Incremental Construction of Three-way Concept Lattice for Knowledge Discovery in Social Networks

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Abstract

Three-way concept analysis (3WCA), a combination of three-way decision and formal concept analysis, is widely used in the field of knowledge discovery. Generally, constructing three-way concept lattices requires the original formal context and its complement context simultaneously. Additionally, the existing three-way concept lattice construction algorithms focus on the static formal context, and cannot cope with the dynamic formal context that is an essential representation in social networks. Toward this end, this paper pioneers a novel problem and method for the incremental construction of three-way concept lattice for knowledge discovery in social networks. Aiming to facilitate the construction efficiency, this paper firstly investigates the three-way concept lattice construction for attribute-incremental/object-incremental formal contexts, respectively. Then, the dynamic formal context of a social network can be viewed as a special formal context with both attribute-increment and object-increment. Further, we develop the AE/OE concept lattice incremental construction algorithms, called SNS-AE and SNS-OE. Extensive experiments are conducted on various formal contexts to evaluate the effectiveness of our incremental algorithms. The experimental results demonstrate that our proposed incremental algorithms can significantly decrease the construction time of three-way concept

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lattice compared to the non-incremental algorithm.

Keywords: Three-way concept analysis, AE/OE concepts, Social networks.

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1. Introduction

Formal Concept Analysis (FCA), a powerful computational intelligence methodology, is playing an increasingly important role in cognitive inference [1], recommendation systems [2], virtual machines scheduling [3], social networks analysis [4, 5] and social internet of things management [6]. However, a critical and common task for achieving the above services is to generate concept lattice efficiently. Therefore, most research focuses on concept lattice construction for both static and dynamic formal contexts [7, 8, 9, 10].

FCA supports the binary decision-making that is to consider accepting and rejecting two choices, that is to say, non-acceptance is equivalent to rejecting, and non-rejection is equivalent to accepting. However, this is not the commonly occurred case in practical applications. For example, political voting in daily life can be "*for*", "*against*" or "*abstaining*", while in medical diagnosis it is to "*treat*" or "*not to treat*", or "*further diagnosis*", the corresponding theory is the three-way decision-making [11, 12].

To overcome the above shortcomings of FCA-assisted binary decision-making, a series of research efforts on combing FCA with three-way decisions are made [13]. Qi *et al.* [14] extended the conventional FCA and proposed the Three-way Concept Analysis (3WCA); 3WCA presents two types of concepts, i.e., Attribute Export three-way concept (AE concept) and Object Export three-way concept (OE concept). Qian *et al.* [15] transformed the given formal context and its complementary context into new formal contexts. Then, the Type I-combinatorial context and Type II-combinatorial context were defined, which are the apposition and subposition of these new formal contexts. Finally, they proposed the approaches of constructing the three-way concept lattices based on the concept lattices of Type I-combinatorial context and Type II-combinatorial context. Wang *et al.* [16] proposed CBO3C algorithm for constructing three-way concept lattices. They demonstrated that the CBO3C can correctly and efficiently calculate all core three-way concepts of a given formal context. Yu *et al.* [17] investigated the properties of three-way concept lattices by considering the properties of their atoms, irreducible elements, and complements in three-way concept lattices. With these properties, they provided special complete lattices which are isomorphic to their associated three-way concept lattices. Recently, Yang *et al.* [18] introduced an AE-oriented composite operator and an OE-oriented composite operator that combine a pair of formal concepts coming from the concept lattices of original formal context and its complement context. They defined the candidate AE concepts and redundant AE concepts based on the AE-oriented composite operator and defined the candidate OE concepts and redundant OE concepts based on the OE-oriented composite operator, respectively. They proved that the resulting AE concepts and OE concepts can be obtained by filtering out the redundant AE concept and OE concepts from candidate AE concepts and OE concepts, respectively. Experimental results demonstrated that their approach has a significant running time reduction compared to Qi and Qian's approaches. Recently, Hao *et.al* [13] introduced the stability of three-way concepts and analyzed the relevant properties. Then, an efficient algorithm for calculating the stability of three-way concepts is developed. In addition, a promising application on natural language generation is explored with the stability of three-way concepts.

In summary, existing approaches to constructing three-way concept lattices mainly apply classical concept lattice construction approaches. Unfortunately, they did not consider the dynamic property of formal contexts. That is to say, if a given formal context is dynamically changing (e.g. formal context constructed from a social network), how to incrementally construct the three-way concept lattice is becoming a huge challenge. To the best of our knowledge, this is the first study to incrementally construct the three-way concept lattice for a social network.

Aiming to efficiently construct three-way concept lattice for the formal context constructed from a social network, this paper firstly investigates the AE/OE concept lattices incremental construction approaches and algorithms, then further presents a social-incremental AE/OE concept lattices incremental construction approach and algorithm. In addition, an optimized social-incremental AE/OE concept lattices incremental construction approach and algorithm is presented. The major contributions of this paper are summarized as follows.

- (**Novel Problem Formulation**) We formulate a novel problem on three-way concept lattice incremental construction for social networks. First, a formal context of a social network G, termed $K(G)$ can be constructed via a modified adjacency matrix. Then, this problem takes $K(G)$ as an input, then generates the three-way concept lattice incrementally. To be specific, our problem takes the vertices as the objects and attributes, and then constructs a symmetric formal context $K(G)$ which is different from the formal context for the conventional three-way concept lattice construction. Importantly, our problem is the first work to study the incremental construction of three-way concept lattice.

- (**AE/OE Concept Lattice Incremental Construction**) To facilitate the construction efficiency, we firstly investigate the three-way concept lattice construction for attribute-incremental/ object-incremental formal contexts, respectively. Specifically, the dynamic formal context of a social network can be viewed as both attribute-incremental and object-incremental formal context, and thus the above three-way concept lattice construction approach for attribute-incremental/ object-incremental formal contexts can be used for constructing AE/OE concept lattice incrementally. Further, we develop the AE concept lattice incremental construction algorithm, called SNS-AE; and the OE concept lattice incremental construction algorithm, called SNS-OE. The originality of our algorithms is that a novel approach based on dynamic formal context is developed and its correctness is proven mathematically. In addition, we optimize the SNS-AE and SNS-OE algorithms by taking the symmetry property of the formal contexts of social networks into account. It is found that OE concepts can be obtained by simply exchanging the extent and intent of AE concepts.

- (**Evaluation**) We utilize the public datasets to carry out extensive experiments and validate the correctness and effectiveness of the proposed SNS-AE and SNS-OE algorithms by comparing them with the non-incremental algorithm. The experimental results demonstrate that our algorithm can quickly construct the three-way concept lattices for both static social networks and dynamic social networks. Specifically, our algorithm has around 18%, 28%, 4.7%, and 6.2% reduction in running time compared to the non-incremental algorithm from the aspects of attribute-incremental, object-incremental, social-incremental, and optimized social-incremental three-way concept lattice construction, respectively.

The rest of this paper is structured as follows. Section 2 provides the preliminary knowledge about FCA and 3WCA and formulates the problem of this paper. Section 3 and Section 4 present AE and OE concept lattice incremental construction approaches and the corresponding algorithms, respectively. Based on AE/OE concept lattice incremental construction approaches, Section 5 develops the fast AE and OE concept lattice incremental construction algorithms, i.e., SNS-AE and SNS-OE for a given social network. Section 6 conducts the comparison experiments for evaluating the effectiveness of the proposed algorithms. Finally, Section 7 concludes this paper.

2. Preliminary Knowledge and Problem Formulation

This section firstly provides the preliminary knowledge on FCA and 3WCA methodologies; and then formulates the problem description about incremental construction of a three-way concept lattice for a symmetric formal context induced from a social network.

2.1. FCA and 3WCA

FCA, as an effective computational intelligence methodology, has been broadly used in data analysis and mining, artificial intelligence, and so forth. It is often utilized to characterize the relation between objects and attributes in the information systems via the defined formal concepts.

The basics of FCA are presented as follows.

Definition 1. *(**Formal Context**) A formal context is organized as a triple $K = (O, A, I)$, where $O = \{o_1, o_2, \cdots, o_n\}$ indicates the set of objects, and $A = \{a_1, a_2, \cdots, a_m\}$ denotes the set of attributes, and I refers to the binary relation between O and A. If an object o has an attribute a, denoted as oIa or $(o, a) \in I$.*

Definition 2. *(**Formal Concept Positive Derivative Operators**) Given a formal context $K = (O, A, I)$, for $\forall X \subseteq O$, $\forall B \subseteq A$, the formal concept positive derivative operators $\uparrow$ and $\downarrow$ are given as follows.*

$$X^{\uparrow} = \{a \in A | \forall x \in X, (x, a) \in I\} \tag{1}$$

$$B^{\downarrow} = \{x \in O | \forall a \in B, (x, a) \in I\} \tag{2}$$

With the above formal concept positive derivative operators, the definition of formal concept or positive concept is formalized.

Definition 3. *(**Formal Concept**) Given a formal context $K = (O, A, I)$, for $\forall X \subseteq O$, $\forall B \subseteq A$, a pair (X, B) is called a formal concept if $X^{\uparrow} = B$ and $B^{\downarrow} = X$, where X and B denote the extent and intent of the formal concept, respectively.*

Definition 4. *(**Partial Relation**) Let $C(K)$ be the set of all formal concepts generated from the formal context $K = (O, A, I)$. If $(X_i, B_i), (X_j, B_j) \in C(K)$, then let*

$$(X_i, B_i) \leq (X_j, B_j) \Leftrightarrow X_i \subseteq X_j (\Leftrightarrow B_i \supseteq B_j) \tag{3}$$

here, " $\leq$ " call a partial relation over $C(K)$.

Definition 5. *(***Concept Lattice***)[4] All formal concepts are organized with the partial order* $\leq$ *to form a concept lattice* $L(K) = (C(K), \leq)$ *. Generally, it can be graphically represented by a Hasse diagram.*

3WCA extends the conventional FCA methodology inspired by the three-way decision model [19]. From the granular computing point of view, 3WCA is a more refined and complete methodology for describing the relation between objects and attributes since it not only characterizes the common attributes/objects owned by the given objects/attributes (positive relation), but also describes the common attributes/objects not owned by the given objects/attributes (negative relation).

Different from the formal concept positive derivative operators in FCA, 3W-CA defines the formal concept negative derivative operators [14] as follows.

Definition 6. *(***Formal Concept Negative Derivative Operators***) For a formal context* $K = (O, A, I)$*, for* $\forall X \subseteq O$*,* $\forall B \subseteq A$*, the formal concept negative derivative operators* $\uparrow^-$ *and* $\downarrow^-$ *are given as follows.*

$$X^{\uparrow^-} = \{a \in A | \forall x \in X, (x, a) \notin I\} = \{a \in A | \forall x \in X, (x, a) \in I^c\} \tag{4}$$

$$B^{\downarrow^-} = \{x \in O | \forall a \in B, (x, a) \notin I\} = \{a \in A | \forall x \in X, (x, a) \in I^c\} \tag{5}$$

where $I^c = (O \times A) - I$*.*

Similar to the above formal concept definition, the negative concept is formalized with the above two formal concept negative derivative operators.

