# Dirichlet $L$-functions and their Derivatives in Function Fields

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I certify that all material in this thesis that is not my own work has been identified and that any material that has previously been submitted and approved for the award of a degree by this or any other university has been acknowledged.

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# Abstract


This thesis is predominantly based on four projects that focus on moments of Dirichlet $L$-functions in the function field setting. That is, the $L$-functions are, in the appropriate region of convergence, defined as sums over monic polynomials in $\mathcal{A} := \mathbb{F}_q[T]$, the polynomial ring over the finite field of order $q$ for some fixed odd prime power $q$.

For the first project, we obtain the main term of $\frac{1}{\phi^*(R)} \sum^*_{\chi \bmod R} |L(1/2, \chi)|^{2k}$ as $\deg R \longrightarrow \infty$, for $k = 1, 2$. Here, $R$ is a monic polynomial in $\mathcal{A}$, the sum is over all primitive Dirichlet characters of modulus $R$, $\phi^*(R)$ is the number of primitive characters of modulus $R$, and $L(s, \chi)$ is the Dirichlet $L$-function associated to the character $\chi$. This is the function field analogue of Soundararajan's result on the fourth moment of Dirichlet $L$-functions in the number field setting, and it extends upon the work of Tamam in the function field setting. Our proofs require us to obtain results on correlations of the divisor function. This, in turn, requires the function field analogue of Shiu's generalised Brun-Titchmarsh theorem, for the special case of the divisor function. Therefore, we also explore the Selberg sieve for function fields.

For the second project, we obtain, for any non-negative integers $l_1, l_2$, the main term of $\frac{1}{\phi^*(Q)} \sum^*_{\chi \bmod Q} |L^{(l_1)}(1/2, \chi)|^2 |L^{(l_2)}(1/2, \chi)|^2$ as $\deg Q$ tends to infinity over the prime polynomials. Here, $L^{(l)}(s, \chi)$ is the $l$-th derivative of $L(s, \chi)$. Typically with such results, one makes use of the functional equation for Dirichlet $L$-functions, but here this is more difficult due to the derivatives. Furthermore, the derivatives introduce non-multiplicative factors that complicate the computations. Our method of addressing this is only applicable to the function field setting, although there are likely other, more complicated approaches one could explore for the number field analogue of the problem. We also obtain similar results involving derivatives for the first and second moments.

For the third project we conjecture, for any non-negative integer $k$, the main term of $\frac{1}{\phi^*(R)} \sum^*_{\chi \bmod R} |L(1/2, \chi)|^{2k}$ as $\deg R \longrightarrow \infty$. We do this by expressing each $L$-function as a hybrid Euler-Hadamard product: $L(s, \chi) = P_X(s, \chi) Z_X(s, \chi)$. The first factor, $P_X(s, \chi)$, resembles a partial product over the primes, and the second factor, $Z_X(s, \chi)$, resembles a partial product over the zeros of the $L$-function. We conjecture that

$$\frac{1}{\phi^*(R)} \sum^*_{\chi \bmod R} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \sim \left( \frac{1}{\phi^*(R)} \sum^*_{\chi \bmod R} \left| P_X\left(\frac{1}{2}, \chi\right) \right|^{2k} \right) \left( \frac{1}{\phi^*(R)} \sum^*_{\chi \bmod R} \left| Z_X\left(\frac{1}{2}, \chi\right) \right|^{2k} \right).$$

We call this the splitting conjecture and, for $k = 1, 2$, we prove it. Also, for all

non-negative integers $k$, we obtain the main term of $\frac{1}{\phi^*(R)}\sum^*_{\chi \bmod R}|P_X(1/2,\chi)|^{2k}$, while for $\frac{1}{\phi^*(R)}\sum^*_{\chi \bmod R}|Z_X(1/2,\chi)|^{2k}$ we use a random matrix theory model to conjecture the main term. These, along with the splitting conjecture, immediately give a conjecture for the main term of the $2k$-th moment of the $L$-functions. This project is based on the work on Gonek, Hughes, and Keating who undertook the above for moments of the Riemann-zeta function in the number field setting. This was extended to moments of Dirichlet $L$-functions by Bui and Keating, and our work is the function field analogue of this.

For the fourth project we conjecture, for any non-negative integer $k$, the main term of $\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}|L'(1/2,\chi)|^{2k}$ as $\deg Q$ tends to infinity over the prime polynomials. We do this by differentiating the hybrid Euler-Hadamard product formula: $L'(s,\chi) = P'_X(s,\chi)Z_X(s,\chi) + P_X(s,\chi)Z'_X(s,\chi)$. We then make the following splitting conjecture for the first derivative:

$$\begin{aligned}
&\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}\left|P'_X\Big(\frac{1}{2},\chi\Big)Z_X\Big(\frac{1}{2},\chi\Big)\right|^{2k}\\
&\qquad\qquad \sim \bigg(\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}\left|P'_X\Big(\frac{1}{2},\chi\Big)\right|^{2k}\bigg)\bigg(\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}\left|Z_X\Big(\frac{1}{2},\chi\Big)\right|^{2k}\bigg);\\
&\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}\left|P_X\Big(\frac{1}{2},\chi\Big)Z'_X\Big(\frac{1}{2},\chi\Big)\right|^{2k}\\
&\qquad\qquad \sim \bigg(\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}\left|P_X\Big(\frac{1}{2},\chi\Big)\right|^{2k}\bigg)\bigg(\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}\left|Z'_X\Big(\frac{1}{2},\chi\Big)\right|^{2k}\bigg).
\end{aligned}$$

We prove the splitting conjecture for the case $k = 1$. Also, for all non-negative integers $k$, we obtain the main term of $\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}|P'_X(1/2,\chi)|^{2k}$, while for $\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}|Z'_X(1/2,\chi)|^{2k}$ we use a random matrix theory model to conjecture the main term. These, along with the splitting conjecture for the first derivative and results established in the third project, give a conjecture for the main term of $\frac{1}{\phi^*(Q)}\sum^*_{\chi \bmod Q}|L'(1/2,\chi)|^{2k}$. While the hybrid Euler-Hadamard formula approach has been applied to moments of quadratic Dirichlet $L$-functions in the function field setting (by Bui and Florea), as well as having been applied to discrete moments of the derivative of the Riemann zeta-function in the number field setting (by Bui, Gonek, and Milinovich), this project is the first that we are aware of that uses this approach for moments of derivatives of $L$-functions at the central value (in either the number field or function field setting).

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# Contents

# Author's Declaration

As per the University of Exeter's statement of procedures, the following is a statement of the nature and extent of the my individual contribution to this thesis.

Chapters 3 and 4 consist of research that has been published in [AY20] under the authorship of myself and my supervisor. This work was primarily carried out myself, with my supervisor suggesting and discussing the problem; directing me to, and discussing, related articles that included methods relevant to the project; and making notational and presentational remarks. The decision to include the Brun-Titchmarsh theorem for the divisor function (Chapter 3) and its consequences, as well as Theorem 2.2.2, was my own.

Chapter 5 consists of research that has been submitted for publication (ArXiv: [AY19]) under the authorship of myself and my supervisor. This work was primarily carried out myself, with my supervisor suggesting and discussing the problem; directing me to, and discussing, related articles that included methods relevant to the project; and making notational and presentational remarks. The decision to extend the fourth moment result to mixed arbitrary derivatives, was my own.

Chapter 6 consists of research that is intended to be published in the future under the sole authorship of myself. My supervisor suggested and discussed the problem that I would work on, before I began work on the project almost independently as I had gained the necessary experience to do so.

Chapter 7 consists of research that is intended to be published in the future under the sole authorship of myself. The problems that are addressed in this chapter were developed on my own initiative and were carried out independently.

Some of the content in Chapters 1 and 2, and Appendix A appear in [AY20] and [AY19].

# Notation

In this thesis, most notation will be introduced in Chapter 1; or, if used in only one chapter or section, it will be introduced there. Nonetheless, there are some notational introductions for which it is more natural to make them here.

- We denote the set of positive, negative, non-negative, and non-positive integers by $\mathbb{Z}_{>0}$, $\mathbb{Z}_{<0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{\leq 0}$, respectively.
- Throughout this thesis, except sections Sections 1.2, 1.3, and 1.5, $q = p^n$ for some odd prime integer $p$ and some positive integer $n$. Our results hold for all $q$ in this range. In Sections 1.2, 1.3, and 1.5, as we describe there, we take $q$ to be a positive integer. This is because we mention results from the number field setting where $q$ typically has a different definition than its typical definition in the function field setting.
- We define $\mathbb{F}_q$ to be the finite field of order $q$, and ${\mathbb{F}_q}^*$ to be its multiplicative group. Furthermore, we define $\mathcal{A} := \mathbb{F}_q[T]$, the polynomial ring over the finite field of order $q$. $\mathcal{M}$ is the subset of $\mathcal{A}$ consisting of all monic polynomials. For $\mathcal{T} \subset \mathcal{A}$, $n \geq 0$ an integer, and $B \in \mathcal{A}$, we define

  $$\mathcal{T}_n := \{A \in \mathcal{T} : \deg A = n\}$$

  and

  $$B\mathcal{T} := \{AB : A \in \mathcal{T}\}.$$

- For $A \in \mathcal{A} \backslash \{0\}$ we define $|A| := q^{\deg A}$, and we define $|0| := 0$. Let $n \geq 0$. Then, the range $\deg A \leq n$ is not taken to include $A = 0$.
- As $\mathcal{A}$ is a Euclidean domain, we have that primality and irreducibility are equivalent, and we have unique factorisation. Unless otherwise stated, a prime in $\mathcal{A}$ is always taken to be monic. The letters $P, Q$ are reserved for prime polynomials, and are to be taken as such even when not explicitly stated. The set of all monic primes is denoted by $\mathcal{P}$.
- For non-negative integers $n$, we define $\mathcal{S}(n) := \{A \in \mathcal{A} : P \mid A \Rightarrow \deg P \leq n\}$ and $\mathcal{S}_{\mathcal{M}}(n) := \{A \in \mathcal{S}(n) : A \text{ is monic}\}$.
- For all positive integers $a \neq 1$ we define $p_+(a)$ to be the largest (positive) prime divisor of $a$ and $p_-(a)$ to be the smallest (positive) prime divisor of $a$. For $A \in \mathcal{M} \backslash \{1\}$, we define $p_+(A)$ to be the largest integer such that $A$ has a prime divisor of degree $p_+(A)$, and we define $p_-(A)$ to be the smallest integer such that $A$ has a prime divisor of degree $p_-(A)$.

- For integers $a, b$ we denote the highest common factor and the lowest common multiple by $(a, b)$ and $[a, b]$, respectively (of course, both $(a, b)$ and $[a, b]$ are taken to be positive). Similarly, for $A, B \in \mathcal{A}$, we define $(A, B)$ and $[A, B]$ to the highest common factor and lowest common multiple of $A, B$, respectively. Here, "highest" and "lowest" are with regards to the degree of a polynomial, and both $(A, B)$ and $[A, B]$ are taken to be monic.

- Suppose we have $a \in \mathbb{Z}$. When we write $\sum_{e|a}$, we are summing over all *positive* divisors $e$ of $a$. When we write $\sum_{ef=a}$, we are summing over all pairs $(e, f)$ such that $e, f$ are *positive* and $ef = a$. Similar restrictions hold over products, etc. (as opposed to sums). While these definitions apply to any $a \in \mathbb{Z}$, it will typically be the case that $a$ is positive. Similar restrictions hold for $A \in \mathcal{A}$ (as opposed to $a \in \mathbb{Z}$), but we require that the divisors are monic (as opposed to positive). While these definitions hold for any $A \in \mathcal{A}$, it will typically be the case that $A$ is monic.

- Let $a$ be a non-zero integer. We say $a$ is square-free if it is not divisible by $p^2$ for any prime integer $p$. We say that it is square-full if $p^2 \mid a$ for all primes $p \mid a$. We define the radical of $a$, denoted by $\mathrm{rad}(a)$, to be the largest positive square-free divisor of $a$. We have similar definitions for elements in $\mathcal{A}$, but we replace "positive" with "monic".

- The multiplicative functions $\omega, \Omega, d_k, \phi, \mu$ on the non-zero integers are taken to have their standard definitions: The number of (positive) prime divisors (without multiplicity), the number of (positive) prime divisors (with multiplicity), the number of ways of expressing an integer as a product of $k$ integers (not counting any multiplication by units), the Euler totient function, and the Möbius function, respectively. One can easily see the analogous definitions for $\mathcal{A}$ (replace "positive" with "monic" where necessary). We only note that for $a \in \mathcal{A}_0 = {\mathbb{F}_q}^*$ we define $\phi(a) = 1$, and for $R \in \mathcal{A}$ with $\deg R \geq 1$ we have

$$\begin{aligned}\phi(R) :=& \left| \left\{ A \in \mathcal{A} : \deg A < \deg R \text{ and } (A, R) = 1 \right\} \right| \\ =& |A| \prod_{P|R} \left( 1 - \frac{1}{|P|} \right).\end{aligned}$$

- On the integers, unless otherwise stated, $\Lambda$ is taken to be the von Mangoldt function. The analogy in $\mathcal{A}$ is

$$\Lambda(A) := \begin{cases} \log|P| & \text{if } A = aP^e \text{ for some } P \in \mathcal{P}, \text{ integer } e \geq 1, \text{ and } a \in {\mathbb{F}_q}^* \\ 0 & \text{otherwise.} \end{cases}$$

- Let $k \geq 0$ be an integer. For a $k$-times differentiable function $f(x)$, we define $f^{(k)}(x)$ to be its $k$-th derivative.

- Let $a \in \mathbb{C}$ and $b \in \mathbb{C} \backslash \{0\}$, and let $f$ be an integrable complex function. The integral $\int_{t=a}^{a+b\infty} f(t) \mathrm{d}t$ is defined to be over the straight line starting at $a$ and in the direction of $b$. That is, $\int_{t=a}^{a+b\infty} f(t) \mathrm{d}t = \int_{s=0}^{\infty} f\big(a + \frac{b}{|b|} s\big) \mathrm{d}s$. If $a = 0$ then we will simply write $\int_{t=0}^{b\infty} f(t) \mathrm{d}t$, and if $b = \pm 1$ then we will write $\int_{t=a}^{a\pm\infty} f(t) \mathrm{d}t$.

- The function $\log$ is always in base $e$. The function $\log_a$ is, of course, in base $a$.

- For a subset $S \subseteq \mathbb{C}$, we define $\mathbb{1}_S(x)$ to be the function that takes the value 1 if $x \in S$ and takes the value 0 otherwise.

- We denote the cardinality of a set $S$ by $|S|$.

- $I_N$ is the $N \times N$ identity matrix.

- Suppose we have complex functions $f$ and $g$ where $g$ is non-negative and the domain of $g$ contains the domain of $f$. We write $f(x) = O\big(g(x)\big)$ if there is a positive constant $c$ such that $|f(x)| \leq c|g(x)|$ for all $x$ in the domain of $f$. We also write $f(x) \ll g(x)$ for $f(x) = O\big(g(x)\big)$. We write $f(x) \asymp g(x)$ if both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold.

- Suppose we have complex functions $f_k$ and $g_k$ where: The function $f_k$ depends on the parameter $k$; the function $g_k$ may or may not depend on the parameter $k$ (if it is the latter, then $g_k$ is the same for all $k$); the function $g_k$ is non-negative; and the domain of $g_k$ contains the domain of $f_k$. We write $f_k(x) = O\big(g_k(x)\big)$ if there is a positive constant $c$ such that $|f_k(x)| \leq c|g_k(x)|$ for all $k$ and all $x$ in the domain of $f_k$. We write $f_k(x) = O_k\big(g_k(x)\big)$ if, for all $k$, there is a positive constant $c_k$ such that $|f_k(x)| \leq c_k|g_k(x)|$ for all $x$ in the domain of $f_k$. We also write $f_k(x) \ll g_k(x)$ for $f_k(x) = O\big(g_k(x)\big)$ and $f_k(x) \ll_k g_k(x)$ for $f_k(x) = O_k\big(g_k(x)\big)$.

- Suppose we have complex functions $f_k$ and $g_k$ where: The function $f_k$ depends on the parameter $k$; the function $g_k$ may or may not depend on the parameter $k$ (if it is the latter, then $g_k$ is the same for all $k$); the function $g_k$ is non-negative; and the domain of $f_k$ is unbounded and is contained in the domain of $g_k$. We write $f_k(x) = O\big(g_k(x)\big)$ as $x \xrightarrow{k} \infty$ if there is a positive constant $c$, and a positive constant $X_k$ that is dependent on $k$, such that $|f_k(x)| \leq c|g_k(x)|$ for all $k$ and all $x$ in the domain of $f_k$ that satisfy $|x| \geq X_k$. We also write $f_k(x) \ll g_k(x)$ as $x \xrightarrow{k} \infty$ for $f_k(x) = O\big(g_k(x)\big)$ as $x \xrightarrow{k} \infty$.

- Suppose $f$ and $g$ are complex functions, where the domain of $f$ is unbounded and is contained in the domain of $g$. We write $f(x) \sim g(x)$ as $x \longrightarrow \infty$ if $\frac{f(x)}{g(x)} \longrightarrow 1$ as $x \longrightarrow \infty$. We write $f(x) = o\big(g(x)\big)$ as $x \longrightarrow \infty$ if $\frac{f(x)}{g(x)} \longrightarrow 0$ as $x \longrightarrow \infty$. Now, let $c \in \mathbb{C}$, and suppose $f$ and $g$ are complex functions where the domain of $f$ contains some open neighbourhood of $c$ and the domain of $g$ contains the domain of $f$. We write $f(x) \sim g(x)$ as $x \longrightarrow c$ if $\frac{f(x)}{g(x)} \longrightarrow 1$ as $x \longrightarrow c$.

- "RHS" and "LHS" are abbreviations of "right-hand side" and "left-hand side".

- $\gamma$ is the Euler-Mascheroni constant, defined by

$$\gamma := \lim_{k \longrightarrow \infty} \left( \sum_{n=1}^{k} \frac{1}{n} - \int_{x=1}^{k} \frac{1}{x} \mathrm{d}x \right).$$

- For a Dirichlet $L$-function, $L(s, \chi)$, where $\chi$ is a Dirichlet Character on $\mathbb{F}_q[T]$, it is known that all zeros lie on the critical line. Therefore, we can write them as $\rho = \frac{1}{2} + i\gamma$. Furthermore, they can be ordered. We denote the $n$-th zero by $\rho_n = \frac{1}{2} + i\gamma_n$, and it is defined by $\ldots \leq \gamma_{-2} \leq \gamma_{-1} < 0 < \gamma_1 \leq \gamma_2 \leq \ldots$. If $L(s, \chi)$ has a zero at $s = 0$ then we define $\gamma_0 = 0$; otherwise we do not define $\gamma_0$. When we are working with more than one Dirichlet $L$-function, we will write $\rho_n(\chi)$ and $\gamma_n(\chi)$ to distinguish. For a Dirichlet $L$-function, $L(s,\chi)$, where $\chi$ is a Dirichlet Character on $\mathbb{Z}$, we can do the same as above, if we assume the generalised Riemann hypothesis.

- An $N \times N$ unitary matrix, $A$, has $N$ eigenvalues and they all lie on the unit circle in $\mathbb{C}$. Therefore, we can write them as $e^{i\theta}$ for some $\theta \in (-\pi, \pi]$, and we call $\theta$ an eigenphase. For each eigenphase $\theta \in (-\pi, \pi]$, the value $\theta + 2m\pi$, for any integer $m \neq 0$, is also an eigenphase (although it gives the same eigenvalue that $\theta$ does), and we call it a periodicised eigenphase. We can order the eigenphases in the following manner: $\ldots \leq \theta_{-2} \leq \theta_{-1} < 0 < \theta_1 \leq \theta_2 \leq \ldots$. If 0 is an eigenphase of $A$ (that is, 1 is an eigenvalue) then we define $\theta_0 = 0$; otherwise we do not define $\theta_0$.When we are working with more than one unitary matrix, we will write $\theta_n(A)$ to distinguish.

- For products, we use the convention that what is inside the first (pair of) parentheses is what is being multiplied. This applies to curved parentheses, square brackets, modulus bars, etc. For example,

$$\prod_a \big(1 + f(a)\big)^2 \prod_b \left[\left(1 + \frac{g(b)}{h(b)}\right)^3 + 1\right] 5w(x) \prod_c \left|\frac{u(c)}{v(c)}\right|^{-1} \big(w(x) + y(x) + 1\big)$$

should be taken to equal

$$\left(\prod_a \big(1 + f(a)\big)^2\right) \cdot \left(\prod_b \left[\left(1 + \frac{g(b)}{h(b)}\right)^3 + 1\right]\right) \cdot \big(5w(x)\big) \cdot \left(\prod_c \left|\frac{u(c)}{v(c)}\right|^{-1}\right) \cdot \Big(\big(w(x) + y(x) + 1\big)\Big).$$

In certain cases, we can avoid the parenthesis and we may do so for presentational purposes. For example, the product $\prod_a \frac{f(a)}{g(a)}$ is the same as $\prod_a \left(\frac{f(a)}{g(a)}\right)$; and this may occur when the product is by itself or when it appears at the end of an expression. However, we would *not* write $\prod_a 1 + \frac{f(a)}{g(a)}$ in the place of $\prod_a \left(1 + \frac{f(a)}{g(a)}\right)$, for example, as this could be misread as $\left(\prod_a 1\right) + \frac{f(a)}{g(a)}$.

# Chapter 1

# Introduction

## 1.1 The Riemann Zeta-function

The very heart of analytic number theory is undoubtedly the Riemann zeta-function. For $\mathrm{Re}(s) > 1$ it is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

This was first studied as a function on the reals by Euler in the eighteenth century, who obtained [Eul44] the following Euler product (a product over the primes):

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}.$$

Here, we can see how the Riemann zeta-function connects the prime numbers with the natural numbers, and it is this that makes it so important in the study of primes. The value of $\zeta(s)$ at $s = 2$ and several other positive even integers was obtained by Euler. For all integers $n \geq 1$ we have

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!},$$

where $B_{2n}$ is the $2n$-th Bernoulli number. The study of $\zeta(s)$ as a function on the complex numbers was first made by Riemann in his manuscript of 1859 [Rie59]. We note that, for all $\epsilon > 0$, (1.1) is uniformly convergent on $\mathrm{Re}(s) > 1 + \epsilon$, from which we deduce that (1.1) is holomorphic on $\mathrm{Re}(s) > 1$. Riemann showed that it has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ with residue equal to 1. Further, he obtained the following functional equation:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Recalling that $\Gamma(s)$ has poles at the non-positive integers, the functional equation shows us that $\zeta(s)$ must have zeros at the negative even integers, known as the trivial zeros. What are of great interest are the other zeros that lie in the critical strip defined by $0 \leq \mathrm{Re}(s) \leq 1$, known as the non-trivial zeros. In his manuscript, Riemann stated that, in this strip, the number of zeros of $\zeta(s)$ with $0 \leq \mathrm{Im}(s) \leq T$ is asymptotic to $\frac{T}{2\pi} \log \frac{T}{2\pi}$ as $T \longrightarrow \infty$, which was proven by von Mangoldt in 1905. In 1896, Hadamard and de la Vallée Poussin independently proved that there are no zeros of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$, which allowed them to deduce an asymptotic for the prime counting function:

$$\pi(x) := |\{p \in \mathbb{N} : p \text{ prime}, p \leq x\}| \sim \mathrm{Li}(x) \sim \frac{x}{\log x} \tag{1.2}$$

as $x \longrightarrow \infty$, where $\mathrm{Li}(x) := \int_{t=2}^{x} \frac{1}{\log t} \mathrm{d}t$ is the logarithmic integral. (The function $\frac{x}{\log x}$ is simpler than $\mathrm{Li}(x)$, but, as we see below, $\mathrm{Li}(x)$ is better when one is concerned with bounding the error term in the approximation). Several decades earlier, Riemann made the famous Riemann hypothesis, asserting that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. The importance of this hypothesis can be seen by the fact that if it is true then one can obtain a strong bound on the error term of the approximation (1.2):

$$\pi(x) = \mathrm{Li}(x) + O\big(\sqrt{x} \log x\big). \tag{1.3}$$

There are many interesting and interrelated topics in the theory of the Riemann zeta-function:

- Zero-free regions: These are (unbounded) regions of the critical strip that we can prove have no zeros of $\zeta(s)$.

- Proportion of zeros on the critical line: Selberg [Sel42] showed that at least a small positive proportion of the non-trivial zeros lie on the critical line. This was improved to $\frac{1}{3}$ by Levinson [Lev74], $\frac{2}{5}$ by Conrey [Con89], and approximately 41% by Bui, Conrey, and Young [BCY11].

- Mean value theorems: In order to understand how large $\zeta(s)$ can be on the critical line we can study mean values, or "moments":

$$\frac{1}{T} \int_{t=0}^{T} \Big|\zeta\Big(\frac{1}{2} + it\Big)\Big|^{2k} \mathrm{d}t.$$

  Currently we have results on the asymptotic behaviour (as $T \longrightarrow \infty$) only for $k = 0, 1, 2$, although conjectures exist for higher powers.

- Universality: Suppose we have a compact subset $K$ of the critical strip $\{s \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(s) < 1\}$ with connected complement, and a function $f$ that is holomophic in the interior of $K$ with no zeros in $K$. For any $\epsilon > 0$, there exists $t \geq 0$ such that

$$|f(s) - \zeta(s + it)| < \epsilon$$

  for all $s \in K$. This was proved by Voronin [Vor75].

- Special values: As mentioned previously, the values that $\zeta(s)$ takes at the positive even integers are known and can be expressed in terms of $\pi$ and the Bernoulli numbers. The values that it takes on the positive odd integers is considerably more difficult to understand. Apéry [Apé79] was able to show that $\zeta(3)$ is irrational, and it is known that infinitely many of $\zeta(2n + 1)$ (for integers $n \geq 0$) are irrational [Riv00], while at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational [Zud01].

We also mention that the derivatives of the Riemann zeta function are important objects to study. Indeed, Speiser [Spe35] showed that the Riemann hypothesis is equivalent to $\zeta'(s)$ having no zeros to the left of the critical line. Now, let $N_k(T)$ be the number of non-real zeros of $\zeta^{(k)}(s)$ with imaginary part in $(0,T)$. Also, let $N_k^-(c,T)$ be the number of non-real zeros of $\zeta^{(k)}(s)$ with imaginary part in $(0,T)$ and real part in $(-\infty,c]$, and let $N_k^+(c,T)$ be the same but with real part in $[c,\infty)$. It was shown by Levinson and Montgomery [LM74] that

$$N_k^+\Big(\frac{1}{2}+\delta,T\Big)+N_k^-\Big(\frac{1}{2}-\delta,T\Big) \ll_k N_k(T)\frac{\log\log T}{\delta\log T}.$$

That is, most of the zeros of $\zeta^{(k)}(s)$ can be found near the critical line. It was by using results on $\zeta'(s)$ from [LM74], that Levinson showed that at least $\frac{1}{3}$ of the zeros of $\zeta(s)$ lie on the critical line [Lev74]. So, we can see that the derivatives of the Riemann zeta-function play a key role in our understanding of the non-trivial zeros of the Riemann zeta-function.

For more details and a good introduction to the theory of the Riemann zeta-function, we recommend Titchmarsh's book, edited by Heath-Brown [Tit87].

## 1.2 $L$-functions

$L$-functions are generalisations of the Riemann zeta-function. While the Riemann zeta-function encodes information about the primes, other $L$-functions encode information about other objects of number theoretic interest. One family of such $L$-functions consists of the Dirichlet $L$-functions. Before defining these $L$-functions, we must first define Dirichlet characters. In this section, as well as Sections 1.3 and 1.5, $q$ is a positive integer (in all other parts of this thesis, $q$ is a positive integer power of an odd prime number). A Dirichlet character of modulus $q$ is a function $\chi:\mathbb{Z}\longrightarrow\mathbb{C}^*$ satisfying, for all $n,m\in\mathbb{Z}$,

1. $\chi(nm)=\chi(n)\chi(m)$;
2. $\chi(n)=\chi(m)$ if $n\equiv m(\operatorname{mod} q)$;
3. $\chi(n)=0$ if and only if $(n,q)\neq 1$ .

There are $\phi(q)$ Dirichlet characters of modulus $q$. We say that $\chi_0$ is the trivial character of modulus $q$ if $\chi_0(n)=1$ for all $(n,q)=1$. Now, suppose $\chi$ is a character of modulus $q$ and $r\mid q$. We say that $r$ is an induced modulus of $\chi$ if there exists a character $\chi_1$ of modulus $r$ such that

$$\chi(n)=\begin{cases}\chi_1(n) & \text{if } (n,q)=1\\ 0 & \text{otherwise.}\end{cases}$$

$\chi$ is said to be primitive if there is no induced modulus strictly smaller than $q$. Otherwise, $\chi$ is said to be non-primitive. $\phi^*(q)$ denotes the number of primitive characters of modulus $q$. We denote a sum over all characters $\chi$ of modulus $q$ by $\sum_{\chi \bmod q}$, and a sum over all primitive characters $\chi$ of modulus $q$ by $\sideset{}{^*}\sum_{\chi \bmod q}$.

A Dirichlet $L$-function is a complex function defined, for $\mathrm{Re}(s) > 1$, by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi$ is some Dirichlet character. Note that, when $\chi$ is the character of modulus 1 (that is, $\chi(n) = 1$ for all $n \in \mathbb{Z}$) we have $L(s,\chi) = \zeta(s)$. As is the case for the Riemann zeta-function, Dirichlet $L$-functions have an Euler product,

$$L(s,\chi) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{\chi(p)}{p^s}},$$

as well as a meromorphic continuation to $\mathbb{C}$, and a functional equation. They encode information about primes in arithmetic progressions. Indeed, in the nineteenth century, by showing that $L(1,\chi) \neq 0$ for all non-trivial characters $\chi$, Dirichlet proved that there are infinitely many primes that are congruent to $a$ modulo $q$ (where $a, q$ are coprime integers).

There are many other families of $L$-functions. In 1989, Selberg [Sel92] introduced an axiomatic definition of $L$-functions. The functions satisfying these axioms form the Selberg class, $S$, which includes many of the previously established $L$-functions, including Dirichlet $L$-functions. A function $F$ is an element of $S$ if it can be expressed as a Dirichlet series,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

that is absolutely convergent for $\mathrm{Re}(s) > 1$ and satisfies the following:

1. $F(s)$ has a meromorphic continuation to $\mathbb{C}$ with the only possible pole being at $s = 1$;

2. $a_1 = 1$ and for all $\epsilon > 0$ we have $a_n \ll_\epsilon n^\epsilon$;

3. $F(s)$ has a functional equation with certain conditions;

4. $F(s)$ has an Euler product with certain conditions.

For a more in-depth look at the Selberg class, see the survey by Perelli [Per05].

## 1.3 Mean Values of $L$-functions

Mean values, or moments, of $L$-functions are the average values that an $L$-function takes on a line or a set of points, or the average value that a family of $L$-functions take at a point. Actually, most of the time we do not only work with $L$-functions, but with powers of $L$-functions, derivatives of $L$-functions, products of $L$-functions with other functions, or some combination of these. Some results that are known are the following. In the early twentieth century, it was shown by Hardy and Littlewood [HL18] that

$$\frac{1}{T}\int_{t=0}^{T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t \sim \log T$$

as $T \longrightarrow \infty$, and it was shown by Ingham [Ing26] that

$$\frac{1}{T}\int_{t=0}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{4}\mathrm{d}t \sim \frac{1}{12}\frac{6}{\pi^2}(\log T)^4$$

as $T \longrightarrow \infty$. Results for higher powers have resisted attempts by mathematicians for almost one hundred years. Nonetheless, it was a "folklore" conjecture that, for integers $k \geq 0$,

$$\lim_{T\longrightarrow\infty}\frac{1}{(\log T)^{k^2}}\frac{1}{T}\int_{t=0}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}\mathrm{d}t = f(k)a(k), \tag{1.4}$$

where $f(k)$ is a real-valued function and [1]

$$a(k) := \prod_{p}\left(\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^m}\right). \tag{1.5}$$

As described above, this has been proved for the cases $k = 0, 1, 2$ where we have $f(0) = 1$, $f(1) = 1$, $f(2) = \frac{1}{12}$. It has been conjectured via number-theoretic means that $f(3) = \frac{42}{9!}$ [CG98] and $f(4) = \frac{24024}{16!}$ [CG99]. By using a random matrix theory model, Keating and Snaith [KS00b] recovered the "folklore" conjecture (1.4) giving an explicit form for $f(k)$ that agrees with the established results for $k = 0, 1, 2$, as well as the conjectures obtained via number-theoretic means for $k = 3, 4$. In fact, the conjecture of Keating and Snaith extends beyond the non-negative integer values for $k$. We consider this in more detail in Section 1.5.

Conrey, Farmer, Keating, Rubinstein, and Snaith [CFK+05] developed a recipe for conjecturing the full main terms for moments of $L$-functions. As they describe, they assume certain cancellations between the off-diagonal terms that arise in moment calculations, but it is not clear how or why these cancellations occur. Nonetheless, their conjectures are consistent with previously established results and conjectures on moments of $L$-functions, as well as the analogous expressions for moments of characteristic polynomials of random matrices.

Proceeding with our discussion on moments of $L$-functions, one can, alternatively, study the following mean values:

$$\frac{1}{N(T)}\sum_{0<\mathrm{Im}(\rho)\leq T}|\zeta'(\rho)|^{2k}.$$

Here, we are considering the derivative of the Riemann zeta-function, and, instead of averaging over the critical line, we are averaging over a set of points. Namely, we are averaging over the zeros of $\zeta(s)$ in the critical strip. Of course, under the assumption of the Riemann hypothesis, all such zeros would lie on the critical line. $N(T)$ is the number of zeros in the critical strip with imaginary part in $(0, T]$. Such mean values are related to, for example, the proportion of simple zeros on the critical line. In

[1] It is not immediately obvious that $a(k)$ is convergent. To see this, we note that $\left(1-\frac{1}{p}\right)^{k^2} = 1 - \frac{k^2}{p} + O_k\left(\frac{1}{p^2}\right)$ and $\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^m} = 1 + \frac{k^2}{p} + O_k\left(\frac{1}{p^2}\right)$, and so $\prod_p\left(\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^m}\right) = \prod_p\left(1+O_k\left(\frac{1}{p^2}\right)\right) < \infty$.

the way of results, Gonek [Gon84] proved under the assumption of the Riemann hypothesis that

$$\frac{1}{N(T)}\sum_{0<\mathrm{Im}(\rho)\leq T}|\zeta'(\rho)|^2 \sim \frac{1}{12}(\log T)^3.$$

Asymptotic formulas for higher moments, even under the assumption of the Riemann hypothesis, have yet to be obtained, although there are various other results such as bounds. We refer the reader to the introduction of Kirila's article [Kir20] for more details.

While the above mean values are concerned with a single $L$-function, the Riemann zeta-function, on the critical line, we can also consider, for example, the mean value of Dirichlet $L$-functions at the central value of $\frac{1}{2}$. Paley [Pal31] is attributed to showing that

$$\frac{1}{\phi^*(q)}\sideset{}{^*}\sum_{\chi \bmod q}\left|L\Big(\frac{1}{2},\chi\Big)\right|^2 \sim \frac{\phi(q)}{q}\log q \tag{1.6}$$

as $q \longrightarrow \infty$. Heath-Brown [HB81] proved that

$$\frac{1}{\phi^*(q)}\sideset{}{^*}\sum_{\chi \bmod q}\left|L\Big(\frac{1}{2},\chi\Big)\right|^4 = \frac{1}{2\pi^2}(\log q)^4\prod_{p|q}\left(\frac{(1-p^{-1})^3}{1+p^{-1}}\right) + O\Big(2^{\omega(q)}\frac{q}{\phi^*(q)}(\log q)^3\Big), \tag{1.7}$$

where $q$ is a positive integer. In order to ensure that the error term is of lower order (as $q \longrightarrow \infty$) than the main term, we must restrict $q$ to

$$\omega(q) \leq \frac{\log\log q - 7\log\log\log q}{\log 2}.$$

Soundararajan [Sou07] addressed this by proving that

$$\begin{aligned}&\frac{1}{\phi^*(q)}\sideset{}{^*}\sum_{\chi \bmod q}\left|L\Big(\frac{1}{2},\chi\Big)\right|^4\\ =&\frac{1}{2\pi^2}(\log q)^4\prod_{p|q}\left(\frac{(1-p^{-1})^3}{1+p^{-1}}\right)\left(1+O\Big(\frac{\omega(q)}{\log q}\sqrt{\frac{q}{\phi(q)}}\Big)\right) + O\Big(\frac{q}{\phi^*(q)}(\log q)^{\frac{7}{2}}\Big).\end{aligned} \tag{1.8}$$

Here, the error terms are of lower order than the main term without the need to have any restriction on $q$. This was further improved by Young [You11] who obtained lower order terms for the case where $q$ is an odd prime:

$$\frac{1}{\phi^*(q)}\sideset{}{^*}\sum_{\chi \bmod q}\left|L\Big(\frac{1}{2},\chi\Big)\right|^4 = \sum_{i=0}^{4}c_i(\log q)^i + O_\epsilon\big(q^{-\frac{5}{512}+\epsilon}\big),$$

where the constants $c_i$ can be given explicitly. The error term was subsequently improved by Blomer *et al.* [BFK$^+$17] who proved that

$$\frac{1}{\phi^*(q)}\sideset{}{^*}\sum_{\chi \bmod q}\left|L\Big(\frac{1}{2},\chi\Big)\right|^4 = \sum_{i=0}^{4}c_i(\log q)^i + O_\epsilon\big(q^{-\frac{1}{32}+\epsilon}\big).$$

Again, results have only been obtained up to the fourth power, but, as is the case with the Riemann zeta-function on the critical line, for conjectures on higher powers one can look to random matrix theory [KS00a, BK07] to obtain the following conjecture:

$$\frac{1}{\phi^*(q)}\sum_{\chi \bmod q}^{*}\left|L\Big(\frac{1}{2},\chi\Big)\right|^{2k} \sim a(k)\frac{G^2(k+1)}{G(2k+1)}(\log q)^{k^2}\prod_{p|q}\bigg(\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^m}\bigg)^{-1} \tag{1.9}$$

as $q \longrightarrow \infty$, where $a(k)$ is as in (1.5), and $G$ is the Barnes $G$-function. For conjectures on the lower order terms, we refer the reader to the work of Conrey, Farmer, Keating, Rubinstein, and Snaith [CFK$^+$05].

One may ask why we are interested in mean values of $L$-functions. They are certainly interesting in that results for higher powers are difficult to obtain, but they also have several applications. For example, the Lindelöf hypothesis states that for all $\epsilon > 0$ we have

$$\zeta\Big(\frac{1}{2}+it\Big) = O_\epsilon\big(t^\epsilon\big).$$

It can be shown that this is equivalent to the statement that for all $\sigma > \frac{1}{2}$ we have

$$|\{\rho \in \mathbb{C} : \mathrm{Re}(\rho) \geq \sigma, \mathrm{Im}(\rho) \in [T, T+1], \zeta(\rho) = 0\}| = o(\log T).$$

The Riemann hypothesis would imply that the RHS is zero, and so we see that the Lindelöf hypothesis is weaker. However, it is still unproven. Another equivalent statement of the Lindelöf hypothesis can be given in terms of mean values: For all $\epsilon > 0$ and all positive integers $k$,

$$\frac{1}{T}\int_{t=0}^{T}\left|\zeta\Big(\frac{1}{2}+it\Big)\right|^{2k}\mathrm{d}t = O_{k,\epsilon}(T^\epsilon).$$

Thus, we see that an understanding of the mean values of $\zeta(s)$ can prove (or disprove, although this seems unlikely) the Lindelöf hypothesis. We refer the reader to [Tit87, Chapter 13] for proofs of these equivalences. For a detailed account of the various applications of mean value theorems to the theory of $\zeta(s)$ and its zeros, we recommend [Gon05].

What about moments of families of $L$-functions at a point? One application is to the non-vanishing of those $L$-functions at that point. For example, it is a folklore conjecture of Chowla that $L\big(\frac{1}{2},\chi\big) \neq 0$ for all Dirichlet characters $\chi$. To illustrate the connection to moments involving Dirichlet $L$-functions, let $N(q)$ be the proportion of primitive characters $\chi$ of modulus $q$ such that $L(s,\chi)$ does not vanish at $s = \frac{1}{2}$. By the Cauchy-Schwarz inequality, we have that

$$\frac{1}{\phi^*(q)}\left|\sum_{\chi \bmod q}^{*}L\Big(\frac{1}{2},\chi\Big)\right| \leq N(q)^{\frac{1}{2}}\bigg(\frac{1}{\phi^*(q)}\sum_{\chi \bmod q}^{*}\left|L\Big(\frac{1}{2},\chi\Big)\right|^2\bigg)^{\frac{1}{2}},$$

giving the lower bound

$$N(q) \geq \bigg(\frac{1}{\phi^*(q)}\left|\sum_{\chi \bmod q}^{*}L\Big(\frac{1}{2},\chi\Big)\right|\bigg)^2\bigg(\frac{1}{\phi^*(q)}\sum_{\chi \bmod q}^{*}\left|L\Big(\frac{1}{2},\chi\Big)\right|^2\bigg)^{-1}.$$

Unfortunately, the numerator is of lower order than the denominator as $q \longrightarrow \infty$, meaning this bound is not very strong. To address this we multiply $L\big(\frac{1}{2},\chi\big)$ by a mollifier $M(\chi)$. The mollifier takes the form $M(\chi) = \sum_{n\leq q^\kappa} \frac{b_n}{\sqrt{n}}$ for some $b_n$ and $0 < \kappa < 1$, and the idea is that this mollifier will negate the effect that the large $L\big(\frac{1}{2},\chi\big)$ have on making

$$\Big(\frac{1}{\phi^*(q)}\Big|\sum_{\chi \bmod q}^{*} L\Big(\frac{1}{2},\chi\Big)\Big|\Big)^2 = o\Big(\frac{1}{\phi^*(q)}\sum_{\chi \bmod q}^{*} \Big|L\Big(\frac{1}{2},\chi\Big)\Big|^2\Big).$$

That is, we obtain

$$N(q) \geq \Big(\frac{1}{\phi^*(q)}\Big|\sum_{\chi \bmod q}^{*} M(\chi)L\Big(\frac{1}{2},\chi\Big)\Big|\Big)^2\Big(\frac{1}{\phi^*(q)}\sum_{\chi \bmod q}^{*} \Big|M(\chi)L\Big(\frac{1}{2},\chi\Big)\Big|^2\Big)^{-1} \longrightarrow c$$

as $q \longrightarrow \infty$, for some constant $0 < c \leq 1$. Hence, we obtain a positive lower bound for the proportion of non-vanishing Dirichlet $L$-functions at $\frac{1}{2}$.

This leads us to twisted moments of $L$-functions, which feature in Chapters 6 and 7. One such example is the second moment of the Riemann zeta function on the critical line where it is twisted by a finite Dirichlet series:

$$I := \int_{t\in\mathbb{R}} \Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^2\Big|A\Big(\frac{1}{2}+it\Big)\Big|^2 f\Big(\frac{t}{T}\Big)\mathrm{d}t, \tag{1.10}$$

where $f(x)$ is usually a smooth function with support in $[1,2]$, and

$$A(s) := \sum_{n\leq T^\theta} \frac{a_n}{n^s}$$

with $a_n \ll_\epsilon n^\epsilon$ for all $\epsilon > 0$, and $0 < \theta < 1$. When $f(x)$ is the indicator function on $[1,2]$ and $\theta < \frac{1}{2}$, it was shown by Balasubramanian, Conrey, and Heath-Brown [CHB85] that

$$I = T \sum_{d,e\leq T^\theta} \frac{a_d\overline{a_e}}{[d,e]}\Big(\log\Big(\frac{T(d,e)^2}{2\pi de}\Big) + 2\gamma + \log 4 - 1\Big) \qquad + o(T),$$

among other related results. One application of this is obtaining a lower bound for the proportion of complex zeros on the critical line. The condition $\theta < \frac{1}{2}$ is significant in that when $\theta < \frac{1}{2}$ only the diagonal terms contribute. When $\theta > \frac{1}{2}$, one must also consider the off-diagonal terms. For $\theta < \frac{17}{33}$, this undertaken by Bettin, Chandee, and Radziwiłł [BCR17]. They proved, among other related results, that when $f(x)$ is a smooth function with support in $[1,2]$, we have

$$I = T \sum_{d,e\leq T^\theta} \frac{a_d\overline{a_e}}{[d,e]} \int_{t\in\mathbb{R}} \Big(\log\Big(\frac{T(d,e)^2}{2\pi de}\Big) + 2\gamma\Big) f\Big(\frac{t}{T}\Big)\mathrm{d}t \qquad + O_\epsilon\Big(T^{\frac{3}{20}+\epsilon}T^{\frac{33}{20}\theta} + T^{\frac{1}{3}+\epsilon}\Big).$$

Interestingly, a corollary of their result is an upper bound of the correct order of magnitude for the third moment of $\zeta(s)$:

$$\int_{t=T}^{2T} \Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^3 \mathrm{d}t \ll T(\log T)^{\frac{9}{4}}.$$

If we are able to obtain an asymptotic formula for (1.10) that is valid for $\theta < 1$ (with $f(x)$ being a smooth function with support in $[1,2]$), then we would be able to resolve the Lindelöf hypothesis.

For the twisted fourth moment, we consider

$$\int_{t=0}^{T} \Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^4 \Big|A\Big(\frac{1}{2}+it\Big)\Big|^2 \mathrm{d}t. \tag{1.11}$$

Again, (1.11) is related to the Lindelöf hypothesis, but also has applications to upper bounds on the number of primes in short intervals. An upper bound for (1.11), when $\theta < \frac{1}{5}$, was obtained by Deshouillers and Iwaniec [DI82], which was later improved to $\theta < \frac{1}{4}$ by Watt [Wat95]. With regards to an asymptotic formula for (1.10), Gaggero Jara obtained such a result for $\theta < \frac{4}{589}$ in his thesis [GJ97]. This was improved to $\theta < \frac{1}{11}$ by Hughes and Young [HY10] and to $\theta < \frac{1}{4}$ by Bettin, Bui, Li, and Radziwiłł [BBLR20]. We mention that Motohashi [Mot09] approaches the problem of obtaining an asymptotic formula for (1.11) via an alternative method: Spectral theory.

One can also study twisted moments of Dirichlet $L$-functions:

$$\sum_{\chi \bmod q}^{*} \Big|L\Big(\frac{1}{2},\chi\Big)\Big|^{2k} \Big|B\Big(\frac{1}{2},\chi\Big)\Big|^2,$$

for $k = 1, 2$. Here, we define

$$B\Big(\frac{1}{2},\chi\Big) := \sum_{n \leq q^{\kappa}} \frac{b_n \chi(n)}{n^s}$$

where the $b_n$ can be chosen arbitrarily given the condition $b_n \ll_{\epsilon} n^{\epsilon}$ for all $\epsilon > 0$, and $0 < \kappa < 1$. For the second moment, an asymptotic formula when $\kappa < \frac{1}{2}$ was obtained by Iwaniec and Sarnak [IS99]. Again, considering $\kappa \geq \frac{1}{2}$ means that off-diagonal terms will be involved. Nonetheless, an asymptotic formula when $\kappa < \frac{51}{101}$ was obtained by Bui, Pratt, Robles, and Zaharescu [BPRZ20], and one of their applications of this was to obtain the correct order of magnitude for the third moment of Dirichlet $L$-functions:

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q}^{*} \Big|L\Big(\frac{1}{2},\chi\Big)\Big|^3 \asymp (\log q)^{\frac{9}{4}}.$$

With regards to the twisted fourth moment, Hough [Hou16] obtained an asymptotic formula for

$$\sum_{\chi \bmod q}^{*} \Big|L\Big(\frac{1}{2},\chi\Big)\Big|^4 \chi(l_1)\overline{\chi}(l_2),$$

where $1 \leq l_1, l_2 \leq q^{\kappa}$ are square-free and satisfy $(l_1,l_2) = 1$, and $\kappa < \frac{1}{32}$. This was extended to cube-free $l_1, l_2$ by Zacharias [Zac19], allowing for the application to non-vanishing results.

In Chapters 6 and 7, we encounter twisted moments of Dirichlet $L$-functions, where the Dirichlet series associated to the twist has an explicit form. Furthermore, this will be in the function field setting, which we introduce in the following section.

## 1.4 Function Fields

There is another setting that one can study number theory. While in the above, we have been concerned with the integers (referred to as the number field setting or classical setting), we can, instead, study polynomials over finite fields (referred to as the function field setting). There are many analogies between the two settings, and the function field setting often acts as a signpost for the number field setting, as we will see later, but first let us make some definitions. Much of the material in this section can be found in Rosen's book [Ros02], and we recommend this for an in-depth account of analytic number theory in function fields. Further details can also be found in the thesis of Andrade [And12].

Let $q := p^n$ for some positive prime integer $p$ and some positive integer $n$, with $q \neq 2$, and let $\mathbb{F}_q$ be the finite field of order $q$. We denote the multiplicative group of $\mathbb{F}_q$ by ${\mathbb{F}_q}^*$. Let $\mathcal{A} := \mathbb{F}_q[T]$ be the polynomial ring over $\mathbb{F}_q$. The notation of $\mathcal{A}$ does not demonstrate dependence on $q$, but as our results hold for all prime powers $q \neq 2$, this is not necessary. For $\mathcal{T} \subseteq \mathcal{A}$, an integer $n \geq 0$, and $B \in \mathcal{A}$, we define $\mathcal{T}_n := \{A \in \mathcal{T} : \deg A = n\}$ and $B\mathcal{T} := \{AB : A \in \mathcal{T}\}$. We identify $\mathcal{A}_0$ with ${\mathbb{F}_q}^*$, and remark that this is the multiplicative group of $\mathcal{A}$ (note this is finite as is the case for $\mathbb{Z}$).

We define $\mathcal{M}$ to be the set of all monic polynomials in $\mathcal{A}$. These play the role of the positive integers. Indeed, any non-zero element of $\mathcal{A}$ can be uniquely expressed as the product of an element of the multiplicative group ${\mathbb{F}_q}^*$ and an element in $\mathcal{M}$, just as any non-zero integer can be uniquely expressed as the product of an element in the multiplicative group $\{1, -1\}$ and an element in $\mathbb{Z}_{>0}$. We can see that $|\mathcal{M}_n| = q^n$ and $|\mathcal{A}_n| = (q-1)q^n$.

For non-zero $A \in \mathcal{A}$ we define $|A| := q^{\deg A}$, and for the zero polynomial we define $|0| := 0$. This is natural in the sense that we define the norm of an element $A \neq 0$ to be the number of equivalence classes modulo $A$, just as in the number field setting we have $|n| = n = |\mathbb{Z}/n\mathbb{Z}|$ for non-zero $n$. Of course, we also have that the norm function on $\mathcal{A}$ is multiplicative.

As is the case with $\mathbb{Z}$, $\mathcal{A}$ is a Euclidean domain. In particular we have unique factorisation and primality is equivalent to irreducibility. When referring to a prime in $\mathcal{A}$, it will always be a monic prime unless stated otherwise. The letters $P, Q$ are reserved for primes in $\mathcal{A}$ and are to be taken as such even when it is not explicitly stated (particularly in the ranges of summations and products). We denote the set of monic primes by $\mathcal{P}$. For non-negative integers $n$ we define
$\mathcal{S}(n) := \{A \in \mathcal{A} : P \mid A \Rightarrow \deg P \leq n\}$ and $\mathcal{S}_{\mathcal{M}}(n) := \{A \in \mathcal{S}(n) : A \text{ is monic}\}$. We define the totient function for $a \in \mathcal{A}_0 = {\mathbb{F}_q}^*$ by $\phi(a) = 1$, and for $R \in \mathcal{A}$ with $\deg R > 0$ by

$$\begin{aligned}\phi(R) :=& \left| \left\{ A \in \mathcal{A} : \deg A < \deg R \text{ and } (A, R) = 1 \right\} \right| \\ =& |A| \prod_{P \mid R} \left( 1 - \frac{1}{|P|} \right).\end{aligned}$$

As we can see, there are many fundamental analogies between $\mathbb{Z}$ and $\mathcal{A}$. Further analogies can be seen in deeper results too, as we will see, but there are also important results that have only been obtained in the function field setting. For example, we have the striking result that

$$|\mathcal{P}_n| = \frac{1}{n} \sum_{d|n} \mu(d) q^{\frac{n}{d}}. \tag{1.12}$$

That is, we have an exact formula for the number of primes of a given degree. Also, taking $x = q^n$, we have

$$\begin{aligned} |\{P \in \mathcal{P} : |P| = x\}| =& |\mathcal{P}_n| = \frac{q^n}{n} + \frac{1}{n} \sum_{\substack{d|n \\ d \neq 1}} \mu(d) q^{\frac{n}{d}} = \frac{q^n}{n} + O\Big(\frac{q^{\frac{n}{2}}}{n}\Big) \\ =& \frac{x}{\log_q x} + O\Big(\frac{x^{\frac{1}{2}}}{\log_q x}\Big). \end{aligned}$$

This illuminates an analogy with the classical prime number theorem, (1.2), and the stronger statement, (1.3), that is dependent on the classical Riemann hypothesis. We refer to (1.12) as the prime polynomial theorem.

We can define, for $\mathrm{Re}(s) > 1$, the Riemann zeta-function on $\mathcal{A}$:

$$\zeta_{\mathcal{A}}(s) := \sum_{A \in \mathcal{M}} \frac{1}{|A|^s} = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{1 - q^{1-s}}.$$

Note that the far RHS provides a meromorphic continuation of $\zeta_{\mathcal{A}}(s)$ to $\mathbb{C}$, with simple poles, of residue $(\log q)^{-1}$, at $1 + \frac{2m\pi i}{\log q}$ for $m \in \mathbb{Z}$. Note also that the Riemann hypothesis is true in that there are no zeros off the critical line $\mathrm{Re}(s) = \frac{1}{2}$ (in fact, there are no zeros at all). We also have the Euler product formula

$$\zeta_{\mathcal{A}}(s) = \prod_{P \in \mathcal{P}} \frac{1}{1 - \frac{1}{|P|^s}}.$$

We can also define Dirichlet $L$-functions on $\mathcal{A}$, but first we make some definitions and point out some results. We go into more detail here for the function field setting, as this thesis is concerned primarily with this setting. A Dirichlet character on $\mathcal{A}$ of modulus $R \in \mathcal{M}$ is a function $\chi : \mathcal{A} \longrightarrow \mathbb{C}^*$ satisfying, for all $A, B \in \mathcal{A}$,

1. $\chi(AB) = \chi(A)\chi(B)$ ;
2. $\chi(A) = \chi(B)$ when $A \equiv B (\mathrm{mod}\, R)$;
3. $\chi(A) = 0$ when $(A, R) \neq 1$.

We say that $\chi_0$ is the trivial character if $\chi_0(A) = 1$ for all $(A, R) = 1$. It is not difficult to see that $\chi(1) = 1$ and $|\chi(A)| = 1$ for all characters $\chi$ and all $(A, R) = 1$. We say that a character $\chi$ is even if $\chi(a) = 1$ for all $a \in {\mathbb{F}_q}^*$; otherwise we say that $\chi$ is odd. The number of Dirichlet characters of modulus $R$ is $\phi(R)$, while the number of even Dirichlet characters of modulus $R$ is, for $R \in \mathcal{M} \backslash \{1\}$, equal to $\frac{\phi(R)}{q-1}$.

Now, suppose $S \mid R$. We say that $S$ is an induced modulus of $\chi$ if there exists a character $\chi_1$ of modulus $S$ such that

$$\chi(A) = \begin{cases} \chi_1(A) & \text{if } (A, R) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$\chi$ is said to be primitive if there is no induced modulus of strictly smaller degree than $R$. Otherwise, $\chi$ is said to be non-primitive. $\phi^*(R)$ denotes the number of primitive characters of modulus $R$. We note that all trivial characters of modulus $R \in \mathcal{M}\backslash\{1\}$ are non-primitive as they are induced by the character of modulus 1. We also note that if $R$ is prime, then the only non-primitive character of modulus $R$ is the trivial character of modulus $R$. We denote a sum over all characters $\chi$ of modulus $R$ by $\sum_{\chi \bmod R}$, and a sum over all primitive characters $\chi$ of modulus $R$ by $\sum^*_{\chi \bmod R}$.

From point 2 above, we can view $\chi$ as a function on $\mathcal{A}/R\mathcal{A}$. This makes expressions such as $\chi(A^{-1})$ well-defined for $A \in (\mathcal{A}/R\mathcal{A})^*$.

It is not difficult to see that the set of characters of a fixed modulus $R$ forms an abelian group under multiplication. The identity element is $\chi_0$. The inverse of $\chi$ is $\overline{\chi}$, which is defined by $\overline{\chi}(A) = \overline{\chi(A)}$ for all $A \in \mathcal{A}$. The subset consisting of even characters forms a subgroup.

Now we state some results on Dirichlet characters.

**Lemma 1.4.1.** *Suppose $\chi$ is a non-trivial character of modulus $R \in \mathcal{M}$. Then,*

$$\sum_{\deg A < \deg R} \chi(A) = 0.$$

*Proof.* The case $R = 1$ is trivial, so suppose $R \neq 1$. We can find some $B \in \mathcal{A}$ with $(B, R) = 1$ and $\chi(B) \neq 1$. From this, and the fact that

$$\sum_{\deg A < \deg R} \chi(A) = \sum_{A \in (\mathcal{A}\backslash R\mathcal{A})^*} \chi(A) = \sum_{A \in (\mathcal{A}\backslash R\mathcal{A})^*} \chi(AB) = \chi(B) \sum_{A \in (\mathcal{A}\backslash R\mathcal{A})^*} \chi(A),$$

the results follows. ☐

**Lemma 1.4.2.** *Suppose that $R \in \mathcal{M}$. Then*

$$\sum_{\chi \bmod R} \chi(A) = \begin{cases} \phi(R) & \text{if } A \equiv 1 (\bmod R) \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* This is clear if $R = 1$ or $A \equiv 1 (\bmod R)$. If $R \neq 1$ and $A \not\equiv 1 (\bmod R)$, then we can find a character $\chi_1$ of modulus $R$ such that $\chi_1(A) \neq 1$. Since all the characters of a given modulus form a multiplicative group, we have

$$\sum_{\chi \bmod R} \chi(A) = \sum_{\chi \bmod R} (\chi \cdot \chi_1)(A) = \chi_1(A) \sum_{\chi \bmod R} \chi(A),$$

from which the result follows. ☐

Similarly, we can prove the following lemma.

**Lemma 1.4.3.** *Suppose $R \in \mathcal{M}$ with $R \neq 1$, then*

$$\sum_{\substack{\chi \bmod R \\ \chi \text{ even}}} \chi(A) = \begin{cases} \frac{\phi(R)}{q-1} & \text{if } A \equiv a (\bmod R) \text{ for some } a \in {\mathbb{F}_q}^* \\ 0 & \text{otherwise.} \end{cases}$$

From Lemmas 1.4.2 and 1.4.3 we can deduce the following.

**Lemma 1.4.4** (Orthogonality Relations)**.** *Let $R \in \mathcal{M}$. Then,*

$$\sum_{\chi \bmod R} \chi(A)\overline{\chi}(B) = \begin{cases} \phi(R) & \text{if } (AB, R) = 1 \text{ and } A \equiv B (\bmod R) \\ 0 & \text{otherwise.} \end{cases}$$

*and, if $R \neq 1$ as well, then*

$$\sum_{\substack{\chi \bmod R \\ \chi \text{ even}}} \chi(A)\overline{\chi}(B)$$
$$= \begin{cases} \frac{1}{q-1}\phi(R) & \text{if } (AB, R) = 1 \text{ and } A \equiv aB (\bmod R) \text{ for some } a \in {\mathbb{F}_q}^* \\ 0 & \text{otherwise.} \end{cases}$$

With regards to primitive characters, we have the following.

**Lemma 1.4.5.** *For $R \in \mathcal{M}$, we have*

$$\sideset{}{^*}\sum_{\chi \bmod R} \chi(A)\overline{\chi}(B) = \begin{cases} \sum_{\substack{EF=R \\ F \mid (A-B)}} \mu(E)\phi(F) & \text{if } (AB, R) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

*and*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod R \\ \chi \text{ even}}} \chi(A)\overline{\chi}(B) = \begin{cases} \frac{1}{q-1} \sum_{a \in {\mathbb{F}_q}^*} \sum_{\substack{EF=R \\ F \mid (A-aB)}} \mu(E)\phi(F) & \text{if } (AB, R) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We prove the first result. The proof of the second result is similar. The case $(AB, R) \neq 1$ is trivial. If $(AB, R) = 1$, then

$$\sum_{\chi \bmod R} \chi(A)\overline{\chi}(B) = \sum_{EF=R} \sideset{}{^*}\sum_{\chi \bmod E} \chi(AB^{-1}).$$

Now we apply the Möbius inversion formula, and make use of Lemma 1.4.4. The result follows ☐

**Corollary 1.4.6.** *By taking $A, B = 1$ in Lemma 1.4.5, we can see that*

$$\phi^*(R) = \sum_{EF=R} \mu(E)\phi(F).$$

We can now define Dirichlet $L$-functions in function fields, and demonstrate some of their properties. A Dirichlet $L$-function on $\mathcal{A}$ is a complex function defined, for $\text{Re}(s) > 1$, by

$$L(s, \chi) := \sum_{A \in \mathcal{M}} \frac{\chi(A)}{|A|^s} = \sum_{n=0}^{\infty} L_n(\chi) q^{-ns},$$

where $\chi$ is a Dirichlet character and $L_n(\chi) := \sum_{A \in \mathcal{M}_n} \chi(A)$. If $\chi_0$ is the trivial Dirichlet character of modulus $R$, then

$$L(s, \chi_0) = \sum_{\substack{A \in \mathcal{M} \\ (A,R)=1}} \frac{1}{|A|^s} = \prod_{\substack{P \in \mathcal{P} \\ P \nmid R}} \frac{1}{1 - \frac{1}{|P|^s}} = \prod_{P \mid R} \left(1 - \frac{1}{|P|^s}\right) \zeta_{\mathcal{A}}(s).$$

We can see that the far RHS provides a meromorphic continuation to $\mathbb{C}$ with simple poles at $1 + \frac{2m\pi i}{\log q}$ for $m \in \mathbb{Z}$. We also see that the Riemann hypothesis holds here as well. Now suppose $\chi$ is a non-trivial character of modulus $R \in \mathcal{M}\backslash\{1\}$ and $n \geq \deg R$. Then, by the periodicity of Dirichlet characters and Lemma 1.4.1, we have

$$L_n(\chi) = \sum_{\substack{B \in \mathcal{M} \\ \deg B \leq n - \deg R}} \sum_{\deg A < \deg R} \chi(A) = 0. \tag{1.13}$$

Hence,

$$L(s, \chi) = \sum_{n=0}^{\deg R - 1} L_n(\chi) q^{-ns}.$$

That is, $L(s, \chi)$ is a finite polynomial in $q^{-s}$, and this provides a holomorphic extension to $\mathbb{C}$. We also have a functional equation (see [Ros02, Theorem 9.24 A]): If $\chi$ is an even primitive character of modulus $R \in \mathcal{M}$, then

$$\big(q^{1-s} - 1\big) L(s, \chi) = W(\chi) q^{\frac{\deg R}{2}} \big(q^{-s} - 1\big) \big(q^{-s}\big)^{\deg R - 1} L(1 - s, \overline{\chi}); \tag{1.14}$$

and if $\chi$ is an odd primitive character of modulus $R \in \mathcal{M}$, then

$$L(s, \chi) = W(\chi) q^{\frac{\deg R - 1}{2}} \big(q^{-s}\big)^{\deg R - 1} L(1 - s, \overline{\chi}); \tag{1.15}$$

where $|W(\chi)| = 1$.

It was conjectured by Weil [Wei49] that Dirichlet $L$-functions in $\mathcal{A}$, as well as various generalisations, satisfy a Riemann hypothesis, asserting that all their zeros lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. This ground-breaking result was first proven by Deligne [Del74]. It is arguably the most important result in the function field setting, and it demonstrates some of the major differences between this setting and the number field setting.

We end this section by stating the analogous conjecture of (1.9) for Dirichlet $L$-functions in $\mathcal{A}$, which will be required later. See [CF00] for details. For all non-negative integers $k$, it is conjectured that

$$\lim_{\deg R \longrightarrow \infty} \frac{1}{(\deg R)^{k^2}} \frac{1}{\phi^*(R)} \prod_{P \mid R} \bigg( \sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \bigg) \sideset{}{^*}\sum_{\chi \bmod R} \Big| L\Big(\frac{1}{2}, \chi\Big) \Big|^{2k} = f(k) a(k), \tag{1.16}$$

where

$$a(k) := \prod_{P \in \mathcal{P}} \left( \left( 1 - \frac{1}{|P|} \right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k\big(P^m\big)^2}{|P|^m} \right).$$

Again, $f(0) = 1$, $f(1) = 1$, $f(2) = \frac{1}{12}$, and as we will see later, it is conjectured that

$$f(k) = \frac{G^2(k+1)}{G(2k+1)} = \prod_{i=0}^{k-1} \frac{i!}{(i+k)!},$$

where $G$ is the Barnes $G$-function.

## 1.5 Random Matrix Theory

It has been known for some time that there is a relationship between the zeros of the Riemann zeta-function and the eigenvalues of random unitary matrices. In 1972 it was observed by Montgomery and Dyson that the pair correlation of the non-trivial zeros of the Riemann zeta-function appears to behave similarly to the pair correlation of eigenvalues of a typical Hermitian matrix. The latter can be seen to be the same as the pair correlation of the eigenphases of a typical unitary matrix. Later, Odlyzko produced numerical evidence in support of this [Odl87].

To be more explicit, assuming the Riemann hypothesis, let us write the non-trivial zeros of the Riemann zeta-function as $\rho_n = \frac{1}{2} + i\gamma_n$. As we know, the number of zeros up to height $T$ on the critical line is asymptotic to $\frac{T}{2\pi}\log\frac{T}{2\pi}$, so let us "unfold" the zeros by taking $w_n := \frac{\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi}$, so that we have unit mean spacing between the zeros: $\lim_{W\longrightarrow\infty}\frac{1}{W}|\{w_n \in [0,W]\}| = 1$. Now define

$$F(\alpha,\beta;W) = \frac{1}{W}|\{w_n, w_m \in [0,W] : \alpha \le w_n - w_m < \beta\}|.$$

Montgomery [Mon73] conjectured that

$$F(\alpha,\beta) := \lim_{W\longrightarrow\infty} F(\alpha,\beta;W)$$

exists. Furthermore, we note that

$$F(\alpha,\beta;W) = \sum_{w_n,w_m\in[0,W]} \mathbb{1}_{[\alpha,\beta)}(w_n - w_m).$$

Let us replace the function $\mathbb{1}_{[\alpha,\beta)}$ with a function $f$ that has Fourier transform with support in $(-1,1)$. Montgomery showed that

$$\lim_{W\longrightarrow\infty} \sum_{\substack{w_n,w_m\in[0,W]\\ w_n\neq w_m}} f(w_n - w_m) = \int_{-\infty}^{\infty} f(x)\left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)\mathrm{d}x,$$

which led him to conjecture that

$$F(\alpha,\beta) = \int_{\alpha}^{\beta}\left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right)\mathrm{d}x, \tag{1.17}$$

where $\delta$ is Dirac's delta-function. Consider now the unitary matrices. For $A \in U(N)$ we can write the eigenvalues of $A$ as $e^{i\theta_n(A)}$ with $1 \le n \le N$. We refer to the $\theta_n(A)$ as the eigenphases of $A$ and we can see that their mean density is $\frac{N}{2\pi}$. We unfold them by defining $\phi_n(A) := \frac{N}{2\pi}\theta_n(A)$. We can now define

$$F_U(\alpha,\beta;N,A) := \frac{1}{N}|\{\phi_n(A), \phi_m(A) : \alpha \le \phi_n(A) - \phi_m(A) < \beta\}|.$$

Dyson [Dys62] showed that

$$F_U(\alpha,\beta) := \lim_{N\longrightarrow\infty}\int_{A\in U(N)} F_U(\alpha,\beta;N,A)\mathrm{d}A$$

exists, where the integral is with respect to the Haar measure, and that

$$F_U(\alpha, \beta) = \int_\alpha^\beta \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 + \delta(x)\right) \mathrm{d}x.$$

We see that this is identical to (1.17). Further results on correlations exist; for a survey regarding random matrix theory and the Riemann zeta function, we recommend [KS03].

Given that the eigenvalues of a matrix are the zeros of its characteristic polynomial, it is reasonable to expect a relationship between $\zeta(s)$ on the critical line and the characteristic polynomials of unitary matrices. Keating and Snaith [KS00b] modeled $\zeta(s)$ at around height $T$ on the critical line by the characteristic polynomial of a random $N \times N$ unitary matrix. (Here, $N$ is chosen such that the mean spacing between the eigenphases of a random $N \times N$ unitary matrix is the same as the mean spacing of the zeros of the Riemann zeta-function at around height $T$ on the critical line). They obtained the following result for $\mathrm{Re}(k) > -1$:

$$\int_{U \in U(N)} |Z(U, \theta)|^{2k} \mathrm{d}U = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+s)}{\left(\Gamma(j + \frac{s}{2})\right)^2} \sim f_{CUE}(k) N^{k^2}. \tag{1.18}$$

Here $U(N)$ is the set of all unitary $N \times N$ matrices; for all $U \in U(N)$, we take $Z(U, \theta) := \det\left(I_N - Ue^{-i\theta}\right)$ to be the characteristic polynomial of $U$; the integral is with respect to the Haar measure on $U(N)$; "CUE" stands for circular unitary ensemble; and

$$f_{CUE}(k) := \frac{G^2(1+k)}{G(1+2k)}.$$

In particular, if $k$ is a non-negative integer, we have $f_{CUE}(k) = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$. (The fact that (1.18) is independent of $\theta$ is not immediately obvious, and so we remark that this lack of dependency is not an error). Now, we note that

$$f_{CUE}(k) = \begin{cases} 1 & \text{if } k = 1; \\ \frac{1}{12} & \text{if } k = 2; \\ \frac{42}{9!} & \text{if } k = 3; \\ \frac{24024}{16!} & \text{if } k = 4. \end{cases}$$

That is, $f_{CUE}(k)$ agrees with the established values of $f(k)$ that are described in (1.4), as well as the values that have been conjectured by number theoretic means (again, see (1.4)). This lends strong support to the connection between the Riemann zeta-function and random matrix theory, and it provides a conjecture for the moments of the Riemann zeta-function.

Note that this conjecture does not introduce the factor $a(k)$ in (1.4) in any natural way. This was addressed by Gonek, Hughes, and Keating [GHK07] who expressed $\zeta(s)$ as a hybrid Euler-Hadamard product: $\zeta(s) \approx P_X(s)Z_X(s)$, where $P_X(s)$ is a roughly a partial Euler product and $Z_X(s)$ is roughly a partial Hadamard product (a product over the zeros of $\zeta(s)$). $X$ is a variable that determines the contribution of each factor. They conjectured that, asymptotically, the main term of the $2k$-th moment of $\zeta(s)$ on the critical line can be factored into the main term of the $2k$-th moment of $P_X(s)$ multiplied by the main term of the $2k$-th moment of $Z_X(s)$ (known as the splitting conjecture); and they showed that the former contributes the factor $a(k)$ in (1.4) and conjectured via random matrix theory that the latter contributes the factor $f(k)$. That is they obtained a conjecture for the $2k$-th moment of $\zeta(s)$ in a way that the factor $a(k)$ appears naturally. They also lent support for the splitting conjecture by demonstrating that it holds for the cases $k = 1, 2$.

The relationship between random matrix theory and the Riemann zeta-function extends to other $L$-functions, particularly certain families of $L$-functions [KS99]. For example, one aspect of the relationship is that the proportion of $L$-functions of a certain family with conductor $q$ that have $j$-th zero in some interval $[a, b]$ appears to be the same as the proportion of matrices of a certain matrix ensemble of size $N \times N$ that have $j$-th eigenphase in $[a, b]$. At least, this appears to be the case as $q \longrightarrow \infty$. $N = N(q)$ is chosen so that the mean spacing of the eigenvalues is the same as the mean spacing of the zeros of the $L$-functions of conductor $q$. The ensemble is dependent on the family. In the number field setting the reason for the connection between the family and the ensemble is not directly evident. One must consider the function field analogue of the family, and in this setting we have a spectral interpretation of the zeros that allows us to determine the associated ensemble. Therefore, function fields play a key role in the relationship between $L$-functions and random matrix theory. We recommend [KS99] and [CF00] for a more detailed discussion on this topic.

Let us consider the family of Dirichlet $L$-functions. The associated ensemble of matrices is the unitary matrices [CF00, page 887]. By making use of this relationship, Bui and Keating [BK07] obtained an analogue of [GHK07] where they considered the $2k$-th moment of Dirichlet $L$-functions at $s = \frac{1}{2}$, averaged over all primitive Dirichlet $L$-functions of modulus $q$, instead of the Riemann zeta-function averaged over the critical line. That is, using a hybrid Euler-Hadamard product for the Dirichlet $L$-functions, they conjectured (among other results) that, for non-negative integers $k$,

$$\frac{1}{\phi^*(q)} \sum_{\chi \bmod q}^{*} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^{2k} \sim a(k) f_{CUE}(k) \, (\log q)^{k^2} \prod_{p|q} \bigg( \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^m} \bigg)^{-1}$$

as $q \longrightarrow \infty$. This had been conjectured previously (see [KS00a]), but this approach allows for all the factors to appear naturally. The similarity with (1.4) is due to the fact that the Riemann-zeta function on the critical line, and the family of Dirichlet $L$-functions evaluated at $\frac{1}{2}$, share the same symmetry and associated matrix ensemble.

In the function field setting, Bui and Florea [BF18] developed the hybrid Euler-Hadamard product model for the family of quadratic Dirichlet $L$-functions. In the number field setting, Bui, Gonek, and Milinovich [BGM15] developed such a model for the discrete moments of the derivative of the Riemann zeta-function. As we will see later, we develop such a model for the family of all Dirichlet $L$-functions, as well as their first derivatives, at the central value of $\frac{1}{2}$ in the function field setting.

# Chapter 2

# Statement and Discussion of Results

In this chapter we state the main results that we prove in this thesis and discuss them with regards to, for example, their applications and their relations to the work of others. The actual proofs and their preliminary lemmas are given in the later chapters. In addition to the background given in Section 1.4, further results on function fields are given in Appendix A. These results are required in this thesis, but they are well known, and hence the reason they are in an appendix.

## 2.1 The Brun-Titchmarsh Theorem for the Divisor Function

This thesis focuses on mean values of Dirichlet $L$-functions and their derivatives at the central value in function fields. A crucial result for the proofs of all fourth moment results in this thesis is the function field analogue of Shiu's generalised Brun-Titchmarsh theorem [Shi80], for the special case of the divisor function:

**Theorem 2.1.1.** *Suppose $\alpha, \beta$ are fixed and satisfy $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. Let $X \in \mathcal{M}$ and $y$ be a positive integer satisfying $\beta \deg X < y \leq \deg X$. Also, let $A \in \mathcal{A}$ and $G \in \mathcal{M}$ satisfy $(A,G) = 1$ and $\deg G < (1-\alpha)y$. Then, we have that*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G)}} d(N) \ll_{\alpha,\beta} \frac{q^y \deg X}{\phi(G)}.$$

**Remark 2.1.2.** *We are summing all monic polynomials, in a certain arithmetic progression, whose difference with $X$ is of degree less than $y$. The condition on the size of $y$ is necessary to ensure that we are working in a large enough interval relative to the size of $X$ so that we can ensure that we see average behaviour. The condition on the size of $G$ is necessary to ensure that our arithmetic progression in the interval has a large enough number of elements; again, this is to ensure that we see average behaviour.*

*The result itself makes intuitive sense. Indeed, the polynomials we are summing over are of degree equal to $\deg X$, and therefore, we expect the divisor function to be, on average, of size $\deg X$. Furthermore, there are $\frac{q^y}{|G|} \approx \frac{q^y}{\phi(G)}$ number of polynomials in* *our sum.*

*It is trivial to extend the theorem to $0 < \alpha < 1$ and $0 < \beta < 1$ once you have proved it for $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. So, it is a bit unusual to use the latter condition. We do this as it is the form Shiu used, as we see in Theorem 2.1.4.*

We also prove another similar result:

**Theorem 2.1.3.** *Suppose $\alpha, \beta$ are fixed and satisfy $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. Let $X \in \mathcal{M}$ and $y$ be a positive integer satisfying $\beta \deg X < y \leq \deg X$. Also, let $A \in \mathcal{A}$ and $G \in \mathcal{M}$ satisfy $(A, G) = 1$ and $\deg G < (1-\alpha)y$. Finally, let $a \in \mathbb{F}_q^*$. Then, we have that*

$$\sum_{\substack{N \in \mathcal{A} \\ \deg(N-X)=y \\ (N-X) \in a\mathcal{M} \\ N \equiv A (\operatorname{mod} G)}} d(N) \ll_{\alpha,\beta} \frac{q^y \deg X}{\phi(G)}.$$

We prove Theorems 2.1.1 and 2.1.3 in Chapter 3. They require the Selberg sieve in function fields. This is an established result in this setting, but a precise statement and proof is difficult to come by. Therefore, we provide this in Appendix B.

For comparison, we give Shiu's theorem in the number field setting:

**Theorem 2.1.4** (Shiu)**.** *Suppose $f$ is a non-negative, multiplicative function satisfying the following two conditions:*

1. *There exists a positive constant $A_1$ such that $f(p^l) \leq {A_1}^l$ for all primes $p$ and all integers $l \geq 1$.*
2. *For every $\epsilon > 0$, there exists a positive constant $A_2(\epsilon)$ such that $f(n) \leq A_2(\epsilon) n^\epsilon$ for all $n \geq 1$.*

*Furthermore, let $0 < \alpha, \beta < \frac{1}{2}$, and let $a, k$ be integers satisfying $0 < a < k$ and $(a, k) = 1$. Then,*

$$\sum_{\substack{x-y<n\leq x \\ n \equiv a (\operatorname{mod} k)}} f(n) \ll \frac{y}{\phi(k)} \frac{1}{\log x} \exp\left( \sum_{\substack{p \leq x \\ p \nmid k}} \frac{f(p)}{p} \right),$$

*uniformly in $a, k, y$ provided that $k < y^{1-\alpha}$ and $x^\beta < y \leq x$.*

## 2.2 The Second and Fourth Moments of Dirichlet $L$-functions

In Chapter 4 we obtain the main term of the second and fourth moment of Dirichlet $L$-functions at the central value, where we average over all primitive characters of modulus $R \in \mathcal{M}$. We also obtain an exact formula for the second moment, where we average over the primitive characters of square-full $R \in \mathcal{M}$:

**Theorem 2.2.1.** *Let $R \in \mathcal{M} \backslash \{1\}$. Then,*

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \operatorname{mod} R} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^2 = \frac{\phi(R)}{|R|} \deg R + O\bigg( \frac{\phi(R)}{|R|} \log \omega(R) \bigg).$$

**Theorem 2.2.2.** *Let $R \in \mathcal{M}\backslash\{1\}$ be a square-full polynomial. Then,*

$$\sum_{\chi \bmod R}^{*} \Big| L\Big(\frac{1}{2}, \chi\Big)\Big|^2 = \frac{\phi(R)^3}{|R|^2} \deg R + 2\frac{\phi(R)^3}{|R|^2} \sum_{P|R} \frac{\deg P}{|P|-1} + \frac{1}{\big(q^{\frac{1}{2}}-1\big)^2} \bigg( -\frac{\phi(R)^3}{|R|^2} + 2\frac{\phi(R)}{|R|^{\frac{1}{2}}} \prod_{P|R} \Big(1 - \frac{1}{|P|^{\frac{1}{2}}}\Big)^2 \bigg).$$

We prove these two theorems in Section 4.1. One may ask why we are able to obtain an exact formula in Theorem 2.2.2 but not in Theorem 2.2.1. To answer this, suppose we have $EF = R$ where $\mu(E) \neq 0$. Sums over such $E, F$ appear in the proofs of Theorems 2.2.1 and 2.2.2. Now, if $R$ is square-full, then $F$ has the same prime factors as $R$. This makes our calculations considerably easier and so we can obtain an exact formula. On the other hand, if $R$ is not square-full, then $F$ will not always have the same prime factors as $R$. This complicates matters and we are required to bound certain terms, which ultimately means we do not obtain an exact formula.

In Section 4.4 we prove the following.

**Theorem 2.2.3.** *Let $R \in \mathcal{M}\backslash\{1\}$. Then,*

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big| L\Big(\frac{1}{2}, \chi\Big)\Big|^4 = \frac{1-q^{-1}}{12} \prod_{P|R} \bigg( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \bigg) (\deg R)^4 \Bigg( 1 + O\bigg( \sqrt{\frac{\omega(R) + (\log \deg R)^6}{\deg R}} \bigg) \Bigg).$$

One should compare Theorems 2.2.1 and 2.2.3 to (1.6) and (1.8), while noting that $\frac{1-q^{-1}}{12} = \frac{1}{12\zeta_{\mathcal{A}}(2)}$ and $\frac{1}{2\pi^2} = \frac{1}{12\zeta(2)}$, and comparing $|R|$ and $\deg R$ above to $q$ and $\log q$ in (1.6) and (1.8) [1].

We dedicate two sections, 4.2 and 4.3, to preliminary results that are required for the proof of Theorem 2.2.3. The first involves preliminary results that are likely to have applications to other problems. The second involves preliminary results for which it is more difficult to find other applications.

Before proceeding, we give a brief outline of the proof of Theorem 2.2.3. By applying the functional equation and then Lemma 1.4.5, we obtain, after some rearrangement, the following:

$$\sum_{\chi \bmod R}^{*} \Big| L\Big(\frac{1}{2}, \chi\Big)\Big|^4 = \Big( \sum_{EF=R} \mu(E)\phi(F) \Big) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB < \deg R \\ \deg CD < \deg R \\ (ABCD,R)=1 \\ AC=BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} + \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB < \deg R \\ \deg CD < \deg R \\ (ABCD,R)=1 \\ AC \equiv BD (\bmod F) \\ AC \neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}}.$$

[1]Please excuse the fact that in the function field setting the parameter $q$ is different to that in the number field setting. This is the case because we wished to preserve the standard notation that is used in the two settings.

The first term on the RHS is called the diagonal term (or terms) and the second term on the RHS is called the off-diagonal term (or terms). The former is considerably easier to address than the latter. Indeed, only the first half of Section 4.2 is required for its preliminary results.

Now, consider the inner sum of the off-diagonal terms. The key aspect is the condition $AC \equiv BD$. Let us write $N := BD$. Note that, given the restrictions, there are at most $d(N)$ ways to choose $B, D$, assuming $N$ is fixed. Let us also write $AC = KF + N$. This comes from the fact that $AC$ is equivalent to $BD$ modulo $F$. Again, note that, given the restrictions, there are at most $d(KF + N)$ ways to choose $A, C$, assuming $KF + N$ is fixed. So, the inner sum of our off-diagonal sum can be written as something similar to

$$\sum_{K} \sum_{N} d(N) d(KF + N). \tag{2.1}$$

That is, we have expressed our off-diagonal terms in terms of sums of the product of a divisor function and a shifted divisor function. Section 4.3 is dedicated to rigorously establishing this.

The second half of Section 4.2 is dedicated to solving sums of the form (2.1). Instead of having a sum over the product of a divisor function and a shifted divisor function, we express this in terms of sums of a single divisor function, but the summation ranges are over arithmetic progressions in intervals. We can then apply our analogue of the Brun-Titchmarsh theorem. This concludes our brief outline of the proof.

## 2.3 The First, Second, and Fourth Moments of Derivatives of Dirichlet $L$-functions

In Chapter 5, we focus on moments of derivatives of Dirichlet $L$-functions at the central value, but we average only over the primitive characters of modulus $Q \in \mathcal{P}$. Since $Q$ is prime, the only non-primitive character of modulus $Q$ is the trivial one, and so $\phi^*(Q) = \phi(Q) - 1 \sim \phi(Q)$ as $\deg Q \longrightarrow \infty$. In Section 5.1, we prove the following two results.

**Theorem 2.3.1.** *For all positive integers $k$, we have that*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} L^{(k)}\Big(\frac{1}{2}, \chi\Big) = \frac{-(-\log q)^k}{q^{\frac{1}{2}} - 1} \frac{(\deg Q)^k}{|Q|^{\frac{1}{2}}} + O_k\bigg( (\log q)^k \frac{(\deg Q)^{k-1}}{|Q|^{\frac{1}{2}}} \bigg).$$

**Theorem 2.3.2.** *For all positive integers $k$ we have that*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \bigg| L^{(k)}\Big(\frac{1}{2}, \chi\Big) \bigg|^2 = \frac{(\log q)^{2k}}{2k+1} (\deg Q)^{2k+1} + O\Big( (\log q)^{2k} (\deg Q)^{2k} \Big).$$

In Section 5.4, we prove the following two results.

