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The compost bomb instability in the continuum limit

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Abstract The 'Compost Bomb' instability refers to a proposed uncontrolled increase in soil temperature. This instability is caused when sufficiently rapid atmospheric warming increases soil heterotrophic respiration which, in turn, heats the soil further. This generates a runaway effect in which soil temperatures rise rapidly. We investigate this process, neglected in Earth system models, but which has thus far been analysed with a conceptual model using ordinary differential equations. That model is deliberately idealised without any representation of the spatial structure of soils. We confirm using a partial differential equation framework, this runaway effect still occurs when accounting for soil depth. Using this newer representation we investigate the forcing parameters that make soils vulnerable to this instability. In particular, we discover that the effect of dangerously large seasonal cycle variations in air temperature can create plausible conditions for a 'compost bomb' thermal instability.

1 Introduction

Coupled climate-carbon cycle Earth system models (ESMs) show that rising temperatures will cause carbon cycle feedbacks that accelerate global warming further [1]. Although the magnitude of the increase remains uncertain, a main contributing factor is the response of the land carbon cycle to increased temperatures [2,3]. Over the last decade, there has therefore been a strong focus on improving models of the expected response of terrestrial carbon to global warming.

The largest uncertainties are associated with the response of soil carbon to warming [4]. One positive carbon cycle feedback related to the soil is the Jenkinson effect [5]. Heterotrophic respiration converts organic matter held in soils into CO_2. At higher temperatures the rate of this reaction increases, leading to larger emissions of CO_2 from soils. However, a key aspect of heterotrophic respiration, ignored by ESMs [3], is that due to respiration being an exothermic reaction, the released heat must raise the temperature of the soils it occurs in. This biogeochemical heating has been shown to be important in the thawing of permafrosts [6,7].

Furthermore, because the rate of respiration increases with temperature, the biogeochemical heating will tend to further increase the rate of respiration. This positive feedback creates the possibility of a tipping point, in which runaway respiration also significantly increases the soil temperature.

This runaway potential was first investigated by Luke and Cox [8]. The Luke and Cox model (hereafter referred to as the LC10 model) showed an instability if the rate of increase of air temperature was large compared to the soil turnover time. They dubbed this instability the 'compost bomb' due to the known capability of compost heaps to self-heat [9–11]. A range of climate tipping points have been observed in paleoclimate records [12], and both expert opinion [13] and ESMs [14] raise the possibility that they may be triggered this century by climate change. Most tipping points associated so far with the behaviour of the Earth system are believed to correspond to bifurcations. The mechanisms underpinning the compost bomb instability are more unusual in that this is an example of rate-dependent tipping [15]. A more mathematical approach to the compost bomb was carried out by [16] where the compost bomb was studied as an 'excitable' system in which critical rates were calculated analytically. They found that when the air temperature was raised sufficiently slowly the system could follow the steady state equilibrium. However, when the air temperature was raised more rapidly, the system was unable to respond quickly enough. Here we focus on the compost bomb instability in response to the seasonal cycle. In this case the timescale of the forcing (1 year) is much faster than the response timescale of the soil carbon (decades), and the soil carbon can be treated as a prescribed time-invariant quantity (i.e. we consider the 'compost bomb limit' of LC10).

Some limitations of the LC10 model are that it neglects important thermal processes and soil structure, and in particular vertical variation. For example, as a 'single box' model, it assumes that the soil is well represented by averaged quantities, such as an average soil temperature, when in fact these quantities can be quite heterogeneous [17].

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By definition, box models neglect processes such as heat diffusion which tend to suppress regions of unusually high temperature. Hence an initial assumption might be that diffusive damping may make the compost bomb harder to trigger. Additionally, the LC10 model assumes a single pool of carbon, rather than a spatially extended distribution, which might increase the possibility for a compost bomb.

Despite these caveats, the LC10 model captures the essence of the system. We aim to add realism to the LC10 model by considering the vertical structure and heat conductivity of the soil. We model a one dimensional soil column in which heat can diffuse and soil carbon decreases exponentially with depth. We investigate whether an instability still exists in this model.

The compost bomb has generally been considered in relation to an upward decadal timescale linear ramp in air temperature, which is an idealisation of the change in air temperature due to human caused climate change. However, there is also the possibility of rate-induced tipping by the sinusoidal variations in air temperature caused by the diurnal and seasonal cycles. We investigate those possibilities here, to see if there exist features of these oscillations that may raise the risk of a compost bomb.