Definition 7. *(***Negative Concept***) Given a formal context* $K = (O, A, I)$*, for* $\forall X \subseteq O$ *and* $\forall B \subseteq A$*, a pair* (X, B) *is called a negative concept if* $X^{\uparrow^-} = B$ *and* $B^{\downarrow^-} = X$*, where* X *and* B *refer to the extent and the intent of the negative concept.*

With the positive and negative derivative operators of formal concept, the three-way operators, *i.e*, attributed-induced operators (AE-operators) and object-induced operators (OE-operators) are further derived as follows.

Definition 8. *(**AE-operators and OE-operators**) Let $K = (O, A, I)$ be a formal context. For any objects subsets $X, Y \subseteq O$ and attributes subset $B \subseteq A$, a pair of AE-operators $\lessdot$ and $\gtrdot$ are given as follows.*

$$B^{\lessdot} = (B^{\downarrow}, B^{\downarrow^{-}}) \tag{6}$$

$$(X, Y)^{\gtrdot} = \{a \subseteq A | a \in X^{\uparrow} \wedge a \in Y^{\uparrow^{-}}\} = X^{\uparrow} \cap Y^{\uparrow^{-}} \tag{7}$$

Similar to the definition of AE-operators, for any objects subset $X \subseteq O$ and attributes subset $B, C \subseteq A$, a pair of OE-operators $\lessdot$ and $\gtrdot$ are given as follows.

$$X^{\lessdot} = (X^{\uparrow}, X^{\uparrow^{-}}) \tag{8}$$

$$(B, C)^{\gtrdot} = \{o \subseteq O | o \in B^{\downarrow} \wedge o \in C^{\downarrow^{-}}\} = B^{\downarrow} \cap C^{\downarrow^{-}} \tag{9}$$

Based on the above AE-operators and OE-operators, the attribute/object-induced three-way concepts (AE-concept/OE-concept) are easily defined, respectively.

Definition 9. *(**AE-concept and OE-concept**) Given a formal context $K = (O, A, I)$, for any two object subsets $X, Y \subseteq O$ and an attribute subset $B \subseteq A$, $((X, Y), B)$ is called AE-concept* iff *$(X, Y)^{\gtrdot} = B$ and $B^{\lessdot} = (X, Y)$. Note that (X, Y) and B are the extent and intent of AE-concept, respectively. We use $AEC(K)$ to indicate the set of all AE-concepts.*

Similarly, for $X \subseteq O$, $B, C \subseteq A$, $(X, (B, C))$ is called OE-concept iff *$(X)^{\lessdot} = (B, C)$ and $(B, C)^{\gtrdot} = X$. Note that X and (B, C) denote the extent and intent of OE-concept, respectively. We use $OEC(K)$ to indicate the set of all OE-concepts.*

Definition 10. *(**AE Lattice and OE Lattice**) An AE lattice $AEL(K) = (AEC(K), \leq)$ can be constructed by organizing all AE-concepts,i.e., $AEC(K)$ of a formal context K with the partial order $\leq$, i.e., for any $((X, Y), B)$ and $((Z, W), C)$, $((X, Y), B) \leq ((Z, W), C) \Leftrightarrow (X, Y) \subseteq (Z, W) \leftrightarrow C \subseteq B$. The geometry structure of $AEL(K)$ is a Hasse diagram.*

Similarly, an OE lattice $OEL(K) = (OEC(K), \leq)$ can be obtained by organizing all AE-concepts, i.e., $OEC(K)$ of a formal context K with the partial order $\leq$, i.e., for any $(X, (B, C))$ and $(Y, (D, E)$, $(X, (B, C)) \leq (Y, (D, E)) \Leftrightarrow X \subseteq Y \leftrightarrow (D, E) \subseteq (B, C)$. The geometry structure of $OEL(K)$ is also a Hasse diagram.

Example 1. *Table 1 exhibits a formal context where $O = \{o_1, o_2, o_3, o_4\}$, $A = \{a, b, c, d, e\}$ and the binary relation between O and A denoted with $\times$. For instance, the object "o_2" has the attributes "a", "b", "c".*

Table 1: A Formal Context $K = (O, A, I)$

	a	b	c	d	e
o_1	1	1		1	1
o_2	1	1	1		
o_3				1	
o_4	1	1	1		

With the concept lattice generation algorithm presented in our previous work [10], the concept lattice of the above formal context is generated and visualized as shown in Figure 1. Clearly, we generate 6 formal concepts, i.e., $(1234, \emptyset), (\emptyset, abcde), (24, abc), (1, abde), (124, ab), (13, d)$.

According to Definition 10, the AE lattice and OE lattice are presented in Figure 2.

2.2. Problem Formulation

First of all, a social network $G = (V, E)$ is taken as our research object, its formal context $K(G)$ is symmetric which has been proved in [10, 20]. The formal context of G can be represented as $K(G) = (V, V, I)$, where individuals V are regarded as the objects and attributes, and $I \subseteq V \times V$ indicates the social relationship between individuals.

This paper address a novel problem which concentrates on a social network, and constructs the symmetric formal context that is the representation of so-

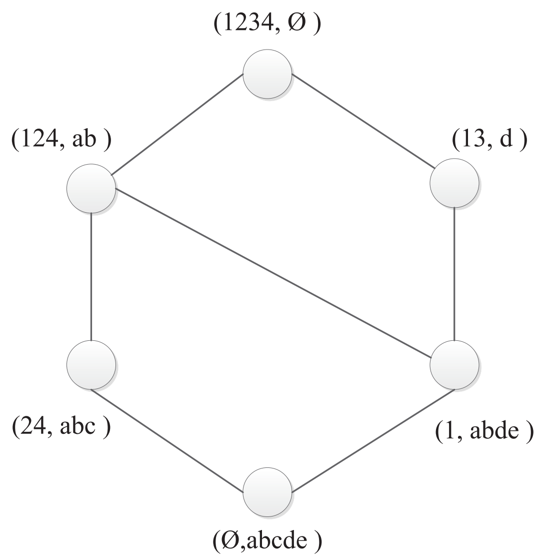


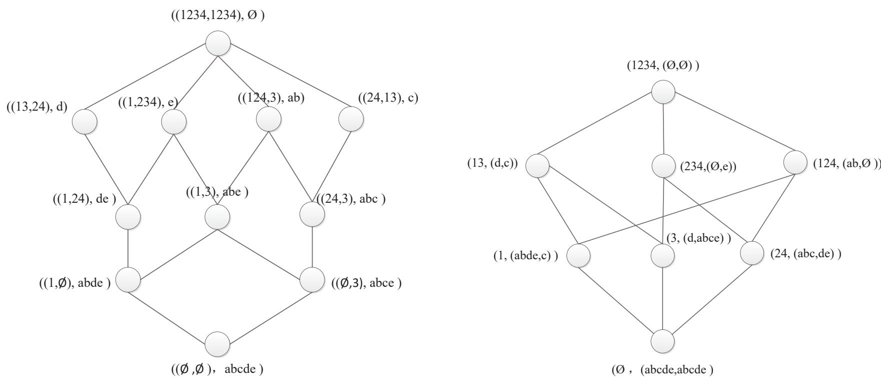
Figure 1: Concept Lattice $C(K)$

cial network; finally generates the three-way concept lattice for the constructed symmetric formal context induced by a social network. In particular, its formal context is often dynamically updated due to the large-scale and dynamical properties of social networks. However, the existing three-way concept lattice construction algorithms focus on a static formal context and cannot cope with this type of dynamic formal context.

Therefore, the formulism on three-way concept lattice construction for a social network is described as follows.

- **Input**: A social network $G = (V, E)$;
- **Output**: It generates a set of AE-concepts $AEC(K(G))$ and OE-concepts $OEC(K(G))$.

The target of this problem is to develop an efficient incremental algorithm on constructing three-way concept lattice for a social network (i.e., objects and attributes can be simultaneously added). To this end, we therefore firstly investigate the incremental AE/OE lattice construction algorithm for a dynamic



(a) AE Lattice $AEL(K)$ (b) OE Lattice $OEL(K)$

Figure 2: AE Lattice $AEL(K)$ and OE Lattice $OEL(K)$

formal context (such as adding attributes/objects).

3. AE Lattice Incremental Construction

Our proposed AE lattice incremental construction algorithm is based on recent work [18]. Therefore, we will introduce the basic idea of this work at first, and then present our algorithm in detail.

Figure 3 illustrates the solution idea of AE lattice construction algorithm presented in [18]. Specifically, they introduced an AE-oriented composite operator $+AE$ that combines formal concepts (X_1, B_1) from original formal context K and formal concepts (X_2, B_2) from complement formal context K^c. Then, the candidate AE concepts and redundant AE concepts are defined based on composite operator $+AE$. Finally, they proposed an AE lattice construction algorithm by filtering redundant AE concepts from the candidate AE concepts.

The above AE lattice construction algorithm can significantly shorten the running time compared to the existing three-way concept lattice construction algorithms [15, 21]. However, their algorithms cannot cope with the dynamic formal contexts (such as attributes/objects are added) and incrementally generate the AE lattice. Motivated by this, this section presents incremental construction algorithms including object/attribute-incremental algorithms (hereinafter

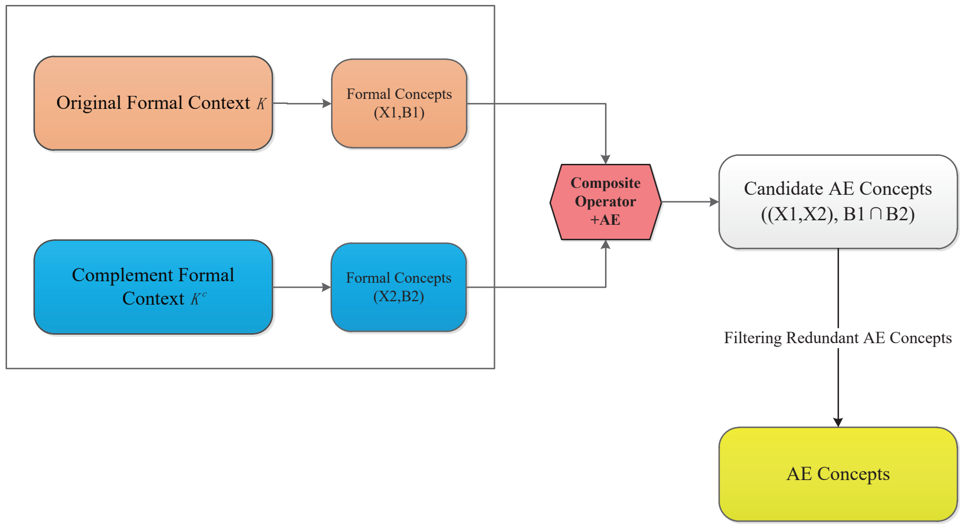


Figure 3: AE Lattice Construction based on AE-oriented Composite Operator

referred to as Add-AE) for AE lattice followed by the above framework.