**Theorem 2.3.3.** *For all non-negative integers $k, l$ we have that*

$$\frac{1}{\phi(Q)} \frac{1}{(\log q)^{2k+2l}} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| L^{(k)}\Big(\frac{1}{2}, \chi\Big) \Big|^2 \Big| L^{(l)}\Big(\frac{1}{2}, \chi\Big) \Big|^2$$

$$
\begin{aligned}
&= (1-q^{-1})(\deg Q)^{2k+2l+4} \\
&\qquad \cdot \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} f_k\big(a_1+a_3, a_1+a_4, 1\big) f_l\big(a_2+a_4, a_2+a_3, 1\big) \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4 \\
&\quad + O_{k,l}\Big((\deg Q)^{2k+2l+\frac{7}{2}}\Big),
\end{aligned}
$$

*where for all non-negative integers $i$ we define*

$$
f_i\big(x,y,z\big) = x^i y^i + (z-x)^i (z-y)^i.
$$

**Theorem 2.3.4.** *For all non-negative integers $m$ we define*

$$
\begin{aligned}
&D_m \\
&:= \lim_{\deg Q \longrightarrow \infty} \frac{1}{(1-q^{-1})(\log q)^{4m}} \frac{1}{\phi(Q)} \frac{1}{(\deg Q)^{4m+4}} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| L^{(m)}\Big(\frac{1}{2}, \chi\Big) \Big|^4 \\
&= \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} \Big( (a_1+a_3)^m (a_1+a_4)^m + (1-a_1-a_3)^m (1-a_1-a_4)^m \Big) \\
&\qquad \cdot \Big( (a_2+a_3)^m (a_2+a_4)^m + (1-a_2-a_3)^m (1-a_2-a_4)^m \Big) \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4.
\end{aligned}
$$

*We have that*

$$
D_m \sim \frac{1}{16m^4}
$$

*as $m \longrightarrow \infty$.*

In proving Theorem 2.3.3, one runs into certain obstacles. One such obstacle is that the functional equation ((1.14) and (1.15)), which is required to express the derivatives of our Dirichlet $L$-functions as short sums, is more difficult to use. This is because we are working with *derivatives* of $L$-functions, and this ultimately means we must take derivatives of the functional equation. For the even case, this is problematic as the $k$-th derivative involves $2^k$ terms on each side, only one of which is what we want. Section 5.2 is dedicated to addressing this. Section 5.3 is dedicated to handling some of the summations that arise in the proof of Theorem 2.3.3.

We now discuss the results. Theorems 2.3.1, 2.3.2, and 2.3.3 are extensions of Tamam's work [Tam14], where she proves them for the cases where $k, l = 0$. There is, however, an error in her work in that she takes $d(A) \ll \deg A$ on page 209, which does not hold. We ultimately address this via our use of the analogue of the Brun-Titchmarsh theorem that we developed in Chapter 3, as well as several subsequent results.

In the number field setting, Conrey, Rubinstein, and Snaith [CRS06] conjectured using random matrix theory that, for positive integers $k$,

$$
\frac{1}{T} \int_{t=0}^{T} \Big| \zeta'\Big(\frac{1}{2}+it\Big) \Big|^{2k} \mathrm{d}t \sim a_k b_k (\log T)^{k^2+2k}
$$

as $\deg T \longrightarrow \infty$, where

$$
a_k := \prod_{p \text{ prime}} \Bigg( \Big(1-\frac{1}{p}\Big)^{k^2} \sum_{m=0}^{\infty} \Big( \frac{\Gamma(m+k)}{m!\Gamma(k)} \Big)^2 p^{-m} \Bigg)
$$

and values for $b_1, b_2, \ldots, b_{15}$ are explicitly given. In particular,

$$\begin{aligned} b_1 =& \frac{1}{3}, \\ b_2 =& \frac{61}{2^5 \cdot 3^2 \cdot 5 \cdot 7}. \end{aligned}$$

Notice the similarity between these conjectures and the corresponding special cases of our results:

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| L'\left(\frac{1}{2}, \chi\right) \right|^2 \sim (\log q)^2 \frac{1}{3} (\deg Q)^3 \tag{2.2}$$

and

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| L'\left(\frac{1}{2}, \chi\right) \right|^4 \sim (\log q)^4 (1 - q^{-1}) \frac{61}{2^5 \cdot 3^2 \cdot 5 \cdot 7} (\deg Q)^8. \tag{2.3}$$

This is not surprising given that the Riemann zeta-function and the family of Dirichlet $L$-functions share the same symmetry [CF00]. While conjectures on moments of the derivatives of $\zeta(s)$ are made in [CRS06], definitive results have been obtained by Conrey [Con88]. While there are some clear differences, there are still similarities between Theorem (3) in [Con88] and our Theorem 2.3.3. Another of Conrey's result states that

$$\frac{\pi^2}{6} C_{2,m} \sim \frac{1}{16 m^4}$$

as $m \longrightarrow \infty$, where

$$C_{2,m} = \lim_{T \longrightarrow \infty} T^{-1} \left( \log \left( \frac{T}{2\pi} \right) \right)^{-4m-4} \int_{t=1}^{T} \left| \zeta^{(m)} \left( \frac{1}{2} + it \right) \right|^4 \mathrm{d}t.$$

This was our motivation for obtaining Theorem 2.3.4. Note the similarity between the two results. The factor of $\zeta(2) = \frac{\pi^2}{6}$ in Conrey's result corresponds to the factor of $\zeta_{\mathcal{A}}(2) = \frac{1}{1-q^{-1}}$ in our definition of $D_m$.

## 2.4 A Random Matrix Theory Model for Moments of Dirichlet $L$-functions

In Chapter 6, analogous to the work of Bui and Keating [BK07] in the number field setting that is described in Section 1.5, we develop a random matrix theory model for the moments of Dirichlet $L$-functions at the central value in function fields, where, again, we average over primitive characters of modulus $R \in \mathcal{M}$. First, in Section 6.1, we prove an Euler-Hadamard hybrid formula:

**Theorem 2.4.1.** *Let $X \geq 1$ be an integer and let $u(x)$ be a positive $C^\infty$-function with support in $[e, e^{1+q^{-X}}]$. Let*

$$v(x) = \int_{t=x}^{\infty} u(t) \mathrm{d}t$$

*and take $u$ to be normalised so that $v(0) = 1$. Furthermore, for $y \in \mathbb{C}\backslash\{0\}$ with $\arg(y) \neq \pi$, we define $E_1(y) := \int_{w=y}^{y+\infty} \frac{e^{-w}}{w} \mathrm{d}w$; and for $z \in \mathbb{C}\backslash\{0\}$ with $\arg(z) \neq \pi$, we define*

$$U(z) := \int_{x=0}^{\infty} u(x) E_1(z \log x) \mathrm{d}x.$$

*Let $\chi$ be a primitive Dirichlet character of modulus $R \in \mathcal{M}\backslash\{1\}$, and let $\rho_n = \frac{1}{2} + i\gamma_n$ be the $n$-th zero of $L(s,\chi)$. Then, for all $s \in \mathbb{C}$ we have*

$$L(s,\chi) = P_X(s,\chi) Z_X(s,\chi), \tag{2.4}$$

*where*

$$P_X(s,\chi) = \exp\Bigg( \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq X}} \frac{\chi(A)\Lambda(A)}{|A|^s \log|A|} \Bigg)$$

*and*

$$Z_X(s,\chi) = \exp\bigg( -\sum_{\rho_n} U\Big( (s-\rho_n)(\log q) X \Big) \bigg).$$

*Strictly speaking, if $s = \rho$ or $\arg(s-\rho) = \pi$ for some zero $\rho$ of $L(s,\chi)$, then $Z_X(s,\chi)$ is not well defined. In this case, we take*

$$Z_X(s,\chi) = \lim_{s_0 \longrightarrow s} Z_X(s_0,\chi)$$

*and we show that this is well defined.*

**Remark 2.4.2.** *We note that our hybrid Euler-Hadamard product formula, (2.4), does not involve an error term, unlike the analogous Theorem 1 in [GHK07] and Theorem 1 in [BK07]. This is due to the fact that we are working in the function field setting.*

*We also note that $Z_X(s,\chi)$ is expressed in terms of $u(x)$. Whereas, $P_X(s,\chi)$ and $L(s,\chi)$ are independent of $u(x)$. Thus, given the equality (2.4), we can see that, as long as $u(x)$ satisfies the conditions in the theorem, the value of $Z_X(s,\chi)$ is independent of any further choice made on $u(x)$. Ultimately, this is due to the fact that we are working in the function field setting and due to our choice of support for $u(x)$. Indeed, this is why our support for $u(x)$ is not quite analogous to the support of $u(x)$ in Theorem 1 of [BK07]. We note that in Theorem 1 in [BK07], $P_X(s,\chi)$ and $L(s,\chi)$ also do not depend on $u(x)$, but this is because the dependency exists in the error term.*

We conjecture that the $2k$-th moment of the $L$-functions can be split into the $2k$-th moment of their partial Euler products multiplied by $2k$-th moment of their partial Hadamard products:

**Conjecture 2.4.3** (Splitting Conjecture)**.** *For integers $k \geq 0$, we have*

$$\begin{aligned} &\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big| L\Big(\frac{1}{2},\chi\Big) \Big|^{2k} \\ \sim &\bigg( \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big| P_X\Big(\frac{1}{2},\chi\Big) \Big|^{2k} \bigg) \cdot \bigg( \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big| Z_X\Big(\frac{1}{2},\chi\Big) \Big|^{2k} \bigg) \end{aligned}$$

*as $X, \deg R \longrightarrow \infty$ with $X \leq \log_q \deg R$.*

We then obtain the $2k$-th moment of the partial Euler products in Section 6.2, and we use a random matrix theory model to conjecture the $2k$-th moment of the Hadamard products in Section 6.3:

**Theorem 2.4.4.** *For positive integers $k$, we have*

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| P_X\Big(\frac{1}{2}, \chi\Big) \right|^{2k} \sim a(k) \Bigg[ \prod_{\substack{\deg P \leq X \\ P|R}} \bigg( \sum_{m=0}^{\infty} \frac{d_k\big(P^m\big)^2}{|P|^m} \bigg)^{-1} \Bigg] \Big(e^{\gamma} X\Big)^{k^2}$$

*as $X, \deg R \longrightarrow \infty$ with $X \leq \log_q \deg R$. Here, $\gamma$ is the Euler-Mascheroni constant, and*

$$a(k) = \prod_{P \in \mathcal{P}} \left( \Big(1 - \frac{1}{|P|}\Big)^{k^2} \sum_{m=0}^{\infty} \frac{d_k\big(P^m\big)^2}{|P|^m} \right).$$

**Conjecture 2.4.5.** *For integers $k \geq 0$, we have*

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| Z_X\Big(\frac{1}{2}, \chi\Big) \right|^{2k} \sim \frac{G^2(k+1)}{G(2k+1)} \Big(\frac{\deg R}{e^{\gamma} X}\Big)^{k^2},$$

*as $\deg R \longrightarrow \infty$, where $\gamma$ is the Euler-Mascheroni constant and $G$ is the Barnes $G$-function. For our purposes, it suffices to note that*

$$\frac{G^2(k+1)}{G(2k+1)} = \prod_{i=0}^{k-1} \frac{i!}{(i+k)!}.$$

**Remark 2.4.6.** *We must point out that our support for Conjecture 2.4.5 follows the method of [GHK07] and relies on results established in [GHK07]. However, as will be described in Remark 6.3.2, there is an error in one of these results. For this reason, we reformulate Conjecture 2.4.5 into Conjecture 2.5.10 and provide support for this in Chapter 7 (specifically Sections 7.1 and 7.2) via a method that differs from that in [GHK07]. Nonetheless, some of the tools developed there can be used to address the error that is described in Remark 6.3.2, and we explain this in Remark 7.2.2.*

In Section 6.4 we rigorously obtain the second moment of the Hadamard product:

**Theorem 2.4.7.** *We have that*

$$\begin{aligned} \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| Z_X\Big(\frac{1}{2}, \chi\Big) \right|^{2} =& \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| L\Big(\frac{1}{2}, \chi\Big) P_X\Big(\frac{1}{2}, \chi\Big)^{-1} \right|^{2} \\ \sim& \frac{\deg R}{e^{\gamma} X} \prod_{\substack{\deg P > X \\ P|R}} \Big(1 - \frac{1}{|P|}\Big) \end{aligned}$$

*as $X, \deg R \longrightarrow \infty$ with $X \leq \log_q \deg R$.*

In Section 6.6 we rigorously obtain the fourth moment of the Hadamard product:

**Theorem 2.4.8.** *We have*

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| Z_X\Big(\frac{1}{2}, \chi\Big) \right|^{4} = \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| L\Big(\frac{1}{2}, \chi\Big) P_X\Big(\frac{1}{2}, \chi\Big)^{-1} \right|^{4}$$

$$\sim \frac{1}{12}\Big(\frac{\deg R}{e^{\gamma}X}\Big)^4 \prod_{\substack{\deg P>X \\ P|R}} \frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}$$

*as* $X, \deg R \longrightarrow \infty$ *with* $X \leq \log_q \log \deg R$.

We can see that Theorems 2.4.7, 2.4.4, and 2.2.1 verify the Splitting Conjecture for the case $k=1$. This can be seen from the fact that $a(1)=1$ and

$$\prod_{\substack{\deg P\leq X \\ P|R}} \left(\sum_{m=0}^{\infty} \frac{d_1\left(P^m\right)^2}{|P|^m}\right)^{-1} = \prod_{\substack{\deg P\leq X \\ P|R}} \left(1-\frac{1}{|P|}\right).$$

Also, we can see that Theorems 2.4.8, 2.4.4, and 2.2.3 verify the Splitting Conjecture for the case $k=2$. This can be seen from the fact that

$$\begin{aligned} a(2) =& \prod_{P\in\mathcal{P}} \left(\left(1-\frac{1}{|P|}\right)^4 \sum_{m=0}^{\infty} \frac{d_2\left(P^m\right)^2}{|P|^m}\right) = \prod_{P\in\mathcal{P}} \left(\left(1-\frac{1}{|P|}\right)^4 \sum_{m=0}^{\infty} \frac{(m+1)^2}{|P|^m}\right) \\ =& \prod_{P\in\mathcal{P}} \left(\left(1-\frac{1}{|P|}\right)^4 \sum_{m=0}^{\infty} \frac{2d_3(P^m)-d_2(P^m)}{|P|^m}\right) \\ =& \prod_{P\in\mathcal{P}} \left(\left(1-\frac{1}{|P|}\right)^4 \left(2\Big(\frac{1}{1-\frac{1}{|P|}}\Big)^3 - \Big(\frac{1}{1-\frac{1}{|P|}}\Big)^2\right)\right) \\ =& \prod_{P\in\mathcal{P}} \left(1-\frac{1}{|P|^2}\right) = \zeta_{\mathcal{A}}(2)^{-1} = 1-q^{-1} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &\prod_{\substack{\deg P\leq X \\ P|R}} \left(\sum_{m=0}^{\infty} \frac{d_2\left(P^m\right)^2}{|P|^m}\right)^{-1} \\ =& \prod_{\substack{\deg P\leq X \\ P|R}} \left(2\Big(\frac{1}{1-\frac{1}{|P|}}\Big)^3 - \Big(\frac{1}{1-\frac{1}{|P|}}\Big)^2\right)^{-1} = \prod_{\substack{\deg P\leq X \\ P|R}} \frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}. \end{aligned}$$

However, in Theorem 2.4.8 we required the condition $X \leq \log_q \log \deg R$ which is stronger than the condition $X \leq \log_q \deg R$ in the Splitting Conjecture.

Furthermore, we note that the Splitting Conjecture, Conjecture 2.4.5, and Theorem 2.4.4 do not together reproduce the conjecture (1.16). Most notably they do if we impose $p_+(R) \leq X$, but also if we impose that $R$ is prime or various other restrictions.

Finally, we note that Theorems 2.4.7 and 2.4.8 are special cases of the twisted second and fourth moments of Dirichlet $L$-functions. Considering the results we referenced at the end of Section 1.3 on twisted moments of Dirichlet $L$-functions in the classical setting, and what would be the function field analogue of the those results, it is likely that one can extend Theorems 2.4.7 and 2.4.8 to hold for a larger range: One of the form $X \leq \kappa \deg R$, where, for the second moment we have $0 < \kappa < \frac{51}{101}$ and for the fourth moment we have $0 < \kappa < \frac{1}{32}$. Further extensions could be possible but would go beyond what is currently established in the classical setting. We could perhaps extend the range of $X$ Conjecture 2.4.3 as well.

## 2.5 A Random Matrix Theory Model for Moments of the First Derivative of Dirichlet $L$-functions

In Chapter 7 we develop a conjecture for the main term of even moments of the first derivative of Dirichlet $L$-functions in $\mathbb{F}_q[T]$ at the central value. As in Section 2.3, where we also worked with derivatives, we are averaging over non-trivial characters of prime modulus $Q$ and taking the limit as $\deg Q \longrightarrow \infty$; although, some of our conjectures apply more generally to $R \in \mathcal{M}$, in which case we write $R$ instead of $Q$. As in Section 2.4, we wish for all factors in the main term to appear naturally. Thus, we will be making use of the Euler-Hadamard hybrid formula and random matrix theory.

We begin by differentiating the formula (2.4) to obtain

$$L'(s,\chi) = P_X'(s,\chi)Z_X(s,\chi) + P_X(s,\chi)Z_X'(s,\chi). \tag{2.6}$$

Similarly as in Section 2.4 we make a splitting conjecture.

**Conjecture 2.5.1** (Splitting Conjecture for the First Derivative). *For all integers $k \geq 0$, we have*

$$\begin{aligned}&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P_X'\Big(\frac{1}{2},\chi\Big) Z_X\Big(\frac{1}{2},\chi\Big) \right|^{2k} \\ \sim &\left( \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P_X'\Big(\frac{1}{2},\chi\Big) \right|^{2k} \right) \cdot \left( \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| Z_X\Big(\frac{1}{2},\chi\Big) \right|^{2k} \right)\end{aligned} \tag{2.7}$$

*and*

$$\begin{aligned}&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P_X\Big(\frac{1}{2},\chi\Big) Z_X'\Big(\frac{1}{2},\chi\Big) \right|^{2k} \\ \sim &\left( \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P_X\Big(\frac{1}{2},\chi\Big) \right|^{2k} \right) \cdot \left( \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| Z_X'\Big(\frac{1}{2},\chi\Big) \right|^{2k} \right)\end{aligned} \tag{2.8}$$

*as $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$.*

Before providing support for this conjecture, we consider its application to the moments of derivatives of Dirichlet $L$-functions. For this, we will require the following conjecture, which is based on random matrix theory. Support for this conjecture is given in Section 7.2, while preliminary results that are required are given in Section 7.1.

**Conjecture 2.5.2.** *Assume that $\max_{x \in \mathbb{R}} \{|u'(x)|\} \ll q^X$. This is certainly possible given that our only requirement on $u$ is that it is a positive, normalised $C^\infty$-function with support in $[e, e^{1+q^{-X}}]$.*

*Let $N(R) := \lfloor (\log q) \deg R \rfloor$. Let $U(N)$ be the set of $N \times N$ unitary matrices and, for $A \in U(N)$, let $\theta_n(A) \in (-\pi, \pi]$ be its $n$-th eigenphase. Let $X \sim \log_q \deg R$. Finally, let*

$$\widehat{\Lambda}_{A,X}(s) := \prod_{|\theta_n(A)| \leq \frac{\pi}{(\log q) e^\gamma X}} \Big(1 - s e^{-i (\log q) e^\gamma X \theta_n(A)}\Big).$$

*Then, for integers $k \geq 0$, as $X, \deg R \longrightarrow \infty$,*

$$\begin{aligned}\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| Z'_X\left(\frac{1}{2}, \chi\right)\right|^{2k} \sim & \int_{A \in U\left(N(R)\right)} \left|\widehat{\Lambda}'_{A,X}(1)\right|^{2k} \mathrm{d}A \\ \sim & b_k N(R)^{2k} \left(\frac{N(R)}{(\log q) e^{\gamma} X}\right)^{k^2} \\ \sim & b_k \big((\log q) \deg R\big)^{2k} \left(\frac{\deg R}{e^{\gamma} X}\right)^{k^2},\end{aligned}$$

*where the integral is with respect to the Haar measure and $b_k$ is as in equation (1.4) of [CRS06]. The first two relations are conjectural, while the last can easily be seen to be true.*

We will also require the following Theorem, which we prove in Section 7.3.

**Theorem 2.5.3.** *Let $k \geq 0$ be an integer. As $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$,*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P'_X\left(\frac{1}{2}, \chi\right)\right|^{2k} \sim a(k) c_k(k) (\log q)^{2k} X^{2k} \left(e^{\gamma} X\right)^{k^2},$$

*where, for $l = 0, \ldots, 2k$, we define*

$$c_k(l) := \sum_{m=0}^{l} \binom{l}{m} \binom{l}{m} m! \frac{1}{2^m} k^{2(l-m)}.$$

(While we only require $c_k(k)$ in the theorem above, we require $c_k(0), \ldots, c_k(2k)$ for the proof of the theorem). We can now give a conjecture for the moments of derivatives of Dirichlet $L$-functions.

**Conjecture 2.5.4.** *By applying Theorem 2.5.3 and Conjecture 2.4.5 to (2.7), we conjecture that*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P'_X\left(\frac{1}{2}, \chi\right) Z_X\left(\frac{1}{2}, \chi\right)\right|^{2k} \sim a(k) c_k(k) \frac{G^2(k+1)}{G(2k+1)} (\log q)^{2k} X^{2k} (\deg Q)^{k^2} \tag{2.9}$$

*as $X, \deg Q \longrightarrow \infty$ with $X = \lfloor \log_q \deg Q \rfloor$.*

*By applying Theorem 2.4.4 and Conjecture 2.5.2 to (2.8), we conjecture that*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P_X\left(\frac{1}{2}, \chi\right) Z'_X\left(\frac{1}{2}, \chi\right)\right|^{2k} \sim a(k) b_k \big((\log q) \deg Q\big)^{2k} (\deg Q)^{k^2} \tag{2.10}$$

*as $X, \deg Q \longrightarrow \infty$ with $X = \lfloor \log_q \deg Q \rfloor$.*

*By (2.6), the Cauchy-Schwarz inequality, and (2.9) and (2.10), we conjecture that*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| L'\left(\frac{1}{2}, \chi\right)\right|^{2k} \sim a(k) b_k \big((\log q) \deg Q\big)^{2k} (\deg Q)^{k^2} \tag{2.11}$$

*as $X, \deg Q \longrightarrow \infty$.*

We note that the cases $k = 1, 2$ of (2.11) are in agreement with our established results (2.2) and (2.3). One may wish to recall (2.5) when verifying the latter.

We now provide support for conjecture 2.5.1 - the splitting conjecture for the first derivative - by establishing that it holds for the case $k = 1$. In Section 7.4 we prove the following theorem.

**Theorem 2.5.5.** *As $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$, we have*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P_X'\Big(\frac{1}{2}, \chi\Big) Z_X\Big(\frac{1}{2}, \chi\Big) \right|^2 \sim \frac{3}{2} (\log q)^2 X^2 \deg Q.$$

This result, along with Theorem 2.4.7 and the case $k = 1$ for Theorem 2.5.3, verify (2.7) for the case $k = 1$.

We also have the following theorem.

**Theorem 2.5.6.** *As $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$, we have*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P_X\Big(\frac{1}{2}, \chi\Big) Z_X'\Big(\frac{1}{2}, \chi\Big) \right|^2 \sim \frac{(\log q)^2 (\deg Q)^3}{3}.$$

This theorem follows immediately from (2.6), the Cauchy Schwarz inequality, Theorem 2.5.5, and the case $k = 1$ of Theorem 2.3.2. Furthermore, we have the following theorem.

**Theorem 2.5.7.** *As $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$, we have*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| Z_X'\Big(\frac{1}{2}, \chi\Big) \right|^2 \sim \frac{(\log q)^2 (\deg Q)^3}{3 e^{\gamma} X}.$$

This theorem follows immediately from the fact that

$$\begin{aligned} Z_X'(s, \chi) =& P_X(s, \chi)^{-1} L'(s, \chi) - \frac{P_X'(s, \chi)}{P_X(s, \chi)} Z_X(s, \chi) \\ =& P_X(s, \chi)^{-1} L'(s, \chi) - \frac{P_X'(s, \chi)}{P_X(s, \chi)} P_X(s, \chi)^{-1} L(s, \chi) \end{aligned}$$

(which follows from (2.6)), the Cauchy-Schwarz inequality, and the following two Propositions that we prove in Section 7.5.

**Proposition 2.5.8.** *As $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$, we have*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P_X\Big(\frac{1}{2}, \chi\Big)^{-1} L'\Big(\frac{1}{2}, \chi\Big) \right|^2 \sim \frac{(\log q)^2 (\deg Q)^3}{3 e^{\gamma} X}.$$

**Proposition 2.5.9.** *As $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$,*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \frac{P_X'\big(\frac{1}{2}, \chi\big)}{P_X\big(\frac{1}{2}, \chi\big)} P_X\Big(\frac{1}{2}, \chi\Big)^{-1} L\Big(\frac{1}{2}, \chi\Big) \right|^2 \sim \frac{X \deg Q}{2 e^{\gamma}}.$$

We can now see that Theorems 2.5.6, 2.5.7, and 2.4.4 verify that (2.8) is true for the case $k = 1$.

Finally, as explained in Remark 2.4.6 we reformulate Conjecture 2.4.5 into the conjecture below.

**Conjecture 2.5.10.** *Assume that* $\max_{x\in\mathbb{R}}\{|u'(x)|\} \ll q^X$. *This is certainly possible given that our only requirement on $u$ is that it is a positive, normalised $C^\infty$-function with support in* $[e, e^{1+q^{-X}}]$.

*Let* $N(R) := \lfloor (\log q) \deg R \rfloor$. *Let $U(N)$ be the set of $N \times N$ unitary matrices and for $A \in U(N)$ let $\theta_n(A) \in (-\pi, \pi]$ be its $n$-th eigenphase. Let $X \sim \log_q \deg R$. Finally, let*

$$\widehat{\Lambda}_{A,X}(s) := \prod_{|\theta_n(A)| \leq \frac{\pi}{(\log q)e^{\gamma}X}} \Big(1 - se^{-i(\log q)e^{\gamma}X\theta_n(A)}\Big).$$

*Then, for integers $k \geq 0$, as* $X, \deg R \longrightarrow \infty$,

$$\begin{aligned}\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \Big|Z_X\Big(\frac{1}{2}, \chi\Big)\Big|^{2k} \sim& \int_{A\in U\left(N(R)\right)} \big|\widehat{\Lambda}_{A,X}(1)\big|^{2k} \mathrm{d}A \\ \sim& \frac{G^2(1+k)}{G(1+2k)} \bigg(\frac{N(R)}{(\log q)e^{\gamma}X}\bigg)^{k^2} \\ \sim& \frac{G^2(1+k)}{G(1+2k)} \bigg(\frac{\deg R}{e^{\gamma}X}\bigg)^{k^2},\end{aligned}$$

*where the integral is with respect to the Haar measure and $G$ is the Barnes $G$-function. The first two relations are conjectural, while the last can easily be seen to be true.*

Support for this conjecture is given in Section 7.2, along with the support for Conjecture 2.5.2. Indeed, both are similar.

As we did in Section 2.4, we remark that one could likely extend the range of $X$ in Theorems 2.5.5 to 2.5.7, and Propositions 2.5.8 and 2.5.9. This is because we are again dealing with twisted moments. This is clear for Propositions 2.5.8 and 2.5.9, and for Theorems 2.5.5 to 2.5.7 it can be seen from their proofs. We note, however, that Conjectures 2.5.4 and 2.5.10 require that $X \sim \log_q \deg R$.

# Chapter 3

# The Brun-Titchmarsh Theorem for the Divisor Function

In this chapter, we prove a result on sums of the divisor function over arithmetic progressions in intervals of $\mathbb{F}_q[T]$. For the ring of integers, such a result (slightly stronger, actually) was obtained for a certain class of multiplicative functions by Shiu [Shi80]. However, we only require the case of the divisor function, and so our proof is slightly easier. Technically, we prove two results. We restate them before proving them, for ease of reference. The first theorem, Theorem 2.1.1, is the following.

**Theorem.** *Suppose $\alpha, \beta$ are fixed and satisfy $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. Let $X \in \mathcal{M}$ and $y$ be a positive integer satisfying $\beta \deg X < y \leq \deg X$. Also, let $A \in \mathcal{A}$ and $G \in \mathcal{M}$ satisfy $(A, G) = 1$ and $\deg G < (1-\alpha)y$. Then, we have that*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G)}} d(N) \ll_{\alpha,\beta} \frac{q^y \deg X}{\phi(G)}. \tag{3.1}$$

We refer the reader to Remark 2.1.2 for an intuitive explanation of this result.

**Remark 3.0.1.** *We recall that, for $A \in \mathcal{A}$ with $\deg A \geq 1$, the degree of the largest prime divisor of $A$ is denoted by $p_+(A)$, while the degree of the smallest prime divisor of $A$ is denoted by $p_-(A)$. Now, an important aspect of the proof of Theorem 2.1.1 is considering the size of the prime divisors of the polynomials. Indeed, suppose that $\frac{\deg X}{r} < p_-(N) \leq \frac{\deg X}{r-1}$, for some integer $r \geq 2$. Then,*

$$d(N) \leq 2^{\Omega(N)} \leq 2^{\frac{\deg X}{p_-(N)}} \leq 2^r.$$

*The problem here is when $r$ is large. That is, when the smallest prime divisor of $N$ is small. In order to address this, the proof makes use of the following key technique:*

*Let $z < \deg X$ be real number whose exact value is not required at this time. Let us write*

$$N = {P_1}^{e_1} \dots {P_j}^{e_j} {P_{j+1}}^{e_{j+1}} \dots {P_n}^{e_n},$$

*where*

$$\deg(P_1) \leq \deg(P_2) \leq \dots \leq \deg(P_n)$$

*(there is some freedom here in the ordering of the prime divisors of a given degree, but our results are identical for any ordering) and $j$ is chosen so that*

$$\deg\left(P_1^{e_1}\dots P_j^{e_j}\right) \le z < \deg\left(P_{j+1}{}^{e_{j+1}}\dots P_n{}^{e_n}\right),$$

*and let*

$$\begin{aligned} B_N &:= P_1{}^{e_1}\dots P_j{}^{e_j} \\ D_N &:= P_{j+1}{}^{e_{j+1}}\dots P_n{}^{e_n}. \end{aligned} \tag{3.2}$$

*We then split the sum over $N$ into two. One is a sum over the possible $B_N$, and another is a sum over the possible $D_N$. The benefit of this is that if $p_-(D_N)$ is large, then $d(D_N)$ is small and so the latter sum is small, and if $p_-(D_N)$ is small, then $p_+(B_N) < p_-(D_N)$ is small and this allows us to make the former sum small. So, this technique allows us, in any case, to have some control over the size of the whole sum over $N$.*

The second theorem, Theorem 2.1.3, is the following.

**Theorem.** *Suppose $\alpha, \beta$ are fixed and satisfy $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. Let $X \in \mathcal{M}$ and $y$ be a positive integer satisfying $\beta \deg X < y \le \deg X$. Also, let $A \in \mathcal{A}$ and $G \in \mathcal{M}$ satisfy $(A, G) = 1$ and $\deg G < (1-\alpha)y$. Finally, let $a \in \mathbb{F}_q^*$. Then, we have that*

$$\sum_{\substack{N \in \mathcal{A} \\ \deg(N-X)=y \\ (N-X)\in a\mathcal{M} \\ N \equiv A (\mathrm{mod}\, G)}} d(N) \ll_{\alpha,\beta} \frac{q^y \deg X}{\phi(G)}.$$

Before proving these two theorems, we prove a corollary of Theorem B.2.2 (The Selberg sieve in function fields) and two lemmas.

**Corollary 3.0.2.** *Let $X \in \mathcal{M}$ and $y$ be a positive integer satisfying $y \le \deg X$. Also, let $K \in \mathcal{M}$ and $A \in \mathcal{A}$ satisfy $(A, K) = 1$. Finally, let $z$ be a positive integer such that $\deg K + z \le y$. Then,*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\mathrm{mod}\, K) \\ p_-(N)>z}} 1 \le \frac{2q^y}{\phi(K)z}.$$

*Proof.* Let us define

$$\mathcal{S} = \{N \in \mathcal{M} : \deg(N-X) < y, N \equiv A (\mathrm{mod}\, K)\}$$

and

$$\mathcal{Q} = \{P \text{ prime} : P \nmid K\}.$$

Then, following the notation of Section B.2 we have that

$$|\mathcal{S}_{\mathcal{Q},>z}| = \sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\mathrm{mod}\, K) \\ p_-(N)>z}} 1,$$

which is what we want to bound.

For $D \mid \Pi_{\mathcal{Q},\leq z}$ with $\deg D \leq z$ we have that

$$|\mathcal{S}_D| = |\{N \in \mathcal{M} : \deg(N - X) < y, N \equiv A(\operatorname{mod} K), N \equiv 0(\operatorname{mod} D)\}| = \frac{q^y}{|KD|}.$$

Therefore, for such $D$, we have $\omega(D) = 1$ and $|r(D)| = 0$. We also have that $\psi(D) = \phi(D)$. We can now see that

$$\sum_{\substack{G \in \mathcal{M}_{\leq \frac{z}{2}} \\ G \mid \Pi_{\mathcal{Q},\leq z}}} \frac{\mu^2(G)}{\psi(G)} = \sum_{\substack{G \in \mathcal{M}_{\leq \frac{z}{2}} \\ (G,K)=1}} \frac{\mu(G)^2}{\phi(G)},$$

and we have that

$$\sum_{\substack{G \in \mathcal{M}_{\leq \frac{z}{2}} \\ (G,K)=1}} \frac{\mu^2(G)}{\psi(G)} \sum_{H \mid K} \frac{\mu(H)^2}{\phi(H)} \geq \sum_{G \in \mathcal{M}_{\leq \frac{z}{2}}} \frac{\mu(H)^2}{\phi(H)}.$$

To this we apply Lemma A.3.5 and the fact that

$$\sum_{H \mid K} \frac{\mu(H)^2}{\phi(H)} = \prod_{P \mid K} \left(1 + \frac{1}{|P| - 1}\right) = \prod_{P \mid K} \left(1 - |P|^{-1}\right)^{-1} = \frac{|K|}{\phi(K)},$$

to obtain

$$\sum_{\substack{G \in \mathcal{M}_{\leq \frac{z}{2}} \\ G \mid \Pi_{\mathcal{Q},\leq z}}} \frac{\mu^2(G)}{\psi(G)} \geq \frac{\phi(K)}{|K|} \frac{z}{2}.$$

Also, we have that

$$\sum_{\substack{E,F \in \mathcal{M}_{\leq \frac{z}{2}} \\ E,F \mid \Pi_{\mathcal{Q},\leq z}}} \Big| r\big([E, F]\big) \Big| = 0.$$

The result now follows by applying Theorem B.2.2. □

**Lemma 3.0.3.** *Let us define $\mathfrak{c}$ to be such that*

$$|\{A \in \mathbb{F}_q[T] : \deg A = n, A \textit{ is prime}\}| \leq \mathfrak{c}\frac{q^n}{n}$$

*for all prime powers $q \neq 2$ and all positive integers $n$. Let $\mathfrak{d}$ be such that*

$$\frac{1}{|P|^{\frac{1}{8}} - 1} \leq \mathfrak{d}\frac{1}{|P|^{\frac{1}{8}}}$$

*for all prime powers $q \neq 2$ and all primes $P \in \mathbb{F}_q[T]$. Let $\mathfrak{e}$ be such that*

$$\sum_{n=1}^{x} \frac{q^{\delta n}}{n} \leq \mathfrak{e}\frac{q^{\delta x}}{x}$$

*for all primes powers $q \neq 2$ and all $\frac{3}{4} \leq \delta < 1$. Finally, let $z_0 \geq 3$ be such that*

$$\frac{1}{8}z + \mathfrak{c}\mathfrak{d}\mathfrak{e}\frac{z^{\frac{7}{8}}}{\log z} \leq \frac{1}{4}z$$

*for all $z \geq z_0$. Then, for all $z \geq z_0$, we have*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ p_+(N) \leq \log_q z}} 1 \leq q^{\frac{1}{4}z}.$$

*Proof.* We have

$$\begin{aligned}
\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ p_+(N) \leq \log_q z}} 1 \leq q^{\frac{1}{8}z} \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ p_+(N) \leq \log_q z}} |N|^{-\frac{1}{8}} \leq q^{\frac{1}{8}z} \sum_{\substack{N \in \mathcal{M} \\ p_+(N) \leq \log_q z}} |N|^{-\frac{1}{8}} \\
=q^{\frac{1}{8}z} \prod_{\deg P \leq \log_q z} \left(1 + \frac{1}{|P|^{\frac{1}{8}} - 1}\right) \leq q^{\frac{1}{8}z} \prod_{\deg P \leq \log_q z} \left(\exp\left(\frac{1}{|P|^{\frac{1}{8}} - 1}\right)\right) \\
\leq q^{\frac{1}{8}z} \exp\left(\mathfrak{d} \sum_{\deg P \leq \log_q z} \frac{1}{|P|^{\frac{1}{8}}}\right) \leq q^{\frac{1}{8}z} \exp\left(\mathfrak{c}\mathfrak{d} \sum_{n=1}^{\log_q z} \frac{q^{\frac{7}{8}n}}{n}\right) \\
\leq \exp\left(\frac{\log q}{8}z + \mathfrak{c}\mathfrak{d}\mathfrak{e}\frac{z^{\frac{7}{8}}}{\log_q z}\right) \leq q^{\frac{1}{4}z}.
\end{aligned}$$

□

**Lemma 3.0.4.** *Let $z$ and $r$ be a positive integers satisfying $r \log_q r \leq z$. Then,*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N \geq \frac{z}{2} \\ p_+(N) \leq \frac{z}{r}}} \frac{d(N)}{|N|} \ll z^2 \exp\left(-\frac{r \log r}{9}\right).$$

*Proof.* Let $\frac{3}{4} \leq \delta < 1$. We will optimise on the value of $\delta$ later. We have that

$$\begin{aligned}
\sum_{\substack{N \in \mathcal{M} \\ \deg N \geq \frac{z}{2} \\ p_+(N) \leq \frac{z}{r}}} \frac{d(N)}{|N|} \leq& q^{(\delta-1)\frac{z}{2}} \sum_{\substack{N \in \mathcal{M} \\ \deg N \geq \frac{z}{2} \\ p_+(N) \leq \frac{z}{r}}} \frac{d(N)}{|N|^\delta} \leq q^{(\delta-1)\frac{z}{2}} \sum_{\substack{N \in \mathcal{M} \\ p_+(N) \leq \frac{z}{r}}} \frac{d(N)}{|N|^\delta} \\
\leq& q^{(\delta-1)\frac{z}{2}} \prod_{\deg P \leq \frac{z}{r}} \left(1 + \frac{2}{|P|^\delta} + \sum_{l=2}^{\infty} \frac{l+1}{|P|^{l\delta}}\right) \\
\leq& \exp\left((\log q)(\delta - 1)\frac{z}{2} + 2 \sum_{\deg P \leq \frac{z}{r}} \frac{1}{|P|^\delta} + \sum_{\deg P \leq \frac{z}{r}} \sum_{l=2}^{\infty} \frac{l+1}{|P|^{l\delta}}\right)
\end{aligned} \tag{3.3}$$

where the last relation uses the Taylor series for the exponential function.

Note that

$$\begin{aligned}
\sum_{\deg P \leq \frac{z}{r}} \sum_{l=2}^{\infty} \frac{l+1}{|P|^{l\delta}} \leq& \sum_{P \in \mathcal{P}} \frac{3}{|P|^{2\delta}} \sum_{l=0}^{\infty} \frac{l+1}{|P|^{l\delta}} = \sum_{P \in \mathcal{P}} \frac{3}{|P|^{2\delta}} \left(\frac{1}{1 - \frac{1}{|P|^\delta}}\right)^2 \\
=& 3 \sum_{P \in \mathcal{P}} \left(\frac{1}{|P|^\delta - 1}\right)^2 = O(1),
\end{aligned} \tag{3.4}$$

where the last relation uses the fact that $\delta \geq \frac{3}{4}$. Also, we can write $\frac{1}{|P|^{\delta}} = \frac{1}{|P|} + \frac{1}{|P|}\Big(|P|^{1-\delta} - 1\Big)$.

We have that

$$\sum_{\deg P \leq \frac{z}{r}} \frac{1}{|P|} = \sum_{n=1}^{\frac{z}{r}} \frac{1}{q^n}\left(\frac{q^n}{n} + O\Big(\frac{q^{\frac{n}{2}}}{n}\Big)\right) \leq \log z - \log r + O(1) \leq \log(z) + O(1), \tag{3.5}$$

and that

$$\begin{aligned}\sum_{\deg P \leq \frac{z}{r}} \frac{1}{|P|}\Big(|P|^{1-\delta} - 1\Big) &= \sum_{\deg P \leq \frac{z}{r}} \frac{1}{|P|} \sum_{n=1}^{\infty} \frac{\big((1-\delta)\log|P|\big)^n}{n!} \\ &\leq \sum_{n=1}^{\infty} \frac{(1-\delta)^n \big((\log q)\frac{z}{r}\big)^{n-1}}{n!} \sum_{\deg P \leq \frac{z}{r}} \frac{(\log q)\deg P}{|P|} \\ &\leq \mathfrak{c} \sum_{n=1}^{\infty} \frac{(1-\delta)^n \big((\log q)\frac{z}{r}\big)^{n}}{n!} = \mathfrak{c} q^{(1-\delta)\frac{z}{r}},\end{aligned} \tag{3.6}$$

where the second-to-last relation follows from a similar calculation as (3.5).

We substitute (3.4), (3.5), and (3.6) into (3.3) to obtain

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N \geq \frac{z}{2} \\ p_+(N) \leq \frac{z}{r}}} \frac{d(N)}{|N|} \ll z^2 \exp\left(\log q(\delta - 1)\frac{z}{2} + 2\mathfrak{c} q^{(1-\delta)\frac{z}{r}}\right).$$

We can now take $\delta = 1 - \frac{r \log_q r}{4z}$ (by the conditions on $r$ given in theorem, we have that $\frac{3}{4} \leq \delta < 1$, as required). Then,

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N \geq \frac{z}{2} \\ p_+(N) \leq \frac{z}{r}}} \frac{d(N)}{|N|} \ll z^2 \exp\left(-\frac{r \log r}{8} + 2\mathfrak{c} r^{\frac{1}{4}}\right) \ll z^2 \exp\left(-\frac{r\log r}{9}\right).$$

□

We can now prove our first theorem, Theorem 2.1.1.

*Proof of Theorem 2.1.1.* Let $z := \frac{\alpha}{10} y$ and for each $N$ in the summation range of (3.1) we define $B_N$ and $D_N$ as in (3.2). We break up the problem into four cases:

1. $p_-(D_N) > \frac{1}{2}z$;

2. $p_-(D_N) \leq \frac{1}{2}z$ and $\deg B_N \leq \frac{1}{2}z$;

3. $p_-(D_N) < w(z)$ and $\deg B_N > \frac{1}{2}z$;

4. $w(z) \leq p_-(D_N) \leq \frac{1}{2}z$ and $\deg B_N > \frac{1}{2}z$;

where

$$w(z) := \begin{cases} 1 & \text{if } z < z_0 \\ \log_q z & \text{if } z \geq z_0. \end{cases}$$

and $z_0$ is as in Lemma 3.0.3.

Consider case 1. Because $\deg B_N \leq z < y$, we can find a monic polynomial $X_{B_N}$ such that $\deg(X - B_N X_{B_N}) < y$. Then, the following three statements are equivalent:

- $\deg(N - X) < y$;
- $\deg(B_N D_N - B_N X_{B_N}) < y$;
- $\deg(D_N - X_{B_N}) < y - \deg B_N$.

Also, because $(B_N, G) = 1$, we can find some $A_{B_N} \in \mathcal{A}$ such that $(A_{B_N}, G) = 1$ and $B_N A_{B_N} \equiv A \pmod G$. Then, again because $(B_N, G) = 1$, the following three statements are equivalent:

- $N \equiv A \pmod G$;
- $B_N D_N \equiv B_N A_{B_N} \pmod G$;
- $D_N \equiv A_{B_N} \pmod G$.

We also have that

$$d(D_N) \leq 2^{\Omega(D_N)} \leq 2^{\frac{\deg D_N}{p_-(D_N)}} \leq 2^{\frac{20 \deg X}{\alpha y}} \leq 2^{\frac{20}{\alpha\beta}}.$$

So, using Corollary 3.0.2 for the fourth relation below, we have

$$\begin{aligned} \sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G) \\ p_-(D_N) > \frac{1}{2}z}} d(N) = \sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G) \\ p_-(D_N) > \frac{1}{2}z}} d(B_N) d(D_N) &\leq \sum_{\substack{B \in \mathcal{M} \\ \deg B \leq z \\ (B,G)=1}} d(B) \sum_{\substack{D \in \mathcal{M} \\ \deg(D-X_B)<y-\deg B \\ D \equiv A_B (\operatorname{mod} G) \\ p_-(D) > \frac{1}{2}z}} d(D) \\ &\ll_{\alpha,\beta} \sum_{\substack{B \in \mathcal{M} \\ \deg B \leq z \\ (B,G)=1}} d(B) \sum_{\substack{D \in \mathcal{M} \\ \deg(D-X_B)<y-\deg B \\ D \equiv A_B (\operatorname{mod} G) \\ p_-(D) > \frac{1}{2}z}} 1 \ll \frac{q^y}{\phi(G) z} \sum_{\substack{B \in \mathcal{M} \\ \deg B \leq z \\ (B,G)=1}} \frac{d(B)}{|B|} \\ &\leq \frac{q^y}{\phi(G) z} \Bigg( \sum_{\substack{B \in \mathcal{M} \\ \deg B \leq z}} \frac{1}{|B|} \Bigg)^2 \ll \frac{q^y z}{\phi(G)} \leq \frac{q^y \deg X}{\phi(G)}. \end{aligned}$$

Now suppose $N$ satisfies case 2. Then, the associated $P_{j+1}$ from (3.2) satisfies the following three conditions:

- $\deg P_{j+1} \leq \frac{1}{2}z$;
- $\deg {P_{j+1}}^{e_{j+1}} > \frac{1}{2}z$;
- $(P_{j+1}, G) = 1$ (since $P_{j+1} \mid N$, $N \equiv A \pmod G$, and $(A, G) = 1$).

For a general prime $P$ with $\deg P \leq \frac{1}{2}z$, we denote $e_P \geq 2$ to be the smallest integer such that $\deg P^{e_P} > \frac{1}{2}z$. We note that $\deg P^{e_P} \leq z$. Furthermore,

$$\sum_{\deg P \leq \frac{1}{2}z} \frac{1}{|P|^{e_P}} \leq \sum_{\deg P \leq \frac{1}{4}z} q^{-\frac{1}{2}z} + \sum_{\frac{1}{4}z < \deg P \leq \frac{1}{2}z} \frac{1}{|P|^2} \ll q^{-\frac{1}{4}z},$$

and

$$d(N) \ll_{\alpha,\beta} |N|^{\frac{\alpha\beta}{80}} \leq |X|^{\frac{\alpha\beta}{80}} \leq q^{\frac{\alpha}{80}y} = q^{\frac{1}{8}z}.$$

Hence,

$$\begin{aligned} \sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G) \\ p_-(D_N) \leq \frac{1}{2}z \\ \deg B_N \leq \frac{1}{2}z}} d(N) &\leq \sum_{\substack{\deg P \leq \frac{1}{2}z \\ (P,G)=1}} \sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G) \\ N \equiv 0 (\operatorname{mod} P^{e_P})}} d(N) \ll_{\alpha,\beta} q^{\frac{1}{8}z} \sum_{\substack{\deg P \leq \frac{1}{2}z \\ (P,G)=1}} \sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G) \\ N \equiv 0 (\operatorname{mod} P^{e_P})}} 1 \\ &= q^{\frac{1}{8}z} \sum_{\substack{\deg P \leq \frac{1}{2}z \\ (P,G)=1}} \frac{q^y}{|GP^{e_P}|} \leq \frac{q^y q^{\frac{1}{8}z}}{|G|} \sum_{\deg P \leq \frac{1}{2}z} \frac{1}{|P^{e_P}|} \ll \frac{q^y q^{-\frac{1}{8}z}}{|G|}. \end{aligned} \tag{3.7}$$

Now suppose $N$ satisfies case 3. If $z < z_0$, then $p_-(D_N) < w(z) = 1$, and so

$$\sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G) \\ p_-(D_N) < w(z) \\ \frac{1}{2}z < \deg B_N \leq z}} d(N) = 0.$$

If $z \geq z_0$, then, using Lemma 3.0.3 for the last relation below, we have,

$$\begin{aligned} \sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G) \\ p_-(D_N) < w(z) \\ \frac{1}{2}z < \deg B_N \leq z}} d(N) &\ll_{\alpha,\beta} q^{\frac{1}{8}z} \sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G) \\ p_-(D_N) < w(z) \\ \frac{1}{2}z < \deg B_N \leq z}} 1 \leq q^{\frac{1}{8}z} \sum_{\substack{B \in \mathcal{M} \\ \frac{1}{2}z < \deg B \leq z \\ (B,G)=1 \\ p_+(B) < w(z)}} \sum_{\substack{N \in \mathcal{M} \\ \deg(N-X)<y \\ N \equiv A (\operatorname{mod} G) \\ N \equiv 0 (\operatorname{mod} B)}} 1 \\ &\leq q^{\frac{1}{8}z} \sum_{\substack{B \in \mathcal{M} \\ \frac{1}{2}z < \deg B \leq z \\ p_+(B) < w(z)}} \frac{q^y}{|GB|} \leq \frac{q^y}{|G|} q^{-\frac{3}{8}z} \sum_{\substack{B \in \mathcal{M} \\ \frac{1}{2}z < \deg B \leq z \\ p_+(B) < \log_q z}} 1 \ll \frac{q^y}{|G|} q^{-\frac{1}{8}z}. \end{aligned}$$

Now suppose $N$ satisfies case 4. If $z < z_0$, we break up the possible values of $p_-(D_N)$ into the following cases:

$$\frac{1}{r+1} z < p_-(D_N) \leq \frac{1}{r} z,$$

for $2 \leq r \leq z$. We then see that

$$\Omega(D_N) \leq \frac{\deg X}{p_-(D_N)} \leq \frac{\deg X}{\frac{1}{r+1} z} \leq \frac{10(r+1)}{\alpha\beta} \leq \frac{20r}{\alpha\beta},$$

and so, taking $a := 2^{\frac{20}{\alpha\beta}}$, we have $d(D_N) \le a^r$. Hence, using Corollary 3.0.2 for the third relation below, we have

$$\begin{aligned}
\sum_{\substack{N\in\mathcal{M}\\ \deg(N-X)<y\\ N\equiv A(\mathrm{mod}\,G)\\ w(z)\le p_-(D_N)\le\frac{1}{2}z\\ \deg B_N>\frac{1}{2}z}} d(N) &\le \sum_{r=2}^{z} a^r \sum_{\substack{B\in\mathcal{M}\\ \frac{1}{2}z<\deg B\le z\\ (B,G)=1\\ p_+(B)\le\frac{1}{r}z}} d(B) \sum_{\substack{N\in\mathcal{M}\\ \deg(N-X)<y\\ N\equiv A(\mathrm{mod}\,G)\\ N\equiv 0(\mathrm{mod}\,B)\\ \frac{1}{r+1}z<p_-(D_N)\le\frac{1}{r}z}} 1\\
&\le \sum_{r=2}^{z} a^r \sum_{\substack{B\in\mathcal{M}\\ \frac{1}{2}z<\deg B\le z\\ (B,G)=1\\ p_+(B)\le\frac{1}{r}z}} d(B) \sum_{\substack{D\in\mathcal{M}\\ \deg(D-X_B)<y-\deg B\\ D\equiv A_B(\mathrm{mod}\,G)\\ p_-(D)>\frac{1}{r+1}z}} 1\\
&\ll \frac{q^y}{\phi(G)z}\sum_{r=2}^{z}(r+1)a^r \sum_{\substack{B\in\mathcal{M}\\ \frac{1}{2}z<\deg B\le z\\ (B,G)=1\\ p_+(B)\le\frac{1}{r}z}} \frac{d(B)}{|B|}\\
&\le \frac{q^y}{\phi(G)}z\sum_{r=2}^{z_0}(r+1)a^r \ll_{\alpha,\beta} \frac{q^y}{\phi(G)}\deg X.
\end{aligned}$$

Now suppose $z \ge z_0$. Then, we break up the possible values of $p_-(D_N)$ into the following cases:

$$\frac{1}{r+1}z < p_-(D_N) \le \frac{1}{r}z,$$

for $2 \le r \le r_0 := \min\left\{z, \frac{z}{\log_q z}\right\}$. Note that this covers all $D_N$ with $w(z) \le p_-(D_N) \le \frac{z}{2}$. We also have $r\log_q r \le z$, and so we can apply Lemma 3.0.4 for the fourth relation below. We have,

$$\begin{aligned}
\sum_{\substack{N\in\mathcal{M}\\ \deg(N-X)<y\\ N\equiv A(\mathrm{mod}\,G)\\ w(z)\le p_-(D_N)\le\frac{1}{2}z\\ \deg B_N>\frac{1}{2}z}} d(N) &\le \sum_{r=2}^{r_0} a^r \sum_{\substack{B\in\mathcal{M}\\ \frac{1}{2}z<\deg B\le z\\ (B,G)=1\\ p_+(B)\le\frac{1}{r}z}} d(B) \sum_{\substack{N\in\mathcal{M}\\ \deg(N-X)<y\\ N\equiv A(\mathrm{mod}\,G)\\ N\equiv 0(\mathrm{mod}\,B)\\ \frac{1}{r+1}z<p_-(D_N)\le\frac{1}{r}z}} 1\\
&\le \sum_{r=2}^{r_0} a^r \sum_{\substack{B\in\mathcal{M}\\ \frac{1}{2}z<\deg B\le z\\ (B,G)=1\\ p_+(B)\le\frac{1}{r}z}} d(B) \sum_{\substack{D\in\mathcal{M}\\ \deg(D-X_B)<y-\deg B\\ D\equiv A_B(\mathrm{mod}\,G)\\ p_-(D)>\frac{1}{r+1}z}} 1\\
&\ll \frac{q^y}{\phi(G)z}\sum_{r=2}^{r_0}(r+1)a^r \sum_{\substack{B\in\mathcal{M}\\ \frac{1}{2}z<\deg B\le z\\ (B,G)=1\\ p_+(B)\le\frac{1}{r}z}} \frac{d(B)}{|B|}\\
&\le \frac{q^y z}{\phi(G)}\sum_{r=2}^{\infty}(r+1)a^r \exp\left(-\frac{r\log r}{9}\right)\\
&\ll_{\alpha,\beta} \frac{q^y}{\phi(G)}\deg X.
\end{aligned}$$

□

The proof of Theorem 2.1.3 is very similar to that of Theorem 2.1.1, although some minor changes must be made to Corollary 3.0.2 as well. To see why the proofs are so similar, consider the following. In Theorem 2.1.3, $N$ is such that its coefficients in positions $y+1, y+2, \dots, \deg N$ are the same as those of $X$; its $y$-th coefficient differs from that of $X$ by some fixed $a \neq 0$; and the coefficients in positions $0, 1, \dots, y-1$ are free to take any value in $\mathbb{F}_q$. In Theorem 2.1.1, $N$ is such that its coefficients in positions $y+1, y+2, \dots, \deg N$ are the same as those of $X$; its $y$-th coefficient differs from that of $X$ by $a = 0$ (i.e. the $y$-th coefficients are also the same); and the coefficients in positions $0, 1, \dots, y-1$ are free to take any value in $\mathbb{F}_q$. Thus, the only difference between the $N$ in the two theorems is that the $y$-th coefficient in Theorem 2.1.1 differs from the $y$-th coefficient of $X$ by $a = 0$, while for Theorem 2.1.3 it is some fixed $a \neq 0$. Ultimately, this makes no difference to the bound we obtain, but it does require us to use different notation, and thus the requirement of having two different theorems for essentially the same result. The difference in notation is only that in Theorem 2.1.3 we have $\deg(N - X) < y$, while in Theorem 2.1.1 we have $\deg(N - X) = y$ but $(N - X) \in a\mathcal{M}$.

# Chapter 4

# The Second and Fourth Moments of Dirichlet $L$-functions, Averaged over Primitive Characters

In this chapter, we prove Theorems 2.2.1, 2.2.2, and 2.2.3. For a discussion on these results and a brief, intuitive explanation of their proofs, we refer the reader to Section 2.2.

## 4.1 The Second Moment

For ease of reference, we restate the theorems before proving them. Theorem 2.2.1 is the following.

**Theorem.** *Let $R \in \mathcal{M}\backslash\{1\}$. Then,*

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^2 = \frac{\phi(R)}{|R|} \deg R + O\bigg( \frac{\phi(R)}{|R|} \log \omega(R) \bigg).$$

*Proof of Theorem 2.2.1.* From Lemma A.1.2, Lemma A.1.3, and (A.8) in Appendix A, we have

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^2 = \frac{2}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} + \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} c(\chi).$$

For the first term on the RHS, by Lemma 1.4.5 and Corollaries 1.4.6 and A.3.3, we

have

$$\begin{aligned}
&\frac{2}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} = \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F)}} \frac{1}{|AB|^{\frac{1}{2}}} \\
=&2 \sum_{\substack{A \in \mathcal{M} \\ \deg A < \frac{1}{2} \deg R \\ (A,R)=1}} \frac{1}{|A|} + \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F) \\ A \neq B}} \frac{1}{|AB|^{\frac{1}{2}}} \\
=&\frac{\phi(R)}{|R|} \deg R + O\left( \frac{\phi(R)}{|R|} \log \omega(R) \right) + \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F) \\ A \neq B}} \frac{1}{|AB|^{\frac{1}{2}}} .
\end{aligned} \tag{4.1}$$

We will look at the third term on the far RHS of (4.1). Consider the case where $\deg AB = z$ and $\deg A > \deg B$. Then, $\deg B < \frac{z}{2}$ and we can write $A = LF + B$ for monic $L$ with $\deg L = z - \deg B - \deg F$. So,

$$\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = z \\ \deg A > \deg B \\ (AB,R)=1 \\ A \equiv B (\bmod F)}} \frac{1}{|AB|^{\frac{1}{2}}} \leq \sum_{\substack{B \in \mathcal{M} \\ \deg B < \frac{z}{2}}} \frac{1}{|B|^{\frac{1}{2}}} \sum_{\substack{L \in \mathcal{M} \\ \deg L = z - \deg B - \deg F}} \frac{1}{|LF|^{\frac{1}{2}}} \leq \frac{q^{\frac{z}{2}}}{|F|} \sum_{\substack{B \in \mathcal{M} \\ \deg B < \frac{z}{2}}} \frac{1}{|B|} = \frac{zq^{\frac{z}{2}}}{2|F|}.$$

The case where $\deg A < \deg B$ is similar. For the case $\deg A = \deg B$, we have $\deg B = \frac{z}{2}$ and $A = LF + B$ for $L \in \mathcal{A}$ with $\deg L < \deg B - \deg F$. So,

$$\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = z \\ \deg A = \deg B \\ (AB,R)=1 \\ A \equiv B (\bmod F) \\ A \neq B}} \frac{1}{|AB|^{\frac{1}{2}}} \leq \sum_{\substack{B \in \mathcal{M} \\ \deg B = \frac{z}{2}}} \frac{1}{|B|^{\frac{1}{2}}} \sum_{\substack{L \in \mathcal{A} \\ \deg L < \deg B - \deg F}} \frac{1}{|LF|^{\frac{1}{2}}} \ll \frac{q^{\frac{1}{2}}}{|F|} \sum_{\substack{B \in \mathcal{M} \\ \deg B = \frac{z}{2}}} 1 = \frac{q^{\frac{z+1}{2}}}{|F|}.$$

Hence,

$$\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F) \\ A \neq B}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{1}{|F|} \sum_{z=0}^{\deg R - 1} z q^{\frac{z+1}{2}} \ll \frac{|R|^{\frac{1}{2}} \deg R}{|F|},$$

and so

$$\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F) \\ A \neq B}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{|R|^{\frac{1}{2}} \deg R}{\phi^*(R)} \sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|}$$

$$\leq \frac{2^{\omega(R)}|R|^{\frac{1}{2}} \deg R}{\phi^*(R)} \ll |R|^{-\frac{1}{3}},$$

where the last equality follows from (A.19). Applying this to (4.1) gives

$$\frac{2}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} = \frac{\phi(R)}{|R|} \deg R + O\bigg( \frac{\phi(R)}{|R|} \log \omega(R) \bigg).$$

Finally,

$$\begin{aligned} \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} c(\chi) =& \frac{1}{\phi^*(R)} \bigg( \sum_{\substack{\chi \bmod R \\ \chi \text{ odd}}} c_o(\chi) + \sum_{\substack{\chi \bmod R \\ \chi \text{ even}}} c_e(\chi) \bigg) \\ =& \frac{1}{\phi^*(R)} \bigg( \sum_{\chi \bmod R} c_o(\chi) + \sum_{\substack{\chi \bmod R \\ \chi \text{ even}}} c_e(\chi) - c_o(\chi) \bigg). \end{aligned}$$

By similar methods as previously in the proof, we can see that the above is $O(1)$. The result follows. □

We now prove Theorem 2.2.2:

**Theorem.** *Let $R \in \mathcal{M} \backslash \{1\}$ be a square-full polynomial. Then,*

$$\begin{aligned} \sideset{}{^*}\sum_{\chi \bmod R} \Big| L\Big(\frac{1}{2}, \chi\Big) \Big|^2 =& \frac{\phi(R)^3}{|R|^2} \deg R + 2 \frac{\phi(R)^3}{|R|^2} \sum_{P|R} \frac{\deg P}{|P| - 1} \\ &+ \frac{1}{\big(q^{\frac{1}{2}} - 1\big)^2} \bigg( - \frac{\phi(R)^3}{|R|^2} + 2 \frac{\phi(R)}{|R|^{\frac{1}{2}}} \prod_{P|R} \Big( 1 - \frac{1}{|P|^{\frac{1}{2}}} \Big)^2 \bigg). \end{aligned}$$

*Proof of Theorem 2.2.2.* We have that

$$\begin{aligned} \sideset{}{^*}\sum_{\chi \bmod R} \Big| L\Big(\frac{1}{2}, \chi\Big) \Big|^2 =& \sideset{}{^*}\sum_{\chi \bmod R} \sum_{\substack{A,B \in \mathcal{M} \\ \deg A, \deg B < \deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \\ =& \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{M} \\ \deg A, \deg B < \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F)}} \frac{1}{|AB|^{\frac{1}{2}}} \\ =& \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A \in \mathcal{M} \\ \deg A < \deg R \\ (A,R)=1}} \frac{1}{|A|^{\frac{1}{2}}} \sum_{\substack{B \in \mathcal{M} \\ \deg B < \deg R \\ B \equiv A (\bmod F)}} \frac{1}{|B|^{\frac{1}{2}}}. \end{aligned}$$

The second equality follows from Lemma 1.4.5. For the last equality we note that if $R$ is square-full, $EF = R$, and $\mu(E) \neq 0$, then $F$ and $R$ have the same prime factors. Therefore, if we also have that $(A, R) = 1$ and $B \equiv A (\bmod F)$, then $(B, R) = 1$.

Continuing,

$$\begin{aligned}
&\sum_{\chi \bmod R}^{*} \Big| L\Big(\frac{1}{2},\chi\Big)\Big|^2 \\
=&\sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A\in\mathcal{M} \\ \deg A<\deg R}} \frac{1}{|A|^{\frac{1}{2}}} \sum_{G|(A,R)} \mu(G) \sum_{\substack{B\in\mathcal{M} \\ \deg B<\deg R \\ B\equiv A (\bmod F)}} \frac{1}{|B|^{\frac{1}{2}}} \\
=&\sum_{EF=R} \mu(E)\phi(F) \sum_{G|R} \mu(G) \sum_{\substack{A\in\mathcal{M} \\ \deg A<\deg R \\ G|A}} \frac{1}{|A|^{\frac{1}{2}}} \sum_{\substack{B\in\mathcal{M} \\ \deg B<\deg R \\ B\equiv A (\bmod F)}} \frac{1}{|B|^{\frac{1}{2}}} \\
=&\sum_{EF=R} \mu(E)\phi(F) \sum_{G|R} \mu(G) \sum_{\substack{K\in\mathcal{A} \\ \deg K<\deg F-\deg G \\ \text{or} \\ K=0}} \Bigg( \sum_{\substack{A\in\mathcal{M} \\ \deg A<\deg R \\ A\equiv GK (\bmod F)}} \frac{1}{|A|^{\frac{1}{2}}} \Bigg)\Bigg( \sum_{\substack{B\in\mathcal{M} \\ \deg B<\deg R \\ B\equiv GK (\bmod F)}} \frac{1}{|B|^{\frac{1}{2}}} \Bigg).
\end{aligned} \tag{4.2}$$

The last equality follows from the fact that $F$ and $R$ have the same prime factors, and so, if $\mu(G) \neq 0$, then $G \mid F$. Hence, if $G \mid A$, then $A \equiv GK (\bmod F)$ for some $K \in \mathcal{A}$ with $\deg K < \deg F - \deg G$ or $K = 0$.

Now, we note that if $K \in \mathcal{A}\backslash\mathcal{M}$, then

$$\begin{aligned}
\sum_{\substack{A\in\mathcal{M} \\ \deg A<\deg R \\ A\equiv GK (\bmod F)}} \frac{1}{|A|^{\frac{1}{2}}} = \sum_{\substack{L\in\mathcal{M} \\ \deg L<\deg R-\deg F}} \frac{1}{|LF+GK|^{\frac{1}{2}}} &= \sum_{\substack{L\in\mathcal{M} \\ \deg L<\deg R-\deg F}} \frac{1}{|LF|^{\frac{1}{2}}} \\
&= \frac{1}{q^{\frac{1}{2}}-1} \Bigg( \frac{|R|^{\frac{1}{2}}}{|F|} - \frac{1}{|F|^{\frac{1}{2}}} \Bigg).
\end{aligned}$$

Whereas, if $K \in \mathcal{M}$, then

$$\begin{aligned}
\sum_{\substack{A\in\mathcal{M} \\ \deg A<\deg R \\ A\equiv GK (\bmod F)}} \frac{1}{|A|^{\frac{1}{2}}} &= \frac{1}{|GK|^{\frac{1}{2}}} + \sum_{\substack{L\in\mathcal{M} \\ \deg L<\deg R-\deg F}} \frac{1}{|LF+GK|^{\frac{1}{2}}} \\
&= \frac{1}{|GK|^{\frac{1}{2}}} + \frac{1}{q^{\frac{1}{2}}-1} \Bigg( \frac{|R|^{\frac{1}{2}}}{|F|} - \frac{1}{|F|^{\frac{1}{2}}} \Bigg).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{\substack{K\in\mathcal{A} \\ \deg K<\deg F-\deg G \\ \text{or} \\ K=0}} \Bigg( \sum_{\substack{A\in\mathcal{M} \\ \deg A<\deg R \\ A\equiv GK (\bmod F)}} \frac{1}{|A|^{\frac{1}{2}}} \Bigg)\Bigg( \sum_{\substack{B\in\mathcal{M} \\ \deg B<\deg R \\ B\equiv GK (\bmod F)}} \frac{1}{|B|^{\frac{1}{2}}} \Bigg) \\
=&\frac{1}{\big(q^{\frac{1}{2}}-1\big)^2} \Bigg( \frac{|R|^{\frac{1}{2}}}{|F|} - \frac{1}{|F|^{\frac{1}{2}}} \Bigg)^2 \sum_{\substack{K\in\mathcal{A} \\ \deg K<\deg F-\deg G \\ \text{or} \\ K=0}} 1 \\
&+ \frac{2}{q^{\frac{1}{2}}-1} \Bigg( \frac{|R|^{\frac{1}{2}}}{|F|} - \frac{1}{|F|^{\frac{1}{2}}} \Bigg) \frac{1}{|G|^{\frac{1}{2}}} \sum_{\substack{K\in\mathcal{M} \\ \deg K<\deg F-\deg G}} \frac{1}{|K|^{\frac{1}{2}}}
\end{aligned}$$

$$+\frac{1}{|G|}\sum_{\substack{K\in\mathcal{M}\\ \deg K<\deg F-\deg G}}\frac{1}{|K|}$$
$$=\frac{1}{\left(q^{\frac{1}{2}}-1\right)^2}\left(\frac{|R|}{|FG|}-2\frac{|R|^{\frac{1}{2}}}{|F||G|^{\frac{1}{2}}}-\frac{1}{|G|}+\frac{2}{|F|^{\frac{1}{2}}|G|^{\frac{1}{2}}}\right)+\frac{\deg F}{|G|}-\frac{\deg G}{|G|}.$$

By applying this to (4.2), and using (A.24) to (A.27), we see that

$$\sum_{\chi \bmod R}^{*}\left|L\left(\frac{1}{2},\chi\right)\right|^2=\frac{\phi(R)^3}{|R|^2}\deg R+2\frac{\phi(R)^3}{|R|^2}\sum_{P|R}\frac{\deg P}{|P|-1}$$
$$+\frac{1}{\left(q^{\frac{1}{2}}-1\right)^2}\left(-\frac{\phi(R)^3}{|R|^2}+2\frac{\phi(R)}{|R|^{\frac{1}{2}}}\prod_{P|R}\left(1-\frac{1}{|P|^{\frac{1}{2}}}\right)^2\right).$$

□

## 4.2 The Fourth Moment: General Preliminary Results

In this section, we prove some general preliminary results that are required for Section 4.4. They are general in that it would not be unusual for such results to have applications to other problems.

**Lemma 4.2.1** (Perron's Formula)**.** *Let $c$ be a positive real number, and let $k\geq 2$ be an integer. Then,*

$$\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s^k}\mathrm{d}s=\begin{cases}0 & \text{if } 0\leq y<1;\\ \frac{2\pi i}{(k-1)!}(\log y)^{k-1} & \text{if } y\geq 1.\end{cases}$$

*If $k=1$, then*

$$\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s}\mathrm{d}s=\begin{cases}0 & \text{if } 0\leq y<1;\\ \pi i & \text{if } y=1;\\ 2\pi i & \text{if } y>1.\end{cases}$$

*Proof.* Suppose $k\geq 2$. We will first look at the case when $y\geq 1$. Let $n$ be a positive integer, and define the following curves:

$$\begin{aligned}
l_1(n):=&[c-ni,c+ni];\\
l_2(n):=&[c+ni,ni];\\
l_3(n):=&\left\{ne^{it}:t\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right\} \text{ (orientated anticlockwise)};\\
l_4(n):=&[-ni,c-ni];\\
L(n):=&l_1(n)\cup l_2(n)\cup l_3(n)\cup l_4(n).
\end{aligned}$$

We can see that

$$\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s^k}\mathrm{d}s=\lim_{n\to\infty}\left(\int_{L(n)}\frac{y^s}{s^k}\mathrm{d}s-\int_{l_2(n)}\frac{y^s}{s^k}\mathrm{d}s-\int_{l_3(n)}\frac{y^s}{s^k}\mathrm{d}s-\int_{l_4(n)}\frac{y^s}{s^k}\mathrm{d}s\right).$$

For the first integral we apply the residue theorem to obtain that

$$\lim_{n\to\infty}\int_{L(n)}\frac{y^s}{s^k}\mathrm{d}s=\frac{2\pi i}{(k-1)!}(\log y)^{k-1}.$$

For $j\in\{2,4\}$ we have that

$$\lim_{n\to\infty}\left|\int_{l_j(n)}\frac{y^s}{s^k}\mathrm{d}s\right|\le\lim_{n\to\infty}\frac{y^c}{n^k}\int_{l_j(n)}1\mathrm{d}s=\lim_{n\to\infty}\frac{cy^c}{n^k}=0.$$

For the third integral we note that when $s\in l_3(n)$ we have $|y^s|\le 1$ (since $\operatorname{Re}s\le 0$ and $y\ge 1$). Hence,

$$\lim_{n\to\infty}\left|\int_{l_3(n)}\frac{y^s}{s^k}\mathrm{d}s\right|\le\lim_{n\to\infty}\frac{1}{n^k}\int_{l_3(n)}1\mathrm{d}s=\lim_{n\to\infty}\frac{\pi}{n^{k-1}}=0.$$

So, for $y\ge 1$ we deduce that

$$\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s^k}\mathrm{d}s=\frac{2\pi i}{(k-1)!}(\log y)^{k-1}.$$

Now we will look at the case when $0\le y<1$. Again, let $n$ be a positive integer, and define the following curves:

$$\begin{aligned}
l_1(n) :=&[c-ni,c+ni];\\
l_2(n) :=&\Big\{c+ne^{it}:t\in\Big[\frac{-\pi}{2},\frac{\pi}{2}\Big]\Big\}\text{ (orientated clockwise)};\\
L(n) :=&l_1(n)\cup l_2(n).
\end{aligned}$$

We can see that

$$\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s^k}\mathrm{d}s=\lim_{n\to\infty}\left(\int_{L(n)}\frac{y^s}{s^k}\mathrm{d}s-\int_{l_2(n)}\frac{y^s}{s^k}\mathrm{d}s\right).$$

The limit of the first integral is equal to zero by the residue theorem, because there are no poles inside $L(n)$. The limit of the second integral is also zero, and this can be shown by a method similar to that applied for the curve $l_3(n)$ in the case $y\ge 1$. So, for $0\le y<1$ we deduce that

$$\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s^k}\mathrm{d}s=0.$$

Now suppose $k=1$. The proof for the case $y>1$ is identical to the corresponding proof when $k\ge 2$, except for how we evaluate the integral over $l_3(n)$. Instead, we do the following:

$$\begin{aligned}
\lim_{n\to\infty}\left|\int_{l_3(n)}\frac{y^s}{s}\mathrm{d}s\right| &\le\lim_{n\to\infty}\left(\int_{\substack{s\in l_3(n)\\ \mathrm{Re}(s)\le-\sqrt{n}}}\frac{y^{\operatorname{Re}s}}{|s|}\mathrm{d}s+\int_{\substack{s\in l_3(n)\\ -\sqrt{n}\le\mathrm{Re}(s)\le 0}}\frac{y^{\operatorname{Re}s}}{|s|}\mathrm{d}s\right)\\
&\le\lim_{n\to\infty}\left(\frac{y^{-\sqrt{n}}}{n}\int_{\substack{s\in l_3(n)\\ \mathrm{Re}(s)\le-\sqrt{n}}}1\mathrm{d}s+\frac{1}{n}\int_{\substack{s\in l_3(n)\\ -\sqrt{n}\le\mathrm{Re}(s)\le 0}}1\mathrm{d}s\right) \qquad (4.3)\\
&\le\lim_{n\to\infty}\Big(\frac{\pi n y^{-\sqrt{n}}}{n}+\frac{n\sin(n^{-\frac{1}{2}})}{n}\Big)=0.
\end{aligned}$$

The proof for the case $0 \leq y < 1$ is also identical to the corresponding proof when $k \geq 2$, except for how we evaluate the integral over $l_2(n)$, which is, instead, done in a similar method as (4.3). For the case $y = 1$, we have the following:

$$\int_{c-i\infty}^{c+i\infty} \frac{1}{s} \mathrm{d}s = \frac{1}{2}\bigg( \int_{c-i\infty}^{c+i\infty} \frac{1}{s} \mathrm{d}s + \int_{-c+i\infty}^{-c-i\infty} \frac{1}{s} \mathrm{d}s \bigg) = \pi i.$$

The last equality follows by similar means as how we evaluated integrals previously in this proof: Express the integrals in terms of a limit of an integral over a rectangle and then apply the residue theorem and some simple bounds. □

**Lemma 4.2.2.** *For all $R \in \mathcal{M}$ and all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -1$ we define*

$$f_R(s) := \prod_{P \mid R} \frac{1 - |P|^{-s-1}}{1 + |P|^{-s-1}}.$$

*Then, for all $R \in \mathcal{A}$ and $j = 1, 2, 3, 4$ we have that*

$$f_R^{(j)}(0) \ll \big( \log_q \log_q |R| \big)^j \prod_{P \mid R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}}.$$

**Remark 4.2.3.** *We must mention that, in the lemma and the proof, the implied constants may depend on $j$, for example; but because there are only finitely many cases of $j$ that we are interested in, we can take the implied constants to be independent. Furthermore, strictly speaking, we require that $\deg R \geq 2$ so that $\log_q \log_q |R|$ is well defined and non-zero.*

*Proof.* First, we note that

$$f_R'(s) = g_R(s) f_R(s), \tag{4.4}$$

where

$$g_R(s) := \sum_{P \mid R} 2 \log |P| \bigg( \frac{1}{|P|^{s+1} + 1} + \frac{1}{|P|^{2s+2} - 1} \bigg).$$

We note further that

$$\begin{aligned} f_R''(s) &= \Big( g_R(s)^2 + g_R'(s) \Big) f_R(s), \\ f_R'''(s) &= \Big( g_R(s)^3 + 3 g_R(s) g_R'(s) + g_R''(s) \Big) f_R(s), \\ f_R''''(s) &= \Big( g_R(s)^4 + 6 g_R(s)^2 g_R'(s) + 4 g_R(s) g_R''(s) + 3 g_R'(s)^2 + g_R'''(s) \Big) f_R(s). \end{aligned} \tag{4.5}$$

For all $R \in \mathcal{A}$ and $k = 0, 1, 2, 3$ it is not difficult to deduce that

$$g_R^{(k)}(0) \ll \sum_{P \mid R} \frac{\big( \log |P| \big)^{k+1}}{|P| - 1}. \tag{4.6}$$

The function $\frac{(\log x)^{k+1}}{x-1}$ is decreasing at large enough $x$, and the limit as $x \longrightarrow \infty$ is 0. Therefore, there exists an independent constant $c \geq 1$ such that for $k = 0, 1, 2, 3$ and all $A, B \in \mathcal{A}$ with $\deg A \leq \deg B$ we have that

$$c \frac{\big( \log |A| \big)^{k+1}}{|A| - 1} \geq \frac{\big( \log |B| \big)^{k+1}}{|B| - 1}.$$

Hence, taking $n=\omega(R)$, and $R_n$ and $m_n$ defined as in Definition A.2.1, we see that

$$\begin{aligned}\sum_{P|R}\frac{\big(\log|P|\big)^{k+1}}{|P|-1}\ll\sum_{P|R_n}\frac{\big(\log|P|\big)^{k+1}}{|P|-1}\ll\sum_{r=1}^{m_n+1}\frac{q^r}{r}\frac{r^{k+1}}{q^r-1}\ll\sum_{r=1}^{m_n+1}r^k\\ \ll(m_n+1)^{k+1}\ll\big(\log_q\log_q|R_n|\big)^{k+1}\ll\big(\log_q\log_q|R|\big)^{k+1},\end{aligned}\tag{4.7}$$

where we have used the prime polynomial theorem and Lemma A.2.2. So, by (4.4)–(4.7), we deduce that

$$f_R^{(j)}(0)\ll\big(\log_q\log_q|R|\big)^j\prod_{P|R}\frac{1-|P|^{-1}}{1+|P|^{-1}}.$$

□

**Lemma 4.2.4.** *Let $R\in\mathcal{M}$, and define ${z_R}':=\deg R-\log_q 9^{\omega(R)}$. We have that*

$$\begin{aligned}&\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq {z_R}'\\ (N,R)=1}}\frac{2^{\omega(N)}}{|N|}({z_R}'-\deg N)^2\\ =&\frac{(1-q^{-1})}{12}\prod_{P|R}\bigg(\frac{1-|P|^{-1}}{1+|P|^{-1}}\bigg)(\deg R)^4\\ &\quad+O\Bigg(\prod_{P|R}\bigg(\frac{1-|P|^{-1}}{1+|P|^{-1}}\bigg)\Big((\deg R)^3\omega(R)+(\deg R)^3\log\deg R\Big)\Bigg).\end{aligned}$$

**Remark 4.2.5.** *This result is to be expected. Indeed, consider the function $F$ defined for* $\operatorname{Re}s>1$ *by*

$$F(s)=\sum_{\substack{N\in\mathcal{M}\\ (N,R)=1}}\frac{2^{\omega(N)}}{|N|^s}.$$

*We can see that*

$$\begin{aligned}F(s)=&\prod_{\substack{P\in\mathcal{P}\\ P\nmid R}}\bigg(1+\frac{2}{|P|^s}+\frac{2}{|P|^{2s}}+\frac{2}{|P|^{3s}}\cdots\bigg)=\prod_{\substack{P\in\mathcal{P}\\ P\nmid R}}\bigg(\frac{2}{1-|P|^{-s}}-1\bigg)\\ =&\prod_{P\in\mathcal{P}}\bigg(\frac{1+|P|^{-s}}{1-|P|^{-s}}\bigg)\prod_{P|R}\bigg(\frac{1-|P|^{-s}}{1+|P|^{-s}}\bigg)=\frac{\zeta_{\mathcal{A}}(s)^2}{\zeta_{\mathcal{A}}(2s)}\prod_{P|R}\bigg(\frac{1-|P|^{-s}}{1+|P|^{-s}}\bigg).\end{aligned}\tag{4.8}$$

*Now, due to the pole of $\zeta_{\mathcal{A}}(s)$ at $s=1$, $F(1)=\sum_{\substack{N\in\mathcal{M}\\ (N,R)=1}}\frac{2^{\omega(N)}}{|N|}$ is not well-defined. However, we are interested in $\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq {z_R}'\\ (N,R)=1}}\frac{2^{\omega(N)}}{|N|}$, so let us replace $\zeta_{\mathcal{A}}(1)$ by*

$$\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq {z_R}'}}\frac{1}{|N|}={z_R}'\asymp\deg R.$$

*Then, from (4.8), and the fact that $\zeta_{\mathcal{A}}(2) = 1 - q^{-1}$ and $z_R{}' - \deg N \asymp \deg R$ for small $N$, we expect*

$$\begin{aligned}\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R{}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|}(z_R{}' - \deg N)^2 \approx &(\deg R)^2 \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R{}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \\ \approx &(1-q^{-1}) \prod_{P \mid R} \left( \frac{1-|P|^{-1}}{1+|P|^{-1}} \right) (\deg R)^4.\end{aligned}$$

*Proof of Lemma 4.2.4.* **STEP 1:** Let $F$ be defined as in Remark 4.2.5, let $c$ be a positive real number, and define $y_R := q^{z_R{}'}$. On the one hand, we have that

$$\begin{aligned}&\frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} F(s+1) \frac{{y_R}^s}{s^3} \mathrm{d}s = \frac{1}{\pi i} \sum_{\substack{N \in \mathcal{M} \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \int_{c-i\infty}^{c+i\infty} \frac{{y_R}^s}{|N|^s s^3} \mathrm{d}s \\ =& \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R{}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \log \left( \frac{y_R}{|N|} \right)^2 = (\log q)^2 \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R{}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} (z_R{}' - \deg N)^2,\end{aligned} \tag{4.9}$$

where the second equality follows from Lemma 4.2.1. On the other hand, for all positive integers $n$, define the following curves:

$$\begin{aligned}l_1(n) :=& \Big[ c - \frac{(2n+1)\pi i}{\log q}, c + \frac{(2n+1)\pi i}{\log q} \Big]; \\ l_2(n) :=& \Big[ c + \frac{(2n+1)\pi i}{\log q}, -\frac{1}{4} + \frac{(2n+1)\pi i}{\log q} \Big]; \\ l_3(n) :=& \Big[ -\frac{1}{4} + \frac{(2n+1)\pi i}{\log q}, -\frac{1}{4} - \frac{(2n+1)\pi i}{\log q} \Big]; \\ l_4(n) :=& \Big[ -\frac{1}{4} - \frac{(2n+1)\pi i}{\log q}, c - \frac{(2n+1)\pi i}{\log q} \Big]; \\ L(n) :=& l_1(n) \cup l_2(n) \cup l_3(n) \cup l_4(n).\end{aligned}$$

Then, we have that

$$\begin{aligned}&\frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} F(s+1) \frac{{y_R}^s}{s^3} \mathrm{d}s \\ =& \frac{1}{\pi i} \lim_{n \to \infty} \Bigg( \int_{L(n)} F(s+1) \frac{{y_R}^s}{s^3} \mathrm{d}s - \int_{l_2(n)} F(s+1) \frac{{y_R}^s}{s^3} \mathrm{d}s \\ &- \int_{l_3(n)} F(s+1) \frac{{y_R}^s}{s^3} \mathrm{d}s - \int_{l_4(n)} F(s+1) \frac{{y_R}^s}{s^3} \mathrm{d}s \Bigg).\end{aligned} \tag{4.10}$$

**STEP 2:** For the first integral in (4.10) we note that $F(1+s)\frac{{y_R}^s}{s^3}$ has a fifth-order pole at $s=0$ and double poles at $s = \frac{2m\pi i}{\log q}$ for $m = \pm 1, \pm 2, \dots, \pm n$. By applying Cauchy's residue theorem we see that

$$\lim_{n \to \infty} \frac{1}{\pi i} \int_{L(n)} F(s+1) \frac{{y_R}^s}{s^3} \mathrm{d}s = 2 \operatorname{Res}_{s=0} F(s+1) \frac{{y_R}^s}{s^3} + 2 \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \operatorname{Res}_{s=\frac{2m\pi i}{\log q}} F(s+1) \frac{{y_R}^s}{s^3}. \tag{4.11}$$

**STEP 2.1:** For the first residue term we have that

$$\begin{aligned}&\mathrm{Res}_{s=0}\, F(s+1)\frac{{y_R}^s}{s^3}\\ =&\frac{1}{4!}\lim_{s\longrightarrow 0}\frac{\mathrm{d}^4}{\mathrm{d}s^4}\left(\zeta_{\mathcal{A}}(s+1)^2 s^2 \frac{1}{\zeta_{\mathcal{A}}(2s+2)}\prod_{P|R}\left(\frac{1-|P|^{-s-1}}{1+|P|^{-s-1}}\right){y_R}^s\right).\end{aligned} \tag{4.12}$$

If we apply the product rule for differentiation, then one of the terms will be

$$\begin{aligned}&\frac{1}{4!}\lim_{s\longrightarrow 0}\left(\zeta_{\mathcal{A}}(s+1)^2 s^2 \frac{1}{\zeta_{\mathcal{A}}(2s+2)}\left(\prod_{P|R}\frac{1-|P|^{-s-1}}{1+|P|^{-s-1}}\right)\frac{\mathrm{d}^4}{\mathrm{d}s^4}{y_R}^s\right)\\ =&\frac{(1-q^{-1})(\log q)^2}{24}\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)({z_R}')^4\\ =&\frac{(1-q^{-1})(\log q)^2}{24}\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)(\deg R)^4\\ &+O\left(\log q\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)(\deg R)^3\omega(R)\right).\end{aligned}$$