2 LC10 single box conceptual model

The compost bomb instability is based on the idea that heterotrophic respiration in the soil is both an exothermic reaction [18] and also a reaction whose rate increases with temperature. Hence this reaction could lead to a scenario of thermal runaway, where respiration warms the soil which increases respiration further. The rate of respiration is often modelled with a Q_{10} form [19] for temperature dependence. In this representation the reaction rate increases by a factor Q_{10} for every 10°C of temperature increase. Hence we can model the specific rate of respiration, $r(T_\mathrm{s})$ ($\mathrm{m^{-2}s^{-1}}$) as $r(T_\mathrm{s}) = r_0 Q_{10}^{(T_\mathrm{s}-T_\mathrm{ref})/10}$ where T_s (°C) is the soil temperature and r_0 is the reaction rate at T_ref. The rate of reaction also increases in proportion to the available substrate, here soil carbon C_s (kgC $\mathrm{m^{-2}}$). The Q_{10} form implies an exponential dependence of the specific respiration rate on temperature: $r(T_\mathrm{s}) = r_0 \exp(\alpha (T_\mathrm{s} - T_\mathrm{ref}))$, where $\alpha = \ln Q_{10}/10$. The soil carbon is increased by net primary production (NPP) Π (kgC $\mathrm{m^{-2}\,s^{-1}}$) and decreased by heterotrophic respiration. Introducing the parameters A (J $\mathrm{kgC^{-1}}$), the specific heat of respiration, μ_A (J $\mathrm{m^{-2}\,{}^\circ C^{-1}}$) the areal soil heat capacity and λ ($\mathrm{Wm^{-2}\,{}^\circ C^{-1}}$) the soil-to-atmosphere heat transfer coefficient we have the LC10 model

$$\mu_\mathrm{A}\frac{\mathrm{d}T_\mathrm{s}}{\mathrm{d}t} = -\lambda(T_\mathrm{s} - T_\mathrm{a}) + AC_\mathrm{s}r_0\mathrm{e}^{\alpha(T_\mathrm{s}-T_\mathrm{ref})}, \quad (1a)$$

$$\frac{\mathrm{d}C_\mathrm{s}}{\mathrm{d}t} = \Pi - C_\mathrm{s}r_0\mathrm{e}^{\alpha(T_\mathrm{s}-T_\mathrm{ref})}. \quad (1b)$$

The model assumes that in the absence of biogeochemical heating the soil temperature equilibrates to the atmospheric temperature T_a. The amount of soil carbon is set by the equilibrium balance between Π and heterotrophic respiration. If the decrease in soil carbon is too slow to offset an increase in air temperature, the compost bomb instability is triggered, corresponding mathematically to an instability in which T_s is 'excited' to a very large value, well in excess of 100°C. We further note that this model implies a value for the equilibrium soil carbon

$$C_\mathrm{s}^\mathrm{eq} = \frac{1}{r_0}\mathrm{e}^{-\alpha(T_\mathrm{a}-T_\mathrm{ref})}\Pi\mathrm{e}^{-\Pi/\Pi_c} \quad (2)$$

where $\Pi_c = \lambda/\alpha A$. This equilibrium value is obtained by setting the derivatives in Eq. (1) to zero.

3 Continuum model with vertical depth

We investigate the effects of the representation of spatial variability in the vertical z (m) direction, which extends from the surface at $z = 0$ down to $z = -\infty$. We assume soil carbon falls exponentially with depth, over a characteristic distance of H (m). We model soil temperature as a reaction-diffusion system, in which heat is generated by heterotrophic respiration and diffused vertically. The conductivity of the soil is given by κ ($\mathrm{Wm^{-1}\,{}^\circ C^{-1}}$) [20], it has heat capacity μ_V ($\mathrm{Jm^{-3}\,{}^\circ C^{-1}}$) and contains a total amount C_s of soil carbon. We can therefore write the heat diffusion equation for the soil temperature $T_\mathrm{s}(z,t)$ as:

$$\mu_\mathrm{V}\frac{\partial T_\mathrm{s}}{\partial t} = \kappa\frac{\partial^2 T_s}{\partial z^2} + \frac{AC_\mathrm{s}r_0}{H}\mathrm{e}^{\alpha(T_\mathrm{s}-T_\mathrm{ref})}\mathrm{e}^{z/H}. \quad (3)$$