3.1. Attribute-incremental AE Concept Lattice Construction

To develop an attribute-incremental AE concept lattice construction algorithm, one key issue is to incrementally generate the formal concepts for K and K^c. As shown in Figure 4, given an original formal context $K_1 = (O, A_1, I_1)$, it will be updated to a new formal context $K = (O, A, I)$ if the attributes are added, *i.e.*, a new formal context $K_2 = (O, A_2, I_2)$ is appended to K_1 and forms K, *i.e.*, $K = K_1 \cup K_2$ where $A = A_1 \cup A_2$ and $I = I_1 \cup I_2$.

Definition 11. *(**Attribute-incremental Formal Concept Composite Operator**) We represent the concept lattices of K_1 and K_2 as $L(K_1)$ and $L(K_2)$. For any formal concepts $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, then (X_1, B_1) and (X_2, B_2) can be combined in the form of $((X_1 \cap X_2), (X_1 \cap X_2)^{\uparrow})$, this composition is called a formal concept of $L(K_1 \cup K_2)$, denoted by $+_C$.*

Proof. 1. $X_1 \in L(K_1)$, $X_2 \in L(K_2)$, $\exists B_1 \subseteq A_1, \exists B_2 \subseteq A_2$, then $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$. $X_1 \cap X_2 = B_1^{\downarrow} \cap B_2^{\downarrow} = (B_1 \cup B_2)^{\downarrow}$, due to $B_1 \cup B_2 \subseteq$

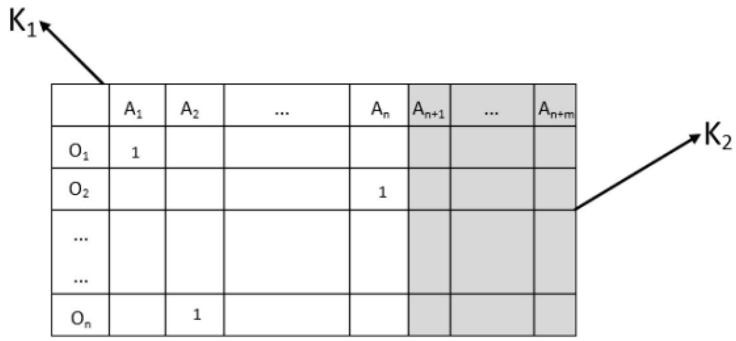


Figure 4: Attribute-incremental Formal Context

$A_1 \cup A_2$, we have $((X_1 \cap X_2), (B_1 \cap B_2)^{\downarrow\uparrow}) = ((B_1 \cap B_2)^{\downarrow}, (B_1 \cap B_2)^{\downarrow\uparrow}) = L(K)$, hence, $X_1 \cap X_2 \subseteq L(K)$.

Besides, $X \in L(K)$, $\exists B \subseteq A_1 \cup A_2$, then $(X, B) \in L(K)$, $X = B^{\downarrow} = (B \cap (A_1 \cup A_2))^{\downarrow} = ((B \cap A_1) \cup (B \cap A_2))^{\downarrow} = (B \cap A_1)^{\downarrow} \cap (B \cap A_2)$, due to $B \cap A_1 \subseteq A_1$, we have $(B \cap A_1)^{\downarrow} \in L(K_1)$, and $B \cap A_2 \subset A_2$ we have $(B \cap A_2)^{\downarrow} \in L(K_2)$, therefore, $L(K) = \{X_1 \cap X_2; X_1 \in L(K_1), X_2 \in L(K_2)\}$.

2. When $A_2 = \{m\}$, $K_2 = \{O, m, I_2\}$, $L(O, A \cup \{m\}, I) = EL(K) \cup \{X \cap m^{\downarrow}, X \in EL(K)\}$, EL(K) indicates the set of extents of formal concepts. According to 1), $EL(O, \{m\}, I_2) = \{m^{\downarrow}, \emptyset^{\downarrow}\} = \{m^{\downarrow}, O\}$.

□

We extend the above operator $+_{AE}$ to attribute-incremental formal concept composite operator $+^{a}_{AE}$ with the following definition. Note that $L(K_1 \cup K_2)$ and $L(K_1^c \cup K_2^c)$ be the concept lattices of a formal context K and its complement formal context K^c.

Definition 12. ***(Attribute-incremental AE-oriented Composite Operator)*** *For any formal concepts* $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$ $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, *then* $(X_1, B_1) +_C (X_2, B_2)$=$((X_1 \cap X_2), (X_1 \cap X_2)^{\uparrow})$ *refers to a formal concept of* $L(K_1 \cup K_2)$ *which is the concept lattice of* K.

Similarly, then $(X_3, B_3) +_C (X_4, B_4)$=$((X_3 \cap X_4), (X_3 \cap X_4)^{\uparrow})$ *is called a formal concept of* $L(K_1^c \cup K_2^c)$ *which is the concept lattice of* K^c. *Then,* $((X_1 \cap X_2), (X_1 \cap X_2)^{\uparrow})$ *and* $((X_3 \cap X_4), (X_3 \cap X_4)^{\uparrow})$ *can be combined in the form of* $(((X_1 \cap X_2), (X_3 \cap X_4)), (X_1 \cap X_2)^{\uparrow} \cap (X_3 \cap X_4)^{\uparrow})$, *and this composition is called an attribute-incremental AE-oriented composite operator, denoted by* $+^a_{AE}$.

According to Definition 12, we define the attribute-incremental candidate AE concepts as follows.

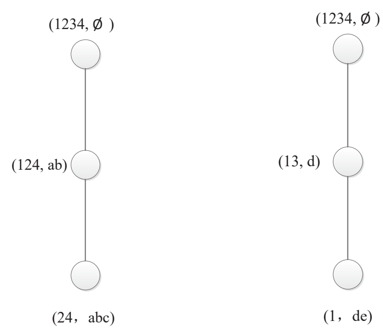
Definition 13. ***(Attribute-incremental Candidate AE Concept)*** *For any formal concepts* $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, *then* $((X_1 \cap X_2), (X_1 \cap X_2)^{\uparrow}) +^a_{AE} ((X_3 \cap X_4), (X_3 \cap X_4)^{\uparrow})$=$(((X_1 \cap X_2), (X_3 \cap X_4)), (X_1 \cap X_2)^{\uparrow} \cap (X_3 \cap X_4)^{\uparrow})$ *is called attribute-incremental candidate AE concept. We use* AE^c *to store the set of attribute-incremental candidate AE concepts of* K.

Example 2. *Table 2 shows the formal context* K *including the original formal context* K_1 *and the incremental formal context* K_2 *(gray columns). Figure 5 presents the concept lattices for* K_1 *and* K_2. *Table 3 shows the complement formal context* K^c *including the original complement formal context* K_1^c *and the incremental complement formal context* K_2^c *(gray columns). Figure 6 presents the concept lattices for* K_1^c *and* K_2^c.

Table 2: Formal Context $K = K_1 \cup K_2$

$O \times A$	a	b	c	d	e
1	1	1		1	1
2	1	1	1		
3				1	
4	1	1	1		

Here, for a formal concept $(124, ab) \in L(K_1)$, $(13, d) \in L(K_2)$, $(13, e) \in L(K_1^c)$, $(234, e) \in L(K_2^c)$, *then* $((\{124\} \cap \{13\}), (\{124\} \cap \{13\})^{\uparrow}) +^a_{AE} ((\{13\} \cap \{234\}), (\{13\} \cap \{234\})^{\uparrow})$=$(((\{124\} \cap \{13\}), (\{13\} \cap \{234\})), (\{124\} \cap \{13\})^{\uparrow} \cap$



(a) $L(K_1)$ (b) $L(K_2)$

Figure 5: Concept Lattices of K_1 and K_2.

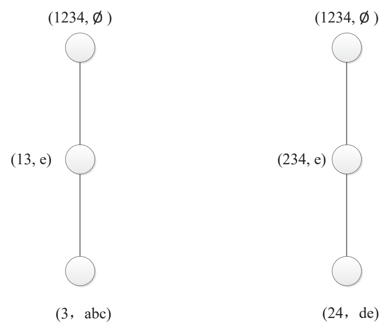
Table 3: Complement Formal Context $K^c = K_1^c \cup K_2^c$

$O \times A$	a	b	c	d	e
1			1		
2				1	1
3	1	1	1		1
4				1	1

$(\{13\} \cap \{234\})^{\uparrow})$=$((1,3), abe)$ *is called an attribute-incremental candidate AE concept.*

Definition 14. ***(Attribute-incremental Redundant AE Concept)*** *Let* $L(K_1 \cup K_2)$ *and* $L(K_1^c \cup K_2^c)$ *be the concept lattices of a formal context* K *and its complement formal context* K^c*. For any formal concepts* $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, *if* $((X_1 \cap X_2)^{\uparrow} \cap (X_3 \cap X_4)^{\uparrow})^{\downarrow} \supset (X_1 \cap X_2)$, *or* $((X_1 \cap X_2)^{\uparrow} \cap (X_3 \cap X_4)^{\uparrow})^{\downarrow^{-}} \supset (X_3 \cap X_4)$, *then* $((X_1, B_1) +_C (X_2, B_2)) +^{a}_{AE} ((X_3, B_3) +_C (X_4, B_4))$=$((X_1 \cap X_2), (X_1 \cap X_2)^{\uparrow}) +^{a}_{AE} ((X_3 \cap X_4), (X_3 \cap X_4)^{\uparrow})$=$(((X_1 \cap X_2), (X_3 \cap X_4)), (X_1 \cap X_2)^{\uparrow} \cap (X_3 \cap X_4)^{\uparrow})$ *is called an attribute-incremental redundant AE concept. We use* AE^r *to store the set of attribute-incremental redundant AE concepts of* K.

Example 3. *Continue the Example 2, for a formal concept* $(124, ab) \in L(K_1)$,



(a) $L(K_1^c)$ (b) $L(K_2^c)$

Figure 6: Concept Lattices of K_1^c and K_2^c.

$(1234, \emptyset) \in L(K_2)$, $(3, abc) \in L(K_1^c)$, $(24, de) \in L(K_2^c)$, due to $((X_1 \cap X_2)^{\uparrow} \cap (X_3 \cap X_4)^{\uparrow})^{\downarrow^-} = \{3\} \supset \emptyset$, therefore $(((X_1 \cap X_2), (X_3 \cap X_4)), (X_1 \cap X_2)^{\uparrow} \cap (X_3 \cap X_4)^{\uparrow})$ is an attribute-incremental redundant AE concept.

Inspired by the findings of research [18], this paper presents a relational theorem on attribute-incremental AE concept lattice, candidate AE concept lattice, and redundant AE concept lattice.

Theorem 1. *Let $K = K_1 \cup K_2$ be the formal context K, the attribute-incremental AE concept lattice of K denoted as AE^a, then*

$$AE^a = AE^c - AE^r \tag{10}$$

3.2. Object-incremental AE Concept Lattice Construction

Similar to attribute-incremental AE concept lattice construction approach, the key issue is to incrementally generate the formal concepts for K and K^c. As shown in Figure 7, given an original formal context $K_1 = (O_1, A, I_1)$, it will be updated as a new formal context $K = (O, A, I)$ if the attributes are added, *i.e.*, a new formal context $K_2 = (O_2, A, I_2)$ is appended to K_1 and forms a new formal context K, *i.e.*, $K = K_1 \cup K_2$ where $O = O_1 \cup O_2$ and $I = I_1 \cup I_2$.