Now we look at the remaining terms that arise from the product rule. By using the fact that $\zeta_{\mathcal{A}}(s+1)=\frac{1}{1-q^{-s}}$ and the Taylor series for $q^{-s}$, we have for $k=0,1,2,3,4$ that

$$\lim_{s\to 0}\frac{1}{(\log q)^{k-1}}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\zeta(s+1)s=O(1). \tag{4.13}$$

Similarly,

$$\lim_{s\to 0}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\zeta(2s+2)^{-1}=\lim_{s\to 0}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\left(1-q^{-1-2s}\right)=O(1). \tag{4.14}$$

By (4.13), (4.14), and Lemma 4.2.2, we see that the remaining terms are of order

$$(\log q)^2\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)(\deg R)^3\log\deg R.$$

Hence,

$$\begin{aligned}&2\,\mathrm{Res}_{s=0}\, F(s+1)\frac{{y_R}^s}{s^3}\\ =&\frac{(1-q^{-1})(\log q)^2}{12}\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)(\deg R)^4\\ &+O\left((\log q)^2\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\left((\deg R)^3\omega(R)+(\deg R)^3\log\deg R\right)\right).\end{aligned} \tag{4.15}$$

**STEP 2.2:** Now we look at the remaining residue terms in (4.11). By similar (but simpler) means as above we can show that

$$\mathrm{Res}_{s=\frac{2m\pi i}{\log q}}\, F(s+1)\frac{{y_R}^s}{s^3}=O\left(\frac{1}{m^3}(\log q)^2\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\deg R\right),$$

and so

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \mathrm{Res}_{s=\frac{2m\pi i}{\log q}} F(s+1)\frac{{y_R}^s}{s^3} = O\left( (\log q)^2 \prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right) \deg R \right). \tag{4.16}$$

**STEP 2.3:** By (4.11), (4.15) and (4.16), we see that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\pi i}\int_{L(n)} F(s+1)\frac{{y_R}^s}{s^3}\mathrm{d}s \\
=&\frac{(1-q^{-1})(\log q)^2}{12}\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)(\deg R)^4 \\
&+O\left((\log q)^2\left[\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\left((\deg R)^3\omega(R)+(\deg R)^3\log\deg R\right)\right)\right).
\end{aligned} \tag{4.17}$$

**STEP 3:** We now look at the integrals over $l_2(n)$ and $l_4(n)$. There exists an absolute constant $\kappa$ such that for all positive integers $n$ and all $s\in l_2(n), l_4(n)$ we have that $F(s+1){y_R}^s \leq \kappa |R|^{c+1}$. One can now easily deduce for $i=2,4$ that

$$\lim_{n\to\infty}\left|\frac{1}{\pi i}\int_{l_i(n)} F(s+1)\frac{{y_R}^s}{s^3}\mathrm{d}s\right| = 0. \tag{4.18}$$

**STEP 4:** We now look at the integral over $l_3(n)$. For all positive integers $n$ and all $s\in l_3(n)$ we have that

$$\frac{\zeta_{\mathcal{A}}(s+1)^2}{\zeta_{\mathcal{A}}(2s+2)} = O(1)$$

and

$$\begin{aligned}
\left|\prod_{P|R}\left(\frac{1-|P|^{-s-1}}{1+|P|^{-s-1}}\right){y_R}^s\right| &\ll \prod_{P|R}\left(\frac{1+|P|^{-\frac{3}{4}}}{1-|P|^{-\frac{3}{4}}}\right)\prod_{P|R}(9^{\frac{1}{4}})\ \ |R|^{-\frac{1}{4}} \\
&\ll \prod_{P|R}\left(1+\frac{2}{2^{\frac{3}{4}}-1}\right)\prod_{P|R}(2)\ \ |R|^{-\frac{1}{4}} \\
&\ll \prod_{P|R}(8)\ \ |R|^{-\frac{1}{4}} \ll \prod_{P|R}\left(\frac{8}{|P|^{\frac{1}{4}}}\right) \ll 1.
\end{aligned}$$

We can now easily deduce that

$$\lim_{n\to\infty}\left|\frac{1}{\pi i}\int_{l_3(n)} F(s+1)\frac{{y_R}^s}{s^3}\mathrm{d}s\right| = O(1). \tag{4.19}$$

**STEP 5:** By (4.9), (4.10), (4.17), (4.18) and (4.19), we deduce that

$$\begin{aligned}
&\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq {z_R}'\\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|}({z_R}'-\deg N)^2 \\
=&\frac{(1-q^{-1})}{12}\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)(\deg R)^4
\end{aligned}$$

$$+O\left(\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\left((\deg R)^3\omega(R)+(\deg R)^3\log\deg R\right)\right).$$

□

**Lemma 4.2.6.** *We have that*

$$\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq x}}\frac{2^{\omega(N)}}{|N|}=\frac{q-1}{2q}x^2+\frac{3q+1}{2q}x+1=O(x^2).$$

*Proof.* For $s>1$ we define

$$F(s):=\sum_{N\in\mathcal{M}}\frac{2^{\omega(N)}}{|N|^{s+1}}.$$

We can see that

$$F(s)=\prod_{P\in\mathcal{P}}\left(1+\frac{2}{|P|^{s+1}}+\frac{2}{|P|^{2(s+1)}}+\frac{2}{|P|^{3(s+1)}}+\dots\right)=\prod_{P\in\mathcal{P}}\left(\frac{2}{1-\frac{1}{|P|^{s+1}}}-1\right)$$
$$=\prod_{P\in\mathcal{P}}\frac{1-\frac{1}{|P|^{2(s+1)}}}{\left(1-\frac{1}{|P|^{s+1}}\right)^2}=\frac{\zeta(s+1)^2}{\zeta(2s+2)}=\left(\sum_{n=0}^{\infty}q^{-ns}\right)^2\left(1-q^{-1-2s}\right).$$

By comparing the coefficients of powers of $q^{-s}$, we see that

$$\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq x}}\frac{2^{\omega(N)}}{|N|}=\left(\sum_{n=0}^{x}n+1\right)-\frac{1}{q}\left(\sum_{n=2}^{x}n-1\right)=\frac{q-1}{2q}x^2+\frac{3q+1}{2q}x+1.$$

□

**Lemma 4.2.7.** *Let $R\in\mathcal{M}$. We have that*

$$\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq\deg R\\ (N,R)=1}}\frac{2^{\omega(N)}}{|N|}\ll\prod_{P|R}\left(\frac{1}{1+2|P|^{-1}}\right)\left(\deg R\right)^2\asymp\prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\left(\deg R\right)^2.$$

*Proof.* We have that

$$\left(\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq\deg R\\ (N,R)=1}}\frac{2^{\omega(N)}}{|N|}\right)\left(\sum_{E|R}\frac{2^{\omega(E)}}{|E|}\right)\leq\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq 2\deg R}}\frac{2^{\omega(N)}}{|N|}\ll\left(\deg R\right)^2.$$

where the last relation follows from Lemma 4.2.6. We also note that

$$\sum_{E|R}\frac{2^{\omega(E)}}{|E|}\geq\sum_{E|R}\frac{\mu(E)^2 2^{\omega(E)}}{|E|}=\prod_{P|R}1+\frac{2}{|P|}.$$

This proves the first relation in the lemma. The second relation follows from (A.17) and (A.18).
□

**Lemma 4.2.8.** *Let $F, K \in \mathcal{M}$, $x \geq 0$, and $a \in \mathbb{F}_q^*$. Suppose also that $\frac{1}{2}x < \deg KF \leq \frac{3}{4}x$. Then,*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N = x - \deg KF \\ (N,F)=1}} d(N) d(KF + aN) \ll q^x x^2 \frac{1}{|KF|} \sum_{\substack{H \mid K \\ \deg H \leq \frac{x - \deg KF}{2}}} \frac{d(H)}{|H|}.$$

**Remark 4.2.9.** *The factor of $q^x x^2 \frac{1}{|KF|}$ is to be expected. Indeed, we expect $d(N)d(KF + aN) \approx x^2$ because, generally speaking, $d(A)$ is on average equal to $\deg A$, and we have $\deg N, \deg(KF + aN) \asymp x$.The other factor is due to the fact that there are $\frac{q^x}{|KF|}$ elements of degree $x - \deg KF$. Of course, we have not considered that $(N, F) = 1$.*

*Proof of Lemma 4.2.8.* We have that,

$$\begin{aligned}
&\sum_{\substack{N \in \mathcal{M} \\ \deg N = x - \deg KF \\ (N,F)=1}} d(N) d(KF + aN) \\
\leq 2 &\sum_{\substack{N \in \mathcal{M} \\ \deg N = x - \deg KF \\ (N,F)=1}} \sum_{\substack{G \mid N \\ \deg G \leq \frac{x - \deg KF}{2}}} d(KF + aN) \\
\ll &\sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x - \deg KF}{2} \\ (G,F)=1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = x - \deg KF \\ G \mid N}} d(KF + aN) \\
= &\sum_{\substack{H \mid K \\ \deg H \leq \frac{x - \deg KF}{2}}} \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x - \deg KF}{2} \\ (G,F)=1 \\ (G,K)=H}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = x - \deg KF \\ G \mid N}} d(KF + aN) \\
= &\sum_{\substack{H \mid K \\ \deg H \leq \frac{x - \deg KF}{2}}} \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x - \deg KF}{2} \\ (G,F)=1 \\ (G,K)=H}} \sum_{\substack{N' \in \mathcal{M} \\ \deg N' = x - \deg KF - \deg H \\ G' \mid N'}} d(HK'F + aHN')
\end{aligned}$$

where $N', G', K'$ are defined by $HN' = N, HG' = G, HK' = K$. Continuing, we have that

$$\begin{aligned}
&\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq x - \deg KF \\ (N,F)=1}} d(N) d(KF + aN) \\
\ll &\sum_{\substack{H \mid K \\ \deg H \leq \frac{x - \deg KF}{2}}} d(H) \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x - \deg KF}{2} \\ (G,F)=1 \\ (G,K)=H}} \sum_{\substack{N' \in \mathcal{M} \\ \deg N' = x - \deg KF - \deg H \\ G' \mid N'}} d(K'F + aN') \\
\leq &\sum_{\substack{H \mid K \\ \deg H \leq \frac{x - \deg KF}{2}}} d(H) \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x - \deg KF}{2} \\ (G,F)=1 \\ (G,K)=H}} \sum_{\substack{M \in \mathcal{M} \\ \deg(M - K'F) = x - \deg KF - \deg H \\ (M - K'F) \in a\mathcal{M} \\ M \equiv K'F (\mathrm{mod}\, G')}} d(M)
\end{aligned}$$

$$\ll q^x x \frac{1}{|KF|} \sum_{\substack{H|K \\ \deg H \leq \frac{x-\deg KF}{2}}} \frac{d(H)}{|H|} \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x-\deg KF}{2} \\ (G,F)=1 \\ (G,K)=H}} \frac{1}{\phi(G')}$$

$$\ll q^x x^2 \frac{1}{|KF|} \sum_{\substack{H|K \\ \deg H \leq \frac{x-\deg KF}{2}}} \frac{d(H)}{|H|}.$$

The third relation holds by Theorem 2.1.3 with $\beta = \frac{1}{6}$ and $\alpha = \frac{1}{4}$ (one may wish to note that $(K'F, G') = 1$ and that the other conditions of the theorem are satisfied because $\frac{1}{2}x < \deg KF \leq \frac{3}{4}x$). The last relation follows from Lemma A.3.4. □

**Lemma 4.2.10.** *Let $F \in \mathcal{M}$, $K \in \mathcal{A}\backslash\{0\}$, and $x \geq 0$ satisfy $\deg KF < x$. Then,*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ (N,F)=1}} d(N) d(KF+N) \ll q^x x^2 \sum_{\substack{H|K \\ \deg H \leq \frac{x}{2}}} \frac{d(H)}{|H|}.$$

*Proof.* The proof is similar to the proof of Lemma 4.2.8. We have that

$$\begin{aligned}\sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ (N,F)=1}} d(N) d(KF+N) \leq & 2 \sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ (N,F)=1}} \sum_{\substack{G|N \\ \deg G \leq \frac{x}{2}}} d(KF+N) \ll \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x}{2} \\ (G,F)=1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ G|N}} d(KF+N) \\ = & \sum_{\substack{H|K \\ \deg H \leq \frac{x}{2}}} \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x}{2} \\ (G,F)=1 \\ (G,K)=H}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ G|N}} d(KF+N) \\ = & \sum_{\substack{H|K \\ \deg H \leq \frac{x}{2}}} \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x}{2} \\ (G,F)=1 \\ (G,K)=H}} \sum_{\substack{N' \in \mathcal{M} \\ \deg N' = x - \deg H \\ G'|N'}} d(HK'F + HN'),\end{aligned}$$

where $N', G', K'$ are defined by $HN' = N, HG' = G, HK' = K$. Continuing, we have that

$$\begin{aligned}\sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ (N,F)=1}} d(N) d(KF+N) \ll & \sum_{\substack{H|K \\ \deg H \leq \frac{x}{2}}} d(H) \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x}{2} \\ (G,F)=1 \\ (G,K)=H}} \sum_{\substack{N' \in \mathcal{M} \\ \deg N' = x - \deg H \\ G'|N'}} d(K'F + N') \\ \leq & \sum_{\substack{H|K \\ \deg H \leq \frac{x}{2}}} d(H) \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x}{2} \\ (G,F)=1 \\ (G,K)=H}} \sum_{\substack{M \in \mathcal{M} \\ \deg(M-X) < x - \deg H \\ M \equiv K'F \pmod{G'}}} d(M),\end{aligned}$$

where we define $X := T^{x - \deg H}$. We can now apply Theorem 2.1.1 to obtain that

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ (N,F)=1}} d(N) d(KF+N) \ll q^x x \sum_{\substack{H|K \\ \deg H \leq \frac{x}{2}}} \frac{d(H)}{|H|} \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x}{2} \\ (G,F)=1 \\ (G,K)=H}} \frac{1}{\phi(G')} \ll q^x x^2 \sum_{\substack{H|K \\ \deg H \leq \frac{x}{2}}} \frac{d(H)}{|H|}.$$

□

The proof of the following lemma is very similar to that of Lemma 4.2.10.

**Lemma 4.2.11.** *Let $a \in {\mathbb{F}_q}^*$ with $a \neq 1$, and $x \geq 0$. Furthermore, let $F \in \mathcal{M}$ and $K \in (1-a)\mathcal{M}$ satisfy* $\deg KF = x$. *Then,*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ (N,F)=1}} d(N) d(KF + aN) \ll q^x x^2 \sum_{\substack{H \mid K \\ \deg H \leq \frac{x}{2}}} \frac{d(H)}{|H|}.$$

## 4.3 The Fourth Moment: Specific Preliminary Results

In this section, we prove some specific preliminary results that are required for Section 4.4. They are specific in that it is not so easy to find applications of these results to other problems.

**Lemma 4.3.1.** *Let $F \in \mathcal{M}$ and $z_1, z_2$ be non-negative integers. Then, for all $\epsilon > 0$ we have that*

$$\sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = z_1 \\ \deg CD = z_2 \\ (ABCD, F)=1 \\ AC \equiv BD (\operatorname{mod} F) \\ AC \neq BD}} 1 \begin{cases} \ll_\epsilon \frac{1}{|F|} \left( q^{z_1} q^{z_2} \right)^{1+\epsilon} & \text{if } z_1 + z_2 \leq \frac{19}{10} \deg F \\ \ll \frac{1}{\phi(F)} q^{z_1} q^{z_2} (z_1 + z_2)^3 & \text{if } z_1 + z_2 > \frac{19}{10} \deg F. \end{cases}$$

*Proof.* We can split the sum into the cases $\deg AC > \deg BD$, $\deg AC < \deg BD$, and $\deg AC = \deg BD$ with $AC \neq BD$. The first two cases are identical by symmetry.

When $\deg AC > \deg BD$, we have that $AC = KF + BD$ where $K \in \mathcal{M}$ and $\deg KF > \deg BD$. Furthermore,

$$2 \deg KF = 2 \deg AC > \deg AC + \deg BD = \deg AB + \deg CD = z_1 + z_2,$$

from which we deduce that $\frac{z_1+z_2}{2} < \deg KF \leq z_1 + z_2$; and

$$\deg KF + \deg BD = \deg AC + \deg BD = z_1 + z_2,$$

from which we deduce that $\deg BD = z_1 + z_2 - \deg KF$.

When $\deg AC = \deg BD$, we must have that $\deg AC = \deg BD = \frac{z_1+z_2}{2}$ (in particular, this case applies only when $z_1 + z_2$ is even). Also, we can write $AC = KF + BD$, where $\deg KF < \deg BD = \frac{z_1+z_2}{2}$ and $K$ need not be monic.

So, writing $N = BD$, we have that

$$\begin{aligned} \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = z_1 \\ \deg CD = z_2 \\ (ABCD, F)=1 \\ AC \equiv BD (\operatorname{mod} F) \\ AC \neq BD}} 1 \leq & 2 \sum_{\substack{K \in \mathcal{M} \\ \frac{z_1+z_2}{2} < \deg KF \leq z_1 + z_2}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 - \deg KF \\ (N,F)=1}} d(N) d(KF + N) \\ & + \sum_{\substack{K \in \mathcal{A} \\ \deg KF < \frac{z_1+z_2}{2}}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = \frac{z_1+z_2}{2} \\ (N,F)=1}} d(N) d(KF + N). \end{aligned} \tag{4.20}$$

**STEP 1:** Let us consider the case when $z_1+z_2 \leq \frac{19}{10}\deg F$. By using the well known bound that $d(N) \ll_\epsilon |N|^\epsilon$, we have that

$$\begin{aligned}
&\sum_{\substack{K\in\mathcal{M} \\ \frac{z_1+z_2}{2}<\deg KF\leq z_1+z_2}} \sum_{\substack{N\in\mathcal{M} \\ \deg N=z_1+z_2-\deg KF \\ (N,F)=1}} d(N)d(KF+N) \\
\ll_\epsilon &\left(q^{z_1}q^{z_2}\right)^{\frac{\epsilon}{2}} \sum_{\substack{K\in\mathcal{M} \\ \frac{z_1+z_2}{2}<\deg KF\leq z_1+z_2}} \sum_{\substack{N\in\mathcal{M} \\ \deg N=z_1+z_2-\deg KF \\ (N,F)=1}} 1 \\
\leq &\left(q^{z_1}q^{z_2}\right)^{1+\frac{\epsilon}{2}} \sum_{\substack{K\in\mathcal{M} \\ \frac{z_1+z_2}{2}<\deg KF\leq z_1+z_2}} \frac{1}{|KF|} \\
\leq &\left(q^{z_1}q^{z_2}\right)^{1+\frac{\epsilon}{2}} \frac{z_1+z_2}{|F|} \\
\ll_\epsilon &\left(q^{z_1}q^{z_2}\right)^{1+\epsilon} \frac{1}{|F|}.
\end{aligned}$$

As for the sum

$$\sum_{\substack{K\in\mathcal{A} \\ \deg KF<\frac{z_1+z_2}{2}}} \sum_{\substack{N\in\mathcal{M} \\ \deg N=\frac{z_1+z_2}{2} \\ (N,F)=1}} d(N)d(KF+N),$$

we note that it does not apply to this case where $z_1+z_2 \leq \frac{19}{10}\deg F$ because $\deg KF \geq \deg F \geq \frac{20}{19}\frac{z_1+z_2}{2}$, which does not overlap with range $\deg KF < \frac{z_1+z_2}{2}$ in the sum.

Hence,

$$\sum_{\substack{A,B,C,D\in\mathcal{M} \\ \deg AB=z_1 \\ \deg CD=z_2 \\ (ABCD,F)=1 \\ AC\equiv BD (\mathrm{mod}\, F) \\ AC\neq BD}} 1 \ll_\epsilon \left(q^{z_1}q^{z_2}\right)^{1+\epsilon} \frac{1}{|F|}.$$

**STEP 2:** We now consider the case when $z_1+z_2 > \frac{19}{10}\deg F$.

**STEP 2.1:** We consider the subcase where $\frac{z_1+z_2}{2} < \deg KF \leq \frac{3(z_1+z_2)}{4}$. This allows us to apply Lemma 4.2.8 for the first relation below.

$$\begin{aligned}
&\sum_{\substack{K\in\mathcal{M} \\ \frac{z_1+z_2}{2}<\deg KF\leq \frac{3(z_1+z_2)}{4}}} \sum_{\substack{N\in\mathcal{M} \\ \deg N=z_1+z_2-\deg KF \\ (N,F)=1}} d(N)d(KF+N) \\
\ll &q^{z_1}q^{z_2}(z_1+z_2)^2\frac{1}{|F|} \sum_{\substack{K\in\mathcal{M} \\ \frac{z_1+z_2}{2}<\deg KF\leq \frac{3(z_1+z_2)}{4}}} \frac{1}{|K|} \sum_{\substack{H|K \\ \deg H\leq \frac{z_1+z_2-\deg KF}{2}}} \frac{d(H)}{|H|} \\
\leq &q^{z_1}q^{z_2}(z_1+z_2)^2\frac{1}{|F|} \sum_{\substack{K\in\mathcal{M} \\ \deg K\leq z_1+z_2}} \frac{1}{|K|} \sum_{H|K} \frac{d(H)}{|H|}
\end{aligned}$$

$$
\begin{aligned}
=& q^{z_1} q^{z_2} (z_1+z_2)^2 \frac{1}{|F|} \sum_{\substack{H \in \mathcal{M} \\ \deg H \leq z_1+z_2}} \frac{d(H)}{|H|} \sum_{\substack{K \in \mathcal{M} \\ \deg K \leq z_1+z_2 \\ H|K}} \frac{1}{|K|} \\
\leq & q^{z_1} q^{z_2} (z_1+z_2)^3 \frac{1}{|F|} \sum_{\substack{H \in \mathcal{M} \\ \deg H \leq z_1+z_2}} \frac{d(H)}{|H|^2} \\
\ll & q^{z_1} q^{z_2} (z_1+z_2)^3 \frac{1}{|F|}.
\end{aligned}
$$

**STEP 2.2:** Now we consider the subcase where $\frac{3(z_1+z_2)}{4} < \deg KF \leq z_1+z_2$. We have that

$$
\begin{aligned}
& \sum_{\substack{K \in \mathcal{M} \\ \frac{3(z_1+z_2)}{4} < \deg KF \leq z_1+z_2}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1+z_2 - \deg KF \\ (N,F)=1}} d(N) d(KF+N) \\
= & \sum_{\substack{N \in \mathcal{M} \\ \deg N < \frac{z_1+z_2}{4} \\ (N,F)=1}} \sum_{\substack{K \in \mathcal{M} \\ \deg KF = z_1+z_2 - \deg N}} d(N) d(KF+N) \\
\leq & \sum_{\substack{N \in \mathcal{M} \\ \deg N < \frac{z_1+z_2}{4} \\ (N,F)=1}} d(N) \sum_{\substack{M \in \mathcal{M} \\ \deg(M - X_{(N)}) < z_1+z_2 - \deg N \\ M \equiv N (\mathrm{mod}\, F)}} d(M),
\end{aligned}
$$

where we define $X_{(N)} = T^{z_1+z_2-\deg N}$. We can now apply Theorem 2.1.1. One may wish to note that

$$
y = z_1+z_2 - \deg N \geq \frac{3}{4}(z_1+z_2) \geq \frac{3}{4}\frac{19}{10} \deg F
$$

and so

$$
\deg F \leq \frac{40}{57} y = (1-\alpha) y,
$$

where $0 < \alpha < \frac{1}{2}$, as required. Hence, we have that

$$
\begin{aligned}
& \sum_{\substack{K \in \mathcal{M} \\ \frac{3(z_1+z_2)}{4} < \deg KF \leq z_1+z_2}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1+z_2 - \deg KF \\ (N,F)=1}} d(N) d(KF+N) \\
\ll & q^{z_1} q^{z_2} (z_1+z_2) \frac{1}{\phi(F)} \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq \frac{z_1+z_2}{4} \\ (N,F)=1}} \frac{d(N)}{|N|} \\
\leq & q^{z_1} q^{z_2} (z_1+z_2) \frac{1}{\phi(F)} \Bigg( \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_1+z_2}} \frac{1}{|N|} \Bigg)^2 \\
\ll & q^{z_1} q^{z_2} (z_1+z_2)^3 \frac{1}{\phi(F)}.
\end{aligned}
$$

**STEP 2.3:** We now look at the sum

$$
\sum_{\substack{K \in \mathcal{A} \\ \deg KF < \frac{z_1+z_2}{2}}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = \frac{z_1+z_2}{2} \\ (N,F)=1}} d(N) d(KF+N).
$$

Note that this is zero if $z_1 + z_2$ is odd. So, we assume it is even. In particular, this means that $\deg KF < \frac{z_1+z_2}{2}$ is equivalent to $\deg KF \leq \frac{z_1+z_2}{2} - 1$. Now, by Lemma 4.2.10 we have that

$$\sum_{\substack{K \in \mathcal{A} \\ \deg KF < \frac{z_1+z_2}{2}}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = \frac{z_1+z_2}{2} \\ (N,F)=1}} d(N)d(KF+N) \ll q^{\frac{z_1+z_2}{2}} (z_1+z_2)^2 \sum_{\substack{K \in \mathcal{A} \\ \deg KF < \frac{z_1+z_2}{2}}} \sum_{H|K} \frac{d(H)}{|H|}$$

$$= q^{\frac{z_1+z_2}{2}} (z_1+z_2)^2 \sum_{\substack{H \in \mathcal{M} \\ \deg H < \frac{z_1+z_2}{2} - \deg F - 1}} \frac{d(H)}{|H|} \sum_{\substack{K \in \mathcal{A} \\ \deg K \leq \frac{z_1+z_2}{2} - \deg F - 1 \\ H|K}} 1$$

$$= q^{z_1+z_2} (z_1+z_2)^2 \frac{1}{|F|} \sum_{\substack{H \in \mathcal{M} \\ \deg H < \frac{z_1+z_2}{2} - \deg F - 1}} \frac{d(H)}{|H|^2} \leq q^{z_1+z_2} (z_1+z_2)^2 \frac{1}{|F|}.$$

**STEP 2.4:** We apply steps 2.1, 2.2, and 2.3 to (4.20) and we see that

$$\sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = z_1 \\ \deg CD = z_2 \\ (ABCD,F)=1 \\ AC \equiv BD \pmod F \\ AC \neq BD}} 1 \ll q^{z_1} q^{z_2} (z_1+z_2)^3 \frac{1}{\phi(F)}$$

for $z_1 + z_2 \geq \frac{19}{10} \deg F$. □

In fact, we can prove the following, more general Lemma.

**Lemma 4.3.2.** *Let $F \in \mathcal{M}$, $z_1, z_2$ be non-negative integers, and let $a \in \mathbb{F}_q^*$. Then, for all $\epsilon > 0$ we have that*

$$\sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = z_1 \\ \deg CD = z_2 \\ (ABCD,F)=1 \\ AC \equiv aBD \pmod F \\ AC \neq BD}} 1 \begin{cases} \ll_\epsilon \frac{1}{|F|} \Big( q^{z_1} q^{z_2} \Big)^{1+\epsilon} & \text{if } z_1 + z_2 \leq \frac{19}{10} \deg F \\ \ll \frac{1}{\phi(F)} q^{z_1} q^{z_2} (z_1+z_2)^3 & \text{if } z_1 + z_2 > \frac{19}{10} \deg F. \end{cases}$$

*Proof.* The case where $a = 1$ is just Lemma 4.3.1. The proof of the case where $a \neq 1$ is very similar to the proof of Lemma 4.3.1. The main difference is when $\deg AC = \deg BD$. Again, we would have that $\deg AC = \deg BD = \frac{z_1+z_2}{2}$ and $AC = KF + aBD$, but instead of $K$ being in $\mathcal{A}$ and $\deg KF < \deg BD = \frac{z_1+z_2}{2}$, we would have $K \in (1-a)\mathcal{M}$ and $\deg KF = \deg BD = \frac{z_1+z_2}{2}$. Hence, in Step 2.3, we would use Lemma 4.2.11 instead of Lemma 4.2.10. □

**Lemma 4.3.3.** *Let $R \in \mathcal{M}$ and define $z_R := \deg R - \log_q 2^{\omega(R)}$. Also, let $a \in \mathbb{F}_q^*$. Then,*

$$\sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB \leq z_R \\ \deg CD \leq z_R \\ (ABCD,R)=1 \\ AC \equiv aBD \pmod F \\ AC \neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}}$$

$$\ll |R| \big( \deg R \big)^3 \ll \phi^*(R) \prod_{P|R} \bigg( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \bigg) (\deg R)^3 (\log \deg R)^6.$$

*Proof.* The second relation follows easily from (A.20). So, we proceed to prove the first relation. We apply Lemma 4.3.2 with $\epsilon = \frac{1}{50}$ to deduce that

$$\begin{aligned}
&\sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB \leq z_R \\ \deg CD \leq z_R \\ (ABCD,R)=1 \\ AC \equiv aBD (\operatorname{mod} F) \\ AC \neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} \\
\ll &\frac{1}{|F|} \sum_{\substack{z_1,z_2 \leq z_R \\ z_1+z_2 \leq \frac{19}{10} \deg F}} \left( q^{z_1} q^{z_2} \right)^{\frac{1}{2}+\epsilon} + \frac{1}{\phi(F)} \sum_{\substack{z_1,z_2 \leq z_R \\ \frac{19}{10} \deg F < z_1+z_2 \leq 2 \deg R}} q^{\frac{z_1}{2}} q^{\frac{z_2}{2}} (z_1+z_2)^3 \\
\ll &\frac{1}{|F|^{\frac{1}{20}-2\epsilon}} + \frac{1}{\phi(F)} \big( \deg R \big)^3 \sum_{z_1 < z_R} \sum_{z_2 < z_R} q^{\frac{z_1}{2}} q^{\frac{z_2}{2}} \ll \frac{1}{|F|^{\frac{1}{20}-2\epsilon}} + \frac{1}{|F|} q^{z_R} \big( \deg R \big)^3 .
\end{aligned}$$

So,

$$\begin{aligned}
&\sum_{EF=R} \mu(E) \phi(F) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB \leq z_R \\ \deg CD \leq z_R \\ (ABCD,R)=1 \\ AC \equiv BD (\operatorname{mod} F) \\ AC \neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} \\
\ll &q^{z_R} \big( \deg R \big)^3 \sum_{EF=R} |\mu(E)| + \sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|^{\frac{1}{20}-2\epsilon}} \\
\ll &q^{z_R} \big( \deg R \big)^3 2^{\omega(R)} + |R| \ll |R| \big( \deg R \big)^3 ,
\end{aligned}$$

where the second-to-last relation uses the following.

$$\begin{aligned}
&\sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|^{\frac{1}{20}-2\epsilon}} \\
\leq &\sum_{EF=R} |\mu(E)| \phi(F) = \phi(R) \sum_{EF=R} |\mu(E)| \prod_{\substack{P|E \\ P^2|R}} \left( \frac{1}{|P|} \right) \prod_{\substack{P|E \\ P^2 \nmid R}} \left( \frac{1}{|P|-1} \right) \\
\leq &\phi(R) \sum_{EF=R} |\mu(E)| \prod_{P|E} \left( \frac{1}{|P|-1} \right) = \phi(R) \prod_{P|R} \left( 1 + \frac{1}{|P|-1} \right) = \phi(R) \frac{|R|}{\phi(R)} = |R| .
\end{aligned}$$

□

## 4.4 The Fourth Moment

We are now in a position to prove Theorem 2.2.3, which we restate for ease of reference.

**Theorem.** *Let $R \in \mathcal{M} \backslash \{1\}$. Then,*

$$\begin{aligned}
&\sideset{}{^*}\sum_{\chi \operatorname{mod} R} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^4 \\
= &\frac{1-q^{-1}}{12} \phi^*(R) \prod_{P|R} \left( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \right) (\deg R)^4 \left( 1 + O\left( \sqrt{\frac{\omega(R) + (\log \deg R)^6}{\deg R}} \right) \right) .
\end{aligned}$$

*Proof of Theorem 2.2.3.* Let $\chi$ be a primitive Dirichlet character of modulus $R \in \mathcal{M}\backslash\{1\}$. By lemmas A.1.2 and A.1.3, we have that

$$\left|L\Big(\frac{1}{2},\chi\Big)\right|^2 = 2 \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} + c(\chi) = 2a(\chi) + 2b(\chi) + c(\chi),$$

where

$$\begin{aligned} z_R &:= \deg R - \log_q(2^{\omega(Q)}), \\ a(\chi) &:= \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB\leq z_R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}, \\ b(\chi) &:= \sum_{\substack{A,B\in\mathcal{M} \\ z_R<\deg AB<\deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}, \end{aligned}$$

and $c(\chi)$ is defined as in (A.8). Then,

$$\sum_{\chi \bmod R}^{*} \left|L\Big(\frac{1}{2},\chi\Big)\right|^4 = \sum_{\chi \bmod R}^{*} \Big(2a(\chi) + 2b(\chi) + c(\chi)\Big)^2.$$

We will show that $\sum^{*}_{\chi \bmod R}|a(\chi)|^2$ has an asymptotic main term of higher order than $\sum^{*}_{\chi \bmod R}|b(\chi)|^2$ and $\sum^{*}_{\chi \bmod R}|c(\chi)|^2$. From this and the Cauchy-Schwarz inequality, we deduce that $\sum^{*}_{\chi \bmod R}|a(\chi)|^2$ gives the leading term in the asymptotic formula.

One may ask why we break the moments up into these three pieces. To answer this, consider the proof of Lemma 4.3.3 (which we will use to address the off-diagonal terms of $\sum^{*}_{\chi \bmod R}|a(\chi)|^2$ ). There, we made use of the following calculation:

$$q^{z_R} \sum_{E|R} |\mu(E)| = q^{z_R} 2^{\omega(R)} = |R|.$$

We can see that by using $z_R$ instead of $\deg R$, we ensure that there is cancellation with $\sum_{E|R}|\mu(E)|$, and this is crucial to ensure that our error term is of lower order than the main term.

Naturally, one asks how we then deal with the sum $\sum^{*}_{\chi \bmod R}|b(\chi)|^2$, which involves the range $z_R < \deg AB < \deg R$. Here, we replace the sum $\sum^{*}_{\chi \bmod R}$ with $\sum_{\chi \bmod R}$. This is helpful because we can now avoid the $\sum_{E|R}|\mu(E)|$ factor, but it does mean we are overestimating the contribution of $\sum^{*}_{\chi \bmod R}|b(\chi)|^2$. This, however, is mitigated by the fact that the range $z_R < \deg AB < \deg R$ is relatively small, and so we are left with a lower order term, as required.

So, essentially, as is often the case in analytic number theory, we break the sum into two smaller sums where we can apply a different method to each smaller sum. With regards to the sum $\sum^{*}_{\chi \bmod R}|c(\chi)|^2$, this is simply an outcome of our application of the functional equation that needs to be addressed separately as it does not "fit" nicely with the other two sums.

**STEP 1:** We have that

$$\begin{aligned}\sideset{}{^*}\sum_{\chi \bmod R} |a(\chi)|^2 &= \sideset{}{^*}\sum_{\chi \bmod R} \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB\leq z_R\\ \deg CD\leq z_R}} \frac{\chi(AC)\overline{\chi}(BD)}{|ABCD|^{\frac{1}{2}}}\\ &= \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB\leq z_R\\ \deg CD\leq z_R}} \frac{1}{|ABCD|^{\frac{1}{2}}} \sideset{}{^*}\sum_{\chi \bmod R} \chi(AC)\overline{\chi}(BD)\\ &= \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB\leq z_R\\ \deg CD\leq z_R\\ (ABCD,R)=1}} \frac{1}{|ABCD|^{\frac{1}{2}}} \sum_{\substack{EF=R\\ F|(AC-BD)}} \mu(E)\phi(F),\end{aligned}$$

where the last equality follows from Lemma 1.4.5. Continuing,

$$\begin{aligned}&\sideset{}{^*}\sum_{\chi \bmod R} |a(\chi)|^2 = \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB\leq z_R\\ \deg CD\leq z_R\\ (ABCD,R)=1\\ F|(AC-BD)}} \frac{1}{|ABCD|^{\frac{1}{2}}}\\ &= \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB\leq z_R\\ \deg CD\leq z_R\\ (ABCD,R)=1\\ F|(AC-BD)\\ AC=BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} + \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB\leq z_R\\ \deg CD\leq z_R\\ (ABCD,R)=1\\ F|(AC-BD)\\ AC\neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}}\\ &= \Bigg(\sum_{EF=R} \mu(E)\phi(F)\Bigg) \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB\leq z_R\\ \deg CD\leq z_R\\ (ABCD,R)=1\\ AC=BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} + \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB\leq z_R\\ \deg CD\leq z_R\\ (ABCD,R)=1\\ F|(AC-BD)\\ AC\neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}}.\end{aligned} \tag{4.21}$$

**STEP 1.1:** We will look at the first term on the far RHS of (4.21). Since $AC = BD$, we can write $A = GU, B = GV, C = HV, D = HU$, where $G, H, U, V$ are monic and $U, V$ are coprime. Let us write $N = UV$, and note that there are $2^{\omega(N)}$ ways of writing $N = UV$ with $U, V$ being coprime. Then,

$$\begin{aligned}
&\sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB \leq z_R \\ \deg CD \leq z_R \\ (ABCD,R)=1 \\ AC=BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} \\
=& \sum_{\substack{G,H,U,V \in \mathcal{M} \\ (U,V)=1 \\ \deg G^2UV \leq z_R \\ \deg H^2UV \leq z_R \\ (GHUV,R)=1}} \frac{1}{|GHUV|} = \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{z_R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \Bigg)^2 \\
=& \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R{}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{z_R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \Bigg)^2 + \sum_{\substack{N \in \mathcal{M} \\ z_R{}' < \deg N \leq z_R \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{z_R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \Bigg)^2,
\end{aligned} \tag{4.22}$$

where $z_R{}' := \deg R - \log_q 9^{\omega(R)}$.

Let us look at the first term on the far RHS of (4.22). We can apply Corollary A.3.3 because $x = \frac{z_R - \deg N}{2} \geq \log_q 3^{\omega(R)}$. This gives

$$\begin{aligned}
&\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R{}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{z_R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \Bigg)^2 \\
=& \bigg( \frac{\phi(R)}{2|R|} \bigg)^2 \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R{}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \bigg( z_R - \deg N + O\big( \log \omega(R) \big) \bigg)^2 \\
=& \bigg( \frac{\phi(R)}{2|R|} \bigg)^2 \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R{}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \bigg( \big( z_R{}' - \deg N \big)^2 + O\big( \deg R \log \omega(R) \big) \bigg) \qquad (4.23) \\
=& \frac{1-q^{-1}}{48} \prod_{P|R} \bigg( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \bigg) (\deg R)^4 \\
&+ O\Bigg( \prod_{P|R} \bigg( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \bigg) \Big( (\deg R)^3 \omega(R) + (\deg R)^3 \log \deg R \Big) \Bigg),
\end{aligned}$$

where the last equality follows from Lemma 4.2.4 and Lemma 4.2.7.

Now we look at the second term on the far RHS of (4.22). Because $z_R{}' < \deg N \leq z_R$, we have that $\deg G \leq \log_q \big( \frac{3}{\sqrt{2}} \big)^{\omega(R)}$. Using this and Corollary A.3.3, we have that

$$\sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{z_R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \leq \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \log_q \left( \frac{3}{\sqrt{2}}^{\omega(R)} \right) \\ (G,R)=1}} \frac{1}{|G|} \ll \frac{\phi(R)}{|R|} \omega(R).$$

Also, by similar means as in Lemma 4.2.6, we can see that

$$\sum_{\substack{N\in\mathcal{M}\\ z_R'\leq \deg N\leq z_R\\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \leq \sum_{\substack{N\in\mathcal{M}\\ z_R'\leq \deg N\leq z_R}} \frac{2^{\omega(N)}}{|N|} \ll \omega(R)\deg R.$$

Hence,

$$\begin{aligned}\sum_{\substack{N\in\mathcal{M}\\ z_R'\leq \deg N\leq z_R\\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|}\bigg(\sum_{\substack{G\in\mathcal{M}\\ \deg G\leq \frac{z_R-\deg N}{2}\\ (G,R)=1}} \frac{1}{|G|}\bigg)^2 &\ll \bigg(\frac{\phi(R)}{|R|}\bigg)^2 \omega(R)^3\deg R\\ &\ll \prod_{P|R}\bigg(\frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}}\bigg)(\deg R)^3\omega(R),\end{aligned} \tag{4.24}$$

where the last equality uses (A.21).

By (4.22), (4.23) and (4.24), we have that

$$\begin{aligned}&\bigg(\sum_{EF=R}\mu(E)\phi(F)\bigg)\sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB\leq z_R\\ \deg CD\leq z_R\\ (ABCD,R)=1\\ AC=BD}} \frac{1}{|ABCD|^{\frac{1}{2}}}\\ =&\frac{1-q^{-1}}{48}\phi^*(R)\prod_{P|R}\bigg(\frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}}\bigg)(\deg R)^4\\ &\qquad + O\Bigg(\phi^*(R)\prod_{P|R}\bigg(\frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}}\bigg)\Big((\deg R)^3\omega(R)+(\deg R)^3\log\deg R\Big)\Bigg).\end{aligned}$$

**<u>STEP 1.2:</u>** For the second term on the far RHS of (4.21) we simply apply Lemma 4.3.3. From this, Step 1.1, and (4.21), we deduce that

$$\begin{aligned}&\sum_{\chi \bmod R}{}^{*} |a(\chi)|^2\\ =&\frac{1-q^{-1}}{48}\phi^*(R)\prod_{P|R}\bigg(\frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}}\bigg)(\deg R)^4\\ &\qquad + O\Bigg(\phi^*(R)\prod_{P|R}\bigg(\frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}}\bigg)\Big((\deg R)^3\omega(R)+(\deg R)^3(\log\deg R)^6\Big)\Bigg).\end{aligned}$$

**STEP 2:** We will now look at $\sum^{*}_{\chi \bmod R} |b(\chi)|^2$. Using Lemma 1.4.4, we have that

$$\begin{aligned}\sum^{*}_{\chi \bmod R} |b(\chi)|^2 \leq \sum_{\chi \bmod R} |b(\chi)|^2 = \phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ z_R < \deg AB < \deg R \\ z_R < \deg CD < \deg R \\ (ABCD,R)=1 \\ AC \equiv BD (\bmod R)}} \frac{1}{|ABCD|^{\frac{1}{2}}} \\ = \phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ z_R < \deg AB < \deg R \\ z_R < \deg CD < \deg R \\ (ABCD,R)=1 \\ AC = BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} + \phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ z_R < \deg AB < \deg R \\ z_R < \deg CD < \deg R \\ (ABCD,R)=1 \\ AC \equiv BD (\bmod R) \\ AC \neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}}.\end{aligned} \tag{4.25}$$

**STEP 2.1:** Looking at the first term on the far RHS, we apply the same technique as in (4.22) to obtain

$$\begin{aligned}&\phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ z_R < \deg AB < \deg R \\ z_R < \deg CD < \deg R \\ (ABCD,R)=1 \\ AC = BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} \\ =&\phi(R) \sum_{\substack{N \in \mathcal{M} \\ \deg N < \deg R \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \frac{z_R - \deg N}{2} < \deg G < \frac{\deg R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \Bigg)^2 \\ \leq&\phi(R) \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq {z_R}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \frac{z_R - \deg N}{2} < \deg G < \frac{\deg R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \Bigg)^2 \\ &+ \phi(R) \sum_{\substack{N \in \mathcal{M} \\ {z_R}' < \deg N < \deg R \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \deg G < \frac{\deg R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \Bigg)^2,\end{aligned} \tag{4.26}$$

where ${z_R}' := \deg R - \log_q 9^{\omega(R)}$.

We look at the first term on the far RHS of (4.26). By Corollary A.3.3, we have that

$$\begin{aligned}\sum_{\substack{G \in \mathcal{M} \\ \frac{z_R - \deg N}{2} < \deg G < \frac{\deg R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} = \sum_{\substack{G \in \mathcal{M} \\ \deg G < \frac{\deg R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} - \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{z_R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \\ \ll \frac{\phi(R)}{|R|} \Big( \omega(R) + \log \omega(R) \Big) \ll \frac{\phi(R)}{|R|} \omega(R),\end{aligned}$$

and so

$$\phi(R) \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq {z_R}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \frac{z_R - \deg N}{2} < \deg G < \frac{\deg R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \Bigg)^2$$

$$\ll |R| \left( \frac{\phi(R)}{|R|} \right)^3 \omega(R)^2 \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z_R{}' \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|}$$

$$\ll |R| \left( \frac{\phi(R)}{|R|} \right)^3 \omega(R)^2 \prod_{P|R} \left( \frac{1-|P|^{-1}}{1+|P|^{-1}} \right) \big( \deg R \big)^2.$$

For the last relation we applied Lemma 4.2.7.

Now we look at the second term on the far RHS of (4.26). Because $z_R{}' < \deg N < \deg R$, we have that $\frac{\deg R - \deg N}{2} < \log_q 9^{\frac{\omega(R)}{2}}$. Hence, using Corollary A.3.3 and Lemma 4.2.7, we have

$$\phi(R) \sum_{\substack{N \in \mathcal{M} \\ z_R{}' < \deg N < \deg R \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \deg G < \frac{\deg R - \deg N}{2} \\ (G,R)=1}} \frac{1}{|G|} \Bigg)^2$$

$$\ll |R| \left( \frac{\phi(R)}{|R|} \right)^3 \omega(R)^2 \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq \deg R \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|}$$

$$\ll |R| \left( \frac{\phi(R)}{|R|} \right)^3 \omega(R)^2 \prod_{P|R} \left( \frac{1-|P|^{-1}}{1+|P|^{-1}} \right) \big( \deg R \big)^2.$$

Hence, by (A.22),

$$\phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ z_R < \deg AB < \deg R \\ z_R < \deg CD < \deg R \\ (ABCD,R)=1 \\ AC=BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} \ll |R| \left( \frac{\phi(R)}{|R|} \right)^3 \omega(R)^2 \prod_{P|R} \left( \frac{1-|P|^{-1}}{1+|P|^{-1}} \right) \big( \deg R \big)^2$$

$$\ll \phi^*(R) \prod_{P|R} \left( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \right) (\deg R)^3 \omega(R).$$

**<u>STEP 2.2:</u>** We now look at the second term on the far right-hand-side of (4.25):

$$\phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ z_R < \deg AB < \deg R \\ z_R < \deg CD < \deg R \\ (ABCD,R)=1 \\ AC \equiv BD (\mathrm{mod}\, R) \\ AC \neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}}$$

$$= \phi(R) \sum_{z_R < z_1, z_2 < \deg R} \frac{1}{(q^{z_1+z_2})^{\frac{1}{2}}} \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = z_1 \\ \deg CD = z_2 \\ (ABCD,R)=1 \\ AC \equiv BD (\mathrm{mod}\, R) \\ AC \neq BD}} 1 \ll \sum_{z_R < z_1, z_2 < \deg R} q^{\frac{z_1+z_2}{2}} (z_1+z_2)^3$$

$$\ll |R| \big( \deg R \big)^3 \ll \phi^*(R) \prod_{P|R} \left( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \right) (\deg R)^3 \omega(R). \tag{4.27}$$

The second relation follows from Lemma 4.3.1 with $F := R$. Indeed, for $q \geq 2^{20}$, we have

$$z_1 + z_2 \geq 2z_R = 2\deg R - 2\log_q 2^{\omega(R)} \geq 2\deg R(1 - \log_q 2) > \frac{19}{10}\deg R;$$

and for $q \leq 2^{20}$ we use Lemma A.2.3 to obtain

$$z_1 + z_2 \geq 2z_R = 2\deg R - 2\log_q 2^{\omega(R)} = 2\deg R - O\Big(\frac{\deg R}{\log \deg R}\Big) > \frac{19}{10}\deg R$$

for all $\deg R$ greater than some constant $d$ that is independent of $q$. There are a finite number of cases where $q \leq 2^{20}$ and $\deg R \leq d$, and so the second relation of (4.27) holds for them too.

**STEP 2.3:** Hence, we see that

$$\sum_{\chi \bmod R}^{*} |b(\chi)|^2 \ll \phi^*(R) \prod_{\substack{P \in \mathcal{P} \\ P|R}} \left( \frac{\big(1 - |P|^{-1}\big)^3}{1 + |P|^{-1}} \right) (\deg R)^3 \omega(R).$$

**STEP 3:** We will now look at $\sum_{\chi \bmod R}^{*} |c(\chi)|^2$. We have that

$$\sum_{\chi \bmod R}^{*} |c(\chi)|^2 \leq \sum_{\chi \bmod R} |c(\chi)|^2 = \sum_{\chi \bmod R} |c_o(\chi)|^2 - \sum_{\substack{\chi \bmod R \\ \chi \text{ even}}} |c_o(\chi)|^2 + \sum_{\substack{\chi \bmod R \\ \chi \text{ even}}} |c_e(\chi)|^2.$$

Now,

$$\begin{aligned}
\sum_{\chi \bmod R} |c_o(\chi)|^2 = & \sum_{\chi \bmod R} \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = \deg R - 1 \\ \deg CD = \deg R - 1}} \frac{\chi(AC)\overline{\chi}(BD)}{|ABCD|^{\frac{1}{2}}} \\
= & \phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = \deg R - 1 \\ \deg CD = \deg R - 1 \\ (ABCD,R)=1 \\ AC = BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} + \phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = \deg R - 1 \\ \deg CD = \deg R - 1 \\ (ABCD,R)=1 \\ AC \equiv BD (\bmod R) \\ AC \neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}}.
\end{aligned}$$

For the first term on the far RHS we have that

$$\begin{aligned}
\sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = \deg R - 1 \\ \deg CD = \deg R - 1 \\ (ABCD,R)=1 \\ AC = BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} \leq & \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq \deg R - 1}} \frac{2^{\omega(N)}}{|N|} \Bigg( \sum_{\substack{G \in \mathcal{M} \\ \deg G = \frac{\deg R - \deg N - 1}{2}}} \frac{1}{|G|} \Bigg)^2 \\
= & \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq \deg R - 1}} \frac{2^{\omega(N)}}{|N|} \ll \big(\deg R\big)^2.
\end{aligned}$$

For the second term we have that

$$\sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = \deg R - 1 \\ \deg CD = \deg R - 1 \\ (ABCD,R)=1 \\ AC \equiv BD (\bmod R) \\ AC \neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} = \frac{q}{|R|} \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = \deg R - 1 \\ \deg CD = \deg R - 1 \\ (ABCD,R)=1 \\ AC \equiv BD (\bmod R) \\ AC \neq BD}} 1 \ll \frac{|R|}{\Phi(R)} \big(\deg R\big)^3,$$

where we have used Lemma 4.3.1. (Strictly speaking, with regards to our application of Lemma 4.3.1, we have $2(\deg R - 1) > \frac{19}{10} \deg R$ when $\deg R > 20$. When $\deg R \leq 20$ we can apply Lemma 4.3.1 for the case $z_1 + z_2 \leq \frac{19}{20} \deg R$ with $\epsilon = \frac{1}{20}$, and we still get the desired result, including the cancellation of the $q$ factor). So,

$$\sum_{\chi \bmod R} |c_o(\chi)|^2 \ll |R| \big( \deg R \big)^3 \ll \phi^*(R) \prod_{P|R} \left( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \right) (\deg R)^3 (\log \deg R)^6 .$$

Similarly, by using Lemma 4.3.2 for the even case, we can show, for $a = 0, 1, 2$, that

$$\begin{aligned} &\sum_{\chi \bmod R} \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = \deg R - a \\ \deg CD = \deg R - a}} \frac{\chi(AC)\overline{\chi}(BD)}{|ABCD|^{\frac{1}{2}}} \\ \ll &\phi^*(R) \prod_{P|R} \left( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \right) (\deg R)^3 (\log \deg R)^6 \end{aligned}$$

and

$$\begin{aligned} &\sum_{\substack{\chi \bmod R \\ \chi \text{ even}}} \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = \deg R - a \\ \deg CD = \deg R - a}} \frac{\chi(AC)\overline{\chi}(BD)}{|ABCD|^{\frac{1}{2}}} \\ \ll &\phi^*(R) \prod_{P|R} \left( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \right) (\deg R)^3 (\log \deg R)^6 . \end{aligned}$$

Hence, by using the Cauchy-Schwarz inequality, we can deduce that

$$\sideset{}{^*}\sum_{\chi \bmod R} |c(\chi)|^2 \ll \phi^*(R) \prod_{P|R} \left( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \right) (\deg R)^3 (\log \deg R)^6 .$$

**<u>STEP 4:</u>** From steps 1 to 3, and the use of the Cauchy-Schwarz inequality (as described at the start of the proof), the result follows. $\square$

# Chapter 5

# The First, Second, and Fourth Moments of Derivatives of Dirichlet $L$-functions with Prime Modulus

In this chapter we prove results on moments of derivatives of Dirichlet $L$-functions at the central value, where we average over characters of a prime modulus $Q$. For a discussion of the results and their relation to the work of others, we refer the reader to Section 2.3.

## 5.1 The First and Second Moments of Derivatives

In this section we prove Theorems 2.3.1 and 2.3.2. First we require a lemma.

**Lemma 5.1.1.** *For all positive integers $k$ we have that*

$$\sum_{n=0}^{\deg Q-1} n^k q^{\frac{n}{2}} = \frac{1}{q^{\frac{1}{2}}-1}(\deg Q)^k |Q|^{\frac{1}{2}} + O_k\Big((\deg Q)^{k-1}|Q|^{\frac{1}{2}}\Big).$$

*Proof.* We have that

$$\begin{aligned}
&\sum_{n=0}^{\deg Q-1} n^k q^{\frac{n}{2}} \\
=&\frac{1}{q^{\frac{1}{2}}-1}\sum_{n=0}^{\deg Q-1}\Big(n^k q^{\frac{n+1}{2}} - n^k q^{\frac{n}{2}}\Big) \\
=&\frac{1}{q^{\frac{1}{2}}-1}\sum_{n=0}^{\deg Q-1}\Big((n+1)^k q^{\frac{n+1}{2}} - n^k q^{\frac{n}{2}}\Big) - \frac{1}{q^{\frac{1}{2}}-1}\sum_{n=0}^{\deg Q-1}\Big((n+1)^k q^{\frac{n+1}{2}} - n^k q^{\frac{n+1}{2}}\Big) \\
=&\frac{1}{q^{\frac{1}{2}}-1}(\deg Q)^k|Q|^{\frac{1}{2}} + O\Bigg(\sum_{i=0}^{k-1}\binom{k}{i}(\deg Q)^i\sum_{n=0}^{\deg Q-1} q^{\frac{n+1}{2}}\Bigg) \\
=&\frac{1}{q^{\frac{1}{2}}-1}(\deg Q)^k|Q|^{\frac{1}{2}} + O_k\Big((\deg Q)^{k-1}|Q|^{\frac{1}{2}}\Big).
\end{aligned}$$

□

We now prove Theorem 2.3.1, which we restate for ease of reference.

**Theorem.** *For all positive integers $k$, we have that*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} L^{(k)}\Big(\frac{1}{2}, \chi\Big) = \frac{-(-\log q)^k}{q^{\frac{1}{2}} - 1} \frac{(\deg Q)^k}{|Q|^{\frac{1}{2}}} + O_k\bigg( (\log q)^k \frac{(\deg Q)^{k-1}}{|Q|^{\frac{1}{2}}} \bigg).$$

*Proof of Theorem 2.3.1.* We can easily see that

$$L^{(k)}(s, \chi) = (-\log q)^k \sum_{n=1}^{\deg Q - 1} n^k q^{-ns} \sum_{\substack{A \in \mathcal{M} \\ \deg A = n}} \chi(A),$$

from which we deduce that

$$\begin{aligned} \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} L^{(k)}\Big(\frac{1}{2}, \chi\Big) =& \frac{(-\log q)^k}{\phi(Q)} \sum_{n=1}^{\deg Q - 1} n^k q^{-\frac{n}{2}} \sum_{\substack{A \in \mathcal{M} \\ \deg A = n}} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \chi(A) \\ =& - \frac{(-\log q)^k}{\phi(Q)} \sum_{n=1}^{\deg Q - 1} n^k q^{-\frac{n}{2}} \sum_{\substack{A \in \mathcal{M} \\ \deg A = n}} 1 \\ =& - \frac{(-\log q)^k}{q^{\frac{1}{2}} - 1} \frac{(\deg Q)^k}{|Q|^{\frac{1}{2}}} + O_k\bigg( (\log q)^k \frac{(\deg Q)^{k-1}}{|Q|^{\frac{1}{2}}} \bigg). \end{aligned}$$

For the second equality we used Lemma 1.4.4, and for the last equality we used Lemma 5.1.1 and the fact that $\phi(Q) = |Q| - 1$ (since $Q$ is prime). □

We now prove Theorem 2.3.2, which we restate for ease of reference.

**Theorem.** *For all positive integers $k$ we have that*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \bigg| L^{(k)}\Big(\frac{1}{2}, \chi\Big) \bigg|^2 = \frac{(\log q)^{2k}}{2k+1} (\deg Q)^{2k+1} + O\Big( (\log q)^{2k} (\deg Q)^{2k} \Big).$$

*Proof of Theorem 2.3.2.* For positive integers $k$ we have that

$$\begin{aligned} L^{(k)}\Big(\frac{1}{2}, \chi\Big) =& (-\log q)^k \sum_{n=1}^{\deg Q - 1} n^k q^{-\frac{n}{2}} \sum_{\substack{A \in \mathcal{M} \\ \deg A = n}} \chi(A) \\ =& (-\log q)^k \sum_{\substack{A \in \mathcal{M} \\ \deg A < \deg Q}} \frac{(\log_q |A|)^k \chi(A)}{|A|^{\frac{1}{2}}}, \end{aligned}$$

and so

$$\begin{aligned} &\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \bigg| L^{(k)}\Big(\frac{1}{2}, \chi\Big) \bigg|^2 \\ =& \frac{(\log q)^{2k}}{\phi(Q)} \sum_{\substack{A, B \in \mathcal{M} \\ \deg A, \deg B < \deg Q}} \frac{(\log_q |A| \log_q |B|)^k}{|AB|^{\frac{1}{2}}} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \chi(A) \overline{\chi}(B). \end{aligned}$$

We now apply Lemma 1.4.4 to obtain that

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| L^{(k)}\left(\frac{1}{2}, \chi\right)\right|^2 \\
=&(\log q)^{2k} \sum_{\substack{A \in \mathcal{M} \\ \deg A < \deg Q}} \frac{(\log_q |A|)^{2k}}{|A|} - \frac{(\log q)^{2k}}{\phi(Q)} \sum_{\substack{A,B \in \mathcal{M} \\ \deg A, \deg B < \deg Q}} \frac{(\log_q |A| \log_q |B|)^k}{|AB|^{\frac{1}{2}}}.
\end{aligned}$$

For the first term on the RHS we have that

$$\sum_{\substack{A \in \mathcal{M} \\ \deg A < \deg Q}} \frac{(\log_q |A|)^{2k}}{|A|} = \sum_{n=0}^{\deg Q - 1} n^{2k} = \frac{1}{2k+1} (\deg Q)^{2k+1} + O\Big( (\deg Q)^{2k} \Big),$$

where the final equality uses Faulhaber's formula. For the second term we have that

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{A,B \in \mathcal{M} \\ \deg A, \deg B < \deg Q}} \frac{(\log_q |A| \log_q |B|)^k}{|AB|^{\frac{1}{2}}} = \frac{1}{\phi(Q)} \left( \sum_{n=0}^{\deg Q - 1} n^k q^{\frac{n}{2}} \right)^2 \\
\leq& \frac{1}{\phi(Q)} \left( (\deg Q)^k \sum_{n=0}^{\deg Q - 1} q^{\frac{n}{2}} \right)^2 \ll \frac{1}{\phi(Q)} \Big( (\deg Q)^k |Q|^{\frac{1}{2}} \Big)^2 \ll (\deg Q)^{2k}.
\end{aligned}$$

The result now follows. □

## 5.2 Fourth Moments of Derivatives: Expressing as Manageable Summations

In order to prove Theorem 2.3.3, we will need to express our $L$-functions as shortened sums by using the functional equation. We do this in this section. We begin with the odd character case.

**Lemma 5.2.1.** *Let $\chi$ be an odd character of modulus $Q \in \mathcal{P}$, and let $k$ be a non-negative integer. Then,*

$$\begin{aligned}
&(\log q)^{-2k} \left| L^{(k)}\left(\frac{1}{2}, \chi\right)\right|^2 \\
=& \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{\Big( f_k\big(\deg A, \deg B, \deg Q\big) + g_{O,k}\big(\deg A, \deg B, \deg Q\big)\Big) \chi(A) \overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \\
&+ \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = \deg Q - 1}} \frac{h_{O,k}\big(\deg A, \deg B, \deg Q\big) \chi(A) \overline{\chi}(B)}{|AB|^{\frac{1}{2}}},
\end{aligned}$$

*where*

$$\begin{aligned}
f_k\big(\deg A, \deg B, \deg Q\big) =&(\deg A)^k (\deg B)^k + (\deg Q - \deg A)^k (\deg Q - \deg B)^k; \\
g_{O,k}\big(\deg A, \deg B, \deg Q\big) =&(\deg Q - \deg A - 1)^k (\deg Q - \deg B - 1)^k \\
&- (\deg Q - \deg A)^k (\deg Q - \deg B)^k; \\
h_{O,k}\big(\deg A, \deg B, \deg Q\big) =& - (\deg Q - \deg A - 1)^k (\deg Q - \deg B - 1)^k.
\end{aligned}$$

**Remark 5.2.2.** *The "O" in the subscript is to signify that these polynomials apply to the odd character case. It is important to note that $g_{O,k}\big(\deg A, \deg B, \deg Q\big)$ has degree $2k-1$, whereas $f_k\big(\deg A, \deg B, \deg Q\big)$ has degree $2k$ (hence, the later ultimately contributes the higher order term); and that all three polynomials are independent of $q$.*

*Proof.* The functional equation (1.15) gives us that

$$\begin{aligned}\sum_{n=0}^{\deg Q-1} L_n(\chi)(q^{-s})^n =& W(\chi)q^{\frac{\deg Q-1}{2}}(q^{-s})^{\deg Q-1}\sum_{n=0}^{\deg Q-1} L_n(\overline{\chi})(q^{s-1})^n\\ =& W(\chi)q^{-\frac{\deg Q-1}{2}}\sum_{n=0}^{\deg Q-1} L_n(\overline{\chi})(q^{1-s})^{\deg Q-n-1}.\end{aligned}$$

Taking the $k^{\text{th}}$ derivative of both sides gives

$$\begin{aligned}&(-\log q)^k\sum_{n=0}^{\deg Q-1} n^k L_n(\chi)(q^{-s})^n\\ =&(-\log q)^k W(\chi)q^{-\frac{\deg Q-1}{2}}\sum_{n=0}^{\deg Q-1}(\deg Q-n-1)^k L_n(\overline{\chi})(q^{1-s})^{\deg Q-n-1}.\end{aligned}$$

Let us now take the squared modulus of both sides. In order to make our calculations slightly easier, we restrict our attention to the case where $s\in\mathbb{R}$. We obtain

$$\begin{aligned}&(\log q)^{2k}\sum_{n=0}^{2\deg Q-2}\Bigg(\sum_{\substack{i+j=n\\0\leq i,j<\deg Q}} i^k j^k L_i(\chi)L_j(\overline{\chi})\Bigg)(q^{-s})^n\\ =&(\log q)^{2k}q^{-\deg Q+1}\\ &\cdot\sum_{n=0}^{2\deg Q-2}\Bigg(\sum_{\substack{i+j=n\\0\leq i,j<\deg Q}}(\deg Q-i-1)^k(\deg Q-j-1)^k L_i(\chi)L_j(\overline{\chi})\Bigg)(q^{1-s})^{2\deg Q-n-2}.\end{aligned}$$

Both sides of the above are equal to $\big|L^{(k)}(s,\chi)\big|^2$. By the linear independence of powers of $q^{-s}$, we have that $\big|L^{(k)}(s,\chi)\big|^2$ is the sum of the terms corresponding to $n=0,1,\ldots,\deg Q-1$ from the LHS and $n=0,1,\ldots,\deg Q-2$ from the RHS. This gives

$$\begin{aligned}&(\log q)^{-2k}\big|L^{(k)}(s,\chi)\big|^2\\ =&\sum_{n=0}^{\deg Q-1}\Bigg(\sum_{\substack{i+j=n\\0\leq i,j<\deg Q}} i^k j^k L_i(\chi)L_j(\overline{\chi})\Bigg)(q^{-s})^n\\ &+q^{-\deg Q+1}\sum_{n=0}^{\deg Q-2}\Bigg(\sum_{\substack{i+j=n\\0\leq i,j<\deg Q}}(\deg Q-i-1)^k(\deg Q-j-1)^k L_i(\chi)L_j(\overline{\chi})\Bigg)(q^{1-s})^{2\deg Q-n-2}.\end{aligned}$$

We now substitute $s=\frac{1}{2}$ and simplify the right-hand-side to obtain

$$(\log q)^{-2k}\bigg|L^{(k)}\Big(\frac{1}{2},\chi\Big)\bigg|^2$$

$$
\begin{aligned}
&= \sum_{n=0}^{\deg Q-1} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j<\deg Q}} i^k j^k L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-\frac{n}{2}} \\
&\quad + \sum_{n=0}^{\deg Q-2} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j<\deg Q}} (\deg Q - i - 1)^k (\deg Q - j - 1)^k L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-\frac{n}{2}} \\
&= \sum_{n=0}^{\deg Q-1} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j<\deg Q}} \Big[ i^k j^k + (\deg Q - i - 1)^k (\deg Q - j - 1)^k \Big] L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-\frac{n}{2}} \\
&\quad - \sum_{\substack{i+j=\deg Q-1 \\ 0\leq i,j<\deg Q}} (\deg Q - i - 1)^k (\deg Q - j - 1)^k L_i(\chi) L_j(\overline{\chi}) q^{-\frac{\deg Q-1}{2}}.
\end{aligned}
$$

Finally, we substitute back $L_n(\chi) = \sum_{\substack{A\in\mathcal{M} \\ \deg A=n}} \chi(A)$ to obtain the required result. □

As the functional equation for even characters is more complicated, we must first prove a lemma before being able to express the associated $L$-functions as shortened sums.

**Definition 5.2.3.** *For all $s \in \mathbb{C}$ and all non-trivial even characters, $\chi$, of prime modulus we define*

$$
\hat{L}(s,\chi) := (q^{1-s} - 1) L(s,\chi). \tag{5.1}
$$

**Lemma 5.2.4.** *For all non-trivial even characters, $\chi$, of prime modulus and all non-negative integers $k$ we have that*

$$
\begin{aligned}
L^{(k)}\Big(\frac{1}{2},\chi\Big) =& \frac{1}{q^{\frac{1}{2}}-1} \hat{L}^{(k)}\Big(\frac{1}{2},\chi\Big) + \frac{1}{q^{\frac{1}{2}}-1} \sum_{i=0}^{k-1} (-\log q)^{k-i} p_{k,i}\Big(\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}\Big) \hat{L}^{(i)}\Big(\frac{1}{2},\chi\Big) \\
=& \frac{1}{q^{\frac{1}{2}}-1} \sum_{i=0}^{k} (-\log q)^{k-i} p_{k,i}\Big(\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}\Big) \hat{L}^{(i)}\Big(\frac{1}{2},\chi\Big),
\end{aligned}
$$

*where, for non-negative integers $k, i$ satisfying $i \leq k$, we define the polynomials $p_{k,i}$ by*

$$
\begin{aligned}
p_{k,k}\Big(\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}\Big) =& 1; \\
p_{k,i}\Big(\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}\Big) =& -\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1} \sum_{j=i}^{k-1} \binom{k}{j} p_{j,i}\Big(\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}\Big) \qquad \text{for } i < k.
\end{aligned}
$$

**Remark 5.2.5.** *Because $1 \leq \frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1} < 4$ for all prime powers $q$, we can see that the polynomials $p_{k,i}\Big(\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}\Big)$ can be bounded independently of $q$ (but dependent on $k$ and $i$ of course). The factors $(-\log q)^{k-i}$ are of course still dependent on $q$, as well as $k$ and $i$. These two points are important when we later determine how the lower order terms in our main results are dependent on $q$.*

*Proof of Lemma 5.2.4.* We prove this by strong induction on $k$. The base case, $k = 0$, is obvious by Definition 5.2.3. Now, suppose the claim holds for $j = 0, 1, \ldots, k$. Differentiating, $k+1$ number of times, the equation (5.1) gives

$$\hat{L}^{(k+1)}(s,\chi) = (q^{1-s} - 1)L^{(k+1)}(s,\chi) + q^{1-s}\sum_{j=0}^{k}\binom{k+1}{j}(-\log q)^{k+1-j}L^{(j)}(s,\chi).$$

Substituting $s = \frac{1}{2}$ and rearranging gives

$$L^{(k+1)}\Big(\frac{1}{2},\chi\Big) = \frac{1}{q^{\frac{1}{2}} - 1}\hat{L}^{(k+1)}\Big(\frac{1}{2},\chi\Big) - \frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}} - 1}\sum_{j=0}^{k}\binom{k+1}{j}(-\log q)^{k+1-j}L^{(j)}\Big(\frac{1}{2},\chi\Big).$$

We now apply the inductive hypothesis to obtain

$$\begin{aligned}
&L^{(k+1)}\Big(\frac{1}{2},\chi\Big)\\
=&\frac{1}{q^{\frac{1}{2}} - 1}\hat{L}^{(k+1)}\Big(\frac{1}{2},\chi\Big)\\
&- \frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}} - 1}\sum_{j=0}^{k}\binom{k+1}{j}(-\log q)^{k+1-j}\frac{1}{q^{\frac{1}{2}} - 1}\sum_{i=0}^{j}(-\log q)^{j-i}p_{j,i}\Big(\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}} - 1}\Big)\hat{L}^{(i)}\Big(\frac{1}{2},\chi\Big)\\
=&\frac{1}{q^{\frac{1}{2}} - 1}\hat{L}^{(k+1)}\Big(\frac{1}{2},\chi\Big)\\
&+ \frac{1}{q^{\frac{1}{2}} - 1}\sum_{i=0}^{k}(-\log q)^{k+1-i}\Big( - \frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}} - 1}\sum_{j=i}^{k}\binom{k+1}{j}p_{j,i}\Big(\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}} - 1}\Big)\Big)\hat{L}^{(i)}\Big(\frac{1}{2},\chi\Big).
\end{aligned}$$

The result follows by the definition of the polynomials $p_{k,i}$ . ☐

**Lemma 5.2.6.** *For all non-negative integers $k$, and all non-trivial even characters $\chi$ of modulus $Q \in \mathcal{P}$, we have that*

$$\begin{aligned}
&\frac{1}{(\log q)^{2k}(q^{\frac{1}{2}} - 1)^2}\Big|\hat{L}^{(k)}\Big(\frac{1}{2},\chi\Big)\Big|^2\\
=&\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB<\deg Q}}\frac{\Big(f_k\big(\deg A,\deg B,\deg Q\big) + g_{E,k}\big(\deg A,\deg B,\deg Q\big)\Big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}\\
&+\sum_{\deg Q-2\leq n\leq\deg Q}\ \sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=n}}\frac{h_{E,k,n}\big(\deg A,\deg B,\deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}},
\end{aligned}$$

*where*

$$f_k\big(\deg A,\deg B,\deg Q\big) = (\deg A)^k(\deg B)^k + (\deg Q - \deg A)^k(\deg Q - \deg B)^k,$$

*and $g_{E,k}\big(\deg A,\deg B,\deg Q\big)$ , $h_{E,k,n}\big(\deg A,\deg B,\deg Q\big)$ are polynomials of degrees $2k-1$ and $2k$, respectively, that are symmetric in $\deg A, \deg B$, and whose coefficients can be bounded independently of $q$.*

*Proof.* Let us define $L_{-1}(\chi) := 0$, and recall from (1.13) that $L_{\deg Q}(\chi) = 0$. We can now define, for $n = 0, 1, \ldots, \deg Q$,

$$M_n(\chi) := L_n(\chi) - qL_{n-1}(\chi).$$

Then, the functional equation for even characters, (1.14), can be written as

$$-\sum_{n=0}^{\deg Q} M_n(\chi)(q^{-s})^n = W(\chi)q^{-\frac{\deg Q}{2}} \sum_{n=0}^{\deg Q} M_n(\overline{\chi})(q^{1-s})^{\deg Q-n}. \tag{5.2}$$

Note that both sides of (5.2) are equal to $\hat{L}(s,\chi)$. We proceed similarly to the odd character case. First we differentiate, $k$ number of times, the equation (5.2); and then we take the modulus squared of both sides. . In order to make our calculations slightly easier, we restrict our attention to the case where $s \in \mathbb{R}$. This gives

$$\begin{aligned}&(\log q)^{2k} \sum_{n=0}^{2\deg Q} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j\leq \deg Q}} i^k j^k M_i(\chi)M_j(\overline{\chi}) \Bigg) (q^{-s})^n \\ =&(\log q)^{2k} q^{-\deg Q} \sum_{n=0}^{2\deg Q} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j\leq \deg Q}} (\deg Q - i)^k (\deg Q - j)^k M_i(\chi)M_j(\overline{\chi}) \Bigg) (q^{1-s})^{2\deg Q-n}.\end{aligned}$$

Now we take the terms corresponding to $n = 0, 1, \dots, \deg Q$ from the LHS and $n = 0, 1, \dots, \deg Q - 1$ from the RHS to obtain

$$\begin{aligned}&\hat{L}^{(k)}(s,\chi) \\ =&(\log q)^{2k} \sum_{n=0}^{\deg Q} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j\leq \deg Q}} i^k j^k M_i(\chi)M_j(\overline{\chi}) \Bigg) (q^{-s})^n \\ +&(\log q)^{2k} q^{-\deg Q} \sum_{n=0}^{\deg Q-1} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j\leq \deg Q}} (\deg Q - i)^k (\deg Q - j)^k M_i(\chi)M_j(\overline{\chi}) \Bigg) (q^{1-s})^{2\deg Q-n}.\end{aligned}$$

Substituting $s = \frac{1}{2}$ and simplifying the RHS gives

$$\begin{aligned}&\hat{L}^{(k)}\left(\frac{1}{2},\chi\right) \\ =&(\log q)^{2k} \sum_{n=0}^{\deg Q-1} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j\leq \deg Q}} \Big( i^k j^k + (\deg Q - i)^k (\deg Q - j)^k \Big) M_i(\chi)M_j(\overline{\chi}) \Bigg) q^{-\frac{n}{2}} \\ &+ (\log q)^{2k} \sum_{\substack{i+j=\deg Q \\ 0\leq i,j\leq \deg Q}} i^k j^k M_i(\chi)M_j(\overline{\chi}) q^{-\frac{\deg Q}{2}}.\end{aligned} \tag{5.3}$$

Now, we want factors such as $L_n(\chi)$ in our expression, as opposed to factors like $M_n(\chi)$. To this end, suppose $p(i,j)$ is a finite polynomial. Then,

$$\begin{aligned}&\sum_{n=0}^{\deg Q-1} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j\leq \deg Q}} p(i,j) M_i(\chi)M_j(\overline{\chi}) \Bigg) q^{-\frac{n}{2}} \\ =& \sum_{n=0}^{\deg Q-1} \Bigg( \sum_{\substack{i+j=n \\ 0\leq i,j\leq \deg Q}} p(i,j) \Big( L_i(\chi) - qL_{i-1}(\chi) \Big) \Big( L_j(\overline{\chi}) - qL_{j-1}(\overline{\chi}) \Big) \Bigg) q^{-\frac{n}{2}}\end{aligned}$$

$$= \sum_{n=0}^{\deg Q-1} \Bigg( \sum_{\substack{i+j=n \\ 0\le i,j\le \deg Q}} p(i,j) L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-\frac{n}{2}}$$
$$+ \sum_{n=0}^{\deg Q-3} \Bigg( \sum_{\substack{i+j=n \\ 0\le i,j\le \deg Q}} p(i+1,j+1) L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-\frac{n-2}{2}}$$
$$- \sum_{n=0}^{\deg Q-2} \Bigg( \sum_{\substack{i+j=n \\ 0\le i,j\le \deg Q}} p(i,j+1) L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-\frac{n-1}{2}}$$
$$- \sum_{n=0}^{\deg Q-2} \Bigg( \sum_{\substack{i+j=n \\ 0\le i,j\le \deg Q}} p(i+1,j) L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-\frac{n-1}{2}}.$$

Grouping the terms together gives

$$\sum_{n=0}^{\deg Q-1} \Bigg( \sum_{\substack{i+j=n \\ 0\le i,j\le \deg Q}} p(i,j) M_i(\chi) M_j(\overline{\chi}) \Bigg) q^{-\frac{n}{2}}$$
$$= \sum_{n=0}^{\deg Q-1} \sum_{\substack{i+j=n \\ 0\le i,j\le \deg Q}} \Big[ qp(i+1,j+1) - q^{\frac{1}{2}} p(i,j+1) - q^{\frac{1}{2}} p(i+1,j) + p(i,j) \Big] L_i(\chi) L_j(\overline{\chi}) q^{-\frac{n}{2}}$$
$$- \sum_{\substack{i+j=\deg Q-2 \\ 0\le i,j\le \deg Q}} qp(i+1,j+1) L_i(\chi) L_j(\overline{\chi}) q^{-\frac{\deg Q-2}{2}}$$
$$+ \sum_{\substack{i+j=\deg Q-1 \\ 0\le i,j\le \deg Q}} \Big( q^{\frac{1}{2}} p(i,j+1) + q^{\frac{1}{2}} p(i+1,j) - qp(i+1,j+1) \Big) L_i(\chi) L_j(\overline{\chi}) q^{\frac{\deg Q-1}{2}}.$$

In the case where

$$p(i,j) = i^k j^k + (\deg Q - i)^k (\deg Q - j)^k$$

we have that

$$\begin{aligned} & qp(i+1,j+1) - q^{\frac{1}{2}} p(i,j+1) - q^{\frac{1}{2}} p(i+1,j) + p(i,j) \\ =& (q^{\frac{1}{2}} - 1)^2 \Big( f_k(i,j,\deg Q) + g_{E,k}(i,j,\deg Q) \Big), \end{aligned}$$

where $g_{E,k}(i,j,\deg Q)$ is a polynomial of degree $2k-1$ whose coefficients can be bounded independently of $q$.

We can now see that (5.3) becomes

$$\frac{1}{(\log q)^{2k} (q^{\frac{1}{2}} - 1)^2} \hat{L}^{(k)}\Big(\frac{1}{2}, \chi\Big)$$
$$= \sum_{n=0}^{\deg Q-1} \Bigg( \sum_{\substack{i+j=n \\ 0\le i,j\le \deg Q}} \Big( f_k(i,j,\deg Q) + g_{E,k}(i,j,\deg Q) \Big) L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-\frac{n}{2}}$$
$$+ \sum_{n=\deg Q-2}^{\deg Q} \Bigg( \sum_{\substack{i+j=n \\ 0\le i,j\le \deg Q}} h_{E,k,n}(i,j,\deg Q) L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-\frac{n}{2}},$$

where $h_{E,k,n}(i,j,\deg Q)$ is a polynomial of degree $k$ whose coefficients can be bounded independently of $q$. Finally, we substitute back $L_n(\chi) = \sum_{\substack{A\in\mathcal{M} \\ \deg A=n}} \chi(A)$ to obtain the required result. □

## 5.3 Fourth Moments of Derivatives: Handling the Summations

In this section we demonstrate some techniques for handling the summations that we obtained in Section 5.2.

**Lemma 5.3.1.** *Let $Q \in \mathcal{P}$, and let $p_1\big(\deg A, \deg B, \deg Q\big)$ and $p_2\big(\deg A, \deg B, \deg Q\big)$ be finite polynomials (which, for presentational purposes, we will write as $p_1$ and $p_2$, except when we need to use variables other than* $\deg A$, $\deg B$, $\deg Q$*). Then,*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi\neq\chi_0}} \Bigg( \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q}} \frac{p_1\, \chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \Bigg( \sum_{\substack{C,D\in\mathcal{M} \\ \deg CD<\deg Q}} \frac{p_2\, \chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \Bigg)$$
$$= \sum_{\substack{A,B,C,D\in\mathcal{M} \\ \deg AB<\deg Q \\ \deg CD<\deg Q \\ AC=BD}} \frac{p_1 p_2}{|ABCD|^{\frac{1}{2}}} + \sum_{\substack{A,B,C,D\in\mathcal{M} \\ \deg AB<\deg Q \\ \deg CD<\deg Q \\ AC\equiv BD (\bmod Q) \\ AC\neq BD}} \frac{p_1 p_2}{|ABCD|^{\frac{1}{2}}} - \frac{1}{\phi(Q)} \sum_{\substack{A,B,C,D\in\mathcal{M} \\ \deg AB<\deg Q \\ \deg CD<\deg Q}} \frac{p_1 p_2}{|ABCD|^{\frac{1}{2}}}.$$

*Proof.* This follows by expanding the brackets and applying Lemma 1.4.4. □

**Lemma 5.3.2.** *Let $p\big(\deg A, \deg B, \deg C, \deg D, \deg Q\big)$ be a finite homogeneous polynomial of degree $d$. Then,*

$$\sum_{\substack{A,B,C,D\in\mathcal{M} \\ \deg AB<\deg Q \\ \deg CD<\deg Q \\ AC=BD}} \frac{p\big(\deg A, \deg B, \deg C, \deg D, \deg Q\big)}{|ABCD|^{\frac{1}{2}}}$$
$$=(1-q^{-1})(\deg Q)^{d+4} \int_{\substack{a_1,a_2,a_3,a_4\geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} p\big(a_1+a_3, a_1+a_4, a_2+a_4, a_2+a_3, 1\big) \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4$$
$$+ O_p\big((\deg Q)^{d+3}\big).$$

**Remark 5.3.3.** *The subscript $p$ in $O_p$ should be interpreted as saying that the implied constant is dependent on the coefficients of $p$.*

*Proof.* Consider the function $f$ defined by

$$f(t_1,t_2,t_3,t_4) = \sum_{\substack{A,B,C,D\in\mathcal{M} \\ AC=BD}} \frac{{t_1}^{\deg A} {t_2}^{\deg B} {t_3}^{\deg C} {t_4}^{\deg D}}{|ABCD|^{\frac{1}{2}}} \tag{5.4}$$

with domain $|t_i| < \frac{1}{2} q^{-\frac{1}{2}}$. Note that $AC = BD$ if and only if there exist $G, H, R, S \in$

$\mathcal{M}$ satisfying $(R,S)=1$ and $A=GR$, $B=GS$, $C=HS$, $D=HR$. Hence,

$$\begin{aligned}
&f(t_1,t_2,t_3,t_4)\\
&=\sum_{\substack{G,H,R,S\in\mathcal{M}\\(R,S)=1}}\frac{{t_1}^{\deg GR}{t_2}^{\deg GS}{t_3}^{\deg HS}{t_4}^{\deg HR}}{|GHRS|}\\
&=\sum_{G,H,R,S\in\mathcal{M}}\frac{{t_1}^{\deg GR}{t_2}^{\deg GS}{t_3}^{\deg HS}{t_4}^{\deg HR}}{|GHRS|}\\
&\qquad\qquad -q^{-1}\sum_{G,H,R,S\in\mathcal{M}}\frac{{t_1}^{\deg GR+1}{t_2}^{\deg GS+1}{t_3}^{\deg HS+1}{t_4}^{\deg HR+1}}{|GHRS|}\\
&=\sum_{a_1,a_2,a_3,a_4\geq 0}{t_1}^{a_1+a_3}{t_2}^{a_1+a_4}{t_3}^{a_2+a_4}{t_4}^{a_2+a_3}\\
&\qquad\qquad -q^{-1}\sum_{a_1,a_2,a_3,a_4\geq 0}{t_1}^{a_1+a_3+1}{t_2}^{a_1+a_4+1}{t_3}^{a_2+a_4+1}{t_4}^{a_2+a_3+1},
\end{aligned}\tag{5.5}$$

where the second equality follows by Lemma A.1.1.