At $z = -\infty$ we impose a no flux boundary condition. At the upper boundary the soil temperature is controlled by the turbulent heat flux from the atmosphere which has temperature $T_\mathrm{a0} + \delta T_\mathrm{a}(t)$, where T_a0 represents a background mean temperature and $\delta T_\mathrm{a}(t)$ a time dependent warming. Mathematically:

$$\frac{\partial T_\mathrm{s}}{\partial z} = 0 \quad \text{at} \quad z = -\infty, \quad (4a)$$

$$-\kappa\frac{\partial T}{\partial z} = \lambda\left(T(0,t) - T_\mathrm{a0} - \delta T_\mathrm{a}(t)\right) \quad \text{at} \quad z = 0. \quad (4b)$$

Here the parameter λ characterises the turbulent heat transfer from the atmosphere to the top layer of the soil. We set C_s to the equilibrium value using Eq. (2) where the air temperature is taken to be T_a0. This approximation is justified provided we work on timescales short relative to the turnover time of soil carbon, which is on the order of many decades [4].

Throughout this study we undertake the mathematical investigation using nondimensional values. However, to aid understanding we plot certain figures using

Table 1 Parameter values used to produce the figures in this study

Parameter	Name	Equation	Value
Π	Net primary productivity	(1b)	$0.5\,\mathrm{kgC\ year}^{-1}$
Q_{10}	Q_{10} value for response of specific respiration to temperature	(3)	2.5
H	Characteristic soil depth	(3)	0.4 m
κ	Soil thermal conductivity	(3)	$0.16\,\mathrm{Wm}^{-1}\,{}^\circ\mathrm{C}^{-1}$
A	Specific heat of respiration	(3)	$3.9 \times 10^9\,\mathrm{J}\ \ (\mathrm{kgC})^{-1}$
μ_V	Volumetric heat capacity	(3)	$1.0\,\mathrm{MJm}^{-3}\,{}^\circ\mathrm{C}^{-1}$
λ	Heat transfer coefficient	(4b)	$10\,\mathrm{Wm}^{-2}\,{}^\circ\mathrm{C}^{-1}$
$T_{\mathrm{a}0}$	Average air temperature	(4b)	$0.0\,{}^\circ\mathrm{C}$

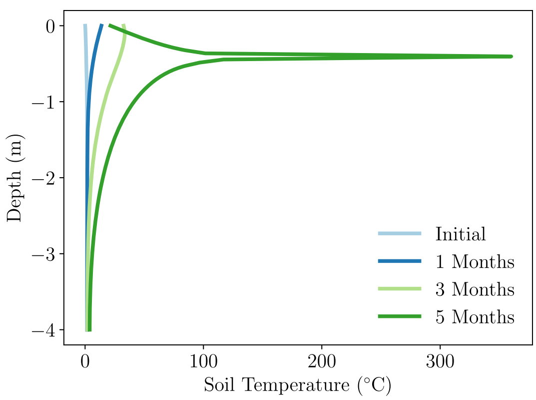

Fig. 1 Vertical temperature profiles of soil undergoing a compost bomb thermal runaway when subject to a seasonal cycle of temperatures with amplitude 32.5 °C

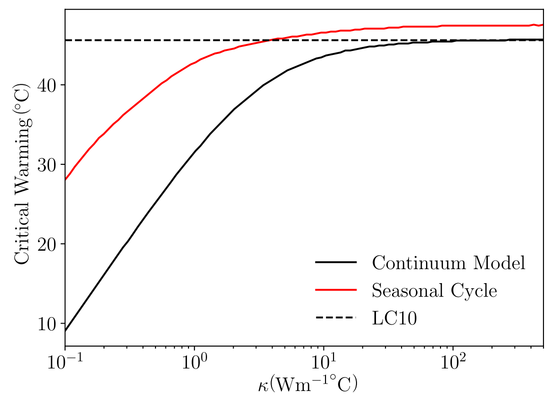

Fig. 2 The warming required to make the continuum model unstable as a function of κ, compared to the LC10 model. We have also plotted on the seasonal cycle amplitude required for this instability

dimensional units, with standard values given in Table 1. It should be noted however that these parameters are choices we have made, and different choices will lead to different figures, whereas the nondimensional plots are valid for all parameter choices.

