{E_n(n)}{M_0(n)}\right),$$

and

$$M_0(n+1)^{\beta} = \left(\frac{1}{\alpha(n+1)}\sum_{j=1}^{n+1} a(1+\varepsilon p(\log(x_j)))\right)^{-1} \ge \frac{\alpha}{a}(1+\varepsilon C)^{-1}.$$

Combining these three estimates and using the fact that $E_0(n) = o(1)$ we know that there exists a constant $A > 0$ such that

$$T|_{I_n}(x) \ge 1 + An^{-1}. \tag{4.6}$$

Moreover, as $\alpha(\beta+1) = 1+\alpha > 2$, we may fix $\varepsilon > 0$ small enough so that $\alpha((\beta + 1))(1-\varepsilon C) - \varepsilon C)(1+\varepsilon C)^{-1} > 2$. Then for all n large enough we have that (4.6) holds for some $A > 2$.

Denoting by D the operator $T \mapsto T'$ we find, by a simple application of the chain rule that for any $n \geq 1$

$$\log(DT^n) = \sum_{j=1}^{n} \log(DT) \circ T^{j-1}.$$

Applying D to both sides of the above we obtain

$$\frac{D^2T^n}{DT^n} = \sum_{j=1}^{n} DT^{j-1} \left(\frac{D^2T}{DT}\right) \circ T^{j-1},$$

and so

$$\frac{|D^2T^n|}{(DT^n)^2} \leq \sum_{j=1}^{n} \left|\frac{DT^{j-1}}{DT^n}\right| \left|\left(\frac{D^2T}{DT}\right) \circ T^{j-1}\right| \tag{4.7}$$

Using the bounds found above for the derivatives of T we see that

$$\left|\left(\frac{D^2T}{DT}\right) \circ T^{j-1}(x)\right| \leq \left|\frac{\sup_{y\in I_{n-j+1}} D^2T(y)}{\inf_{y\in I_{n-j+1}} DT(y)}\right| \leq \frac{B(n-j+1)^{\alpha-1}}{1+\frac{A}{n-j+1}} \leq C'(n-j+1)^{\alpha-1}, \tag{4.8}$$

for some $C' > 0$ independent of n. We then calculate that

$$\begin{aligned}\log\left(\frac{DT^{j-1}}{DT^n}(x)\right) &= -\sum_{k=j}^{n} \log(DT) \circ T^{k-1}(x) \\ &\leq -\sum_{k=j}^{n} \log\left(1+\frac{A}{n-k+1}\right) \\ &\leq -\sum_{k=j}^{n} \frac{A}{n-k+1+A} \leq -A\log(n-j+2) + C'',\end{aligned}$$

for some $C'' > 0$ independent of n. Thus, for $x \in J_n$, we have that

$$\frac{DT^{j-1}}{DT^n}(x) \leq e^{C''} \frac{1}{(n-j+2)^{-A}}. \tag{4.9}$$

Combining (4.8) and (4.9)

$$\frac{|D^2T^n|}{(DT^n)^2}(x) \leq C'e^{C''} \sum_{j=1}^{n} \frac{(n-j+2)^{\alpha-1}}{(n-j+2)^{A}}. = C'e^{C''} \sum_{k=2}^{n+1} n^{\alpha-1-A}$$

From our comments following (4.6) we know that there exists an $A > 2$ such that (4.6) holds for all but finitely many n, so we obtain

$$\frac{|D^2T^n|}{(DT^n)^2}(x) \leq= C'e^{C''}\left(C''' + \sum_{k=2}^{\infty} k^{\alpha-3}\right) < \infty,$$

concluding the proof. □

We may now conclude the proof of Theorem B.

Proof of Theorem B. We have seen that T satisfies the A4. Now we may apply Lemma 3.0.3 to the observable u to see that u_Y satisfies A5 and then we can apply Theorem 3.0.4 to conclude. □

4.3 Proof of Theorem C

Throughout this section we let $T \equiv T_{LSV}$ and let $u(x) = M(x)$ where M is as defined in (2.1). As we have mentioned before we know already from Remark 3.0.1 that T satisfies A4. We will show in this section that u satisfies A5 and use Theorem 3.0.4 in order to conclude Theorem C. We begin by studying the tail distribution of the induced observable u_Y.