Definition 15. ***(Object-incremental Formal Concept Composite Operator)*** *For any formal concepts $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, then*

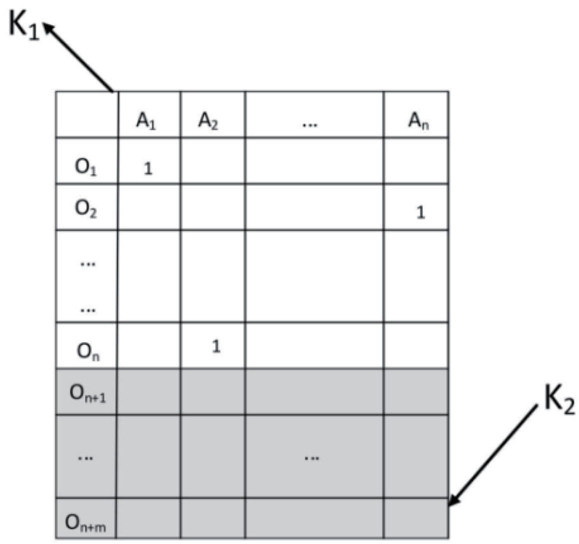


Figure 7: Object-incremental Formal Context

(X_1, B_1) and (X_2, B_2) can be combined in the form of $((B_1 \cap B_2)^{\downarrow}, (B_1 \cap B_2))$, this composition is called a formal concept of $L(K_1 \cup K_2)$, denoted by $+_C$.

Proof. 1. $B_1 \in L(K_1)$ and $B_2 \in L(K_2)$, $\exists X_1 \subseteq A_1$, then (X_1, B_1) is a concept and $(X_1, B_1) \in L(K_1)$, $\exists X_2 \subseteq A_2$, then (X_2, B_2) is a concept, $(X_2, B_2) \in L(K_2)$. $B_1 \cap B_2 = X_1^{\uparrow} \cap X_2^{\uparrow} = (X_1 \cup X_2)^{\uparrow}$, due to $X_1 \cup X_2 \subseteq A_1 \cup A_2$, we have $((X_1 \cap X_2)^{\uparrow\downarrow}, (B_1 \cap B_2)) = ((X_1 \cap X_2)^{\uparrow\downarrow}, (X_1 \cap X_2)^{\uparrow}) = L(K)$, hence, $B_1 \cap B_2 \subseteq L(K)$, the intersection of B_1 and B_2 is the intent of concepts of K.

Besides, $B \in L(K)$, $\exists B \subseteq B_1 \cup B_2$, then $(X, B) \in L(K)$, $B = X^{\uparrow} = (X \cap (A_1 \cup A_2))^{\uparrow} = ((X \cap A_1) \cup (X \cap A_2))^{\uparrow} = (X \cap A_1)^{\uparrow} \cap (X \cap A_2)$, due to $X \cap A_1 \subseteq A_1$, we have $(X \cap A_1)^{\uparrow} \in L(K_1)$, and $X \cap A_2 \subset A_2$ we have $(X \cap A_2)^{\uparrow} \in L(K_2)$, therefore, $L(K) = \{B_1 \cap B_2; B_1 \in L(K_1), B_2 \in L(K_2)\}$.

2. When $X_2 = \{n\}$, $K_2 = \{\{n\}, B, I_2\}$, $L(A \cup \{n\}, B, I) = L(A, B, I) \cup \{B \cap n^{\uparrow}, B \in L(A, B, I)\}$, according to 1), $L(\{n\}, B, I_2) = \{n\}^{\uparrow}$.

□

Similarly, we extend the AE-oriented composite operator $+_{AE}$ to $+^{o}_{AE}$ with the following definition. Note that $L(K_1 \cup K_2)$ and $L(K_1^c \cup K_2^c)$ be the concept lattices of a formal context K and its complement formal context K^c.

Definition 16. ***(Object-incremental AE-oriented Composite Operator)*** *For any formal concepts $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, then $(X_1, B_1) +_C (X_2, B_2)$=$((B_1 \cap B_2)^{\downarrow}, (B_1 \cap B_2))$ called a formal concept of $L(K_1 \cup K_2)$. Similarly, then $(X_3, B_3) +_C (X_4, B_4)$=$((B_3 \cap B_4)^{\downarrow}, (B_3 \cap B_4))$ called a formal concept of $L(K_1^c \cup K_2^c)$. Then, $((B_1 \cap B_2)^{\downarrow}, (B_1 \cap B_2))$ and $((B_3 \cap B_4)^{\downarrow}, (B_3 \cap B_4))$ can be combined in the form of $(((B_1 \cap B_2)^{\downarrow}, (B_3 \cap B_4)^{\downarrow}), (B_1 \cap B_2) \cap (B_3 \cap B_4))$, and this composition is called an object-incremental AE-oriented composite operator, denoted by $+^{o}_{AE}$.*

According to Definition 16, we define the object-incremental candidate AE concepts as follows.

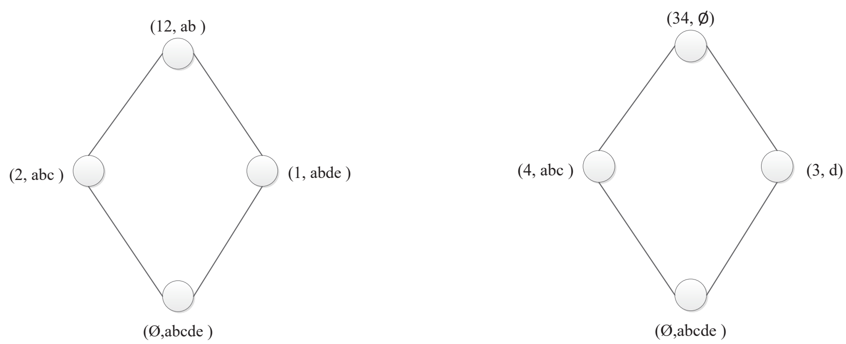
Definition 17. ***(Object-incremental Candidate AE Concept)*** *For any formal concepts $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$ $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, then $((B_1 \cap B_2)^{\downarrow}, (B_1 \cap B_2)) +^{o}_{AE} ((B_3 \cap B_4)^{\downarrow}, (B_3 \cap B_4))$=$(((B_1 \cap B_2)^{\downarrow}, (B_3 \cap B_4)^{\downarrow}), (B_1 \cap B_2) \cap (B_3 \cap B_4))$ is called an object-incremental candidate AE concept. We use AE^c to store the set of object-incremental candidate AE concepts of K.*

Example 4. *Table 4 shows the formal context K including the original formal context K_1 and the incremental formal context K_2 (green rows). Figure 8 presents the concept lattices for K_1 and K_2. Table 5 shows the complement formal context K^c including the original complement formal context K_1^c and the incremental complement formal context K_2^c (green rows). Figure 9 presents the concept lattices for K_1^c and K_2^c.*

Here, for a formal concept $(2, abc) \in L(K_1)$, $(4, abc) \in L(K_2)$, $(2, de) \in L(K_1^c)$, $(3, abce) \in L(K_2^c)$, then $((\{abc\} \cap \{abc\})^{\downarrow}, (\{abc\} \cap \{abc\})) +^{o}_{AE} ((\{de\} \cap \{abce\})^{\downarrow}, (\{de\} \cap \{abcd\}))$=$(((\{abc\} \cap \{abc\})^{\downarrow}, (\{de\} \cap \{abce\})^{\downarrow}), (\{abc\} \cap \{abc\}) \cap (\{de\} \cap \{abce\}))$=$((24, 13), \emptyset)$ is called an object-incremental candidate AE concept.

Table 4: Formal Context $K = K_1 \cup K_2$

$O \times A$	a	b	c	d	e
1	1	1		1	1
2	1	1	1		
3				1	
4	1	1	1		



(a) $L(K_1)$ (b) $L(K_2)$

Figure 8: Concept Lattices of K_1 and K_2.

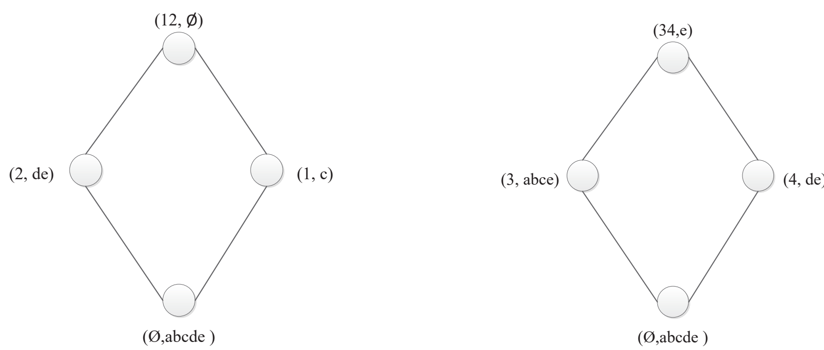
Theorem 2. *For any formal concepts* $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, *and if* $(B_1 \cap B_2)^{\downarrow} = (B_3 \cap B_4)^{\downarrow}$, *then* $((B_1 \cap B_2)^{\downarrow}, (B_1 \cap B_2)) +^{o}_{AE} ((X_3 \cap X_4)^{\downarrow}, (B_3 \cap B_4)) = (((B_1 \cap B_2)^{\downarrow}, (B_3 \cap B_4)^{\downarrow}), (B_1 \cap B_2) \cap (B_3 \cap B_4))$ *is an object-incremental AE concept.*

Definition 18. ***(Object-incremental Redundant AE Concept)*** *For any formal concepts* $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, *if* $((B_1 \cap B_2) \cap (B_3 \cap X_4))^{\uparrow}) \supset ((B_1 \cap B_2)^{\downarrow} \cap (B_1 \cap B_2)^{\downarrow})$, *or* $((B_1 \cap B_2) \cap (B_3 \cap X_4))^{\uparrow -} \supset ((B_3 \cap B_4)^{\downarrow}$, *then* $((X_1, B_1) +_C (X_2, B_2)) +^{o}_{AE} ((X_3, B_3) +_C (X_4, B_4)) = (((B_1 \cap B_2)^{\downarrow}, (B_3 \cap B_4)^{\downarrow}), (B_1 \cap B_2) \cap (B_3 \cap B_4))$ *is called an object-incremental redundant AE concept. We utilize* AE^r *to store the set of object-incremental redundant AE concepts of* K.

Similarly to Theorem 2, this paper presents a relational theorem on object-

Table 5: Complement Formal Context $K^c = K_1^c \cup K_2^c$

$O \times A$	a	b	c	d	e
1			1		
2				1	1
3	1	1	1		1
4				1	1



(a) $L(K_1^c)$ (b) $L(K_2^c)$

Figure 9: Concept Lattices of K_1^c and K_2^c.

incremental AE concept lattice, candidate AE concept lattice, and redundant AE concept lattice.

Theorem 3. *Let $K = K_1 \cup K_2$ be the formal context K, the object-incremental AE concept lattice of K denoted as AE^o, then*

$$AE^o = AE^c - AE^r \tag{11}$$

Based on Theorem 1 and Theorem 3, the AE concept lattice incremental construction algorithm is developed as shown in Algorithm 1.