3.1 Numerical investigation

We numerically integrated the continuum model in both space and time, and discovered that it can give rise to a compost bomb instability. The PDE, Eq. (3), was solved using the 'method of lines' technique [21]. It was discretized spatially into 100 equally spaced intervals. The spatial derivatives were approximated using central differences. This was then integrated using the backwards differentiation formula "BDF" method from the scipy library [22]. The "BDF" method was chosen as it is well suited to stiff problems. We considered the instability to occur when scipy's solver could not find a solution.

This is our first piece of evidence that the results of the LC10 model remain when soil depth is taken into account. In Fig. 1 we plot the temperature profile of soil at different times, initialised to be in equilibrium, undergoing the compost bomb thermal runaway after five months. We set δT_a to a a sinusoid of frequency 1 year. Figure 1 shows that the instability remains in the continuous case, and it is not prevented by diffusion.

3.2 Consistency of the continuum model with LC10

In this subsection we compare the LC10 and continuum models numerically. The level of soil carbon is chosen such that the soil temperature is initially in an equilibrium state with $\delta T_a = 0$. We then integrate forward in time with δT_a constant and greater than zero. We find for sufficiently large values of δT_a the system has an instability. We refer to the smallest value of δT_a for which this is true as the 'critical warming level', as if air temperatures were increased by this amount quickly with respect to the soil carbon turn over time, an instability would be triggered.

We plot this critical warming, as a function of κ in Fig. 2. We also overlay the warming required to cause a compost bomb in the LC10 model. Also shown is the amplitude of the seasonal cycle needed to cause a compost bomb. We see that as $\kappa \to \infty$ the vertical structure matters less and the continuum result asymptotes to the LC10 case. In addition, we see that the soil is more stable to seasonal cycle oscillations than instantaneous jumps. This 'overshoot' behaviour [23,24] is due to the fact that soil temperatures do not instantly respond to the seasonal cycle. Surprisingly for smaller values of κ the critical warming is actually lower in the continuum case than the LC10 case, despite the damping effects of diffusion.

To investigate this further, we derive the LC10 model from the continuum case. We do this by vertically averaging (3). We denote vertically averaged quantities as $\langle q\rangle = \frac{1}{H}\int_{-\infty}^{0} q\mathrm{d}z$. Neglecting the vertical dependence of soil carbon, and taking a vertical average, applying boundary conditions, and finally replacing μ_{A} with $\mu_{\mathrm{V}}H$ we get

$$\mu_{\mathrm{A}}\frac{\mathrm{d}\langle T_{\mathrm{s}}\rangle}{\mathrm{d}t} = -\lambda\left(T_{\mathrm{s}}(0,t) - T_{\mathrm{a0}} - \delta T_{\mathrm{a}}(t)\right) + AC_{\mathrm{s}}r_0\langle \mathrm{e}^{\alpha(T_{\mathrm{s}}-T_{\mathrm{ref}})}\rangle. \quad (5)$$

This is equivalent to the LC10 model if $T_{\mathrm{s}}(0,t)\approx\langle T_{\mathrm{s}}\rangle$ and $\langle \mathrm{e}^{\alpha T_{\mathrm{s}}}\rangle \approx \mathrm{e}^{\alpha\langle T_{\mathrm{s}}\rangle}$. These approximations become exact when T_{s} is not a function of z, which occurs when $\kappa\to\infty$, as shown in Fig. 2. Note however that because $\langle \mathrm{e}^{\alpha T_{\mathrm{s}}}\rangle \geq \mathrm{e}^{\langle\alpha T_{\mathrm{s}}\rangle}$ the LC10 model underpredicts the amount of respiration in soils when κ is finite, and so the LC10 model will need a higher amount of warming to trigger a compost bomb.

3.3 Existence of the compost bomb in the continuum case

Now that we have numerical evidence for a compost bomb in the continuum case, we investigate this analytically. In the approximation where soil carbon is constant, we lose the rate dependent tipping feature of the model, and instead it reverts to a classical bifurcation-induced tipping problem. We consider the case where δT_{a} is a constant, given by the temperature increase relative to the long term background state, T_{a0}, which we refer to as an atmospheric warming. In this case δT_{a} is the bifurcation control parameter. We begin by making the following nondimensionalisations:

$$\theta = \alpha\left(T_{\mathrm{s}} - T_{\mathrm{a0}}\right), \quad (6a)$$
$$\delta\theta_{\mathrm{a}} = \alpha\delta T_{\mathrm{a}}, \quad (6b)$$
$$x = \frac{\lambda z}{\kappa}, \quad (6c)$$
$$\tau = \frac{\lambda^2 t}{\kappa\mu_{\mathrm{V}}}, \quad (6d)$$
$$\widetilde{\Pi} = \Pi/\Pi_c. \quad (6e)$$