4.3.1 Calculating the tail distribution of the induced observable

In this section we aim to show that the induced observable satisfies (2) of A5. Let us first remark that we cannot apply Lemma 3.0.3 to u, in particular we note that u is not Hölder for any $\nu \in (0, 1]$. To see this fix $x, y \in (0, 1]$ with $|u(x) - u(y)| = C > 0$. Then for any $\nu \in (0, 1]$ and any $k \in \mathbb{N}$ we have that

$$\frac{|u(c^{-k/\alpha}x) - u(c^{-k/\alpha}y)|}{|c^{-k/\alpha}x - c^{-k/\alpha}y|} = c^{\nu k/\alpha}\frac{C}{|x-y|}.$$

As the above converges to ∞ as $k \to \infty$ we see that u is not ν-Hölder.

In what follows we define M_0 for $x \geq 1$ by setting $M_0(x) := \frac{1}{x}\int_0^x u\left(\frac{1}{2}\alpha^\alpha z^{-\alpha}\right) dz$. We note that M_0 is log-periodic with period c^{1/α^2}:

$$M_0(c^{1/\alpha^2}x) = \frac{1}{c^{1/\alpha^2}x}\int_0^{c^{1/\alpha^2}x} u\left(\frac{1}{2}\alpha^\alpha z^{-\alpha}\right) dz = M_0(x),$$

where in the final equality we make the substitution $z = c^{1/\alpha^2}w$.

Our first step towards verifying A5 is to establish the following Lemma.

Lemma 4.3.1. *For any $x \in J_n$ we have*

$$u_Y(x) = n(M_0(n) + O((\log(n))^2/n)).$$

Before proving this Lemma we give a series of intermediate results.

For $x \in Y$ let us denote by $(y_j = y_j(x))_{j\geq 0}$ the orbit of x under T. For $x \in \{\tau = n\} = J_n$ we have by definition that

$$u_Y(x) = \sum_{j=0}^{n-1} u(y_j).$$

Lemma 4.3.2. *If $y \in I_n$ then $|u(x_{n+1}) - u(y)| = O(n^{-1})$.*

Proof. First we calculate $|\log(x_n) - \log(y)|$. As $y \in [x_{n+1}, x_n)$ we may write $y = x_{n+1} + t(x_n - x_{n+1})$ for some $t \in [0,1)$. Thus

$$\log(x_{n+1}) - \log(y) = -\log\left(1 + t\left(\frac{x_n}{x_{n+1}} - 1\right)\right).$$

We recall from Section 1.2.4 that $x_n = \frac{1}{2}\alpha^\alpha n^{-\alpha}(1 + O(\frac{\log n}{n}))$ and we also recall from

[LSV99] that $|x_n - x_{n+1}| = O(n^{-(1+\alpha)})$. Since $\frac{x_n}{x_{n+1}} = 1 + \frac{1}{x_{n+1}}(x_n - x_{n+1})$ we have that

$$\log(x_{n+1}) - \log(y) = \log(1 + O(n^{-1})) = O(n^{-1}).$$

Then, as p is Lipschitz by assumption, we have that

$$|u(x_{n+1}) - u(y)| = O(n^{-1}),$$

as required. □

Using Lemma 4.3.2 we may write

$$u_Y(x) = \sum_{j=1}^{n} u(x_j) + O(j^{-1}) = \sum_{j=1}^{n} u(x_j) + O(\log(n)). \tag{4.10}$$

For $x > 1$ let $\overline{M}_0(x) := \frac{1}{x}\int_1^x u\left(\frac{1}{2}\alpha^\alpha z^{-\alpha}\right) dz$ and note that

$$\overline{M}_0(x) = M_0(x) + O(1/x). \tag{4.11}$$

Let us put $\tilde{M}_0(x) = \frac{1}{\lfloor x \rfloor}\sum_{j=1}^{\lfloor x \rfloor} u(x_j)$. We now wish to compare $\tilde{M}_0$ and $\overline{M}_0$.