Algorithm 1 Add-AE: AE Concept Lattice Incremental Construction Algorithm

Input:

$K_1 = (X, B_1, I_1)$

$K_2 = (X, B_2, I_2)$

Output:

Set of AE concepts $AE(K_1 \cup K_2)$ (here $K = K_1 \cup K_2$)

K_1^c=$\emptyset$;

K_2^c=$\emptyset$;

$L(K_1) = \emptyset$;

$L(K_2) = \emptyset$;

$K_1^c = \emptyset$;

$K_2^c = \emptyset$;

$AE^a = \emptyset$;

$AE^c = \emptyset$;

$AE^r = \emptyset$;

while ($x \in O$ and $a \in A$) **do**

  if($(x, a) \notin I$) **then**

    $K^c = K^c \cup \{(x, a)\}$

  end if

end while

Generate the concept sets of K_1, K_2, K_1^c, K_2^c, denoted as $L(K_1), L(K_2), L(K_1^c), L(K_2^c)$

Incrementally generate the concept sets of $K_1 \cup K_2$, and $K_1^c \cup K_2^c$, denoted as $L(K), L(K^c)$

$AE(K_1 \cup K_2) = AE^c - AE^r$

end

4. OE Lattice Incremental Construction

We also adopt the OE lattice construction framework [18] to develop our OE lattice incremental construction algorithm.

As shown in Figure 10, they introduced an OE-oriented composite operator $+OE$ that combines formal concepts (X_1, B_1) from the original formal context and formal concepts (X_2, B_2) from its complement context. Then, the candidate OE concepts and redundant OE concepts are defined based on the OE-oriented composite operator. Finally, they proposed an OE lattice construction algorithms by filtering redundant OE concepts from the candidate OE concepts.

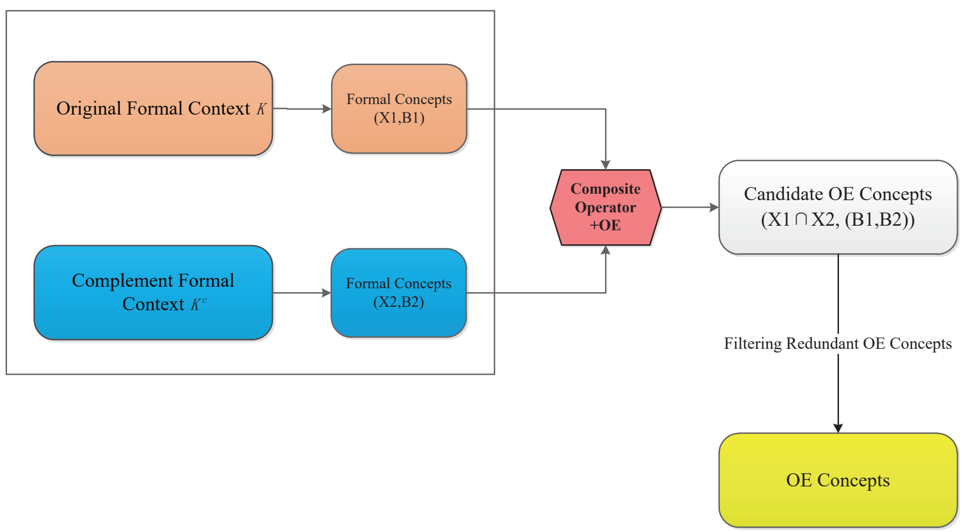


Figure 10: OE Lattice Construction based on OE-oriented Composite Operator

The above OE lattice construction algorithm also suffers a severe time efficiency problem and cannot cope with the dynamic formal contexts (such as attributes/objects are added) and incrementally generate the OE lattice. Towards this end, this section presents incremental construction algorithms including object/attribute-incremental algorithms (hereinafter referred to as Add-OE) for OE lattice.

4.1. Attribute-incremental OE Concept Lattice Construction

We extend the above operator $+_{OE}$ to an attribute-incremental formal concept composite operator $+^a_{OE}$ with the following definition.

Definition 19. ***(Attribute-incremental OE-oriented Composite Operator)*** *For any formal concepts* $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, *then* $(X_1, B_1) +_C (X_2, B_2)$=$((X_1 \cap X_2), (X_1 \cap X_2)^{\uparrow})$ *called a formal concept of* $L(K_1 \cup K_2)$. *Similarly, then* $(X_3, B_3) +_C (X_4, B_4)$=$((X_3 \cap X_4), (X_3 \cap X_4)^{\uparrow})$ *called a formal concept of* $L(K_1^c \cup K_2^c)$. *Then,* $((X_1 \cap X_2), (X_1 \cap X_2)^{\uparrow})$ *and* $((X_3 \cap X_4), (X_3 \cap X_4)^{\uparrow})$ *can be combined in the form of* $((X_1 \cap X_2) \cap (X_3 \cap X_4), ((X_1 \cap X_2)^{\uparrow}, (X_3 \cap X_4)^{\uparrow}))$, *and this composition is called an attribute-incremental OE-oriented composite operator, denoted by* $+^a_{OE}$.

According to Definition 19, we define the attribute-incremental candidate OE concepts as follows.

Definition 20. ***(Attribute-incremental Candidate OE Concept)*** *For any formal concepts* $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, *then* $((X_1 \cap X_2), (X_1 \cap X_2)^{\uparrow}) +^a_{AE} ((X_3 \cap X_4), (X_3 \cap X_4)^{\uparrow})$=$((X_1 \cap X_2) \cap (X_3 \cap X_4), ((X_1 \cap X_2)^{\uparrow}, (X_3 \cap X_4)^{\uparrow}))$ *is called attribute-incremental candidate OE concept. We use* OE^c *to store the set of attribute-incremental candidate OE concepts of* K.

Example 5. *Continue the Example 2, for a formal concept* $(124, ab) \in L(K_1)$, $(13, d) \in L(K_2)$, $(13, e) \in L(K_1^c)$, $(234, e) \in L(K_2^c)$, *then* $((\{124\} \cap \{13\}), (\{124\} \cap \{13\})^{\uparrow}) +^a_{OE} ((\{13\} \cap \{234\}), (\{13\} \cap \{234\})^{\uparrow})$=$((\{124\} \cap \{13\}) \cap (\{13\} \cap \{234\}), (\{124\} \cap \{13\})^{\uparrow}, (\{13\} \cap \{234\})^{\uparrow}))$=$(\emptyset, (abde, abce))$ *is called an attribute-incremental candidate OE concept.*

Definition 21. ***(Attribute-incremental Redundant OE Concept)*** *For any formal concepts* $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, *if* $((X_1 \cap X_2)^{\uparrow} \cap (X_3 \cap X_4)^{\uparrow})^{\downarrow} \supset (X_1 \cap X_2)$, *or* $((X_1 \cap X_2)^{\uparrow} \cap (X_3 \cap X_4)^{\uparrow})^{\downarrow^-} \supset (X_3 \cap X_4)$, *then* $((X_1, B_1) +_C (X_2, B_2)) +^a_{AE} ((X_3, B_3) +_C (X_4, B_4))$=$((X_1 \cap X_2), (X_1 \cap X_2)^{\uparrow}) +^a_{OE} ((X_3 \cap X_4), (X_3 \cap X_4)^{\uparrow})$=$((X_1 \cap X_2) \cap$

$(X_3 \cap X_4), (X_1 \cap X_2)^{\uparrow}, (X_3 \cap X_4)^{\uparrow}))$ *is called an attribute-incremental redundant OE concept. The set of attribute-incremental redundant OE concepts of* K *is* OE^r.

Inspired by the findings of research [18], this paper presents a relational theorem on attribute-incremental OE concept lattice, candidate OE concept lattice, and redundant OE concept lattice.

Theorem 4. *Let* $K = K_1 \cup K_2$ *be the formal context* K*, the attribute-incremental OE concept lattice of* K *denoted as* OE^a*, then*

$$OE^a = OE^c - OE^r \tag{12}$$

4.2. Object-incremental OE Concept Lattice Construction

Based on Definition 15, we extend operator $+_{OE}$ to object-incremental OE-oriented composite operator $+^o_{OE}$ as follows.

Definition 22. ***(Object-incremental OE-oriented Composite Operator)*** *For any formal concepts* $(X_1, B_1) \in L(K_1)$*,* $(X_2, B_2) \in L(K_2)$*,* $(X_3, B_3) \in L(K_1^c)$*,* $(X_4, B_4) \in L(K_2^c)$*, then* $(X_1, B_1) +_C (X_2, B_2)$=$((B_1 \cap B_2)^{\downarrow}, (B_1 \cap B_2))$ *called a formal concept of* $L(K_1 \cup K_2)$*. Similarly, then* $(X_3, B_3) +_C (X_4, B_4)$=$((B_3 \cap B_4)^{\downarrow}, (B_3 \cap B_4))$ *called a formal concept of* $L(K_1^c \cup K_2^c)$*. Then,* $((B_1 \cap B_2)^{\downarrow}, (B_1 \cap B_2))$ *and* $((B_3 \cap B_4)^{\downarrow}, (B_3 \cap B_4))$ *can be combined in the form of* $((B_1 \cap B_2)^{\downarrow} \cap (B_3 \cap B_4)^{\downarrow}, (B_1 \cap B_2), (B_3 \cap B_4))$*, and this composition is called an object-incremental OE-oriented composite operator denoted by* $+^o_{OE}$.

Based on Definition 22, we define the object-incremental candidate OE concepts as follows.

Definition 23. ***(Object-incremental Candidate OE Concept)*** *For any formal concepts* $(X_1, B_1) \in L(K_1)$*,* $(X_2, B_2) \in L(K_2)$*,* $(X_3, B_3) \in L(K_1^c)$*,* $(X_4, B_4) \in L(K_2^c)$*, then* $((B_1 \cap B_2)^{\downarrow}, (B_1 \cap B_2)) +^o_{OE} ((B_3 \cap B_4)^{\downarrow}, (B_3 \cap B_4))$=$((B_1 \cap B_2)^{\downarrow} \cap (B_3 \cap B_4)^{\downarrow}, ((B_1 \cap B_2), (B_3 \cap B_4))$ *is called an object-incremental candidate OE concept. And,* OE^c *represents the set of object-incremental candidate OE concepts of a formal context* K.