then equations (3),(4) become

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial x^2} + \mathcal{W}\mathrm{e}^{\theta}\mathrm{e}^{x/D}, \quad (7a)$$
$$\frac{\partial\theta}{\partial x} = 0 \quad \text{when} \quad x=-\infty, \quad (7b)$$
$$-\frac{\partial\theta}{\partial x} = \theta(0,t) - \delta\theta_{\mathrm{a}}(t) \quad \text{when} \quad x=0. \quad (7c)$$

The nondimensionalisation process reveals two parameter clusters $D = H\lambda/\kappa$ and $\mathcal{W} = \widetilde{\Pi}\mathrm{e}^{-\widetilde{\Pi}}/D$, which correspond to a nondimensional soil thermal depth and a nondimensional respiration strength. Note that θ and $\delta\theta_{\mathrm{a}}$ should be interpreted as (nondimensional) temperatures relative to the background air temperature, so in particular $\delta\theta_{\mathrm{a}}$ is a temperature anomaly.

We now show that the model defined by Eq. (7) only has an equilibrium state for low levels of atmospheric warming, $\delta\theta_{\mathrm{a}}$. We make a change of variables which lets us find the first integral of the steady-state of Eq. (7a). Then a second integration followed by a change of variables reveals a standard integral with a well-known solution.

We assume $\delta\theta_{\mathrm{a}}$ is constant, set $\partial_\tau\theta = 0$, and then let $\psi = \theta + x/D$. Then Eq. (7a) becomes

$$\frac{d^2\psi}{dx^2} + \mathcal{W}\mathrm{e}^{\psi} = 0. \quad (8)$$

Multiplying this by $\mathrm{d}\psi/\mathrm{d}x$ and integrating gives

$$\frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)^2 + \mathcal{W}\mathrm{e}^{\psi} = \frac{1}{2}c_1 \quad (9)$$

with c_1 a constant. Rearranging and integrating gives

$$\int\frac{\mathrm{d}\psi}{\left(1-\frac{2\mathcal{W}}{c_1}\mathrm{e}^{\psi}\right)^{1/2}} = \pm\sqrt{c_1}\int\mathrm{d}x = c_2 \pm\sqrt{c_1}x \quad (10)$$

where c_2 is another integration constant. This is then reduced to a standard integral [25] with the substitution $u = \left(1-\frac{2\mathcal{W}}{c_1}\mathrm{e}^{\psi}\right)^{1/2}$. This substitution gives the solution

$$\theta = \ln\frac{c_1}{2\mathcal{W}} + 2\ln\operatorname{sech}\left(\frac{c_2\pm\sqrt{c_1}x}{2}\right) - \frac{x}{D}. \quad (11)$$

Applying the boundary condition at $x=-\infty$ implies that $c_1 = 1/D^2$. We can write the solution:

$$\theta = \ln\frac{1}{2\mathcal{W}D^2} + 2\ln\operatorname{sech}\left(\frac{1}{2}\left(c_2\pm\frac{x}{D}\right)\right) - \frac{x}{D}. \quad (12)$$

Now applying the boundary condition at $x=0$ gives

$$\pm\frac{1}{D}\tanh\frac{c_2}{2} - 2\ln\operatorname{sech}\frac{c_2}{2} = \frac{-1}{D} - \ln 2\mathcal{W}D^2 - \delta\theta_{\mathrm{a}}. \quad (13)$$

Denoting the left-hand side of this equation as $F(c_2)$, it can be shown that F has a minimum. Hence for sufficiently large $\delta\theta_{\mathrm{a}}$, the right-hand side will always be less than the left-hand side. This means there is no equilibrium solution, consistent with a compost bomb having been triggered. To determine the critical level of $\delta\theta_{\mathrm{a}}$ to cause a compost bomb we find the c_2 that satisfies $F'(c_2)=0$ and substitute this back into equation (13):

$$\delta\theta_{\mathrm{a}}^{\mathrm{crit}} = \ln\left(2D\sqrt{D^2+1} - 2D^2\right) - 1 + \frac{1}{D}\sqrt{D^2+1} - \frac{1}{D} - \ln 2\mathcal{W}D^2 \quad (14)$$