Lemma 4.3.3.

$$|\overline{M}_0(x) - \tilde{M}_0(x)| = O\left(\frac{(\log x)^2}{x}\right)$$

Proof. First let us calculate the difference between $u(x_n)$ and $u(\frac{1}{2}\alpha^\alpha n^{-\alpha})$. Writing $x_n = \frac{1}{2}\alpha^\alpha n^{-\alpha}(1 + e_n)$ where $e_n = o(1)$ we have that $|\log(x_n) - \log(\frac{1}{2}\alpha^\alpha n^{-\alpha})| = O(e_n)$. So we find that

$$\left| u(x_n) - u\left(\frac{1}{2}\alpha^\alpha n^{-\alpha}\right) \right| = O(e_n).$$

Letting $h(x) := u(\frac{1}{2}\alpha^\alpha x^{-\alpha})$ for $x > 1$ know that h is log-periodic with period c^{1/α^2}. Moreover h has finitely many intervals of monotonicity in $[c^{1/\alpha^2}, c^{2/\alpha^2}]$. Thus h has

$O(\log x)$ intervals of monotonicity in $[1, x]$ and so as in the case of T_w in the previous section we have that

$$\overline{M}_0(x) = \frac{1}{x}\int_1^x h(u)du = \frac{1}{x}\left(\sum_{j=1}^{\lfloor x\rfloor} h(j) + O(\log(x))\right).$$

Thus

$$\overline{M}_0(x) - \tilde{M}_0(x) = \frac{1}{x}\sum_{j=1}^{\lfloor x\rfloor} u\left(\frac{1}{2}\alpha^{\alpha} j^{-\alpha}\right) - u(x_j) + O\left(\frac{\log x}{x}\right) = \frac{1}{x}\sum_{j=1}^{\lfloor x\rfloor} O(e_j) + O\left(\frac{\log x}{x}\right).$$

Since $e_n = O(\log(n)/n)$ we obtain that

$$|\overline{M}_0(x) - \tilde{M}_0(x)| = O\left(\frac{\log(x)}{x}\right),$$

as required. □

Proof of Lemma 4.3.1. From (4.10) we know that

$$u_Y|_{J_n} = n(\tilde{M}_0(n) + O(\log(n)/n)),$$

thus using Lemma 4.3.3 and (4.11)

$$u_Y|_{J_n}(x) = n(M_0(n) + O(\log(n)^2/n)), \tag{4.12}$$

which concludes the proof. □

Since the map $x \mapsto xM_0(x)$ is continuous and strictly increasing (at least for $\varepsilon > 0$ sufficiently small) we know that it is invertible. Let g be the inverse of $x \mapsto xM_0(x)$ so that if $xM_0(x) = y$ then $g(y) = x$. If $xM_0(x) = y$ then $c^{1/\alpha^2}xM_0(c^{1/\alpha^2}x) = c^{1/\alpha^2}y$ and thus we obtain $g(c^{1/\alpha^2}y) = c^{1/\alpha^2}g(y)$. Writing $g(y) = yf(y)$ we then see that f must be log-periodic with period c^{1/α^2}, and also f must be bounded as $f(y) = 1/M_0(g(y))$.

Then we may compute that

$$m_Y(u_Y > y) \sim m_Y(\tau > g(y)) = \frac{1}{2}\alpha^\alpha g(y)^{-\alpha}(1+o(1)) = \frac{1}{2}\alpha^\alpha y^{-\alpha} f(y)^{-\alpha}(1+o(1)).$$

This concludes that u_Y satisfies (2) of A5.

4.3.2 Concluding the proof

In the previous section we established (2) of A5. In this section we will verify the remaining assumptions in A5 and then apply Theorem 3.0.4 to conclude.