Example 6. *Continue the Example 4, for a formal concept* $(2, abc) \in L(K_1)$, $(4, abc) \in L(K_2)$, $(2, de) \in L(K_1^c)$, $(3, abce) \in L(K_2^c)$, *then* $((\{abc\}\cap\{abc\})^{\downarrow}, (\{abc\}\cap\{abc\})) +_{OE}^{o} ((\{de\} \cap \{abce\})^{\downarrow}, (\{de\} \cap \{abcd\}))$=$(((\{abc\} \cap \{abc\})^{\downarrow} \cap (\{de\} \cap \{abce\})^{\downarrow}, (\{abc\}\cap\{abc\}), (\{de\}\cap\{abce\}))$=$(24, (abc, e))$ *is called an object-incremental candidate OE concept.*

Definition 24. ***(Object-incremental Redundant OE Concept)*** *For any formal concepts* $(X_1, B_1) \in L(K_1)$, $(X_2, B_2) \in L(K_2)$, $(X_3, B_3) \in L(K_1^c)$, $(X_4, B_4) \in L(K_2^c)$, *if* $((B_1 \cap B_2) \cap (B_3 \cap X_4))^{\uparrow}) \supset ((B_1 \cap B_2)^{\downarrow} \cap (B_1 \cap B_2)^{\downarrow})$, *or* $((B_1 \cap B_2) \cap (B_3 \cap X_4))^{\uparrow-} \supset ((B_3 \cap B_4)^{\downarrow}$, *then* $((X_1, B_1) +_C (X_2, B_2)) +_{OE}^{o} ((X_3, B_3) +_C (X_4, B_4))$=$(((B_1 \cap B_2)^{\downarrow} \cap (B_3 \cap B_4)^{\downarrow}, ((B_1 \cap B_2), (B_3 \cap B_4))$ *is called an object-incremental redundant OE concept. And,* OE^r *indicates the set of object-incremental redundant OE concepts of* K*.*

Similarly, a relational theorem on object-incremental OE concept lattice, candidate OE concept lattice, and redundant OE concept lattice is presented.

Theorem 5. *Let* $K = K_1 \cup K_2$ *be the formal context* K*, the object-incremental OE concept lattice of* K *denoted as* OE^o*, then*

$$OE^o = OE^c - OE^r \tag{13}$$

Based on Theorem 4 and Theorem 5, the OE concept lattice incremental construction algorithm is developed as shown in Algorithm 2.

Algorithm 2 Add-OE: OE Concept Lattice Incremental Construction Algorithm

Input:

$K_1 = (X, B_1, I_1)$

$K_2 = (X, B_2, I_2)$

Output:

Set of OE concepts $OE(K_1 \cup K_2)$ (here $K = K_1 \cup K_2$)

K_1^c=$\emptyset$;
K_2^c=$\emptyset$;
$L(K_1) = \emptyset$;
$L(K_2) = \emptyset$;
$K_1^c = \emptyset$;
$K_2^c = \emptyset$;
$AE^a = \emptyset$;
$AE^c = \emptyset$;
$AE^r = \emptyset$;
while $(x \in O$ and $a \in A)$ **do**
    if$((x, a) \notin I)$ **then**
        $K^c = K^c \cup \{(x, a)\}$
    end if
end while
Generate the concept sets of K_1, K_2, K_1^c, K_2^c, denoted as $L(K_1), L(K_2), L(K_1^c), L(K_2^c)$
Incrementally generate the concept sets of $K_1 \cup K_2$, and $K_1^c \cup K_2^c$, denoted as $L(K), L(K^c)$
$OE(K_1 \cup K_2) = OE^c - OE^r$
end

5. Constructing AE Lattice and OE Lattice for a Social Network

The previous section presents the approaches on constructing the AE concept lattice and OE concept lattice for attribute-incremental and object-incremental formal contexts, respectively. However, there exists a special type of formal context constructed from a social network which includes both attribute-increment and object-increment (as shown in Figure 11).

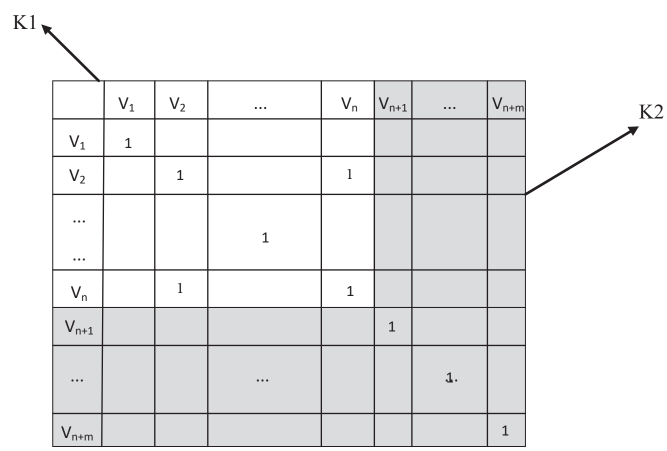


Figure 11: A Formal Context Constructed from a Social Network

Intuitively, a social network is usually expanding as the users joining the network and having the social interactions with other existing users. In our previous works [4, 10], we can use a formal context to represent a social network where the nodes are viewed as the objects as well as attributes of it. Therefore, constructing the three-way concept lattice (*i.e.*, AE lattice and OE lattice) for a social network is a great and new challenge.

Table 6 shows the differences between the attribute(object)-incremental three-way concept lattice construction and the three-way concept lattice construction for a social network, termed social-incremental three-way concept lattice construction. Note that $AE^a \rightarrow AE^o$ and $OE^a \rightarrow OE^o$ indicate that AE concept lattice and OE concept lattice are constructed based on the joint of attribute-incremental and object-incremental.

Table 6: Differences between the attribute(object)-incremental three-way concept lattice construction and the three-way concept lattice construction for a social network

	Dynamic Formal Context		Social Network
Type	Attribute-incremental	Object-incremental	Social-incremental
AE Lattice	$AE^a = AE^c - AE^r$	$AE^o = AE^c - AE^r$	$AE^a \rightarrow AE^o$
OE Lattice	$OE^a = OE^c - OE^r$	$OE^o = OE^c - OE^r$	$OE^a \rightarrow OE^o$

Before the construction of AE lattice and OE lattice for a social network, we firstly present the formal context construction approach for a social network.

5.1. Formal Context Construction for a Social Network

A social network G is represented as a graph with the vertices indicating a set of individuals and the edges indicating the relations between vertices. In this paper, we adopt the modified adjacency matrix to represent the formal context of G, that is, $K(G)$=(V, V, I), where I is the binary relation between two vertices.

Definition 25. ***(Modified Adjacency Matrix)****[4]. Let a social network be a graph with n vertices that are assumed to be ordered from v_1 to v_n. The $n \times n$ matrix M is called a modified adjacency matrix, in which*

$$M = \begin{cases} m_{ij} = 1 & \textit{if} \quad (v_i, v_j) \in E \\ m_{ij} = 1 & \textit{if} \quad i = j \\ m_{ij} = 0 & \textit{otherwise} \end{cases}$$

We can easily construct the following formal context (Table 7) of a social network G as shown in Figure 12 by using Definition 25. Further, the corresponding concept lattice is built as shown in Figure 13.

For online social networks, a social-incremental construction approach on three-way concept lattice is elaborated in the following sections. Specifically, AE concept lattice and OE concept lattice construction approaches are discussed, respectively.

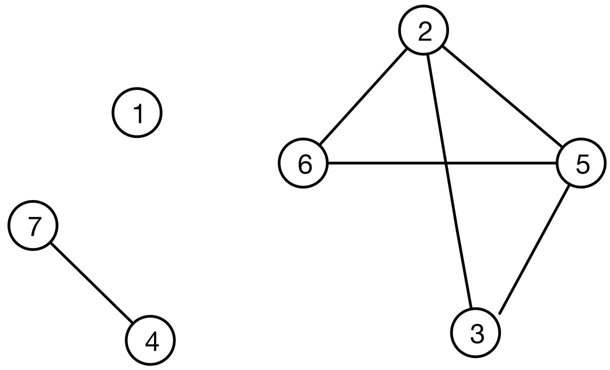


Figure 12: A Social Network G

Table 7: The Formal Context of a Social Network G.

V× V	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	1						
V_2		1	1		1	1	
V_3		1	1		1		
V_4				1			1
V_5		1	1		1	1	
V_6		1			1	1	
V_7				1			1

5.2. AE Concept Lattice Construction for a Social Network

Given a social network $G = (V, E)$, its formal context is constructed as $K_1 = (V, V, I)$. If some users $v_{n+1}, \cdots, v_{n+m}$ join the network and have the interactions with other existing users, then these users and interactions form an social-incremental formal context K_2 as shown in the gray area of Figure 11. We implement the AE concept lattice construction by dividing the K_2 as K_3 and K_4. The steps for AE concept lattice construction are listed as follows.

Step 1: As shown in Figure 14, we take K_1 and K_3 as the input of the ADD-AE algorithm, then construct the attribute-incremental AE concept lattice;

Step 2: As shown in Figure 15, we initialize $K_1 \leftarrow K_1 \cup K_3$, then take K_1 and

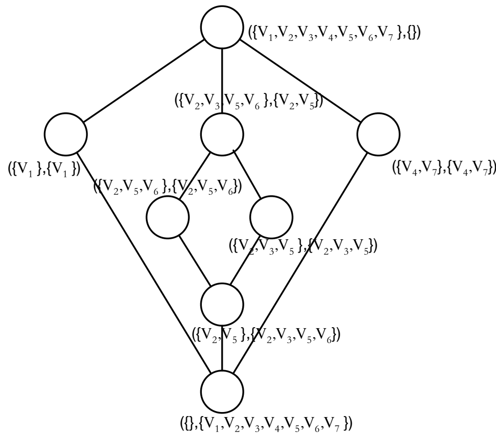


Figure 13: Concept Lattice of Social Network G

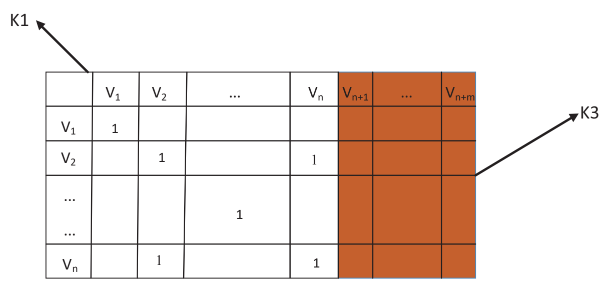


Figure 14: Step 1 of AE Concept Lattice Construction for a Social Network

K_4 as the input of the ADD-AE algorithm, then construct the object-incremental AE concept lattice;

Step 3: Store the object-incremental AE concept lattice generated from Step 2 into resulting AE concept lattice AE^s which is a social-incremental AE concept lattice.

Base on the above steps, the AE concept lattice dynamic construction algorithm for a social network, termed SNS-AE is developed in Algorithm 3.

5.3. OE Concept Lattice Construction for a Social Network

Similar with AE concept lattice construction for a social network, we also implement the OE concept lattice construction by dividing the K_2 as K_3 and

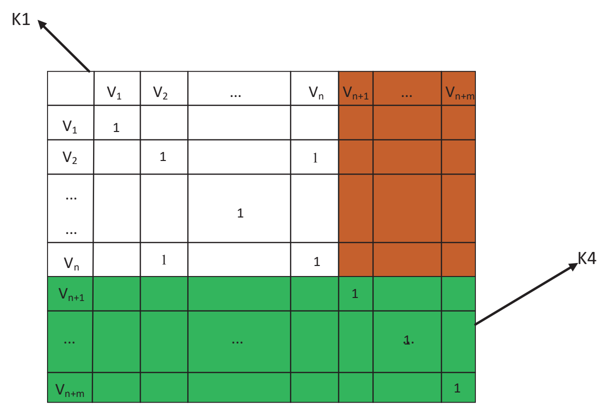


Figure 15: Step 2 of AE Concept Lattice Construction for a Social Network

K_4. The steps for OE concept lattice construction are listed as follows.