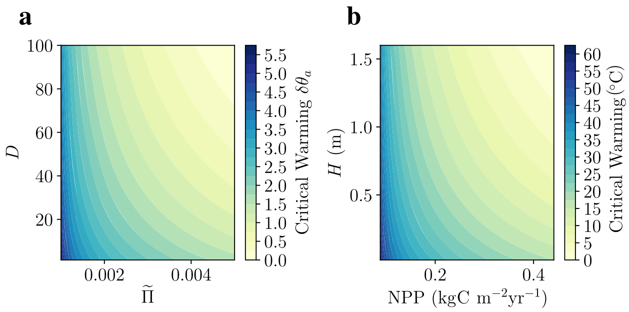

Fig. 3 **a** The relationship between nondimensional soil carbon e-folding depth D, nondimensional NPP $\widetilde{\Pi}$ and the critical warming required to cause a compost bomb $\delta\theta_{\mathrm{a}}$. **b** We also plot a dimensional version using the standard parameters in Table 1

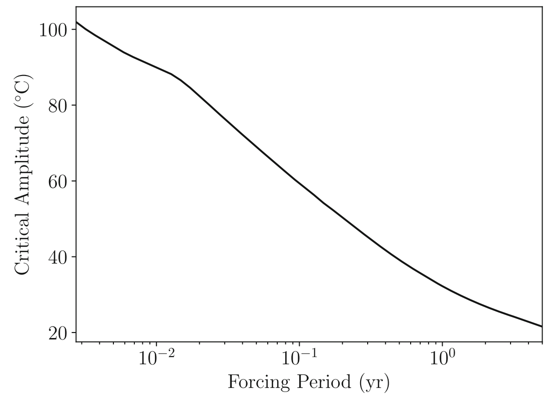

Fig. 4 The relationship between forcing period T and the amplitude of the minimum near surface air temperature changes required to trigger a compost bomb

where we have used the fact that the logarithm of a negative number is not real. Note that for sufficiently large values of $\mathcal{W}$, the right-hand side of this equation is negative, which we can interpret as the soil being inherently unstable without any warming.

As $\mathcal{W}$ is a function of $\widetilde{\Pi}$ and D, specifying $\widetilde{\Pi}$ and D is enough to determine $\delta\theta_{\mathrm{a}}^{\mathrm{crit}}$, through Eq. (14). We plot the critical maximum warming obtained in Fig. 3 for a range of $\widetilde{\Pi}$ and D values. Figure 3 shows the critical warming, above which no equilibrium solutions exist, corresponding to triggering a compost bomb. Increasing the 'fuel', $\widetilde{\Pi}$, decreases the warming required to trigger a compost bomb. Furthermore, the figure shows that in this model, soils with larger D, corresponding to soils that are well insulated and have a soil carbon with a weak dependence on depth, are more unstable.

4 Vulnerability to seasonal cycle

The compost bomb instability is a rate-induced instability. To trigger the compost bomb instability, the rate of increase in air temperature needs to be fast relative to the rate of decrease of soil carbon. Prior work [8] examined a linear increase in air temperature, corresponding to the increase in mean air temperature being caused by anthropogenic influence.

However, air temperature varies on multiple timescales. Two important modes of rapid air temperature change (relative to the timescale of soil carbon) are the diurnal and seasonal cycles. We let δT_{a} vary sinusoidally,

$$\delta T_{\mathrm{a}}(t) = \Delta T_{\mathrm{a}} \sin \frac{2\pi t}{T} \tag{15}$$

and numerically integrate Eq. (7a) for a range of forcing periods, using the standard parameters in Table 1. Sufficiently large forcing amplitudes will lead to a compost bomb. We plot these in Fig. 4.

Whilst the rate dependence alone would suggest that higher frequency oscillations are more unstable, high frequency oscillations can be too rapid to affect the soil temperature. Such high frequencies are in effect 'averaged out', as the air cannot heat the soil rapidly enough to have an effect deeper in the soil. Hence we do not expect compost bombs except at very high and unrealistic amplitudes.

In particular from Fig. 4, we see that the amplitudes required to cause a compost bomb from the very high frequency diurnal cycle are about 100°C, which is implausibly large. However for oscillations corresponding to timescales of the annual seasonal cycle, the magnitude is around 30°C. This remains large, but such variations occur in high latitude, soil carbon rich ecosystems, for instance in parts of Siberia [26].