Proof of Theorem C . As in the proof of Lemma — we begin by noting that as the induced map is uniformly expanding: $\lambda := \inf_{x\in Y} T_Y'(x)$, we know that $|x-y| \leq C\lambda^{-s(x,y)}$ where s is the separation time under T_Y for some $C>0$. Letting $x, y \in J_n$ with $s(x,y)=k$ we have $|T^j x - T^j y| \leq C\lambda^{-k}$.

Thus we see that for any $0 \leq j \leq n-1$

$$\log(T^j x) - \log(T^j y) = O(|T^j x - T^j y|/T^j y) = O(\lambda^{-k}(n-j)^\alpha),$$

and so

$$\begin{aligned} |u_Y(x) - u_Y(y)| &\leq \sum_{j=0}^{n-1} |u \circ T^j(x) - u \circ T^j(y)| = a\varepsilon \sum_{j=0}^{n-1} |p(\log T^j(x)) - p(\log T^j(y))| \\ &\leq C'\lambda^{-k} n^{1+\alpha}, \end{aligned}$$

for some $C' > 0$. Taking $\theta = \lambda^{-1}$ we obtain

$$D_\theta u_Y(J_n) \leq C' n^{1+\alpha}.$$

As u is bounded we also have trivially that for $x \in J_n$

$$|u_Y(x)| \leq Bn.$$

Let us now take $\delta \in (0,1)$ close enough to 0 so that $(\alpha+1)(\delta-1) < -1$. Then we see that since $m_Y(J_n) \sim n^{-(1+\alpha)}$ we have

$$\sum_{n=1}^{\infty} \|u_Y|_{J_n}\|_\infty^\delta m_Y(J_n) \leq B \sum_{n=1}^{\infty} n^\delta n^{-1-\alpha} < \infty,$$

and

$$\sum_{n=1}^{\infty} D_\theta u_Y(J_n)^\delta m_Y(J_n) \leq B \sum_{n=1}^{\infty} n^{(1+\alpha)\delta} n^{-1-\alpha} < \infty,$$

by our choice of δ. This concludes that u_Y satisfies A5. We then conclude the proof by an application of Theorem 3.0.4. □

Part III

Appendix

4.4 Some properties of distribution functions in $\mathbb{D}_{gp}(G)$

Proposition 4.4.1. *Let G be the distribution function of a non-negative semi-stable random variable of index $\alpha \in (0,2)$ and period $c > 1$. Let $F \in \mathbb{D}_{gp}(G_\alpha, k_n, A_n)$ satisfying A2 be given by*

$$1 - F(x) = x^{-\alpha}(M(\delta(x)) + h(x)).$$

Suppose that F is continuous and suppose that $1 - F$ is strictly decreasing, then the sequence $x_n := F(n)$ satisfies

$$\lim_{n\to\infty} \frac{x_{n-1}}{x_n} = 1.$$

Proof. As we are assuming that M is continuous we know that $h(x) \to 0$ as $x \to \infty$ and thus if

$$\lim_{n\to\infty} \frac{M(\delta(n-1))}{M_R(\delta(n))} = 1,$$

we can conclude that

$$\lim_{n\to\infty} \frac{x_{n-1}}{x_n} = 1,$$

Let us restrict the domain of M to $[1, c^{1/\alpha} + \varepsilon]$ for some $\varepsilon > 0$ and let us note that as M_R is continuous on this compact set it must also be uniformly continuous on $[1, c^{1/\alpha} + \varepsilon]$. In the remainder of the proof we will assume that n is large enough so that $\delta(n) \in [1, c^{1/\alpha} + \varepsilon]$.

With the function a as in A2 we note that for each n large enough we also have that $a(n-1) \in \{a(n), a(n)c^{-1/\alpha}(1 + \varepsilon(n))\}$ where $\varepsilon(n) \to 0$ as $n \to \infty$. Let separate n into two disjoint subsequences n' and n'' so that $a(n'-1) = a(n')$ and $a(n''-1) = a(n'')c^{-1/\alpha}(1 + \varepsilon(n''))$. Then

$$\frac{M(\delta(n'-1))}{M(\delta(n'))} = \frac{M(\delta(n')(1 - 1/a(n'))}{M(\delta(n'))} \to 1,$$

in the limit as $n' \to \infty$ by the uniform continuity of M and

$$\frac{M(\delta(n''-1))}{M(\delta(n''))} = \frac{M(\delta(n'')(1+\varepsilon(n)))}{M(\delta(n''))} \to 1$$

in the limit as $n'' \to \infty$ concluding the proof. □

Proposition 4.4.2. *Let $F \in \mathbb{D}_{gp}(G_\alpha, k_n, A_{k_n})$ for a non-stable semi-stable distribution G_α so that F is given by (1.9). Then for any continuity point x_0 of M we have the following*