Now, for $i=1,2,3,4$ we define the operator $\Omega_i := t_i\frac{\mathrm{d}}{\mathrm{d}t_i}$. For non-negative integers $k_1,k_2,k_3,k_4$ we can apply the operator ${\Omega_1}^{k_1}{\Omega_2}^{k_2}{\Omega_3}^{k_3}{\Omega_4}^{k_4}$ to (5.4) and (5.5) to obtain

$$\begin{aligned}
&\sum_{\substack{A,B,C,D\in\mathcal{M}\\AC=BD}}\frac{(\deg A)^{k_1}(\deg B)^{k_2}(\deg C)^{k_3}(\deg D)^{k_4}}{|ABCD|^{\frac{1}{2}}}{t_1}^{\deg A}{t_2}^{\deg B}{t_3}^{\deg C}{t_4}^{\deg D}\\
&=\sum_{a_1,a_2,a_3,a_4\geq 0}(a_1+a_3)^{k_1}(a_1+a_4)^{k_2}(a_2+a_4)^{k_3}(a_2+a_3)^{k_4}{t_1}^{a_1+a_3}{t_2}^{a_1+a_4}{t_3}^{a_2+a_4}{t_4}^{a_2+a_3}\\
&\quad -q^{-1}\sum_{a_1,a_2,a_3,a_4\geq 0}(a_1+a_3+1)^{k_1}(a_1+a_4+1)^{k_2}(a_2+a_4+1)^{k_3}(a_2+a_3+1)^{k_4}\\
&\qquad\qquad\qquad\cdot {t_1}^{a_1+a_3+1}{t_2}^{a_1+a_4+1}{t_3}^{a_2+a_4+1}{t_4}^{a_2+a_3+1}\\
&=(1-q^{-1})\sum_{a_1,a_2,a_3,a_4\geq 0}(a_1+a_3)^{k_1}(a_1+a_4)^{k_2}(a_2+a_4)^{k_3}(a_2+a_3)^{k_4}\\
&\qquad\qquad\qquad\cdot {t_1}^{a_1+a_3}{t_2}^{a_1+a_4}{t_3}^{a_2+a_4}{t_4}^{a_2+a_3}\\
&\quad +q^{-1}\sum_{\substack{(a_1,a_2)=(0,0),(0,1),(1,0)\\a_3,a_4\geq 0}}(a_1+a_3)^{k_1}(a_1+a_4)^{k_2}(a_2+a_4)^{k_3}(a_2+a_3)^{k_4}\\
&\qquad\qquad\qquad\cdot {t_1}^{a_1+a_3}{t_2}^{a_1+a_4}{t_3}^{a_2+a_4}{t_4}^{a_2+a_3}.
\end{aligned}$$

From this we can deduce that if $p\big(\deg A,\deg B,\deg C,\deg D,\deg Q\big)$ is a finite homogeneous polynomial of degree $d$, then

$$\begin{aligned}
&\sum_{\substack{A,B,C,D\in\mathcal{M}\\AC=BD}}\frac{p\big(\deg A,\deg B,\deg C,\deg D,\deg Q\big)}{|ABCD|^{\frac{1}{2}}}{t_1}^{\deg A}{t_2}^{\deg B}{t_3}^{\deg C}{t_4}^{\deg D}\\
&=(1-q^{-1})\sum_{a_1,a_2,a_3,a_4\geq 0}p\big(a_1+a_3,a_1+a_4,a_2+a_4,a_2+a_3,\deg Q\big)\\
&\qquad\qquad\qquad\cdot {t_1}^{a_1+a_3}{t_2}^{a_1+a_4}{t_3}^{a_2+a_4}{t_4}^{a_2+a_3}\\
&\quad +q^{-1}\sum_{\substack{(a_1,a_2)=(0,0),(0,1),(1,0)\\a_3,a_4\geq 0}}p\big(a_1+a_3,a_1+a_4,a_2+a_4,a_2+a_3,\deg Q\big)
\end{aligned}$$

$$\cdot \, {t_1}^{a_1+a_3}{t_2}^{a_1+a_4}{t_3}^{a_2+a_4}{t_4}^{a_2+a_3}.$$

Now, we can extract and sum the coefficients of ${t_1}^{i_1}{t_2}^{i_2}{t_3}^{i_3}{t_4}^{i_4}$ for which $i_1+i_2<\deg Q$ and $i_3+i_4<\deg Q$ to obtain

$$\begin{aligned}
&\sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB<\deg Q\\ \deg CD<\deg Q\\ AC=BD}} \frac{p\big(\deg A,\deg B,\deg C,\deg D,\deg Q\big)}{|ABCD|^{\frac{1}{2}}}\\
&=(1-q^{-1})\sum_{\substack{a_1,a_2,a_3,a_4\geq 0\\ 2a_1+a_3+a_4<\deg Q\\ 2a_2+a_3+a_4<\deg Q}} p\big(a_1+a_3,a_1+a_4,a_2+a_4,a_2+a_3,\deg Q\big)\\
&\quad+q^{-1}\sum_{\substack{(a_1,a_2)=(0,0),(0,1),(1,0)\\ a_3,a_4\geq 0\\ 2a_1+a_3+a_4<\deg Q\\ 2a_2+a_3+a_4<\deg Q}} p\big(a_1+a_3,a_1+a_4,a_2+a_4,a_2+a_3,\deg Q\big)\\
&=(1-q^{-1})\int_{\substack{a_1,a_2,a_3,a_4\geq 0\\ 2a_1+a_3+a_4<\deg Q\\ 2a_2+a_3+a_4<\deg Q}} p\big(a_1+a_3,a_1+a_4,a_2+a_4,a_2+a_3,\deg Q\big)\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4\\
&\quad+O_p\big((\deg Q)^{d+3}\big)+O_p\big((\deg Q)^{d+2}\big)\\
&=(1-q^{-1})(\deg Q)^{d+4}\int_{\substack{a_1,a_2,a_3,a_4\geq 0\\ 2a_1+a_3+a_4<1\\ 2a_2+a_3+a_4<1}} p\big(a_1+a_3,a_1+a_4,a_2+a_4,a_2+a_3,1\big)\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4\\
&\quad+O_p\big((\deg Q)^{d+3}\big).
\end{aligned}$$

□

**Lemma 5.3.4.** *Let $p\big(\deg A,\deg B,\deg C,\deg D,\deg Q\big)$ be a finite polynomial of degree $d$. Then,*

$$\sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB<\deg Q\\ \deg CD<\deg Q\\ AC\equiv BD(\mathrm{mod}\, Q)\\ AC\neq BD}} \frac{p\big(\deg A,\deg B,\deg C,\deg D,\deg Q\big)}{|ABCD|^{\frac{1}{2}}} \ll_p (\deg Q)^{d+3}.$$

*Proof.* Because $\deg AB,\deg CD<\deg Q$, we have that

$$p\big(\deg A,\deg B,\deg C,\deg D,\deg Q\big)\ll_p(\deg Q)^d.$$

Hence,

$$\begin{aligned}
&\sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB<\deg Q\\ \deg CD<\deg Q\\ AC\equiv BD(\mathrm{mod}\, Q)\\ AC\neq BD}} \frac{p\big(\deg A,\deg B,\deg C,\deg D,\deg Q\big)}{|ABCD|^{\frac{1}{2}}}\\
&\ll_p(\deg Q)^d \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB<\deg Q\\ \deg CD<\deg Q\\ AC\equiv BD(\mathrm{mod}\, Q)\\ AC\neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} = (\deg Q)^d \sum_{0\leq z_1,z_2<\deg Q} q^{-\frac{z_1+z_2}{2}} \sum_{\substack{A,B,C,D\in\mathcal{M}\\ \deg AB=z_1\\ \deg CD=z_2\\ AC\equiv BD(\mathrm{mod}\, Q)\\ AC\neq BD}} 1.
\end{aligned} \tag{5.6}$$

We now apply Lemma 4.3.1 with $\epsilon < \frac{1}{38}$ to obtain

$$\begin{aligned}
&\sum_{0 \leq z_1, z_2 < \deg Q} q^{-\frac{z_1+z_2}{2}} \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB = z_1 \\ \deg CD = z_2 \\ AC \equiv BD (\bmod Q) \\ AC \neq BD}} 1 \\
\ll &\frac{1}{|Q|} \sum_{\substack{0 \leq z_1, z_2 < \deg Q \\ z_1 + z_2 \leq \frac{19}{10} \deg Q}} \left(q^{\frac{1}{2}+\epsilon}\right)^{z_1+z_2} + \frac{1}{\phi(Q)} \sum_{\substack{0 \leq z_1, z_2 < \deg Q \\ z_1 + z_2 > \frac{19}{10} \deg Q}} q^{\frac{z_1+z_2}{2}} (z_1+z_2)^3 \\
\ll &\frac{|Q|}{\phi(Q)} (\deg Q)^3 \ll (\deg Q)^3.
\end{aligned}$$

The result follows by applying this to (5.6). □

**Lemma 5.3.5.** *Let $p\big(\deg A, \deg B, \deg C, \deg D, \deg Q\big)$ be a finite polynomial of degree $d$. Then,*

$$\frac{1}{\phi(Q)} \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB < \deg Q \\ \deg CD < \deg Q}} \frac{p\big(\deg A, \deg B, \deg C, \deg D, \deg Q\big)}{|ABCD|^{\frac{1}{2}}} \ll_p (\deg Q)^{d+2}.$$

*Proof.* Because $\deg AB, \deg CD < \deg Q$, we have that

$$p\big(\deg A, \deg B, \deg C, \deg D, \deg Q\big) \ll_p (\deg Q)^d.$$

Hence,

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{A,B,C,D \in \mathcal{M} \\ \deg AB < \deg Q \\ \deg CD < \deg Q}} \frac{p\big(\deg A, \deg B, \deg C, \deg D, \deg Q\big)}{|ABCD|^{\frac{1}{2}}} \\
\ll_p &\frac{(\deg Q)^d}{\phi(Q)} \Bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{1}{|AB|^{\frac{1}{2}}} \Bigg) \Bigg( \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD < \deg Q}} \frac{1}{|CD|^{\frac{1}{2}}} \Bigg) \\
= &\frac{(\deg Q)^d}{\phi(Q)} \Bigg( \sum_{\substack{n,m \geq 0 \\ n+m < \deg Q}} q^{\frac{m+n}{2}} \Bigg)^2 \ll (\deg Q)^{d+2}.
\end{aligned}$$

□

From Lemmas 5.3.1 to 5.3.5 we can deduce the following:

**Lemma 5.3.6.** *Let $Q \in \mathcal{P}$, and let $p_1\big(\deg A, \deg B, \deg Q\big)$ and $p_2\big(\deg C, \deg D, \deg Q\big)$ be finite homogeneous polynomials of degree $d_1$ and $d_2$, respectively. Then,*

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{p_1\, \chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \Bigg( \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD < \deg Q}} \frac{p_2\, \chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \Bigg) \\
= &(1-q^{-1})(\deg Q)^{d_1+d_2+4} \\
&\cdot \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4 < 1 \\ 2a_2+a_3+a_4 < 1}} p_1\big(a_1+a_3, a_1+a_4, 1\big) p_2\big(a_2+a_4, a_2+a_3, 1\big) \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4 \\
&+ O_{p_1,p_2}\big((\deg Q)^{d_1+d_2+3}\big).
\end{aligned}$$

Similarly, the following can be proved:

**Lemma 5.3.7.** *Let $Q \in \mathcal{P}$, and let $p_1\big(\deg A, \deg B, \deg Q\big)$ and $p_2\big(\deg C, \deg D, \deg Q\big)$ be finite homogeneous polynomials of degree $d_1$ and $d_2$, respectively. Then,*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{p_1\, \chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \Bigg( \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD < \deg Q}} \frac{p_2\, \chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \Bigg)$$
$$= q^{-1} (\deg Q)^{d_1+d_2+4}$$
$$\cdot \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} p_1\big(a_1+a_3, a_1+a_4, 1\big) p_2\big(a_2+a_4, a_2+a_3, 1\big) \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4$$
$$+ O_{p_1,p_2}\big((\deg Q)^{d_1+d_2+3}\big).$$

The proof of Lemma 5.3.7 is similar to the proof of Lemma 5.3.6. We use Lemma 4.3.2 instead of Lemma 4.3.1.

We can similarly prove the following:

**Lemma 5.3.8.** *Let $Q \in \mathcal{P}$, let $p_1\big(\deg A, \deg B, \deg Q\big)$ and $p_2\big(\deg C, \deg D, \deg Q\big)$ be finite homogeneous polynomials of degree $d_1$ and $d_2$, respectively, and let $a \in \{0, 1, 2\}$. Then,*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = \deg Q - a}} \frac{p_1\, \chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \Bigg( \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD = \deg Q - a}} \frac{p_2\, \chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \Bigg)$$
$$= O_{p_1,p_2}\big((\deg Q)^{d_1+d_2+3}\big),$$

*and*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = \deg Q - a}} \frac{p_1\, \chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \Bigg( \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD = \deg Q - a}} \frac{p_2\, \chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \Bigg)$$
$$= O_{p_1,p_2}\big((\deg Q)^{d_1+d_2+3}\big).$$

## 5.4 Fourth Moments of Derivatives

We are now equipped to prove Theorems 2.3.3 and 2.3.4. For ease of reference, we restate Theorem 2.3.3:

**Theorem.** *For all non-negative integers $k, l$ we have that*

$$\frac{1}{\phi(Q)} \frac{1}{(\log q)^{2k+2l}} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| L^{(k)}\Big(\frac{1}{2}, \chi\Big) \Big|^2 \Big| L^{(l)}\Big(\frac{1}{2}, \chi\Big) \Big|^2$$
$$= (1 - q^{-1}) (\deg Q)^{2k+2l+4}$$
$$\cdot \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} f_k\big(a_1+a_3, a_1+a_4, 1\big) f_l\big(a_2+a_4, a_2+a_3, 1\big) \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4$$
$$+ O_{k,l}\Big((\deg Q)^{2k+2l+\frac{7}{2}}\Big),$$

*where for all non-negative integers $i$ we define*

$$f_i\big(x,y,z\big) = x^i y^i + (z-x)^i (z-y)^i .$$

*Proof of Theorem 2.3.3.* We have that

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| L^{(k)}\Big(\frac{1}{2},\chi\Big)\Big|^2 \Big| L^{(l)}\Big(\frac{1}{2},\chi\Big)\Big|^2 \\
&= \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ odd}}} \Big| L^{(k)}\Big(\frac{1}{2},\chi\Big)\Big|^2 \Big| L^{(l)}\Big(\frac{1}{2},\chi\Big)\Big|^2 + \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Big| L^{(k)}\Big(\frac{1}{2},\chi\Big)\Big|^2 \Big| L^{(l)}\Big(\frac{1}{2},\chi\Big)\Big|^2 .
\end{aligned} \tag{5.7}$$

Using Lemma 5.2.1, we have, for the first term on the RHS, that

$$\begin{aligned}
&\frac{1}{\phi(Q)} \frac{1}{(\log q)^{2k+2l}} \sum_{\substack{\chi \bmod Q \\ \chi \text{ odd}}} \Big| L^{(k)}\Big(\frac{1}{2},\chi\Big)\Big|^2 \Big| L^{(l)}\Big(\frac{1}{2},\chi\Big)\Big|^2 \\
&= \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ odd}}} \Bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{\big(f_k + g_{O,k}\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} + \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = \deg Q - 1}} \frac{h_{O,k}\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \\
&\qquad \cdot \Bigg( \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD < \deg Q}} \frac{\big(f_l + g_{O,l}\big)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} + \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD = \deg Q - 1}} \frac{h_{O,l}\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \Bigg).
\end{aligned} \tag{5.8}$$

By using Lemmas 5.3.6 and 5.3.7, we have that

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ odd}}} \Bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{\big(f_k + g_{O,k}\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \\
&\qquad\qquad \cdot \Bigg( \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD < \deg Q}} \frac{\big(f_l + g_{O,l}\big)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \Bigg) \\
&= \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{\big(f_k + g_{O,k}\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \\
&\qquad\qquad \cdot \Bigg( \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD < \deg Q}} \frac{\big(f_l + g_{O,l}\big)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \Bigg) \\
&\quad - \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{\big(f_k + g_{O,k}\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \\
&\qquad\qquad \cdot \Bigg( \sum_{\substack{C,D \in \mathcal{M} \\ \deg CD < \deg Q}} \frac{\big(f_l + g_{O,l}\big)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \Bigg)
\end{aligned}$$

$$= (1-2q^{-1})(\deg Q)^{2k+2l+4} \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} f_k\big(a_1+a_3, a_1+a_4, 1\big) f_l\big(a_2+a_4, a_2+a_3, 1\big) \cdot \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4 + O_{k,l}\Big((\deg Q)^{2k+2l+3}\Big).$$

Strictly speaking, Lemmas 5.3.6 and 5.3.7 require that the polynomials $f_k + g_{O,k}$ and $f_l + g_{O,l}$ are homogeneous, which is not the case. However, these polynomials can be written as sums of homogeneous polynomials, with the terms of highest degree being $f_k$ and $f_l$, respectively. We can then apply the lemmas term-by-term to obtain the result above.

We now have the main term of (5.8). Indeed, for the remaining terms we can apply the Cauchy-Schwarz inequality and Lemmas 5.3.6, 5.3.7, and 5.3.8 to see that they are equal to $O_{k,l}\Big((\deg Q)^{2k+2l+\frac{7}{2}}\Big)$. Hence,

$$\frac{1}{\phi(Q)} \frac{1}{(\log q)^{2k+2l}} \sum_{\substack{\chi \bmod Q \\ \chi \text{ odd}}} \Big| L^{(k)}\Big(\frac{1}{2}, \chi\Big)\Big|^2 \Big| L^{(l)}\Big(\frac{1}{2}, \chi\Big)\Big|^2$$
$$= (1-2q^{-1})(\deg Q)^{2k+2l+4} \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} f_k\big(a_1+a_3, a_1+a_4, 1\big) f_l\big(a_2+a_4, a_2+a_3, 1\big) \cdot \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4 + O_{k,l}\Big((\deg Q)^{2k+2l+\frac{7}{2}}\Big). \tag{5.9}$$

We now look at the second term on the RHS of (5.7). By using Lemma 5.2.6 and similar means as those used to deduce (5.9), we can show for all non-negative integers $i, j$ that

$$\frac{1}{\phi(Q)} \frac{1}{(\log q)^{2i+2j}} \frac{1}{(q^{\frac{1}{2}}-1)^4} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Big| \hat{L}^{(i)}\Big(\frac{1}{2}, \chi\Big)\Big|^2 \Big| \hat{L}^{(j)}\Big(\frac{1}{2}, \chi\Big)\Big|^2$$
$$= q^{-1}(\deg Q)^{2i+2j+4} \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} f_i\big(a_1+a_3, a_1+a_4, 1\big) f_j\big(a_2+a_4, a_2+a_3, 1\big) \cdot \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4 + O_{i,j}\Big((\deg Q)^{2i+2j+\frac{7}{2}}\Big).$$

Using Lemma 5.2.4 and the Cauchy-Schwarz inequality, we obtain that

$$\frac{1}{\phi(Q)} \frac{1}{(\log q)^{2k+2l}} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Big| L^{(k)}\Big(\frac{1}{2}, \chi\Big)\Big|^2 \Big| L^{(l)}\Big(\frac{1}{2}, \chi\Big)\Big|^2$$
$$= q^{-1}(\deg Q)^{2k+2l+4} \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} f_k\big(a_1+a_3, a_1+a_4, 1\big) f_l\big(a_2+a_4, a_2+a_3, 1\big) \cdot \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4 + O_{k,l}\Big((\deg Q)^{2k+2l+\frac{7}{2}}\Big). \tag{5.10}$$

The proof follows from (5.7), (5.9), (5.10). □

We now proceed to prove Theorem 2.3.4:

**Theorem.** *For all non-negative integers $m$ we define*

$$\begin{aligned} &D_m \\ &:= \lim_{\deg Q \longrightarrow \infty} \frac{1}{(1-q^{-1})(\log q)^{4m}} \frac{1}{\phi(Q)} \frac{1}{(\deg Q)^{4m+4}} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| L^{(m)}\Big(\frac{1}{2}, \chi\Big) \right|^4 \\ &= \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} \Big( (a_1+a_3)^m (a_1+a_4)^m + (1-a_1-a_3)^m (1-a_1-a_4)^m \Big) \\ &\qquad \cdot \Big( (a_2+a_3)^m (a_2+a_4)^m + (1-a_2-a_3)^m (1-a_2-a_4)^m \Big) \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4. \end{aligned} \tag{5.11}$$

*We have that*

$$D_m \sim \frac{1}{16m^4}$$

*as $m \longrightarrow \infty$.*

First, we require a lemma.

**Lemma 5.4.1.** *Let $m \geq 8$ be an integer. For all non-negative $x$ we have that*

$$\Big(1 - \frac{x}{m}\Big)^m \leq e^{-x},$$

*and for all $x \in [0, 2m^{\frac{1}{3}}]$ we have that*

$$\Big(1 - \frac{x}{m}\Big)^m \geq e^{-x} e^{-8m^{-\frac{1}{3}}}.$$

*Proof.* By using the Taylor series for log we have that

$$\log\Big( \big(1 - \frac{x}{m}\big)^m \Big) = -x - \frac{x^2}{2m} - \frac{x^3}{3m^2} - \frac{x^4}{4m^3} - \dots.$$

Clearly, the RHS is $\leq -x$, which proves the first inequality. For the second inequality we use the bounds on $x$ to obtain that

$$\begin{aligned} \frac{x^2}{2m} + \frac{x^3}{3m^2} + \frac{x^4}{4m^3} + \dots \leq & \frac{x^2}{m} \sum_{i=0}^{\infty} \Big(\frac{x}{m}\Big)^i = \frac{x^2}{m} \Bigg( \frac{1}{1-\frac{x}{m}} \Bigg) \\ \leq & \Bigg( \frac{4}{m^{\frac{1}{3}} - 2m^{-\frac{1}{3}}} \Bigg) \leq 8m^{-\frac{1}{3}}, \end{aligned}$$

from which the result follows. □

*Proof of Theorem 2.3.4.* Let us expand the brackets in (5.11) and multiply by $m^4$. One of the terms is the following:

$$m^4 \int_{\substack{a_1,a_2,a_3,a_4 \geq 0 \\ 2a_1+a_3+a_4<1 \\ 2a_2+a_3+a_4<1}} (1-a_1-a_3)^m (1-a_1-a_4)^m (1-a_2-a_3)^m (1-a_2-a_4)^m \cdot \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4$$

$$= \int_{\substack{a_1,a_2,a_3,a_4\geq 0\\ 2a_1+a_3+a_4<m\\ 2a_2+a_3+a_4<m}} \Big(1-\frac{a_1+a_3}{m}\Big)^m \Big(1-\frac{a_1+a_4}{m}\Big)^m \Big(1-\frac{a_2+a_3}{m}\Big)^m \Big(1-\frac{a_2+a_4}{m}\Big)^m \cdot \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4,$$

where we have used the substitutions $a_i \to \frac{a_i}{m}$. On one hand, by using Lemma 5.4.1, we have that

$$\int_{\substack{a_1,a_2,a_3,a_4\geq 0\\ 2a_1+a_3+a_4<m\\ 2a_2+a_3+a_4<m}} \Big(1-\frac{a_1+a_3}{m}\Big)^m \Big(1-\frac{a_1+a_4}{m}\Big)^m \Big(1-\frac{a_2+a_3}{m}\Big)^m \Big(1-\frac{a_2+a_4}{m}\Big)^m \cdot \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4$$
$$\geq \int_{\substack{0\leq a_1,a_2,a_3,a_4\leq m^{\frac{1}{3}}\\ 2a_1+a_3+a_4<m\\ 2a_2+a_3+a_4<m}} \Big(1-\frac{a_1+a_3}{m}\Big)^m \Big(1-\frac{a_1+a_4}{m}\Big)^m \Big(1-\frac{a_2+a_3}{m}\Big)^m \Big(1-\frac{a_2+a_4}{m}\Big)^m \cdot \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4$$
$$\geq e^{-8m^{-\frac{1}{3}}} \int_{0\leq a_1,a_2,a_3,a_4\leq m^{\frac{1}{3}}} e^{-2(a_1+a_2+a_3+a_4)}\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4 \longrightarrow \frac{1}{16}$$

as $m \longrightarrow \infty$. On the other hand, by the same lemma, we have that

$$\int_{\substack{a_1,a_2,a_3,a_4\geq 0\\ 2a_1+a_3+a_4<m\\ 2a_2+a_3+a_4<m}} \Big(1-\frac{a_1+a_3}{m}\Big)^m \Big(1-\frac{a_1+a_4}{m}\Big)^m \Big(1-\frac{a_2+a_3}{m}\Big)^m \Big(1-\frac{a_2+a_4}{m}\Big)^m \cdot \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4$$
$$\leq \int_{0\leq a_1,a_2,a_3,a_4\leq m} e^{-2(a_1+a_2+a_3+a_4)}\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4 \longrightarrow \frac{1}{16}$$

as $m \longrightarrow \infty$. So, we see that

$$m^4 \int_{\substack{a_1,a_2,a_3,a_4\geq 0\\ 2a_1+a_3+a_4<1\\ 2a_2+a_3+a_4<1}} (1-a_1-a_3)^m(1-a_1-a_4)^m(1-a_2-a_3)^m(1-a_2-a_4)^m \cdot \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4 \longrightarrow \frac{1}{16} \tag{5.12}$$

as $m \longrightarrow \infty$.

Now, after we expanded the brackets in (5.11) and multiplied by $m^4$, there were other terms. These can be seen to tend to 0 as $m \longrightarrow \infty$. We prove one case below; the rest are similar.

$$m^4 \int_{\substack{a_1,a_2,a_3,a_4\geq 0\\ 2a_1+a_3+a_4<1\\ 2a_2+a_3+a_4<1}} (1-a_1-a_3)^m(1-a_1-a_4)^m(a_2+a_3)^m(a_2+a_4)^m\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3\mathrm{d}a_4$$
$$\leq m^4 \int_{\substack{a_1,a_2,a_3,a_4\geq 0\\ 2a_1+a_3+a_4<1\\ 2a_2+a_3+a_4<1}} (a_2+a_3)^m(a_2+a_4)^m \ll \frac{m^4}{4^m},$$

where we have used the following: The maximum value that $(a_2+a_3)(a_2+a_4)$ can take subject to the conditions in the integral is at most equal to the maximum value that $f(x,y) := xy$ can take subject to the conditions $x, y \geq 0$ and $x+y<1$. By plotting this range and looking at contours of $f(x,y)$ we can see that the maximum value is $\frac{1}{4}$. The result follows. □

# Chapter 6

# A Random Matrix Theory Model for Moments of Dirichlet $L$-functions

Many of the results in this Chapter are based on the number field analogues in [GHK07] and [BK07].

## 6.1 The Hybrid Euler-Hadamard Product Formula

In this section, we prove Theorem 2.4.1. That is, for primitive Dirichlet characters of modulus $R \in \mathcal{M}\backslash\{1\}$, we express $L(s,\chi)$ as two factors. The first is a partial Euler product, while the second is a partial Hadamard product (a product over the zeros of the $L$-function). We will briefly give an intuitive explanation of how this is obtained.

We use the logarithmic derivative of our $L$-function. Namely we will use a function of the form $\frac{\tilde{u}(w+1)}{w}\frac{L'}{L}(s+aw,\chi)$, where $s,w$ are variables and the details regarding $a$ and $\tilde{u}$ are not needed at this time. We will integrate this function over an appropriate line.

Due to the $\frac{1}{w}$ factor, this function has a pole at $w=0$. With an appropriate contour shift we can obtain its residue: $\frac{L'}{L}(s,\chi)$.

The function also has poles at $w=\frac{\rho-s}{a}$ for all zeros, $\rho$, of our $L$-function. These can be captured by our contour shift and we can obtain a sum over the zeros.

Of course, we require a product over the zeros, but this is obtained by integrating our result and taking exponential: The sum over the zeros becomes a product, and $\frac{L'}{L}(s,\chi)$ becomes $L(s,\chi)$.

If we directly calculate our original integral of $\frac{\tilde{u}(w+1)}{w}\frac{L'}{L}(s+aw,\chi)$, without taking contour shifts, and then integrate and take exponentials, we obtain an Euler product. Equating this with the results we obtained by taking contour shifts, we obtain our Hybrid Euler-Hadamard product formula.

The key is the function $\frac{\tilde{u}(w+1)}{w}\frac{L'}{L}(s+aw,\chi)$: It allows us to obtain a sum over the zeros via its poles, and this can be made into a product by reversing the logarithmic differentiation.

Now, before proving Theorem 2.4.1, we prove several lemmas.

**Lemma 6.1.1.** *For all Dirichlet characters* $\chi$ *and all* $\mathrm{Re}(s) > 1$ *we have*

$$-\frac{L'}{L}(s,\chi) = \sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s}.$$

*Proof.* Taking the logarithmic derivative of

$$L(s,\chi) = \prod_{P \in \mathcal{P}} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1}$$

gives

$$\frac{L'}{L}(s,\chi) = -\sum_{P \in \mathcal{P}} \frac{\chi(P) \log|P|}{|P|^s} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1} = -\sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s}.$$

□

**Lemma 6.1.2.** *Let* $\chi$ *be a non-trivial character. As* $\mathrm{Re}(s) \longrightarrow \infty$,

$$\frac{L'}{L}(-s,\chi) = O_\chi(1).$$

*Proof.* As $\chi$ is non-trivial, there is some maximal integer $N \geq 0$ with $L_N(\chi) \neq 0$. Hence,

$$L(-s,\chi) = \sum_{n=0}^{N} L_n(\chi) q^{ns} \gg_\chi q^{N \mathrm{Re}(s)}$$

and

$$L'(-s,\chi) = -\log q \sum_{n=0}^{N} n L_n(\chi) q^{ns} \ll_\chi q^{N \mathrm{Re}(s)}.$$

The proof follows. □

**Lemma 6.1.3.** *Let* $X$ *be a positive integer, and let* $u(x)$ *be a positive* $C^\infty$*-function with support in* $[e, e^{1+q^{-X}}]$. *Let* $\tilde{u}(s)$ *be its Mellin transform. That is,*

$$\tilde{u}(s) = \int_{x=0}^{\infty} x^{s-1} u(x) \mathrm{d}x$$

*and*

$$u(x) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} x^{-s} \tilde{u}(s) \mathrm{d}s,$$

*where* $c$ *can take any value in* $\mathbb{R}$ *(due to our restrictions on the support of* $u$*, we can see that* $\tilde{u}(s)$ *is well-defined for all* $s \in \mathbb{C}$*, and so, by the Mellin inversion theorem,* $c$ *can take any value in* $\mathbb{R}$*). Then,*

$$\tilde{u}(s) \ll \begin{cases} \frac{1}{|s|+1} \max_x\{|u'(x)|\} e^{2\mathrm{Re}(s)} & \text{if } \mathrm{Re}(s) > 0 \\ \frac{1}{|s|+1} \max_x\{|u'(x)|\} e^{\mathrm{Re}(s)} & \text{if } \mathrm{Re}(s) \leq 0. \end{cases}$$

*Proof.* We have, by integration by parts, that

$$\tilde{u}(s) = \int_{x=e}^{e^{1+q^{-X}}} x^{s-1} u(x) \mathrm{d}x = -\frac{1}{s} \int_{x=e}^{e^{1+q^{-X}}} x^s u'(x) \mathrm{d}x.$$

If $|s| > 1$, then it is not difficult to deduce that the above is

$$\ll \begin{cases} \frac{1}{|s|+1} \max_x\{|u'(x)|\} e^{2\operatorname{Re}(s)} & \text{if } \operatorname{Re}(s) > 0 \\ \frac{1}{|s|+1} \max_x\{|u'(x)|\} e^{\operatorname{Re}(s)} & \text{if } \operatorname{Re}(s) \leq 0. \end{cases}$$

If $|s| \leq 1$, then, by using the fact that $\int_{x=e}^{e^{1+q^{-X}}} u'(x)\mathrm{d}x = 0$, we obtain

$$\begin{aligned} \tilde{u}(s) = \int_{x=e}^{e^{1+q^{-X}}} \frac{1-x^s}{s} u'(x) \mathrm{d}x &= -\int_{x=e}^{e^{1+q^{-X}}} \left( \int_{y=1}^{x} y^{s-1} \mathrm{d}y \right) u'(x) \mathrm{d}x \\ \ll \int_{x=e}^{e^{1+q^{-X}}} |u'(x)| \mathrm{d}x \ll \max_x\{|u'(x)|\}, \end{aligned}$$

from which the result follows. □

**Lemma 6.1.4.** *Let $X$ be a positive integer, and let $u(x)$ be a positive $C^\infty$-function with support in $[e, e^{1+q^{-X}}]$, and let $\tilde{u}(s)$ be its Mellin transform. Let*

$$v(x) = \int_{t=x}^{\infty} u(t) \mathrm{d}t$$

*and take $u$ to be normalised so that $v(0) = 1$. Note that its Mellin transform is*

$$\tilde{v}(s) = \frac{\tilde{u}(s+1)}{s}.$$

*Let $\chi$ be a primitive Dirichlet character of modulus $R \in \mathcal{M}\backslash\{1\}$. Then, for $s \in \mathbb{C}$ not being a zero of $L(s,\chi)$, we have*

$$-\frac{L'}{L}(s,\chi) = \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq X}} \frac{\chi(A)\Lambda(A)}{|A|^s} + \sum_{\rho_n} \frac{\tilde{u}\big(1 + (\rho_n - s)(\log q)X\big)}{\rho_n - s}, \tag{6.1}$$

*where $\rho_n = \frac{1}{2} + i\gamma_n$ is the $n$-th zero of $L(s,\chi)$. Note that, by Lemma 6.1.3, we can see that the sum over the zeros is absolutely convergent.*

*Proof.* Let $c > \max\{0, \big(1 - \operatorname{Re}(s)\big)(\log q)X\}$. By the Mellin inversion theorem, we have

$$\begin{aligned} \sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} v\Big(e^{\frac{\deg A}{X}}\Big) =& \frac{1}{2\pi i} \sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} \int_{\operatorname{Re}(w)=c} \frac{\tilde{u}(w+1)}{w} |A|^{-\frac{w}{(\log q)X}} \mathrm{d}w \\ =& \frac{1}{2\pi i} \int_{\operatorname{Re}(w)=c} \frac{\tilde{u}(w+1)}{w} \sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^{s+\frac{w}{(\log q)X}}} \mathrm{d}w \\ =& -\frac{1}{2\pi i} \int_{\operatorname{Re}(w)=c} \frac{\tilde{u}(w+1)}{w} \frac{L'}{L}\Big(s + \frac{w}{(\log q)X}, \chi\Big) \mathrm{d}w. \end{aligned}$$

The interchange of integral and summation is justified by absolute convergence, which holds because $c > \big(1 - \operatorname{Re}(s)\big)(\log q)X$ and by Lemma 6.1.3.

We now shift the line of integration to $\mathrm{Re}(w) = -M$, for some $M > \max\{0, \mathrm{Re}(s)(\log q)X\}$, giving

$$\sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} v\left(e^{\frac{\deg A}{X}}\right) = -\frac{L'}{L}(s,\chi) - \sum_{\rho_n} \frac{\tilde{u}\big(1 + (\rho_n - s)(\log q)X\big)}{\rho_n - s}$$
$$- \frac{1}{2\pi i} \int_{\mathrm{Re}(w)=-M} \frac{\tilde{u}(w+1)}{w} \frac{L'}{L}\Big(s + \frac{w}{(\log q)X}, \chi\Big) \mathrm{d}w,$$

where the sum over the zeros counts multiplicities. This requires some justification. We make use of the contour that is the rectangle with vertices at

$$c \pm i\Big(\big(d - \mathrm{Im}(s)\big)(\log q)X + 2\pi n X\Big),$$
$$-M \pm i\Big(\big(d - \mathrm{Im}(s)\big)(\log q)X + 2\pi n X\Big).$$

Here, $d > 0$ is such that $\frac{1}{2} + id$ is not a pole of $\frac{L'}{L}(s,\chi)$ (that is, not a zero of $L(s,\chi)$). It is clear that as $n \longrightarrow \infty$ we capture all the poles and the left edge tends to the integral over $\mathrm{Re}(w) = -M$. Due to the vertical periodicity of $\frac{L'}{L}$, and our choice of $d$, we can see that the top and bottom integrals are equal to $O_{c,M}(n^{-1})$, which vanishes as $n \longrightarrow \infty$.

By Lemmas 6.1.2 and 6.1.3, if we let $M \longrightarrow \infty$ then we see that the integral over $\mathrm{Re}(w) = -M$ vanishes.

Finally, we note that

$$v\left(e^{\frac{\deg A}{X}}\right) = \begin{cases} 1 & \text{if } \deg A \leq X \\ 0 & \text{if } \deg A \geq X(1 + q^{-X}). \end{cases}$$

Also, since $X$ is a positive integer, there are no integers in the interval $\big(X, X(1+q^{-X})\big) \subseteq \big(X, X + \frac{1}{2}\big)$, and so there are no $A \in \mathcal{A}$ that have degree in this interval. It follows that

$$\sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} v\left(e^{\frac{\log|A|}{(\log q)X}}\right) = \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq X}} \frac{\chi(A)\Lambda(A)}{|A|^s}.$$

□

**Lemma 6.1.5.** *Suppose $u(x)$ has support in $[e, e^{1+q^{-X}}]$. For all $z \in \mathbb{C}\backslash\{0\}$ with $\arg(z) \neq \pi$ we define*

$$U(z) := \int_{x=0}^{\infty} u(x) E_1(z \log x) \mathrm{d}x.$$

*(Recall, for $y \in \mathbb{C}\backslash\{0\}$ with $\arg(y) \neq \pi$, we define $E_1(y) := \int_{w=y}^{y+\infty} \frac{e^{-w}}{w} \mathrm{d}w$). Let $\chi$ be a primitive Dirichlet character of modulus $R \in \mathcal{M}\backslash\{1\}$, and suppose $\rho$ is a zero of $L(s,\chi)$ and $s \in \mathbb{C}\backslash\{\rho\}$ with $\arg(s - \rho) \neq \pi$. Then,*

$$\int_{s_0=s}^{s+\infty} \frac{\tilde{u}\big(1 + (\rho - s_0)(\log q)X\big)}{\rho - s_0} \mathrm{d}s_0 = -U\Big((s-\rho)(\log q)X\Big).$$

*Proof.* We have

$$\begin{aligned}
\int_{s_0=s}^{s+\infty} \frac{\tilde{u}\big(1+(\rho-s_0)(\log q)X\big)}{\rho-s_0}\mathrm{d}s_0 &= \int_{s_0=s}^{s+\infty} \frac{1}{\rho-s_0}\int_{x=0}^{\infty} x^{(\rho-s_0)(\log q)X}u(x)\mathrm{d}x\mathrm{d}s_0 \\
&= \int_{x=0}^{\infty} u(x)\int_{s_0=s}^{s+\infty} \frac{e^{(\rho-s_0)(\log q)X\log x}}{\rho-s_0}\mathrm{d}s_0\mathrm{d}x \\
&= -\int_{x=0}^{\infty} u(x)\int_{w=(s-\rho)(\log q)X\log x}^{(s-\rho)(\log q)X\log x+\infty} \frac{e^{-w}}{w}\mathrm{d}w\mathrm{d}x \\
&= -\int_{x=0}^{\infty} u(x)E_1\Big((s-\rho)(\log q)X\log x\Big)\mathrm{d}x \\
&= -U\Big((s-\rho)(\log q)X\Big).
\end{aligned}$$

The interchange of integration is justified by absolute convergence, which holds for $X > 1$. ☐

We can now proceed with the proof of Theorem 2.4.1, which we restate for ease of reference.

**Theorem.** *Let $X \geq 1$ be an integer and let $u(x)$ be a positive $C^\infty$-function with support in $[e, e^{1+q^{-X}}]$. Let*

$$v(x) = \int_{t=x}^{\infty} u(t)\mathrm{d}t$$

*and take $u$ to be normalised so that $v(0) = 1$. Furthermore, for $y \in \mathbb{C}\backslash\{0\}$ with $\arg(y) \neq \pi$, we define $E_1(y) := \int_{w=y}^{y+\infty} \frac{e^{-w}}{w}\mathrm{d}w$; and for $z \in \mathbb{C}\backslash\{0\}$ with $\arg(z) \neq \pi$, we define*

$$U(z) := \int_{x=0}^{\infty} u(x)E_1(z\log x)\mathrm{d}x.$$

*Let $\chi$ be a primitive Dirichlet character of modulus $R \in \mathcal{M}\backslash\{1\}$, and let $\rho_n = \frac{1}{2}+i\gamma_n$ be the $n$-th zero of $L(s,\chi)$. Then, for all $s \in \mathbb{C}$ we have*

$$L(s,\chi) = P_X(s,\chi)Z_X(s,\chi),$$

*where*

$$P_X(s,\chi) = \exp\Bigg(\sum_{\substack{A\in\mathcal{M}\\ \deg A\leq X}} \frac{\chi(A)\Lambda(A)}{|A|^s\log|A|}\Bigg)$$

*and*

$$Z_X(s,\chi) = \exp\bigg(-\sum_{\rho_n} U\Big((s-\rho_n)(\log q)X\Big)\bigg).$$

*Strictly speaking, if $s=\rho$ or $\arg(s-\rho)=\pi$ for some zero $\rho$ of $L(s,\chi)$, then $Z_X(s,\chi)$ is not defined. In this case, we take*

$$Z_X(s,\chi) = \lim_{s_0\longrightarrow s} Z_X(s_0,\chi)$$

*and we show that this is well defined.*

*Proof of Theorem 2.4.1.* Suppose $s \in \mathbb{C}$ is not a zero of $L(s,\chi)$ and $\arg(s-\rho) \neq \pi$ for all zeros $\rho$ of $L(s,\chi)$. We recall that (6.1) gives us

$$-\frac{L'}{L}(s_0,\chi) = \sum_{\substack{A\in\mathcal{M}\\ \deg A\leq X}} \frac{\chi(A)\Lambda(A)}{|A|^{s_0}} + \sum_{\rho_n} \frac{\tilde{u}\big(1+(\rho_n-s_0)(\log q)X\big)}{\rho_n - s_0},$$

to which we apply the integral $\int_{s_0=s}^{s+\infty} \mathrm{d}s_0$ to both sides to obtain

$$\log L(s,\chi) = \sum_{\substack{A\in\mathcal{M}\\ \deg A\leq X}} \frac{\chi(A)\Lambda(A)}{|A|^{s}\log|A|} - \sum_{\rho} U\Big((s-\rho)(\log q)X\Big). \tag{6.2}$$

For the integral over the sum over zeros, we applied Lemma 6.1.5, after an interchange of summation and integration that is justified by Lemma 6.1.3. We now take exponentials of both sides of (6.2) to obtain

$$\begin{aligned} L(s,\chi) =& \exp\Bigg( \sum_{\substack{A\in\mathcal{M}\\ \deg A\leq X}} \frac{\chi(A)\Lambda(A)}{|A|^{s}\log|A|}\Bigg) \exp\Bigg( -\sum_{\rho} U\Big((s-\rho)(\log q)X\Big)\Bigg) \\ =& P_X(s,\chi) Z_X(s,\chi). \end{aligned}$$

Now suppose we have $s \in \mathbb{C}$, not being a zero of $L(s,\chi)$, but with $\arg(s-\rho) = \pi$ for some zero $\rho$ of $L(s,\chi)$. We can see that $\lim_{s_0\longrightarrow s} L(s_0,\chi) = L(s,\chi)$ and $\lim_{s_0\longrightarrow s} P_X(s_0,\chi) = P_X(s,\chi) \neq 0$. That latter is non-zero as $P_X(s,\chi)$ is the exponential of a polynomial. From this, we can deduce that $\lim_{s_0\longrightarrow s} Z_X(s_0,\chi) = L(s,\chi)\big(P_X(s,\chi)\big)^{-1} \in \mathbb{C}$.

Similarly, if $s$ is a zero of $L(s,\chi)$, then we can see that $\lim_{s_0\longrightarrow s} Z_X(s_0,\chi) = L(s,\chi)\big(P_X(s,\chi)\big)^{-1} = 0$.

This completes the proof. □

## 6.2 Moments of the Euler Product

Before proving Theorem 2.4.4, we prove a lemma.

**Lemma 6.2.1.** *For all* $\mathrm{Re}(s) > 0$ *and primitive characters* $\chi$ *we define*

$$P_X^*(s,\chi) := \prod_{\deg P\leq X} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1} \prod_{\frac{X}{2}<\deg P\leq X} \left(1 + \frac{\chi(P)^2}{2|P|^{2s}}\right)^{-1}, \tag{6.3}$$

*and for positive integers* $k$ *and* $A \in \mathcal{S}_{\mathcal{M}}(X)$ *we define* $\alpha_k(A)$ *by*

$$P_X^*(s,\chi)^k = \sum_{A\in\mathcal{S}_{\mathcal{M}}(X)} \frac{\alpha_k(A)\chi(A)}{|A|^s}.$$

*Then, for positive integers* $k$*, we have*

$$\begin{aligned} P_X\Big(\frac{1}{2},\chi\Big)^k =& \Big(1+O_k\big(X^{-1}\big)\Big) P_X^*\Big(\frac{1}{2},\chi\Big)^k \\ =& \Big(1+O_k\big(X^{-1}\big)\Big) \sum_{A\in\mathcal{S}_{\mathcal{M}}(X)} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}}. \end{aligned} \tag{6.4}$$

*We also have that*

$$\begin{aligned} \alpha_k(A) =& d_k(A) && \text{if } A \in \mathcal{S}_{\mathcal{M}}\Big(\frac{X}{2}\Big) \text{ or } A \text{ is prime} \\ 0 \leq \alpha_k(A) \leq& d_k(A) && \text{if } A \notin \mathcal{S}_{\mathcal{M}}\Big(\frac{X}{2}\Big) \text{ and } A \text{ is not prime.} \end{aligned} \tag{6.5}$$

*Proof.* First we note that

$$P_X\Big(\frac{1}{2},\chi\Big) = \exp\Big( \sum_{\substack{A\in\mathcal{M}\\ \deg A\leq X}} \frac{\chi(A)\Lambda(A)}{|A|^{\frac{1}{2}}\log|A|}\Big) = \exp\Big( \sum_{\deg P\leq X} \sum_{j=1}^{N_P} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}}\Big),$$

where

$$N_P := \Big\lfloor \frac{X}{\deg P}\Big\rfloor.$$

Also, by using the Taylor series for log, we have

$$P^*_X\Big(\frac{1}{2},\chi\Big) = \exp\Big( \sum_{\deg P\leq X} \sum_{j=1}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} + \sum_{\frac{X}{2}<\deg P\leq X} \sum_{j=1}^{\infty} \frac{(-1)^j\chi(P)^{2j}}{j2^j|P|^j}\Big).$$

Hence,

$$\begin{aligned} &P_X\Big(\frac{1}{2},\chi\Big)P^*_X\Big(\frac{1}{2},\chi\Big)^{-1} \\ =&\exp\Big( - \sum_{\deg P\leq X} \sum_{j=N_p+1}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} - \sum_{\frac{X}{2}<\deg P\leq X} \sum_{j=1}^{\infty} \frac{(-1)^j\chi(P)^{2j}}{j2^j|P|^j}\Big). \end{aligned}$$

We now show that the terms inside the exponential are equal to $O\big(X^{-1}\big)$, from which we easily deduce

$$P_X\Big(\frac{1}{2},\chi\Big)^k = \Big(1+O_k\big(X^{-1}\big)\Big)P^*_X\Big(\frac{1}{2},\chi\Big)^k.$$

To this end, using the prime polynomial theorem for the last line below, we have

$$\begin{aligned} &\sum_{\deg P\leq X} \sum_{j=N_P+1}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} + \sum_{\frac{X}{2}<\deg P\leq X} \sum_{j=1}^{\infty} \frac{(-1)^j\chi(P)^{2j}}{j2^j|P|^j} \\ =& \sum_{\deg P\leq \frac{X}{2}} \sum_{j=N_P+1}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} + \sum_{\frac{X}{2}<\deg P\leq X} \sum_{j=2}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} + \sum_{\frac{X}{2}<\deg P\leq X} \sum_{j=1}^{\infty} \frac{(-1)^j\chi(P)^{2j}}{j2^j|P|^j} \\ =& \sum_{\deg P\leq \frac{X}{2}} \sum_{j=N_P+1}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} + \sum_{\frac{X}{2}<\deg P\leq X} \sum_{j=3}^{\infty} \frac{\chi(P)^j}{j|P|^{\frac{j}{2}}} + \sum_{\frac{X}{2}<\deg P\leq X} \sum_{j=2}^{\infty} \frac{(-1)^j\chi(P)^{2j}}{j2^j|P|^j} \\ \ll& \sum_{\deg P\leq \frac{X}{2}} |P|^{-\frac{N_P+1}{2}} + \sum_{\frac{X}{2}<\deg P\leq X} |P|^{-\frac{3}{2}} \ll q^{-\frac{X}{2}} \sum_{\deg P\leq \frac{X}{2}} 1 + \sum_{\frac{X}{2}<n\leq X} \frac{q^{-\frac{n}{2}}}{n} \ll \frac{1}{X}. \end{aligned} \tag{6.6}$$

We now proceed to prove (6.5). The first case is clear, so assume that $A \notin \mathcal{S}_{\mathcal{M}}\big(\frac{X}{2}\big)$ and $A$ is not prime. We note that

$$\Big(1-\frac{\chi(P)}{|P|^{\frac{1}{2}}}\Big)^{-1}\Big(1+\frac{\chi(P)^2}{2|P|}\Big)^{-1}$$

$$=\left(1+\frac{\chi(P)}{|P|^{\frac{1}{2}}}+\frac{\chi(P)^2}{|P|}+\dots\right)\left(1-\frac{\chi(P)^2}{2|P|}+\frac{\chi(P)^4}{2^2|P|^2}-\dots\right)$$
$$=\sum_{r=0}^{\infty}\left(\sum_{\substack{r_1,r_2\geq 0\\ r_1+2r_2=r}}\left(-\frac{1}{2}\right)^{r_2}\right)\frac{\chi(P)^r}{|P|^{\frac{r}{2}}}=\sum_{r=0}^{\infty}\frac{2}{3}\left(1-\left(-\frac{1}{2}\right)^{\lfloor\frac{r}{2}\rfloor+1}\right)\frac{\chi(P)^r}{|P|^{\frac{r}{2}}}.$$

Since

$$0\leq\frac{2}{3}\left(1-\left(-\frac{1}{2}\right)^{\lfloor\frac{r}{2}\rfloor+1}\right)\leq 1$$

for all $r\geq 0$, the result follows. □

We can now prove Theorem 2.4.4, which we restate for ease of reference.

**Theorem.** *For positive integers $k$, we have*

$$\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\left|P_X\left(\frac{1}{2},\chi\right)\right|^{2k}\sim a(k)\prod_{\substack{\deg P\leq X\\ P|R}}\left(\sum_{m=0}^{\infty}\frac{d_k\left(P^m\right)^2}{|P|^m}\right)^{-1}\left(e^{\gamma}X\right)^{k^2}$$

*as $X,\deg R\longrightarrow\infty$ with $X\leq\log_q\deg R$. Here, $\gamma$ is the Euler-Mascheroni constant, and*

$$a(k)=\prod_{P\in\mathcal{P}}\left(\left(1-\frac{1}{|P|}\right)^{k^2}\sum_{m=0}^{\infty}\frac{d_k\left(P^m\right)^2}{|P|^m}\right).$$

*Proof of Theorem 2.4.4.* Throughout this proof, any asymptotic relations are to be taken as $X,\deg R\xrightarrow{q,k}\infty$ with $X\leq\log_q\deg R$. By Lemma 6.2.1 it suffices to prove that

$$\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\left|\sum_{A\in\mathcal{S}_{\mathcal{M}}(X)}\frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}}\right|^2\sim a(k)\prod_{\substack{\deg P\leq X\\ P|R}}\left(\sum_{m=0}^{\infty}\frac{d_k\left(P^m\right)^2}{|P|^m}\right)^{-1}\left(e^{\gamma}X\right)^{k^2}.$$

We will truncate our Dirichlet series. This will allow us to bound the lower order terms later. We have

$$\sum_{A\in\mathcal{S}_{\mathcal{M}}(X)}\frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}}=\sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A\leq\frac{1}{4}\deg R}}\frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}}+O\left(|R|^{-\frac{1}{17}}\right).\tag{6.7}$$

This makes use of the following:

$$\sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A>\frac{1}{4}\deg R}}\frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}}\leq|R|^{-\frac{1}{16}}\sum_{A\in\mathcal{S}_{\mathcal{M}}(X)}\frac{d_k(A)}{|A|^{\frac{1}{4}}}=|R|^{-\frac{1}{16}}\prod_{\deg P\leq X}\left(1-\frac{1}{|P|^{\frac{1}{4}}}\right)^{-k}$$
$$=|R|^{-\frac{1}{16}}\exp\left(\sum_{\deg P\leq X}-k\log\left(1-\frac{1}{|P|^{\frac{1}{4}}}\right)\right)=|R|^{-\frac{1}{16}}\exp\left(kO\left(\sum_{\deg P\leq X}\frac{1}{|P|^{\frac{1}{4}}}\right)\right)$$
$$=|R|^{-\frac{1}{16}}\exp\left(kO\left(\frac{q^{\frac{3}{4}X}}{X}\right)\right)=|R|^{-\frac{1}{16}}\exp\left(kO\left(\frac{\deg R}{\log_q\deg R}\right)\right)=O\left(|R|^{-\frac{1}{17}}\right).\tag{6.8}$$

By the Cauchy-Schwarz inequality, it suffices to prove that

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \Bigg| \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \leq \frac{1}{4} \deg R}} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}} \Bigg|^2 \sim a(k) \prod_{\substack{\deg P \leq X \\ P \mid R}} \left( \sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right)^{-1} \left( e^{\gamma} X \right)^{k^2}.$$

Now, we have that

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \Bigg| \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \leq \frac{1}{4} \deg R}} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}} \Bigg|^2 \\
=&\frac{1}{\phi^*(R)} \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A, \deg B \leq \frac{1}{4} \deg R \\ (AB,R)=1}} \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}} \sum_{\substack{EF=R \\ F \mid (A-B)}} \mu(E)\phi(F) \\
=&\frac{1}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A, \deg B \leq \frac{1}{4} \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F)}} \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}} \qquad (6.9) \\
=& \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \leq \frac{1}{4} \deg R \\ (A,R)=1}} \frac{\alpha_k(A)^2}{|A|} + \frac{1}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A, \deg B \leq \frac{1}{4} \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F) \\ A \neq B}} \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}}.
\end{aligned}$$

We first consider the second term on the far RHS: The off-diagonal terms. We note that the inner sum is zero if $\deg F > \frac{1}{4} \deg R$, and we also make use of (6.5), to obtain

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A, \deg B \leq \frac{1}{4} \deg R \\ (AB,R)=1 \\ A \equiv B (\bmod F) \\ A \neq B}} \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}} \\
\ll&\frac{1}{\phi^*(R)} \sum_{\substack{EF=R \\ \deg F \leq \frac{1}{4} \deg R}} \phi(F) \sum_{A,B \in \mathcal{S}_{\mathcal{M}}(X)} \frac{d_k(A)d_k(B)}{|AB|^{\frac{1}{2}}} \\
\leq&\frac{1}{\phi^*(R)} \prod_{\deg P \leq X} \left(1 - |P|^{-\frac{1}{2}}\right)^{-2k} \sum_{\substack{EF=R \\ \deg F \leq \frac{1}{4} \deg R}} \phi(F) \\
\leq&\frac{1}{\phi^*(R)} \prod_{\deg P \leq X} \left(1 - |P|^{-\frac{1}{2}}\right)^{-2k} \sum_{\substack{F \in \mathcal{M} \\ \deg F \leq \frac{1}{4} \deg R}} |R|^{\frac{1}{4}} \\
\leq&\frac{|R|^{\frac{1}{2}}}{\phi^*(R)} \exp\left( O\Big(2k \frac{q^{\frac{X}{2}}}{X}\Big) \right) = o(1).
\end{aligned}$$

The second-to-last relation makes use of a similar result to (6.8), and the last relation follows from the fact that $X, \deg R \longrightarrow \infty$ with $X \leq \log_q \deg R$. Now we consider the first term on the far RHS of (6.9): The diagonal terms. We required a truncated sum only for the off-diagonal terms, and so we extend our sum using similar means as in (6.8):

$$\sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A\leq\frac{1}{4}\deg R\\ (A,R)=1}} \frac{\alpha_k(A)^2}{|A|} = \sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ (A,R)=1}} \frac{\alpha_k(A)^2}{|A|} + O\Big(|R|^{-\frac{1}{9}}\Big).$$

Now, using (6.5) for the first relation below (and part of the second relation), we have that

$$\begin{aligned}
\sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ (A,R)=1}} \frac{\alpha_k(A)^2}{|A|} &= \prod_{\substack{\deg P\leq X\\ P\nmid R}} \left(\sum_{m=0}^{\infty} \frac{\alpha_k\big(P^m\big)^2}{|P|^m}\right) \\
&= \prod_{\substack{\deg P\leq X\\ P\nmid R}} \left(\sum_{m=0}^{\infty} \frac{d_k\big(P^m\big)^2}{|P|^m}\right) \prod_{\substack{\frac{X}{2}<\deg P\leq X\\ P\nmid R}} \left(\frac{1+\frac{d_k(P)^2}{|P|}+\sum_{m=2}^{\infty}\frac{\alpha_k\left(P^m\right)^2}{|P|^m}}{1+\frac{d_k(P)^2}{|P|}+\sum_{m=2}^{\infty}\frac{d_k\left(P^m\right)^2}{|P|^m}}\right) \\
&= \prod_{\substack{\deg P\leq X\\ P\mid R}} \left(\sum_{m=0}^{\infty} \frac{d_k\big(P^m\big)^2}{|P|^m}\right)^{-1} \prod_{\deg P\leq X} \left(\left(1-\frac{1}{|P|}\right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k\big(P^m\big)^2}{|P|^m}\right) \qquad (6.10)\\
&\quad \cdot \prod_{\deg P\leq X} \left(1-\frac{1}{|P|}\right)^{-k^2} \prod_{\frac{X}{2}<\deg P\leq X} \left(1+O_k\Big(\frac{1}{|P|^2}\Big)\right) \\
&= \big(1+o(1)\big)a(k) \prod_{\substack{\deg P\leq X\\ P\mid R}} \left(\sum_{m=0}^{\infty} \frac{d_k\big(P^m\big)^2}{|P|^m}\right)^{-1} \Big(e^{\gamma}X\Big)^{k^2}.
\end{aligned}$$

For the last equality, we used Lemma A.2.7. The proof follows. □

## 6.3 Moments of the Hadamard Product

In this section we provide support for the Conjecture 2.4.5, which we restate for ease of reference.

**Conjecture.** *We have*

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left|Z_X\Big(\frac{1}{2},\chi\Big)\right|^{2k} \sim \frac{G^2(k+1)}{G(2k+1)} \left(\frac{\deg R}{e^{\gamma}X}\right)^{k^2},$$

*as* $\deg R \longrightarrow \infty$*, where* $\gamma$ *is the Euler-Mascheroni constant and* $G$ *is the Barnes* $G$*-function. For our purposes, it suffices to note that*

$$\frac{G^2(k+1)}{G(2k+1)} = \prod_{i=0}^{k-1} \frac{i!}{(i+k)!}.$$

First we give a lemma.

**Lemma 6.3.1.** *For real $y > 0$ define*

$$\mathrm{Ci}(y) := -\int_{t=y}^{\infty} \frac{\cos(t)}{t} \mathrm{d}t,$$

*and let $x$ be real and non-zero. Then,*

$$\operatorname{Re} E_1(ix) = -\operatorname{Ci}(|x|).$$

*Proof.* If $x > 0$, then

$$\operatorname{Re} E_1(ix) = \operatorname{Re} \int_{w=ix}^{ix+\infty} \frac{e^{-w}}{w} \mathrm{d}w = \operatorname{Re} \int_{w=ix}^{i\infty} \frac{e^{-w}}{w} \mathrm{d}w = \operatorname{Re} \int_{t=x}^{\infty} \frac{e^{-it}}{t} \mathrm{d}t = -\operatorname{Ci}(|x|),$$

where the second relation follows from a contour shift. Similarly, if $x < 0$, then

$$\operatorname{Re} E_1(ix) = \operatorname{Re} \int_{w=ix}^{ix+\infty} \frac{e^{-w}}{w} \mathrm{d}w = \operatorname{Re} \int_{w=ix}^{-i\infty} \frac{e^{-w}}{w} \mathrm{d}w = \operatorname{Re} \int_{t=|x|}^{\infty} \frac{e^{it}}{t} \mathrm{d}t = -\operatorname{Ci}(|x|).$$

□

Now, writing $\gamma_n(\chi)$ for the imaginary part of the $n$-th zero of $L(s, \chi)$, we can see that

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \left| Z_X\left(\frac{1}{2}, \chi\right)\right|^{2k} \\
=&\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \exp\left( -2k \operatorname{Re} \sum_{\gamma_n(\chi)} U\Big( -i\gamma_n(\chi)(\log q)X\Big)\right) \\
=&\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \exp\left( -2k \operatorname{Re} \sum_{\gamma_n(\chi)} \int_{x=0}^{\infty} u(x) E_1\big( -i\gamma_n(\chi)(\log q)X \log x\big) \mathrm{d}x\right) \\
=&\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \exp\left( 2k \sum_{\gamma_n(\chi)} \int_{x=0}^{\infty} u(x) \operatorname{Ci}\big( |\gamma_n(\chi)|(\log q)X \log x\big) \mathrm{d}x\right).
\end{aligned} \tag{6.11}$$

We note that the terms in the exponential tend to zero as $|\gamma_n(\chi)|$ tends to infinity, and so the above is primarily concerned with the zeros close to $\frac{1}{2}$. As described in Section 1.5, there is a relationship between the zeros of Dirichlet $L$-functions near $\frac{1}{2}$ and the eigenphases of random unitary matrices near 0: The proportion of Dirichlet $L$-functions of modulus $R$ that have $j$-th zero (that is, its imaginary part) in some interval $[a, b]$ appears to be the same as the proportion of unitary $N(R) \times N(R)$ matrices that have $j$-th eigenphase in $[a, b]$ (at least, this is the case in an appropriate limit). Naturally, one asks what value $N(R)$ should take in terms of $R$. We note that the mean spacing between zeros of Dirichlet $L$-functions of modulus $R$ is $\frac{2\pi}{\log q \deg R}$, while the mean spacing between eigenphases of unitary $N \times N$ matrices is $\frac{2\pi}{N}$. Therefore, we take $N(R) = \lfloor \log q \deg R \rfloor$. So, by replacing the imaginary parts of the zeros with eigenphases of $N(R) \times N(R)$ unitary matrices, we conjecture that

$$\begin{aligned}&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R}\left|Z_X\left(\frac{1}{2},\chi\right)\right|^{2k}\\ =&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R}\exp\left(2k\sum_{\gamma_n(\chi)}\int_{x=0}^{\infty}u(x)\operatorname{Ci}\left(|\gamma_n(\chi)|(\log q)X\log x\right)\mathrm{d}x\right)\\ \sim&\int_{A\in U\left(N(R)\right)}\exp\left(2k\sum_{\theta_n(A)\in(-\pi,\pi]}\int_{x=0}^{\infty}u(x)\operatorname{Ci}\left(|\theta_n(A)|(\log q)X\log x\right)\mathrm{d}x\right)\mathrm{d}A\end{aligned} \tag{6.12}$$

as $\deg R \longrightarrow \infty$, where the integral is with respect to the Haar measure, and $\theta_n(A)$ is the $n$-th eigenphase of $A$. An asymptotic evaluation of the far RHS can be made identically as in Section 4 of [GHK07]; but we simply replace their $\log X$ with our $(\log q)X$, and we replace their $N = \lfloor \log T \rfloor$ with our $N(R) = \lfloor \log q \deg R \rfloor$. This leads us to the conjecture that

$$\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R}\left|Z_X\left(\frac{1}{2},\chi\right)\right|^{2k} \sim \frac{G^2(k+1)}{G(2k+1)}\left(\frac{\deg R}{e^{\gamma}X}\right)^{k^2}, \tag{6.13}$$

as $\deg R \longrightarrow \infty$. We note that in [GHK07], their $u(x)$ has a slightly different support than the support of our $u(x)$. However, this does not affect the result.

**Remark 6.3.2.** *As described in Remark 2.4.6, there is an error in [GHK07] that we must address. The problem begins in (6.12). We argued that the zeros close to $\frac{1}{2}$ behave similarly to the eigenphases near $0$, and used this to justify our interchange of* all *the zeros with the eigenvalues that lie in $(-\pi,\pi]$. This justification is concerning because it is by no means the case that* all *the zeros are close to $\frac{1}{2}$, and we are also dismissing the periodicised eigenvalues in $(-\infty,-\pi]\cup(\pi,\infty)$.*

*Now, the problem in [GHK07] is that, between their equations (20) and (21), they argue that the far RHS of our (6.12) is*

$$\begin{aligned}&\int_{A\in U\left(N(R)\right)}\exp\left(2k\sum_{\theta_n(A)\in(-\pi,\pi]}\int_{x=0}^{\infty}u(x)\operatorname{Ci}\left(|\theta_n(A)|(\log q)X\log x\right)\mathrm{d}x\right)\mathrm{d}A\\ \sim&\int_{A\in U\left(N(R)\right)}\exp\left(2k\sum_{\theta_n(A)\in(-\pi,\pi]}\int_{x=0}^{\infty}u(x)\left(\sum_{j=-\infty}^{\infty}\operatorname{Ci}\left(|\theta_n(A)+2\pi j|(\log q)X\log x\right)\right)\mathrm{d}x\right)\mathrm{d}A.\end{aligned} \tag{6.14}$$

*That is, the eigenphases are periodic with period $2\pi$, and they have included these periodicised eigenphases in the exponential sums without affecting the main term. Their justification for this is that for each $\theta_n(A)$ the contribution of the terms $j \neq 0$ is $\ll \frac{1}{(\log q)X}$ (see the proof of their Lemma 6). While this is correct for an individual $\theta_n(A)$, an error arises in that there are $N(R)$ number of $\theta_n(A)$, and this ultimately means that (6.14) does not hold. Indeed, we have*

$$\int_{A\in U\left(N(R)\right)}\exp\left(2k\sum_{\theta_n(A)\in(-\pi,\pi]}\int_{x=0}^{\infty}u(x)\operatorname{Ci}\left(|\theta_n(A)|(\log q)X\log x\right)\mathrm{d}x\right)\mathrm{d}A$$

$$\sim \int_{A\in U\left(N(R)\right)} \exp\left(2k \sum_{\theta_n(A)\in(-\pi,\pi]} \left[\int_{x=0}^{\infty} u(x)\Big(\sum_{j=-\infty}^{\infty} \mathrm{Ci}\left(|\theta_n(A)+2\pi j|(\log q)X\log x\right)\Big)\mathrm{d}x + O\Big(\frac{1}{(\log q)X}\Big)\right]\right)\mathrm{d}A$$

$$\sim \int_{A\in U\left(N(R)\right)} \exp\left(2k \sum_{\theta_n(A)\in(-\pi,\pi]} \int_{x=0}^{\infty} u(x)\Big(\sum_{j=-\infty}^{\infty}\mathrm{Ci}\left(|\theta_n(A)+2\pi j|(\log q)X\log x\right)\Big)\mathrm{d}x\right)\mathrm{d}A \cdot \exp\left(O\Big(\frac{kN(R)}{(\log q)X}\Big)\right)$$

$$\sim \frac{G^2(k+1)}{G(2k+1)}\left(\frac{\deg R}{e^{\gamma}X}\right)^{k^2} \exp\left(O\Big(\frac{kN(R)}{(\log q)X}\Big)\right).$$

*The first relation follows from the second-to-last equation in the proof of Lemma 6 in [GHK07], while the last relation follows from Theorem 4 in [GHK07]. As we can see, (6.14) is in contradiction to the third line above. Furthermore, the last line above is in contradiction with (6.13).*

*Nonetheless, (6.13) is ultimately correct because none of the errors described need have arisen. Indeed, we first dismissed the zeros in $(-\infty,-\pi]\cup(\pi,\infty)$ and later included the periodicised eigenphases in $(-\infty,-\pi]\cup(\pi,\infty)$, without correctly addressing the effect on the main term in both situations. Instead, we should argue that we can interchange the zeros in $(-\infty,-\pi]\cup(\pi,\infty)$ with the eigenphases in $(-\infty,-\pi]\cup(\pi,\infty)$. Of course, we mentioned previously that it is only the zeros and eigenphases near their respective central values that behave similarly to each other, and so this cannot justify this interchange. However, it is believed that, for a typical Dirichlet L-function in function fields and a typical unitary matrix, their respective zeros and eigenphases are somewhat equidistributed, and this could justify an interchange and avoid the problematic error terms. It is also important that the range $(-\infty,-\pi]\cup(\pi,\infty)$ avoids the discontinuity of $\mathrm{Ci}(x)$ at $x=0$, and this is why this "equidistribution approach" would not immediately work for the zeros near the central value.*

*Further justification for this is given in Sections 7.1 and 7.2 (see Remark 7.2.2).*

## 6.4 The Second Moment of the Hadamard Product

Before proving Theorem 2.4.7, we prove several lemmas. First, by (6.4) we have

$$P_X\Big(\frac{1}{2},\chi\Big) = \Big(1+O\big(X^{-1}\big)\Big)P_X^*\Big(\frac{1}{2},\chi\Big).$$

Rearranging and using (6.3) gives

$$\begin{aligned} P_X\Big(\frac{1}{2},\chi\Big)^{-1} =& \Big(1+O\big(X^{-1}\big)\Big)P_X^*\Big(\frac{1}{2},\chi\Big)^{-1} \\ =& \Big(1+O\big(X^{-1}\big)\Big)\prod_{\deg P\leq X}\Big(1-\frac{\chi(P)}{|P|^s}\Big)^{-1}\prod_{\frac{X}{2}<\deg P\leq X}\Big(1+\frac{\chi(P)^2}{2|P|^{2s}}\Big)^{-1} \\ =& \Big(1+O\big(X^{-1}\big)\Big)\sum_{A\in\mathcal{S}_{\mathcal{M}}(X)}\frac{\alpha_{-1}(A)\chi(A)}{|A|^{\frac{1}{2}}}, \end{aligned} \tag{6.15}$$

where $\alpha_{-1}$ is defined multiplicatively by

$$\begin{aligned} \alpha_{-1}(P) :=& \begin{cases} -1 & \text{if } \deg P\leq X \\ 0 & \text{if } \deg P > X; \end{cases} \\ \alpha_{-1}(P^2) :=& \begin{cases} 0 & \text{if } \deg P\leq \frac{X}{2} \\ \frac{1}{2} & \text{if } \frac{X}{2} < \deg P\leq X \\ 0 & \text{if } \deg P > X; \end{cases} \\ \alpha_{-1}(P^3) :=& \begin{cases} 0 & \text{if } \deg P\leq \frac{X}{2} \\ -\frac{1}{2} & \text{if } \frac{X}{2} < \deg P\leq X \\ 0 & \text{if } \deg P > X; \end{cases} \\ \alpha_{-1}(P^m) :=& \, 0 \text{ for } m\geq 4. \end{aligned}$$

**Lemma 6.4.1.** *For all $R\in\mathcal{M}$, we have that*

$$\sum_{\substack{HST\in\mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ (HST,R)=1 \\ \deg HS, \deg HT\leq\frac{1}{10}\deg R}}\frac{|\alpha_{-1}(HS)\alpha_{-1}(HT)|}{|HST|}\ll X^3$$

*as $X\longrightarrow\infty$.*

*Proof.* Using Lemma A.2.7, we have that

$$\sum_{\substack{HST\in\mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ (HST,R)=1 \\ \deg HS, \deg HT\leq\frac{1}{10}\deg R}}\frac{|\alpha_{-1}(HS)\alpha_{-1}(HT)|}{|HST|}\ll\Big(\sum_{H\in\mathcal{S}_{\mathcal{M}}(X)}\frac{1}{|H|}\Big)^3=\prod_{\deg P\leq X}\Big(1-|P|^{-1}\Big)^{-3}\ll X^3$$

as $X\longrightarrow\infty$. ∎

**Lemma 6.4.2.** *For all $R\in\mathcal{M}$, we have that*

$$\sum_{\substack{HST\in\mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ (HST,R)=1 \\ \deg HS, \deg HT\leq\frac{1}{10}\deg R}}\frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|}\deg ST\ll X^4$$

*as $X\longrightarrow\infty$.*

*Proof.* We have that

$$\sum_{\substack{HST \in \mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ (HST,R)=1 \\ \deg HS, \deg HT \leq \frac{1}{10} \deg R}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} \deg ST \ll \sum_{H \in \mathcal{S}_{\mathcal{M}}(X)} \frac{1}{|H|} \sum_{S,T \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\deg ST}{|ST|}.$$

Consider

$$f(s) := \sum_{S,T \in \mathcal{S}_{\mathcal{M}}(X)} \frac{1}{|ST|^s} = \Bigg( \sum_{T \in \mathcal{S}_{\mathcal{M}}(X)} \frac{1}{|T|^s} \Bigg)^2 = \prod_{\deg P \leq X} \Big( 1 - |P|^{-s} \Big)^{-2}.$$

Taking the derivative of the above and then evaluating at $s = 1$, we obtain

$$\sum_{S,T \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\deg ST}{|ST|} = 2 \prod_{\deg P \leq X} \Big( 1 - |P|^{-1} \Big)^{-2} \sum_{\deg P \leq X} \frac{\deg P}{|P| - 1} \ll X^3$$

as $X \longrightarrow \infty$, where we have made use of Lemma A.2.7 and the prime polynomial theorem. This, along with the fact that

$$\sum_{H \in \mathcal{S}_{\mathcal{M}}(X)} \frac{1}{|H|} = \prod_{\deg P \leq X} \Big( 1 - |P|^{-1} \Big)^{-1} \ll X$$

as $X \longrightarrow \infty$, proves the lemma. ☐

**Lemma 6.4.3.** *Let $V \in \mathcal{M}$. $V$ may or may not depend on $R$. As $X, \deg R \overset{q}{\longrightarrow} \infty$ with $X \leq \log_q \deg R$, we have*

$$\sum_{\substack{HST \in \mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ (HST,V)=1 \\ \deg HS, \deg HT \leq \frac{1}{10} \deg R}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|}$$

$$= \bigg( 1 + O\Big( q^{-\frac{X}{2}} \Big) \bigg) \prod_{\substack{\deg P \leq X \\ P \nmid V}} \bigg( 1 - \frac{1}{|P|} \bigg) + O\bigg( \frac{1}{|R|^{\frac{1}{21}}} \bigg) \sim \prod_{\substack{\deg P \leq X \\ P \nmid V}} \bigg( 1 - \frac{1}{|P|} \bigg).$$

*Proof.* The second relation in the Lemma follows easily from Lemma A.2.7. We will prove the first. In this proof, all asymptotic relations are to be taken as $X, \deg R \overset{q}{\longrightarrow} \infty$ with $X \leq \log_q \deg R$.

Similar to (6.7), we can remove the conditions $\deg HS, \deg HT \leq \frac{1}{10} \deg R$ from the sum and this only adds an $O\big(|R|^{-\frac{1}{21}}\big)$ term . Now, writing $C = HS$ and $D = HT$, we have

$$\sum_{\substack{HST \in \mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ (HST,V)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} = \sum_{\substack{CD \in \mathcal{S}_{\mathcal{M}}(X) \\ (CD,V)=1}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|CD|} |(C,D)|$$

$$= \sum_{\substack{CD \in \mathcal{S}_{\mathcal{M}}(X) \\ (CD,V)=1}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|CD|} \sum_{G | (C,D)} \phi(G) = \sum_{\substack{G \in \mathcal{S}_{\mathcal{M}}(X) \\ (G,V)=1}} \frac{\phi(G)}{|G|^2} \Bigg( \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ (C,V)=1}} \frac{\alpha_{-1}(CG)}{|C|} \Bigg)^2.$$

Before continuing, let us make a definition: For all $A \in \mathcal{M}$ and all $P \in \mathcal{P}$, let $e_P(A)$ be the largest integer such that $P^{e_P(A)} \mid A$. Continuing, we note that we can restrict the sums to polynomials that are fourth power free. Indeed, $\alpha_{-1}(P^m) = 0$ for all $P \in \mathcal{P}$ and all $m \geq 4$. Note that if $P \mid G$ then we must have that $0 \leq e_P(C) \leq 3 - e_P(G)$, while if $P \nmid G$ then $0 \leq e_P(C) \leq 3$. So, we have

$$\sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ (C,V)=1}} \frac{\alpha_{-1}(CG)}{|C|} = \prod_{P \mid G} \left( \sum_{j=0}^{3-e_P(G)} \frac{\alpha_{-1}(P^{j+e_P(G)})}{|P|^j} \right) \prod_{\substack{\deg P \leq X \\ P \nmid G \\ P \nmid V}} \left( \sum_{j=0}^{3} \frac{\alpha_{-1}(P^j)}{|P|^j} \right)$$

$$= \prod_{\substack{\deg P \leq X \\ P \nmid V}} \left( \sum_{j=0}^{3} \frac{\alpha_{-1}(P^j)}{|P|^j} \right) \prod_{P \mid G} \left( \sum_{j=0}^{3-e_P(G)} \frac{\alpha_{-1}(P^{j+e_P(G)})}{|P|^j} \right) \prod_{P \mid G} \left( \sum_{j=0}^{3} \frac{\alpha_{-1}(P^j)}{|P|^j} \right)^{-1}.$$

So,

$$\sum_{\substack{G \in \mathcal{S}_{\mathcal{M}}(X) \\ (G,V)=1}} \frac{\phi(G)}{|G|^2} \left( \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ (C,V)=1}} \frac{\alpha_{-1}(CG)}{|C|} \right)^2$$

$$= \prod_{\substack{\deg P \leq X \\ P \nmid V}} \left( \sum_{j=0}^{3} \frac{\alpha_{-1}(P^j)}{|P|^j} \right)^2 \prod_{\substack{\deg P \leq X \\ P \nmid V}} \left( \sum_{i=0}^{3} \frac{\phi(P^i)}{|P|^{2i}} \left( \sum_{j=0}^{3-i} \frac{\alpha_{-1}(P^{j+i})}{|P|^j} \right)^2 \left( \sum_{j=0}^{3} \frac{\alpha_{-1}(P^j)}{|P|^j} \right)^{-2} \right)$$

$$= \prod_{\substack{\deg P \leq X \\ P \nmid V}} \left( \sum_{i=0}^{3} \frac{\phi(P^i)}{|P|^{2i}} \left( \sum_{j=0}^{3-i} \frac{\alpha_{-1}(P^{j+i})}{|P|^j} \right)^2 \right)$$

$$= \prod_{\substack{\deg P \leq X \\ P \nmid V}} \left( \sum_{i=0}^{3} \sum_{j=0}^{3-i} \sum_{k=0}^{3-i} \frac{\phi(P^i)\alpha_{-1}(P^{j+i})\alpha_{-1}(P^{k+i})}{|P^{2i+j+k}|} \right)$$

$$= \prod_{\substack{\deg P \leq X \\ P \nmid V}} \left( 1 - \frac{1}{|P|} \right) \prod_{\substack{\frac{X}{2} \leq \deg P \leq X \\ P \nmid V}} \left( 1 + O\Big( \frac{1}{|P^2|} \Big) \right)$$

$$= \left( 1 + O\Big( q^{-\frac{X}{2}} \Big) \right) \prod_{\substack{\deg P \leq X \\ P \nmid V}} \left( 1 - \frac{1}{|P|} \right).$$

The result follows. □

**Lemma 6.4.4.** *Let $R \in \mathcal{M}$. Suppose $Z_1 \leq \deg R$ and $F \mid R$. Further, suppose $C, D \in \mathcal{S}_{\mathcal{M}}(X)$ with $\deg C, \deg D \leq \frac{1}{10} \deg R$. Then, we have*

$$\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = Z_1 \\ AC \equiv BD (\bmod F) \\ AC \neq BD \\ (AB,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{q^{\frac{Z_1}{2}}(Z_1+1)|CD|}{|F|}.$$

*Proof.* Consider the case where $\deg AC > \deg BD$, and suppose that $\deg A = i$. We have that $AC = LF + BD$ for some $L \in \mathcal{M}$ with $\deg L = \deg AC - \deg F =$

$i + \deg C - \deg F$, and $\deg B = Z_1 - \deg A = Z_1 - i$. Hence,

$$\begin{aligned}
&\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = Z_1 \\ AC \equiv BD (\mathrm{mod}\, F) \\ (AB,R)=1 \\ \deg AC > \deg BD}} \frac{1}{|AB|^{\frac{1}{2}}} \\
\leq & q^{-\frac{Z_1}{2}} \sum_{i=0}^{Z_1} \sum_{\substack{L \in \mathcal{M} \\ \deg L = i + \deg C - \deg F}} \sum_{\substack{B \in \mathcal{M} \\ \deg B = Z_1 - i}} 1 \\
= & q^{\frac{Z_1}{2}} \sum_{i=0}^{Z_1} \sum_{\substack{L \in \mathcal{M} \\ \deg L = i + \deg C - \deg F}} q^{-i} = \frac{q^{\frac{Z_1}{2}} |C|}{|F|} \sum_{i=0}^{Z_1} 1 = \frac{q^{\frac{Z_1}{2}} (Z_1 + 1)|C|}{|F|}.
\end{aligned}$$

Similarly, when $\deg BD > \deg AC$ we have

$$\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = Z_1 \\ AC \equiv BD (\mathrm{mod}\, F) \\ (AB,R)=1 \\ \deg AC > \deg BD}} \frac{1}{|AB|^{\frac{1}{2}}} \leq \frac{q^{\frac{Z_1}{2}} (Z_1 + 1)|D|}{|F|}.$$

Suppose now that $\deg AC = \deg BD = i$. Then, $2i = \deg ABCD = Z_1 + \deg CD$. We have $\deg B = i - \deg D = \frac{Z_1 + \deg C - \deg D}{2}$, and $AC = LF + BD$ for some $L \in \mathcal{A}$ with $\deg L < i - \deg F = \frac{Z_1 + \deg CD}{2} - \deg F$. Hence,

$$\begin{aligned}
\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = Z_1 \\ AC \equiv BD (\mathrm{mod}\, F) \\ (AB,R)=1 \\ \deg AC = \deg BD}} \frac{1}{|AB|^{\frac{1}{2}}} \leq & q^{-\frac{Z_1}{2}} \sum_{\substack{B \in \mathcal{M} \\ \deg B = \frac{Z_1 + \deg C - \deg D}{2}}} \sum_{\substack{L \in \mathcal{A} \\ \deg L < \frac{Z_1 + \deg CD}{2} - \deg F}} 1 \\
= & \frac{|CD|^{\frac{1}{2}}}{|F|} \sum_{\substack{B \in \mathcal{M} \\ \deg B = \frac{Z_1 + \deg C - \deg D}{2}}} 1 = \frac{q^{\frac{Z_1}{2}} |C|}{|F|}.
\end{aligned}$$

The result follows. □

We can now prove Theorem 2.4.7, which we restate for ease of reference.

**Theorem.** *We have that*

$$\begin{aligned}
\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| Z_X\left(\frac{1}{2}, \chi\right) \right|^2 = & \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2}, \chi\right) P_X\left(\frac{1}{2}, \chi\right)^{-1} \right|^2 \\
\sim & \frac{\deg R}{e^{\gamma} X} \prod_{\substack{\deg P > X \\ P | R}} \left( 1 - \frac{1}{|P|} \right)
\end{aligned}$$

*as* $X, \deg R \longrightarrow \infty$ *with* $X \leq \log_q \deg R$.

*Proof of Theorem 2.4.7.* Throughout the proof, all asymptotic relations will be taken as $X, \deg R \overset{q}{\longrightarrow} \infty$ with $X \leq \log_q \deg R$. Now, by (6.15), we have

$$\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2}, \chi\right) P_X\left(\frac{1}{2}, \chi\right)^{-1} \right|^2 \sim \frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| L\left(\frac{1}{2}, \chi\right) P_X^*\left(\frac{1}{2}, \chi\right)^{-1} \right|^2. \tag{6.16}$$

Similar to (6.7), we truncate our sum:

$$P_X^*\Big(\frac{1}{2},\chi\Big)^{-1} = \sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg C\leq\frac{1}{10}\deg R}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} + O\Big(|R|^{-\frac{1}{50}}\Big).$$

Using this, the Cauchy-Schwarz inequality, and Theorem 2.2.1, it suffices to prove that

$$\begin{aligned}&\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\Big|L\Big(\frac{1}{2},\chi\Big)\Big|^2 \sum_{\substack{C,D\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \\ \sim&\frac{\deg R}{e^{\gamma}X}\prod_{\substack{\deg P>X \\ P|R}}\Big(1-\frac{1}{|P|}\Big).\end{aligned} \tag{6.17}$$

Now, by Lemmas A.1.2 and A.1.3, we have

$$\begin{aligned}&\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\Big|L\Big(\frac{1}{2},\chi\Big)\Big|^2 \sum_{\substack{C,D\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \\ =&\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\Big(a(\chi)+c(\chi)\Big) \sum_{\substack{C,D\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}},\end{aligned}$$

where

$$a(\chi) := 2\sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}$$

and $c(\chi)$ is defined as in (A.8).