Step 1: We take K_1 and K_3 as the input of the ADD-OE algorithm, then construct the attribute-incremental OE concept lattice;

Step 2: We initialize $K_1 \leftarrow K_1 \cup K_3$, then take K_1 and K_4 as the input of the ADD-OE algorithm, then construct the object-incremental OE concept lattice;

Step 3: Store the object-incremental OE concept lattice generated from Step 2 into resulting OE concept lattice OE^s which is a social-incremental OE concept lattice.

Base on the above steps, the OE concept lattice dynamic construction algorithm for a social network, termed SNS-OE is developed in Algorithm 4.

Algorithm 3 SNS-AE: AE Concept Lattice Dynamic Construction Algorithm for a Social Network

Input:

$K_1 = (V, V, I)$

$K_2 = (V_a, V_a, I_a)$

Output:

Set of AE concepts $AE(K)$, $K = K_1 \cup K_2$

```
Initialize AE(K_2)'=∅, AE(K)=∅
begin
    K_2' ← (V, V ∪ V_a, I ∪ I_a)
    AE(K_2)' ← Invoking attribute-incremental AE concept lattice construction algorithm ADD-AE(K_1, K_2')
    K_2'' ← (V ∪ V_a, V ∪ V_a, I ∪ I_a)
    AE(K) ← Invoking object-incremental AE concept lattice construction algorithm ADD-AE(K_2', K_2'')
end
return AE(K)
end
```

Algorithm 4 SNS-OE: OE Concept Lattice Dynamic Construction Algorithm for a Social Network

Input:

$K_1 = (V, V, I)$

$K_2 = (V_a, V_a, I_a)$

Output:

Set of OE concepts $OE(K)$, $K = K_1 \cup K_2$

```
Initialize OE(K_2)'=∅, OE(K)=∅
begin
    K_2' ← (V, V ∪ V_a, I ∪ I_a)
    OE(K_2)' ← Invoking attribute-incremental OE concept lattice construction algorithm ADD-OE(K_1, K_2')
    K_2'' ← (V ∪ V_a, V ∪ V_a, I ∪ I_a)
    OE(K) ← Invoking object-incremental OE concept lattice construction algorithm ADD-OE(K_2', K_2'')
end
return OE(K)
end
```

5.4. Illustrative Example

Consider a social network $g = (V, E)$ as shown in Figure 16, then two new users v_8 and v_9 join the network and have social interactions with v_4, v_7, and v_2, v_6, respectively.

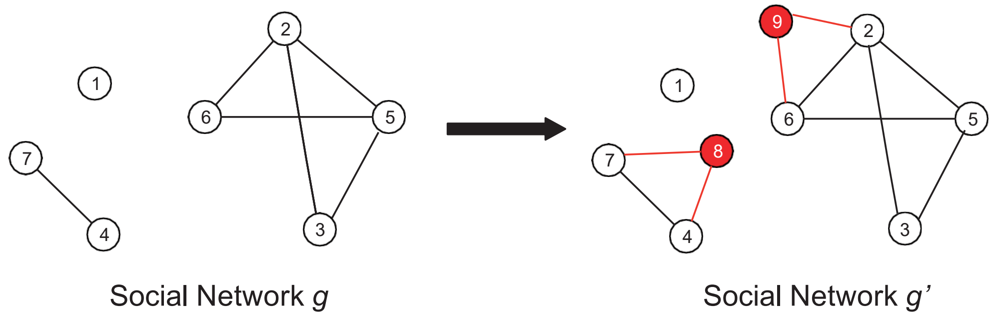


Figure 16: Social Networks g and $g^{'}$

According to Definition 25, we construct the following formal context of a social network g and $g^{'}$.

Table 8: Formal Contexts of Social Networks g (white area only) and $g^{'}$ (whole area).

V× V	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
V_1	1								
V_2		1	1		1	1			1
V_3		1	1		1				
V_4				1			1	1	
V_5		1	1		1	1			
V_6		1			1	1			1
V_7				1			1	1	
V_8				1			1	1	
V_9		1				1			1

By invoking the above SNS-AE and SNS-OE algorithms, we obtain the following 28 AE concepts and 28 OE concepts as shown in Table 9 and Table 10. Clearly, it is observed that the total number of AE concepts and OE concepts are exactly the same. Interestingly, each OE concept can be obtained by exchanging the extent and intent of a certain AE concept. For example, for an AE concept $((2,5,6),(1,4,7,8),(2,5,6))$ ($\sharp$12 AE concept), we can easily exchange the order of extent and intent, then an OE concept $((2,5,6),(2,5,6),(1,4,7,8))$ ($\sharp$20 OE concept) can be obtained.

Table 9: AE Concepts of Social Network g'

Concept ID	Extent	Intent
1	((2, 3, 5, 6, 9), (1, 4, 7, 8))	(2)
2	((2, 6, 9), (1, 4, 7, 8))	(9, 2, 6)
3	((2, 3, 5, 6), (1, 4, 7, 8, 9))	(5)
4	((), (3, 4, 7, 8))	(1, 6, 9)
5	((), (3))	(1, 4, 6, 7, 8, 9)
6	((2, 3, 5, 6), (1, 4, 7, 8))	(2, 5)
7	((), (4, 6, 7, 8, 9))	(1, 3)
8	((), (3, 4, 5, 7, 8))	(1, 9)
9	((), (1, 9))	(3, 4, 5, 7, 8)
10	((), (3, 5))	(1, 4, 7, 8, 9)
11	((1), (2, 3, 4, 5, 6, 7, 8, 9))	(1)
12	((2, 5, 6), (1, 4, 7, 8))	(2, 5, 6)
13	((2, 6), (1, 4, 7, 8))	(9, 2, 5, 6)
14	((), (4, 7, 8, 9))	(1, 3, 5)
15	((), ())	(1, 2, 3, 4, 5, 6, 7, 8, 9)
16	((2), (1, 4, 7, 8))	(2, 3, 5, 6, 9)
17	((2, 3, 5), (1, 4, 7, 8))	(2, 3, 5)
18	((2, 5), (1, 4, 7, 8))	(2, 3, 5, 6)
19	((4, 7, 8), (1, 2, 3, 5, 6, 9))	(8, 4, 7)
20	((), (1, 3))	(4, 6, 7, 8, 9)
21	((), (1))	(2, 3, 4, 5, 6, 7, 8, 9)
22	((2, 3, 5), (1, 4, 6, 7, 8, 9))	(3)
23	((2, 6, 9), (1, 3, 4, 5, 7, 8))	(9)
24	((2, 3, 5), (1, 4, 7, 8, 9))	(3, 5)
25	((2, 5, 6, 9), (1, 3, 4, 7, 8))	(6)
26	((2, 6, 9), (1, 3, 4, 7, 8))	(9, 6)
27	((2, 5, 6, 9), (1, 4, 7, 8))	(2, 6)
28	((1, 2, 3, 4, 5, 6, 7, 8, 9), (1, 2, 3, 4, 5, 6, 7, 8, 9))	()

Table 10: OE Concepts of Social Network $g^{'}$

Concept ID	Extent	Intent
1	(9, 6)	((2, 6, 9), (1, 3, 4, 7, 8))
2	(1)	((1), (2, 3, 4, 5, 6, 7, 8, 9))
3	(3, 5)	((2, 3, 5), (1, 4, 7, 8, 9))
4	(2, 3, 5, 6)	((2, 5), (1, 4, 7, 8))
5	(1, 3)	((), (4, 6, 7, 8, 9))
6	(5)	((2, 3, 5, 6), (1, 4, 7, 8, 9))
7	(8, 4, 7)	((4, 7, 8), (1, 2, 3, 5, 6, 9))
8	(1, 9)	((), (3, 4, 5, 7, 8))
9	(1, 4, 6, 7, 8, 9)	((), (3))
10	(2, 3, 4, 5, 6, 7, 8, 9)	((), (1))
11	(4, 6, 7, 8, 9)	((), (1, 3))
12	(2, 6)	((2, 5, 6, 9), (1, 4, 7, 8))
13	(9)	((2, 6, 9), (1, 3, 4, 5, 7, 8))
14	(3, 4, 5, 7, 8)	((), (1, 9))
15	(1, 2, 3, 4, 5, 6, 7, 8, 9)	((), ())
16	(6)	((2, 5, 6, 9), (1, 3, 4, 7, 8))
17	(3)	((2, 3, 5), (1, 4, 6, 7, 8, 9))
18	(1, 6, 9)	((), (3, 4, 7, 8))
19	(2, 3, 5)	((2, 3, 5), (1, 4, 7, 8))
20	(2, 5, 6)	((2, 5, 6), (1, 4, 7, 8))
21	(2, 5)	((2, 3, 5, 6), (1, 4, 7, 8))
22	(1, 4, 7, 8, 9)	((), (3, 5))
23	(9, 2, 5, 6)	((2, 6), (1, 4, 7, 8))
24	(9, 2, 6)	((2, 6, 9), (1, 4, 7, 8))
25	(2)	((2, 3, 5, 6, 9), (1, 4, 7, 8))
26	(2, 3, 5, 6, 9)	((2), (1, 4, 7, 8))
27	()	((1, 2, 3, 4, 5, 6, 7, 8, 9), (1, 2, 3, 4, 5, 6, 7, 8, 9))
28	(1, 3, 5)	((), (4, 7, 8, 9))

Based on the important observations about AE concepts and OE concepts of a social network, we can induce the following theorem.

Theorem 6. *Given a social network G, if $((X,Y),B)$ is an AE concept, then $(B,(X,Y))$ must be an OE concept.*

Proof. Since $((X,Y),B)$ is an AE concept regarding the formal context of a social network G, i.e., K, that is to say $(X,B_1) \in L(K)$, $X^{\uparrow\downarrow} = B_1$ and $(Y,B_2) \in L(K^c)$, $Y^{\uparrow-\downarrow-} = B_2$, where $B = B_1 \cap B_2$. Since the binary relation in K is symmetric, i.e., for $x_i \in X$, $b_i \in B_1$, $(x_i,b_i) \in I$, then $(b_i,x_i) \in I$; similarly, for $y_i \in Y$, $b_i \in B_2$, $(y_i,b_i) \in I^c$, then $(b_i,y_i) \in I^c$; we can easily obtain the following equation

$$B_1^{\uparrow} = X \quad and \quad B_1^{\uparrow\downarrow} = B_1 \tag{14}$$

$$B_2^{\uparrow-} = Y \quad and \quad B_2^{\uparrow-\downarrow-} = B_2 \tag{15}$$

□

Eqs. (14) and (15) imply that the pairs $(B_1,X) \in L(K)$ and $(B_2,Y) \in L(K^c)$. Therefore, according to the definition of OE concept, i.e., $((B_1 \cap B_2),(X,Y))$ is an OE concept. Obviously, the OE concept $((B_1 \cap B_2),(X,Y))$, i.e., $(B,(X,Y))$ can be obtained by exchanging the order of the extent and intent of AE concept $((X,Y),B)$.

6. Evaluation

In this section, we conduct 4 groups of experiments on 10 formal contexts for evaluating the performance of our proposed incremental algorithms on attribute-incremental formal contexts, object-incremental formal contexts, and social-incremental formal contexts, respectively. In addition, a case study is investigated for illustrating the usefulness of our research.