We investigate the possibility of a seasonal cycle triggered compost bomb in Fig. 5, scanning across our two nondimensional parameters. This shows that for a variety of plausible parameters, a large seasonal cycle could trigger a compost bomb. Soils with larger $\widetilde{\Pi}$ are more susceptible to a seasonal cycle driven compost bomb as they have more 'fuel'. However, in marked contrast to the conditions for the existence of an equilibrium, shallower soils are more susceptible to seasonal cycle driven compost bombs. This susceptibility is because the atmosphere can warm larger fractions of shallower soils per unit of time.

5 Conclusion

The purpose of this research is to determine features of raised near-surface temperature that could trigger thermal runaway in soils, a process called a 'compost bomb'. The compost bomb instability was originally identified in a single box model. Here we present a substantial advance in realism, by introducing the spatial dimension of soil depth to simulations. Our overarching finding is that when accounting for vertical heterogeneity, thermal runaway still occurs. Furthermore, for realistic levels of heat diffusion, warming levels required to initiate the instability are much smaller than in the

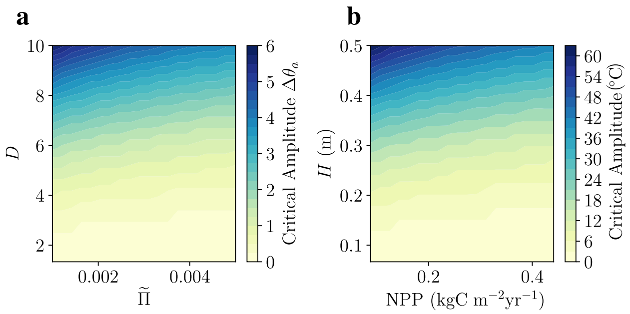

Fig. 5 **a** The relationship between nondimensional respiration, soil depth and the seasonal cycle amplitude required to cause a compost bombs. For typical values of Q_{10}, the seasonal cycle could trigger a compost bomb. **b** We also plot the dimensional version, here we modify the standard value of κ to $0.5\,\mathrm{Wm^{-1}\,°C^{-1}}$ to make the H values more realistic

LC10 case. We attribute this difference, at least in part, to strong nonlinearity in respiration heat release, due to its exponential ('Q_{10}') temperature response. For all finite values of thermal conductivity, we find these temperature thresholds are lower than those in the LC10 model. Nondimensionalisation supports this balance of two dominant factors, by collapsing the governing equations to parameters D and $\mathcal{W}$, that characterise the physical thermal aspects and respiration strength respectively. For our standard set of parameters, a rapid increase followed by a sustained temperature of order 10°C may push the soil system into the unstable regime. Besides discovering dangerous jumps in critical near-surface air temperature, above which there is no stable state, we also investigate the vulnerability to seasonal cycles in air temperature. We find that it is plausible that annual cycles of around 30°C risk uncontrolled warming in soils.

We have created a deliberately simplified model of the compost bomb. This has the advantage of being analytically tractable, yet ignores certain processes. Most obviously, we assume soil carbon is in equilibrium. This is not an unreasonable assumption however as we consider short enough timescales where this is approximately true. More importantly we neglect the role of soil moisture. Although it is beyond the scope of this paper to analyse this thoroughly we offer some justification for why our model captures the essential features of the system. The addition of soil moisture has two principal effects.

Firstly, it affects the amount of respiration in the soils. In the framework of our model, weakening respiration corresponds to decreasing $\mathcal{W}$, a nondimensional parameter that encodes the amount of soil carbon, the heat released from respiration and the thermal properties of the soil. Although this affects the precise value for the air temperature warming that leads to a compost bomb, for any $\mathcal{W} > 0$ there is still a critical level of warming that produces a compost bomb.

Secondly soil moisture affects heat transport through phase changes, which is a second order effect and so we are justified in ignoring it, and also through setting the conductivity of the soil. It is reasonable to neglect seasonal changes in the conductivity when compared to seasonal changes in respiration, as respiration varies exponentially with temperature.

However, even with these approximations, we have added to the evidence base of the plausibility of self-igniting fires in soils, and with an equation set that accounts explicitly for depth variation. Whilst adding a feature that can cause 'blow-up' requires great care, we would encourage ESM developers to examine this effect in more complicated models to discover the situations in which it is important.

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Data Availability All code used in this paper to plot figures and run numerical simulations is available at http://www.github.com/josephjclarke/ContinuumCompostBomb.

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