We first consider the case with $a(\chi)$. We have

$$\begin{aligned}
&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R} a(\chi) \sum_{\substack{C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C,\deg D\leq\frac{1}{10}\deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}}\\
=&\frac{2}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R} \sum_{\substack{A,B\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(AC)\overline{\chi}(BD)}{|ABCD|^{\frac{1}{2}}}\\
=&\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F) \sum_{\substack{A,B\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (ABCD,R)=1\\ AC\equiv BD(\bmod F)}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}\\
=&2\sum_{\substack{A,B\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (ABCD,R)=1\\ AC=BD}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}\\
&+\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F) \sum_{\substack{A,B\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (ABCD,R)=1\\ AC\equiv BD(\bmod F)\\ AC\neq BD}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}.
\end{aligned} \tag{6.18}$$

For the first term on the far RHS, the diagonal terms, we can write $A = GS$, $B = GT$, $C = HT$, $D = HS$ where $G, H, S, T \in \mathcal{M}$ and $(S, T) = 1$, giving

$$\begin{aligned}
&2\sum_{\substack{A,B,C,D\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (ABCD,R)=1\\ AC=BD}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}\\
=&2\sum_{\substack{G\in\mathcal{M}\\ H,S,T\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G^2ST<\deg R\\ \deg HS,\deg HT\leq\frac{1}{10}\deg R\\ (GHST,R)=1\\ (S,T)=1}} \frac{\alpha_{-1}(HT)\alpha_{-1}(HS)}{|GHST|}\\
=&2\sum_{\substack{H,S,T\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg HS,\deg HT\leq\frac{1}{10}\deg R\\ (HST,R)=1\\ (S,T)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|}\sum_{\substack{G\in\mathcal{M}\\ \deg G\leq\frac{\deg R-\deg ST}{2}\\ (G,R)=1}}\frac{1}{|G|}.
\end{aligned} \tag{6.19}$$

By Corollary A.3.3 and Lemmas 6.4.1, 6.4.2, and 6.4.3 we obtain the asymptotic relation below. The final equality uses Lemma A.2.7.

$$\begin{aligned}2\sum_{\substack{A,B\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (ABCD,R)=1\\ AC=BD}}\frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}\sim&\frac{\phi(R)}{|R|}\deg R\prod_{\substack{\deg P\leq X\\ P\nmid R}}\left(1-\frac{1}{|P|}\right)\\ \sim&\frac{\deg R}{e^{\gamma}X}\prod_{\substack{\deg P>X\\ P\mid R}}\left(1-\frac{1}{|P|}\right).\end{aligned} \tag{6.20}$$

For the second term on the far RHS of (6.18), the off-diagonal terms, we use Lemma 6.4.4 to obtain

$$\begin{aligned}&\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB<\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (ABCD,R)=1\\ AC\equiv BD(\operatorname{mod}F)\\ AC\neq BD}}\frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}\\ =&\frac{2}{\phi^*(R)}\sum_{\substack{C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C,\deg D\leq\frac{1}{10}\deg R\\ (CD,R)=1}}\frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|CD|^{\frac{1}{2}}}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB<\deg R\\ (AB,R)=1\\ AC\equiv BD(\operatorname{mod}F)\\ AC\neq BD}}\frac{1}{|AB|^{\frac{1}{2}}}\\ \ll&\frac{|R|^{\frac{1}{2}}\deg R}{\phi^*(R)}\sum_{\substack{C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C,\deg D\leq\frac{1}{10}\deg R}}|CD|^{\frac{1}{2}}\sum_{EF=R}|\mu(E)|\frac{\phi(F)}{|F|}\\ \ll&\frac{|R|^{\frac{4}{5}}2^{\omega(R)}\deg R}{\phi^*(R)}=o(1).\end{aligned} \tag{6.21}$$

Finally, consider the case with $c(\chi)$. We recall that if $\chi$ is odd then it consists of one sum, whereas, if $\chi$ is even it consists of three sums. We will show that one of the sums for the even $\chi$ is of lower order. The other sums for the even $\chi$, and the odd $\chi$, are similar. We then see that the total contribution of the case with $c(\chi)$ is of lower order. We have

$$\begin{aligned}&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\substack{\chi\operatorname{mod}R\\ \chi\text{ even}}}\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}\sum_{\substack{C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C,\deg D\leq\frac{1}{10}\deg R}}\frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}}\\ \leq&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi\operatorname{mod}R}\sum_{\substack{A,B,C,D\in\mathcal{M}\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg AB=\deg R\\ \deg C,\deg D\leq\frac{1}{10}\deg R}}\frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(AC)\overline{\chi}(BD)}{|ABCD|^{\frac{1}{2}}}\ll X^3,\end{aligned} \tag{6.22}$$

where the last relation follows by similar means as the case with $a(\chi)$. □

## 6.5 Preliminary Results for the Fourth Hadamard Moment

In this section we develop the preliminary results that are required for the proof of Theorem 2.4.8. We begin with two results that will simplify the problem.

**Lemma 6.5.1.** *For $X \geq 12$, we have that*

$$P_X\Big(\frac{1}{2},\chi\Big)^{-2} = \big(1+O(X^{-1})\big)P_X{}^{**}\Big(\frac{1}{2},\chi\Big),$$

*where*

$$P_X^{**}\Big(\frac{1}{2},\chi\Big) := \sum_{A\in\mathcal{S}_{\mathcal{M}}(X)} \frac{\beta(A)\chi(A)}{|A|^{\frac{1}{2}}}$$

*and $\beta$ is defined multiplicatively by*

$$\begin{aligned}
&\beta(P) := \begin{cases} -2 & \text{if } \deg P \leq X \\ 0 & \text{if } \deg P > X \end{cases}\\
&\beta(P^2) := \begin{cases} 1 & \text{if } \deg P \leq \frac{X}{2} \\ 2 & \text{if } \frac{X}{2} < \deg P \leq X \\ 0 & \text{if } \deg P > X \end{cases}\\
&\beta(P^k) := 0 \text{ for } k \geq 3.
\end{aligned} \tag{6.23}$$

*Proof.* By Lemma 6.2.1 we have

$$P_X\Big(\frac{1}{2},\chi\Big)^{-2} = \big(1+O(X^{-1})\big)\prod_{\deg P\leq X}\bigg(1-\frac{\chi(P)}{|P|^{\frac{1}{2}}}\bigg)^2\prod_{\frac{X}{2}<\deg P\leq X}\bigg(1+\frac{\chi(P)^2}{2|P|}\bigg)^2.$$

By writing $P_X^{**}\Big(\frac{1}{2},\chi\Big)$ as an Euler product, we see that

$$\begin{aligned}
&\prod_{\deg P\leq X}\bigg(1-\frac{\chi(P)}{|P|^{\frac{1}{2}}}\bigg)^2\prod_{\frac{X}{2}<\deg P\leq X}\bigg(1+\frac{\chi(P)^2}{2|P|}\bigg)^2\\
=&P_X^{**}\Big(\frac{1}{2},\chi\Big)\prod_{\frac{X}{2}<\deg P\leq X}\left(1+\frac{-\frac{2\chi(P)^3}{|P|^{\frac{3}{2}}}+\frac{5\chi(P)^4}{4|P|^2}-\frac{\chi(P)^5}{2|P|^{\frac{5}{2}}}+\frac{\chi(P)^6}{4|P|^6}}{1-\frac{2\chi(P)}{|P|^{\frac{1}{2}}}+\frac{2\chi(P)^2}{|P|}}\right)\\
=&P_X^{**}\Big(\frac{1}{2},\chi\Big)\prod_{\frac{X}{2}<\deg P\leq X}\Big(1+O\Big(|P|^{-\frac{3}{2}}\Big)\Big)\\
=&P_X^{**}\Big(\frac{1}{2},\chi\Big)\exp\bigg(O\bigg(\sum_{\frac{X}{2}<\deg P\leq X}|P|^{-\frac{3}{2}}\bigg)\bigg)\\
=&\Big(1+O\big(X^{-1}q^{-\frac{X}{4}}\big)\Big)P_X^{**}\Big(\frac{1}{2},\chi\Big).
\end{aligned}$$

The result follows. The requirement that $X \geq 12$ is so that the factor $\bigg(1-\frac{2\chi(P)}{|P|^{\frac{1}{2}}}+\frac{2\chi(P)^2}{|P|}\bigg)^{-1}$ in the second line is guaranteed to be non-zero. □

**Lemma 6.5.2.** *We define*

$$\widehat{P_x^{**}}\Big(\frac{1}{2},\chi\Big) := \sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A\leq\frac{1}{8}\log_q\deg R}} \frac{\beta(A)\chi(A)}{|A|^{\frac{1}{2}}}.$$

*Then, as* $X,\deg R\xrightarrow{q}\infty$ *with* $X\leq\log_q\deg R$*,*

$$P_X^{**}\Big(\frac{1}{2},\chi\Big) = \widehat{P_x^{**}}\Big(\frac{1}{2},\chi\Big) + O\Big((\deg R)^{-\frac{1}{33}}\Big).$$

*Proof.* We have, as $X,\deg R\xrightarrow{q}\infty$ with $X\leq\log_q\deg R$,

$$\begin{aligned}
&\sum_{\substack{A\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A>\frac{1}{8}\log_q\deg R}} \frac{\beta(A)\chi(A)}{|A|^{\frac{1}{2}}} \ll \frac{1}{(\deg R)^{\frac{1}{32}}}\sum_{A\in\mathcal{S}_{\mathcal{M}}(X)}\frac{|\beta(A)|}{|A|^{\frac{1}{4}}}\\
=&(\deg R)^{-\frac{1}{32}}\prod_{\deg P\leq X}\Big(1+2|P|^{-\frac{1}{4}}+2|P|^{-\frac{1}{2}}\Big)\\
=&(\deg R)^{-\frac{1}{32}}\exp\left(O\Big(\sum_{\deg P\leq X}|P|^{-\frac{1}{4}}\Big)\right)\\
=&(\deg R)^{-\frac{1}{32}}\exp\left(O\Big(\frac{q^{\frac{3}{4}X}}{X}\Big)\right)\leq(\deg R)^{-\frac{1}{33}}.
\end{aligned}$$

□

We now prove several results that will be used to obtain the main asymptotic term in Theorem 2.4.8.

**Lemma 6.5.3.** *Suppose* $A_1,A_2,A_3,B_1,B_2,B_3\in\mathcal{M}$ *satisfy* $A_1A_2A_3=B_1B_2B_3$*. Then, there are* $G_1,G_2,G_3,V_{1,2},V_{1,3},V_{2,1},V_{2,3},V_{3,1},V_{3,2}\in\mathcal{M}$*, satisfying* $(V_{i,j},V_{k,l})=1$ *when both* $i\neq k$ *and* $j\neq l$ *hold, such that*

$$\begin{aligned}
A_1&=G_1V_{1,2}V_{1,3} & B_1&=G_1V_{2,1}V_{3,1}\\
A_2&=G_2V_{2,1}V_{2,3} & B_2&=G_2V_{1,2}V_{3,2}\\
A_3&=G_3V_{3,1}V_{3,2} & B_3&=G_3V_{1,3}V_{2,3}.
\end{aligned}$$

*Furthermore, this is a bijective correspondence. To clarify,* $G_i$ *is the highest common divisor of* $A_i$ *and* $B_i$*; and in* $V_{i,j}$ *the subscript* $i$ *indicates that* $V_{i,j}$ *divides* $A_i$ *and the subscript* $j$ *indicates that* $V_{i,j}$ *divides* $B_j$*.*

*Proof.* Let us write $A_i=G_iS_i$ and $B_i=G_iT_i$, where

$$\begin{aligned}G_i&=(A_i,B_i)\\ (S_i,T_i)&=1.\end{aligned}\tag{6.24}$$

Since $A_1A_2A_3=B_1B_2B_3$, we must have that

$$S_1S_2S_3=T_1T_2T_3.\tag{6.25}$$

First we note that, due to (6.25) and the coprimality relations in (6.24), we have that $S_i \mid T_j T_k$ and $T_i \mid S_j S_k$ for $i, j, k$ distinct.

Second, again due to to (6.25) and (6.24), we must have that $(S_1, S_2, S_3), (T_1, T_2, T_3) = 1$.

Third, for $i \neq j$, we define $S_{i,j} := (S_i, S_j)$ and $T_{i,j} := (T_i, T_j)$. Again due to to (6.25) and (6.24), we have $(S_{i,j})^2 \mid T_k$ and $(T_{i,j})^2 \mid S_k$ for $i, j, k$ distinct. Furthermore, $(S_{i_1,j_1}, S_{i_2,j_2}) = 1$ and $(T_{i_1,j_1}, T_{i_2,j_2}) = 1$ for all $\{i_1, j_1\} \neq \{i_2, j_2\}$, and $(S_{i_1,j_1}, T_{i_2,j_2}) = 1$ for all $i_1, j_1, i_2, j_2$.

From these three points we can deduce that

$$\begin{aligned} S_1 &= S_{1,2}S_{1,3}(T_{2,3})^2 S_1{}' \qquad & T_1 &= T_{1,2}T_{1,3}(S_{2,3})^2 T_1{}' \\ S_2 &= S_{1,2}S_{2,3}(T_{1,3})^2 S_2{}' \qquad & T_2 &= T_{1,2}T_{2,3}(S_{1,3})^2 T_2{}' \\ S_3 &= S_{1,3}S_{2,3}(T_{1,2})^2 S_3{}' \qquad & T_3 &= T_{1,3}T_{2,3}(S_{1,2})^2 T_3{}' \end{aligned}$$

for some $S_i{}'$ and $T_i{}'$ satisfying $(S_i{}', T_i{}') = 1$ for all $i$ and $(S_i{}', S_j{}'), (T_i{}', T_j{}') = 1$ for $i \neq j$. By (6.25) we have that $S_1{}'S_2{}'S_3{}' = T_1{}'T_2{}'T_3{}'$. From these points we can deduce that

$$\begin{aligned} S_1{}' &= U_{1,2}U_{1,3} \qquad & T_1{}' &= U_{2,1}U_{3,1} \\ S_2{}' &= U_{2,1}U_{2,3} \qquad & T_2{}' &= U_{1,2}U_{3,2} \\ S_3{}' &= U_{3,1}U_{3,2} \qquad & T_3{}' &= U_{1,3}U_{2,3} \end{aligned}$$

where the $U_{i,j}$ are pairwise coprime. Also, for $i, j, k$ distinct, because $U_{i,j} \mid T_j$ and $(S_j, T_j) = 1$, we have that $(U_{i,j}, S_j) = 1$, and hence $(U_{i,j}, S_{j,k}), (U_{i,j}, S_{j,i}) = 1$. Similarly, for $i, j, k$ distinct, we have $(U_{i,j}, T_{i,k}), (U_{i,j}, T_{i,j}) = 1$.

So, by defining

$$\begin{aligned} V_{1,2} &= S_{1,3}T_{2,3}U_{1,2} \qquad & V_{2,1} &= S_{2,3}T_{1,3}U_{2,1} \qquad & V_{3,1} &= S_{2,3}T_{1,2}U_{3,1} \\ V_{1,3} &= S_{1,2}T_{2,3}U_{1,3} \qquad & V_{2,3} &= S_{1,2}T_{1,3}U_{2,3} \qquad & V_{3,2} &= S_{1,3}T_{1,2}U_{3,2} \end{aligned}$$

we complete the proof for the existence claim.

Uniqueness follows from the following observation: If we have $G_i$ and $V_{i,j}$ satisfying the conditions in the Lemma, then we can deduce

$$\begin{aligned} G_i &= (A_i, B_i) \quad \text{for all } i, \text{ and} \\ V_{i,j} &= \Big(V_{i,j}V_{k,j}, \frac{V_{i,j}V_{k,j}V_{j,i}V_{k,i}}{V_{k,i}V_{k,j}}\Big) = \Big(\hat{B}_j, \frac{\hat{B}_i\hat{B}_j}{\hat{A}_k}\Big) \quad \text{for } i, j, k \text{ distinct,} \end{aligned}$$

where we define $\hat{B}_i, \hat{A}_i$ by $B_i = G_i\hat{B}_i = (A_i, B_i)\hat{B}_i$ and $A_i = G_i\hat{A}_i = (A_i, B_i)\hat{A}_i$ for all $i$. Since the far RHS of each line above is expressed entirely in terms of $A_1, A_2, A_3, B_1, B_2, B_3$, we must have uniqueness. □

**Lemma 6.5.4.** *Suppose* $V_{1,3}, V_{2,3}, V_{3,1}, V_{3,2} \in \mathcal{M}$, *and* $(V_{1,3}, V_{3,1}V_{3,2}) = 1$ *and* $(V_{2,3}, V_{3,1}V_{3,2}) = 1$. *Then,*

$$\Big\{(V_{1,2}, V_{2,1}) \in \mathcal{M}^2 : (V_{1,2}, V_{2,3}V_{3,1}) = 1, (V_{2,1}, V_{1,3}V_{3,2}) = 1, (V_{1,2}, V_{2,1}) = 1\Big\}$$

$$= \bigcup_{\substack{V \in \mathcal{M} \\ \big(V,(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\big)=1}} \Big\{(V_{1,2},V_{2,1}) \in \mathcal{M}^2 : V_{1,2}V_{2,1} = V, (V_{1,2},V_{2,3}V_{3,1}) = 1, (V_{2,1},V_{1,3}V_{3,2}) = 1, (V_{1,2},V_{2,1}) = 1\Big\},$$

*and for each such $V$ we have*

$$\#\Big\{(V_{1,2},V_{2,1}) \in \mathcal{M}^2 : V_{1,2}V_{2,1} = V, (V_{1,2},V_{2,3}V_{3,1}) = 1, (V_{2,1},V_{1,3}V_{3,2}) = 1, (V_{1,2},V_{2,1}) = 1\Big\} = 2^{\omega(V)-\omega\Big(\big(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\big)\Big)}.$$

*Proof.* For the first claim we note that $(V_{1,2},V_{2,3}V_{3,1}) = 1$ and $(V_{2,1},V_{1,3}V_{3,2}) = 1$ imply that

$$\Big(V,(V_{1,3},V_{2,3})\cdot(V_{3,1},V_{3,2})\Big) = 1,$$

and, due to the given coprimality relations of $V_{1,3}$,$V_{2,3}$,$V_{3,1}$, and $V_{3,2}$ given in Lemma 6.5.3, we have

$$(V_{1,3},V_{2,3})\cdot(V_{3,1},V_{3,2}) = (V_{1,3}V_{3,1},V_{2,3}V_{3,2}).$$

The first claim follows.

We now look at the second claim. For $A,B \in \mathcal{M}$, we define $A_B$ to be the maximal divisor of $A$ that is coprime to $B$, and we define $A^B$ by $A = A_B A^B$. We then have that

$$V = V_{V_{1,3}V_{2,3}V_{3,1}V_{3,2}} V^{V_{1,3}V_{2,3}V_{3,1}V_{3,2}} = V_{V_{1,3}V_{2,3}V_{3,1}V_{3,2}} V^{V_{1,3}} V^{V_{2,3}} V^{V_{3,1}} V^{V_{3,2}},$$

where the last equality follows from $\big(V,(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\big) = 1$ and the fact that $(V_{1,3},V_{3,1}) = 1$ and $(V_{2,3},V_{3,2}) = 1$. Now, $V = V_{1,2}V_{2,1}$ and by the coprimality relations we must have that $V^{V_{1,3}}V^{V_{3,2}} \mid V_{1,2}$ and $V^{V_{2,3}}V^{V_{3,1}} \mid V_{2,1}$. So, we see that

$$\begin{aligned}
&\#\Big\{(V_{1,2},V_{2,1}) \in \mathcal{M}^2 : V_{1,2}V_{2,1} = V, (V_{1,2},V_{2,3}V_{3,1}) = 1, (V_{2,1},V_{1,3}V_{3,2}) = 1, (V_{1,2},V_{2,1}) = 1\Big\} \\
=&\#\Big\{(V_{1,2},V_{2,1}) \in \mathcal{M}^2 : V_{1,2}V_{2,1} = V_{V_{1,3}V_{2,3}V_{3,1}V_{3,2}} V^{V_{1,3}} V^{V_{2,3}} V^{V_{3,1}} V^{V_{3,2}}, \\
&\qquad V^{V_{1,3}}V^{V_{3,2}} \mid V_{1,2} \;,\; V^{V_{2,3}}V^{V_{3,1}} \mid V_{2,1} \;,\; (V_{1,2},V_{2,1}) = 1\Big\} \\
=&2^{\omega\big(V_{V_{1,3}V_{2,3}V_{3,1}V_{3,2}}\big)} = 2^{\omega(V)-\omega\Big(\big(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\big)\Big)}.
\end{aligned}$$

□

**Lemma 6.5.5.** *For all $R \in \mathcal{M}$ with $\deg R \geq 1$, non-negative integers $k$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -1$ we define*

$$f_{R,k}(s) := \prod_{P \mid R} \Big(1 - |P|^{-s-1}\Big)^k,$$

$$h_{R,k}(s) := \prod_{P|R} \Big( 1 + |P|^{-s-1} \Big)^{-k}.$$

*Then, for all non-negative integers $j$ and all integers $r$ we have*

$$f^{(j)}_{R,k}\Big(\frac{2r\pi i}{\log q}\Big) \ll_j k^j \big( \log_q \deg R + O(1) \big)^j \prod_{P|R} \Big( 1 - |P|^{-1} \Big)^k,$$

$$h^{(j)}_{R,k}\Big(\frac{2r\pi i}{\log q}\Big) \ll_j k^j \big( \log_q \deg R + O(1) \big)^j \prod_{P|R} \Big( 1 + |P|^{-1} \Big)^{-k}.$$

*Generally, we could incorporate the $O(1)$ terms into the relation $\ll_j$, but for the case $\deg R = 1$, where we would have $\log_q \deg R = 0$, the $O(1)$ terms are required.*

*Proof.* We will prove only the claim for $f_{R,k}(s)$ and $r = 0$. The proofs for all $r$ and $h_{R,k}(s)$ are almost identical. First, we note that

$$f'_{R,k}(s) = k \, g_R(s) f_{R,k}(s), \tag{6.26}$$

where

$$g_R(s) := \sum_{P|R} \frac{\log|P|}{|P|^{s+1} - 1}.$$

We note further that, for integers $j \geq 1$,

$$f^{(j)}_{R,k}(s) = G_{R,k,j}(s) f_{R,k}(s), \tag{6.27}$$

where $G_{R,k,j}(s)$ is a sum of terms of the form

$$k^m g_R^{(j_1)}(s) \, g_R^{(j_2)}(s) \ \ldots \ g_R^{(j_m)}(s), \tag{6.28}$$

where $1 \leq m \leq j$ and $\sum_{r=1}^{m} (j_r + 1) = j$. The number of such terms and their coefficients are dependent only on $j$.

Now, for all $R \in \mathcal{M}$, and non-negative integers $l$, it is not difficult to deduce that

$$g_R^{(l)}(0) \ll_l \sum_{P|R} \frac{\big( \log|P| \big)^{l+1}}{|P| - 1}. \tag{6.29}$$

The function $\frac{(\log x)^{l+1}}{x-1}$ is decreasing at large enough $x$, and the limit as $x \longrightarrow \infty$ is 0. Therefore, there exists a constant $c_l > 0$ such that for all $A, B \in \mathcal{A}$ with $1 \leq \deg A \leq \deg B$ we have that

$$c_l \frac{\big( \log|A| \big)^{l+1}}{|A| - 1} \geq \frac{\big( \log|B| \big)^{l+1}}{|B| - 1}.$$

Hence, taking $n = \omega(R)$ and using Definition A.2.1, Lemma A.2.2, and the prime polynomial theorem, we see that

$$\sum_{P|R} \frac{\big( \log|P| \big)^{l+1}}{|P| - 1} \ll_l \sum_{P|R_n} \frac{\big( \log|P| \big)^{l+1}}{|P| - 1} \ll \sum_{r=1}^{m_n+1} \frac{q^r}{r} \frac{r^{l+1}}{q^r - 1} \ll \sum_{r=1}^{m_n+1} r^l \ll (m_n + 1)^{l+1}$$
$$\ll \big( \log_q \log_q |R_n| + O(1) \big)^{l+1} \ll \big( \log_q \deg R_n + O(1) \big)^{l+1} \ll \big( \log_q \deg R + O(1) \big)^{l+1}. \tag{6.30}$$

The result follows by (6.27), (6.28), (6.29), and (6.30) . ☐

**Lemma 6.5.6.** *Let $R, M \in \mathcal{M}$ with $\deg M \leq \deg R$, $k$ be a non-negative integer, and $z$ be an integer-valued function of $R$ such that $z \sim \deg R$ as $\deg R \longrightarrow \infty$. We have that*

$$\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ (N,R)=1}} \frac{2^{\omega(N)-\omega\big((N,M)\big)}}{|N|}(z-\deg N)^k$$
$$=\frac{(1-q^{-1})}{(k+2)(k+1)} \prod_{P \mid MR} \left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right) \prod_{\substack{P \mid M \\ P \nmid R}} \left(\frac{1}{1-|P|^{-1}}\right) \Big(z^{k+2} + O_k\big(z^{k+1} \log \deg R\big)\Big)$$

*as* $\deg R \longrightarrow \infty$.

*Proof.* **Step 1:** Let us define the function $F$, for $\operatorname{Re} s > 1$, by

$$F(s) = \sum_{\substack{N \in \mathcal{M} \\ (N,R)=1}} \frac{2^{\omega(N)-\omega\big((N,M)\big)}}{|N|^s}.$$

We can see that

$$\begin{aligned} F(s) &= \prod_{P \nmid MR} \left(1 + \frac{2}{|P|^s} + \frac{2}{|P|^{2s}} + \dots\right) \prod_{\substack{P \mid M \\ P \nmid R}} \left(1 + \frac{1}{|P|^s} + \frac{1}{|P|^{2s}} + \dots\right) \\ &= \prod_{P \nmid MR} \left(\frac{2}{1-|P|^{-s}} - 1\right) \prod_{\substack{P \mid M \\ P \nmid R}} \left(\frac{1}{1-|P|^{-s}}\right) \\ &= \prod_{P \in \mathcal{P}} \left(\frac{1+|P|^{-s}}{1-|P|^{-s}}\right) \prod_{P \mid MR} \left(\frac{1-|P|^{-s}}{1+|P|^{-s}}\right) \prod_{\substack{P \mid M \\ P \nmid R}} \left(\frac{1}{1-|P|^{-s}}\right) \\ &= \frac{\zeta_{\mathcal{A}}(s)^2}{\zeta_{\mathcal{A}}(2s)} \prod_{P \mid MR} \left(\frac{1-|P|^{-s}}{1+|P|^{-s}}\right) \prod_{\substack{P \mid M \\ P \nmid R}} \left(\frac{1}{1-|P|^{-s}}\right). \end{aligned}$$

Now, let $c$ be a positive real number, and define

$$y := \begin{cases} q^{z+\frac{1}{2}} & \text{if } k = 0 \\ q^z & \text{if } k \neq 0. \end{cases}$$

On the one hand, we have that

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(1+s) \frac{y^s}{s^{k+1}} \mathrm{d}s &= \frac{1}{2\pi i} \sum_{\substack{N \in \mathcal{M} \\ (N,R)=1}} \frac{2^{\omega(N)-\omega\big((N,M)\big)}}{|N|} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{|N|^s s^{k+1}} \mathrm{d}s \\ &= \frac{(\log q)^k}{k!} \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ (N,R)=1}} \frac{2^{\omega(N)-\omega\big((N,M)\big)}}{|N|} \big(z - \deg N\big)^k. \end{aligned} \tag{6.31}$$

For $k \geq 1$, the interchange of integral and summation is justified by absolute convergence, and the second equality follows by Lemma 4.2.1. For $k = 0$, the above holds by Lemma 6.5.7 below. We remark that we take $y = q^{z+\frac{1}{2}}$ when $k = 0$ so that $\left(\frac{y}{|N|}, k+1\right) \neq (1,1)$, which would be a special case of Lemma 4.2.1 that would be tedious to address.

On the other hand, for all positive integers $n$ define the following curves:

$$\begin{aligned}
l_1(n) :=& \left[c - \frac{\left(2n+\frac{1}{2}\right)\pi i}{\log q}, c + \frac{\left(2n+\frac{1}{2}\right)\pi i}{\log q}\right];\\
l_2(n) :=& \left[c + \frac{\left(2n+\frac{1}{2}\right)\pi i}{\log q}, -\frac{1}{4} + \frac{\left(2n+\frac{1}{2}\right)\pi i}{\log q}\right];\\
l_3(n) :=& \left[-\frac{1}{4} + \frac{\left(2n+\frac{1}{2}\right)\pi i}{\log q}, -\frac{1}{4} - \frac{\left(2n+\frac{1}{2}\right)\pi i}{\log q}\right];\\
l_4(n) :=& \left[-\frac{1}{4} - \frac{\left(2n+\frac{1}{2}\right)\pi i}{\log q}, c - \frac{\left(2n+\frac{1}{2}\right)\pi i}{\log q}\right];\\
L(n) :=& l_1(n) \cup l_2(n) \cup l_3(n) \cup l_4(n).
\end{aligned}$$

Then, we have that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(1+s)\frac{y^s}{s^{k+1}}\mathrm{d}s\\
=&\frac{1}{2\pi i}\lim_{n\to\infty}\Bigg(\int_{L(n)} F(1+s)\frac{y^s}{s^{k+1}}\mathrm{d}s - \int_{l_2(n)} F(1+s)\frac{y^s}{s^{k+1}}\mathrm{d}s \\
&\qquad - \int_{l_3(n)} F(1+s)\frac{y^s}{s^{k+1}}\mathrm{d}s - \int_{l_4(n)} F(1+s)\frac{y^s}{s^{k+1}}\mathrm{d}s\Bigg).
\end{aligned} \tag{6.32}$$

**Step 2:** For the first integral in (6.32) we note that $F(1+s)\frac{y^s}{s^{k+1}}$ has a pole at $s=0$ of order $k+3$ and double poles at $s = \frac{2m\pi i}{\log q}$ for $m = \pm 1, \pm 2, \ldots, \pm n$. By applying the residue theorem we see that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{2\pi i}\int_{L(n)} F(1+s)\frac{y^s}{s^{k+1}}\mathrm{d}s\\
=&\operatorname{Res}_{s=0} F(s+1)\frac{y^s}{s^{k+1}} + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\operatorname{Res}_{s=\frac{2m\pi i}{\log q}} F(1+s)\frac{y^s}{s^{k+1}}.
\end{aligned} \tag{6.33}$$

**Step 2.1:** For the first residue term we have

$$\begin{aligned}
&\operatorname{Res}_{s=0} F(s+1)\frac{y^s}{s^{k+1}}\\
=&\frac{1}{(k+2)!}\lim_{s\longrightarrow 0}\frac{\mathrm{d}^{k+2}}{\mathrm{d}s^{k+2}}\\
&\left(\zeta_{\mathcal{A}}(s+1)^2 s^2 \frac{1}{\zeta_{\mathcal{A}}(2s+2)}\prod_{P\mid MR}\left(\frac{1-|P|^{-s-1}}{1+|P|^{-s-1}}\right)\prod_{\substack{P\mid M\\ P\nmid R}}\left(\frac{1}{1-|P|^{-s-1}}\right)y^s\right).
\end{aligned} \tag{6.34}$$

If we apply the product rule for differentiation, then one of the terms will be

$$\frac{1}{(k+2)!}\lim_{s\longrightarrow 0}$$

$$\left(\zeta_{\mathcal{A}}(s+1)^2 s^2 \frac{1}{\zeta_{\mathcal{A}}(2s+2)} \prod_{P|MR} \left(\frac{1-|P|^{-s-1}}{1+|P|^{-s-1}}\right) \prod_{\substack{P|M \\ P\nmid R}} \left(\frac{1}{1-|P|^{-s-1}}\right) \frac{\mathrm{d}^{k+2}}{\mathrm{d}s^{k+2}} y^s\right)$$

$$= \frac{(1-q^{-1})(\log q)^k}{(k+2)!} \prod_{P|MR} \left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right) \prod_{\substack{P|M \\ P\nmid R}} \left(\frac{1}{1-|P|^{-1}}\right) \Big(z+O(1)\Big)^{k+2}.$$

The $O(1)$ term is to account for the case where $y = q^{z+\frac{1}{2}}$ (when $k=0$).

Now we look at the remaining terms that arise from the product rule. By using the fact that $\zeta_{\mathcal{A}}(1+s) = \frac{1}{1-q^{-s}}$, the Taylor series for $q^{-s}$, and the chain rule, we have, for non-negative integers $i$, that

$$\lim_{s \to 0} \frac{1}{(\log q)^{i-1}} \frac{\mathrm{d}^i}{\mathrm{d}s^i} \zeta(s+1)s = O_i(1). \tag{6.35}$$

Similarly, for non-negative integers $i$,

$$\frac{1}{(\log q)^i} \lim_{s \to 0} \frac{\mathrm{d}^i}{\mathrm{d}s^i} \zeta(2s+2)^{-1} = \frac{1}{(\log q)^i} \lim_{s \to 0} \frac{\mathrm{d}^i}{\mathrm{d}s^i} \Big(1-q^{-1-2s}\Big) = O_i(1). \tag{6.36}$$

By (6.35), (6.36), and Lemma 6.5.5 and the fact that $\deg M \leq \deg R$, we see that the remaining terms are

$$\ll_k \frac{(\log q)^k}{(k+2)!} \prod_{P|MR} \left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right) \prod_{\substack{P|M \\ P\nmid R}} \left(\frac{1}{1-|P|^{-1}}\right) z^{k+1} \log \deg R.$$

Hence,

$$\begin{aligned} &\mathrm{Res}_{s=0}\, F(s+1) \frac{y^s}{s^{k+1}} \\ =& \frac{(1-q^{-1})(\log q)^k}{(k+2)!} \prod_{P|MR} \left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right) \prod_{\substack{P|M \\ P\nmid R}} \left(\frac{1}{1-|P|^{-1}}\right) \left(z^{k+2} + O_k\big(z^{k+1} \log \deg R\big)\right) \end{aligned} \tag{6.37}$$

as $\deg R \longrightarrow \infty$.

**Step 2.2:** Now we look at the remaining residue terms in (6.33). By similar (but simpler) means as above we can show that

$$\mathrm{Res}_{s=\frac{2m\pi i}{\log q}}\, F(1+s) \frac{y^s}{s^{k+1}} = O_k\left(\frac{(\log q)^k}{m^{k+1}} \prod_{P|MR} \left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right) \prod_{\substack{P|M \\ P\nmid R}} \left(\frac{1}{1-|P|^{-1}}\right) z\right)$$

as $\deg R \longrightarrow \infty$, and so, for $k \geq 1$,

$$\sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \mathrm{Res}_{s=\frac{2m\pi i}{\log q}}\, F(1+s) \frac{{y_R}^s}{s^{k+1}} = O_k\left((\log q)^k \prod_{P|MR} \left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right) \prod_{\substack{P|M \\ P\nmid R}} \left(\frac{1}{1-|P|^{-1}}\right) z\right) \tag{6.38}$$

as $\deg R \longrightarrow \infty$. When $k=0$ we look at things more precisely and see that the term $\frac{1}{m}$ cancels with the term with $\frac{1}{-m}$, and so (6.38) holds for $k=0$ as well.

**Step 2.3:** By (6.33), (6.37) and (6.38), we see that

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{2\pi i}\int_{L(n)}F(1+s)\frac{{y_R}^s}{s^3}\mathrm{d}s\\ =&\frac{(1-q^{-1})(\log q)^k}{(k+2)!}\prod_{P\mid MR}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\prod_{\substack{P\mid M\\ P\nmid R}}\left(\frac{1}{1-|P|^{-1}}\right)\Big(z^{k+2}+O_k\big(z^{k+1}\log\deg R\big)\Big)\end{aligned} \tag{6.39}$$

as $\deg R \longrightarrow \infty$.

**Step 3:** We now look at the integrals over $l_2(n)$ and $l_4(n)$. For all positive integers $n$ and all $s\in l_2(n), l_4(n)$ we have that $F(s+1)y^s = O_{q,R,c}(1)$. One can now easily deduce for $i=2,4$ that

$$\lim_{n\to\infty}\left|\frac{1}{2\pi i}\int_{l_i(n)}F(1+s)\frac{y^s}{s^{k+1}}\mathrm{d}s\right|=0. \tag{6.40}$$

**Step 4:** We now look at the integral over $l_3(n)$. For all positive integers $n$ and all $s\in l_3(n)$ we have that

$$\frac{\zeta_{\mathcal{A}}(s+1)^2}{\zeta_{\mathcal{A}}(2s+2)}=O(1)$$

and

$$\begin{aligned}&\left|\prod_{P\mid MR}\left(\frac{1-|P|^{-s-1}}{1+|P|^{-s-1}}\right)\prod_{\substack{P\mid M\\ P\nmid R}}\left(\frac{1}{1-|P|^{-s-1}}\right)y^s\right|\\ \ll&\prod_{P\mid R}\left(\frac{1+|P|^{-\frac{3}{4}}}{1-|P|^{-\frac{3}{4}}}\right)\prod_{P\mid M}\left(\frac{1}{1-|P|^{-\frac{3}{4}}}\right)|R|^{-\frac{1}{12}}|R|^{-\frac{1}{12}}|M|^{-\frac{1}{12}}q^{o(\deg R)}\\ \ll&\prod_{P\mid R}\left(|P|^{-\frac{1}{12}}\frac{1+|P|^{-\frac{3}{4}}}{1-|P|^{-\frac{3}{4}}}\right)\prod_{P\mid M}\left(|P|^{-\frac{1}{12}}\frac{1}{1-|P|^{-\frac{3}{4}}}\right)q^{o(\deg R)-\frac{1}{12}\deg R}\\ \ll&O(1)\end{aligned}$$

as $\deg R \longrightarrow \infty$. We now easily deduce that, for $k\geq 1$,

$$\lim_{n\to\infty}\left|\frac{1}{2\pi i}\int_{l_3(n)}F(1+s)\frac{y^s}{s^{k+1}}\mathrm{d}s\right|=O(1) \tag{6.41}$$

as $\deg R \longrightarrow \infty$. For the case $k=0$ we must be more careful. Using the fact that $F(1+s)$ has vertical periodicity with period $\frac{2\pi i}{\log q}$, and the fact that $y=q^{z+\frac{1}{2}}$ where $z$ is an integer, we have that

$$\begin{aligned}&\int_{-\frac{1}{4}}^{-\frac{1}{4}+i\infty}F(1+s)\frac{y^s}{s}\mathrm{d}s\\ =&\sum_{m=0}^{\infty}\int_{-\frac{1}{4}+\frac{4m\pi i}{\log q}}^{-\frac{1}{4}+\frac{(4m+2)\pi i}{\log q}}F(1+s)\frac{y^s}{s}\mathrm{d}s+\int_{-\frac{1}{4}+\frac{(4m+2)\pi i}{\log q}}^{-\frac{1}{4}+\frac{(4m+4)\pi i}{\log q}}F(1+s)\frac{y^s}{s}\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&=\sum_{m=0}^{\infty}\int_{-\frac{1}{4}}^{-\frac{1}{4}+\frac{2\pi i}{\log q}} F(1+s)\frac{y^s}{s+\frac{4m\pi i}{\log q}}\mathrm{d}s-\int_{-\frac{1}{4}}^{-\frac{1}{4}+\frac{2\pi i}{\log q}} F(1+s)\frac{y^s}{s+\frac{(4m+2)\pi i}{\log q}}\mathrm{d}s\\
&=\frac{2\pi i}{\log q}\sum_{m=0}^{\infty}\int_{-\frac{1}{4}}^{-\frac{1}{4}+\frac{2\pi i}{\log q}} F(1+s)\frac{y^s}{\left(s+\frac{4m\pi i}{\log q}\right)\left(s+\frac{(4m+2)\pi i}{\log q}\right)}\mathrm{d}s\\
&=\frac{2\pi i}{\log q}\sum_{m=0}^{\infty}\int_{-\frac{1}{4}+\frac{4m\pi i}{\log q}}^{-\frac{1}{4}+\frac{(4m+2)\pi i}{\log q}} F(1+s)\frac{y^s}{s\left(s+\frac{2\pi i}{\log q}\right)}\mathrm{d}s\\
&\ll\int_{-\frac{1}{4}}^{-\frac{1}{4}+i\infty}\frac{1}{|s|\cdot\left|s+\frac{2\pi i}{\log q}\right|}\mathrm{d}s\ll 1.
\end{aligned}
$$

A similar result can be obtained for the integral from $-\frac{1}{4}$ to $-\frac{1}{4}-i\infty$. Hence, we have that

$$
\lim_{n\to\infty}\left|\frac{1}{2\pi i}\int_{l_3(n)} F(1+s)\frac{y^s}{s}\mathrm{d}s\right|=O(1) \tag{6.42}
$$

as $\deg R\longrightarrow\infty$.

**Step 5:** By (6.31), (6.32), (6.39), (6.40), (6.41) and (6.42), we deduce that

$$
\begin{aligned}
&\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq z\\ (N,R)=1}}\frac{2^{\omega(N)-\omega\left((N,M)\right)}}{|N|}(z-\deg N)^k\\
=&\frac{(1-q^{-1})}{(k+2)(k+1)}\prod_{P|MR}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\prod_{\substack{P|M\\ P\nmid R}}\left(\frac{1}{1-|P|^{-1}}\right)\left(z^{k+2}+O_k\left(z^{k+1}\log\deg R\right)\right)
\end{aligned}
$$

as $\deg R\longrightarrow\infty$. □

**Lemma 6.5.7.** *Let $F(s)$, $z$, and $c$ be as in Lemma 6.5.6, and let $y=q^{z+\frac{1}{2}}$. Then,*

$$
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(1+s)\frac{y^s}{s}\mathrm{d}s=\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq z\\ (N,R)=1}}\frac{2^{\omega(N)-\omega\left((N,M)\right)}}{|N|}.
$$

*Proof.* Let $w>z+\frac{1}{2}$ and define

$$
F_w(s):=\sum_{\substack{N\in\mathcal{M}\\ \deg N>w\\ (N,R)=1}}\frac{2^{\omega(N)-\omega\left((N,M)\right)}}{|N|^s}.
$$

Then,

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(1+s)\frac{y^s}{s}\mathrm{d}s\\
=&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq w\\ (N,R)=1}}\frac{2^{\omega(N)-\omega\left((N,M)\right)}}{|N|^{1+s}}\frac{y^s}{s}\mathrm{d}s+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F_w(s)\frac{y^s}{s}\mathrm{d}s
\end{aligned}
$$

$$= \sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ (N,R)=1}} \frac{2^{\omega(N) - \omega\big((N,M)\big)}}{|N|} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_w(s) \frac{y^s}{s} \mathrm{d}s,$$

where we have used Lemma 4.2.1 for the last equality. We must show that the second term on the far RHS is zero. To this end, we note that

$$F_w(s) \leq \sum_{\substack{N \in \mathcal{M} \\ \deg N > w}} \frac{1}{|N|^{\mathrm{Re}(s)-1}} \ll q^{w(2-\mathrm{Re}(s))},$$

and we define the contours

$$\begin{aligned} l_1(n,m) :=& [c - ni, c + ni]; \\ l_2(n,m) :=& [c + ni, m + ni]; \\ l_3(n,m) :=& [m + ni, m - ni]; \\ l_4(n,m) :=& [m - ni, c - ni]; \\ L(n,m) :=& l_1(n) \cup l_2(n) \cup l_3(n) \cup l_4(n). \end{aligned}$$

We then have that

$$\int_{l_3(n,m)} F_w(s) \frac{y^s}{s} \mathrm{d}s \leq 2 \frac{n}{m} q^{2w} \Big( \frac{y}{q^w} \Big)^m \longrightarrow 0$$

as $m \longrightarrow \infty$, since $q^w > q^{z+\frac{1}{2}} = y$. We also have that

$$\int_{c+ni}^{\infty+ni} F_w(s) \frac{y^s}{s} \mathrm{d}s \leq \frac{q^{2w}}{n} \int_{t=c}^{\infty} \Big( \frac{y}{q^w} \Big)^t \mathrm{d}t \ll O_{z,w,c}(n^{-1}) \longrightarrow 0$$

as $n \longrightarrow \infty$, and, similarly,

$$\int_{c-ni}^{\infty-ni} F_w(s) \frac{y^s}{s} \mathrm{d}s \longrightarrow 0$$

as $n \longrightarrow \infty$. Finally, we note that

$$\int_{L(n,m)} F_w(s) \frac{y^s}{s} \mathrm{d}s = 0$$

for all positive $n, m$, by the residue theorem. Hence, we can see that

$$\int_{c-i\infty}^{c+i\infty} F_w(s) \frac{y^s}{s} \mathrm{d}s = 0.$$

as required. □

We now give a Corollary to Lemma 6.5.6.

**Corollary 6.5.8.** *Let $R, M \in \mathcal{M}$ with $\deg M \leq \deg R$, $k$ be a non-negative integer, and $z$ be an integer-valued function of $R$ such that $z \sim \deg R$ as $\deg R \longrightarrow \infty$. We have that*

$$\begin{aligned} &\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ (N,R)=1}} \frac{2^{\omega(N) - \omega\big((N,M)\big)}}{|N|} (\deg N)^k \\ =& \frac{(1-q^{-1})}{(k+2)} \prod_{P \mid MR} \Big( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \Big) \prod_{\substack{P \mid M \\ P \nmid R}} \Big( \frac{1}{1 - |P|^{-1}} \Big) \Big( z^{k+2} + O_k\big( z^{k+1} \log \deg R \big) \Big) \end{aligned}$$

*as* $\deg R \longrightarrow \infty$.

*Proof.* By the binomial theorem we have

$$(\deg N)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i (z - \deg N)^i z^{k-i},$$

and let us define

$$a(R) := (1 - q^{-1}) \prod_{P \mid MR} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{\substack{P \mid M \\ P \nmid R}} \left( \frac{1}{1 - |P|^{-1}} \right).$$

Then, by Lemma 6.5.6, we have

$$\begin{aligned}
&\sum_{\substack{N \in \mathcal{M} \\ \deg N \leq z \\ (N,R)=1}} \frac{2^{\omega(N) - \omega\left((N,M)\right)}}{|N|} (\deg N)^k \\
=& a(R) z^{k+2} \sum_{i=0}^{k} \binom{k}{i} \frac{1}{(i+2)(i+1)} (-1)^i + O_k\big(a(R) z^{k+1} \log \deg R\big) \\
=& \frac{a(R) z^{k+2}}{(k+2)(k+1)} \sum_{i=2}^{k+2} \binom{k+2}{i} (-1)^i + O_k\big(a(R) z^{k+1} \log \deg R\big) \\
=& \frac{a(R) z^{k+2}}{k+2} + O_k\big(a(R) z^{k+1} \log \deg R\big).
\end{aligned}$$

□

**Lemma 6.5.9.** *Suppose $\nu$ is a multiplicative function on $\mathcal{A}$ and that there exists a non-negative integer $r$ such that $\nu(P^k) = O(k^r)$ for all primes $P$ (the implied constant is independent of $P$). Furthermore, suppose there is an $\eta > 0$ such that $\nu(A) \ll_\eta |A|^\eta$ as $\deg A \stackrel{q}{\longrightarrow} \infty$.*

*Let $R \in \mathcal{M}$ be a variable, $a, b > 0$ be constants, and $X = X(R), y = y(R)$ be non-negative, increasing, integer-valued functions such that $X \leq a \log_q \log \deg R$ and $y \geq b \log_q \deg R$ for large enough $\deg R$.*

*Let $c$ and $\epsilon$ be such that $c > \epsilon > \max\left\{0, 1 - \frac{1}{a}\right\}$ and $c > \eta$, and let $\delta > 0$ be small. Finally, let $S \in \mathcal{M}$; $S$ may depend on $R$. We then have that*

$$\begin{aligned}
&\sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \leq y \\ (A,S)=1}} \frac{\nu(A)}{|A|^c} \\
=& \prod_{\substack{\deg P \leq X \\ (P,S)=1}} \left( 1 + \frac{\nu(P)}{|P|^c} + \frac{\nu(P^2)}{|P|^{2c}} + \ldots \right) + O_{q,a,b,c,r,\epsilon,\delta}\Big( (\deg R)^{-b(c-\epsilon)(1-\delta)} \Big)
\end{aligned}$$

*as $\deg R \longrightarrow \infty$.*

*Proof.* Let $d \geq 2$. By similar means as in Lemma 6.5.7, we have that

$$\frac{1}{2\pi i} \int_{d - i\infty}^{d + i\infty} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y + \frac{1}{2})s}}{s} \mathrm{d}s = \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A \leq y \\ (A,S)=1}} \frac{\nu(A)}{|A|^c}.$$

Now, let $n$ be a positive integer and let us define the following contours in $\mathbb{C}$.

$$\begin{aligned} l_1(n) :=& \left[ d - \frac{2n\pi i}{\log q}, d + \frac{2n\pi i}{\log q} \right]; \\ l_2(n) :=& \left[ d + \frac{2n\pi i}{\log q}, -c + \epsilon + \frac{2n\pi i}{\log q} \right]; \\ l_3(n) :=& \left[ -c + \epsilon + \frac{2n\pi i}{\log q}, -c + \epsilon - \frac{2n\pi i}{\log q} \right]; \\ l_4(n) :=& \left[ -c + \epsilon - \frac{2n\pi i}{\log q}, d - \frac{2n\pi i}{\log q} \right]; \\ L(n) :=& l_1(n) \cup l_2(n) \cup l_3(n) \cup l_4(n). \end{aligned}$$

We can see that

$$\begin{aligned} &\frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} \mathrm{d}s \\ =& \frac{1}{2\pi i} \lim_{n \longrightarrow \infty} \Bigg( \int_{L(n)} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} \mathrm{d}s - \int_{l_2(n)} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} \mathrm{d}s \\ &- \int_{l_3(n)} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} \mathrm{d}s - \int_{l_4(n)} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} \mathrm{d}s \Bigg). \end{aligned}$$

For the integral over $L(n)$ there is a simple pole at $s = 0$. So, we have

$$\begin{aligned} &\frac{1}{2\pi i} \int_{L(n)} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} \mathrm{d}s \\ =& \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{c}} = \prod_{\substack{\deg P \leq X \\ (P,S)=1}} \left( 1 + \frac{\nu(P)}{|P|^c} + \frac{\nu(P^2)}{|P|^{2c}} + \dots \right). \end{aligned}$$

We can see that for all $s \in l_2(n)$ and all $s \in l_4(n)$ we have that $\sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}}$ and $q^{(y+\frac{1}{2})s}$ are uniformly bounded, independently of $n$. Hence, we can see that the integrals over $l_2(n), l_4(n)$ tend to 0 as $n \longrightarrow \infty$.

Now consider the integral over $l_3(n)$. Suppose $\epsilon < 1$. Then, for all positive integers $n$ and all $s \in l_3(n)$ we have that

$$\begin{aligned} &\sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \ll \prod_{\substack{\deg P \leq X \\ (P,S)=1}} \left( 1 + \frac{|\nu(P)|}{|P|^{\epsilon}} + \frac{|\nu(P^2)|}{|P|^{2\epsilon}} + \dots \right) \\ \leq& \prod_{\substack{\deg P \leq X \\ (P,S)=1}} \left( 1 + O_{r,\epsilon}\Big( \frac{1}{|P|^{\epsilon}} \Big) \right) \leq \exp \left( O_{r,\epsilon} \Big( \sum_{\substack{\deg P \leq X \\ (P,S)=1}} \frac{1}{|P|^{\epsilon}} \Big) \right) \\ \leq& \exp \left( O_{r,\epsilon} \Big( \sum_{i=1}^{X} \frac{q^{i(1-\epsilon)}}{i} \Big) \right) \leq \exp \left( O_{r,\epsilon,a} \Big( \frac{(\log \deg R)^{a(1-\epsilon)}}{\log_q \log \deg R} \Big) \right) \ll (\deg R)^{b(c-\epsilon)\delta} \end{aligned}$$

as $\deg R \overset{a,b,c,q,r,\epsilon,\delta}{\longrightarrow} \infty$. Now suppose $\epsilon \geq 1$, then we can show that

$$\sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \ll \exp\left( O_{r,\epsilon,a}\Big( a \log_q \log \deg R \Big) \right) \ll (\deg R)^{b(c-\epsilon)\delta}$$

as $\deg R \overset{a,b,c,q,r,\epsilon,\delta}{\longrightarrow} \infty$. We also have that

$$q^{(y+\frac{1}{2})s} \ll (\deg R)^{-b(c-\epsilon)},$$

from which we deduce that

$$\frac{1}{2\pi i} \int_{l_3(n)} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ (A,S)=1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} \mathrm{d}s \ll (\deg R)^{-b(c-\epsilon)(1-\delta)}$$

as $\deg R \overset{a,b,c,q,r,\epsilon,\delta}{\longrightarrow} \infty$. ☐

We now prove a result that is required to bound the lower order terms in the proof of Theorem 2.4.8.

**Lemma 6.5.10.** *Let $F \in \mathcal{M}$, $A_3, B_3 \in \mathcal{S}_{\mathcal{M}}(X)$ with $(A_3B_3, F) = 1$, and $z_1, z_2$ be non-negative integers. Also, we define*

$$\widehat{\deg}(A) := \begin{cases} 1 & \text{if } \deg A = 0 \\ \deg A & \text{if } \deg A \geq 1. \end{cases}$$

*Then, for all $\epsilon > 0$ we have the following:*

$$\sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ \deg A_1B_1 = z_1 \\ \deg A_2B_2 = z_2 \\ (A_1A_2B_1B_2, F)=1 \\ A_1A_2A_3 \equiv B_1B_2B_3 (\mathrm{mod}\, F) \\ A_1A_2A_3 \neq B_1B_2B_3}} 1 \ll_\epsilon \Big( q^{z_1} q^{z_2} \Big)^{1+\epsilon} |A_3B_3| \frac{\widehat{\deg}(A_3B_3)}{|F|}$$

*if $z_1 + z_2 + \deg A_3B_3 \leq \frac{19}{10} \deg F$; and*

$$\sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ \deg A_1B_1 = z_1 \\ \deg A_2B_2 = z_2 \\ (A_1A_2B_1B_2, F)=1 \\ A_1A_2A_3 \equiv B_1B_2B_3 (\mathrm{mod}\, F) \\ A_1A_2A_3 \neq B_1B_2B_3}} 1 \ll q^{z_1+z_2} |A_3B_3| (z_1 + z_2 + \deg A_3B_3)^3 \frac{1}{\phi(F)}$$

*if $z_1 + z_2 + \deg A_3B_3 > \frac{19}{10} \deg F$.*

*Proof.* We can split the sum into the cases $\deg A_1A_2A_3 > \deg B_1B_2B_3$, $\deg A_1A_2A_3 < \deg B_1B_2B_3$, and $\deg A_1A_2A_3 = \deg B_1B_2B_3$ with $A_1A_2A_3 \neq B_1B_2B_3$.

When $\deg A_1A_2A_3 > \deg B_1B_2B_3$, we have that $A_1A_2A_3 = KF + B_1B_2B_3$ where $K \in \mathcal{M}$ and $\deg KF > \deg B_1B_2B_3$. Furthermore,

$$2 \deg KF = 2 \deg A_1A_2A_3 > \deg A_1A_2A_3 + \deg B_1B_2B_3$$

$$= \deg A_1 B_1 + \deg A_2 B_2 + \deg A_3 B_3 = z_1 + z_2 + \deg A_3 B_3,$$

from which we deduce that

$$a_0 := \frac{z_1 + z_2 + \deg A_3 B_3}{2} < \deg KF \leq z_1 + z_2 + \deg A_3 =: a_1.$$

Also,

$$\deg KF + \deg B_1 B_2 = \deg A_1 A_2 A_3 + \deg B_1 B_2 = z_1 + z_2 + \deg A_3,$$

from which we deduce that

$$\deg B_1 B_2 = z_1 + z_2 + \deg A_3 - \deg KF.$$

Similarly, if $\deg A_1 A_2 A_3 < \deg B_1 B_2 B_3$, we can show that

$$b_0 := \frac{z_1 + z_2 + \deg A_3 B_3}{2} < \deg KF \leq z_1 + z_2 + \deg B_3 =: b_1$$

and

$$\deg A_1 A_2 = z_1 + z_2 + \deg B_3 - \deg KF.$$

When $\deg A_1 A_2 A_3 = \deg B_1 B_2 B_3$, we must have that

$$\deg A_1 A_2 = \frac{z_1 + z_2 + \deg B_3 - \deg A_3}{2},$$
$$\deg B_1 B_2 = \frac{z_1 + z_2 + \deg A_3 - \deg B_3}{2}.$$

Also, we can write $A_1 A_2 A_3 = KF + B_1 B_2 B_3$, where $\deg KF < \deg B_1 B_2 B_3 = \frac{z_1+z_2+\deg A_3 B_3}{2}$ and $K \neq 0$ need not be monic.

So, writing $N = B_1 B_2$ when $\deg A_1 A_2 A_3 \geq \deg B_1 B_2 B_3$, and $N = A_1 A_2$ when $\deg A_1 A_2 A_3 < \deg B_1 B_2 B_3$, we have that

$$\begin{aligned}
&\sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ \deg A_1 B_1 = z_1 \\ \deg A_2 B_2 = z_2 \\ (A_1 A_2 B_1 B_2, F)=1 \\ A_1 A_2 A_3 \equiv B_1 B_2 B_3 (\operatorname{mod} F) \\ A_1 A_2 A_3 \neq B_1 B_2 B_3}} 1 \\
\leq &\sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg A_3 - \deg KF \\ (N,F)=1}} d(N) d\Big( (KF + NB_3) {A_3}^{-1} \Big) \\
&+ \sum_{\substack{K \in \mathcal{M} \\ b_0 < \deg KF \leq b_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg B_3 - \deg KF \\ (N,F)=1}} d(N) d\Big( (KF + NA_3) {B_3}^{-1} \Big) \\
&+ \sum_{\substack{K \in \mathcal{A} \backslash \{0\} \\ \deg KF < a_0}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = \frac{z_1 + z_2 + \deg A_3 - \deg B_3}{2} \\ (N,F)=1}} d(N) d\Big( (KF + NB_3) {A_3}^{-1} \Big).
\end{aligned} \tag{6.43}$$

We must remark that if $A_3 \mid (KF + NB_3)$ then we define $(KF+NB_3){A_3}^{-1}$ by $(KF+NB_3){A_3}^{-1} \cdot A_3 = (KF+NB_3)$. If $A_3 \nmid (KF+NB_3)$, then we ignore the term with $(KF+NB_3){A_3}^{-1}$ in the sum; that is, we take the definition $d\Big((KF+NB_3){A_3}^{-1}\Big) := 0$. We do the same for $(KF+NA_3){B_3}^{-1}$.

**Step 1:** Let us consider the case when $z_1 + z_2 + \deg A_3B_3 \leq \frac{19}{10}\deg F$. By using well known bounds on the divisor function, we have that

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1+z_2+\deg A_3 - \deg KF \\ (N,F)=1}} d(N) d\Big((KF+NB_3){A_3}^{-1}\Big) \\
\ll_\epsilon & \Big(q^{z_1}q^{z_2}\Big)^{\frac{\epsilon}{2}} \sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1+z_2+\deg A_3 - \deg KF \\ (N,F)=1}} 1 \\
\leq & \Big(q^{z_1}q^{z_2}\Big)^{1+\frac{\epsilon}{2}} |A_3| \sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq a_1}} \frac{1}{|KF|} \\
\ll & \Big(q^{z_1}q^{z_2}\Big)^{1+\frac{\epsilon}{2}} |A_3| \frac{z_1+z_2+\deg A_3}{|F|} \ll_\epsilon \Big(q^{z_1}q^{z_2}\Big)^{1+\epsilon} |A_3| \frac{\widehat{\deg} A_3}{|F|}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{M} \\ b_0 < \deg KF \leq b_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1+z_2+\deg B_3 - \deg KF \\ (N,F)=1}} d(N) d\Big((KF+NA_3){B_3}^{-1}\Big) \\
\ll_\epsilon & \Big(q^{z_1}q^{z_2}\Big)^{1+\epsilon} |B_3| \frac{\widehat{\deg} B_3}{|F|}.
\end{aligned}$$

As for the sum

$$\sum_{\substack{K \in \mathcal{A}\backslash\{0\} \\ \deg KF < a_0}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = \frac{z_1+z_2+\deg A_3 - \deg B_3}{2} \\ (N,F)=1}} d(N) d\Big((KF+NB_3){A_3}^{-1}\Big),$$

we note that it does not apply to this case where $z_1 + z_2 + \deg A_3B_3 \leq \frac{19}{10}\deg F$ because this would imply $\deg KF \geq \deg F \geq \frac{20}{19} a_0$, which does not overlap with range $\deg KF < a_0$ in the sum.

Hence,

$$\sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ \deg A_1B_1 = z_1 \\ \deg A_2B_2 = z_2 \\ (A_1A_2B_1B_2, F)=1 \\ A_1A_2A_3 \equiv B_1B_2B_3 (\mathrm{mod}\, F) \\ A_1A_2A_3 \neq B_1B_2B_3}} 1 \ll_\epsilon \Big(q^{z_1}q^{z_2}\Big)^{1+\epsilon} |A_3B_3| \frac{\widehat{\deg}(A_3B_3)}{|F|}$$

for $z_1 + z_2 + \deg A_3B_3 \leq \frac{19}{10}\deg F$.

**Step 2:** We now consider the case when $z_1 + z_2 + \deg A_3B_3 > \frac{19}{10}\deg F$.

**Step 2.1:** We consider the subcase where $a_0 < \deg KF \leq \frac{3}{2}a_0$. This allows us to apply Lemma 4.2.8 for the second relation below.

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq \frac{3}{2}a_0}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg A_3 - \deg KF \\ (N,F)=1}} d(N) d\Big( (KF + NB_3){A_3}^{-1} \Big) \\
\leq &\sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq \frac{3}{2}a_0}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = 2a_0 - \deg KF \\ (N,F)=1}} d(N) d\Big( KF + N \Big) \\
\ll &q^{z_1} q^{z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^2 \frac{1}{|F|} \sum_{\substack{K \in \mathcal{M} \\ a_0 < \deg KF \leq \frac{3}{2}a_0}} \frac{1}{|K|} \sum_{\substack{H | K \\ \deg H \leq \frac{2a_0 - \deg KF}{2}}} \frac{d(H)}{|H|} \\
\leq &q^{z_1} q^{z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^2 \frac{1}{|F|} \sum_{\substack{K \in \mathcal{M} \\ \deg KF \leq 2a_0}} \frac{1}{|K|} \sum_{H | K} \frac{d(H)}{|H|} \\
\leq &q^{z_1} q^{z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^2 \frac{1}{|F|} \sum_{\substack{H \in \mathcal{M} \\ \deg H \leq 2a_0}} \frac{d(H)}{|H|} \sum_{\substack{K \in \mathcal{M} \\ \deg K \leq 2a_0 \\ H | K}} \frac{1}{|K|} \\
\leq &q^{z_1} q^{z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^3 \frac{1}{|F|} \sum_{\substack{H \in \mathcal{M} \\ \deg H \leq 2a_0}} \frac{d(H)}{|H|^2} \\
\ll &q^{z_1} q^{z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^3 \frac{1}{|F|}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{M} \\ b_0 < \deg KF \leq \frac{3}{2}b_0}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg B_3 - \deg KF \\ (N,F)=1}} d(N) d\Big( (KF + NA_3){B_3}^{-1} \Big) \\
\ll &q^{z_1} q^{z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^3 \frac{1}{|F|}.
\end{aligned}$$

**Step 2.2:** Now we consider the subcase where $\frac{3}{2}a_0 < \deg KF \leq a_1$. We have that

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{M} \\ \frac{3}{2}a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg A_3 - \deg KF \\ (N,F)=1}} d(N) d\Big( (KF + NB_3){A_3}^{-1} \Big) \\
\leq &\sum_{\substack{K \in \mathcal{M} \\ \frac{3}{2}a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = 2a_0 - \deg KF \\ (N,F)=1}} d(N) d(KF + N) \\
\leq &\sum_{\substack{N \in \mathcal{M} \\ \deg N < \frac{a_0}{2} \\ (N,F)=1}} \sum_{\substack{K \in \mathcal{M} \\ \deg KF = 2a_0 - \deg N}} d(N) d(KF + N) \\
\leq &\sum_{\substack{N \in \mathcal{M} \\ \deg N < \frac{a_0}{2} \\ (N,F)=1}} d(N) \sum_{\substack{M \in \mathcal{M} \\ \deg(M - X_{(N)}) < 2a_0 - \deg N \\ M \equiv N (\mathrm{mod}\, F)}} d(M)
\end{aligned}$$

where we define $X_{(N)} := T^{2a_0 - \deg N}$ (The monic polynomial of degree $2a_0 - \deg N$ with all non-leading coefficients equal to 0).

We can now apply Theorem 2.1.1. One may wish to note that

$$y := 2a_0 - \deg N \geq \frac{3}{4}(z_1 + z_2 + \deg A_3B_3) \geq \frac{3}{4}\frac{19}{10} \deg F$$

and so

$$\deg F \leq \frac{40}{57} y = (1-\alpha)y$$

where $0 < \alpha < \frac{1}{2}$, as required. Hence, we have that

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{M} \\ \frac{3}{2}a_0 < \deg KF \leq a_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg A_3 - \deg KF \\ (N,F)=1}} d(N) d\Big((KF + NB_3){A_3}^{-1}\Big) \\
\leq & q^{z_1} q^{z_2} |A_3B_3| (z_1 + z_2 + \deg A_3B_3) \frac{1}{\phi(F)} \sum_{\substack{N \in \mathcal{M} \\ \deg N < \frac{a_0}{2} \\ (N,F)=1}} \frac{d(N)}{|N|} \\
\leq & q^{z_1} q^{z_2} |A_3B_3| (z_1 + z_2 + \deg A_3B_3)^3 \frac{1}{\phi(F)}.
\end{aligned}$$

Similarly, if $\frac{3}{2}b_0 < \deg KF \leq b_1$ then

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{M} \\ \frac{3}{2}b_0 < \deg KF \leq b_1}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = z_1 + z_2 + \deg B_3 - \deg KF \\ (N,F)=1}} d(N) d\Big((KF + NA_3){B_3}^{-1}\Big) \\
\leq & q^{z_1} q^{z_2} |A_3B_3| (z_1 + z_2 + \deg A_3B_3)^3 \frac{1}{\phi(F)}.
\end{aligned}$$

**Step 2.3:** We now look at the sum

$$\sum_{\substack{K \in \mathcal{A} \backslash \{0\} \\ \deg KF < a_0}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = \frac{z_1 + z_2 + \deg A_3 - \deg B_3}{2} \\ (N,F)=1}} d(N) d\Big((KF + NB_3){A_3}^{-1}\Big).$$

By Lemma 4.2.10 we have that

$$\begin{aligned}
&\sum_{\substack{K \in \mathcal{A} \backslash \{0\} \\ \deg KF < a_0}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = \frac{z_1 + z_2 + \deg A_3 - \deg B_3}{2} \\ (N,F)=1}} d(N) d\Big((KF + NB_3){A_3}^{-1}\Big) \\
\leq & \sum_{\substack{K \in \mathcal{A} \backslash \{0\} \\ \deg KF < a_0}} \sum_{\substack{N \in \mathcal{M} \\ \deg N = a_0 \\ (N,F)=1}} d(N) d(KF + N) \\
\ll & q^{\frac{z_1 + z_2}{2}} |A_3B_3|^{\frac{1}{2}} (z_1 + z_2 + \deg A_3B_3)^2 \sum_{\substack{K \in \mathcal{A} \backslash \{0\} \\ \deg KF < a_0}} \sum_{H \mid K} \frac{d(H)}{|H|} \\
\leq & q^{z_1 + z_2 - 1} |A_3B_3| (z_1 + z_2 + \deg A_3B_3)^2 \frac{1}{|F|} \sum_{\substack{K \in \mathcal{A} \backslash \{0\} \\ \deg KF < a_0}} \frac{1}{|K|} \sum_{H \mid K} \frac{d(H)}{|H|}
\end{aligned}$$

$$\leq q^{z_1+z_2}|A_3B_3|(z_1+z_2+\deg A_3B_3)^3\frac{1}{|F|},$$

where the second-to-last relation uses the fact that $a_0$ is an integer (since $\deg A_1A_2A_3 = \deg B_1B_2B_3$) and so $\deg KF < a_0$ implies $\deg KF \leq a_0 - 1$, and the last relation uses a similar calculation as that in Step 2.1.

**Step 2.4:** We apply steps 2.1, 2.2, and 2.3 to (6.43) and we see that

$$\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ \deg A_1B_1=z_1\\ \deg A_2B_2=z_2\\ (A_1A_2B_1B_2,F)=1\\ A_1A_2A_3\equiv B_1B_2B_3 (\mathrm{mod}\, F)\\ A_1A_2A_3\neq B_1B_2B_3}} 1 \ll q^{z_1+z_2}|A_3B_3|(z_1+z_2+\deg A_3B_3)^3\frac{1}{\phi(F)}$$

for $z_1+z_2+\deg A_3B_3 > \frac{19}{10}\deg F$. □

## 6.6 The Fourth Hadamard Moment

In this section we prove Theorem 2.4.8, which we restate for ease of reference.

**Theorem.** *We have*

$$\begin{aligned}\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R}\left|Z_X\Big(\frac{1}{2},\chi\Big)\right|^4 =&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R}\left|L\Big(\frac{1}{2},\chi\Big)P_X\Big(\frac{1}{2},\chi\Big)^{-1}\right|^4\\ &\sim\frac{1}{12}\Big(\frac{\deg R}{e^\gamma X}\Big)^4\prod_{\substack{\deg P>X\\ P|R}}\left(\frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}}\right)\end{aligned}$$

*as $X, \deg R \longrightarrow \infty$ with $X \leq \log_q \log \deg R$.*

*Proof of Theorem 2.4.8.* In this proof, we assume all asymptotic relations are as $X, \deg R \stackrel{q}{\longrightarrow} \infty$ with $X \leq \log_q \log \deg R$. Using Lemmas 6.5.1 and 6.5.2, we have

$$\begin{aligned}&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R}\left|L\Big(\frac{1}{2},\chi\Big)P_X\Big(\frac{1}{2},\chi\Big)^{-1}\right|^4\\ \sim&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R}\left|L\Big(\frac{1}{2},\chi\Big)\right|^4\left|P_X^{**}\Big(\frac{1}{2},\chi\Big)\right|^2\\ =&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R}\left|L\Big(\frac{1}{2},\chi\Big)\right|^4\left|\widehat{P_X^{**}}\Big(\frac{1}{2},\chi\Big)+O\Big((\deg R)^{-\frac{1}{33}}\Big)\right|^2.\end{aligned}$$

By the Cauchy-Schwarz inequality, Theorem 2.2.3, and Lemma A.2.7, it suffices to prove

$$\begin{aligned}&\frac{1}{\phi^*(R)}\sideset{}{^*}\sum_{\chi \bmod R}\left|L\Big(\frac{1}{2},\chi\Big)\right|^4\left|\widehat{P_X^{**}}\Big(\frac{1}{2},\chi\Big)\right|^2\\ \sim&\frac{1}{12}(\deg R)^4\prod_{\substack{\deg P>X\\ P|R}}\left(\frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}}\right)\prod_{\deg P\leq X}\big(1-|P|^{-1}\big)^4.\end{aligned}$$

By Lemmas A.1.2 and A.1.3, we have

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big| L\Big(\frac{1}{2}, \chi\Big) \Big|^4 \Big| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) \Big|^2 \\
=&\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big( 2a(\chi) + 2b(\chi) + c(\chi) \Big)^2 \Big| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) \Big|^2,
\end{aligned}$$

where $c(\chi)$ is as in (A.8) and

$$\begin{aligned}
z_R :=& \deg R - \log_q 2^{\omega(R)}; \\
a(\chi) :=& \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB \leq z_R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}; \\
b(\chi) :=& \sum_{\substack{A,B \in \mathcal{M} \\ z_R < \deg AB < \deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}.
\end{aligned}$$

Note that, by symmetry in $A, B$, the terms $a(\chi)$, $b(\chi)$, and $c(\chi)$ are equal to their conjugates and, therefore, they are real. Hence, by the Cauchy-Schwarz inequality, it suffices to obtain the asymptotic main term of

$$\frac{4}{\phi^*(R)} \sum_{\chi \bmod R}^{*} a(\chi)^2 \Big| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) \Big|^2 \tag{6.44}$$

and show that

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} b(\chi)^2 \Big| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) \Big|^2 \quad \text{and} \quad \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} c(\chi)^2 \Big| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) \Big|^2$$

are of lower order. The reason we express the sum in terms of $a(\chi)$ and $b(\chi)$ is because the fact that $a(\chi)$ is truncated allows us to bound the lower order terms that it contributes. We cannot do this with $b(\chi)$ but, because $b(\chi)$ is a relatively short sum, we can apply others methods to bound it.

**Step 1; the asymptotic main term of $\frac{4}{\phi^*(R)} \sum_{\chi \bmod R}^{*} a(\chi)^2 \Big| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big) \Big|^2$:**

By Lemma 1.4.5 and Corollary 1.4.6, we have that

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} a(\chi)^2 \Big| \widehat{P_X^{**}}\Big(\frac{1}{2}, \chi\Big)\Big|^2 \\
=&\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1B_1, \deg A_2B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R}} \frac{\beta(A_3)\beta(B_3)\chi(A_1A_2A_3)\overline{\chi}(B_1B_2B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \\
=& \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1B_1, \deg A_2B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_1A_2A_3B_1B_2B_3, R)=1 \\ A_1A_2A_3 = B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \\
&+ \frac{1}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1B_1, \deg A_2B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_1A_2A_3B_1B_2B_3, R)=1 \\ A_1A_2A_3 \equiv B_1B_2B_3 (\bmod F) \\ A_1A_2A_3 \neq B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}.
\end{aligned} \tag{6.45}$$

**Step 1.1:** We consider the first term on the far RHS of (6.45): the diagonal terms. By Lemma 6.5.3 we have

$$\begin{aligned}
&\sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1B_1, \deg A_2B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_1A_2A_3B_1B_2B_3, R)=1 \\ A_1A_2A_3 = B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \\
=& \sum_{\substack{G_1,G_2,V_{1,2},V_{2,1} \in \mathcal{M} \\ G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2} \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg (G_1)^2 V_{1,2}V_{1,3}V_{2,1}V_{3,1} \leq z_R \\ \deg (G_2)^2 V_{2,1}V_{2,3}V_{1,2}V_{3,2} \leq z_R \\ \deg G_3V_{3,1}V_{3,2} \leq \frac{1}{8}\log_q \deg R \\ \deg G_3V_{1,3}V_{2,3} \leq \frac{1}{8}\log_q \deg R \\ (G_i,R),(V_{j,k},R)=1 \ \forall i,j,k \\ (V_{i,j},V_{k,l})=1 \text{ for } (i \neq k \wedge j \neq l)}} \frac{\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})}{|G_1G_2G_3V_{1,2}V_{1,3}V_{2,1}V_{2,3}V_{3,1}V_{3,2}|} \\
=& \sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2} \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G_3V_{3,1}V_{3,2} \leq \frac{1}{8}\log_q \deg R \\ \deg G_3V_{1,3}V_{2,3} \leq \frac{1}{8}\log_q \deg R \\ (G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}, R)=1 \\ (V_{1,3}V_{2,3}, V_{3,1}V_{3,2})=1}} \frac{\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})}{|G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}|} \\
&\cdot \sum_{\substack{V_{1,2},V_{2,1} \in \mathcal{M} \\ \deg V_{1,2}V_{2,1} \leq z_R - \deg V_{1,3}V_{3,1} \\ \deg V_{1,2}V_{2,1} \leq z_R - \deg V_{2,3}V_{3,2} \\ (V_{1,2}V_{2,1}, R)=1 \\ (V_{1,2}, V_{2,3}V_{3,1})=1 \\ (V_{2,1}, V_{3,2}V_{1,3})=1 \\ (V_{1,2}, V_{2,1})=1}} \frac{1}{|V_{1,2}V_{2,1}|} \sum_{\substack{G_1,G_2 \in \mathcal{M} \\ \deg G_1 \leq \frac{z_R - \deg V_{1,2}V_{2,1}V_{1,3}V_{3,1}}{2} \\ \deg G_2 \leq \frac{z_R - \deg V_{1,2}V_{2,1}V_{2,3}V_{3,2}}{2} \\ (G_1G_2, R)=1}} \frac{1}{|G_1G_2|}.
\end{aligned}$$

By Lemma 6.5.4 we have

$$\sum_{\substack{V_{1,2},V_{2,1}\in\mathcal{M}\\ \deg V_{1,2}V_{2,1}\leq z_R-\deg V_{1,3}V_{3,1}\\ \deg V_{1,2}V_{2,1}\leq z_R-\deg V_{2,3}V_{3,2}\\ (V_{1,2}V_{2,1},R)=1\\ (V_{1,2},V_{2,3}V_{3,1})=1\\ (V_{2,1},V_{3,2}V_{1,3})=1\\ (V_{1,2},V_{2,1})=1}} \frac{1}{|V_{1,2}V_{2,1}|} \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1\leq\frac{z_R-\deg V_{1,2}V_{2,1}V_{1,3}V_{3,1}}{2}\\ \deg G_2\leq\frac{z_R-\deg V_{1,2}V_{2,1}V_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}$$

$$= \sum_{\substack{V\in\mathcal{M}\\ \deg V\leq z_R-\deg V_{1,3}V_{3,1}\\ \deg V\leq z_R-\deg V_{2,3}V_{3,2}\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{1}{|V|} \sum_{\substack{V_{1,2},V_{2,1}\in\mathcal{M}\\ V_{1,2}V_{2,1}=V\\ (V_{1,2},V_{2,1})=1\\ (V_{1,2},V_{2,3}V_{3,1})=1\\ (V_{2,1},V_{3,2}V_{1,3})=1}} \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1\leq\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\\ \deg G_2\leq\frac{z_R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}$$

$$= \sum_{\substack{V\in\mathcal{M}\\ \deg V\leq z_R-\deg V_{1,3}V_{3,1}\\ \deg V\leq z_R-\deg V_{2,3}V_{3,2}\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V)-\omega\left(\left(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|} \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1\leq\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\\ \deg G_2\leq\frac{z_R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}.$$

So, we have

$$\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A_1B_1,\deg A_2B_2\leq z_R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3=B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}$$

$$= \sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3V_{3,1}V_{3,2}\leq\frac{1}{8}\log_q\deg R\\ \deg G_3V_{1,3}V_{2,3}\leq\frac{1}{8}\log_q\deg R\\ (G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2},R)=1\\ (V_{1,3}V_{2,3},V_{3,1}V_{3,2})=1}} \frac{\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})}{|G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}|}$$

$$\cdot \sum_{\substack{V\in\mathcal{M}\\ \deg V\leq z_R-\deg V_{1,3}V_{3,1}\\ \deg V\leq z_R-\deg V_{2,3}V_{3,2}\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V)-\omega\left(\left(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|} \sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1\leq\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\\ \deg G_2\leq\frac{z_R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}. \tag{6.46}$$

Now, by Corollary A.3.3, if

$$\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\geq\log_q 3^{\omega(R)}$$

- that is,

$$\deg V\leq\deg R-\log_q 18^{\omega(R)}-\deg V_{1,3}V_{3,1}$$

- then

$$\sum_{\substack{G_1 \in \mathcal{M} \\ \deg G_1 \leq \frac{z_R - \deg V V_{1,3} V_{3,1}}{2} \\ (G_1, R)=1}} \frac{1}{|G_1|} = \frac{\phi(R)}{2|R|}(z_R - \deg V V_{1,3} V_{3,1}) + O\Big( \frac{\phi(R)}{|R|} \log \omega(R) \Big)$$
$$= \frac{\phi(R)}{2|R|} \Big( \deg R - \deg V + O\big( \log \deg R + \omega(R) \big) \Big). \tag{6.47}$$

If

$$\deg V > \deg R - \log_q 18^{\omega(R)} - \deg V_{1,3} V_{3,1},$$

then

$$\sum_{\substack{G_1 \in \mathcal{M} \\ \deg G_1 \leq \frac{z_R - \deg V V_{1,3} V_{3,1}}{2} \\ (G_1, R)=1}} \frac{1}{|G_1|} \leq \sum_{\substack{G_1 \in \mathcal{M} \\ \deg G_1 \leq \log_q 3^{\omega(R)} \\ (G_1, R)=1}} \frac{1}{|G_1|} \ll \frac{\phi(R)}{|R|} \omega(R). \tag{6.48}$$

Similar results hold for the sum over $G_2$.

So, let us define

$$m_0 := \min \big\{ \deg R - \log_q 18^{\omega(R)} - \deg V_{1,3} V_{3,1} \ , \ \deg R - \log_q 18^{\omega(R)} - \deg V_{2,3} V_{3,2} \big\};$$
$$m_1 := \max \big\{ \deg R - \log_q 18^{\omega(R)} - \deg V_{1,3} V_{3,1} \ , \ \deg R - \log_q 18^{\omega(R)} - \deg V_{2,3} V_{3,2} \big\}.$$

Then, by (6.47) and (6.48), we have

$$\sum_{\substack{V \in \mathcal{M} \\ \deg V \leq z_R - \deg V_{1,3} V_{3,1} \\ \deg V \leq z_R - \deg V_{2,3} V_{3,2} \\ \big(V, R(V_{1,3} V_{3,1}, V_{2,3} V_{3,2})\big)=1}} \frac{2^{\omega(V) - \omega\Big( \big( V, V_{1,3} V_{2,3} V_{3,1} V_{3,2} \big) \Big)}}{|V|} \sum_{\substack{G_1, G_2 \in \mathcal{M} \\ \deg G_1 \leq \frac{z_R - \deg V V_{1,3} V_{3,1}}{2} \\ \deg G_2 \leq \frac{z_R - \deg V V_{2,3} V_{3,2}}{2} \\ (G_1 G_2, R)=1}} \frac{1}{|G_1 G_2|}$$
$$= \frac{\phi(R)^2}{4|R|^2} \sum_{\substack{V \in \mathcal{M} \\ \deg V \leq m_0 \\ \big(V, R(V_{1,3} V_{3,1}, V_{2,3} V_{3,2})\big)=1}} \frac{2^{\omega(V) - \omega\Big( \big( V, V_{1,3} V_{2,3} V_{3,1} V_{3,2} \big) \Big)}}{|V|}$$
$$\cdot \Big( \deg R - \deg V + O\big( \log \deg R + \omega(R) \big) \Big)^2$$
$$+ l_1(R, V_{1,3}, V_{3,1}, V_{2,3}, V_{3,2}), \tag{6.49}$$

where

$$\begin{aligned} & l_1(R, V_{1,3}, V_{3,1}, V_{2,3}, V_{3,2}) \\ \ll & \frac{\phi(R)^2 \omega(R) \deg R}{2|R|^2} \sum_{\substack{V \in \mathcal{M} \\ m_0 < \deg V \leq m_1 \\ \big(V, R(V_{1,3}V_{3,1}, V_{2,3}V_{3,2})\big)=1}} \frac{2^{\omega(V) - \omega\Big(\big(V, V_{1,3}V_{2,3}V_{3,1}V_{3,2}\big)\Big)}}{|V|} \\ & + \frac{\phi(R)^2 \omega(R)^2}{|R|^2} \sum_{\substack{V \in \mathcal{M} \\ m_1 < \deg V \leq \deg R \\ \big(V, R(V_{1,3}V_{3,1}, V_{2,3}V_{3,2})\big)=1}} \frac{2^{\omega(V) - \omega\Big(\big(V, V_{1,3}V_{2,3}V_{3,1}V_{3,2}\big)\Big)}}{|V|}. \end{aligned} \tag{6.50}$$

We now apply Corollary 6.5.8 to both terms on the RHS of (6.49). For the second term, we use (6.50) and it is then just two direct applications. For the first term, we must expand $\Big( \deg R - \deg V + O\big( \log \deg R + \omega(R)\big)\Big)^2$ and use Corollary 6.5.8 on each of the resulting terms. We obtain

$$\begin{aligned} & \sum_{\substack{V \in \mathcal{M} \\ \deg V \leq z_R - \deg V_{1,3}V_{3,1} \\ \deg V \leq z_R - \deg V_{2,3}V_{3,2} \\ \big(V, R(V_{1,3}V_{3,1}, V_{2,3}V_{3,2})\big)=1}} \frac{2^{\omega(V) - \omega\Big(\big(V, V_{1,3}V_{2,3}V_{3,1}V_{3,2}\big)\Big)}}{|V|} \sum_{\substack{G_1, G_2 \in \mathcal{M} \\ \deg G_1 \leq \frac{z_R - \deg V V_{1,3}V_{3,1}}{2} \\ \deg G_2 \leq \frac{z_R - \deg V V_{2,3}V_{3,2}}{2} \\ (G_1 G_2, R)=1}} \frac{1}{|G_1 G_2|} \\ = & \frac{1 - q^{-1}}{48} (\deg R)^4 \bigg( 1 + O\Big( \frac{\omega(R) + \log \deg R}{\deg R} \Big) \bigg) \prod_{P \mid R} \left( \frac{\big(1 - |P|^{-1}\big)^3}{1 + |P|^{-1}} \right) \\ & \cdot \prod_{P \mid V_{1,3}V_{2,3}V_{3,1}V_{3,2}} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{\substack{P \mid V_{1,3}V_{2,3}V_{3,1}V_{3,2} \\ P \nmid (V_{1,3}, V_{2,3}), (V_{3,1}, V_{3,2})}} \left( \frac{1}{1 - |P|^{-1}} \right) \\ =: & l_2(R, V_{1,3}, V_{2,3}, V_{3,1}, V_{3,2}). \end{aligned} \tag{6.51}$$

Before proceeding let us make the following definitions: For $A \in \mathcal{A} \backslash \{0\}$ and $P \in \mathcal{P}$ we define $e_P(A)$ to be the largest non-negative integer such that $P^{e_P(A)} \mid A$, and

$$\gamma(A) := \prod_{P \mid A} \left( 1 + e_P(A) \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right). \tag{6.52}$$

Then, we can see that

$$\begin{aligned}
&\sum_{\substack{V_{1,3},V_{2,3}\in\mathcal{S}_{\mathcal{M}}(X)\\ V_{1,3}V_{2,3}=B_3{}'}}\prod_{P|V_{1,3}V_{2,3}}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\prod_{\substack{P|V_{1,3}V_{2,3}\\ P\nmid(V_{1,3},V_{2,3})}}\left(\frac{1}{1-|P|^{-1}}\right)\\
&=\prod_{P|B_3{}'}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\sum_{\substack{W_1W_2=B_3{}'\\ (W_1,W_2)=1}}\sum_{\substack{V_{1,3},V_{2,3}\in\mathcal{S}_{\mathcal{M}}(X)\\ V_{1,3}V_{2,3}=B_3{}'\\ \operatorname{rad}(V_{1,3},V_{2,3})=\operatorname{rad}W_1}}\prod_{P|W_2}\left(\frac{1}{1-|P|^{-1}}\right)\\
&=\prod_{P|B_3{}'}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\sum_{\substack{W_1W_2=B_3{}'\\ (W_1,W_2)=1}}\prod_{P|W_2}\left(\frac{1}{1-|P|^{-1}}\right)2^{\omega(W_2)}\prod_{P|W_1}\Big(e_P(B_3{}')-1\Big) \qquad (6.53)\\
&=\prod_{P|B_3{}'}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\prod_{P|B_3{}'}\left(\frac{2}{1-|P|^{-1}}+\Big(e_P(B_3{}')-1\Big)\right)\\
&=\prod_{P|B_3{}'}\left(1+e_P(B_3{}')\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)=\gamma(B_3{}').
\end{aligned}$$

Similarly,

$$\sum_{\substack{V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ V_{3,1}V_{3,2}=A_3{}'}}\prod_{P|V_{3,1}V_{3,2}}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)\prod_{\substack{P|V_{3,1}V_{3,2}\\ P\nmid(V_{3,1},V_{3,2})}}\left(\frac{1}{1-|P|^{-1}}\right)=\gamma(A_3{}'). \qquad (6.54)$$

We now substitute (6.51) to (6.46) and apply (6.53) and (6.54) to obtain

$$\begin{aligned}
&\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A_1B_1,\deg A_2B_2\le z_R\\ \deg A_3,\deg B_3\le\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3=B_1B_2B_3}}\frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}\\
=&\sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3V_{3,1}V_{3,2}\le\frac{1}{8}\log_q\deg R\\ \deg G_3V_{1,3}V_{2,3}\le\frac{1}{8}\log_q\deg R\\ (G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2},R)=1\\ (V_{1,3}V_{2,3},V_{3,1}V_{3,2})=1}}\frac{\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})}{|G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}|}l_2(R,V_{1,3},V_{2,3},V_{3,1},V_{3,2})\\
=&\sum_{\substack{G_3,A_3{}',B_3{}'\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3A_3{}'\le\frac{1}{8}\log_q\deg R\\ \deg G_3B_3{}'\le\frac{1}{8}\log_q\deg R\\ (G_3A_3{}'B_3{}',R)=1\\ (A_3{}',B_3{}')=1}}\frac{\beta(G_3A_3{}')\beta(G_3B_3{}')}{|G_3A_3{}'B_3{}'|}\sum_{\substack{V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ V_{3,1}V_{3,2}=A_3{}'}}\sum_{\substack{V_{1,3},V_{2,3}\in\mathcal{S}_{\mathcal{M}}(X)\\ V_{1,3}V_{2,3}=B_3{}'}}l_2(R,V_{1,3},V_{2,3},V_{3,1},V_{3,2})\\
=&\frac{1-q^{-1}}{48}\prod_{P|R}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)(\deg R)^4\sum_{\substack{G_3,A_3{}',B_3{}'\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3A_3{}'\le\frac{1}{8}\log_q\deg R\\ \deg G_3B_3{}'\le\frac{1}{8}\log_q\deg R\\ (G_3A_3{}'B_3{}',R)=1\\ (A_3{}',B_3{}')=1}}\frac{\beta(G_3A_3{}')\beta(G_3B_3{}')}{|G_3A_3{}'B_3{}'|}\gamma(A_3{}')\gamma(B_3{}')\\
&\qquad+l_3(R),
\end{aligned} \qquad (6.55)$$

where

$$l_3(R) \ll \prod_{P \mid R} \left( \frac{\left(1 - |P|^{-1}\right)^3}{1 + |P|^{-1}} \right) (\deg R)^3 \big(\omega(R) + \log \deg R\big) \cdot \sum_{\substack{G_3, A_3', B_3' \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G_3 A_3' \leq \frac{1}{8} \log_q \deg R \\ \deg G_3 B_3' \leq \frac{1}{8} \log_q \deg R \\ (G_3 A_3' B_3', R) = 1 \\ (A_3', B_3') = 1}} \frac{|\beta(G_3 A_3') \beta(G_3 B_3')|}{|G_3 A_3' B_3'|} \gamma(A_3') \gamma(B_3'). \tag{6.56}$$

Consider the first term on the far RHS of (6.55). We recall that $\beta(A) = 0$ if $A$ is divisible by $P^3$ for any prime $P$. Hence, defining $\Pi_{\mathcal{P},X} := \prod_{\deg P \leq X} P$, we may assume that $G_3 = IJ^2$ where $I, J \mid \Pi_{\mathcal{P},X}$, $(IJ, R) = 1$, and $(I, J) = 1$. By similar reasoning, we may assume that $A_3' = K A_3''$ where $K \mid I$, $(A_3'', RIJ) = 1$; and $B_3' = L B_3''$ where $L \mid I$, $(L, K) = 1$ and $(B_3'', RIJA_3'') = 1$. Then, by the multiplicativity of $\beta$ and $\gamma$, we have

$$\begin{aligned} &\sum_{\substack{G_3, A_3', B_3' \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G_3 A_3' \leq \frac{1}{8} \log_q \deg R \\ \deg G_3 B_3' \leq \frac{1}{8} \log_q \deg R \\ (G_3 A_3' B_3', R) = 1 \\ (A_3', B_3') = 1}} \frac{\beta(G_3 A_3') \beta(G_3 B_3')}{|G_3 A_3' B_3'|} \gamma(A_3') \gamma(B_3') \\ = &\sum_{\substack{I \mid \Pi_{\mathcal{P},X} \\ \deg I \leq \frac{1}{8} \log_q \deg R \\ (I, R) = 1}} \frac{\beta(I)^2}{|I|} \sum_{\substack{J \mid \Pi_{\mathcal{P},X} \\ \deg J \leq \frac{1}{16} \log_q \deg R - \frac{\deg I}{2} \\ (J, RI) = 1}} \frac{\beta(J^2)^2}{|J|^2} \sum_{K \mid I} \frac{\beta(K^2) \gamma(K)}{\beta(K) |K|} \sum_{\substack{L \mid I \\ (L, K) = 1}} \frac{\beta(L^2) \gamma(L)}{\beta(L) |L|} \\ &\cdot \sum_{\substack{A_3'' \mid (\Pi_{\mathcal{P},X})^2 \\ \deg A_3'' \leq \frac{1}{8} \log_q \deg R - \deg IJ^2 K \\ (A_3'', RIJ) = 1}} \frac{\beta(A_3'') \gamma(A_3'')}{|A_3''|} \sum_{\substack{B_3'' \mid (\Pi_{\mathcal{P},X})^2 \\ \deg B_3'' \leq \frac{1}{8} \log_q \deg R - \deg IJ^2 L \\ (B_3'', RIJA_3'') = 1}} \frac{\beta(B_3'') \gamma(B_3'')}{|B_3''|}. \end{aligned} \tag{6.57}$$

Consider the case where $\deg I > \frac{1}{64} \log_q \deg R$ or $\deg J > \frac{1}{64} \log_q \deg R$. Without loss of generality, suppose the former. Then, all the sums above, except that over $I$, can be bounded by $O\big((\log_q \log \deg R)^c\big)$ for some constant $c > 0$, while the sum over $I$ can be bounded by $O\big((\deg R)^{-\frac{1}{66}}\big)$ (this is obtained in the same way we have done several times before, such as in (6.8)). So, with these restrictions, we have that the above is $O\big((\deg R)^{-\frac{1}{67}}\big)$.