1. $\lim_{n\to\infty} M(\delta(A_{k_n}x_0)) = \lim_{n\to\infty} M(\delta(A_{k_n}x_0(1+o(1))) = M(x_0)$.

2. $\lim_{n\to\infty} h(A_{k_n}x_0(1+o(1))) = 0$.

Proof. Let x_0 be a continuity point of M.

1. First let us suppose that $x_0 \in [1, c^{1/\alpha})$ then clearly $M(\delta(A_{k_n}x_0)) = M(x_0)$ for every $n \in \mathbb{N}$ large enough. Now suppose that x_0 is an arbitrary continuity point of M and let us write $x_0 = c^{p/\alpha}x_0'$ where $p \in \mathbb{Z}$ and $x_0'' \in [1, c^{1/\alpha})$. Then using the fact that $A_{k_{n+1}}/A_{k_n} \to c^{1/\alpha}$ we may deduce that for in the limit as $n \to \infty$

$$A_{k_n}x_0 = A_{k_{n+p}}x_0'(1+o(1)).$$

For large n we deduce that

$$a(A_{k_n}x_0) \in \{A_{k_{n+p-1}}, A_{k_{n+p}}, A_{k_{n+p+1}}\},$$

and whence the set of limit points of the sequence

$$\delta(A_{k_n}x_0)$$

is certainly contained in

$$\{c^{p-1/\alpha}x_0', c^{p/\alpha}x_0', c^{p+1/\alpha}x_0'\},$$

yielding, by the log-periodicity of M, that

$$\lim_{n\to\infty} M(\delta(A_{k_n}x_0)) = M(x_0') = M(x_0).$$

An almost identical argument establishes that

$$\lim_{n\to\infty} M(\delta(A_{k_n}(x_0 + o(1))) = M(x_0).$$

2. First let us suppose that x_n converges to x_0 from above. Then by part 1 above we know that

$$\lim_{n\to\infty} M(\delta(A_{k_n}x_n)) = M(x_0),$$

and so

$$\frac{\bar{F}(A_{k_n}x_n)}{\ell^*(A_{k_n}x_n)(A_{k_n}x_n)^{-\alpha}} - h(A_{k_n}x_n) = M(x_0) + o(1).$$

We also know that

$$M(x_0) = \lim_{n\to\infty} \frac{\bar{F}(A_{k_n}x_0)}{\ell^*(A_{k_n}x_0)(A_{k_n}x_0)^{-\alpha}}.$$

Thus

$$h(A_{k_n}x_n) = \frac{(A_{k_n}x_n)^\alpha \bar{F}(A_{k_n}x_n)}{\ell^*(A_{k_n}x_n)} - \frac{(A_{k_n}x_0)^\alpha \bar{F}(A_{k_n}x_0)}{\ell^*(A_{k_n}x_0)} + o(1) \tag{4.13}$$

$$\leq \frac{[(A_{k_n}x_n)^\alpha \ell^*(A_{k_n}x_0) - (A_{k_n}x_0)^\alpha \ell^*(A_{k_n}x_n)]}{\ell^*(A_{k_n}x_n)\ell^*(A_{k_n}x_0)} \bar{F}(A_{k_n}x_0) + o(1), \tag{4.14}$$

$$= \left[\frac{\ell^*(A_{k_n}x_0)}{\ell^*(A_{k_n}x_n)}(A_{k_n}x_n)^\alpha - \frac{\ell^*(A_{k_n}x_n)}{\ell^*(A_{k_n}x_0)}(A_{k_n}x_0)\right] \bar{F}(A_{k_n}x_0) + o(1) \tag{4.15}$$