6.1. Datasets

The experimental datasets utilized by 4 groups of experiments are described as follows.

- **Group1:** We adopt data2 [22], data3, data4 [23], as well as sushi_1 [1] for evaluation on attribute-incremental three-way concept lattice generation;
- **Group2:** We adopt data1, data2 [22], data3 ,data4 [23], and sushi_2 [2] for evaluation on object-incremental three-way concept lattice generation:
- **Group3:** We extract 4 sub social networks from Karate dataset [3], denoted as karate_1, karate_2, karate_3, karate_4 for evaluation on social-incremental three-way concept lattice generation:
- **Group4:** Regarding evaluation on optimized Social-incremental three-way concept lattice generation, the dataset utilized here is the same with Group3.

6.2. Comparison Algorithms

This paper mainly compares our proposed algorithm with the latest algorithm for the three-way concept lattice generation [18].

- **Non-incremental algorithm**: As a non-incremental algorithm, Yang's algorithm [18] defines AE/OE composite operators based on the concept lattices of the original formal context and its complement context, then further constructs the three-way concept lattices. However, it cannot handle with the dynamical formal contexts as well as social networks scenario.
- **Incremental algorithm**: Our proposed incremental algorithm incorporates social-incremental idea and improves Yang's algorithm. The basic

[1] http://www.kamishima.net/sushi/
[2] http://www.kamishima.net/sushi/
[3] http://networkrepository.com/soc-karate.php

idea is to devise the object-incremental and attribute-incremental three-way concept lattice generation algorithms, respectively. Then, a social network oriented three-way concept lattice generation algorithm is further presented.

6.3. Results

The experimental environment is MAC operating system, Mojave processor@2.3GHz Intel Core i5, 16GRAM, and Python 3.7 programming language. We run the above comparison algorithms and obtain the experimental results are as follows.

6.3.1. Performance Evaluation on Attribute-incremental Three-way Concept Lattice Construction

We take datasets data2, data3, data4 and sushi_1 as our experimental formal contexts, and set the original formal contexts as $K_1 = (O, A_1, I_1)$. By adding the attributes and the binary relations I_2, their formal contexts are updated as $K = (O, A_1 \cup A_2, I_1 \cup I_2)$. Table 11 reports the experimental results on 4 formal contexts, where $|O|, |A_1|$ refer to the cardinality of the set of objects, and the set of attributes in original formal context, respectively. And, $|A_2|$ denotes the number of added attributes. $|AE^a|$ and $|OE^a|$ indicate the number of AE concepts and OE concepts. The running time T for generating three-way concepts under two algorithms are shown in Table 11.

Table 11: Experimental Results for Attribute-incremental Three-way Concept Lattice Construction

Dataset	$\|O\|$	$\|A_1\|$	$\|A_2\|$	$\|AE^a\|$	$\|OE^a\|$	T(ms)	
						Non-incremental Alg.	Incremental Alg.
data2	4	4	2	16	13	7.115	**5.635**
data3	8	5	3	39	42	20.894	**15.907**
data4	12	5	3	23	12	9.408	**7.563**
sushi_1	10	10	10	420	268	350.527	**324.954**

As can be seen from Table 11, our proposed incremental algorithm for three-way concept lattice generation is faster than the non-incremental algorithm (Yang's method) under various datasets. Specifically, our algorithm has around 18% reduction in running time compared to the non-incremental algorithm.

6.3.2. Performance Evaluation on Object-incremental Three-way Concept Lattice Construction

Similarly, we take datasets data1, data2, data3, data4 and sushi_2 as our experimental formal contexts, and set the original formal contexts as $K_1 = (O_1, A, I_1)$. By adding the objects and the binary relations I_2, their formal contexts are updated as $K = (O_1 \cup O_2, A, I_1 \cup I_2)$. Table 12 reports the results on 5 formal contexts, where $|O_1|, |A|$ refer to the cardinality of the set of objects, the set of attributes of original formal context, respectively. And, $|O_2|$ denotes the number of added objects. $|AE^o|$ and $|OE^o|$ indicate the number of AE concepts and OE concepts. The running time T for generating three-way concepts under two algorithms are shown in Table 12.

Table 12: Experimental Results for Object-incremental Three-way Concept Lattice Construction

Dataset	$\|O_1\|$	$\|O_2\|$	$\|A\|$	$\|AE^o\|$	$\|OE^o\|$	T(ms)	
						Non-incremental Alg.	Incremental Alg.
data1	2	2	5	11	8	3.674	**2.491**
data2	1	3	6	16	13	5.329	**4.400**
data3	6	2	8	39	42	23.365	**15.658**
data4	9	3	8	23	12	14.130	**9.028**
sushi_2	10	10	10	211	395	845.780	**671.186**

From Table 12, it is clearly to observe that our proposed incremental algorithm for the three-way concept lattice generation is faster than the non-incremental algorithm (Yang's method) under various datasets. Specifically, our algorithm has around 28% reduction in running time compared to non-incremental algorithm.

6.3.3. Performance Evaluation on Social-incremental Three-way Concept Lattice Construction

In order to validate the effectiveness of the social-incremental three-way concept lattice generation algorithm, we first extract 4 sub social networks from Karate social network and construct the corresponding formal contexts, i.e., karate_1, karate_2, karate_3 and karate_4, as our experimental formal contexts in this section. We set the original formal contexts as $K_1 = (V_1, V_1, I_1)$. By adding the users V_2 and the binary social relationships I_2, their formal contexts are updated as $K = (V_1 \cup V_2, V_1 \cup V_2, I_1 \cup I_2)$. Table 13 reports the results on 4 formal contexts. $|AE^s|$ and $|OE^s|$ indicate the number of AE concepts and OE concepts. The running time T for generating three-way concepts under two algorithms are shown in Table 13.

Table 13: Experimental Results for Social-incremental Three-way Concept Lattice Construction

Dataset	$\|V_1\|$	$\|V_2\|$	$\|AE^s\|$	$\|OE^s\|$	T(h:m:s.ms)	
					Non-incremental Alg.	Incremental Alg.
karate_1	10	4	374	374	0:00:00.700163	**0:00:00.663997**
karate_2	10	6	1388	1388	0:00:28.570954	**0:00:27.992967**
karate_3	11	5	11853	11853	0:15:24:366909	**0:15:03.199154**
karate_4	15	4	5409	5409	0:39:47.988519	**0:36:02.947095**

The above experimental results illustrate that our proposed incremental algorithm for the three-way concept lattice generation is faster than the non-incremental algorithm (Yang's method) under various social network datasets. Specifically, our algorithm has around a 4.7% reduction in running time compared to the non-incremental algorithm.

6.3.4. Performance Evaluation on Optimized Social-incremental Three-way Concept Lattice Construction

In this section, we will evaluate the optimized social-incremental three-way concept lattice construction for a given social network. According to Theorem 6, we can only generate the AE concepts or OE concepts, then the corresponding OE concepts or AE concepts can be obtained by exchanging the extent and intent of them. Intuitively, the total running time for generating AE concepts as well as OE concepts can be significantly reduced.

We evaluate the performance of our optimized algorithm by using the same datasets as presented in the previous section. Table 14 reports the results on 4 formal contexts. $|AE^s|$ and $|OE^s|$ indicate the number of AE concepts and OE concepts. The running time T for generating three-way concepts under two algorithms are shown in Table 14.

Table 14: Experimental Results for Social-incremental Three-way Concept Lattice Construction

Dataset	$\|V_1\|$	$\|V_2\|$	$\|AE^s\|$	$\|OE^s\|$	T(h:m:s.ms)	
					Non-incremental Alg.	Incremental Alg.
karate_1	10	4	374	374	0:00:00.620862	**0:00:00.576490**
karate_2	10	6	1388	1388	0:00:25.872786	**0:00:24.510769**
karate_3	11	5	11853	11853	0:14:55:448044	**0:13:44.989466**
karate_4	15	4	5409	5409	0:36:11.945512	**0:34:30.655896**

The above experimental results show that our proposed optimized incremental algorithm for the three-way concept lattice generation is faster than the optimized non-incremental algorithm (optimized Yang's method) under various social network datasets. Specifically, our algorithm has around 6.2% reduction in running time compared to non-incremental algorithm. Clearly, our proposed optimized algorithms are better than the non-incremental and incremental algorithms for the construction of the three-way concept lattice of a social network.

6.4. Case Study

To illustrate the usefulness of our research, a case study about the detection of polarized groups of synonyms and antonyms in a special signed social network, i.e., AdjWordNet [4], is conducted. The AdjWordNet signed social network contains 117,000 words for synonyms and antonyms. From the network point of view, there exist the positive edges among synonyms and the negative edges among antonyms, and there exists no edge between unrelated words. From the 3WCA point of view, the synonyms and antonyms are viewed as "jointly possessed" and "jointly not possessed" sets once the initial set of Adjwords is given. Therefore, detecting the polarized groups of synonyms and antonyms is equivalent to finding the AE-concepts or OE-concepts. Specifically, the extent and intent of the AE-concept/OE-concept are corresponding to the polarized groups of synonyms and antonyms.

In this case study, for simplicity, we select 25 words from AdjWordNet as shown in Table 15.

Table 15: Partial Words from AdjWordNet

raw	rough	rude	relaxing	refined
interior	smooth	assumed	false	intimate
uneasy	ungratified	restful	reposeful	unsatisfied
outer	existent	veridical	actual	sour
participating	undynamic	active	undermentioned	treasured

According to Algorithm 3 and Algorithm 4, we can easily detect 3 polarized groups as shown in Table 16. Obviously, each polarized group contains two parts G_L and G_R. The words within G_L or G_R have similar meanings while the words between G_L and G_R are opposite. This case study verifies that three-way concept analysis for knowledge discovery from social networks can be applied in the applications to find synonym and antonym polarized groups on dictionary

[4]http://wordnet.princeton.edu

Table 16: The polarized groups of AdjWordNet

	G_L	G_R
♯1 polarized group	refined, smooth	raw, rough, rude
♯2 polarized group	assumed, false, sour	existent, veridical, actual
♯3 polarized group	relaxing, restful, reposeful	uneasy, ungratified, unsatisfied

data.

7. Conclusions

The construction of three-way concept lattice is a quite new and critical research issue in the field of 3WCA. This paper attempts to present an efficient algorithm on the generation of three-way concept lattice for knowledge discovery in social networks. To be specific, we incrementally generate the concept lattices of original formal context and its complement context, and further construct three-way concept lattices for attribute/object-incremental formal contexts by the virtue of the extended attribute/object-incremental AE/OE composite operators. Regarding a social network, this paper develops two incremental three-way concept lattice generation algorithms, called SNS-AE and SNS-OE. By considering the symmetry of the formal context constructed from a social network, an optimized technique for rapid constructing three-way concept lattice of a social network is further provided. Experimental results demonstrate that our algorithm can greatly reduce the redundant computation, which saves the running time and improves the efficiency of the three-way concept lattice construction. Additionally, a case study on detecting synonym and antonym polarized groups is conducted for validating the potential usefulness of this research. In the future, we will utilize the proposed approach to conducting the guidance of public opinions over social media.

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