Now consider the case where $\deg I \leq \frac{1}{64} \log_q \deg R$ and $\deg J \leq \frac{1}{64} \log_q \deg R$. Then,

$$\frac{1}{8} \log_q \deg R - \deg IJ^2 K \geq \frac{1}{16} \log_q \deg R$$

and

$$\frac{1}{8} \log_q \deg R - \deg IJ^2 L \geq \frac{1}{16} \log_q \deg R.$$

In particular, we can apply Lemma 6.5.9 to the last two summations of (6.57):

$$\begin{aligned}
&\sum_{\substack{A_3''|(\Pi_{\mathcal{P},X})^2 \\ \deg A_3'' \leq \frac{1}{8}\log_q \deg R - \deg IJ^2K \\ (A_3'',RIJ)=1}} \frac{\beta(A_3'')\gamma(A_3'')}{|A_3''|} \sum_{\substack{B_3''|(\Pi_{\mathcal{P},X})^2 \\ \deg B_3'' \leq \frac{1}{8}\log_q \deg R - \deg IJ^2L \\ (B_3'',RIJA_3'')=1}} \frac{\beta(B_3'')\gamma(B_3'')}{|B_3''|} \\
=& \prod_{\substack{\deg P \leq X \\ (P,R)=1}} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right) \prod_{P|IJ} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1} \\
&\cdot \sum_{\substack{A_3''|(\Pi_{\mathcal{P},X})^2 \\ \deg A_3'' \leq \frac{1}{8}\log_q \deg R - \deg IJ^2K \\ (A_3'',RIJ)=1}} \frac{\beta(A_3'')\gamma(A_3'')}{|A_3''|} \prod_{P|A_3''} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1} \\
&+ O\Big((\deg R)^{-\frac{1}{17}}\Big) \\
=& \prod_{\substack{\deg P \leq X \\ (P,R)=1}} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right) \prod_{P|IJ} \left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1} \\
&\cdot \prod_{\substack{\deg P \leq X \\ (P,R)=1}} \left(1 + \left(\frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)\left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1}\right) \\
&\cdot \prod_{P|IJ} \left(1 + \left(\frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)\left(1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1}\right)^{-1} \\
&+ O\Big((\deg R)^{-\frac{1}{17}}\Big) \\
=& \prod_{\substack{\deg P \leq X \\ (P,R)=1}} \left(1 + \frac{2\beta(P)\gamma(P)}{|P|} + \frac{2\beta(P^2)\gamma(P^2)}{|P^2|}\right) \\
&\cdot \prod_{P|IJ} \left(1 + \frac{2\beta(P)\gamma(P)}{|P|} + \frac{2\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1} \\
&+ O\Big((\deg R)^{-\frac{1}{17}}\Big).
\end{aligned} \tag{6.58}$$

Consider now the two middle summations on the RHS of (6.57). We have

$$\begin{aligned}
&\sum_{K|I} \frac{\beta(K^2)\gamma(K)}{\beta(K)|K|} \sum_{\substack{L|I \\ (L,K)=1}} \frac{\beta(L^2)\gamma(L)}{\beta(L)|L|} \\
&= \prod_{P|I} \left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|}\right) \sum_{K|I} \frac{\beta(K^2)\gamma(K)}{\beta(K)|K|} \prod_{P|K} \left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)^{-1} \\
&= \prod_{P|I} \left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|}\right) \prod_{P|I} \left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|}\left(1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)^{-1}\right) \\
&= \prod_{P|I} \left(1 + \frac{2\beta(P^2)\gamma(P)}{\beta(P)|P|}\right).
\end{aligned} \tag{6.59}$$

Applying (6.58) and (6.59) to (6.57), we obtain

$$
\begin{aligned}
&\sum_{\substack{G_3,A_3',B_3'\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3A_3'\leq\frac{1}{8}\log_q\deg R\\ \deg G_3B_3'\leq\frac{1}{8}\log_q\deg R\\ (G_3A_3'B_3',R)=1\\ (A_3',B_3')=1}} \frac{\beta(G_3A_3')\beta(G_3B_3')}{|G_3A_3'B_3'|}\gamma(A_3')\gamma(B_3')\\
=&\prod_{\substack{\deg P\leq X\\ (P,R)=1}}\left(1+\frac{2\beta(P)\gamma(P)}{|P|}+\frac{2\beta(P^2)\gamma(P^2)}{|P^2|}\right)\\
&\qquad\cdot\sum_{\substack{I|\Pi_{\mathcal{P},X}\\ \deg I\leq\frac{1}{64}\log_q\deg R\\ (I,R)=1}}\frac{\beta(I)^2}{|I|}\sum_{\substack{J|\Pi_{\mathcal{P},X}\\ \deg J\leq\frac{1}{64}\log_q\deg R\\ (J,RI)=1}}\frac{\beta(J^2)^2}{|J|^2}\\
&\qquad\cdot\prod_{P|I}\left(1+\frac{2\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)\prod_{P|IJ}\left(1+\frac{2\beta(P)\gamma(P)}{|P|}+\frac{2\beta(P^2)\gamma(P^2)}{|P^2|}\right)^{-1}\\
&\quad+O\Big((\deg R)^{-\frac{1}{67}}\Big)\\
=&\prod_{\substack{\deg P\leq X\\ (P,R)=1}}\left(1+\frac{2\beta(P)\gamma(P)}{|P|}+\frac{2\beta(P^2)\gamma(P^2)}{|P^2|}+\frac{\beta(P^2)^2}{|P|^2}\right)\\
&\qquad\cdot\sum_{\substack{I|\Pi_{\mathcal{P},X}\\ \deg I\leq\frac{1}{64}\log_q\deg R\\ (I,R)=1}}\frac{\beta(I)^2}{|I|}\\
&\qquad\cdot\prod_{P|I}\left(\left(1+\frac{2\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)\left(1+\frac{2\beta(P)\gamma(P)}{|P|}+\frac{2\beta(P^2)\gamma(P^2)}{|P^2|}+\frac{\beta(P^2)^2}{|P|^2}\right)^{-1}\right)\\
&\quad+O\Big((\deg R)^{-\frac{1}{67}}\Big)\\
=&\prod_{\substack{\deg P\leq X\\ (P,R)=1}}\left(1+\frac{2\beta(P)\gamma(P)}{|P|}+\frac{2\beta(P^2)\gamma(P^2)}{|P^2|}+\frac{\beta(P^2)^2}{|P|^2}+\frac{\beta(P)^2}{|P|}\left(1+\frac{2\beta(P^2)\gamma(P)}{\beta(P)|P|}\right)\right)\\
&\quad+O\Big((\deg R)^{-\frac{1}{67}}\Big).
\end{aligned}
$$

Now, recalling the definitions of $\beta,\gamma$ (equations (6.23) and (6.52), respectively) we see that the product above is equal to

$$
\begin{aligned}
&\prod_{\substack{\deg P\leq X\\ P\nmid R}}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)\prod_{\substack{\frac{X}{2}<\deg P\leq X\\ P\nmid R}}\Big(1+O\big(|P|^{-2}\big)\Big)\\
\sim&\prod_{P|R}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)^{-1}\prod_{\substack{\deg P>X\\ P|R}}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)\prod_{\deg P\leq X}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)\\
=&\prod_{P|R}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)^{-1}\prod_{\substack{\deg P>X\\ P|R}}\left(\frac{\left(1-|P|^{-1}\right)^3}{1+|P|^{-1}}\right)
\end{aligned}
$$

$$\cdot \prod_{\deg P \leq X} \big(1-|P|^{-1}\big)^4 \prod_{\deg P \leq X} \big(1-|P|^{-2}\big)^{-1}$$

$$\sim \big(1-q^{-1}\big)^{-1} \prod_{P \mid R} \bigg( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \bigg)^{-1} \prod_{\substack{\deg P > X \\ P \mid R}} \bigg( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \bigg) \Big( \frac{1}{e^{\gamma} X} \Big)^4 ,$$

where we have used Lemma A.2.7 for the last equality. Recall that the above is to be applied to the first term on the far RHS of (6.55). We now consider $l_3(R)$: the second term on the far RHS of (6.55). By means similar to those described in the paragraph after (6.57), we can show that there is some constant $c > 0$ such that

$$\sum_{\substack{G_3, A_3', B_3' \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G_3 A_3' \leq \frac{1}{8} \log_q \deg R \\ \deg G_3 B_3' \leq \frac{1}{8} \log_q \deg R \\ (G_3 A_3' B_3', R) = 1 \\ (A_3', B_3') = 1}} \frac{|\beta(G_3 A_3') \beta(G_3 B_3')|}{|G_3 A_3' B_3'|} \gamma(A_3') \gamma(B_3') \ll X^c \ll \big( \log_q \log \deg R \big)^c .$$

We apply this to (6.56) to obtain a bound for $l_3(R)$.

Hence, considering all of the above, (6.55) becomes

$$\begin{aligned} &\sum_{\substack{A_1, A_2, B_1, B_2 \in \mathcal{M} \\ A_3, B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1 B_1, \deg A_2 B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_1 A_2 A_3 B_1 B_2 B_3, R) = 1 \\ A_1 A_2 A_3 = B_1 B_2 B_3}} \frac{\beta(A_3) \beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^{\frac{1}{2}}} \\ &\sim \frac{1}{48} \Big( \frac{\deg R}{e^{\gamma} X} \Big)^4 \prod_{\substack{\deg P > X \\ P \mid R}} \bigg( \frac{\big(1-|P|^{-1}\big)^3}{1+|P|^{-1}} \bigg) \end{aligned} \tag{6.60}$$

**Step 1.2:** We consider the second term on the far RHS of (6.45): the off-diagonal terms. We have

$$\begin{aligned} &\sum_{EF = R} \mu(E) \phi(F) \sum_{\substack{A_1, A_2, B_1, B_2 \in \mathcal{M} \\ A_3, B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1 B_1, \deg A_2 B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_1 A_2 A_3 B_1 B_2 B_3, R) = 1 \\ A_1 A_2 A_3 \equiv B_1 B_2 B_3 (\mathrm{mod}\, F) \\ A_1 A_2 A_3 \neq B_1 B_2 B_3}} \frac{\beta(A_3) \beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^{\frac{1}{2}}} \\ \leq &\sum_{\substack{A_3, B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_3, \deg B_3 \leq \frac{1}{8} \log_q \deg R \\ (A_3 B_3, R) = 1}} \frac{|\beta(A_3) \beta(B_3)|}{|A_3 B_3|^{\frac{1}{2}}} \sum_{EF = R} |\mu(E)| \phi(F) \sum_{z_1, z_2 = 0}^{z_R} q^{-\frac{z_1 + z_2}{2}} \sum_{\substack{A_1, A_2, B_1, B_2 \in \mathcal{M} \\ \deg A_1 B_1 = z_1 \\ \deg A_2 B_2 = z_2 \\ (A_1 A_2 B_1 B_2, R) = 1 \\ A_1 A_2 A_3 \equiv B_1 B_2 B_3 (\mathrm{mod}\, F) \\ A_1 A_2 A_3 \neq B_1 B_2 B_3}} 1. \end{aligned}$$

By Lemma 6.5.10 we have, for $\epsilon = \frac{1}{40}$,

$$\sum_{z_1, z_2 = 0}^{z_R} q^{-\frac{z_1 + z_2}{2}} \sum_{\substack{A_1, A_2, B_1, B_2 \in \mathcal{M} \\ \deg A_1 B_1 = z_1 \\ \deg A_2 B_2 = z_2 \\ (A_1 A_2 B_1 B_2, R) = 1 \\ A_1 A_2 A_3 \equiv B_1 B_2 B_3 (\mathrm{mod}\, F) \\ A_1 A_2 A_3 \neq B_1 B_2 B_3}} 1$$

$$\ll \frac{|A_3B_3|^{1+\frac{\epsilon}{2}}}{|F|} \sum_{\substack{z_1,z_2=0 \\ z_1+z_2+\deg A_3B_3 \leq \frac{19}{10}\deg F}}^{z_R} q^{(z_1+z_2)\left(\frac{1}{2}+\frac{\epsilon}{2}\right)}$$
$$+ \frac{|A_3B_3|}{\phi(F)} \sum_{\substack{z_1,z_2=0 \\ z_1+z_2+\deg A_3B_3 > \frac{19}{10}\deg F}}^{z_R} q^{\frac{z_1+z_2}{2}} (z_1+z_2+\deg A_3B_3)^3$$
$$\ll \frac{|A_3B_3|^{1+\epsilon}}{|F|^{\frac{1}{20}-\epsilon}} + \frac{|A_3B_3|}{\phi(F)} q^{z_R} (\deg R)^3.$$

We also have

$$\sum_{EF=R} |\mu(E)|\phi(F)\left(\frac{|A_3B_3|^{1+\epsilon}}{|F|^{\frac{1}{20}-\epsilon}} + \frac{|A_3B_3|}{\phi(F)} q^{z_R}(\deg R)^3\right)$$
$$= |A_3B_3|^{1+\epsilon} \sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|^{\frac{1}{20}-\epsilon}} + |A_3B_3| q^{z_R} (\deg R)^3 \sum_{EF=R} |\mu(E)|$$
$$\ll |A_3B_3|^{1+\epsilon}|R| + |A_3B_3R|(\deg R)^3,$$

where the last relation uses the following results:

$$\sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|^{\frac{1}{20}-\epsilon}} \leq \sum_{EF=R} |\mu(E)|\phi(F)$$
$$= \phi(R) \sum_{EF=R} |\mu(E)| \prod_{\substack{P|E \\ P^2|R}} \left(\frac{1}{|P|}\right) \prod_{\substack{P|E \\ P^2 \nmid R}} \left(\frac{1}{|P|-1}\right) \leq \phi(R) \sum_{EF=R} |\mu(E)| \prod_{P|E} \left(\frac{1}{|P|-1}\right)$$
$$= \phi(R) \prod_{P|R} \left(1 + \frac{1}{|P|-1}\right) = \phi(R) \frac{|R|}{\phi(R)} = |R|.$$

Finally, using the fact that

$$\sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_3B_3,R)=1}} |\beta(A_3)\beta(B_3)||A_3B_3|^{\frac{1}{2}+\epsilon} \leq \left(\sum_{\substack{A\in\mathcal{M} \\ \deg A \leq \frac{1}{8}\log_q \deg R}} |\beta(A)||A|^{\frac{1}{2}+\epsilon}\right)^2$$
$$\leq \left(\sum_{\substack{A\in\mathcal{M} \\ \deg A \leq \frac{1}{8}\log_q \deg R}} 2^{\omega(A)}|A|^{\frac{1}{2}+\epsilon}\right)^2 \leq \left(\sum_{\substack{A\in\mathcal{M} \\ \deg A \leq \frac{1}{8}\log_q \deg R}} d(A)|A|^{\frac{1}{2}+\epsilon}\right)^2$$
$$\leq \left(\sum_{\substack{A\in\mathcal{M} \\ \deg A \leq \frac{1}{8}\log_q \deg R}} |A|^{\frac{1}{2}+\epsilon}\right)^4 \leq (\deg R)^{\frac{7}{8}},$$

we see that

$$\frac{1}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M} \\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1B_1, \deg A_2B_2 \leq z_R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_1A_2A_3B_1B_2B_3,R)=1 \\ A_1A_2A_3 \equiv B_1B_2B_3 (\mathrm{mod}\, F) \\ A_1A_2A_3 \neq B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \ll \frac{|R|}{\phi^*(R)}(\deg R)^{3+\frac{7}{8}}.$$

This is indeed of lower order than (6.60). This can be seen by applying (A.17) and Lemma A.2.4 to the product in (6.60), and by using Lemmas A.2.5 and A.2.4 on the factor $\frac{|R|}{\phi^*(R)}$ above.

**Step 2; the asymptotic main term of $\frac{1}{\phi^*(R)}\sum^*_{\chi \bmod R} b(\chi)^2\left|\widehat{P^{**}_X}\left(\frac{1}{2},\chi\right)\right|^2$:**

We have that

$$\begin{aligned}
&\frac{1}{\phi^*(R)}\sum^*_{\chi \bmod R} b(\chi)^2\left|\widehat{P^{**}_X}\left(\frac{1}{2},\chi\right)\right|^2 \leq \frac{1}{\phi^*(R)}\sum_{\chi \bmod R} b(\chi)^2\left|\widehat{P^{**}_X}\left(\frac{1}{2},\chi\right)\right|^2\\
\leq&\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z_R<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R}} \frac{\beta(A_3)\beta(B_3)\chi(A_1A_2A_3)\overline{\chi}(B_1B_2B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}\\
=&\frac{\phi(R)}{\phi^*(R)}\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z_R<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3=B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}\\
&+\frac{\phi(R)}{\phi^*(R)}\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z_R<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3\equiv B_1B_2B_3 \pmod R\\ A_1A_2A_3\neq B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}.
\end{aligned} \tag{6.61}$$

**Step 2.1:** For the diagonal term, by similar means as in (6.46), we obtain

$$\begin{aligned}
&\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ z_R<\deg A_1B_1,\deg A_2B_2<\deg R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3=B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}\\
=&\sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3V_{3,1}V_{3,2}\leq\frac{1}{8}\log_q\deg R\\ \deg G_3V_{1,3}V_{2,3}\leq\frac{1}{8}\log_q\deg R\\ (G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2},R)=1\\ (V_{1,3}V_{2,3},V_{3,1}V_{3,2})=1}} \frac{\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})}{|G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}|}\\
&\cdot\sum_{\substack{V\in\mathcal{M}\\ \deg V\leq\deg R-\deg V_{1,3}V_{3,1}\\ \deg V\leq\deg R-\deg V_{2,3}V_{3,2}\\ \left(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\right)=1}} \frac{2^{\omega(V)-\omega\left(\left(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\right)\right)}}{|V|}\\
&\cdot\sum_{\substack{G_1,G_2\in\mathcal{M}\\ \max\left\{0,\frac{z_R-\deg VV_{1,3}V_{3,1}}{2}\right\}<\deg G_1<\frac{\deg R-\deg VV_{1,3}V_{3,1}}{2}\\ \max\left\{0,\frac{z_R-\deg VV_{2,3}V_{3,2}}{2}\right\}<\deg G_2<\frac{\deg R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}.
\end{aligned} \tag{6.62}$$

Now, if $\frac{z_R - \deg VV_{1,3}V_{3,1}}{2} \leq \log_q 3^{\omega(R)}$ then

$$\frac{\deg R - \deg VV_{1,3}V_{3,1}}{2} \leq \log_q 3^{\omega(R)} + \frac{1}{2}\log_q 2^{\omega(R)} < \log_q 6^{\omega(R)},$$

and so, by Corollary A.3.3, we have

$$\sum_{\substack{G_1 \in \mathcal{M} \\ \max\left\{0, \frac{z_R - \deg VV_{1,3}V_{3,1}}{2}\right\} < \deg G_1 < \frac{\deg R - \deg VV_{1,3}V_{3,1}}{2} \\ (G_1,R)=1}} \frac{1}{|G_1|} \leq \sum_{\substack{G_1 \in \mathcal{M} \\ \deg G_1 < \log_q 6^{\omega(R)} \\ (G_1,R)=1}} \frac{1}{|G_1|} \ll \frac{\phi(R)}{|R|}\omega(R).$$

If $\frac{z_R - \deg VV_{1,3}V_{3,1}}{2} > \log_q 3^{\omega(R)}$ then

$$\begin{aligned}
&\sum_{\substack{G_1 \in \mathcal{M} \\ \max\left\{0, \frac{z_R - \deg VV_{1,3}V_{3,1}}{2}\right\} < \deg G_1 < \frac{\deg R - \deg VV_{1,3}V_{3,1}}{2} \\ (G_1,R)=1}} \frac{1}{|G_1|} \\
=& \sum_{\substack{G_1 \in \mathcal{M} \\ \deg G_1 < \frac{\deg R - \deg VV_{1,3}V_{3,1}}{2} \\ (G_1,R)=1}} \frac{1}{|G_1|} - \sum_{\substack{G_1 \in \mathcal{M} \\ \deg G_1 < \frac{z_R - \deg VV_{1,3}V_{3,1}}{2} \\ (G_1,R)=1}} \frac{1}{|G_1|} \ll \frac{\phi(R)}{|R|}\omega(R),
\end{aligned}$$

where we have used Corollary A.3.3 twice for the last relation. Similar results hold for the sum over $G_2$. Hence, proceeding similarly as we did for the diagonal terms of $\frac{1}{\phi^*(R)} \sum^*_{\chi \bmod R} a(\chi)^2 \left| \widehat{P^{**}_X}\left(\frac{1}{2}, \chi\right)\right|^2$, we see that there is a constant $c$ such that

$$\begin{aligned}
&\frac{\phi(R)}{\phi^*(R)} \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ z < \deg A_1B_1, \deg A_2B_2 < \deg R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_1A_2A_3B_1B_2B_3,R)=1 \\ A_1A_2A_3 = B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \\
\ll& \frac{\phi(R)^3}{|R|^2\phi^*(R)}\omega(R)^2(\deg R)^2 \prod_{P|R}\left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}}\right)(\log_q \log \deg R)^c.
\end{aligned}$$

**Step 2.2:** We now look at the second term on the far RHS of (6.61): the off-diagonal terms. Using Lemma 6.5.10, we have

$$\begin{aligned}
&\frac{\phi(R)}{\phi^*(R)} \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ z_R < \deg A_1B_1, \deg A_2B_2 < \deg R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_1A_2A_3B_1B_2B_3,R)=1 \\ A_1A_2A_3 \equiv B_1B_2B_3 (\bmod R) \\ A_1A_2A_3 \neq B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \\
=& \frac{\phi(R)}{\phi^*(R)} \sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_3B_3,R)=1}} \frac{\beta(A_3)\beta(B_3)}{|A_3B_3|^{\frac{1}{2}}} \sum_{z_R < z_1, z_2 < \deg R} q^{-\frac{z_1+z_2}{2}} \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ \deg A_1B_1 = z_1 \\ \deg A_2B_2 = z_2 \\ (A_1A_2A_3B_1B_2B_3,R)=1 \\ A_1A_2A_3 \equiv B_1B_2B_3 (\bmod R) \\ A_1A_2A_3 \neq B_1B_2B_3}} 1
\end{aligned}$$

$$\ll \frac{(\deg R)^3}{\phi^*(R)} \sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_3B_3,R)=1}} |\beta(A_3)\beta(B_3)||A_3B_3|^{\frac{1}{2}} \sum_{z_R < z_1, z_2 < \deg R} q^{\frac{z_1+z_2}{2}}$$

$$\ll \frac{|R|(\deg R)^3}{\phi^*(R)} \sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_3B_3,R)=1}} |\beta(A_3)\beta(B_3)||A_3B_3|^{\frac{1}{2}} \ll \frac{|R|(\deg R)^{3+\frac{3}{4}}}{\phi^*(R)}.$$

**Step 3; the asymptotic main term of $\frac{1}{\phi^*(R)} \sum^*_{\chi \bmod R} c(\chi)^2 \left| \widehat{P^{**}_X}\left(\frac{1}{2}, \chi\right) \right|^2$:**

We recall that $c(\chi)$ differs, depending on whether $\chi$ is even or odd. Furthermore, if $\chi$ is even, then there are three terms to consider. However, by the Cauchy-Schwarz inequality, it suffices to bound the following for $i = 0, 1, 2,$:

$$\frac{1}{\phi^*(R)} \sum^*_{\substack{\chi \bmod R \\ \chi \text{ even}}} d_i(\chi)^2 \left| \widehat{P^{**}_X}\left(\frac{1}{2}, \chi\right) \right|^2,$$

where

$$d_i(\chi) := \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = \deg R - i}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}.$$

We will bound

$$\frac{1}{\phi^*(R)} \sum^*_{\substack{\chi \bmod R \\ \chi \text{ even}}} d_0(\chi)^2 \left| \widehat{P^{**}_X}\left(\frac{1}{2}, \chi\right) \right|^2.$$

The other cases for $d_i(\chi)$ and the odd case are similar.

Now, we have that

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sum^*_{\substack{\chi \bmod R \\ \chi \text{ even}}} d_0(\chi)^2 \left| \widehat{P^{**}_X}\left(\frac{1}{2}, \chi\right) \right|^2 \leq \frac{1}{\phi^*(R)} \sum_{\chi \bmod R} d_0(\chi)^2 \left| \widehat{P^{**}_X}\left(\frac{1}{2}, \chi\right) \right|^2 \\
\leq &\frac{1}{\phi^*(R)} \sum_{\chi \bmod R} \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1B_1, \deg A_2B_2 = \deg R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R}} \frac{\beta(A_3)\beta(B_3)\chi(A_1A_2A_3)\overline{\chi}(B_1B_2B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \\
= &\frac{\phi(R)}{\phi^*(R)} \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1B_1, \deg A_2B_2 = \deg R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_1A_2A_3B_1B_2B_3,R)=1 \\ A_1A_2A_3 = B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}} \\
&+ \frac{\phi(R)}{\phi^*(R)} \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg A_1B_1, \deg A_2B_2 = \deg R \\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_1A_2A_3B_1B_2B_3,R)=1 \\ A_1A_2A_3 \equiv B_1B_2B_3 (\bmod R) \\ A_1A_2A_3 \neq B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}.
\end{aligned} \tag{6.63}$$

For the first term on the far RHS of (6.63), we have, similarly to Step 2.1,

$$\begin{aligned}
&\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A_1B_1,\deg A_2B_2=\deg R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3=B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}\\
=&\sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3V_{3,1}V_{3,2}\leq\frac{1}{8}\log_q\deg R\\ \deg G_3V_{1,3}V_{2,3}\leq\frac{1}{8}\log_q\deg R\\ (G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2},R)=1\\ (V_{1,3}V_{2,3},V_{3,1}V_{3,2})=1}} \frac{\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})}{|G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}|}\\
&\cdot\sum_{\substack{V\in\mathcal{M}\\ \deg V\leq\deg R-\deg V_{1,3}V_{3,1}\\ \deg V\leq\deg R-\deg V_{2,3}V_{3,2}\\ \big(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\big)=1}} \frac{2^{\omega(V)-\omega\Big(\big(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\big)\Big)}}{|V|}\\
&\cdot\sum_{\substack{G_1,G_2\in\mathcal{M}\\ \deg G_1=\frac{\deg R-\deg VV_{1,3}V_{3,1}}{2}\\ \deg G_2=\frac{\deg R-\deg VV_{2,3}V_{3,2}}{2}\\ (G_1G_2,R)=1}} \frac{1}{|G_1G_2|}\\
\ll&\sum_{\substack{G_3,V_{1,3},V_{2,3},V_{3,1},V_{3,2}\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg G_3V_{3,1}V_{3,2}\leq\frac{1}{8}\log_q\deg R\\ \deg G_3V_{1,3}V_{2,3}\leq\frac{1}{8}\log_q\deg R\\ (G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2},R)=1\\ (V_{1,3}V_{2,3},V_{3,1}V_{3,2})=1}} \frac{|\beta(G_3V_{3,1}V_{3,2})\beta(G_3V_{1,3}V_{2,3})|}{|G_3V_{1,3}V_{2,3}V_{3,1}V_{3,2}|}\\
&\cdot\sum_{\substack{V\in\mathcal{M}\\ \deg V\leq\deg R-\deg V_{1,3}V_{3,1}\\ \deg V\leq\deg R-\deg V_{2,3}V_{3,2}\\ \big(V,R(V_{1,3}V_{3,1},V_{2,3}V_{3,2})\big)=1}} \frac{2^{\omega(V)-\omega\Big(\big(V,V_{1,3}V_{2,3}V_{3,1}V_{3,2}\big)\Big)}}{|V|}\\
\ll&(\deg R)^2\prod_{P|R}\left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}}\right)(\log_q\log\deg R)^c,
\end{aligned}$$

for some positive constant $c$.

For the second term on the far-RHS of (6.63), we have, similarly to Step 2.2,

$$\frac{\phi(R)}{\phi^*(R)}\sum_{\substack{A_1,A_2,B_1,B_2\in\mathcal{M}\\ A_3,B_3\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg A_1B_1,\deg A_2B_2=\deg R\\ \deg A_3,\deg B_3\leq\frac{1}{8}\log_q\deg R\\ (A_1A_2A_3B_1B_2B_3,R)=1\\ A_1A_2A_3\equiv B_1B_2B_3(\mathrm{mod}\, R)\\ A_1A_2A_3\neq B_1B_2B_3}} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^{\frac{1}{2}}}$$

$$
\begin{aligned}
=&\frac{\phi(R)}{|R|\phi^*(R)} \sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X)\\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_3B_3,R)=1}} \frac{\beta(A_3)\beta(B_3)}{|A_3B_3|^{\frac{1}{2}}} \sum_{\substack{A_1,A_2,B_1,B_2 \in \mathcal{M} \\ \deg A_1B_1 = \deg R \\ \deg A_2B_2 = \deg R \\ (A_1A_2A_3B_1B_2B_3,R)=1 \\ A_1A_2A_3 \equiv B_1B_2B_3 (\mathrm{mod}\, R) \\ A_1A_2A_3 \neq B_1B_2B_3}} 1 \\
\ll&\frac{|R|(\deg R)^3}{\phi^*(R)} \sum_{\substack{A_3,B_3 \in \mathcal{S}_{\mathcal{M}}(X)\\ \deg A_3, \deg B_3 \leq \frac{1}{8}\log_q \deg R \\ (A_3B_3,R)=1}} |\beta(A_3)\beta(B_3)||A_3B_3|^{\frac{1}{2}} \ll \frac{|R|(\deg R)^{3+\frac{3}{4}}}{\phi^*(R)}.
\end{aligned}
$$

□

# Chapter 7

# A Random Matrix Theory Model for the First Derivative of Dirichlet $L$-functions

## 7.1 Preliminary Results for the Moments of the Hadamard Product and its Derivative

In this section we give some preliminary results that are required for Section 7.2 where we provide support for Conjectures 2.5.2 and 2.5.10. We begin with a discussion on the equidistribution of the zeros of a typical Dirichlet $L$-function and an application of this.

For a suitable function $\phi$ and a primitive Dirichlet character $\chi$ of modulus $R \in \mathcal{M}\backslash\{1\}$, we define

$$\Delta(\chi, \phi) := \sum_{\gamma_n(\chi)} \phi\Big(\frac{\gamma_n(\chi)(\log q) \deg R}{2\pi}\Big)$$

and

$$W(R, \phi) := \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Delta(\chi, \phi).$$

That is, $\Delta(\chi, \phi)$ is the function $\phi$ evaluated and summed at the normalised (to have unit mean spacing) zeros of $L(s, \chi)$, while $W(R, \phi)$ averages this over primitive characters of modulus $R$. In [KS99], particularly $(31')$ for the unitary case and the discussion after equation (55), support is given for the idea that

$$\lim_{\deg R \longrightarrow \infty} W(R, \phi) = \int_{x=-\infty}^{\infty} \phi(x) \mathrm{d}x,$$

given certain restrictions on $\phi$.

For much of the remainder of this section, we suggest an approach to generalising the above, particularly in a way that would have applications to conjecturing moments of derivatives of Dirichlet $L$-functions. This approach is based on some initial considerations of the matter, but we make no claims on its accuracy; it should be viewed simply as a suggestion on how to proceed.

Let $\chi$ be a character of modulus $R \in \mathcal{M}\backslash\{1\}$, and suppose we have a positive valued, increasing function $c : \mathbb{Z}_{>0} \longrightarrow \mathbb{R}$ with $c(\deg R) = o(\deg R)$ as $\deg R \longrightarrow \infty$. For example, such a function could be $X = \lfloor \log_q \deg R \rfloor$. For an appropriate real function $\phi$, we define

$$\widehat{\Delta}(\chi, \phi, c) := \frac{2\pi \; c(\deg R)}{(\log q) \deg R} \sum_{\gamma_n(\chi)} \phi\big(c(\deg R) \; \gamma_n(\chi)\big) \tag{7.1}$$

and

$$\widehat{W}(R, \phi, c) := \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \widehat{\Delta}(\chi, \phi, c). \tag{7.2}$$

For $\widehat{\Delta}(\chi, \phi, c)$ we are evaluating and summing $\phi$ at scaled zeros of $L(s, \chi)$ (scaled by $c(\deg R)$). Given that $c(\deg R) = o(\deg R)$ and that the mean spacing between the zeros is $\frac{2\pi}{(\log q) \deg R}$, we see that the mean spacing between the scaled zeros still tends to 0 as $\deg R \longrightarrow \infty$. Therefore, if the zeros of $L(s, \chi)$ are equidistributed in some manner, we would expect $\widehat{\Delta}(\chi, \phi, c)$ to be roughly equal to $\int_{x=-\infty}^{\infty} \phi(x) \mathrm{d}x$, at least for large $\deg R$. While such an equidistribution of zeros is not expected for every $L(s, \chi)$, it is expected for most $L(s, \chi)$ with primitive characters. Therefore, since $\widehat{W}(R, \phi, c)$ averages $\widehat{\Delta}(\chi, \phi, c)$ over primitive characters of modulus $R$, we expect

$$\lim_{\deg R \longrightarrow \infty} \widehat{W}(R, \phi, c) = \int_{x=-\infty}^{\infty} \phi(x) \mathrm{d}x.$$

We hypothesise the following, more general result:

$$\begin{aligned} &\widehat{W}(R, \phi, c) \\ &= \int_{x=-\infty}^{\infty} \phi(x) \mathrm{d}x + \frac{\pi \; c(\deg R)}{(\log q) \deg R} \int_{x=-\infty}^{\infty} \phi'(x) \mathrm{d}x + O\left( \left( \frac{L_\phi \; c(\deg R)}{(\log q) \deg R} \right)^2 \right) \end{aligned} \tag{7.3}$$

as $\deg R \longrightarrow \infty$, where $L_\phi$ is a constant that is dependent on $\phi$. Furthermore, let $C(\mathcal{A})$ be the set of all primitive Dirichlet characters on $\mathcal{A}$, of any modulus in $\mathcal{M}\backslash\{1\}$. We further hypothesise that there is a subset $C(\mathcal{A}, c) \subseteq C(\mathcal{A})$ satisfying the following two conditions:

1. $C(\mathcal{A}, c)$ contains almost all elements of $C(\mathcal{A})$ in that
$$\lim_{n \longrightarrow \infty} \frac{|\{\chi \in C(\mathcal{A}, c) : \deg \operatorname{mod}(\chi) \leq n\}|}{|\{\chi \in C(\mathcal{A}) : \deg \operatorname{mod}(\chi) \leq n\}|} = 1,$$
where $\operatorname{mod}(\chi)$ is the modulus of the character $\chi$.

2. For all sequences $(\chi_m)_{m \in \mathbb{Z}_{>0}}$ in $C(\mathcal{A}, c)$ with $\deg \operatorname{mod}(\chi_m) \longrightarrow \infty$ as $m \longrightarrow \infty$, we have
$$\begin{aligned} &\widehat{\Delta}(\chi_m, \phi, c) \\ &= \int_{x=-\infty}^{\infty} \phi(x) \mathrm{d}x + \frac{\pi \; c\big(\deg \operatorname{mod}(\chi_m)\big)}{(\log q) \deg \operatorname{mod}(\chi_m)} \int_{x=-\infty}^{\infty} \phi'(x) \mathrm{d}x \\ &\qquad + O\left( \left( \frac{L_\phi \; c\big(\deg \operatorname{mod}(\chi_m)\big)}{(\log q) \deg \operatorname{mod}(\chi_m)} \right)^2 \right). \end{aligned} \tag{7.4}$$

This effectively says that $\widehat{\Delta}(\chi, \phi, c)$ tends to $\int_{x=-\infty}^{\infty} \phi(x)\mathrm{d}x$ as $\deg R \longrightarrow \infty$, as long as we avoid an almost empty set of characters $\chi$. One may ask why we hypothesise this. The reason is that instead of working with

$$\widehat{W}(R, \phi, c) := \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \widehat{\Delta}(\chi, \phi, c),$$

we may instead be working with, for example,

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \exp\Big(\widehat{\Delta}(\chi, \phi, c)\Big).$$

We cannot immediately pass the sum through the exponential, but we do have that

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \exp\Big(\widehat{\Delta}(\chi, \phi, c)\Big) \\
=&\frac{1}{\phi^*(R)} \sum_{\substack{\chi \bmod R \\ \chi \in C(\mathcal{A}, c)}}^{*} \exp\Big(\widehat{\Delta}(\chi, \phi, c)\Big) + \frac{1}{\phi^*(R)} \sum_{\substack{\chi \bmod R \\ \chi \notin C(\mathcal{A}, c)}}^{*} \exp\Big(\widehat{\Delta}(\chi, \phi, c)\Big) \\
\approx&\frac{1}{\phi^*(R)} \sum_{\substack{\chi \bmod R \\ \chi \in C(\mathcal{A}, c)}}^{*} \exp\Big(\int_{x=-\infty}^{\infty} \phi(x)\mathrm{d}x \ + o(1)\Big) \quad + o(1) \\
\approx&\exp\Big(\int_{x=-\infty}^{\infty} \phi(x)\mathrm{d}x\Big),
\end{aligned} \tag{7.5}$$

assuming that $C(\mathcal{A}, c)$ is large enough compared to $C(\mathcal{A})$.

One may ask what exactly "for an appropriate $\phi$" means. This will likely require that, at the very least, $\phi$ is an infinitely differentiable function except at a finite number of points (which are not singularities), as well as some bounds on its derivatives.

Now, our application of (7.4) will be in the following manner, with $c(\deg R) = (\log q)X$ and $X \sim \log_q \deg R$. Let $\chi \in C(\mathcal{A}, c)$ with modulus $R$, and let $[a, b]$ be an interval. Then, as $\deg R \longrightarrow \infty$ we have

$$\begin{aligned}
&\sum_{\gamma_n(\chi) \in [a,b]} \phi\big((\log q)X\gamma_n(\chi)\big) \\
=&\frac{\deg R}{2\pi X} \int_{t \in \big[a(\log q)X \ , \ b(\log q)X\big]} \phi(x)\mathrm{d}x + \frac{1}{2} \int_{t \in \big[a(\log q)X \ , \ b(\log q)X\big]} \phi'(x)\mathrm{d}x \\
&+ O\Big(\frac{L_\phi X}{\deg R}\Big).
\end{aligned} \tag{7.6}$$

In Section 7.2 we will use (7.6) for several different functions for $\phi$. Thus, we will need to establish some results regarding their integrals. Therefore, we give the following four lemmas.

**Lemma 7.1.1.** *We have that*

$$\int_{x=0}^{\frac{\pi}{2}} x \cot x \mathrm{d}x = \frac{\pi}{2} \log 2.$$

*Proof.* First we note that the singularity of $\cot x$ at 0 is negated by the factor of $x$, meaning the integral is well defined. Now, by the Taylor series for sin, we have

$$\lim_{x\longrightarrow 0} x\log\sin x = \lim_{x\longrightarrow 0} x\log\Big(x\big(1+O(x^2)\big)\Big) = \lim_{x\longrightarrow 0} x\log x = \lim_{x\longrightarrow\infty} -\frac{1}{x}\log x = 0. \tag{7.7}$$

From this, we have that

$$\int_{x=0}^{\frac{\pi}{2}} x\cot x\mathrm{d}x = \Big[x\log\sin x\Big]_{x=0}^{\frac{\pi}{2}} - \int_{x=0}^{\frac{\pi}{2}} \log\sin x\mathrm{d}x = -\int_{x=0}^{\frac{\pi}{2}} \log\sin x\mathrm{d}x.$$

Now, we note that, via the substitution $y=\frac{\pi}{2}-x$, we have

$$\int_{x=0}^{\frac{\pi}{2}} \log\sin x\mathrm{d}x = -\int_{y=\frac{\pi}{2}}^{0} \log\sin\Big(\frac{\pi}{2}-y\Big)\mathrm{d}y = \int_{y=0}^{\frac{\pi}{2}} \log\cos(y)\mathrm{d}y;$$

and, via the substitution $y=\pi-x$, we have

$$\int_{x=0}^{\frac{\pi}{2}} \log\sin x\mathrm{d}x = -\int_{y=\pi}^{\frac{\pi}{2}} \log\sin(\pi-y)\mathrm{d}y = \int_{y=\frac{\pi}{2}}^{\pi} \log\sin(y)\mathrm{d}y.$$

Using these two results, we have

$$\begin{aligned}
2\int_{x=0}^{\frac{\pi}{2}} \log\sin(x)\mathrm{d}x &= \int_{x=0}^{\frac{\pi}{2}} \log\sin(x)\mathrm{d}x + \int_{x=0}^{\frac{\pi}{2}} \log\cos(x)\mathrm{d}x \\
= \int_{x=0}^{\frac{\pi}{2}} \log(\sin x\cos x)\mathrm{d}x &= \int_{x=0}^{\frac{\pi}{2}} \log\Big(\frac{\sin 2x}{2}\Big)\mathrm{d}x = \int_{x=0}^{\frac{\pi}{2}} \log\sin(2x)\mathrm{d}x - \frac{\pi}{2}\log 2 \\
= \frac{1}{2}\int_{x=0}^{\pi} \log\sin(x)\mathrm{d}x - \frac{\pi}{2}\log 2 &= \int_{x=0}^{\frac{\pi}{2}} \log\sin(x)\mathrm{d}x - \frac{\pi}{2}\log 2,
\end{aligned}$$

and so

$$\int_{x=0}^{\frac{\pi}{2}} x\cot x\mathrm{d}x = -\int_{x=0}^{\frac{\pi}{2}} \log\sin x\mathrm{d}x = \frac{\pi}{2}\log 2.$$

□

For the following Lemma, recall that for real $y>0$ we have defined $\mathrm{Ci}(y) := -\int_{t=y}^{\infty}\frac{\cos(t)}{t}\mathrm{d}t$.

**Lemma 7.1.2.** *We have that*

$$\int_{x=0}^{\pi} \log(2-2\cos x)\mathrm{d}x = 0$$

*and*

$$\int_{x=0}^{\infty} \mathrm{Ci}(x)\mathrm{d}x = 0.$$

*We also have that*

$$\int_{\substack{t\in\mathbb{R}\\ |t|\le\frac{\pi}{e^\gamma}}} \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\mathrm{Ci}(|t|) - \frac{1}{2}\log\big(2-2\cos(e^\gamma t)\big)\bigg)\mathrm{d}t = 0$$

*and*

$$\int_{\substack{t\in\mathbb{R}\\ |t|>\frac{\pi}{e^\gamma}}} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Ci}(|t|)\Big)\mathrm{d}t = 0.$$

*Proof.* Using a similar result as (7.7), as well as Lemma 7.1.1, we have that

$$\begin{aligned}\int_{x=0}^{\pi} \log(2-2\cos x)\mathrm{d}x =& \Big[x\log(2-2\cos x)\Big]_{x=0}^{\pi} - 2\int_{x=0}^{\pi} \frac{x\sin x}{2-2\cos x}\mathrm{d}x \\ =& \pi\log 4 - \int_{x=0}^{\pi} \frac{x\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}}\mathrm{d}x \\ =& \pi\log 4 - 4\int_{x=0}^{\frac{\pi}{2}} x\cot x\mathrm{d}x = 0.\end{aligned}$$

For the second result, we have that

$$\begin{aligned}&\lim_{x\longrightarrow 0}\big|x\,\mathrm{Ci}(x)-\sin(x)\big| \\ =& \lim_{x\longrightarrow 0}\left| -x\int_{w=x}^{\infty}\frac{\cos(w)}{w}\mathrm{d}w + x\int_{w=x}^{\infty}\frac{\cos(w)}{w} - \frac{\sin(w)}{w^2}\mathrm{d}w\right| \\ =& \lim_{x\longrightarrow 0}\left|x\int_{w=x}^{\infty}\frac{\sin(w)}{w^2}\mathrm{d}w\right| \\ \ll& \lim_{x\longrightarrow 0}\left(x\int_{w=x}^{\frac{1}{2}}\frac{1}{w}\mathrm{d}w + x\int_{w=\frac{1}{2}}^{\infty}\frac{1}{w^2}\mathrm{d}w\right) = \lim_{x\longrightarrow 0} x\log x = 0\end{aligned}$$

and, similarly,

$$\lim_{x\longrightarrow\infty}\big(x\,\mathrm{Ci}(x)-\sin(x)\big) = \lim_{x\longrightarrow\infty}\left(x\int_{w=x}^{\infty}\frac{\sin(w)}{w^2}\mathrm{d}w\right) = \lim_{x\longrightarrow\infty}\frac{1}{x} = 0.$$

From this we deduce that

$$\int_{x=0}^{\infty}\mathrm{Ci}(x)\mathrm{d}x = \Big[x\,\mathrm{Ci}(x)-\sin(x)\Big]_{x=0}^{\infty} = 0.$$

For the third result, we must first justify why the integral is well defined. Indeed, $-\frac{1}{2}\log\big(2-2\cos(e^{\gamma}t)$ has singularity at $t=0$. Furthermore, from the well-known result that $\mathrm{Ci}(|t|)\sim\gamma+\log|t|$ as $t\longrightarrow 0$, we can see that $\mathrm{Ci}(|t|)$ also has a singularity at $t=0$. However, these two singularities cancel and so the integral is indeed well-defined. The result follows from the fact that the integrand is an odd function. Similarly, the fourth result follows from the fact that the integrand is an odd function. □

**Lemma 7.1.3.** *Suppose* $y\in\left(0,\frac{1}{2}\right)$ *and* $2\le a\le\frac{1}{y}$. *Then,*

$$\int_{|t|\le a} e^{it}\frac{e^{iyt}-1}{iyt}\mathrm{d}t \ll \log a$$

*and*

$$\int_{|t|\le a}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{it}\frac{e^{iyt}-1}{iyt}\right)\mathrm{d}t \ll \log a,$$

*where the implied constants are independent of* $y$.

*Proof.* Let $\gamma_a := \{ae^{i\theta}:\theta\in[0,\pi]\}$, and let $\Gamma_a=\gamma_a\cup[-a,a]$. We have that

$$\int_{|t|\le a} e^{it}\frac{e^{iyt}-1}{iyt}\mathrm{d}t = \int_{t\in\Gamma_a} e^{it}\frac{e^{iyt}-1}{iyt}\mathrm{d}t - \int_{t\in\gamma_a} e^{it}\frac{e^{iyt}-1}{iyt}\mathrm{d}t = -\int_{t\in\gamma_a} e^{it}\frac{e^{iyt}-1}{iyt}\mathrm{d}t,$$

since our integrand has no poles. Now, because $a \leq \frac{1}{y}$, we have

$$\max_{t\in\gamma_a}\left|\frac{e^{iyt}-1}{iyt}\right| \leq \max_{|t|\leq\frac{1}{y}}\left|\frac{e^{iyt}-1}{iyt}\right| = \max_{|t|\leq 1}\left|\frac{e^{it}-1}{it}\right| \ll 1,$$

where the implied constant is independent of $y$. Also, for $t\in\gamma_a$ with $\operatorname{Im} t > \log a$, we have $e^{it} \ll a^{-1}$. Therefore,

$$\begin{aligned}\int_{t\in\gamma_a} e^{it}\frac{e^{iyt}-1}{iyt}\mathrm{d}t &= \int_{\substack{t\in\gamma_a\\ \operatorname{Im} t\leq\log a}} e^{it}\frac{e^{iyt}-1}{iyt}\mathrm{d}t + \int_{\substack{t\in\gamma_a\\ \operatorname{Im} t>\log a}} e^{it}\frac{e^{iyt}-1}{iyt}\mathrm{d}t\\ &\ll \int_{\substack{t\in\gamma_a\\ \operatorname{Im} t\leq\log a}} 1\mathrm{d}t + a^{-1}\int_{\substack{t\in\gamma_a\\ \operatorname{Im} t>\log a}} 1\mathrm{d}t \ll \log a.\end{aligned}$$

For the second result, we similarly have that

$$\int_{|t|\leq a}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{it}\frac{e^{iyt}-1}{iyt}\right)\mathrm{d}t = -\int_{t\in\gamma_a}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{it}\frac{e^{iyt}-1}{iyt}\right)\mathrm{d}t.$$

We note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{it}\frac{e^{iyt}-1}{iyt}\right) = e^{it}\left(\frac{e^{iyt}-1}{yt} + iy\left(\frac{e^{iyt}-1-iyte^{iyt}}{(yt)^2}\right)\right),$$

and

$$\begin{aligned}&\max_{t\in\gamma_a}\left|\frac{e^{iyt}-1}{yt} + iy\left(\frac{e^{iyt}-1-iyte^{iyt}}{(yt)^2}\right)\right|\\ \leq&\max_{|t|\leq\frac{1}{y}}\left|\frac{e^{iyt}-1}{yt} + iy\left(\frac{e^{iyt}-1-iyte^{iyt}}{(yt)^2}\right)\right|\\ \leq&\max_{|t|\leq 1}\left|\frac{e^{it}-1}{t} + iy\left(\frac{e^{it}-1-ite^{it}}{t^2}\right)\right| \ll 1.\end{aligned}$$

Hence,

$$\int_{t\in\gamma_a}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{it}\frac{e^{iyt}-1}{iyt}\right)\mathrm{d}t \ll \int_{\substack{t\in\gamma_a\\ \operatorname{Im} t\leq\log a}} 1\mathrm{d}t + a^{-1}\int_{\substack{t\in\gamma_a\\ \operatorname{Im} t>\log a}} 1\mathrm{d}t \ll \log a.$$

□

**Lemma 7.1.4.** *We have that*

$$\int_{y=-\pi}^{\pi}\frac{e^{ie^{-\gamma}y}}{iy} + \frac{e^{iy}}{1-e^{iy}}\mathrm{d}y + \int_{|y|>\pi}\frac{e^{ie^{-\gamma}y}}{iy}\mathrm{d}y = 0,$$

*where we are working with Riemann integrals. We note that the first integral is well defined as the singularities at $y=0$ cancel. We also have that*

$$\int_{y=-\pi}^{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{e^{ie^{-\gamma}y}}{iy} + \frac{e^{iy}}{1-e^{iy}}\right)\mathrm{d}y + \int_{|y|>\pi}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{e^{ie^{-\gamma}y}}{iy}\right)\mathrm{d}y = 0.$$

*Proof.* We consider the first result. We have that

$$\frac{e^{iy}}{1-e^{iy}} = \frac{\cos(y)+i\sin(y)}{1-\cos(y)-i\sin(y)} \cdot \frac{1-\cos(y)+i\sin(y)}{1-\cos(y)+i\sin(y)} = \frac{\cos(y)+i\sin(y)-1}{2-2\cos(y)}$$
$$= \frac{1}{4}\frac{\cos^2\left(\frac{y}{2}\right)-\sin^2\left(\frac{y}{2}\right)+2i\sin\left(\frac{y}{2}\right)\cos\left(\frac{y}{2}\right)-1}{\sin^2\left(\frac{y}{2}\right)} = \frac{i\cos\left(\frac{y}{2}\right)}{2\sin\left(\frac{y}{2}\right)} - \frac{1}{2},$$

from which we deduce that

$$\int_{y=-\pi}^{\pi} \frac{e^{ie^{-\gamma}y}}{iy} + \frac{e^{iy}}{1-e^{iy}}\mathrm{d}y + \int_{|y|>\pi} \frac{e^{ie^{-\gamma}y}}{iy}\mathrm{d}y$$
$$= i\int_{y=-\pi}^{\pi} \frac{\cos\left(\frac{y}{2}\right)}{2\sin\left(\frac{y}{2}\right)} - \frac{\cos(e^{-\gamma}y)}{y}\mathrm{d}y + \int_{|y|>\pi} \frac{\cos(e^{-\gamma}y)}{iy}\mathrm{d}y + \int_{y=-\infty}^{\infty} \frac{\sin(e^{-\gamma}y)}{y}\mathrm{d}y - \pi.$$

For the first integral on the RHS, we see that the singularities cancel, and we have an odd function, meaning the integral evaluates to 0. The second integral is also odd and so it also evaluates to 0. The final integral is equal to $\int_{y=-\infty}^{\infty} \frac{\sin(y)}{y}\mathrm{d}y$ - one of several integrals given the name of "Dirichlet integral" - and evaluates to $\pi$. Hence, the above is 0, as required. We note that the third integral is Riemann integrable but not Lebesgue integrable.

For the second result, we have that

$$\int_{y=-\pi}^{\pi} \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{e^{ie^{-\gamma}y}}{iy} + \frac{e^{iy}}{1-e^{iy}}\right)\mathrm{d}y + \int_{|y|>\pi} \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{e^{ie^{-\gamma}y}}{iy}\right)\mathrm{d}y$$
$$= \left[\frac{e^{ie^{-\gamma}y}}{iy} + \frac{e^{iy}}{1-e^{iy}}\right]_{y=-\pi}^{\pi} + \left(\left.\frac{e^{ie^{-\gamma}y}}{iy}\right|_{y=-\pi}\right) - \left(\left.\frac{e^{ie^{-\gamma}y}}{iy}\right|_{y=\pi}\right) = 0.$$

□

We end this section with a brief discussion on the low lying zeros of Dirichlet $L$-functions. In [KS99] and [CF00], support is given for the idea that the the low lying zeros of Dirichlet $L$-functions behave similarly to the eigenphases of unitary matrices that are near 0. To be more precise, let $k \neq 0$ be an integer and let us define

$$z_k(R)[a,b] := \frac{1}{\phi^*(R)}\left|\left\{\chi \bmod R : \frac{(\log q)\deg R}{2\pi}\gamma_k(\chi) \in [a,b]\right\}\right|$$

and

$$v_k(N)[a,b] := \mathrm{Haar}\left\{A \in U(N) : \frac{N}{2\pi}\theta_k(A) \in [a,b]\right\},$$

where $\gamma_k(\chi)$ is the $k$-th zero of $L(s,\chi)$ and $\theta_k(A)$ is the $k$-th eigenphase of $A$. It can be shown that there is a measure $v_k$ such that

$$\lim_{N\longrightarrow\infty} v_k(N)[a,b] = v_k[a,b],$$

and there is support for

$$\lim_{\deg R\longrightarrow\infty} z_k(R)[a,b] \longrightarrow v_k[a,b].$$

In particular, taking $N(R) := (\log q) \deg R$, it is believed that

$$\lim_{\deg R \longrightarrow \infty} \Big( z_k(R)[a,b] - v_k\big(N(R)\big)[a,b] \Big) = 0.$$

Here, $k$ is fixed. In particular, the $k$-th zero and the $k$-th eigenphase tend to their respective central value as $\deg R, N(R) \longrightarrow \infty$. An interesting question is whether the above holds for $k$ that depends on $R$. To this end let $k(R)$ be a function of $R$. If $k(R)$ is a constant, integer-valued function, then we have just reproduced the above. If $k(R) = \left\lfloor \frac{\deg R}{2} \right\rfloor$, then we do not expect that

$$\lim_{\deg R \longrightarrow \infty} \Big( z_{k(R)}(R)[a,b] - v_{k(R)}\big(N(R)\big)[a,b] \Big) = 0,$$

because (generally) the $k(R)$-th zero and the $k(R)$-th eigenphase remain at a fixed distance from their respective central values (even as $\deg R, N(R) \longrightarrow \infty$), and we only expect similar behaviour as we approach the central values. Suppose instead that $k(R) = \left\lfloor \frac{\deg R}{2X} \right\rfloor$ where $X \sim \log_q \deg R$. While this is an increasing function, the $k(R)$-th zero and the $k(R)$-th eigenphase are (generally) within $O(X^{-1})$ of their respective central values, and so we might expect that

$$\lim_{\deg R \longrightarrow \infty} \Big( z_{k(R)}(R)[a,b] - v_{k(R)}\big(N(R)\big)[a,b] \Big) = 0.$$

We make the following, stronger hypothesis:

Let $k(R) = \left\lfloor \frac{\deg R}{2X} \right\rfloor$ where $X \sim \log_q \deg R$. Then,

$$\lim_{\deg R \longrightarrow \infty} \max_{0 < |k| \leq k(R)} \Big| z_k(R)[a,b] - v_k\big(N(R)\big)[a,b] \Big| = 0. \tag{7.8}$$

In the hypothesis above, there is nothing special about the requirement that $X \sim \log_q \deg R$. All that is required is that $k(R) = \left\lfloor \frac{\deg R}{2X} \right\rfloor = o(\deg R)$ as $\deg R \longrightarrow \infty$, so that the $k(R)$-th zero tends to the central value as $\deg R \longrightarrow \infty$. (Recall that the mean spacing of the zeros/eigenphases is $\frac{2\pi}{(\log q) \deg R}$, and so, typically, the $k(R)$-th zero is within $O_q\Big(\frac{k(R)}{\deg R}\Big)$ distance of the central value).

The hypothesis is based on some considerations of the matter, and, as before, we make no claims on its accuracy. It should be viewed as a suggestion on what to investigate if one wishes to provide stronger support for our conjectures on moments of derivatives of Dirichlet $L$-functions.

## 7.2 Moments of the Hadamard Product and its First Derivative

In this section, we provide support for Conjectures 2.5.2 and 2.5.10. We begin with Conjecture 2.5.10. First we give a lemma that simplifies our problem.

We recall from (6.11) that

$$\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big| Z_X\Big(\frac{1}{2}, \chi\Big) \Big|^{2k}$$

$$= \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \exp\left(2k \sum_{\gamma_n(\chi)} \int_{x=0}^{\infty} u(x) \operatorname{Ci}\big(|\gamma_n(\chi)|(\log q)X \log x\big) \mathrm{d}x\right).$$

**Lemma 7.2.1.** *Let $X \sim \log_q \deg R$ and let $c(\deg R) = (\log q)X$. Let $\chi \in C(\mathcal{A}, c)$ with modulus $R$. Also, as stated in Conjecture 2.5.10, assume that $\max_{x \in \mathbb{R}}\{|u'(x)|\} \ll q^X$. Then, as $\deg R \longrightarrow \infty$, we have*

$$\sum_{\gamma_n(\chi)} \int_{x=e}^{e^{1+q^{-X}}} u(x) \operatorname{Ci}\big(|\gamma_n(\chi)|(\log q)X \log x\big) \mathrm{d}x$$
$$= \sum_{\gamma_n(\chi)} \operatorname{Ci}\big(|\gamma_n(\chi)|(\log q)X\big) + O\Big(\frac{\deg R}{q^X X}\Big).$$

*In particular, the error term is $o(1)$.*

This is not surprising as $u(x)$ is normalised, and $\log x \approx 1$ for $x$ in the integration range.

*Proof.* It suffices to prove that

$$\sum_{\gamma_n(\chi)>0} \int_{x=e}^{e^{1+q^{-X}}} u(x) \operatorname{Ci}\big(\gamma_n(\chi)(\log q)X \log x\big) \mathrm{d}x$$
$$= \sum_{\gamma_n(\chi)>0} \operatorname{Ci}\big(\gamma_n(\chi)(\log q)X\big) + O\Big(\frac{\deg R}{q^X X}\Big),$$

as the case where $\gamma_n(\chi) < 0$ is almost identical.

We have that

$$\sum_{\gamma_n(\chi)>0} \int_{x=e}^{e^{1+q^{-X}}} u(x) \operatorname{Ci}\big(\gamma_n(\chi)(\log q)X \log x\big) \mathrm{d}x - \sum_{\gamma_n(\chi)>0} \operatorname{Ci}\big(\gamma_n(\chi)(\log q)X\big)$$
$$= \sum_{\gamma_n(\chi)>0} \int_{x=e}^{e^{1+q^{-X}}} u(x) \Big( \operatorname{Ci}\big(\gamma_n(\chi)(\log q)X \log x\big) - \operatorname{Ci}\big(\gamma_n(\chi)(\log q)X\big)\Big) \mathrm{d}x$$
$$= \sum_{\gamma_n(\chi)>0} \int_{x=e}^{e^{1+q^{-X}}} u(x) \int_{w=\gamma_n(\chi)(\log q)X}^{\gamma_n(\chi)(\log q)X \log x} \frac{\cos(w)}{w} \mathrm{d}w \mathrm{d}x.$$

First, consider the case where $\gamma_n(\chi) \leq q^X$. Using (7.6) for the third relation below, we have

$$\sum_{0<\gamma_n(\chi)\leq q^X} \int_{x=e}^{e^{1+q^{-X}}} u(x) \int_{w=\gamma_n(\chi)(\log q)X}^{\gamma_n(\chi)(\log q)X \log x} \frac{\cos(w)}{w} \mathrm{d}w \mathrm{d}x$$
$$= \sum_{0<\gamma_n(\chi)\leq q^X} \int_{x=e}^{e^{1+q^{-X}}} u(x) \int_{y=1}^{\log x} \frac{\cos(\gamma_n(\chi)(\log q)Xy)}{y} \mathrm{d}y \mathrm{d}x$$
$$= \int_{x=e}^{e^{1+q^{-X}}} u(x) \int_{y=1}^{\log x} \frac{1}{y} \sum_{0<\gamma_n(\chi)\leq q^X} \cos(\gamma_n(\chi)(\log q)Xy) \mathrm{d}y \mathrm{d}x$$

$$
\begin{aligned}
&\ll \frac{\deg R}{X} \int_{x=e}^{e^{1+q^{-X}}} u(x) \int_{y=1}^{\log x} \frac{1}{y} \int_{t=0}^{(\log q) X q^X} \cos(yt) \mathrm{d}t \mathrm{d}y \mathrm{d}x \\
&\ll \frac{\deg R}{X} \int_{x=e}^{e^{1+q^{-X}}} u(x) \int_{y=1}^{\log x} \frac{1}{y} \mathrm{d}y \mathrm{d}x \ll \frac{\deg R}{q^X X}.
\end{aligned}
$$

Now consider the case where $\gamma_n(\chi) > q^X$. By using integration by parts twice, we have

$$
\begin{aligned}
\int_{w=\gamma_n(\chi)(\log q)X}^{\gamma_n(\chi)(\log q)X \log x} \frac{\cos(w)}{w} \mathrm{d}w =& \frac{\sin\big(\gamma_n(\chi)(\log q)X \log x\big)}{\gamma_n(\chi)(\log q)X \log x} - \frac{\sin\big(\gamma_n(\chi)(\log q)X\big)}{\gamma_n(\chi)(\log q)X} \\
&+ \frac{\cos\big(\gamma_n(\chi)(\log q)X\big)}{\big(\gamma_n(\chi)(\log q)X\big)^2} - \frac{\cos\big(\gamma_n(\chi)(\log q)X \log x\big)}{\big(\gamma_n(\chi)(\log q)X \log x\big)^2} \\
&- 2 \int_{w=\gamma_n(\chi)(\log q)X}^{\gamma_n(\chi)(\log q)X \log x} \frac{\cos(w)}{w^3} \mathrm{d}w.
\end{aligned}
\tag{7.9}
$$

For the second term on the RHS, using (7.6), we have

$$
\begin{aligned}
&\sum_{\gamma_n(\chi) > q^X} \int_{x=0}^{\infty} u(x) \frac{\sin\big(\gamma_n(\chi)(\log q)X\big)}{\gamma_n(\chi)(\log q)X} \mathrm{d}x \\
=& \sum_{\gamma_n(\chi) > q^X} \frac{\sin\big(\gamma_n(\chi)(\log q)X\big)}{\gamma_n(\chi)(\log q)X} \ll \frac{\deg R}{X} \int_{t=(\log q)Xq^X}^{\infty} \frac{\sin(t)}{t} \mathrm{d}t \ll \frac{\deg R}{(\log q) X^2 q^X}.
\end{aligned}
$$

For the third, fourth, and fifth terms on the RHS of (7.9), we see that they are of order $\big(\gamma_n(\chi)(\log q)X\big)^{-2}$, and, again using (7.6), we have

$$
\begin{aligned}
&\sum_{\gamma_n(\chi) > q^X} \int_{x=0}^{\infty} u(x) \frac{1}{\big(\gamma_n(\chi)(\log q)X\big)^2} \mathrm{d}x \\
=& \sum_{\gamma_n(\chi) > q^X} \frac{1}{\big(\gamma_n(\chi)(\log q)X\big)^2} \ll \frac{\deg R}{X} \int_{t > (\log q)Xq^X} \frac{1}{t^2} \mathrm{d}t \ll \frac{\deg R}{(\log q) X^2 q^X}.
\end{aligned}
$$

We now look at the first term on the RHS of (7.9). We have

$$
\begin{aligned}
&\int_{x=e}^{e^{1+q^{-X}}} u(x) \frac{\sin\big(\gamma_n(\chi)(\log q)X \log x\big)}{\gamma_n(\chi)(\log q)X \log x} \mathrm{d}x \\
=& \int_{y=1}^{1+q^{-X}} \frac{u(e^y)e^y}{y} \cdot \frac{\sin\big(\gamma_n(\chi)(\log q)Xy\big)}{\gamma_n(\chi)(\log q)X} \mathrm{d}y \\
=& - \left[ \frac{u(e^y)e^y}{y} \cdot \frac{\cos\big(\gamma_n(\chi)(\log q)Xy\big)}{\big(\gamma_n(\chi)(\log q)X\big)^2} \right]_{y=1}^{1+q^{-X}} \\
&+ \int_{y=1}^{1+q^{-X}} \frac{\mathrm{d}}{\mathrm{d}y} \Big( \frac{u(e^y)e^y}{y} \Big) \frac{\cos\big(\gamma_n(\chi)(\log q)Xy\big)}{\big(\gamma_n(\chi)(\log q)X\big)^2} \mathrm{d}y.
\end{aligned}
$$

Due to the conditions on $u(x)$, the first term on the far RHS is zero. Given that $\max_{x \in \mathbb{R}}\{|u'(x)|\} \ll q^X$ and the integral is of length $q^{-X}$, we see that the second term on the far RHS is of order $\big(\gamma_n(\chi)(\log q)X\big)^{-2}$ and, as before, we have

$$
\sum_{\gamma_n(\chi) > q^X} \frac{1}{\big(\gamma_n(\chi)(\log q)X\big)^2} \ll \frac{\deg R}{(\log q) X^2 q^X}.
$$

□

Now, by Lemma 7.2.1 and similar reasoning as in (7.5), we obtain

$$\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\left|Z_X\left(\frac{1}{2},\chi\right)\right|^{2k} \sim \frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\exp\left(2k\sum_{\gamma_n(\chi)}\mathrm{Ci}\left(|\gamma_n(\chi)|(\log q)X\right)\right),$$

which we rewrite as

$$\begin{aligned}&\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\left|Z_X\left(\frac{1}{2},\chi\right)\right|^{2k}\\ \sim&\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\prod_{|\gamma_n(\chi)|\leq\frac{\pi}{(\log q)e^{\gamma}X}}\left[\left|1-e^{i(\log q)e^{\gamma}X\gamma_n(\chi)}\right|^{2k}\exp\left(2k\sum_{\gamma_n(\chi)}\mathrm{Ci}\left(|\gamma_n(\chi)|(\log q)X\right)\right.\right.\\ &\left.\left.\qquad -k\sum_{|\gamma_n(\chi)|\leq\frac{\pi}{(\log q)e^{\gamma}X}}\log\left(2-2\cos\left((\log q)e^{\gamma}X\gamma_n(\chi)\right)\right)\right)\right],\end{aligned} \tag{7.10}$$

where we have used the fact that

$$\left|1-e^{i(\log q)e^{\gamma}X\gamma_n(\chi)}\right| = \left(2-2\cos\left((\log q)e^{\gamma}X\gamma_n(\chi)\right)\right)^{\frac{1}{2}}.$$

One may ask why we have introduced the factor $e^{\gamma}$. The reason is because

$$\frac{1}{2}\log\left(2-2\cos(e^{\gamma}x)\right)\sim\gamma+\log|x|$$

as $x\longrightarrow 0$; and, as we have mentioned previously,

$$\mathrm{Ci}(|x|)\sim\gamma+\log|x|$$

as $x\longrightarrow 0$. In particular, for

$$\phi(x)=\mathrm{Ci}(|x|)-\frac{1}{2}\log\left(2-2\cos(e^{\gamma}x)\right),$$

the singularities at $x=0$ cancel, and so we can apply (7.6) to the term in the exponential in (7.10). That is, if $\chi \in C(\mathcal{A}, c)$ where $c(\deg R) = (\log q)X$, we have

$$\begin{aligned}
&\exp\Bigg(2k\sum_{\gamma_n(\chi)} \mathrm{Ci}\left(|\gamma_n(\chi)|(\log q)X\right) - k\sum_{|\gamma_n(\chi)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \log\Big(2-2\cos\left((\log q)e^{\gamma}X\gamma_n(\chi)\right)\Big)\Bigg)\\
=&\exp\Bigg(2k\Bigg(\sum_{|\gamma_n(\chi)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \mathrm{Ci}\left(|\gamma_n(\chi)|(\log q)X\right) - \frac{1}{2}\log\Big(2-2\cos\left((\log q)e^{\gamma}X\gamma_n(\chi)\right)\Big)\Bigg)\\
&\qquad + 2k\sum_{|\gamma_n(\chi)|> \frac{\pi}{(\log q)e^{\gamma}X}} \mathrm{Ci}\left(|\gamma_n(\chi)|(\log q)X\right)\Bigg)\\
=&\exp\Bigg(\frac{k\deg R}{\pi X}\int_{|t|\leq\frac{\pi}{e^{\gamma}}} \mathrm{Ci}\left(|t|\right) - \frac{1}{2}\log\Big(2-2\cos\left(e^{\gamma}t\right)\Big)\mathrm{d}t\\
&\qquad + \frac{k\deg R}{\pi X}\int_{|t|>\frac{\pi}{e^{\gamma}}} \mathrm{Ci}\left(|t|\right)\mathrm{d}t \quad + o(1)\Bigg)\\
=&\exp\Bigg(\frac{k\deg R}{\pi X}\int_{t=-\infty}^{\infty} \mathrm{Ci}\left(|t|\right)\mathrm{d}t - \frac{k\deg R}{2\pi e^{\gamma}X}\int_{|t|\leq\pi}\log\left(2-2\cos(t)\right)\mathrm{d}t \quad + o(1)\Bigg)\\
=&\exp\big(o(1)\big) = 1+o(1),
\end{aligned} \tag{7.11}$$

where the second relation uses (7.6) and the last two results in Lemma 7.1.2, and the fourth relation uses the first two results in Lemma 7.1.2. Applying this to (7.10) and using the reasoning of (7.5), we obtain

$$\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\left|Z_X\Big(\frac{1}{2},\chi\Big)\right|^{2k} \sim \frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\prod_{|\gamma_n(\chi)|\leq\frac{\pi}{(\log q)e^{\gamma}X}}\left|1-e^{i(\log q)e^{\gamma}X\gamma_n(\chi)}\right|^{2k}. \tag{7.12}$$

Equation (7.8) provides support for replacing the zeros in the product with the corresponding eigenphases of $N(R)\times N(R)$ unitary matrices, where $N(R) = \lfloor(\log q)\deg R\rfloor$. Therefore, we obtain

$$\begin{aligned}
\frac{1}{\phi^*(R)}\sum_{\chi \bmod R}^{*}\left|Z_X\Big(\frac{1}{2},\chi\Big)\right|^{2k} &\sim \int_{A\in U\left(N(R)\right)}\prod_{|\theta_n(A)|\leq\frac{\pi}{(\log q)e^{\gamma}X}}\left(1-e^{i(\log q)e^{\gamma}X\theta_n(A)}\right)\mathrm{d}A\\
&= \int_{A\in U\left(N(R)\right)}\left|\widehat{\Lambda}_{A,X}(1)\right|^{2k}\mathrm{d}A.
\end{aligned}$$

That is, we have provided support for the first relation in Conjecture 2.5.10. To summarise, in (7.10) we introduced the product which, after interchanging the zeros with eigenphases, is exactly what we want; while everything else inside the exponential cancels and contributes a lower order term, and this is possible because in the exponential we have removed the singularity at 0.

The second relation in Conjecture 2.5.10, namely that the above is

$$\sim \frac{G^2(1+k)}{G(1+2k)}\left(\frac{\deg R}{e^{\gamma}X}\right)^{k^2},$$

may follow by similar means as in [KS00b], although in this case it is more difficult as we lose structure by only considering the eigenphases up to $\frac{\pi}{(\log q)e^{\gamma}X}$. We do not investigate this further in this thesis. However, we do note that this second relation should hold if we are to have consistency with Conjecture 2.4.5.

**Remark 7.2.2.** *We recall that in Remarks 2.4.6 and 6.3.2 we describe an error in the support of Conjecture 2.4.5. This error arises by first incorrectly dismissing the zeros in* $(-\infty,\pi]\cup(\pi,\infty)$ *in (6.12), and then incorrectly including the periodicised eigenphases in* $(-\infty,\pi]\cup(\pi,\infty)$ *in (6.14). Our remedy for this is to, instead, simply replace these zeros with these eigenphases; this is justified by the fact that the zeros and eigenphases generally appear to be equidistributed and so they have the same effect in (6.12) and (6.14), respectively. Much of what we describe here has been carried out in this section. Indeed, our applications of (7.6) has allowed us to replace sums over zeros with integrals. In the same manner, we could then replace these integrals with sums over eigenphases, as required. All of this is based on the equidistribution of the zeros of a typical Dirichlet L-function and of the eigenphases of a typical unitary matrix. Of course, for this to be valid we must avoid any singularities of* $\mathrm{Ci}(x)$*, but we addressed this in this section.*

We now proceed to give support for Conjecture 2.5.2. We first note that

$$\Big|Z_X'\Big(\frac{1}{2},\chi\Big)\Big|^{2k}=\Big|\sum_{\gamma_n(\chi)}\int_{x=0}^{\infty}u(x)\frac{e^{i\gamma_n(\chi)(\log q)X\log x}}{i\gamma_n(\chi)}\mathrm{d}x\Big|^{2k}\Big|Z_X\Big(\frac{1}{2},\chi\Big)\Big|^{2k}. \tag{7.13}$$

As in Lemma 7.2.1, we wish to simplify our problem.

**Lemma 7.2.3.** *Let* $X=\lfloor\log_q\deg R\rfloor$ *and let* $c(\deg R)=(\log q)X$*. Let* $\chi\in C(\mathcal{A},c)$ *with modulus* $R$*. Also, as stated in Conjecture 2.5.2, assume that* $\max_{x\in\mathbb{R}}\{|u'(x)|\}\ll q^X$*. Then, as* $\deg R\longrightarrow\infty$*, we have*

$$\sum_{\gamma_n(\chi)}\int_{x=e}^{e^{1+q^{-X}}}u(x)\frac{e^{i\gamma_n(\chi)(\log q)X\log x}}{i\gamma_n(\chi)}\mathrm{d}x=\sum_{\gamma_n(\chi)}\frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)}+O\Big(\frac{(\log q)^2X\deg R}{q^X}\Big).$$

*In particular the error term is* $O\big((\log q)^2X\big)$*.*

*Proof.* We have that

$$\begin{aligned}&\sum_{\gamma_n(\chi)}\int_{x=e}^{e^{1+q^{-X}}}u(x)\frac{e^{i\gamma_n(\chi)(\log q)X\log x}}{i\gamma_n(\chi)}\mathrm{d}x-\sum_{\gamma_n(\chi)}\frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)}\\&=(\log q)X\sum_{\gamma_n(\chi)}\int_{y=0}^{q^{-X}}u(e^{y+1})e^{y+1}y\frac{e^{i\gamma_n(\chi)(\log q)X(y+1)}-e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q)Xy}\mathrm{d}y.\end{aligned}$$

For $|\gamma_n(\chi)|\leq\frac{q^X}{(\log q)X}$, using (7.6) and Lemma 7.1.3, we have

$$\begin{aligned}&(\log q)X\sum_{|\gamma_n(\chi)|\leq\frac{q^X}{(\log q)X}}\int_{y=0}^{q^{-X}}u(e^{y+1})e^{y+1}y\frac{e^{i\gamma_n(\chi)(\log q)X(y+1)}-e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q)Xy}\mathrm{d}y\\&=(\log q)X\int_{y=0}^{q^{-X}}u(e^{y+1})e^{y+1}y\sum_{|\gamma_n(\chi)|\leq\frac{q^X}{(\log q)X}}\frac{e^{i\gamma_n(\chi)(\log q)X(y+1)}-e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q)Xy}\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&\ll (\log q) \deg R \int_{y=0}^{q^{-X}} u(e^{y+1}) e^{y+1} y \int_{|t| \leq q^X} e^{it} \frac{e^{iyt} - 1}{iyt} \mathrm{d}t \mathrm{d}y \\
&\ll (\log q)^2 X \deg R \int_{y=0}^{q^{-X}} u(e^{y+1}) e^{y+1} y \mathrm{d}y \\
&\ll \frac{(\log q)^2 X \deg R}{q^X} \int_{y=0}^{q^{-X}} u(e^{y+1}) e^{y+1} \mathrm{d}y \ll \frac{(\log q)^2 X \deg R}{q^X}.
\end{aligned}
$$

For $|\gamma_n(\chi)| > \frac{q^X}{(\log q)X}$, using integration by parts, the fact that $\max_{x \in \mathbb{R}} \{|u'(x)|\} \ll q^X$, and (7.6), we have

$$
\begin{aligned}
&(\log q) X \sum_{|\gamma_n(\chi)| > \frac{q^X}{(\log q)X}} \int_{y=0}^{q^{-X}} u(e^{y+1}) e^{y+1} \frac{e^{i\gamma_n(\chi)(\log q)X(y+1)}}{i\gamma_n(\chi)(\log q)X} \mathrm{d}y \\
=& -(\log q) X \sum_{|\gamma_n(\chi)| > \frac{q^X}{(\log q)X}} \int_{y=0}^{q^{-X}} \frac{\mathrm{d}}{\mathrm{d}y} \big( u(e^{y+1}) e^{y+1} \big) \frac{e^{i\gamma_n(\chi)(\log q)X(y+1)}}{\big( i\gamma_n(\chi)(\log q)X \big)^2} \mathrm{d}y \\
=& (\log q) X \sum_{|\gamma_n(\chi)| > \frac{q^X}{(\log q)X}} \frac{1}{\big( \gamma_n(\chi)(\log q)X \big)^2} \int_{y=0}^{q^{-X}} \frac{\mathrm{d}}{\mathrm{d}y} \big( u(e^{y+1}) e^{y+1} \big) e^{i\gamma_n(\chi)(\log q)X(y+1)} \mathrm{d}y \\
\ll& (\log q) X \sum_{|\gamma_n(\chi)| > \frac{q^X}{(\log q)X}} \frac{1}{\big( \gamma_n(\chi)(\log q)X \big)^2} \ll (\log q) \deg R \int_{|t| > q^X} \frac{1}{t^2} \mathrm{d}t \\
\ll& \frac{(\log q) \deg R}{q^X}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(\log q) X \sum_{|\gamma_n(\chi)| > \frac{q^X}{(\log q)X}} \int_{y=0}^{q^{-X}} u(e^{y+1}) e^{y+1} \frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q)X} \mathrm{d}y \\
=& (\log q) X \sum_{|\gamma_n(\chi)| > \frac{q^X}{(\log q)X}} \frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q)X} \\
\ll& (\log q) \deg R \int_{|t| > q^X} \frac{e^{it}}{it} \mathrm{d}t \ll \frac{(\log q) \deg R}{q^X}.
\end{aligned}
$$

The proof follows. □

Now, by (7.13), Lemma 7.2.3, and similar reasoning as in (7.5), we obtain

$$
\begin{aligned}
&\frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big| Z'_X\Big(\frac{1}{2}, \chi\Big) \Big|^{2k} \\
\sim& \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big| \sum_{\gamma_n(\chi)} \frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)} + O\Big( \frac{(\log q)^2 X \deg R}{q^X} \Big) \Big|^{2k} \Big| Z_X\Big(\frac{1}{2}, \chi\Big) \Big|^{2k} \qquad (7.14) \\
\sim& \frac{\big( (\log q) e^{\gamma} X \big)^{2k}}{\phi^*(R)} \sum_{\chi \bmod R}^{*} \Big| \sum_{\gamma_n(\chi)} \frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q) e^{\gamma} X} \Big|^{2k} \Big| Z_X\Big(\frac{1}{2}, \chi\Big) \Big|^{2k}.
\end{aligned}
$$

The second relation uses the Cauchy-Schwarz inequality and Conjecture 2.5.10 (and, strictly speaking, it also uses some of the results that we establish below). Now, for $\chi \in C(\mathcal{A}, c)$, where $c(\deg R) = (\log q)X$, we have that

$$\begin{aligned}
&\sum_{\gamma_n(\chi)} \frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q)e^{\gamma}X} + \sum_{|\gamma_n(\chi)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \frac{e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}}{1-e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}} \\
=&\sum_{|\gamma_n(\chi)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \left( \frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q)e^{\gamma}X} + \frac{e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}}{1-e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}} \right) \\
&\qquad + \sum_{|\gamma_n(\chi)|> \frac{\pi}{(\log q)e^{\gamma}X}} \frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q)e^{\gamma}X} \\
=&\frac{\deg R}{2\pi X}\int_{|t|\leq \frac{\pi}{e^{\gamma}}} \left( \frac{e^{it}}{ie^{\gamma}t} + \frac{e^{ie^{\gamma}t}}{1-e^{ie^{\gamma}t}} \right)\mathrm{d}t + \frac{\deg R}{2\pi X}\int_{|t|>\frac{\pi}{e^{\gamma}}} \frac{e^{it}}{ie^{\gamma}t}\mathrm{d}t \quad + o(1) \\
=&\frac{\deg R}{2\pi e^{\gamma}X}\int_{|t|\leq \pi} \left( \frac{e^{ie^{-\gamma}t}}{it} + \frac{e^{it}}{1-e^{it}} \right)\mathrm{d}t + \frac{\deg R}{2\pi e^{\gamma}X}\int_{|t|>\pi} \frac{e^{ie^{-\gamma}t}}{it}\mathrm{d}t \quad + o(1) \\
=&o(1),
\end{aligned}$$

where the second equality uses (7.6) and the second result in Lemma 7.1.4, and the last equality uses the first result in Lemma 7.1.4. Hence, we have

$$\sum_{\gamma_n(\chi)} \frac{e^{i\gamma_n(\chi)(\log q)X}}{i\gamma_n(\chi)(\log q)e^{\gamma}X} = -\sum_{|\gamma_n(\chi)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \frac{e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}}{1-e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}} + o(1).$$

Applying this to (7.14) and using similar reasoning as in (7.5), we obtain

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| Z_X'\left(\frac{1}{2}, \chi\right) \right|^{2k} \\
\sim&\frac{\big((\log q)e^{\gamma}X\big)^{2k}}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| -\sum_{|\gamma_n(\chi)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \frac{e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}}{1-e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}} \right|^{2k} \left| Z_X\left(\frac{1}{2}, \chi\right) \right|^{2k} \\
=&\frac{\big((\log q)e^{\gamma}X\big)^{2k}}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| \sum_{|\gamma_n(\chi)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \frac{e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}}{1-e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}} \right|^{2k} \left| Z_X\left(\frac{1}{2}, \chi\right) \right|^{2k}.
\end{aligned}$$

By similar means used to obtain (7.12), and by (7.8), we obtain

$$\begin{aligned}
&\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| Z_X'\left(\frac{1}{2}, \chi\right) \right|^{2k} \\
\sim&\frac{\big((\log q)e^{\gamma}X\big)^{2k}}{\phi^*(R)} \sideset{}{^*}\sum_{\chi \bmod R} \left| \sum_{|\gamma_n(\chi)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \frac{e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}}{1-e^{i\gamma_n(\chi)(\log q)e^{\gamma}X}} \right|^{2k} \\
&\qquad\qquad \cdot \prod_{|\gamma_n(\chi)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \left| 1-e^{i\gamma_n(\chi)(\log q)e^{\gamma}X} \right|^{2k} \\
\sim&\big((\log q)e^{\gamma}X\big)^{2k} \int_{A\in U\left(N(R)\right)} \left| \sum_{|\theta_n(A)|\leq \frac{\pi}{(\log q)e^{\gamma}X}} \frac{e^{i\theta_n(A)(\log q)e^{\gamma}X}}{1-e^{i\theta_n(A)(\log q)e^{\gamma}X}} \right|^{2k}
\end{aligned}$$

$$\cdot \prod_{|\theta_n(A)| \leq \frac{\pi}{(\log q)e^\gamma X}} \left| 1 - e^{i\theta_n(A)(\log q)e^\gamma X} \right|^{2k} \mathrm{d}A$$

$$= \int_{A \in U\left(N(R)\right)} \left|\widehat{\Lambda}'_{A,X}(1)\right|^{2k} \mathrm{d}A.$$

So, we have provided support for the first relation in Conjecture 2.5.2. The second relation in Conjecture 2.5.2, namely that the above is

$$\sim b_k N(R)^{2k} \left( \frac{N(R)}{(\log q)e^\gamma X} \right)^{k^2},$$

is based on Theorem 1 of [CRS06]. One may be able to use the methods in [CRS06] to obtain this rigorously, although it would be more difficult as we lose structure by only considering the eigenphases up to $\frac{\pi}{(\log q)e^\gamma X}$. We do not investigate this further in this thesis. However, we do note that the second relation is consistent with Theorem 2.5.7 (which is rigorously established and not conjecture), as well as being consistent with what we would expect the fourth moment,

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| Z'_X\left(\frac{1}{2}, \chi\right) \right|^4,$$

to be if we are to reproduce (2.3) by applying (2.6), Conjecture 2.5.1, and the Cauchy-Schwarz inequality.