$$= \left[A_{k_n}^\alpha (x_n - x_0)(1 + o(1))\right] \bar{F}(A_{k_n}x_0) \underset{n\to\infty}{\to} 0, \tag{4.16}$$

where in the second line we have used the monotonicity of $\bar{F}$ and in the final line we have employed Potter's Theorem to conclude that

$$\lim_{n\to\infty} \frac{\ell^*(A_{k_n}x_n)}{\ell^*(A_{k_n}x_0)} = 1.$$

Proceeding in the same way one obtains a lower bound $h(A_{k_n}x_0)$. Finally assuming that x_n converges to x_0 from below we obtain the same bounds as above but reversed which concludes the proof.

□

4.5 Going from continuous to discrete tails

Lemma 4.5.1. *Let G_α be a non-stable semi-stable distribution and let $F \in \mathbb{D}_{gp}(G_\alpha, k_n, A_{k_n})$ be given by (1.9). Suppose now that X is a non-negative integer valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose distribution satisfies*

$$\mathbb{P}(X > n) = \bar{F}(n),$$

for every n sufficiently large. Then the distribution function F_X of X is in $\mathbb{D}_{gp}(G_\alpha, k_n, A_{k_n})$.

Proof. We show that $F_X(x) = \mathbb{P}(X \leq x)$ satisfies the criteria of Corollary 3 in [Meg00] in order to conclude that $F_X \in \mathbb{G}_{gp}(G_\alpha, k_n, A_{k_n})$.

By our hypothesis for each n large enough we have

$$1 - F_X(x) = 1 - F(\lfloor x \rfloor) \tag{4.17}$$

$$= x^{-\alpha}\tilde{\ell}^*(x)(M(\delta(x)) + \tilde{h}(x)), \tag{4.18}$$

where in the final line of the above we have set

$$\tilde{h}(x) := M(\delta(\lfloor x \rfloor)) - M(\delta(x)) + h(\lfloor x \rfloor),$$

and

$$\tilde{\ell}^*(x) := \left(\frac{\lfloor x \rfloor}{x}\right)^{-\alpha} \ell^*(\lfloor x \rfloor).$$

We then know that it is sufficient to check that $\tilde{l}^*$ is slowly varying at ∞ and that for every continuity point x_0 of M

$$\lim_{n\to\infty} \tilde{h}(A_{k_n} x_0) = 0.$$

Indeed, whenever x_0 is a continuity point of M we have

$$M(A_{k_n}x_0) = M\left(\delta\left(A_{k_n}x_0\left(1-\frac{\{A_{k_n}x_0\}}{A_{k_n}x_0}\right)\right)\right) - M(\delta(A_{k_n}x_0)) + h\left(A_{k_n}x_0\left(1-\frac{\{A_{k_n}x_0\}}{A_{k_n}x_0}\right)\right)$$

which converges to 0 in the limit as $n \to \infty$ by an application of Proposition 4.4.2.

By definition $\tilde{\ell}^*$ is right continuous, to see that $\tilde{\ell}^*(x)$ is slowly varying at ∞ we note that it is sufficient to show that

$$\lim_{x\to\infty} \frac{\ell^*(x)}{\ell^*(\lfloor x \rfloor)} = 1,$$

which holds true by Potter's Theorem. □

4.6 Sums of two log period functions

Here we give the proof of Lemma 1.1.13.

First we prove the following.

Lemma 4.6.1. *Let $M : \mathbb{R}_{>0} \to \mathbb{R}$ be a right continuous function and let $a, b > 0$ be two numbers where b is not a rational power of a. Suppose that $M(ax) = M(x)$ and $M(bx) = M(x)$ for all x. Then the M is a constant function.*

Proof. Suppose that M is log-periodic with periods a, b, and suppose moreover that b is not a rational power of a. Then $\log_a(b) \in \mathbb{R} \setminus \mathbb{Q}$. Let $\psi_a = \log_a \mod 1$, then ψ_a is a bijection $\psi_a : \mathbb{R}/a^{\mathbb{Z}} \to \mathbb{R}/\mathbb{Z}$ and so we can write

$$M(x) = \tilde{M} \circ \psi_a(x),$$

where $\tilde{M}$ satisfies

$$\tilde{M}(x) = \tilde{M}(x+1),$$

for every $x > 0$. By assumption for every integer n,

$$M(x) = M(b^n x) = \tilde{M} \circ \psi_a(b^n x).$$

Since $\log_a(b)$ is irrational we have that for every x the set $\{\psi_a(b^n x) = n \log_a b + \psi_a(x) \mod 1\}_{n \in \mathbb{Z}}$ is dense in $[0, 1]$ - irrational circle rotations have dense orbits. Thus for every $x > 0$ there is a dense set E_x on which M is constant. Letting $y \in [0, 1)$ be arbitrary we may find for a given x a sequence n_r of integers such that $\lim_{r \to \infty} \psi_a(b^{n_r} x) = y$ and moreover may take $\psi_a(b^{n_r} x)$ to be monotone decreasing. Then by right continuity we find that M must indeed be constant on all of $\mathbb{R}_{>0}$. □

Proof of Lemma 1.1.13. If b is a rational power of a, i.e. $a^{p/q} = b$ then letting $c = b^q = a^p$ we see that $M(cx) = M(x)$ for every x.

Now suppose that b is not a rational power of a and suppose that there is some c such that $M(cx) = M(x)$ for all x. Then we have that

$$M_1(cx) - M_1(x) = M_2(x) - M_2(cx).$$

The function $x \mapsto M_1(cx) - M_1(x)$ is log-periodic with period a and the function $x \mapsto M_2(x) - M_2(cx)$ is log-periodic with period b. Thus

$$H(x) = M_1(cx) - M_1(x) = M_2(x) - M_2(cx),$$

satisfies $H(x) = H(ax) = H(bx)$ for every x and so by Lemma 4.6.1 must be constant. □

4.7 Integrating regularly varying functions with index less than −1

Proposition 4.7.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be regularly varying at ∞ with index $p < -1$ so that $f(x) = x^p L(x)$ with $L : \mathbb{R} \to \mathbb{R}$ slowly varying. Then*

$$\int_x^\infty f(t) = O(x^{p+1} L(x)),$$

as $x \to \infty$.

Proof. Let $A > 1$ and let $\delta > 0$ be small enough so that $p + \delta < -1$. Then by Potter's bounds we know that there exists $x_0 > 0$ such that

$$\frac{L(y)}{L(x)} \leq A \max\left\{ \left(\frac{y}{x}\right)^\delta, \left(\frac{x}{y}\right)^\delta \right\},$$

for every $x, y > x_0$. Then we may calculate that

$$
\begin{aligned}
(x^{p+1}L(x))^{-1}\int_x^\infty f(t)dt &= x^{-1}\int_x^\infty \left(\frac{t}{x}\right)^p \frac{L(t)}{L(x)}dt \\
&= \int_1^\infty z^p \frac{L(zx)}{L(x)}dt \\
&\leq A\int_1^\infty t^{p+\delta} = const,
\end{aligned}
$$

where in the second line we make the substitution $z = t/x$. □

4.8 An estimate on $|e^{ia} - 1|$

Proposition 4.8.1. *For any $\delta \in (0,1)$ there exists $C > 1$ such that*

$$|e^{ia} - 1| \leq C|a|^\delta.$$

Proof. Fix $\delta \in (0,1)$ and let $C > 1$ be such that

$$|e^{ia} - 1| \leq C|a|^\delta,$$

for each $a \notin (1,-1)$. Then for each $a \in (1,-1)$ we note that $|e^{ia} - 1| \leq |a|$ as the arc of the circle spanned by the angle a is of length $|a|$ and this is clearly larger than $|e^{ia} - 1|$ which allows us to conclude that $|e^{ia} - 1| \leq C|a|^\delta$. □

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