## 7.3 Moments of the First Derivative of the Euler Product

In this section we prove Theorem 2.5.3. For Theorem 2.4.4, where we were not working with derivatives, we showed that

$$\begin{aligned} P_X\left(\frac{1}{2}, \chi\right)^k &= \left(1 + O\left(\frac{k}{X}\right)\right) P^*_X\left(\frac{1}{2}, \chi\right)^k \\ &= \left(1 + O\left(\frac{k}{X}\right)\right) \sum_{A \in \mathcal{S}_\mathcal{M}(X)} \frac{\alpha_k(A)\chi(A)}{|A|^{\frac{1}{2}}}, \end{aligned}$$

where

$$P^*_X(s, \chi) := \prod_{\deg P \leq X} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1} \prod_{\frac{X}{2} < \deg P \leq X} \left(1 + \frac{\chi(P)^2}{2|P|^{2s}}\right)^{-1}$$

and

$$\alpha_k(A) \begin{cases} = d_k(A) & \text{if } A \in \mathcal{S}_\mathcal{M}\left(\frac{X}{2}\right) \text{ or } A \text{ is prime} \\ \leq d_k(A) & \text{if } A \notin \mathcal{S}_\mathcal{M}\left(\frac{X}{2}\right) \text{ and } A \text{ is not prime.} \end{cases}$$

We prove a similar lemma.

**Lemma 7.3.1.** *For primitive characters $\chi$ of modulus $R \in \mathcal{M}\backslash\{1\}$, we have that*

$$
\begin{aligned}
&P'_X\Big(\frac{1}{2},\chi\Big)^k \\
=&\Big( -\sum_{\deg P \leq X} \frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}} - \sum_{\deg P \leq \frac{X}{2}} \frac{\log|P|\chi(P)^2}{|P|} + O(1) \Big)^k P^*_X\Big(\frac{1}{2},\chi\Big)^k \Big(1 + O\Big(\frac{k}{X}\Big)\Big).
\end{aligned}
$$

*Proof.* Defining $N_P := \lfloor \frac{X}{\deg P} \rfloor$, we have that

$$
\begin{aligned}
&P'_X\Big(\frac{1}{2},\chi\Big)^k \\
=&\Big( - \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq X}} \frac{\chi(A)\Lambda(A)}{|A|^{\frac{1}{2}}} \Big)^k P_X\Big(\frac{1}{2},\chi\Big)^k \\
=&\Big( - \sum_{\deg P \leq X} \log|P| \sum_{i=1}^{N_P} \frac{\chi(P)^i}{|P|^{\frac{i}{2}}} \Big)^k P_X\Big(\frac{1}{2},\chi\Big)^k \\
=&\Big( - \sum_{\deg P \leq X} \frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}} - \sum_{\deg P \leq \frac{X}{2}} \frac{\log|P|\chi(P)^2}{|P|} + O(1) \Big)^k P_X\Big(\frac{1}{2},\chi\Big)^k \\
=&\Big( - \sum_{\deg P \leq X} \frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}} - \sum_{\deg P \leq \frac{X}{2}} \frac{\log|P|\chi(P)^2}{|P|} + O(1) \Big)^k P^*_X\Big(\frac{1}{2},\chi\Big)^k \Big(1 + O\Big(\frac{k}{X}\Big)\Big),
\end{aligned}
$$

where the last equality uses Lemma 6.2.1. □

We also require the following lemma.

**Lemma 7.3.2.** *As $X, \deg R \xrightarrow{q,k} \infty$ with $X \leq \log_q \deg R$, we have*

$$
\sum_{\substack{\deg P_1 \leq X \\ \vdots \\ \deg P_l \leq X}} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg P_1 \dots P_l A > \frac{1}{2} \deg R}} \frac{\log|P_1| \dots \log|P_l|}{|P_1 \dots P_l|^{\frac{1}{2}}} \frac{\alpha_k(A)}{|A|^{\frac{1}{2}}} \chi(P_1 \dots P_l A) \ll |R|^{-\frac{1}{9}}.
$$

*Proof.* We have that

$$
\begin{aligned}
&\sum_{\substack{\deg P_1 \leq X \\ \vdots \\ \deg P_l \leq X}} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg P_1 \dots P_l A > \frac{1}{2} \deg R}} \frac{\log|P_1| \dots \log|P_l|}{|P_1 \dots P_l|^{\frac{1}{2}}} \frac{\alpha_k(A)}{|A|^{\frac{1}{2}}} \chi(P_1 \dots P_l A) \\
\ll& |R|^{-\frac{1}{8}} \sum_{\substack{\deg P_1 \leq X \\ \vdots \\ \deg P_l \leq X}} \sum_{A \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\log|P_1| \dots \log|P_l|}{|P_1 \dots P_l|^{\frac{1}{4}}} \frac{d_k(A)}{|A|^{\frac{1}{4}}} \\
\ll& |R|^{-\frac{1}{8}} \Big( \sum_{\deg P \leq X} \frac{\log|P|}{|P|^{\frac{1}{4}}} \Big)^l \Big( \sum_{A \in \mathcal{S}_{\mathcal{M}}(X)} \frac{1}{|A|^{\frac{1}{4}}} \Big)^k \\
\ll& |R|^{-\frac{1}{8}} (\deg R)^l \exp\Big( O\Big(k \frac{\deg R}{X}\Big)\Big) \ll |R|^{-\frac{1}{9}}.
\end{aligned}
$$

□

We now prove Theorem 2.5.3, which we restate for ease of reference.

**Theorem.** *Let $k \geq 0$ be an integer. As $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$,*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| P'_X\Big(\frac{1}{2}, \chi\Big)\Big|^{2k} \sim a(k) c_k(k) (\log q)^{2k} X^{2k} \Big(e^{\gamma} X\Big)^{k^2},$$

*where, for $l = 0, \ldots, 2k$, we define*

$$c_k(l) := \sum_{m=0}^{l} \binom{l}{m} \binom{l}{m} m! \frac{1}{2^m} k^{2(l-m)}.$$

While we only require $c_k(k)$ in the theorem above, we require $c_k(0), \ldots, c_k(2k)$ for the proof of the theorem.

*Proof of Theorem 2.5.3.* Throughout this proof, all asymptotic relations should be taken as $X, \deg Q \xrightarrow{q,k} \infty$ with $X \leq \log_q \deg Q$.

By Lemma 7.3.1, we have

$$\begin{aligned} &\sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| P'_X\Big(\frac{1}{2}, \chi\Big)\Big|^{2k} \\ \sim &\sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| \sum_{\deg P \leq X} \frac{\log|P| \chi(P)}{|P|^{\frac{1}{2}}} + \sum_{\deg P \leq X} \frac{\log|P| \chi(P)^2}{|P|} + O(1)\Big|^{2k} \Big| P^*_X\Big(\frac{1}{2}, \chi\Big)\Big|^{2k}. \end{aligned}$$

By the Cauchy-Schwarz inequality, it suffices to show that, for $l = 1, \ldots, 2k$,

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| \sum_{\deg P \leq X} \frac{\log|P| \chi(P)}{|P|^{\frac{1}{2}}}\Big|^{2l} \Big| P^*_X\Big(\frac{1}{2}, \chi\Big)\Big|^{2k} \sim a(k) c_k(l) (\log q)^{2l} X^{2l} \Big(e^{\gamma} X\Big)^{k^2};$$

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| \sum_{\deg P \leq X} \frac{\log|P| \chi(P)^2}{|P|}\Big|^{2l} \Big| P^*_X\Big(\frac{1}{2}, \chi\Big)\Big|^{2k} \ll_k X^{k^2};$$

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| P^*_X\Big(\frac{1}{2}, \chi\Big)\Big|^{2k} \ll_k X^{k^2}.$$

We will prove the first result. The second is similar to the first, and the third follows immediately from Theorem 2.4.4. By Lemma 7.3.2, it suffices to show that

$$\begin{aligned} &\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| \sum_{\substack{\deg P_1 \leq X \\ \vdots \\ \deg P_l \leq X}} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg P_1 \ldots P_l A \leq \frac{1}{2} \deg Q}} \frac{\log|P_1| \ldots \log|P_l|}{|P_1 \ldots P_l|^{\frac{1}{2}}} \frac{\alpha_k(A)}{|A|^{\frac{1}{2}}} \chi(P_1 \ldots P_l A)\Big|^2 \\ \sim &a(k) c_k(l) (\log q)^{2l} X^{2l} \Big(e^{\gamma} X\Big)^{k^2}. \end{aligned}$$

To this end, we have that

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Big| \sum_{\substack{\deg P_1 \leq X \\ \vdots \\ \deg P_l \leq X}} \sum_{\substack{A \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg P_1 \ldots P_l A \leq \frac{1}{2} \deg Q}} \frac{\log|P_1| \ldots \log|P_l|}{|P_1 \ldots P_l|^{\frac{1}{2}}} \frac{\alpha_k(A)}{|A|^{\frac{1}{2}}} \chi(P_1 \ldots P_l A)\Big|^2$$

$$= \sum_{\substack{\deg P_1 \leq X \\ \vdots \\ \deg P_l \leq X}} \sum_{\substack{\deg Q_1 \leq X \\ \vdots \\ \deg Q_l \leq X}} \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg P_1 \dots P_l A \leq \frac{1}{2}\deg Q \\ \deg Q_1 \dots Q_l B \leq \frac{1}{2}\deg Q \\ P_1 \dots P_l A \equiv Q_1 \dots Q_l B (\mathrm{mod}\, Q)}} \prod_{i=1}^{l} \left(\frac{\log|P_i|}{|P_i|^{\frac{1}{2}}}\right) \prod_{i=1}^{l} \left(\frac{\log|Q_i|}{|Q_i|^{\frac{1}{2}}}\right) \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}}$$

$$- \frac{1}{\phi(Q)} \sum_{\substack{\deg P_1 \leq X \\ \vdots \\ \deg P_l \leq X}} \sum_{\substack{\deg Q_1 \leq X \\ \vdots \\ \deg Q_l \leq X}} \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg P_1 \dots P_l A \leq \frac{1}{2}\deg Q \\ \deg Q_1 \dots Q_l B \leq \frac{1}{2}\deg Q}} \prod_{i=1}^{l} \left(\frac{\log|P_i|}{|P_i|^{\frac{1}{2}}}\right) \prod_{i=1}^{l} \left(\frac{\log|Q_i|}{|Q_i|^{\frac{1}{2}}}\right) \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}}.$$

We note that, when $\deg P_1 \dots P_l A \leq \frac{1}{2}\deg Q$ and $\deg Q_1 \dots Q_l B \leq \frac{1}{2}\deg Q$, the condition $P_1 \dots P_l A \equiv Q_1 \dots Q_l B (\mathrm{mod}\, Q)$ can only be satisfied if $P_1 \dots P_l A = Q_1 \dots Q_l B$. Then, for this case of equality, we can remove the conditions $\deg P_1 \dots P_l A \leq \frac{1}{2}\deg Q$ and $\deg Q_1 \dots Q_l B \leq \frac{1}{2}\deg Q$ by similar means as in Lemma 7.3.2. Also,

$$\begin{aligned} &\frac{1}{\phi(Q)} \sum_{\substack{\deg P_1 \leq X \\ \vdots \\ \deg P_l \leq X}} \sum_{\substack{\deg Q_1 \leq X \\ \vdots \\ \deg Q_l \leq X}} \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg P_1 \dots P_l A \leq \frac{1}{2}\deg Q \\ \deg Q_1 \dots Q_l B \leq \frac{1}{2}\deg Q}} \prod_{i=1}^{l} \left(\frac{\log|P_i|}{|P_i|^{\frac{1}{2}}}\right) \prod_{i=1}^{l} \left(\frac{\log|Q_i|}{|Q_i|^{\frac{1}{2}}}\right) \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}} \\ \ll &\frac{1}{\phi(Q)} \left( \sum_{\deg P \leq X} \frac{\log|P|}{|P|^{\frac{1}{2}}} \right)^{2l} \left( \sum_{A \in \mathcal{S}_{\mathcal{M}}(X)} \frac{1}{|A|^{\frac{1}{2}}} \right)^{2k} \\ \ll &\frac{(\deg Q)^{2l}}{\phi(Q)} \exp\left( O_k\left(\frac{\deg Q}{X}\right)\right) = o(1). \end{aligned}$$

By these three points, it suffices to show that

$$\begin{aligned} &\sum_{\substack{\deg P_1 \leq X \\ \vdots \\ \deg P_l \leq X}} \sum_{\substack{\deg Q_1 \leq X \\ \vdots \\ \deg Q_l \leq X}} \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ P_1 \dots P_l A = Q_1 \dots Q_l B}} \prod_{i=1}^{l} \left(\frac{\log|P_i|}{|P_i|^{\frac{1}{2}}}\right) \prod_{i=1}^{l} \left(\frac{\log|Q_i|}{|Q_i|^{\frac{1}{2}}}\right) \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}} \\ \sim &a(k) c_k(l) (\log q)^{2l} X^{2l} \left(e^{\gamma} X\right)^{k^2}. \end{aligned}$$

Consider the case where $P_1, \dots, P_l$ are distinct and $Q_1, \dots, Q_l$ are distinct. As we will see, this case contributes the main term. We will condition on the number of $P_i$ that are equal to some $Q_j$. Suppose there are exactly $m$ such $P_i$. There are $\binom{l}{m}$ ways of choosing such $P_i$; there are $\binom{l}{m}$ ways of choosing such $Q_j$; and there are $m!$ ways of equating them. By symmetry, each such case is, without loss of generality, equal to the case where $P_i = Q_i$ for $i = 1, \dots, m$. The requirement that $P_1 \dots P_l A = Q_1 \dots Q_l B$ then becomes $P_{m+1} \dots P_l A = Q_{m+1} \dots Q_l B$. This implies that $A = Q_{m+1} \dots Q_l C$ and $B = P_{m+1} \dots P_l C$ for some $C \in \mathcal{S}_{\mathcal{M}}(X)$. That is, we

have

$$\sum_{\substack{\deg P_1 \le X \\ \vdots \\ \deg P_l \le X \\ P_1,\dots,P_l \text{ distinct}}} \sum_{\substack{\deg Q_1 \le X \\ \vdots \\ \deg Q_l \le X \\ Q_1,\dots,Q_l \text{ distinct}}} \sum_{\substack{A,B \in \mathcal{S}_{\mathcal{M}}(X) \\ P_1 \dots P_l A = Q_1 \dots Q_l B}} \prod_{i=1}^{l} \left( \frac{\log|P_i|}{|P_i|^{\frac{1}{2}}} \right) \prod_{i=1}^{l} \left( \frac{\log|Q_i|}{|Q_i|^{\frac{1}{2}}} \right) \frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}}$$

$$= \sum_{m=0}^{l} \binom{l}{m}\binom{l}{m} m! \sum_{\substack{\deg P_1 \le X \\ \vdots \\ \deg P_l \le X \\ P_1,\dots,P_l \text{ distinct}}} \sum_{\substack{\deg Q_{m+1} \le X \\ \vdots \\ \deg Q_l \le X \\ P_1,\dots,P_l,Q_{m+1},\dots,Q_l \text{ distinct}}} \sum_{C \in \mathcal{S}_{\mathcal{M}}(X)} \prod_{i=1}^{m} \left( \frac{(\log|P_i|)^2}{|P_i|} \right) \prod_{i=m+1}^{l} \left( \frac{\log|P_i|}{|P_i|} \right) \prod_{i=m+1}^{l} \left( \frac{\log|Q_i|}{|Q_i|} \right) \cdot \frac{\alpha_k(Q_{m+1} \dots Q_l C)\alpha_k(P_{m+1} \dots P_l C)}{|C|}$$

$$= \sum_{m=0}^{l} \binom{l}{m}\binom{l}{m} m! \sum_{\substack{\deg P_1 \le X \\ \vdots \\ \deg P_l \le X \\ P_1,\dots,P_l \text{ distinct}}} \sum_{\substack{\deg Q_{m+1} \le X \\ \vdots \\ \deg Q_l \le X \\ P_1,\dots,P_l,Q_{m+1},\dots,Q_l \text{ distinct}}} \prod_{i=1}^{m} \left( \frac{(\log|P_i|)^2}{|P_i|} \right) \prod_{i=m+1}^{l} \left( \log|P_i| \frac{\sum_{j=0}^{\infty} \frac{\alpha_k({P_i}^j)\alpha_k({P_i}^{j+1})}{|P_i|^{j+1}}}{\sum_{j=0}^{\infty} \frac{\alpha_k({P_i}^j)^2}{|P_i|^j}} \right) \prod_{i=m+1}^{l} \left( \log|Q_i| \frac{\sum_{j=0}^{\infty} \frac{\alpha_k({Q_i}^j)\alpha_k({Q_i}^{j+1})}{|Q_i|^{j+1}}}{\sum_{j=0}^{\infty} \frac{\alpha_k({Q_i}^j)^2}{|Q_i|^j}} \right) \sum_{C \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\alpha_k(C)^2}{|C|}. \tag{7.15}$$

From (6.10), we have

$$\sum_{C \in \mathcal{S}_{\mathcal{M}}(X)} \frac{\alpha_k(C)^2}{|C|} = \big(1 + o(1)\big) a(k) \Big(e^{\gamma} X\Big)^{k^2}. \tag{7.16}$$

Also, for any prime $P$,

$$\frac{\sum_{j=0}^{\infty} \frac{\alpha_k(P^j)\alpha_k(P^{j+1})}{|P|^{j+1}}}{\sum_{j=0}^{\infty} \frac{\alpha_k(P^j)^2}{|P|^j}} = \frac{\frac{k}{|P|} + O_k\Big(\frac{1}{|P|^2}\Big)}{1 + O_k\Big(\frac{1}{|P|}\Big)} = \frac{k}{|P|} + O_k\Big(\frac{1}{|P|^2}\Big), \tag{7.17}$$

and so

$$\sum_{\substack{\deg Q_l \le X \\ P_1,\dots,P_l,Q_{m+1},\dots,Q_l \text{ distinct}}} \log|Q_l| \frac{\sum_{j=0}^{\infty} \frac{\alpha_k({Q_l}^j)\alpha_k({Q_l}^{j+1})}{|Q_l|^{j+1}}}{\sum_{j=0}^{\infty} \frac{\alpha_k({Q_l}^j)^2}{|Q_l|^j}}$$

$$= \sum_{\substack{\deg Q_l \le X \\ P_1,\dots,P_l,Q_{m+1},\dots,Q_l \text{ distinct}}} \log|Q_l| \bigg( \frac{k}{|Q_l|} + O_k\Big(\frac{1}{|Q_l|^2}\Big) \bigg) = \big(k(\log q)X + O_k(1)\big).$$

Applying the method above inductively, we obtain

$$\sum_{\substack{\deg Q_{m+1}\leq X \\ \vdots \\ \deg Q_l \leq X \\ P_1,\ldots,P_l,Q_{m+1},\ldots,Q_l \text{ distinct}}} \prod_{i=m+1}^{l}\left(\log|Q_i|\frac{\sum_{j=0}^{\infty}\frac{\alpha_k({Q_i}^j)\alpha_k({Q_i}^{j+1})}{|Q_i|^{j+1}}}{\sum_{j=0}^{\infty}\frac{\alpha_k({Q_i}^j)^2}{|Q_i|^j}}\right)$$
$$=\left(k(\log q)X+O_k(1)\right)^{l-m}.$$

Similarly

$$\sum_{\substack{\deg P_{m+1}\leq X \\ \vdots \\ \deg P_l \leq X \\ P_1\ldots P_l \text{ distinct}}} \prod_{i=m+1}^{l}\left(\log|P_i|\frac{\sum_{j=0}^{\infty}\frac{\alpha_k({P_i}^j)\alpha_k({P_i}^{j+1})}{|P_i|^{j+1}}}{\sum_{j=0}^{\infty}\frac{\alpha_k({P_i}^j)^2}{|P_i|^j}}\right)=\left(k(\log q)X+O_k(1)\right)^{l-m}$$

and

$$\sum_{\substack{\deg P_1\leq X \\ \vdots \\ \deg P_m \leq X \\ P_1\ldots P_m \text{ distinct}}} \prod_{i=1}^{m}\left(\frac{(\log|P_i|)^2}{|P_i|}\right)=\left((\log q)^2\frac{X^2}{2}\left(1+O(X^{-1})\right)\right)^m.$$

So, we have

$$\sum_{\substack{\deg P_1\leq X \\ \vdots \\ \deg P_l \leq X \\ P_1,\ldots,P_l \text{ distinct}}}\ \sum_{\substack{\deg Q_1\leq X \\ \vdots \\ \deg Q_l \leq X \\ Q_1,\ldots,Q_l \text{ distinct}}}\ \sum_{\substack{A,B\in\mathcal{S}_{\mathcal{M}}(X) \\ P_1\ldots P_lA=Q_1\ldots Q_lB}} \prod_{i=1}^{l}\left(\frac{\log|P_i|}{|P_i|^{\frac{1}{2}}}\right)\prod_{i=1}^{l}\left(\frac{\log|Q_i|}{|Q_i|^{\frac{1}{2}}}\right)\frac{\alpha_k(A)\alpha_k(B)}{|AB|^{\frac{1}{2}}}$$
$$\sim a(k)\left(\sum_{m=0}^{l}\binom{l}{m}\binom{l}{m}m!\frac{1}{2^m}k^{2(l-m)}\right)(\log q)^{2l}X^{2l}\left(e^{\gamma}X\right)^{k^2}$$
$$=a(k)c_k(l)(\log q)^{2l}X^{2l}\left(e^{\gamma}X\right)^{k^2},$$

as required. We must remark that, when $m=l$, the sums

$$\sum_{\substack{\deg Q_{m+1}\leq X \\ \vdots \\ \deg Q_l \leq X \\ P_1,\ldots,P_l,Q_{m+1},\ldots,Q_l \text{ distinct}}} \quad \text{and} \quad \sum_{\substack{\deg P_{m+1}\leq X \\ \vdots \\ \deg P_l \leq X \\ P_1\ldots P_l \text{ distinct}}}$$

should be taken as there being no summation; they should not be taken as the empty sum which evaluates to zero. Similarly, when $m=0$, the sum

$$\sum_{\substack{\deg P_1\leq X \\ \vdots \\ \deg P_m \leq X \\ P_1\ldots P_m \text{ distinct}}}$$

should be taken as there being no sum.

Now consider the case where at least two of $P_1, \dots, P_l$ are equal or at least two of $Q_1, \dots, Q_l$ are equal. This case contributes lower order terms, ultimately because we have at least one less degree of freedom in the $P_i$ or in the $Q_j$, and this translates to at least one less factor of $X$ in the final expression. Upon initial considerations, this is certainly believable, but the proof is tedious as we must condition on the number of distinct $P \mid P_1 \dots P_l Q_1 \dots Q_l$, among other things. Nonetheless, we provide a proof for completeness.

We first condition on the number of distinct $P \mid P_1 \dots P_l Q_1 \dots Q_l$, for which the possibilities are $r = 1, \dots, 2l-1$. Now, if $P \mid P_1 \dots P_l Q_1 \dots Q_l$, then $P$ must fall into one and only one of the following five categories:

1. $P \nmid (P_1 \dots P_l, Q_1 \dots Q_l)$ and $P \mid P_1 \dots P_l$;
2. $P \mid (P_1 \dots P_l, Q_1 \dots Q_l)$ and $P \mid \frac{P_1 \dots P_l}{(P_1 \dots P_l, Q_1 \dots Q_l)}$ and $P \nmid \frac{Q_1 \dots Q_l}{(P_1 \dots P_l, Q_1 \dots Q_l)}$ ;
3. $P \mid (P_1 \dots P_l, Q_1 \dots Q_l)$ and $P \nmid \frac{P_1 \dots P_l}{(P_1 \dots P_l, Q_1 \dots Q_l)}$ and $P \nmid \frac{Q_1 \dots Q_l}{(P_1 \dots P_l, Q_1 \dots Q_l)}$ .
4. $P \mid (P_1 \dots P_l, Q_1 \dots Q_l)$ and $P \nmid \frac{P_1 \dots P_l}{(P_1 \dots P_l, Q_1 \dots Q_l)}$ and $P \mid \frac{Q_1 \dots Q_l}{(P_1 \dots P_l, Q_1 \dots Q_l)}$ ;
5. $P \nmid (P_1 \dots P_l, Q_1 \dots Q_l)$ and $P \mid Q_1 \dots Q_l$.

We now condition on the number of distinct $P$ that fall into each category. Let $r_i$ be the number of distinct $P$ in category $i$. We must have that

$$r_1 + r_2 + r_3 + r_4 + r_5 = r.$$

Furthermore, if $P \mid P_1 \dots P_l$ then $P$ must be in categories 1, 2, 3, or 4; and if $P \mid Q_1 \dots Q_l$ then $P$ must be in categories 2, 3, 4, or 5. Also, if $P$ is in category 2 or 4, then $P^2 \mid P_1 \dots P_l$ or $P^2 \mid Q_1 \dots Q_l$, respectively. From this, we see that we must have

$$\begin{aligned} r_1 + 2r_2 + r_3 + r_4 &\leq l, \\ r_2 + r_3 + 2r_4 + r_5 &\leq l. \end{aligned} \tag{7.18}$$

Also, since at least two of $P_1, \dots, P_l$ are equal or at least two of $Q_1, \dots, Q_l$ are equal, we must have that

$$\begin{aligned} &r_1 + 2r_2 + r_3 + r_4 \leq l-1 \\ &\text{or} \\ &r_2 + r_3 + 2r_4 + r_5 \leq l-1. \end{aligned}$$

From this and (7.18), we must have

$$r_1 + 3r_2 + 2r_3 + 3r_4 + r_5 \leq 2l-1. \tag{7.19}$$

So, the possible values of $(r_1, r_2, r_3, r_4, r_5)$ are elements (but not necessarily all of them) of the set

$$\begin{aligned} D_{l,r} := \{ (r_1, r_2, r_3, r_4, r_5) \in {\mathbb{Z}_{\geq 0}}^5 : r_1 + r_2 + r_3 + r_4 + r_5 = r \ , \ r_1 + 2r_2 + r_3 + r_4 \leq l \\ r_2 + r_3 + 2r_4 + r_5 \leq l \ , \ r_1 + 3r_2 + 2r_3 + 3r_4 + r_5 \leq 2l-1 \}. \end{aligned}$$

To help keep track of notation, the $D$ stands for "distinct primes".

Now we condition on the number of $P$ (not necessarily distinct) that fall into each category. For $i = 1, 3, 5$, let $n_i \geq r_i$ be the number of $P$ (not necessarily distinct) that are in category $i$. Now, for the distinct $P_{i_1}, \ldots, P_{i_{r_2}}$ in category 2, there are maximal integers $a_1, \ldots, a_{r_2} \geq 1$ and $a'_1, \ldots, a'_{r_2} \geq 1$ such that

$$P_{i_1}{}^{a_1} \ldots P_{i_{r_2}}{}^{a_{r_2}} \mid (P_1 \ldots P_l, Q_1 \ldots Q_l)$$

and

$$P_{i_1}{}^{a_1+a'_1} \ldots P_{i_{r_2}}{}^{a_{r_2}+a'_{r_2}} \mid P_1 \ldots P_l.$$

Let $n_2 := a_1 + \cdots + a_{r_2} \geq r_2$ and $n'_2 := a'_1 + \cdots + a'_{r_2} \geq r_2$. We define $n_4, n'_4 \geq r_4$ similarly. Note that we must have

$$\begin{aligned} n_1 + n_2 + n'_2 + n_3 + n_4 &\leq l, \\ n_2 + n_3 + n'_4 + n_4 + n_5 &\leq l. \end{aligned}$$

So, the possible values of $(n_1, n_2, n'_2, n_3, n'_4, n_4, n_5)$ are elements (but not necessarily all of them) of the set

$$\begin{aligned} &N_{l,(r_1,r_2,r_3,r_4,r_5)} \\ :=& \{(n_1, n_2, n'_2, n_3, n'_4, n_4, n_5) \in \mathbb{Z}_{\geq 0}{}^5 : n_i \geq r_i \ \forall i \ , \ n'_i \geq r_i \ \forall i \ , \\ & n_1 + n_2 + n'_2 + n_3 + n_4 \leq l \ , \ n_2 + n_3 + n'_4 + n_4 + n_5 \leq l\}. \end{aligned}$$

To help keep track of notation, the $N$ stands for "not necessarily distinct primes".

Finally, for each category, $i$, we condition on the number of ways that we can have $n_i$ (or $n'_i$) primes when exactly $r_i$ are distinct. That is, the number of ways of compositioning $n_i$ (or $n'_i$) into $r_i$ terms. Specifically, we are interested in the sets

$$\begin{aligned} C_{i,n_i,r_i} :=& \{(a_{i,1}, \ldots, a_{i,r_i}) \in \mathbb{Z}_{>0}{}^{r_i} : a_{i,1} + \ldots + a_{i,r_i} = n_i\} \\ C'_{j,n'_j,r_j} :=& \{(a'_{j,1}, \ldots, a'_{j,r_j}) \in \mathbb{Z}_{>0}{}^{r_j} : a'_{j,1} + \ldots + a'_{j,r_j} = n'_j\} \end{aligned} \tag{7.20}$$

for $i = 1, 2, 3, 4, 5$ and $j = 2, 4$. To help keep track of notation, the $C$ stands for "composition".

So, we have distinct primes

$$\begin{aligned} &P_{1,1}, \ldots, P_{1,r_1}, \\ &P_{2,1}, \ldots, P_{2,r_2}, \\ &P_{3,1}, \ldots, P_{3,r_3}, \\ &P_{4,1}, \ldots, P_{4,r_4}, \\ &P_{5,1}, \ldots, P_{5,r_5} \end{aligned}$$

(the first subscript in the primes represents which category they belong to) such that

$$\begin{aligned} &(P_1 \ldots P_l, Q_1 \ldots Q_l) \\ =& \left(P_{2,1}{}^{a_{2,1}} \ldots P_{2,r_2}{}^{a_{2,r_2}}\right)\left(P_{3,1}{}^{a_{3,1}} \ldots P_{3,r_3}{}^{a_{3,r_3}}\right)\left(P_{4,1}{}^{a_{4,1}} \ldots P_{4,r_4}{}^{a_{4,r_4}}\right) \end{aligned}$$

and

$$\frac{P_1\dots P_l Q_1\dots Q_l}{(P_1\dots P_l, Q_1\dots Q_l)}$$
$$=\left({P_{1,1}}^{a_{1,1}}\dots {P_{1,r_1}}^{a_{1,r_1}}\right)\left({P_{2,1}}^{a'_{2,1}}\dots {P_{2,r_2}}^{a'_{2,r_2}}\right)\left({P_{4,1}}^{a'_{4,1}}\dots {P_{4,r_4}}^{a'_{4,r_4}}\right)\left({P_{5,1}}^{a_{5,1}}\dots {P_{5,r_5}}^{a_{5,r_5}}\right).$$

(The powers are as in (7.20)). In particular, the requirement that $P_1\dots P_l A = Q_1\dots Q_l B$ becomes

$$A=\left({P_{4,1}}^{a'_{4,1}}\dots {P_{4,r_4}}^{a'_{4,r_4}}\right)\left({P_{5,1}}^{a_{5,1}}\dots {P_{5,r_5}}^{a_{5,r_5}}\right)C$$

and

$$B=\left({P_{1,1}}^{a_{1,1}}\dots {P_{1,r_1}}^{a_{1,r_1}}\right)\left({P_{2,1}}^{a'_{2,1}}\dots {P_{2,r_2}}^{a'_{2,r_2}}\right)C$$

for some $C\in\mathcal{S}_{\mathcal{M}}(X)$.

Before proceeding, let us define $\delta(a)=1$ if $a=1$ and $\delta(a)=0$ otherwise. Now, from the above we can see that the contribution of the case where at least two of $P_1,\dots,P_l$ are equal or at least two of $Q_1,\dots,Q_l$ are equal is

$$\leq\sum_{r=1}^{2l-1}\sum_{(r_1,r_2,r_3,r_4,r_5)\in D_{l,r}}\sum_{(n_1,n_2,n'_2,n_3,n'_4,n_4,n_5)\in N_{l,(r_1,r_2,r_3,r_4,r_5)}}$$
$$\sum_{\substack{(a_{1,1},\dots,a_{1,r_1})\in C_{1,n_1,r_1}\\(a_{2,1},\dots,a_{2,r_2})\in C_{2,n_2,r_2}\\(a'_{2,1},\dots,a'_{2,r_2})\in C'_{2,n'_2,r_2}\\(a_{3,1},\dots,a_{3,r_3})\in C_{3,n_3,r_3}\\(a'_{4,1},\dots,a'_{4,r_4})\in C'_{4,n'_4,r_4}\\(a_{4,1},\dots,a_{4,r_4})\in C_{4,n_4,r_4}\\(a_{5,1},\dots,a_{5,r_5})\in C_{5,n_5,r_5}}}\ \sum_{\substack{\deg P_{1,1},\dots,\deg P_{1,r_1}\leq X\\\deg P_{2,1},\dots,\deg P_{2,r_2}\leq X\\\deg P_{3,1},\dots,\deg P_{3,r_3}\leq X\\\deg P_{4,1},\dots,\deg P_{4,r_4}\leq X\\\deg P_{5,1},\dots,\deg P_{5,r_5}\leq X\\\text{all primes above distinct}}}\ \sum_{C\in\mathcal{S}_{\mathcal{M}}(X)}$$
$$\prod_{i_1=1}^{r_1}\frac{(\log|P_{1,i_1}|)^{a_{1,i_1}}}{|P_{1,i_1}|^{a_{1,i_1}}}\prod_{i_5=1}^{r_5}\frac{(\log|P_{5,i_5}|)^{a_{5,i_5}}}{|P_{5,i_5}|^{a_{5,i_5}}}$$
$$\prod_{i_2=1}^{r_2}\frac{(\log|P_{2,i_2}|)^{2a_{2,i_2}+a'_{2,i_2}}}{|P_{2,i_2}|^{a_{2,i_2}+a'_{2,i_2}}}\prod_{i_4=1}^{r_4}\frac{(\log|P_{4,i_4}|)^{2a_{4,i_4}+a'_{4,i_4}}}{|P_{4,i_4}|^{a_{4,i_4}+a'_{4,i_4}}}\prod_{i_3=1}^{r_3}\frac{(\log|P_{3,i_3}|)^{2a_{3,i_3}}}{|P_{3,i_3}|^{a_{3,i_3}}}$$
$$\frac{\alpha_k\left(C\prod_{i_1=1}^{r_1}|P_{1,i_1}|^{a_{1,i_1}}\prod_{i_2=1}^{r_2}|P_{2,i_2}|^{a'_{2,i_2}}\right)\alpha_k\left(C\prod_{i_5=1}^{r_5}|P_{5,i_5}|^{a_{5,i_5}}\prod_{i_4=1}^{r_4}|P_{4,i_4}|^{a'_{4,i_4}}\right)}{|C|}$$

$$\ll_k X^{k^2}\sum_{r=1}^{2l-1}\sum_{(r_1,r_2,r_3,r_4,r_5)\in D_{l,r}}\sum_{(n_1,n_2,n'_2,n_3,n'_4,n_4,n_5)\in N_{l,(r_1,r_2,r_3,r_4,r_5)}}$$
$$\sum_{\substack{(a_{1,1},\dots,a_{1,r_1})\in C_{1,n_1,r_1}\\(a_{2,1},\dots,a_{2,r_2})\in C_{2,n_2,r_2}\\(a'_{2,1},\dots,a'_{2,r_2})\in C'_{2,n'_2,r_2}\\(a_{3,1},\dots,a_{3,r_3})\in C_{3,n_3,r_3}\\(a'_{4,1},\dots,a'_{4,r_4})\in C'_{4,n'_4,r_4}\\(a_{4,1},\dots,a_{4,r_4})\in C_{4,n_4,r_4}\\(a_{5,1},\dots,a_{5,r_5})\in C_{5,n_5,r_5}}}\Big((\log q)X\Big)^{\sum_{i_1=1}^{r_1}\delta(a_{1,i_1})+\sum_{i_5=1}^{r_5}\delta(a_{5,i_5})+2\sum_{i_3=1}^{r_3}\delta(a_{3,i_3})}$$

$$\ll_k(\log q)^{2l-1}X^{k^2+2l-1}\sum_{r=1}^{2l-1}\sum_{(r_1,r_2,r_3,r_4,r_5)\in D_{l,r}}\sum_{(n_1,n_2,n'_2,n_3,n'_4,n_4,n_5)\in N_{l,(r_1,r_2,r_3,r_4,r_5)}}$$

$$\sum_{\substack{(a_{1,1},\dots,a_{1,r_1})\in C_{1,n_1,r_1}\\ (a_{2,1},\dots,a_{2,r_2})\in C_{2,n_2,r_2}\\ (a'_{2,1},\dots,a'_{2,r_2})\in C'_{2,n'_2,r_2}\\ (a_{3,1},\dots,a_{3,r_3})\in C_{3,n_3,r_3}\\ (a'_{4,1},\dots,a'_{4,r_4})\in C'_{4,n'_4,r_4}\\ (a_{4,1},\dots,a_{4,r_4})\in C_{4,n_4,r_4}\\ (a_{5,1},\dots,a_{5,r_5})\in C_{5,n_5,r_5}}} 1$$
$$\ll_k (\log q)^{2l-1} X^{k^2+2l-1},$$

as required. For the first relation, to address the factors involving $\alpha_k$, we apply a similar reasoning as in (7.15), (7.16), and (7.17). Also, the first relation uses the fact that, for integers $c \geq 0$ and $d \geq 2$,

$$\sum_{\deg P \leq X} \frac{(\log|P|)^c}{|P|^d} \ll_c 1;$$

and for integers $c \geq 1$,

$$\sum_{\deg P \leq X} \frac{(\log|P|)^c}{|P|} \ll (\log q)^c X^c.$$

The second relation uses the fact that

$$\sum_{i_1=1}^{r_1} \delta(a_{1,i_1}) + \sum_{i_5=1}^{r_5} \delta(a_{5,i_5}) + 2\sum_{i_3=1}^{r_3} \delta(a_{3,i_3}) \leq r_1 + r_2 + 2r_3 \leq 2l-1,$$

which follows from (7.19). We must remark that if $C_{i,n_i,r_i}$ or $C'_{i,n'_i,r_i}$ are empty for some $i$ (for example, if $r_i = 0$), then the sums

$$\sum_{(a_{i,1},\dots,a_{i,r_i})\in C_{i,n_i,r_i}} \qquad \text{or} \qquad \sum_{(a'_{i,1},\dots,a'_{i,r_i})\in C'_{i,n'_i,r_i}},$$

respectively, should be taken as there not being any summation and should not be taken as the empty sum that evaluates to 0.

As mentioned before, and as we can see above, if at least two of $P_1, \dots, P_l$ are equal or at least two of $Q_1, \dots, Q_l$ are equal, then in whatever way this may manifest (which we addressed by conditioning) we always lose at least one factor of $X$. The conditioning is just a technicality that is required so we can establish that this really is the case. □

## 7.4 The Second Moment of $P'_X\left(\frac{1}{2}, \chi\right) Z_X\left(\frac{1}{2}, \chi\right)$

In this section we prove Theorem 2.5.5, which we restate for ease of reference.

**Theorem.** *As $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$, we have*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q\\ \chi \neq \chi_0}} \left| P'_X\left(\frac{1}{2}, \chi\right) Z_X\left(\frac{1}{2}, \chi\right) \right|^2 \sim \frac{3}{2} (\log q)^2 X^2 \deg Q.$$

*Proof of Theorem 2.5.5.* Throughout this proof, all asymptotic relations should be taken as $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$.

As in Lemma 7.3.1, we have that

$$\sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P'_X\Big(\frac{1}{2},\chi\Big) Z_X\Big(\frac{1}{2},\chi\Big) \right|^2 = \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \frac{P'_X\left(\frac{1}{2},\chi\right)}{P_X\left(\frac{1}{2},\chi\right)} L\Big(\frac{1}{2},\chi\Big) \right|^2$$
$$\sim \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \Bigg( \sum_{\deg P \leq X} \frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}} + \sum_{\deg P \leq \frac{X}{2}} \frac{\log|P|\chi(P)^2}{|P|} + O(1) \Bigg) L\Big(\frac{1}{2},\chi\Big) \right|^2.$$

Therefore, it suffices to show that

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \Bigg( \sum_{\deg P \leq X} \frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}} \Bigg) L\Big(\frac{1}{2},\chi\Big) \right|^2 \sim \frac{3}{2} (\log q)^2 X^2 \deg Q \big(1 + O(X^{-1})\big);$$
$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \Bigg( \sum_{\deg P \leq \frac{X}{2}} \frac{\log|P|\chi(P)^2}{|P|} \Bigg) L\Big(\frac{1}{2},\chi\Big) \right|^2 \ll \deg Q;$$
$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| L\Big(\frac{1}{2},\chi\Big) \right|^2 \ll \deg Q. \tag{7.21}$$

We will prove the first result. The second can be obtained similarly. The third follows immediately from Theorem 2.2.1. Now, by Lemmas A.1.2 and A.1.3, we have

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \Bigg( \sum_{\deg P \leq X} \frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}} \Bigg) L\Big(\frac{1}{2},\chi\Big) \right|^2$$
$$= \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \sum_{\deg P \leq X} \frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}} \right|^2 a(\chi) + \frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \sum_{\deg P \leq X} \frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}} \right|^2 c(\chi),$$

where

$$a(\chi) := 2 \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}$$

and $c(\chi)$ is defined as in (A.8). We will show that the sum involving $a(\chi)$ contributes the main term. One can similarly show that the sum involving $c(\chi)$ is equal to $O\big((\log q)^2 X^2\big)$.

To this end, we have that

$$\begin{aligned}
&\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi\neq\chi_0}}\left|\sum_{\deg P\leq X}\frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}}\right|^2 a(\chi)\\
=&\frac{2}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi\neq\chi_0}}\sum_{\substack{\deg P_1\leq X\\ \deg P_2\leq X\\ A,B\in\mathcal{M}\\ \deg AB<\deg Q}}\frac{(\log|P_1|)(\log|P_2|)}{|P_1P_2AB|^{\frac{1}{2}}}\chi(P_1A)\overline{\chi}(P_2B)\\
=&2\sum_{\substack{\deg P_1\leq X\\ \deg P_2\leq X\\ A,B\in\mathcal{M}\\ \deg AB<\deg Q\\ P_1A\equiv P_2B(\bmod Q)}}\frac{(\log|P_1|)(\log|P_2|)}{|P_1P_2AB|^{\frac{1}{2}}}-\frac{2}{\phi(Q)}\sum_{\substack{\deg P_1\leq X\\ \deg P_2\leq X\\ A,B\in\mathcal{M}\\ \deg AB<\deg Q}}\frac{(\log|P_1|)(\log|P_2|)}{|P_1P_2AB|^{\frac{1}{2}}}.
\end{aligned}\tag{7.22}$$

It is not difficult to see that the second term on the far RHS is $O\big(|Q|^{-\frac{1}{3}}\big)$. For the first term, we first consider the diagonal terms. That is, when $P_1A=P_2B$.

If $P_1=P_2$, then we must also have that $A=B$, and so

$$\begin{aligned}
2\sum_{\substack{\deg P_1\leq X\\ \deg P_2\leq X\\ A,B\in\mathcal{M}\\ \deg AB<\deg Q\\ P_1A=P_2B\\ P_1=P_2}}\frac{(\log|P_1|)(\log|P_2|)}{|P_1P_2AB|^{\frac{1}{2}}}=&2\sum_{\substack{\deg P\leq X\\ A\in\mathcal{M}\\ \deg A<\frac{\deg Q}{2}}}\frac{(\log|P|)^2}{|PA|}\\
=&\frac{(\log q)^2}{2}X^2\deg Q\big(1+O(X^{-1})\big).
\end{aligned}$$

If $P_1\neq P_2$, then we must have that $A=P_2C$ and $B=P_1C$ for some $C\in\mathcal{M}$. Therefore,

$$\begin{aligned}
&2\sum_{\substack{\deg P_1\leq X\\ \deg P_2\leq X\\ A,B\in\mathcal{M}\\ \deg AB<\deg Q\\ P_1A=P_2B(\bmod Q)\\ P_1\neq P_2}}\frac{(\log|P_1|)(\log|P_2|)}{|P_1P_2AB|^{\frac{1}{2}}}\\
=&2\sum_{\deg P_1\leq X}\frac{\log|P_1|}{|P_1|}\sum_{\substack{\deg P_2\leq X\\ P_1\neq P_2}}\frac{\log|P_2|}{|P_2|}\sum_{\substack{C\in\mathcal{M}\\ \deg C<\frac{\deg Q-\deg P_1P_2}{2}}}\frac{1}{|C|}\\
=&\sum_{\deg P_1\leq X}\frac{\log|P_1|}{|P_1|}\sum_{\substack{\deg P_2\leq X\\ P_1\neq P_2}}\frac{\log|P_2|}{|P_2|}\big(\deg Q-\deg P_1P_2+O(1)\big)\\
=&(\log q)\sum_{\deg P_1\leq X}\frac{\log|P_1|}{|P_1|}\big(X+O(1)\big)\big(\deg Q-\deg P_1+O(1)\big)\\
&-\frac{\log q}{2}\sum_{\deg P_1\leq X}\frac{\log|P_1|}{|P_1|}\big(X(X+1)+O(1)\big)\\
=&(\log q)^2X^2\deg Q\big(1+O(X^{-1})\big).
\end{aligned}$$

Finally, we consider the off-diagonal terms on the far RHS of (7.22). That is, when $P_1A\equiv P_2B(\bmod Q)$ but $P_1A\neq P_2B$. Using Lemma 6.4.4 with $C=P_1$ and $D=P_2$, we have, for $Z_1 = 0, \dots, \deg Q - 1$,

$$\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = Z_1 \\ P_1 A \equiv P_2 B (\operatorname{mod} Q) \\ P_1 A \neq P_2 B}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{q^{\frac{Z_1}{2}} (Z_1 + 1) |P_1 P_2|}{|Q|}.$$

Therefore,

$$\sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ P_1 A \equiv P_2 B (\operatorname{mod} Q) \\ P_1 A \neq P_2 B}} \frac{(\log|P_1|)(\log|P_2|)}{|P_1 P_2 AB|^{\frac{1}{2}}} \ll \frac{\deg Q}{|Q|^{\frac{1}{2}}} \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X}} (\log|P_1|)(\log|P_2|) \, |P_1 P_2|^{\frac{1}{2}} \ll |Q|^{-\frac{1}{3}}.$$

The proof follows. □

## 7.5 The Second Moment of the Derivative of the Hadamard Product

In this section, we prove Propositions 2.5.8 and 2.5.9, which, as described in Section 2.5, are required for the proof of Theorem 2.5.7. We begin with Proposition 2.5.8, for which we require a lemma.

**Lemma 7.5.1.** *As $X, \deg Q \overset{q}{\longrightarrow} \infty$ with $X \leq \log_q \deg Q$, we have*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \operatorname{mod} Q \\ \chi \neq \chi_0}} \left( \Bigg| \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C) \chi(C)}{|C|^{\frac{1}{2}}} \Bigg|^2 \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{f\big(\deg A, \deg B, \deg Q\big) \chi(A) \overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \right)$$
$$= \frac{(\deg Q)^3}{3 e^{\gamma} X} \Big( 1 + O\big(q^{-\frac{X}{2}}\big) \Big),$$

*where $\alpha_{-1}$ is defined as in (6.15) and*

$$f\big(\deg A, \deg B, \deg Q\big) = (\deg A)(\deg B) + (\deg Q - \deg A)(\deg Q - \deg B).$$

*Proof.* Throughout this proof, all asymptotic relations should be taken as $X, \deg Q \overset{q}{\longrightarrow} \infty$ with $X \leq \log_q \deg Q$.

Now, we have that

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left( \Bigg| \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \Bigg|^2 \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{f\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \right) \\
= &\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q \\ AC \equiv BD (\bmod Q)}} \frac{f\big(\deg A, \deg B, \deg Q\big)\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \\
&- \frac{1}{\phi(Q)} \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q}} \frac{f\big(\deg A, \deg B, \deg Q\big)\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}.
\end{aligned} \tag{7.23}$$

For the second term on the RHS we have

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q}} \frac{f\big(\deg A, \deg B, \deg Q\big)\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \\
\ll &\frac{(\deg Q)^2}{\phi(Q)} \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q}} \frac{1}{|ABCD|^{\frac{1}{2}}} \ll \frac{(\deg Q)^3}{|Q|^{\frac{1}{2}}} \exp\left( O\Big( \frac{\deg Q}{X} \Big) \right) \ll |Q|^{-\frac{1}{3}}.
\end{aligned}$$

For the off-diagonal terms on the RHS of (7.23), we have

$$\begin{aligned}
&\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C,D \leq \frac{1}{10} \deg Q \\ AC \equiv BD (\bmod Q) \\ AC \neq BD}} \frac{f\big(\deg A, \deg B, \deg Q\big)\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \\
\ll &(\deg Q)^2 \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q \\ AC \equiv BD (\bmod Q) \\ AC \neq BD}} \frac{1}{|ABCD|^{\frac{1}{2}}} \ll \frac{(\deg Q)^3}{|Q|^{\frac{1}{2}}} \sum_{\substack{C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q}} |CD|^{\frac{1}{2}} \\
\ll &\frac{(\deg Q)^3 |Q|^{\frac{3}{10}}}{|Q|^{\frac{1}{2}}} \ll |Q|^{-\frac{1}{10}},
\end{aligned}$$

where the second relation follows from Lemma 6.4.4. Finally, we address the diagonal terms on the RHS of (7.23). When $AC = BD$ we can find unique $GHST \in \mathcal{M}$ such that $A = GS$, $B = GT$, $C = HT$, $D = HS$, and $(S,T) = 1$. Hence,

$$
\begin{aligned}
&\sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q \\ C,D\in\mathcal{S}_{\mathcal{M}}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg Q \\ AC=BD}} \frac{f\big(\deg A, \deg B, \deg Q\big)\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \\
=& \sum_{\substack{G\in\mathcal{M} \\ H,S,T\in\mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ \deg G^2ST<\deg Q \\ \deg HS,\deg HT\leq\frac{1}{10}\deg Q}} \frac{f\big(\deg GS, \deg GT, \deg Q\big)\alpha_{-1}(HT)\alpha_{-1}(HS)}{|GHST|} \\
=& \sum_{\substack{G\in\mathcal{M} \\ H,S,T\in\mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ \deg G^2ST<\deg Q \\ \deg HS,\deg HT\leq\frac{1}{10}\deg Q}} \frac{\alpha_{-1}(HT)\alpha_{-1}(HS)}{|GHST|} \\
&\cdot \Big(2(\deg G)^2 + 2(\deg ST - \deg Q)(\deg G) \\
&\quad + 2(\deg S)(\deg T) + (\deg Q)(\deg Q - \deg ST)\Big) \\
=& \sum_{\substack{H,S,T\in\mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ \deg ST<\deg Q \\ \deg HS,\deg HT\leq\frac{1}{10}\deg Q}} \frac{\alpha_{-1}(HT)\alpha_{-1}(HS)}{|HST|} \\
&\cdot \Big(\frac{(\deg Q - \deg ST)^3}{12} - \frac{(\deg Q - \deg ST)^3}{4} \\
&\quad + (\deg S)(\deg T)(\deg Q - \deg ST) \\
&\quad + \frac{(\deg Q)(\deg Q - \deg ST)^2}{2} + O\big((\deg Q)^2\big)\Big),
\end{aligned}
\tag{7.24}
$$

where, for the last line, we evaluated the sum over $G$. Now, by similar means as in Lemma 6.4.2, we can show that, for $i = 0, 1, 2, 3$

$$
\sum_{\substack{H,S,T\in\mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ \deg ST<\deg Q \\ \deg HS,\deg HT\leq\frac{1}{10}\deg Q}} \frac{|\alpha_{-1}(HT)\alpha_{-1}(HS)|}{|HST|}(\deg ST)^i \ll X^{3+i},
$$

and, for $i = 0, 1$,

$$
\begin{aligned}
&\sum_{\substack{H,S,T\in\mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ \deg ST<\deg Q \\ \deg HS,\deg HT\leq\frac{1}{10}\deg Q}} \frac{|\alpha_{-1}(HT)\alpha_{-1}(HS)|}{|HST|}(\deg S)(\deg T)(\deg ST)^i \\
\leq& \sum_{\substack{H,S,T\in\mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ \deg ST<\deg Q \\ \deg HS,\deg HT\leq\frac{1}{10}\deg Q}} \frac{|\alpha_{-1}(HT)\alpha_{-1}(HS)|}{|HST|}(\deg ST)^{i+2} \ll X^6.
\end{aligned}
$$

Also, by Lemmas 6.4.3 and A.2.7 we have that

$$\sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (S,T)=1 \\ \deg ST < \deg Q \\ \deg HS, \deg HT \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(HT)\alpha_{-1}(HS)}{|HST|}(\deg Q)^3 = \frac{(\deg Q)^3}{e^{\gamma} X}\Big(1 + O\big(q^{-\frac{X}{2}}\big)\Big).$$

Applying these three points to (7.24), we obtain

$$\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q \\ AC=BD}} \frac{f\big(\deg A, \deg B, \deg Q\big)\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} = \frac{(\deg Q)^3}{3e^{\gamma} X}\Big(1 + O\big(q^{-\frac{X}{2}}\big)\Big).$$

The proof follows. □

We can now prove Proposition 2.5.8, which we restate for ease of reference.

**Proposition.** *As* $X, \deg Q \longrightarrow \infty$ *with* $X \leq \log_q \deg Q$*, we have*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P_X\Big(\frac{1}{2}, \chi\Big)^{-1} L'\Big(\frac{1}{2}, \chi\Big) \right|^2 \sim \frac{(\log q)^2 (\deg Q)^3}{3e^{\gamma} X}.$$

*Proof of Proposition 2.5.8.* Throughout this proof, all asymptotic relations should be taken as $X, \deg Q \xrightarrow{q} \infty$ with $X \leq \log_q \deg Q$.

By (6.15) and a method similar to (6.7), we have

$$\begin{aligned} P_X\Big(\frac{1}{2}, \chi\Big)^{-1} &= \Big(1 + O\big(X^{-1}\big)\Big) P^*_X\Big(\frac{1}{2}, \chi\Big)^{-1} \\ &= \Big(1 + O\big(X^{-1}\big)\Big) \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} + O\Big(|Q|^{-\frac{1}{50}}\Big), \end{aligned}$$

where $\alpha_{-1}$ is defined as in (6.15). Hence, by the Cauchy-Schwarz inequality and Theorem 2.3.2, it suffices to show that

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \Bigg( \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \Bigg) L'\Big(\frac{1}{2}, \chi\Big) \right|^2 \sim \frac{(\log q)^2 (\deg Q)^3}{3e^{\gamma} X}.$$

We first consider the case where $\chi$ is odd:

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ odd}}} \left| \Bigg( \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \Bigg) L'\Big(\frac{1}{2}, \chi\Big) \right|^2.$$

By Lemma 5.2.1 we have

$$(\log q)^{-2} \left| L'\Big(\frac{1}{2}, \chi\Big) \right|^2 = \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{f\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}$$

$$+ \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q}} \frac{g_O\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}$$
$$+ \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB=\deg Q-1}} \frac{h_O\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}},$$

where

$$f\big(\deg A, \deg B, \deg Q\big) =(\deg A)(\deg B) + (\deg Q - \deg A)(\deg Q - \deg B),$$

and $g_O\big(\deg A, \deg B, \deg Q)\big)$ is a polynomial of degree 1 that is symmetric in $\deg A, \deg B$ and $h_O\big(\deg A, \deg B, \deg Q\big)$ is a polynomial of degree 2 that is symmetric in $\deg A, \deg B$.

Now,

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ odd}}} \Bigg( \Big| \sum_{\substack{C\in\mathcal{S}_\mathcal{M}(X) \\ \deg C\leq\frac{1}{10}\deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \Big|^2 \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q}} \frac{g_O\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \\
\ll&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi\neq\chi_0}} \Bigg( \Big| \sum_{\substack{C\in\mathcal{S}_\mathcal{M}(X) \\ \deg C\leq\frac{1}{10}\deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \Big|^2 \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q}} \frac{g_O\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \\
=& \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q \\ C,D\in\mathcal{S}_\mathcal{M}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg Q \\ AC\equiv BD(\bmod Q)}} \frac{g_O\big(\deg A, \deg B, \deg Q\big)\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \\
&- \frac{1}{\phi(Q)} \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q \\ C,D\in\mathcal{S}_\mathcal{M}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg Q}} \frac{g_O\big(\deg A, \deg B, \deg Q\big)\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \\
\ll&(\deg Q) \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q \\ C,D\in\mathcal{S}_\mathcal{M}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg Q \\ AC\equiv BD(\bmod Q)}} \frac{|\alpha_{-1}(C)||\alpha_{-1}(D)|}{|ABCD|^{\frac{1}{2}}} + O\big(|Q|^{-\frac{1}{3}}\big) \\
=&(\deg Q) \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q \\ C,D\in\mathcal{S}_\mathcal{M}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg Q \\ AC=BD}} \frac{|\alpha_{-1}(C)||\alpha_{-1}(D)|}{|ABCD|^{\frac{1}{2}}} + (\deg Q) \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg Q \\ C,D\in\mathcal{S}_\mathcal{M}(X) \\ \deg C,\deg D\leq\frac{1}{10}\deg Q \\ AC\equiv BD(\bmod Q) \\ AC\neq BD}} \frac{|\alpha_{-1}(C)||\alpha_{-1}(D)|}{|ABCD|^{\frac{1}{2}}} + O\big(|Q|^{-\frac{1}{3}}\big) \\
=&(\deg Q) \sum_{\substack{H,S,T\in\mathcal{S}_\mathcal{M}(X) \\ \deg HS,\deg HT\leq\frac{1}{10}\deg Q \\ (S,T)=1}} \frac{|\alpha_{-1}(HS)||\alpha_{-1}(HT)|}{|HST|} \sum_{\substack{G\in\mathcal{M} \\ \deg G\leq\frac{\deg Q-\deg ST}{2}}} \frac{1}{|G|} + o(1) + O\big(|Q|^{-\frac{1}{3}}\big) \\
\ll&(\deg Q)^2 X^3,
\end{aligned} \tag{7.25}$$

where, for the second-to-last relation, the first term follows by similar reasoning as in (6.19) and the second term follows by similar reasoning as in (6.21) (although it is easier here as $Q$ is prime); and for the last relation we use a similar result as Lemma 6.4.1. Similarly,

$$\begin{aligned}&\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi \text{ odd}}}\Bigg(\Bigg|\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Bigg|^2\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg Q-1}}\frac{h_O\big(\deg A,\deg B,\deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}\Bigg)\\ \ll&(\deg Q)^2\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg Q-1\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C,\deg D\leq\frac{1}{10}\deg Q\\ AC\equiv BD(\bmod Q)}}\frac{1}{|ABCD|^{\frac{1}{2}}}+O\big(|Q|^{-\frac{1}{3}}\big)\ll(\deg Q)^2X^3,\end{aligned} \tag{7.26}$$

where the last relation follows almost identically as (6.22).

So, by Lemma 7.5.1, (7.25), (7.26), and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi \text{ odd}}}\Bigg|\Bigg(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Bigg)L'\Big(\frac{1}{2},\chi\Big)\Bigg|^2\\ =&\frac{(\log q)^2}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi \text{ odd}}}\Bigg(\Bigg|\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Bigg|^2\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB<\deg Q}}\frac{f\big(\deg A,\deg B,\deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}\Bigg)\\ &+O\big((\log q)^2(\deg Q)^{\frac{5}{2}}X\big).\end{aligned} \tag{7.27}$$

We next consider the case where $\chi$ is even:

$$\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi \text{ even}\\ \chi\neq\chi_0}}\Bigg|\Bigg(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Bigg)L'\Big(\frac{1}{2},\chi\Big)\Bigg|^2. \tag{7.28}$$

By (5.1) we have that

$$L'\Big(\frac{1}{2},\chi\Big)=\frac{1}{q^{\frac{1}{2}}-1}\hat{L}'\Big(\frac{1}{2},\chi\Big)+\frac{(\log q)q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}L\Big(\frac{1}{2},\chi\Big). \tag{7.29}$$

First, we recall from (6.17) that

$$\begin{aligned}&\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi \text{ even}\\ \chi\neq\chi_0}}\Bigg|\Bigg(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Bigg)L\Big(\frac{1}{2},\chi\Big)\Bigg|^2\\ \ll&\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi\neq\chi_0}}\Bigg|\Bigg(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Bigg)L\Big(\frac{1}{2},\chi\Big)\Bigg|^2\ll\frac{\deg Q}{X}.\end{aligned} \tag{7.30}$$

Also, by Lemma 5.2.6, we have that

$$\begin{aligned}
&\frac{1}{(\log q)^2 (q^{\frac{1}{2}}-1)^2} \Big| \hat{L}'\Big(\frac{1}{2},\chi\Big)\Big|^2 \\
=& \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{f\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \\
&+ \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{g_E\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \\
&+ \sum_{\deg Q - 2 \leq n \leq \deg Q} \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = n}} \frac{h_{E,n}\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}},
\end{aligned}$$

where

$$f\big(\deg A, \deg B, \deg Q\big) = (\deg A)(\deg B) + (\deg Q - \deg A)(\deg Q - \deg B),$$

and $g_E\big(\deg A, \deg B, \deg Q\big)\big)$ is a polynomial of degree 1 that is symmetric in $\deg A, \deg B$ and $h_{E,n}\big(\deg A, \deg B, \deg Q\big)$ is a polynomial of degree 2 that is symmetric in $\deg A, \deg B$.

Similarly to (7.27), we can show that

$$\begin{aligned}
&\frac{1}{\phi(Q)} \frac{1}{(q^{\frac{1}{2}}-1)^2} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Bigg| \Bigg( \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \Bigg) \hat{L}'\Big(\frac{1}{2},\chi\Big) \Bigg|^2 \\
=& \frac{(\log q)^2}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Bigg( \Bigg| \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \Bigg|^2 \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{f\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \\
&+ O\Big((\log q)^2 (\deg Q)^{\frac{5}{2}} X\Big).
\end{aligned} \tag{7.31}$$

Now, if we apply (7.29) to (7.28), then one of the terms will be (7.31) directly above. Another term will be (7.30) multiplied by a factor of $\Big(\frac{(\log q) q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}\Big)^2$. There are then two terms remaining which we can bound using the Cauchy Schwarz inequality, (7.30), and a bound for (7.31) which we obtain from Lemma 7.5.1 (note that the LHS of the result in Lemma 7.5.1 is not restricted to even $\chi$ as in (7.31), but it is still an upper bound for (7.31)). Thus, we obtain

$$\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Bigg| \Bigg( \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \Bigg) L'\Big(\frac{1}{2},\chi\Big) \Bigg|^2 \\
=& \frac{(\log q)^2}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \text{ even} \\ \chi \neq \chi_0}} \Bigg( \Bigg| \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \Bigg|^2 \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{f\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \\
&+ O\Big((\log q)^2 (\deg Q)^{\frac{5}{2}} X\Big).
\end{aligned} \tag{7.32}$$

Therefore, by (7.27), (7.32), and Lemma 7.5.1, we have

$$
\begin{aligned}
&\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \bigg| \bigg( \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \bigg) L'\Big(\frac{1}{2}, \chi\Big) \bigg|^2 \\
=&\frac{(\log q)^2}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \Bigg( \bigg| \sum_{\substack{C \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C \leq \frac{1}{10} \deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} \bigg|^2 \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q}} \frac{f\big(\deg A, \deg B, \deg Q\big)\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \Bigg) \\
&+ O\Big((\log q)^2 (\deg Q)^{\frac{5}{2}} X\Big) \\
\sim&\frac{(\log q)^2 (\deg Q)^3}{3e^{\gamma} X}.
\end{aligned}
$$

The proof follows. □

We now proceed to prove Proposition 2.5.9. We will require the following lemma.

**Lemma 7.5.2.** *Suppose $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$ and $\deg P_1, \deg P_2 \leq X$. As $X \longrightarrow \infty$, we have*

$$
\begin{aligned}
&\sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,S)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} \\
\sim &\prod_{\deg P \leq X} \Big(1 - \frac{1}{|P|}\Big) \Big(1 + O\big(|P_1|^{-1}\big)\Big) \Big(1 + O\big(|P_2|^{-1}\big)\Big)
\end{aligned}
$$

*and*

$$
\begin{aligned}
&\sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,S)=1 \\ (P_2,T)=1}} \frac{\alpha_{-1}(HP_2S)\alpha_{-1}(HP_1T)}{|HST|} \\
\sim &\prod_{\deg P \leq X} \Big(1 - \frac{1}{|P|}\Big) \Big(1 + O\big(|P_1|^{-1}\big)\Big) \Big(1 + O\big(|P_2|^{-1}\big)\Big)
\end{aligned}
$$

*and*

$$
\begin{aligned}
&\sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,T)=1}} \frac{\alpha_{-1}(HP_2S)\alpha_{-1}(HT)}{|HST|} \\
\sim - &\prod_{\deg P \leq X} \Big(1 - \frac{1}{|P|}\Big) \Big(1 + O\big(|P_1|^{-1}\big)\Big) \Big(1 + O\big(|P_2|^{-1}\big)\Big).
\end{aligned}
$$

*Here, the "big O" terms $O\big(|P_1|^{-1}\big)$ and $O\big(|P_2|^{-1}\big)$ are with respect to $P_1$ and $P_2$ as variables, respectively, and not as $X \longrightarrow \infty$.*

*Proof.* For $P \in \mathcal{P}$, $e \geq 1$, and $A \in \mathcal{A}$, we write $P^e \mid\mid A$ if $P^e \mid A$ and $P^{e+1} \nmid A$. Now, recalling that $\alpha_{-1}(P^4) = 0$ for all $P \in \mathcal{P}$, we have that

$$\sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,S)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} = \sum_{i_1,i_2=0}^{3} \sum_{j_1=0}^{3-i_1} \sum_{j_2=0}^{3-i_2} \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,S)=1 \\ {P_1}^{i_1} {P_2}^{i_2} || H \\ {P_1}^{j_1} || S \\ {P_2}^{j_2} || T}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|}$$

$$= \sum_{i_1,i_2=0}^{3} \sum_{j_1=0}^{3-i_1} \sum_{j_2=0}^{3-i_2} \frac{\alpha_{-1}({P_1}^{i_1+j_1} {P_2}^{i_2})\alpha_{-1}({P_1}^{i_1} {P_2}^{i_2+j_2})}{|P_1|^{i_1+j_1} |P_2|^{i_2+j_2}} \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1 P_2, HST)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|}$$

$$\sim \prod_{\deg P \leq X} \left(1 - \frac{1}{|P|}\right) \left(1 - \frac{1}{|P_1|}\right)^{-1} \left(1 - \frac{1}{|P_2|}\right)^{-1}$$
$$\cdot \sum_{i_1,i_2=0}^{3} \sum_{j_1=0}^{3-i_1} \sum_{j_2=0}^{3-i_2} \frac{\alpha_{-1}({P_1}^{i_1+j_1} {P_2}^{i_2})\alpha_{-1}({P_1}^{i_1} {P_2}^{i_2+j_2})}{|P_1|^{i_1+j_1} |P_2|^{i_2+j_2}}$$
$$= \prod_{\deg P \leq X} \left(1 - \frac{1}{|P|}\right) \Big(1 + O\big(|P_1|^{-1}\big)\Big) \Big(1 + O\big(|P_2|^{-1}\big)\Big),$$

where the second relation follows almost identically as the proof of Lemma 6.4.3. Similarly,

$$\sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,S)=1 \\ (P_2,T)=1}} \frac{\alpha_{-1}(HP_2 S)\alpha_{-1}(HP_1 T)}{|HST|}$$
$$= \sum_{i_1,i_2=0}^{2} \sum_{j_1=0}^{2-i_1} \sum_{j_2=0}^{2-i_2} \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,S)=1 \\ (P_2,T)=1 \\ {P_1}^{i_1} {P_2}^{i_2} || H \\ {P_1}^{j_1} || T \\ {P_2}^{j_2} || S}} \frac{\alpha_{-1}(HP_2 S)\alpha_{-1}(HP_1 T)}{|HST|}$$
$$= \sum_{i_1,i_2=0}^{2} \sum_{j_1=0}^{2-i_1} \sum_{j_2=0}^{2-i_2} \frac{\alpha_{-1}({P_1}^{i_1} {P_2}^{i_2+j_2+1})\alpha_{-1}({P_1}^{i_1+j_1+1} {P_2}^{i_2})}{|P_1|^{i_1+j_1} |P_2|^{i_2+j_2}} \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1 P_2, HST)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|}$$
$$\sim \prod_{\deg P \leq X} \left(1 - \frac{1}{|P|}\right) \Big(1 + O\big(|P_1|^{-1}\big)\Big) \Big(1 + O\big(|P_2|^{-1}\big)\Big),$$

and

$$\sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,T)=1}} \frac{\alpha_{-1}(HP_2 S)\alpha_{-1}(HT)}{|HST|}$$

$$= \sum_{i_1=0}^{3} \sum_{j_1=0}^{3-i_1} \sum_{i_2=0}^{2} \sum_{j_2=0}^{2-i_2} \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,T)=1 \\ {P_1}^{i_1} {P_2}^{i_2} || H \\ {P_1}^{j_1} || S \\ {P_2}^{j_2} || S}} \frac{\alpha_{-1}(HP_2S)\alpha_{-1}(HT)}{|HST|}$$

$$= \sum_{i_1=0}^{3} \sum_{j_1=0}^{3-i_1} \sum_{i_2=0}^{2} \sum_{j_2=0}^{2-i_2} \frac{\alpha_{-1}({P_1}^{i_1+j_1} {P_2}^{i_2+j_2+1})\alpha_{-1}({P_1}^{i_1} {P_2}^{i_2})}{|P_1|^{i_1+j_1}|P_2|^{i_2+j_2}} \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1P_2,HST)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|}$$

$$\sim - \prod_{\deg P \leq X} \Big(1 - \frac{1}{|P|}\Big)\Big(1 + O\big(|P_1|^{-1}\big)\Big)\Big(1 + O\big(|P_2|^{-1}\big)\Big).$$

□

We can now prove Proposition 2.5.9, which we restate for ease of reference.

**Proposition.** *As $X, \deg Q \longrightarrow \infty$ with $X \leq \log_q \deg Q$,*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \frac{P'_X\Big(\frac{1}{2}, \chi\Big)}{P_X\Big(\frac{1}{2}, \chi\Big)} P_X\Big(\frac{1}{2}, \chi\Big)^{-1} L\Big(\frac{1}{2}, \chi\Big) \right|^2 \sim (\log q)^2 \frac{X \deg Q}{2e^{\gamma}}.$$

*Proof of Proposition 2.5.9.* Throughout this proof, unless otherwise stated, all asymptotic relations should be taken as $X, \deg Q \overset{q,k}{\longrightarrow} \infty$ with $X \leq \log_q \deg Q$.

Similar to Lemma 7.3.1, we have

$$\sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \frac{P'_X\Big(\frac{1}{2}, \chi\Big)}{P_X\Big(\frac{1}{2}, \chi\Big)} P_X\Big(\frac{1}{2}, \chi\Big)^{-1} L\Big(\frac{1}{2}, \chi\Big) \right|^2$$
$$= \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \bigg( \sum_{\deg P \leq X} \frac{\log|P| \chi(P)}{|P|^{\frac{1}{2}}} + \sum_{\deg P \leq \frac{X}{2}} \frac{\log|P| \chi(P)^2}{|P|} + O(1) \bigg) P^*_X\Big(\frac{1}{2}, \chi\Big)^{-1} L\Big(\frac{1}{2}, \chi\Big) \right|^2.$$

By the Cauchy-Schwarz inequality, it suffices to show that

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \bigg( \sum_{\deg P \leq X} \frac{\log|P| \chi(P)}{|P|^{\frac{1}{2}}} \bigg) P^*_X\Big(\frac{1}{2}, \chi\Big)^{-1} L\Big(\frac{1}{2}, \chi\Big) \right|^2 \sim (\log q)^2 \frac{X \deg Q}{2e^{\gamma}}$$
$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| \bigg( \sum_{\deg P \leq \frac{X}{2}} \frac{\log|P| \chi(P)^2}{|P|} \bigg) P^*_X\Big(\frac{1}{2}, \chi\Big)^{-1} L\Big(\frac{1}{2}, \chi\Big) \right|^2 \ll \frac{\deg Q}{X}$$
$$\frac{1}{\phi(Q)} \sum_{\substack{\chi \bmod Q \\ \chi \neq \chi_0}} \left| P^*_X\Big(\frac{1}{2}, \chi\Big)^{-1} L\Big(\frac{1}{2}, \chi\Big) \right|^2 \ll \frac{\deg Q}{X}.$$

We will prove the first result. The second can be shown by similar means as the first. The third follows immediately from (6.16) and Theorem 2.4.7. Similar to (6.7), we

have

$$P_X^*\Big(\frac{1}{2},\chi\Big)^{-1} = \sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}} \frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}} + O\Big(|Q|^{-\frac{1}{50}}\Big).$$

Hence, by the Cauchy-Schwarz inequality and the first equation in (7.21), it suffices to show that

$$\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi\neq\chi_0}}\bigg|\Big(\sum_{\deg P\leq X}\frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}}\Big)\Big(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Big)L\Big(\frac{1}{2},\chi\Big)\bigg|^2$$
$$\sim\frac{X\deg Q}{2e^{\gamma}}.$$

Now, by Lemmas A.1.2 and A.1.3, we have

$$\begin{aligned}
&\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi\neq\chi_0}}\bigg|\Big(\sum_{\deg P\leq X}\frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}}\Big)\Big(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Big)L\Big(\frac{1}{2},\chi\Big)\bigg|^2\\
=&\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi\neq\chi_0}}\bigg|\Big(\sum_{\deg P\leq X}\frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}}\Big)\Big(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Big)\bigg|^2 a(\chi)\\
&+\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi\neq\chi_0}}\bigg|\Big(\sum_{\deg P\leq X}\frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}}\Big)\Big(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Big)\bigg|^2 c(\chi),
\end{aligned}$$

where

$$a(\chi):=2\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB<\deg Q}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}$$

and $c(\chi)$ is defined as in (A.8). We will show that the sum involving $a(\chi)$ contributes the main term. By similar means, the term involving $c(\chi)$ can be seen to be equal to $O\big(X^5\big)$.

To this end, we have that

$$\begin{aligned}
&\frac{1}{\phi(Q)}\sum_{\substack{\chi \bmod Q\\ \chi\neq\chi_0}}\bigg|\Big(\sum_{\deg P\leq X}\frac{\log|P|\chi(P)}{|P|^{\frac{1}{2}}}\Big)\Big(\sum_{\substack{C\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C\leq\frac{1}{10}\deg Q}}\frac{\alpha_{-1}(C)\chi(C)}{|C|^{\frac{1}{2}}}\Big)\bigg|^2 a(\chi)\\
=&2\sum_{\substack{\deg P_1\leq X\\ \deg P_2\leq X\\ A,B\in\mathcal{M}\\ \deg AB<\deg Q\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C,\deg D\leq\frac{1}{10}\deg Q\\ P_1AC\equiv P_2BD(\bmod Q)}}\frac{(\log|P_1|)(\log|P_2|)\alpha_{-1}(C)\alpha_{-1}(D)}{|P_1P_2ABCD|^{\frac{1}{2}}}\\
&-\frac{2}{\phi(Q)}\sum_{\substack{\deg P_1\leq X\\ \deg P_2\leq X\\ A,B\in\mathcal{M}\\ \deg AB<\deg Q\\ C,D\in\mathcal{S}_{\mathcal{M}}(X)\\ \deg C,\deg D\leq\frac{1}{10}\deg Q}}\frac{(\log|P_1|)(\log|P_2|)\alpha_{-1}(C)\alpha_{-1}(D)}{|P_1P_2ABCD|^{\frac{1}{2}}}.
\end{aligned}\tag{7.33}$$

The second term on the RHS can be seen to be $O\big(|Q|^{-\frac{1}{3}}\big)$.

For the first term on the RHS we consider the off-diagonal terms first. That is, when $P_1 AC \equiv P_2 BD (\operatorname{mod} Q)$, but $P_1 AC \neq P_2 BD$. Almost identical to Lemma 6.4.4, we have, for $Z_1 = 0, \ldots, \deg Q - 1$, that

$$\sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = Z_1 \\ P_1 AC \equiv P_2 BD (\operatorname{mod} Q) \\ AC \neq BD}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{q^{\frac{Z_1}{2}} (Z_1 + 1) |P_1 P_2 CD|}{|Q|}.$$

Hence,

$$\begin{aligned} &2 \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q \\ P_1 AC \equiv P_2 BD (\operatorname{mod} Q) \\ P_1 AC \neq P_2 BD}} \frac{(\log|P_1|)(\log|P_1|)\alpha_{-1}(C)\alpha_{-1}(D)}{|P_1 P_2 ABCD|^{\frac{1}{2}}} \\ \sim &2 \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q}} \frac{(\log|P_1|)(\log|P_1|)\alpha_{-1}(C)\alpha_{-1}(D)}{|P_1 P_2 CD|^{\frac{1}{2}}} \sum_{Z_1=0}^{\deg Q - 1} \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB = Z_1 \\ P_1 AC \equiv P_2 BD (\operatorname{mod} Q) \\ AC \neq BD \\ (AB,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \\ \ll &\frac{\deg Q}{|Q|^{\frac{1}{2}}} \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q}} (\log|P_1|)(\log|P_1|)|P_1 P_2 CD|^{\frac{1}{2}} \ll \frac{(\deg Q)^4 |Q|^{\frac{3}{10}}}{|Q|^{\frac{1}{2}}} = O\big(|Q|^{-\frac{1}{10}}\big). \end{aligned}$$

We now consider the diagonal terms in (7.33). That is, when $P_1 AC = P_2 BD$. Consider the case where $P_1 = P_2$. Then, the condition $P_1 AC = P_2 BD$ becomes $AC = BD$, and we have

$$\begin{aligned} &2 \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q \\ P_1 AC = P_2 BD \\ P_1 = P_2}} \frac{(\log|P_1|)(\log|P_2|)\alpha_{-1}(C)\alpha_{-1}(D)}{|P_1 P_2 ABCD|^{\frac{1}{2}}} \\ =&2 \bigg( \sum_{\deg P \leq X} \frac{(\log|P|)^2}{|P|} \bigg) \bigg( \sum_{\substack{A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q \\ AC = BD}} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \bigg) \\ \sim &(\log q)^2 \frac{X(X+1)}{2} \frac{\deg Q}{e^{\gamma} X} \sim (\log q)^2 \frac{X \deg Q}{2 e^{\gamma}}, \end{aligned}$$

where the second relation uses (6.20).

Now consider the case where $P_1 \neq P_2$. We can write $A = GA'$, $B = GB'$, $C = HC'$, and $D = HD'$, where $(A', B') = 1$ and $(C', D') = 1$, and $G, A', B' \in \mathcal{M}$ and $H, C', D' \in \mathcal{S}_{\mathcal{M}}(X)$ are unique . The condition $P_1AC = P_2BD$ becomes $P_1A'C' = P_2B'D'$. Now, we must have that $P_1 \mid B'$ or $P_1 \mid D'$, but we cannot have both; otherwise, the coprimality relations would imply that $P_1 \nmid A'C'$ which means that ${P_1}^2 \nmid P_1A'C'$ and ${P_1}^2 \mid P_2B'D'$, a contradiction. Similarly, $P_2 \mid A'$ or $P_2 \mid C'$, but not both. So, there are four cases we must consider:

- If $P_1 \mid B'$ and $P_2 \mid A'$, then we can write $B' = P_1B''$ and $A' = P_2A''$. Then, the condition $P_1A'C' = P_2B'D'$ becomes $A''C' = B''D'$, and the coprimality relations tell us that $A'' = D'$ and $C' = B''$. Let us define $T := A'' = D'$ and $S := C' = B''$ . Note that we must have $(T, S) = 1$, $(P_1, T) = 1$, and $(P_2, S) = 1$.

- If $P_1 \mid D'$ and $P_2 \mid C'$, then we can write $D' = P_1D''$ and $C' = P_2C''$. Then, the condition $P_1A'C' = P_2B'D'$ becomes $A'C'' = B'D''$, and the coprimality relations tell us that $A' = D''$ and $C'' = B'$. Let us define $T := A' = D''$ and $S := C'' = B'$ . Note that we must have $(T, S) = 1$, $(P_1, S) = 1$, and $(P_2, T) = 1$.

- The case where $P_1 \mid B'$ and $P_2 \mid C'$, or, where $P_1 \mid D'$ and $P_2 \mid A'$, are identical by symmetry. So, suppose we have the former. Then we can write $B' = P_1B''$ and $C' = P_2C''$. Then, the condition $P_1A'C' = P_2B'D'$ becomes $A'C'' = B''D'$, and the coprimality relations tell us that $A' = D'$ and $C'' = B''$. Let us define $T := A' = D'$ and $S := C'' = B''$ . Note that we must have $(T, S) = 1$, $(P_1, T) = 1$, and $(P_2, T) = 1$.

So, we have that

$$\sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q \\ P_1 AC = P_2 BD \\ P_1 \neq P_2}} \frac{(\log|P_1|)(\log|P_2|)\alpha_{-1}(C)\alpha_{-1}(D)}{|P_1 P_2 ABCD|^{\frac{1}{2}}}$$

$$= \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ P_1 \neq P_2}} \frac{(\log|P_1|)(\log|P_2|)}{|P_1 P_2|} \sum_{\substack{G \in \mathcal{M} \\ H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G^2 P_1 P_2 ST < \deg Q \\ \deg HS, \deg HT \leq \frac{1}{10} \deg Q \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,S)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|GHST|}$$

$$+ \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ P_1 \neq P_2}} \frac{(\log|P_1|)(\log|P_2|)}{|P_1 P_2|} \sum_{\substack{G \in \mathcal{M} \\ H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G^2 ST < \deg Q \\ \deg HP_2S, \deg HP_1T \leq \frac{1}{10} \deg Q \\ (T,S)=1 \\ (P_1,S)=1 \\ (P_2,T)=1}} \frac{\alpha_{-1}(HP_2S)\alpha_{-1}(HP_1T)}{|GHST|}$$

$$+ 2 \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ P_1 \neq P_2}} \frac{(\log|P_1|)(\log|P_2|)}{|P_1 P_2|} \sum_{\substack{G \in \mathcal{M} \\ H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G^2 P_1 ST < \deg Q \\ \deg HP_2S, \deg HT \leq \frac{1}{10} \deg Q \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,T)=1}} \frac{\alpha_{-1}(HP_2S)\alpha_{-1}(HT)}{|GHST|}. \tag{7.34}$$

Consider the inner sum of the first term on the RHS. We have

$$\sum_{\substack{G \in \mathcal{M} \\ H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg G^2 P_1 P_2 ST < \deg Q \\ \deg HS, \deg HT \leq \frac{1}{10} \deg Q \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,S)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|GHST|}$$

$$= \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg P_1 P_2 ST < \deg Q \\ \deg HS, \deg HT \leq \frac{1}{10} \deg Q \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,S)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} \sum_{\substack{G \in \mathcal{M} \\ \deg G < \frac{\deg Q - \deg P_1 P_2 ST}{2}}} \frac{1}{|G|}$$

$$= \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg P_1 P_2 ST < \deg Q \\ \deg HS, \deg HT \leq \frac{1}{10} \deg Q \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,S)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} \left\lfloor \frac{\deg Q - \deg P_1 P_2 ST}{2} \right\rfloor.$$

By results similar to Lemmas 6.4.1 and 6.4.2, replacing $\left\lfloor \frac{\deg Q - \deg P_1 P_2 ST}{2} \right\rfloor$ with $\frac{\deg Q}{2}$

contributes only $O\big(X^6\big)$ to our final result, which does not affect the main term. Furthermore, we can remove the condition $\deg P_1 P_2 ST < \deg Q$ as this follows from the conditions $\deg HS, \deg HT \leq \frac{1}{10} \deg Q$ and $\deg P_1, \deg P_2 \leq X$. Then, similar to (6.7), we can remove the condition $\deg HS, \deg HT \leq \frac{1}{10} \deg Q$ and this will only alter our final result by $O\big(|Q|^{-\frac{1}{11}}\big)$. We can apply similar reasoning to the inner sums of the second and third terms on the RHS of (7.34), and so we see that

$$
\begin{aligned}
&\sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ A,B \in \mathcal{M} \\ \deg AB < \deg Q \\ C,D \in \mathcal{S}_{\mathcal{M}}(X) \\ \deg C, \deg D \leq \frac{1}{10} \deg Q \\ P_1 AC = P_2 BD \\ P_1 \neq P_2}} \frac{(\log|P_1|)(\log|P_2|)\alpha_{-1}(C)\alpha_{-1}(D)}{|P_1 P_2 ABCD|^{\frac{1}{2}}} \\
=&\frac{\deg Q}{2} \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ P_1 \neq P_2}} \frac{(\log|P_1|)(\log|P_2|)}{|P_1 P_2|} \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,S)=1}} \frac{\alpha_{-1}(HS)\alpha_{-1}(HT)}{|HST|} \\
&+ \frac{\deg Q}{2} \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ P_1 \neq P_2}} \frac{(\log|P_1|)(\log|P_2|)}{|P_1 P_2|} \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,S)=1 \\ (P_2,T)=1}} \frac{\alpha_{-1}(HP_2 S)\alpha_{-1}(HP_1 T)}{|HST|} \\
&+ \deg Q \sum_{\substack{\deg P_1 \leq X \\ \deg P_2 \leq X \\ P_1 \neq P_2}} \frac{(\log|P_1|)(\log|P_2|)}{|P_1 P_2|} \sum_{\substack{H,S,T \in \mathcal{S}_{\mathcal{M}}(X) \\ (T,S)=1 \\ (P_1,T)=1 \\ (P_2,T)=1}} \frac{\alpha_{-1}(HP_2 S)\alpha_{-1}(HT)}{|HST|} \\
\ll&(\log q) \deg Q,
\end{aligned}
$$

where the last relation uses Lemma 7.5.2. This completes the proof. □

# Appendix A

# Function Fields Background

In this appendix we prove results that are required in this thesis, but are well known. We begin with a few results involving $L$-functions; and then look at the growth of the functions $\omega$, $\phi$, and $\phi^*$, as well as Mertens' third theorem in $\mathbb{F}_q[T]$; before ending with some results on sums of multiplicative functions.

## A.1 A Few Results on $L$-functions

**Lemma A.1.1.** *Let $t_1, t_2, t_3, t_4 \in \mathbb{C}$ be such that $|t_1 t_4| < 1$ and $|t_2 t_3| < 1$. Then,*

$$\sum_{\substack{R,S \in \mathcal{M} \\ (R,S)=1}} \frac{(t_1 t_4)^{\deg R} (t_2 t_3)^{\deg S}}{|RS|}$$
$$= \sum_{R,S \in \mathcal{M}} \frac{(t_1 t_4)^{\deg R} (t_2 t_3)^{\deg S}}{|RS|} - q^{-1} \sum_{R,S \in \mathcal{M}} \frac{(t_1 t_4)^{\deg R+1} (t_2 t_3)^{\deg S+1}}{|RS|}$$

*Proof.* We have that

$$\sum_{R,S \in \mathcal{M}} \frac{(t_1 t_4)^{\deg R} (t_2 t_3)^{\deg S}}{|RS|} = \sum_{I \in M} \sum_{\substack{R,S \in \mathcal{M} \\ (R,S)=I}} \frac{(t_1 t_4)^{\deg R} (t_2 t_3)^{\deg S}}{|RS|}$$
$$= \sum_{I \in M} \frac{(t_1 t_2 t_3 t_4)^{\deg I}}{|I|^2} \sum_{\substack{R,S \in \mathcal{M} \\ (R,S)=1}} \frac{(t_1 t_4)^{\deg R} (t_2 t_3)^{\deg S}}{|RS|},$$

from which we deduce that

$$\sum_{\substack{R,S \in \mathcal{M} \\ (R,S)=1}} \frac{(t_1 t_4)^{\deg R} (t_2 t_3)^{\deg S}}{|RS|}$$
$$= \left( \sum_{R,S \in \mathcal{M}} \frac{(t_1 t_4)^{\deg R} (t_2 t_3)^{\deg S}}{|RS|} \right) \left( \sum_{I \in M} \frac{(t_1 t_2 t_3 t_4)^{\deg I}}{|I|^2} \right)^{-1}.$$

The result follows by noting that

$$\sum_{I \in M} \frac{(t_1 t_2 t_3 t_4)^{\deg I}}{|I|^2} = \sum_{n=0}^{\infty} (q^{-1} t_1 t_2 t_3 t_4)^n = \frac{1}{1 - q^{-1} t_1 t_2 t_3 t_4}.$$

□

In the next two lemmas, we use the functional equation for Dirichlet $L$-functions to express $\left|L\left(\frac{1}{2},\chi\right)\right|^2$ as a shortened sum.

**Lemma A.1.2.** *Let $\chi$ be a primitive odd character of modulus $R \in \mathcal{M}\backslash\{1\}$. Then,*

$$\left|L\left(\frac{1}{2},\chi\right)\right|^2 = 2 \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB<\deg R}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} + c_o(\chi),$$

*where we define*

$$c_o(\chi) := - \sum_{\substack{A,B\in\mathcal{M} \\ \deg AB=\deg R-1}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}.$$

*Proof.* The functional equation for odd primitive characters gives

$$\begin{aligned} L(s,\chi) = \sum_{n=0}^{\deg R-1} L_n(\chi) q^{-ns} =& W(\chi) q^{\frac{\deg R-1}{2}} \left(q^{-s}\right)^{\deg R-1} \sum_{n=0}^{\deg R-1} L_n(\overline{\chi}) q^{-n(1-s)} \\ =& W(\chi) q^{-\frac{\deg R-1}{2}} \sum_{n=0}^{\deg R-1} L_n(\overline{\chi}) q^{(1-s)(\deg R-1-n)}. \end{aligned}$$

That is,

$$L(s,\chi) = \sum_{n=0}^{\deg R-1} L_n(\chi) q^{-ns} \tag{A.1}$$

and

$$L(s,\chi) = W(\chi) q^{-\frac{\deg R-1}{2}} \sum_{n=0}^{\deg R-1} L_n(\overline{\chi}) q^{(1-s)(\deg R-1-n)}. \tag{A.2}$$

We now take the squared modulus of both sides of (A.1) and of (A.2). In order to make our calculations slightly easier, we restrict our attention to the case where $s \in \mathbb{R}$. We obtain

$$|L(s,\chi)|^2 = \sum_{n=0}^{2\deg R-2} \Bigg( \sum_{\substack{0\leq i,j<\deg R \\ i+j=n}} L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-ns} \tag{A.3}$$

and

$$|L(s,\chi)|^2 = q^{-\deg R+1} \sum_{n=0}^{2\deg R-2} \Bigg( \sum_{\substack{0\leq i,j<\deg R \\ i+j=n}} L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{(1-s)(2\deg R-2-n)}. \tag{A.4}$$

By the linear independence of powers of $q^{-s}$ we can see that $|L(s,\chi)|^2$ is equal to the sum of the terms $n = 0, 1, \ldots, \deg R - 1$ on the RHS of (A.3) and the terms $n = 0, 1, \ldots, \deg R - 2$ on the RHS of (A.4). That is,

$$|L(s,\chi)|^2 = \sum_{n=0}^{\deg R-1} \Bigg( \sum_{\substack{0\leq i,j<\deg R \\ i+j=n}} L_i(\chi) L_j(\overline{\chi}) \Bigg) q^{-ns}$$

$$+q^{-\deg R+1}\sum_{n=0}^{\deg R-2}\bigg(\sum_{\substack{0\leq i,j<\deg R\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\bigg)q^{(1-s)(2\deg R-2-n)}.$$

Hence,

$$\begin{aligned}&\Big|L\Big(\frac{1}{2},\chi\Big)\Big|^2\\ =&\sum_{n=0}^{\deg R-1}\bigg(\sum_{\substack{0\leq i,j<\deg R\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}+\sum_{n=0}^{\deg R-2}\bigg(\sum_{\substack{0\leq i,j<\deg R\\ i+j=n}} L_i(\chi)L_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}\\ =&2\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB<\deg R}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}-\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-1}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}},\end{aligned}$$

as required. □

**Lemma A.1.3.** *Let $\chi$ a primitive even character of modulus $R\in\mathcal{M}\backslash\{1\}$ (note that this requires $\deg R\geq 2$). Then,*

$$\Big|L\Big(\frac{1}{2},\chi\Big)\Big|^2=2\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB<\deg R}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}+c_e(\chi),$$

*where*

$$\begin{aligned}c_e(\chi):=&-\frac{q}{\big(q^{\frac{1}{2}}-1\big)^2}\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-2}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}-\frac{2q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-1}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}\\ &+\frac{1}{\big(q^{\frac{1}{2}}-1\big)^2}\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}.\end{aligned}$$

*Proof.* The functional equation for even primitive characters gives us that

$$\begin{aligned}\big(q^{1-s}-1\big)L(s,\chi)=&\big(q^{1-s}-1\big)\sum_{n=0}^{\deg R-1}L_n(\chi)q^{-ns}\\ =&W(\chi)q^{\frac{\deg R}{2}}\big(q^{-s}-1\big)\big(q^{-s}\big)^{\deg R-1}\sum_{n=0}^{\deg R-1}L_n(\overline{\chi})q^{-n(1-s)} \qquad\text{(A.5)}\\ =&W(\chi)q^{-\frac{\deg R}{2}}\big(q^{1-s}-q\big)\sum_{n=0}^{\deg R-1}L_n(\overline{\chi})q^{(1-s)(\deg R-1-n)}.\end{aligned}$$

Let us define $L_{-1}(\chi):=0$ and recall that $L_{\deg R}(\chi)=0$. Then, we can define

$$M_i(\chi):=qL_{i-1}(\chi)-L_i(\chi)$$

for $i=0,1,\dots,\deg R$, and so (A.5) gives us that

$$\big(q^{1-s}-1\big)L(s,\chi)=\sum_{n=0}^{\deg R}M_n(\chi)q^{-ns} \qquad\text{(A.6)}$$

and

$$\big(q^{1-s}-1\big)L(s,\chi)=-W(\chi)q^{-\frac{\deg R}{2}}\sum_{n=0}^{\deg R}M_n(\overline{\chi})q^{(1-s)(\deg R-n)}. \tag{A.7}$$

Similarly as in the proof of Lemma A.1.2, we take the squared modulus of both sides of (A.6) and (A.7), and use the linear independence of powers of $q^{-s}$, to obtain

$$\begin{aligned}\big(q^{1-s}-1\big)^2|L(s,\chi)|^2=&\sum_{n=0}^{\deg R}\bigg(\sum_{\substack{0\leq i,j\leq \deg R\\ i+j=n}}M_i(\chi)M_j(\overline{\chi})\bigg)q^{-ns}\\&+q^{-\deg R}\sum_{n=0}^{\deg R-1}\bigg(\sum_{\substack{0\leq i,j\leq \deg R\\ i+j=n}}M_i(\chi)M_j(\overline{\chi})\bigg)q^{(1-s)(2\deg R-n)}.\end{aligned}$$

Again, to make our calculations slightly easier, we have restricted our attention to the case where $s\in\mathbb{R}$. We now take $s=\frac{1}{2}$ and simplify to obtain

$$\begin{aligned}&\big(q^{\frac{1}{2}}-1\big)^2\Big|L\Big(\frac{1}{2},\chi\Big)\Big|^2\\=&2\sum_{n=0}^{\deg R-1}\bigg(\sum_{\substack{0\leq i,j\leq \deg R\\ i+j=n}}M_i(\chi)M_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}+\sum_{\substack{0\leq i,j\leq \deg R\\ i+j=\deg R}}M_i(\chi)M_j(\overline{\chi})q^{-\frac{\deg R}{2}}.\end{aligned}$$

Now,

$$\begin{aligned}&\sum_{n=0}^{\deg R-1}\bigg(\sum_{\substack{0\leq i,j\leq \deg R\\ i+j=n}}M_i(\chi)M_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}\\=&\sum_{n=0}^{\deg R-1}q^2\bigg(\sum_{\substack{0\leq i,j\leq \deg R\\ i+j=n}}L_{i-1}(\chi)L_{j-1}(\overline{\chi})\bigg)q^{-\frac{n}{2}}-\sum_{n=0}^{\deg R-1}q\bigg(\sum_{\substack{0\leq i,j\leq \deg R\\ i+j=n}}L_{i-1}(\chi)L_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}\\&-\sum_{n=0}^{\deg R-1}q\bigg(\sum_{\substack{0\leq i,j\leq \deg R\\ i+j=n}}L_i(\chi)L_{j-1}(\overline{\chi})\bigg)q^{-\frac{n}{2}}+\sum_{n=0}^{\deg R-1}\bigg(\sum_{\substack{0\leq i,j\leq \deg R\\ i+j=n}}L_i(\chi)L_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}\\=&\sum_{n=0}^{\deg R-3}q\bigg(\sum_{\substack{0\leq i,j\leq \deg R-1\\ i+j=n}}L_i(\chi)L_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}-\sum_{n=0}^{\deg R-2}q^{\frac{1}{2}}\bigg(\sum_{\substack{0\leq i,j\leq \deg R-1\\ i+j=n}}L_i(\chi)L_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}\\&-\sum_{n=0}^{\deg R-2}q^{\frac{1}{2}}\bigg(\sum_{\substack{0\leq i,j\leq \deg R-1\\ i+j=n}}L_i(\chi)L_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}+\sum_{n=0}^{\deg R-1}\bigg(\sum_{\substack{0\leq i,j\leq \deg R-1\\ i+j=n}}L_i(\chi)L_j(\overline{\chi})\bigg)q^{-\frac{n}{2}}\\=&\big(q^{\frac{1}{2}}-1\big)^2\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB<\deg R}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}\\&-q\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-2}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}+\big(2q^{\frac{1}{2}}-q\big)\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-1}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}},\end{aligned}$$

and similarly,

$$\begin{aligned}
&\sum_{\substack{0\le i,j\le \deg R\\ i+j=\deg R}} M_i(\chi)M_j(\overline{\chi})q^{-\frac{\deg R}{2}}\\
=&q\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-2}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}-2q^{\frac{1}{2}}\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-1}}\frac{\chi(B)\overline{\chi}(A)}{|AB|^{\frac{1}{2}}}+\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|L\Big(\frac{1}{2},\chi\Big)\right|^2=&2\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB<\deg R}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}-\frac{q}{\big(q^{\frac{1}{2}}-1\big)^2}\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-2}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}\\
&-\frac{2q^{\frac{1}{2}}}{q^{\frac{1}{2}}-1}\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R-1}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}}+\frac{1}{\big(q^{\frac{1}{2}}-1\big)^2}\sum_{\substack{A,B\in\mathcal{M}\\ \deg AB=\deg R}}\frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}},
\end{aligned}$$

as required. □

It is convenient to define

$$c(\chi):=\begin{cases}c_e(\chi) & \text{if } \chi \text{ is even}\\ c_o(\chi) & \text{if } \chi \text{ is odd.}\end{cases} \tag{A.8}$$

## A.2 The Growth of the Functions $\omega$, $\phi$, and $\phi^*$

In this section we obtain bounds on the functions $\omega$, $\phi$, and $\phi^*$. Some of these bounds involve factors of the form $\log_q \deg R$. Thus, for the implied constants to be independent of $q$, we require that $\deg R > q$. We are in fact able to avoid this, although this requires us to change the bounds somewhat. We also give some $\liminf$ and $\limsup$ results. Naturally, due to the limit, these results avoid the need to consider $\deg R > q$. Before proceeding, we must define the primorial polynomials and prove a result on their growth.

**Definition A.2.1** (Primorial Polynomials)**.** *Let $(S_i)_{i\in\mathbb{Z}_{>0}}$ be a fixed ordering of $\mathcal{P}$ such that $\deg S_i \le \deg S_{i+1}$ for all $i\ge 1$ (the order of the primes of a given degree is not of importance here). For all positive integers $n$ we define*

$$R_n := \prod_{i=1}^{n} S_i.$$

*We will refer to $R_n$ as the $n$-th* primorial*. For each positive integer $n$ we have unique non-negative integers $m_n$ and $r_n$ such that*

$$R_n = \Big(\prod_{\deg P\le m_n} P\Big)\Big(\prod_{i=1}^{r_n} Q_i\Big), \tag{A.9}$$

*where the $Q_i$ are distinct primes of degree $m_n+1$. This definition of primorial is not standard.*

**Lemma A.2.2.** *For all positive integers $n$ we have that*

$$\log_q \log_q |R_n| = m_n + O(1).$$

*From this we can deduce that*

$$m_n \ll \log_q \log_q |R_n|$$

*for $n$ satisfying $m_n \geq 1$. In particular, the implied constant is independent of $q$.*

*Proof.* For the first claim, by (A.9) and (1.12), we see that

$$\log_q |R_n| = \deg R_n \leq \sum_{i=1}^{m_n+1} \left( q^i + O\left(q^{\frac{i}{2}}\right) \right) \ll q^{m_n+1}$$

and

$$\log_q |R_n| = \deg R_n \geq \sum_{i=1}^{m_n} \left( q^i + O\left(q^{\frac{i}{2}}\right) \right) \gg q^{m_n}.$$

By taking logarithms of both equations above, we deduce that

$$\log_q \log_q |R_n| = m_n + O(1).$$

For the second claim, if $m_n \geq 1$ then $\log_q \log_q |R_n| \geq 1$, and so by the first claim we have

$$\frac{m_n}{\log_q \log_q |R_n|} \ll 1 + \frac{1}{\log_q \log_q |R_n|} \ll 1.$$

□

Using this result we can obtain results about the growth of the $\omega$, $\phi$, and $\phi^*$ functions.

**Lemma A.2.3.** *For $\deg R > 1$ we have*

$$\omega(R) \ll \frac{\log_q |R|}{\log_q \log_q |R|}.$$

*We also have*

$$\limsup_{\deg R \to \infty} \omega(R) \frac{\log_q \log_q |R|}{\log_q |R|} = 1.$$

*We emphasise that the implied constant in the first result is independent of $q$, which is why the first result does not follow immediately from the second.*

*Proof.* For the first claim, if $\deg R > 1$ with $\omega(R) = 1$, then the result clearly holds. So, suppose $\omega(R) > 1$. It suffices to prove the result for the primorials. Indeed, if $\omega(R) = n$, we then have that

$$\omega(R) \frac{\log_q \log_q |R|}{\log_q |R|} \ll \omega(R_n) \frac{\log_q \log_q |R_n|}{\log_q |R_n|} \ll 1.$$

Now, if $m_n = 0$ then we can easily see that $\omega(R_n) \leq \frac{\log_q|R_n|}{\log_q \log_q|R_n|}$. If $m_n \geq 1$, then by (1.12) we have

$$\omega(R_n) = \sum_{i=1}^{m_n} \frac{1}{i} \sum_{d|i} \mu(d) q^{\frac{i}{d}} \ + r_n = \sum_{d=1}^{m_n} \frac{\mu(d)}{d} \sum_{i=1}^{\lfloor \frac{m_n}{d} \rfloor} \frac{1}{i} q^i \ + r_n \ll \frac{q^{m_n}}{m_n} + r_n, \tag{A.10}$$

and

$$\begin{aligned} \log_q|R_n| = \deg R_n &= \sum_{i=1}^{m_n} \sum_{d|i} \mu(d) q^{\frac{i}{d}} \ + (m_n+1) r_n \\ &= \sum_{d=1}^{m_n} \mu(d) \sum_{i=1}^{\lfloor \frac{m_n}{d} \rfloor} q^i \ + (m_n+1) r_n \gg q^{m_n} + m_n r_n. \end{aligned} \tag{A.11}$$

Thus, using Lemma A.2.2 for the second relation,

$$\frac{\omega(R_n)}{\log_q|R_n|} \ll \frac{1}{m_n} \ll \frac{1}{\log_q \log_q|R_n|}.$$

For the second claim we begin by considering the primorials. By similar means as in (A.10) and (A.11) we have

$$\begin{aligned} \omega(R_n) &= \sum_{i=1}^{m_n} \frac{1}{i} \sum_{d|i} \mu(d) q^{\frac{i}{d}} \ + r_n = \sum_{i=1}^{m_n} \frac{q^i}{i} \ + O\big(q^{\frac{m_n}{2}}\big) + r_n \\ &= \frac{1}{q-1} \sum_{i=1}^{m_n} \Big( \frac{q^{i+1}}{i+1} - \frac{q^i}{i} + \frac{q^{i+1}}{i(i+1)} \Big) + O\big(q^{\frac{m_n}{2}}\big) + r_n \\ &= \frac{q}{q-1} \frac{q^{m_n}}{m_n+1} + r_n + O\Big( \frac{q^{m_n}}{(m_n)^2} \Big) \end{aligned}$$

and

$$\log_q|R_n| = \sum_{d=1}^{m_n} \mu(d) \sum_{i=1}^{\lfloor \frac{m_n}{d} \rfloor} q^i \ + (m_n+1) r_n = \frac{q}{q-1} q^{m_n} + (m_n+1) r_n + O\big(q^{\frac{m_n}{2}}\big).$$

From these two results and lemma A.2.2, we deduce that

$$\omega(R_n) \frac{\log_q \log_q|R_n|}{\log_q|R_n|} \longrightarrow 1 \tag{A.12}$$

as $n \longrightarrow \infty$, which shows that

$$\limsup_{\deg R \to \infty} \omega(R) \frac{\log_q \log_q|R|}{\log_q|R|} \geq 1.$$

We now proceed to prove the other inequality. We are required to work with $S \in \mathcal{M}$ to avoid a clash of notation with "$R$" which will appear in our use of the primorials. We consider two cases:

$$\omega(S) \leq \Big( \frac{\deg S}{\log_q \deg S} \Big)^{\frac{1}{2}} \qquad \text{and} \qquad \omega(S) > \Big( \frac{\deg S}{\log_q \deg S} \Big)^{\frac{1}{2}}$$

For the first case,we have

$$\omega(S)\frac{\log_q \log_q|S|}{\log_q|S|} \leq \Big(\frac{\log_q \deg S}{\deg S}\Big)^{\frac{1}{2}} \longrightarrow 0$$

as $\deg S \longrightarrow \infty$. For the second case we can see that $\omega(S) \longrightarrow \infty$ as $\deg S \longrightarrow \infty$, and so, by (A.12),

$$\omega(S)\frac{\log_q \log_q|S|}{\log_q|S|} \leq \omega\big(R_{\omega(S)}\big)\frac{\log_q \log_q|R_{\omega(S)}|}{\log_q|R_{\omega(S)}|} \longrightarrow 1$$

as $\deg S \longrightarrow \infty$. Thus,

$$\limsup_{\deg S \to \infty} \omega(S)\frac{\log_q \log_q|S|}{\log_q|S|} \leq 1$$

and the result follows . □

**Lemma A.2.4.** *For* $\deg R > q$*,*

$$\phi(R) \gg \frac{|R|}{\log_q \log_q|R|}.$$

*We also have that*

$$\liminf_{\deg R \to \infty} \left(\frac{\phi(R)}{|R|}\log_q \log_q|R|\right) = e^{-\gamma}.$$

*Proof.* We begin with the first claim. Again, we are required to work with $S \in \mathcal{M}$ to avoid a clash of notation with "$R$" which will appear in our use of the primorials. We first consider the case where $\omega(S) \leq q$ (recall that, by our assumption, we also have $\deg S > q$). We have

$$\frac{\phi(S)}{|S|}\log_q \log_q|S| \geq \frac{\phi(S)}{|S|} \geq \frac{\phi(R_q)}{|R_q|} \geq \big(1-q^{-1}\big)^{-q} \gg 1.$$

Now consider the case where $\omega(S) > q$. it suffices to prove claim for the primorials. Indeed, assuming this holds (for the third relation below), we have

$$\frac{\phi(S)}{|S|}\log_q \log_q|S| \geq \frac{\phi\big(\operatorname{rad}(S)\big)}{|\operatorname{rad}(S)|}\log_q \log_q|\operatorname{rad}(S)| \geq \frac{\phi(R_{\omega(S)})}{|R_{\omega(S)}|}\log_q \log_q|R_{\omega(S)}| \geq c$$

for some positive constant $c$ that is independent of $q$. Now, for $n > q$ we have

$$\begin{aligned}\frac{\phi(R_n)}{|R_n|} &= \prod_{P|R_n}\Big(1-\frac{1}{|P|}\Big) \geq \prod_{\deg P \leq m_n+1}\Big(1-\frac{1}{|P|}\Big) = \exp\Bigg(-\sum_{\deg P \leq m_n+1}\sum_{k=1}^{\infty}\frac{1}{k|P|^k}\Bigg)\\ &= \exp\Bigg(-\sum_{\deg P \leq m_n+1}\frac{1}{|P|} + O\Big(\sum_{\deg P \leq m_n+1}\frac{1}{|P|^2}\Big)\Bigg)\\ &= \exp\Bigg(-\sum_{i=1}^{m_n+1}\frac{1}{i} + O\Big(\sum_{i=1}^{m_n+1}\frac{1}{iq^{\frac{i}{2}}}\Big) + O(1)\Bigg)\\ &= \exp\big(-\log m_n + O(1)\big) \gg \frac{1}{m_n} \gg \frac{1}{\log_q \log_q|R_n|}.\end{aligned}$$

We now consider the second claim in the lemma. For all $n \geq 1$, we have that

$$\prod_{\deg P \leq m_n} \Big(1 - \frac{1}{|P|}\Big) \leq \frac{\phi(R_n)}{|R_n|} \leq \prod_{\deg P \leq m_n + 1} \Big(1 - \frac{1}{|P|}\Big),$$

and so by Lemmas A.2.7 and A.2.2 we deduce that

$$\frac{\phi(R_n)}{|R_n|} \sim \frac{e^{-\gamma}}{\log_q \log_q |R_n|} \tag{A.13}$$

as $n \longrightarrow \infty$. Now, we write $\mathcal{M}$ as the union of two disjoint sets $\mathcal{B}$ and $\mathcal{C}$, where

$$\begin{aligned} \mathcal{B} :=& \big\{ S \in \mathcal{A} : \log_q \log_q |S| \geq \big( \log_q \log_q |R_{\omega(S)}| \big)^2 \big\} \\ \mathcal{C} :=& \big\{ S \in \mathcal{A} : \log_q \log_q |S| < \big( \log_q \log_q |R_{\omega(S)}| \big)^2 \big\}. \end{aligned}$$

Suppose $S \in \mathcal{B}$. Then,

$$\begin{aligned} \frac{\phi(S)}{|S|} \log_q \log_q |S| =& \frac{\phi\big(\operatorname{rad}(S)\big)}{|\operatorname{rad}(S)|} \log_q \log_q |S| \geq \frac{\phi(R_{\omega(S)})}{|R_{\omega(S)}|} \log_q \log_q |S| \\ \geq& \bigg[ \frac{\phi(R_{\omega(S)})}{|R_{\omega(S)}|} \log_q \log_q |R_{\omega(S)}| \bigg] \big( \log_q \log_q |S| \big)^{\frac{1}{2}} \longrightarrow \infty \end{aligned} \tag{A.14}$$

as $\deg S \longrightarrow \infty$. The limit follows from the fact that the term inside the square brackets is bounded below by some positive number when $\omega(S) > 1$ (the case $\omega(S) = 1$ is trivial); this, in turn, follows from (A.13). Now suppose $S \in \mathcal{C}$ and $\deg S \longrightarrow \infty$. Then, we can see that $\omega(S) \longrightarrow \infty$. Keeping this in mind, we obtain

$$\frac{\phi(S)}{|S|} \log_q \log_q |S| \geq \frac{\phi(R_{\omega(S)})}{|R_{\omega(S)}|} \log_q \log_q |S| \geq \frac{\phi(R_{\omega(S)})}{|R_{\omega(S)}|} \log_q \log_q |R_{\omega(S)}| \longrightarrow e^{-\gamma} \tag{A.15}$$

as $\deg S \longrightarrow \infty$. From (A.14) and (A.15) we see that

$$\liminf_{\deg R \to \infty} \bigg( \frac{\phi(R)}{|R|} \log_q \log_q |R| \bigg) \geq e^{-\gamma}.$$

Equality follows by considering the primorials. □

**Lemma A.2.5.** *For* $\deg R > q$,

$$\phi^*(R) \gg \frac{\phi(R)}{\log_q \log_q |R|}.$$

*We also have that*

$$\liminf_{\deg R \to \infty} \bigg( \frac{\phi^*(R)}{\phi(R)} \log_q \log_q |R| \bigg) = e^{-\gamma} \prod_{P \in \mathcal{P}} \bigg( 1 - \frac{1}{\big(|P| - 1\big)^2} \bigg).$$

*Proof.* We begin with the first claim. From Corollary 1.4.6, we have that

$$\begin{aligned} \frac{\phi^*(R)}{\phi(R)} =& \sum_{EF=R} \mu(E) \frac{\phi(F)}{\phi(R)} = \sum_{EF=R} \mu(E) \prod_{\substack{P|E \\ P^2 \nmid R}} \bigg( \frac{1}{|P| - 1} \bigg) \prod_{\substack{P|E \\ P^2 | R}} \bigg( \frac{1}{|P|} \bigg) \\ =& \prod_{\substack{P|R \\ P^2 \nmid R}} \bigg( 1 - \frac{1}{|P| - 1} \bigg) \prod_{\substack{P|R \\ P^2 | R}} \bigg( 1 - \frac{1}{|P|} \bigg). \end{aligned} \tag{A.16}$$

Hence, for $\deg R \geq q$, we have

$$\frac{\phi^*(R)}{\phi(R)} \geq \prod_{P|R}\left(1 - \frac{1}{|P|-1}\right) \gg \prod_{P|R}\left(1 - \frac{1}{|P|}\right) = \frac{\phi(R)}{|R|} \gg \frac{1}{\log_q \log_q |R|}$$

where the last relation follows from Lemma A.2.4.

For the second claim, we begin with the primorials. As $n \longrightarrow \infty$, we have

$$\begin{aligned}\frac{\phi^*(R_n)}{\phi(R_n)} \log_q \log_q |R_n| &= \prod_{P|R_n}\left(1 - \frac{1}{|P|-1}\right) \log_q \log_q |R_n| \\ &= \prod_{P|R_n}\left(1 - \frac{1}{(|P|-1)^2}\right) \frac{\phi(R_n)}{|R_n|} \log_q \log_q |R_n| \\ &\sim \prod_{P \in \mathcal{P}}\left(1 - \frac{1}{(|P|-1)^2}\right) e^{-\gamma}\end{aligned}$$

where the last equality uses (A.13). This proves that

$$\liminf_{\deg R \to \infty} \left(\frac{\phi^*(R)}{\phi(R)} \log_q \log_q |R|\right) \leq e^{-\gamma} \prod_{P \in \mathcal{P}}\left(1 - \frac{1}{(|P|-1)^2}\right).$$

To prove the other inequality, We must again work with $S \in \mathcal{M}$ to avoid a clash of notation with "$R$" which will appear in our use of the primorials. There are three possible cases:

1. $\omega(S) \leq q$;
2. $\log_q \log_q |S| > \left(\log_q \log_q |R_{\omega(S)}|\right)^2$ and $\omega(S) > q$;
3. $\log_q \log_q |S| \leq \left(\log_q \log_q |R_{\omega(S)}|\right)^2$.

For the first case, we can easily see that

$$\frac{\phi^*(S)}{\phi(S)} \log_q \log_q |S| \geq \frac{\phi^*(R_q)}{\phi(R_q)} \log_q \log_q |S| \longrightarrow \infty$$

as $\deg S \longrightarrow \infty$. For the second case, we have

$$\begin{aligned}&\frac{\phi^*(S)}{\phi(S)} \log_q \log_q |S| \\ \geq &\prod_{P|S}\left(1 - \frac{1}{|P|-1}\right) \log_q \log_q |S| \\ = &\prod_{P|S}\left(1 - \frac{1}{(|P|-1)^2}\right) \frac{\phi(S)}{|S|} \log_q \log_q |S| \\ \geq &\prod_{P \in \mathcal{P}}\left(1 - \frac{1}{(|P|-1)^2}\right) \frac{\phi(R_{\omega(S)})}{|R_{\omega(S)}|} \left(\log_q \log_q |R_{\omega(S)}|\right) \left(\log_q \log_q |S|\right)^{\frac{1}{2}} \longrightarrow \infty\end{aligned}$$

as $\deg S \longrightarrow \infty$, where the limit uses Lemma A.2.4. For the third case we have that $\omega(S) \longrightarrow \infty$ as $\deg S \longrightarrow \infty$, and so, by (A.13),

$$\frac{\phi^*(S)}{\phi(S)} \log_q \log_q |S|$$

$$
\begin{aligned}
&\geq \prod_{P|S}\left(1-\frac{1}{(|P|-1)^2}\right)\frac{\phi(S)}{|S|}\log_q\log_q|S| \\
&\geq \prod_{P\in\mathcal{P}}\left(1-\frac{1}{\left(|P|-1\right)^2}\right)\frac{\phi\big(R_{\omega(S)}\big)}{|R_{\omega(S)}|}\left(\log_q\log_q|R_{\omega(S)}|\right) \\
&\longrightarrow e^{-\gamma}\prod_{P\in\mathcal{P}}\left(1-\frac{1}{\left(|P|-1\right)^2}\right)
\end{aligned}
$$

as $\deg S \longrightarrow \infty$. Thus,

$$
\liminf_{\deg S\to\infty}\left(\frac{\phi^*(S)}{\phi(S)}\log_q\log_q|S|\right) \geq e^{-\gamma}\prod_{P\in\mathcal{P}}\left(1-\frac{1}{\left(|P|-1\right)^2}\right)
$$

and the result follows. □

We also have the following lemma on the asymptotics of certain functions involving $\phi$ and $\omega$.

**Lemma A.2.6.** *For $R \in \mathcal{M}$,*

$$
\prod_{P|R}\left(\frac{1}{1+|P|^{-1}}\right) \asymp \prod_{P|R}\left(1-|P|^{-1}\right); \tag{A.17}
$$

$$
\prod_{P|R}\left(\frac{1}{1+2|P|^{-1}}\right) \asymp \prod_{P|R}\left(\frac{1}{1+|P|^{-1}}\right)^2; \tag{A.18}
$$

$$
\frac{2^{\omega(R)}|R|^{\frac{1}{2}}\deg R}{\phi^*(R)} \ll |R|^{-\frac{1}{3}}; \tag{A.19}
$$

$$
|R| \ll \phi^*(R)\prod_{P|R}\left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}}\right)(\log\deg R)^6; \tag{A.20}
$$

$$
\omega(R)^2 \ll \prod_{P|R}\left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)(\deg R)^2; \tag{A.21}
$$

$$
|R|\omega(R)\prod_{P|R}\left(1-|P|^{-1}\right) \ll \phi^*(R)\deg R. \tag{A.22}
$$

*The fourth result requires that* $\deg R > 1$.

*Proof.* For the first result, it suffices to note that

$$
\begin{aligned}
1 &\geq \prod_{P|R}\left(1+|P|^{-1}\right)\left(1-|P|^{-1}\right) = \prod_{P|R}\left(1-|P|^{-2}\right) \\
&\geq \prod_{P\in\mathcal{P}}\left(1-|P|^{-2}\right) = \frac{1}{\zeta_{\mathcal{A}}(2)} = 1-q^{-1} \geq \frac{1}{2}.
\end{aligned}
$$

For the second result, it suffices to note that

$$1 \leq \prod_{P|R} \left( \frac{(1+|P|^{-1})^2}{1+2|P|^{-1}} \right) = \prod_{P|R} \left( 1 + \frac{1}{|P|^2+2|P|} \right) \leq \prod_{P|R} \left( 1 + \frac{1}{|P|^2} \right)$$
$$= \prod_{P|R} \left( \frac{1-\frac{1}{|P|^4}}{1-\frac{1}{|P|^2}} \right) = \frac{\zeta_{\mathcal{A}}(2)}{\zeta_{\mathcal{A}}(4)} \leq 2.$$

To prove the third result, we first note that, for $\deg R \leq q$,

$$\begin{aligned} 1 \geq \frac{\phi^*(R)}{|R|} &= \frac{\phi^*(R)}{\phi(R)} \frac{\phi(R)}{|R|} = \prod_{\substack{P|R \\ P^2 \nmid R}} \left(1 - \frac{1}{|P|-1}\right) \prod_{\substack{P|R \\ P^2|R}} \left(1 - \frac{1}{|P|}\right) \prod_{P|R} \left(1 - \frac{1}{|P|}\right) \\ &\geq \prod_{P|R} \left(1 - \frac{1}{|P|-1}\right)^2 \geq \prod_{\deg P=1} \left(1 - \frac{1}{|P|-1}\right)^2 \geq \left(1 - \frac{1}{q-1}\right)^{2q} \\ &\geq \left(1 - \frac{2}{q}\right)^{2q} \gg 1, \end{aligned} \tag{A.23}$$

where the third relation uses (A.16) and the last realtion a variation of the well known result that $\lim_{n \longrightarrow \infty}(1+n^{-1})^n = e$. Thus, for $\deg R \leq q$, we have

$$\frac{2^{\omega(R)}|R|^{\frac{1}{2}} \deg R}{\phi^*(R)} \ll \frac{2^{\omega(R)} \deg R}{|R|^{\frac{1}{2}}} \ll \frac{2^{\omega(R)}}{|R|^{\frac{3}{8}}}.$$

Clearly, for large enough $q$ the above is $\ll |R|^{-\frac{1}{3}}$. There are only finitely many other $q$, and so, by Lemma A.2.3, we can deduce that the above is $\ll |R|^{-\frac{1}{3}}$ for all $q$. For $\deg R > q$ we use Lemmas A.2.3, A.2.4, and A.2.5 to obtain

$$\frac{2^{\omega(R)}|R|^{\frac{1}{2}} \deg R}{\phi^*(R)} \ll \frac{2^{\frac{\log_q \deg R}{\deg R}} (\deg R)(\log_q \deg R)^2}{|R|^{\frac{1}{2}}} \ll \frac{2^{\frac{\log_q \deg R}{\deg R}}}{|R|^{\frac{3}{8}}}.$$

Similarly as before, the above is $\ll |R|^{-\frac{1}{3}}$.

The fourth, fifth, and sixth result can be proved similarly as the third result: For $\deg R > q$ we use Lemmas A.2.3, A.2.4, and A.2.5; while for $\deg R \leq q$ we use (A.23). □

Finally, we end this section with Mertens' Third Theorem in $\mathbb{F}_q[T]$.

**Lemma A.2.7** (Mertens' Third Theorem in $\mathbb{F}_q[T]$)**.** *We have*

$$\prod_{\deg P \leq n} \left(1 - \frac{1}{|P|}\right)^{-1} \sim e^{\gamma} n.$$

*Proof.* The proof is very similar to that of Theorem 3 in [Ros99]. □

## A.3 Sums Involving Multiplicative Functions

First, we note that, for $R \in \mathcal{M}$,

$$\sum_{E|R} \frac{\mu(E)}{|E|^s} = \prod_{P|R} \left(1 - \frac{1}{|P|^s}\right) \tag{A.24}$$

and

$$\sum_{E|R} \frac{\mu(E) \deg E}{|E|^s} = -\prod_{P|R} \left(1 - \frac{1}{|P|^s}\right) \sum_{P|R} \frac{\deg P}{|P|^s - 1}, \tag{A.25}$$

where the first equation holds for all $s \in \mathbb{C}$. The second equation is obtained by differentiating the first with respect to $s$ and, while it holds for a larger domain, we are only interested when $\mathrm{Re}(s) \neq 0$.

Also, for all square-full $R \in \mathcal{M}$ we have that

$$\begin{aligned} \sum_{EF=R} \frac{\mu(E)\phi(F)}{|F|^s} &= \sum_{EF=R} \mu(E) \frac{\phi(R)}{|E|} \frac{|E|^s}{|R|^s} = \frac{\phi(R)}{|R|^s} \sum_{EF=R} \frac{\mu(E)}{|E|^{1-s}} \\ &= \frac{\phi(R)}{|R|^s} \prod_{P|R} \left(1 - \frac{1}{|P|^{1-s}}\right) \end{aligned} \tag{A.26}$$

and

$$\sum_{EF=R} \frac{\mu(E)\phi(F) \deg F}{|F|^s} = \frac{\phi(R)}{|R|^s} \prod_{P|R} \left(1 - \frac{1}{|P|^{1-s}}\right) \left(\deg R + \sum_{P|R} \frac{\deg P}{|P|^{1-s} - 1}\right). \tag{A.27}$$

The first equation holds for all $s \in \mathbb{C}$. The second equation is obtained by differentiating the first with respect to $s$ and, while it holds for a larger domain, we are only interested when $\mathrm{Re}(s) \neq 1$.

**Lemma A.3.1.** *Let $R \in \mathcal{M}$. We have that*

$$\sum_{P|R} \frac{\deg P}{|P| - 1} = O\big(\log \omega(R)\big).$$

*Proof.* It suffices to prove the claim for the primorials:

$$\sum_{P|R_n} \frac{\deg P}{|P| - 1} \ll \log n.$$

From the prime polynomial theorem, we can deduce that there is a constant $c \in (0,1)$, which is independent of $q$, such that $|\mathcal{P}_{\leq m}| \geq c q^{\frac{m}{2}}$ for all positive integers $m$. In particular, if we take $m = \lceil \frac{2}{\log q} \log \frac{n}{c} \rceil$, then $|\mathcal{P}_{\leq m}| \geq n$ . So,

$$\sum_{P|R_n} \frac{\deg P}{|P| - 1} \leq \sum_{i=1}^{\frac{2}{\log q} \log \frac{n}{c} + 1} \sum_{\substack{P \in \mathcal{P} \\ \deg P = i}} \frac{\deg P}{|P| - 1} \ll \sum_{i=1}^{\frac{2}{\log q} \log \frac{n}{c} + 1} \frac{i}{q^i - 1} \frac{q^i}{i} \ll \log n,$$

where the second relation follows from the prime polynomial theorem again. ☐

**Lemma A.3.2.** *Let $R \in \mathcal{M}$ and let $x$ be a positive integer. Then,*

$$\sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x \\ (A,R)=1}} \frac{1}{|A|} = \begin{cases} \frac{\phi(R)}{|R|}x + O\Big(\frac{\phi(R)}{|R|}\log \omega(R)\Big) & \text{if } x \geq \deg R \\ \frac{\phi(R)}{|R|}x + O\Big(\frac{\phi(R)}{|R|}\log \omega(R)\Big) + O\Big(\frac{2^{\omega(R)}x}{q^x}\Big) & \text{if } x < \deg R. \end{cases}$$

*Proof.* For all positive integers $x$ we have that

$$\begin{aligned} \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x \\ (A,R)=1}} \frac{1}{|A|} &= \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x}} \frac{1}{|A|} \sum_{E|(A,R)} \mu(E) = \sum_{E|R} \mu(E) \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x \\ E|A}} \frac{1}{|A|} \\ &= \sum_{\substack{E|R \\ \deg E \leq x}} \frac{\mu(E)}{|E|} \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x - \deg E}} \frac{1}{|A|} = \sum_{\substack{E|R \\ \deg E \leq x}} \frac{\mu(E)}{|E|}(x - \deg E + 1) \\ &= \sum_{E|R} \frac{\mu(E)}{|E|}(x - \deg E + 1) - \sum_{\substack{E|R \\ \deg E > x}} \frac{\mu(E)}{|E|}(x - \deg E + 1). \end{aligned}$$

By (A.24), (A.25), and Lemma A.3.1, we see that

$$\sum_{E|R} \frac{\mu(E)}{|E|}(x - \deg E + 1) = \frac{\phi(R)}{|R|}x + O\bigg(\frac{\phi(R)}{|R|}\log \omega(R)\bigg).$$

When $x \geq \deg R$, it is clear that

$$\sum_{\substack{E|R \\ \deg E > x}} \frac{\mu(E)}{|E|}(x - \deg E + 1) = 0.$$

Whereas, when $x < \deg R$, we have that

$$\sum_{\substack{E|R \\ \deg E > x}} \frac{\mu(E)}{|E|}(x - \deg E + 1) \ll \sum_{\substack{E|R \\ \deg E > x}} \frac{|\mu(E)| \deg E}{|E|} \ll \frac{x}{q^x} \sum_{\substack{E|R \\ \deg E > x}} |\mu(E)| \ll \frac{2^{\omega(R)}x}{q^x}.$$

The proof follows. □

**Corollary A.3.3.** *If $a > 0$ and $x = a \deg R$, then,*

$$\sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x \\ (A,R)=1}} \frac{1}{|A|} = \frac{\phi(R)}{|R|}x + O_a\Big(\frac{\phi(R)}{|R|}\log \omega(R)\Big).$$

*If $b > 2$ and $x = \log_q b^{\omega(R)}$, then*

$$\sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x \\ (A,R)=1}} \frac{1}{|A|} = \frac{\phi(R)}{|R|}x + O_b\Big(\frac{\phi(R)}{|R|}\log \omega(R)\Big).$$

*Proof.* First consider the case where $x = a \deg R$. If $q > e^{\frac{4\log 2}{a}}$, then

$$\frac{2^{\omega(R)}x}{q^x} \ll \frac{2^{\omega(R)}}{q^{\frac{x}{2}}} \leq q^{\frac{\log 2}{\log q}\deg R - \frac{a}{2}\deg R} < q^{-\frac{a}{4}\deg R} \ll_a \frac{\phi(R)}{|R|}.$$

If $q \leq e^{\frac{4\log 2}{a}}$, then

$$\frac{2^{\omega(R)}x}{q^x} \ll \frac{2^{\omega(R)}}{q^{\frac{x}{2}}} = q^{O\left(\frac{\deg R}{\log\deg R}\right) - \frac{a}{2}\deg R} \leq q^{-\frac{a}{4}\deg R} \ll_a \frac{\phi(R)}{|R|},$$

where the second relation holds for $\deg R > c_a$, where $c_a$ is some constant that is dependent on $a$, but independent of $q$. Finally, there are only a finite number of cases where $q \leq e^{\frac{4\log 2}{a}}$ and $\deg R \leq c_a$, and so

$$\frac{2^{\omega(R)}x}{q^x} \ll_a \frac{\phi(R)}{|R|}$$

for these cases too. The proof follows from Lemma A.3.2.

Now consider the case where $x = \log_q b^{\omega(R)}$. We have that

$$\begin{aligned}\frac{2^{\omega(R)}x}{q^x} =& \frac{2^{\omega(R)}(\log_q b)\omega(R)}{b^{\omega(R)}} \ll_b \frac{2^{\omega(R)}}{\left(\frac{b+2}{2}\right)^{\omega(R)}} = \left(\frac{4}{b+2}\right)^{\omega(R)} \\ =& \prod_{P|R}\left(\frac{4}{b+2}\right) \ll_b \prod_{P|R}\left(1 - \frac{1}{|P|}\right) \ll_b \frac{\phi(R)}{|R|}.\end{aligned}$$

Again, the proof follows from Lemma A.3.2. □

**Lemma A.3.4.** *We have that*

$$\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq x}} \frac{1}{\phi(N)} = \Big(\sum_{E\in\mathcal{M}} \frac{\mu(E)^2}{\phi(E)|E|}\Big)x + O(1).$$

*Proof.* For all $N \in \mathcal{A}$ we have that

$$\sum_{E|N}\frac{\mu(E)^2}{\phi(E)} = \prod_{P|N}\left(1 + \frac{1}{|P|-1}\right) = \prod_{P|N}\left(\frac{1}{1-|P|^{-1}}\right) = \frac{|N|}{\phi(N)}.$$

So,

$$\begin{aligned}\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq x}} \frac{1}{\phi(N)} &= \sum_{\substack{N\in\mathcal{M}\\ \deg N\leq x}} \frac{1}{|N|}\frac{|N|}{\phi(N)} = \sum_{\substack{N\in\mathcal{M}\\ \deg N\leq x}} \frac{1}{|N|}\sum_{E|N}\frac{\mu(E)^2}{\phi(E)} = \sum_{\substack{E\in\mathcal{M}\\ \deg E\leq x}} \frac{\mu(E)^2}{\phi(E)} \sum_{\substack{N\in\mathcal{M}\\ \deg N\leq x\\ E|N}} \frac{1}{|N|} \\ &= \sum_{\substack{E\in\mathcal{M}\\ \deg E\leq x}} \frac{\mu(E)^2(x-\deg E+1)}{\phi(E)|E|} = \Big(\sum_{E\in\mathcal{M}} \frac{\mu(E)^2}{\phi(E)|E|}\Big)x + O(1).\end{aligned}$$

□

**Lemma A.3.5.** *We have that*

$$\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq x}} \frac{\mu^2(N)}{\phi(N)} \geq x.$$

*Proof.* For square-free $N$ we have that

$$\frac{1}{\phi(N)} = \frac{1}{|N|}\prod_{P|N}\left(1-|P|^{-1}\right)^{-1} = \frac{1}{|N|}\prod_{P|N}\left(1+\frac{1}{|P|}+\frac{1}{|P|^2}+\dots\right) = \sum_{\substack{M\in\mathcal{M}\\ \mathrm{rad}(M)=N}}\frac{1}{|M|},$$

and so

$$\sum_{\substack{N\in\mathcal{M}\\ \deg N\leq x}}\frac{\mu(N)^2}{\phi(N)} = \sum_{\substack{N\in\mathcal{M}\\ N \text{ is square-free}\\ \deg N\leq x}}\sum_{\substack{M\in\mathcal{M}\\ \mathrm{rad}(M)=N}}\frac{1}{|M|} \geq \sum_{\substack{M\in\mathcal{M}\\ \deg M\leq x}}\frac{1}{|M|} = x.$$

□

# Appendix B

# The Selberg Sieve in Function Fields

In this appendix we first give an introductory example of the Selberg sieve for the ring of integers, which will provide clarity and perspective, before rigorously proving the general Selberg sieve in function fields. Publications relating to the Selberg sieve in function fields do exist, but they are difficult to come by and do not always contain what is desired. An article by Webb [Web83] is the most comprehensive that we could find, but we still feel it is necessary to include this appendix for clarity.

## B.1 An Introduction to the Selberg Sieve

Suppose $x$ and $y$ are positive integers with $y$ being small compared to $x$. We wish to obtain an upper bound for the number of primes in the interval $S := [x, x+y)$. That is, we require an upper bound for the size of the set

$$S' := \{a \in S : a \text{ is prime}\}.$$

If $\sqrt{x+y} \leq x$, which is to be expected since $y$ is small compared to $x$, then a necessary and sufficient condition for an element $a \in S$ to be prime is that $p \nmid a$ for all primes $p < \sqrt{x+y}$. However, since we require only an upper bound for $|S'|$, it turns out to be more convenient to work with the necessary condition that $p \nmid a$ for all primes $p \leq \sqrt{x}$. That is, in order to obtain an upper bound for $|S'|$, it suffices to obtain an upper bound for the size of the set

$$S'' := \{a \in S : p \nmid a \text{ for all primes } p \leq \sqrt{x}\}.$$

Now, for positive integers $d$, we define

$$S_d := \{a \in S : d \mid a\}.$$

By the inclusion-exclusion principle, we have that

$$|S'| \leq |S''| \leq |S| + \sum_{j=1}^{\pi(\sqrt{x})} (-1)^j \sum_{2 \leq p_{i_1} < \ldots < p_{i_j} \leq \sqrt{x}} |S_{p_{i_1} \cdots p_{i_j}}|.$$

Of course, when $d \geq x+y$, we have $|S_d| = 0$, and so the above can be improved to

$$|S'| \leq |S| + \sum_{j=1}^{\pi(\sqrt{x})} (-1)^j \sum_{\substack{2 \leq p_{i_1} < \ldots < p_{i_j} \leq \sqrt{x} \\ p_{i_1} \cdots p_{i_j} < x+y}} |S_{p_{i_1} \cdots p_{i_j}}| = \sum_{\substack{d < x+y \\ p_+(d) \leq \sqrt{x}}} \mu(d)|S_d|. \tag{B.1}$$

For $d$ in the summation range, we have

$$|S_d| = \frac{y}{d} + O(1), \tag{B.2}$$

which gives

$$|S'| \leq y \sum_{\substack{d<x+y \\ p_+(d)\leq\sqrt{x}}} \frac{\mu(d)}{d} + O\bigg(\sum_{\substack{d<x+y \\ p_+(d)\leq\sqrt{x}}} \mu(d)^2\bigg).$$

For the error term we can see that this is larger than $|\{d \leq \sqrt{x} : \ d \text{ is square-free}\}|$, which is about $\zeta(2)^{-1}\sqrt{x}$ in size. Whereas, for the main term we can see that the sum is certainly $\ll \log(x+y)$. Therefore, for the error term to be smaller than the main term, we require, roughly, that $y \gg \sqrt{x}$. Ideally, we would like a result that applies to $y$ that is even smaller compared to $x$.

In order to improve upon this, let us take $1 \leq z < x+y$, which we are free to choose optimally later. Further, for $1 \leq d \leq z$ with $p_+(d) \leq \sqrt{x}$, let $\lambda_d$ be real variables, and for all other $d$ let $\lambda_d := 0$. It is important to keep in mind that $\lambda_d = 0$ if $p_+(d) > \sqrt{x}$. Indeed, we will only express this when we require it. Now, consider

$$\sum_{d\leq z} \lambda_d |S_d| = \sum_{a\in S} \sum_{\substack{d|a \\ d\leq z}} \lambda_d. \tag{B.3}$$

We wish to use the above to bound $|S'|$. Note that, by taking $z = x + y - 1$ and $\lambda_d$ to equal $\mu(d)$ for $1 \leq d \leq z$ satisfying $p_+(d) \leq \sqrt{x}$, we obtain (B.1), and so our new approach should not give us anything worse. Now, if $a \in S$ is prime, then (B.3) counts it $\lambda_1$ times. If $a \in S$ is not prime, then it is counted $\sum_{\substack{d|a \\ d\leq z}} \lambda_d$ times. So, for (B.3) to be a valid upper bound for $|S'|$, we require that

$$\begin{aligned} &\lambda_1 \geq 1 \\ &\sum_{\substack{d|a \\ d\leq z}} \lambda_d \geq 0 \qquad \text{for } a \in [x, x+y). \end{aligned} \tag{B.4}$$

Now, one of the main aspects of the Selberg sieve is the following change of variables. For $1 \leq e \leq z$ let $\Lambda_e$ be a real variable, except when $p_+(e) > \sqrt{x}$ or $e > \sqrt{z}$, in which case we define $\Lambda_e := 0$. Then, for all positive integers $d$, we define $\lambda_d$ by

$$\lambda_d = \sum_{e,f:[e,f]=d} \Lambda_e \Lambda_f.$$

Note that, by this definition, we satisfy the condition that $\lambda_d = 0$ if $p_+(d) > \sqrt{x}$ or if $d > z$. The first condition in (B.4) is satisfied if we impose ${\Lambda_1}^2 \geq 1$. The second condition is automatically satisfied:

$$\sum_{\substack{d|a \\ d\leq z}} \lambda_d = \sum_{d|a} \sum_{e,f:[e,f]=d} \Lambda_e \Lambda_f = \sum_{e,f|a} \Lambda_e \Lambda_f = \bigg(\sum_{\substack{e|a \\ e\leq\sqrt{z}}} \Lambda_e\bigg)^2 \geq 0.$$

We remark that this change of variables loses us some generality in that not all possible values for the $\lambda_d$ can be expressed in this way. Of course, while we do lose generality, we benefit in that the conditions on the variables are more easily satisfied.

So, assuming the condition ${\Lambda_1}^2 \geq 1$ is satisfied, we have

$$\begin{aligned} |S'| &\leq \sum_{a\in[x,x+y)} \sum_{\substack{d|a \\ d\leq z}} \lambda_d = \sum_{a\in[x,x+y)} \Big(\sum_{\substack{e|a \\ e\leq\sqrt{z}}} \Lambda_e\Big)^2 = \sum_{e,f\leq\sqrt{z}} \Lambda_e\Lambda_f |S_{[e,f]}| \\ &= \sum_{e,f\leq\sqrt{z}} \Lambda_e\Lambda_f\Big(\frac{y}{[e,f]} + O(1)\Big) = y \sum_{e,f\leq\sqrt{z}} \Lambda_e\Lambda_f \frac{(e,f)}{ef} + O\Big(\Big(\sum_{e\leq\sqrt{z}} |\Lambda_e|\Big)^2\Big). \end{aligned} \tag{B.5}$$

Now, one may attempt to minimise the far RHS subject to the condition $\Lambda_1^2 \geq 1$, perhaps via the method of Lagrange multipliers. This would be difficult though, due to the error term where the modulus function has been used. Therefore, we will have to deal with the main term and the error term separately. Applying the method of Lagrange multipliers to only the main term is certainly achievable. In order to ensure the error is small, we have the freedom to choose $z$ to be appropriately small, as we will later see. (The whole point of introducing $z$ was to give us a way to reduce the size of the error term).

Let us proceed with the main term. We can make it easier to apply the method of Lagrange multipliers. Indeed, currently, most of the terms in the sum are the product of two variables. We will simplify the problem:

$$\sum_{e,f\leq\sqrt{z}} \Lambda_e\Lambda_f \frac{(e,f)}{ef} = \sum_{e,f\leq\sqrt{z}} \frac{\Lambda_e\Lambda_f}{ef} \sum_{g|(e,f)} \phi(g) = \sum_{g\leq\sqrt{z}} \phi(g) \sum_{\substack{e,f\leq\sqrt{z} \\ g|(e,f)}} \frac{\Lambda_e\Lambda_f}{ef} = \sum_{g\leq\sqrt{z}} \phi(g){\Theta_g}^2, \tag{B.6}$$

where, for $g \leq \sqrt{z}$,

$$\Theta_g := \sum_{\substack{e\leq\sqrt{z} \\ g|e}} \frac{\Lambda_e}{e} = \sum_{e\leq\frac{\sqrt{z}}{g}} \frac{\Lambda_{eg}}{eg}.$$

By the Möbius inversion formula (Lemma B.1.1) we have

$$\frac{\Lambda_e}{e} = \sum_{g\leq\frac{\sqrt{z}}{e}} \mu(g)\Theta_{eg}.$$

Note that the condition, that $\Lambda_e = 0$ if $p_+(e) > \sqrt{x}$ or $e > \sqrt{z}$, is equivalent to the condition that $\Theta_g = 0$ if $p_+(g) > \sqrt{x}$ or $g > \sqrt{z}$. Also, the condition ${\Lambda_1}^2 \geq 1$ becomes

$$\Big(\sum_{g\leq\sqrt{z}} \mu(g)\Theta_g\Big)^2 \geq 1. \tag{B.7}$$

Now, minimising the far RHS of (B.6), subject to (B.7), via the method of Lagrange multipliers is much easier because each term in the sum is dependent on one variable only. We can easily use this method, although the following is slightly quicker. If the condition (B.7) is to hold, then, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}1 \leq & \Big( \sum_{g \leq \sqrt{z}} \mu(g)\Theta_g \Big)^2 = \Big( \sum_{\substack{g \leq \sqrt{z} \\ p_+(g) \leq \sqrt{x}}} \frac{\mu(g)}{\sqrt{\phi(g)}} \sqrt{\phi(g)}\Theta_g \Big)^2 \\ \leq & \Big( \sum_{\substack{g \leq \sqrt{z} \\ p_+(g) \leq \sqrt{x}}} \frac{\mu(g)^2}{\phi(g)} \Big) \Big( \sum_{\substack{g \leq \sqrt{z} \\ p_+(g) \leq \sqrt{x}}} \phi(g){\Theta_g}^2 \Big).\end{aligned} \tag{B.8}$$

So,

$$\sum_{g \leq \sqrt{z}} \phi(g){\Theta_g}^2 = \sum_{\substack{g \leq \sqrt{z} \\ p_+(g) \leq \sqrt{x}}} \phi(g){\Theta_g}^2 \geq \Big( \sum_{\substack{g \leq \sqrt{z} \\ p_+(g) \leq \sqrt{x}}} \frac{\mu(g)^2}{\phi(g)} \Big)^{-1}.$$

The smallest value this could possibly take is when we have equality, which would require equality in both inequalities of (B.8). For the second inequality of (B.8), that means we have a constant $c$ such that, for $g \leq \sqrt{z}$ satisfying $p_+(g) \leq \sqrt{x}$, we have

$$\Theta_g = c\frac{\mu(g)}{\phi(g)}. \tag{B.9}$$

In order to have equality in the first inequality of (B.8), we need

$$c = \pm \Big( \sum_{\substack{g \leq \sqrt{z} \\ p_+(g) \leq \sqrt{x}}} \frac{\mu(g)^2}{\phi(g)} \Big)^{-1}. \tag{B.10}$$

From (B.6), (B.9), and (B.10), we have

$$\sum_{e,f \leq \sqrt{z}} \Lambda_e \Lambda_f \frac{(e,f)}{ef} = \sum_{g \leq \sqrt{z}} \phi(g){\Theta_g}^2 = \Big( \sum_{\substack{g \leq \sqrt{z} \\ p_+(g) \leq \sqrt{x}}} \frac{\mu(g)^2}{\phi(g)} \Big)^{-1}.$$

For convenience, let us assume $z \leq x$ so that $g \leq \sqrt{z}$ implies $p_+(g) \leq \sqrt{x}$ (initially, we had $z < x + y$, but we do not lose much in our new assumption because, by our our initial conditions on $y$, $x$ is not much smaller than $x + y$). Then, using Lemma B.1.2, the above becomes

$$\sum_{e,f \leq \sqrt{z}} \Lambda_e \Lambda_f \frac{(e,f)}{ef} = \Big( \sum_{g \leq \sqrt{z}} \frac{\mu(g)^2}{\phi(g)} \Big)^{-1} \leq \frac{2}{\log z}, \tag{B.11}$$

which concludes our minimisation of the main term on the far RHS of (B.5).

Now we consider the error term. We have

$$\begin{aligned}\Lambda_e =& e \sum_{g \leq \frac{\sqrt{z}}{e}} \mu(g)\Theta_{eg} = \pm \Big( \sum_{g \leq \sqrt{z}} \frac{\mu(g)^2}{\phi(g)} \Big)^{-1} e \sum_{g \leq \frac{\sqrt{z}}{e}} \mu(g) \frac{\mu(eg)}{\phi(eg)} \\ =& \pm \Big( \sum_{g \leq \sqrt{z}} \frac{\mu(g)^2}{\phi(g)} \Big)^{-1} \frac{e\mu(e)}{\phi(e)} \sum_{\substack{g \leq \frac{\sqrt{z}}{e} \\ (e,g)=1}} \frac{\mu(g)^2}{\phi(g)},\end{aligned}$$

and so, using Lemma B.1.3,

$$\sum_{e\le\sqrt{z}}|\Lambda_e| \le \frac{2}{\log z}\sum_{e\le\sqrt{z}}\frac{e}{\phi(e)}\sum_{g\le\frac{\sqrt{z}}{e}}\frac{1}{\phi(g)} = \frac{2}{\log z}\sum_{g\le\sqrt{z}}\frac{1}{\phi(g)}\sum_{e\le\frac{\sqrt{z}}{g}}\frac{e}{\phi(e)} \ll \frac{\sqrt{z}}{\log z}\sum_{g\le\sqrt{z}}\frac{1}{g\phi(g)}$$
$$\ll \frac{\sqrt{z}}{\log z}. \tag{B.12}$$

Finally, applying (B.12) and (B.11) to (B.5), we obtain

$$|S'| \ll \frac{y}{\log z} + \frac{z}{(\log z)^2}.$$

This is optimised roughly when $z = y$, giving our main result:

$$|S'| \ll \frac{y}{\log y}.$$

Because we took $y = z \le x$, we require that $y \le x$. This is fine given that we are interested in the case where $y \ll \sqrt{x}$.

**Lemma B.1.1.** *Let $f, g : \mathbb{Z}_{>0} \longrightarrow \mathbb{C}$ be functions such that*

$$g(n) = \sum_{d\ge1} f(dn)$$

*for all positive integers $n$. Suppose also that*

$$\sum_{e\ge1}\sum_{d\ge1}\mu(e)f(den)$$

*is absolutely convergent for all positive integers $n$. Then,*

$$f(n) = \sum_{e\ge1}\mu(e)g(en)$$

*for all positive integers $n$.*

*Proof.* For all positive integers $n$ we have that

$$\sum_{e\ge1}\mu(e)g(en) = \sum_{e\ge1}\sum_{d\ge1}\mu(e)f(den) = \sum_{h\ge1}\sum_{de=h}\mu(e)f(hn) = \sum_{h\ge1}f(hn)\sum_{de=h}\mu(e)$$
$$= f(n),$$

where for the second equality we used the fact that absolute convergence allows us to change the order of summation. □

**Lemma B.1.2.** *We have that*

$$\sum_{g\le z}\frac{\mu^2(g)}{\phi(g)} \ge \log(z).$$

*Proof.* Suppose $g$ is square free. Then,

$$\frac{1}{\phi(g)} = \frac{1}{g}\prod_{p|g}\left(\frac{1}{1-p^{-1}}\right) = \frac{1}{g}\prod_{p|g}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right) = \sum_{\mathrm{rad}(n)=g}\frac{1}{n}.$$

So, we have that

$$\sum_{g\leq z}\frac{\mu^2(g)}{\phi(g)} = \sum_{\substack{g\leq z\\ g \text{ is square-free}}}\sum_{\mathrm{rad}(n)=g}\frac{1}{n} \geq \sum_{n\leq z}\frac{1}{n} \geq \log(z).$$

□

**Lemma B.1.3.** *We have that*

$$\sum_{n\leq z}\frac{n}{\phi(n)} \ll z.$$

*Proof.* We have that

$$\frac{n}{\phi(n)} = \prod_{p|n}\left(\frac{p}{p-1}\right) = \prod_{p|n}\left(1+\frac{1}{p-1}\right) = \sum_{d|n}\frac{\mu^2(d)}{\phi(d)},$$

and so

$$\sum_{n\leq z}\frac{n}{\phi(n)} = \sum_{n\leq z}\sum_{d|n}\frac{\mu^2(d)}{\phi(d)} = \sum_{d\leq z}\left[\frac{z}{d}\right]\frac{\mu^2(d)}{\phi(d)} = z\sum_{d\leq z}\frac{\mu^2(d)}{d\phi(d)} + O(z) \ll z.$$

□

## B.2 The General Selberg Sieve in Function Fields

We now wish to prove the function field analogue of Section B.1. Let $\mathcal{S} \subseteq \mathcal{M}$ be a finite subset, and for $D \in \mathcal{M}$ let $\mathcal{S}_D := \{A \in \mathcal{S} : D \mid A\}$. Furthermore, let $\mathcal{Q} \subseteq \mathcal{P}$, $z$ be a positive integer, and $\Pi_{\mathcal{Q},\leq z} := \prod_{\substack{P\in\mathcal{Q}\\ \deg P\leq z}} P$. We are interested in the size of

$$\mathcal{S}_{\mathcal{Q},>z} := \mathcal{S}\setminus \cup_{P|\Pi_{\mathcal{Q},\leq z}} \mathcal{S}_P = \{A \in \mathcal{S} : (P \mid A \text{ and } P \in \mathcal{Q}) \Rightarrow \deg P > z\}.$$

Note that we are generalising the Selberg sieve in that $\mathcal{S}$ is not necessarily an interval in $\mathcal{M}$, and $\mathcal{Q}$ is not necessarily the set of all primes in $\mathcal{A}$. However, if we are to do this then we require the following condition:

Suppose there exists a multiplicative function $\omega$, with $0 < \omega(D) < |D|$, and a function $r$ such that

$$|\mathcal{S}_D| = \frac{\omega(D)}{|D|}|\mathcal{S}| + r(D).$$

While our result will hold given the above condition, it will only be helpful if the function $r$ is small enough. Intuitively, $\frac{\omega(D)}{|D|}$ represents the proportion of elements in $\mathcal{S}$ that are divisible by $D$, and $r$ is an error term. In our number field example, we had $\omega(d) = 1$ and $r(d) = O(1)$. Before stating our result, we prove a lemma.

**Lemma B.2.1.** *Define* $\psi$ *multiplicatively by* $\psi(P^e) := \frac{|P|^e}{\omega(P)^e} - \frac{|P|^{e-1}}{\omega(P)^{e-1}}$. *Then, for all* $A \in \mathcal{M}$, *we have*

$$\frac{|A|}{\omega(A)} = \sum_{E|A} \psi(E).$$

*Proof.* Suppose $A = {P_1}^{e_1} \dots {P_n}^{e_n}$. Then,

$$\begin{aligned}\sum_{E|A}\psi(E) =& \prod_{i=1}^{n}\left(1 + \left(\frac{|P_i|}{\omega(P_i)} - 1\right) + \ldots + \left(\frac{|P_i|^{e_i}}{\omega(P_i)^{e_i}} - \frac{|P_i|^{e_i-1}}{\omega(P_i)^{e_i-1}}\right)\right) = \prod_{i=1}^{n}\frac{|P_i|^{e_i}}{\omega({P_i}^{e_i})}\\ =&\frac{|A|}{\omega(A)}.\end{aligned}$$

□

**Theorem B.2.2** (The Selberg Sieve for Function Fields)**.** *We have that*

$$|\mathcal{S}_{\mathcal{Q},>z}| \leq \Bigg(\sum_{\substack{G\in\mathcal{M}_{\leq \frac{z}{2}}\\ G|\Pi_{\mathcal{Q},\leq z}}} \frac{\mu^2(G)}{\psi(G)}\Bigg)^{-1}|\mathcal{S}| + \sum_{\substack{E,F\in\mathcal{M}_{\leq \frac{z}{2}}\\ E,F|\Pi_{\mathcal{Q},\leq z}}} \Big|r\big([E,F]\big)\Big|.$$

*Proof.* For $D \mid \Pi_{\mathcal{Q},\leq z}$ with $\deg D \leq z$ we let $\lambda_D$ be a real variable. For all other $D \in \mathcal{M}$ we define $\lambda_D := 0$. It is important to keep in mind that $\lambda_D = 0$ for $D \nmid \Pi_{\mathcal{Q},\leq z}$. Indeed, for convenience, it will only be expressed when required. Consider

$$\sum_{D\in\mathcal{M}_{\leq z}} \lambda_D|\mathcal{S}_D| = \sum_{A\in\mathcal{S}}\sum_{\substack{D\in\mathcal{M}_{\leq z}\\ D|A}} \lambda_D. \tag{B.13}$$

In order for (B.13) to be an upper bound for $|\mathcal{S}_{\mathcal{Q},>z}|$ we require that

$$\begin{aligned}&\lambda_1 \geq 1\\ &\sum_{\substack{D\in\mathcal{M}_{\leq z}\\ D|A}} \lambda_D \geq 0 \qquad \text{for all } A \in \mathcal{S}.\end{aligned} \tag{B.14}$$

Now, for $E \mid \Pi_{\mathcal{Q},\leq z}$ with $\deg E \leq \frac{z}{2}$, let $\Lambda_E$ be a real variable, and for all other $E \in \mathcal{M}$ let $\Lambda_E := 0$. Let us define the $\lambda_D$ by

$$\lambda_D := \sum_{\substack{E,F\in\mathcal{M}\\ [E,F]=D}} \Lambda_E\Lambda_F.$$

Note that, by this definition, we still satisfy the conditions that $\lambda_D = 0$ if $D \nmid \Pi_{\mathcal{Q},\leq z}$ or $\deg D > z$. Further, the first condition in (B.14) is equivalent to

$${\Lambda_1}^2 \geq 1, \tag{B.15}$$

while the second condition is satisfied:

$$\sum_{\substack{D\in\mathcal{M}_{\leq z}\\ D|A}} \lambda_D = \sum_{D|A}\lambda_D = \sum_{D|A}\sum_{\substack{E,F\in\mathcal{M}\\ [E,F]=D}} \Lambda_E\Lambda_F = \Bigg(\sum_{\substack{E\in\mathcal{M}_{\leq \frac{z}{2}}\\ E|A}} \Lambda_E\Bigg)^2.$$

So, given (B.15), we have

$$\begin{aligned}|\mathcal{S}_{\mathcal{Q},>z}| =&\sum_{A\in\mathcal{S}}\sum_{\substack{D\in\mathcal{M}_{\leq z}\\ D|A}}\lambda_D = \sum_{A\in\mathcal{S}}\Big(\sum_{\substack{E\in\mathcal{M}_{\leq\frac{z}{2}}\\ E|A}}\Lambda_E\Big)^2 = \sum_{E,F\in\mathcal{M}_{\leq\frac{z}{2}}}\Lambda_E\Lambda_F\cdot|\mathcal{S}_{[E,F]}| \\ =&|\mathcal{S}|\sum_{E,F\in\mathcal{M}_{\leq\frac{z}{2}}}\Lambda_E\Lambda_F\frac{\omega\big([E,F]\big)}{\big|[E,F]\big|} + \sum_{E,F\in\mathcal{M}_{\leq\frac{z}{2}}}\Big|\Lambda_E\Lambda_F r\big([E,F]\big)\Big|.\end{aligned} \tag{B.16}$$

First, we consider the first term on the far RHS. By Lemma B.2.1, we have

$$\begin{aligned}&\sum_{E,F\in\mathcal{M}_{\leq\frac{z}{2}}}\Lambda_E\Lambda_F\frac{\omega\big([E,F]\big)}{\big|[E,F]\big|} = \sum_{E,F\in\mathcal{M}_{\leq\frac{z}{2}}}\Lambda_E\Lambda_F\frac{\omega(E)\omega(F)}{|EF|}\frac{\big|(E,F)\big|}{\omega\big((E,F)\big)} \\ =&\sum_{E,F\in\mathcal{M}_{\leq\frac{z}{2}}}\Lambda_E\Lambda_F\frac{\omega(E)\omega(F)}{|EF|}\sum_{G|(E,F)}\psi(G) \\ =&\sum_{G\in\mathcal{M}_{\leq\frac{z}{2}}}\psi(G)\sum_{\substack{E,F\in\mathcal{M}_{\leq\frac{z}{2}}\\ G|E,F}}\Lambda_E\Lambda_F\frac{\omega(E)\omega(F)}{|EF|} = \sum_{G\in\mathcal{M}_{\leq\frac{z}{2}}}\psi(G){\Theta_G}^2,\end{aligned} \tag{B.17}$$

where

$$\Theta_G := \sum_{\substack{E\in\mathcal{M}_{\leq\frac{z}{2}}\\ G|E}}\Lambda_E\frac{\omega(E)}{|E|} = \sum_{\substack{E\in\mathcal{M}\\ \deg EG\leq\frac{z}{2}}}\Lambda_{EG}\frac{\omega(EG)}{|EG|}.$$

By the Möbius inversion formula (the proof of which is very similar to Lemma B.1.1), we have that

$$\Lambda_E = \frac{|E|}{\omega(E)}\sum_{\substack{G\in\mathcal{M}\\ \deg EG\leq\frac{z}{2}}}\mu(G)\Theta_{EG}.$$

Note that, by definition, $\Theta_G = 0$ if $\deg G > \frac{z}{2}$ or $G \nmid \Pi_{\mathcal{Q},\leq z}$. Also, the condition (B.15) is equivalent to

$$\Big(\sum_{G\in\mathcal{M}_{\leq\frac{z}{2}}}\mu(G)\Theta_G\Big)^2 \geq 1. \tag{B.18}$$

So, we wish to minimise the far RHS of (B.17) subject to (B.18). If (B.18) is to hold then, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}1 \leq&\Big(\sum_{G\in\mathcal{M}_{\leq\frac{z}{2}}}\mu(G)\Theta_G\Big)^2 = \Big(\sum_{\substack{G\in\mathcal{M}_{\leq\frac{z}{2}}\\ G|\Pi_{\mathcal{Q},\leq z}}}\frac{\mu(G)}{\sqrt{\psi(G)}}\sqrt{\psi(G)}\Theta_G\Big)^2 \\ \leq&\Big(\sum_{\substack{G\in\mathcal{M}_{\leq\frac{z}{2}}\\ G|\Pi_{\mathcal{Q},\leq z}}}\frac{\mu(G)^2}{\psi(G)}\Big)\Big(\sum_{\substack{G\in\mathcal{M}_{\leq\frac{z}{2}}\\ G|\Pi_{\mathcal{Q},\leq z}}}\psi(G){\Theta_G}^2\Big).\end{aligned} \tag{B.19}$$

Hence,

$$\sum_{\substack{G\in\mathcal{M}_{\leq\frac{z}{2}}\\ G|\Pi_{\mathcal{Q},\leq z}}}\psi(G){\Theta_G}^2 \geq \Big(\sum_{\substack{G\in\mathcal{M}_{\leq\frac{z}{2}}\\ G|\Pi_{\mathcal{Q},\leq z}}}\frac{\mu(G)^2}{\psi(G)}\Big)^{-1}.$$

The smallest value the LHS can take is when we have equality, which requires equality in both inequalities of (B.19). For the second inequality this requires some $c \in \mathbb{R}$ such that

$$\Theta_G = c\frac{\mu(G)}{\psi(G)} \tag{B.20}$$

for all $G \in \mathcal{M}_{\leq \frac{z}{2}}$ with $G \mid \Pi_{\mathcal{Q},\leq z}$. Then, we have equality in the first inequality of (B.19) if

$$c = \pm\Bigg(\sum_{\substack{G\in\mathcal{M}_{\leq \frac{z}{2}}\\ G\mid\Pi_{\mathcal{Q},\leq z}}} \frac{\mu(G)^2}{\psi(G)}\Bigg)^{-1}. \tag{B.21}$$

So, from (B.17), (B.20), and (B.21), we have

$$\sum_{E,F\in\mathcal{M}_{\leq \frac{z}{2}}} \Lambda_E\Lambda_F\frac{\omega\big([E,F]\big)}{\big|[E,F]\big|} = \Bigg(\sum_{\substack{G\in\mathcal{M}_{\leq \frac{z}{2}}\\ G\mid\Pi_{\mathcal{Q},\leq z}}} \frac{\mu(G)^2}{\psi(G)}\Bigg)^{-1}, \tag{B.22}$$

which concludes our minimisation of the first term on the far RHS of (B.16).

We now consider the second term. First, for $E \mid \Pi_{\mathcal{Q},\leq z}$ with $\deg E \leq \frac{z}{2}$, we have

$$\begin{aligned}\Lambda_E =& \frac{|E|}{\omega(E)}\sum_{\substack{G\in\mathcal{M}\\ \deg EG\leq\frac{z}{2}}}\mu(G)\Theta_{EG} = \pm\Bigg(\sum_{\substack{G\in\mathcal{M}_{\leq \frac{z}{2}}\\ G\mid\Pi_{\mathcal{Q},\leq z}}} \frac{\mu(G)^2}{\psi(G)}\Bigg)^{-1}\frac{|E|}{\omega(E)}\sum_{\substack{G\in\mathcal{M}\\ EG\mid\Pi_{\mathcal{Q},\leq z}\\ \deg EG\leq\frac{z}{2}}}\frac{\mu(G)\mu(EG)}{\psi(EG)}\\ =& \pm\Bigg(\sum_{\substack{G\in\mathcal{M}_{\leq \frac{z}{2}}\\ G\mid\Pi_{\mathcal{Q},\leq z}}} \frac{\mu(G)^2}{\psi(G)}\Bigg)^{-1}\frac{|E|\mu(E)}{\omega(E)\psi(E)}\sum_{\substack{G\in\mathcal{M}\\ EG\mid\Pi_{\mathcal{Q},\leq z}\\ \deg EG\leq\frac{z}{2}\\ (E,G)=1}}\frac{\mu(G)^2}{\psi(G)}.\end{aligned}$$

Since $E$ is square-free, we have

$$\frac{|E|\mu(E)}{\omega(E)\psi(E)} = \prod_{P\mid E}\left(\frac{|P|}{\omega(P)}\left(1-\frac{|P|}{\omega(P)}\right)^{-1}\right) = \prod_{P\mid E}\left(\frac{\omega(P)}{|P|}-1\right)^{-1}.$$

We also have

$$\begin{aligned}\sum_{\substack{G\in\mathcal{M}_{\leq \frac{z}{2}}\\ G\mid\Pi_{\mathcal{Q},\leq z}}} \frac{\mu(G)^2}{\psi(G)} &= \sum_{HI=E}\sum_{\substack{G\in\mathcal{M}_{\leq \frac{z}{2}}\\ G\mid\Pi_{\mathcal{Q},\leq z}\\ (E,G)=H}} \frac{\mu(G)^2}{\psi(G)} = \sum_{HI=E}\sum_{\substack{G\in\mathcal{M}\\ HG\mid\Pi_{\mathcal{Q},\leq z}\\ \deg HG\leq\frac{z}{2}\\ (E,G)=1}}\frac{\mu(HG)^2}{\psi(HG)}\\ &\geq \sum_{HI=E}\frac{\mu(H)^2}{\psi(H)}\sum_{\substack{G\in\mathcal{M}\\ EG\mid\Pi_{\mathcal{Q},\leq z}\\ \deg EG\leq\frac{z}{2}\\ (E,G)=1}}\frac{\mu(G)^2}{\psi(G)} = \prod_{P\mid E}\left(1-\frac{\omega(P)}{|P|}\right)^{-1}\sum_{\substack{G\in\mathcal{M}\\ EG\mid\Pi_{\mathcal{Q},\leq z}\\ \deg EG\leq\frac{z}{2}\\ (E,G)=1}}\frac{\mu(G)^2}{\psi(G)}.\end{aligned}$$

Hence, we see that $|\Lambda_E| \leq 1$, and so

$$\sum_{E,F\in\mathcal{M}_{\leq \frac{z}{2}}}\Big|\Lambda_E\Lambda_F r\big([E,F]\big)\Big| \leq \sum_{\substack{E,F\in\mathcal{M}_{\leq \frac{z}{2}}\\ E,F\mid\Pi_{\mathcal{Q},\leq z}}}\Big|r\big([E,F]\big)\Big|. \tag{B.23}$$

The proof follows by applying (B.22) and (B.23) to (B.16). ☐

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