A CRYSTALLINE INCARNATION OF BERTHELOT'S CONJECTURE AND KÜNNETH FORMULA FOR ISOCRYSTALS

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ABSTRACT. Berthelot's conjecture predicts that under a proper and smooth morphism of schemes in characteristic p, the higher direct images of an overconvergent F-isocrystal are overconvergent F-isocrystals. In this paper we prove that this is true for crystals up to isogeny. As an application we prove the Künneth formula for the crystalline fundamental group scheme.

INTRODUCTION

One of the expectations for a good cohomology theory for schemes is that there exists a pushforward functor f_* associated to a proper and smooth morphism $f \colon X \to S$ such that $\mathbf{R}^q f_*$ (for $q \geqslant 0$) sends a coefficient for the cohomology on X to a coefficient for the cohomology on S. This expectation is reality in various contexts.

Let k be a field of characteristic 0, $f \colon X \to S$ be a proper and smooth morphism between two k-varieties, and let $\mathcal{E}$ be a module with integrable connection on X; then the relative de Rham cohomology $\mathbf{R}^q f_*(\mathcal{E})$ comes endowed with an integrable connection, the Gauss–Manin connection (see for example [Kat70], [Har75]), so that it is indeed a coefficient for the cohomology on S.

When k is a field of characteristic $p > 0$, $f \colon X \to S$ is a proper and smooth morphism between two k-varieties, and $\mathcal{E}$ an ℓ-adic lisse sheaf ($\ell \neq p$), then $\mathbf{R}^q f_*(\mathcal{E})$ is an ℓ-adic lisse sheaf ([Del77]).

As for the case $\ell = p$, the expectation for an overconvergent F-isocrystal $\mathcal{E}$ is known as Berthelot's conjecture ([Ber86, (4.3)], [Tsu03]). The conjecture is still open, but several results have been obtained in the last years ([Tsu03], [Shi08a], [Shi08b], [Shi08c] [Car15], [Ete12], . . .). For a survey about this conjecture see [Laz16].

As remarked by Ladza in [Laz16], Berthelot's conjecture can have many incarnations, depending on what kind of coefficients and pushforward one considers. In this paper we deal with a crystalline incarnation of Berthelot's conjecture, working with the category of crystals up to isogeny on the crystalline site.

Let k be a perfect field of characteristic $p > 0$, let W be the ring of Witt vectors of k and let K be the fraction field of W. Set $\mathbf{W} \coloneqq \operatorname{Spec} W$. For a k-scheme X, Berthelot defined the crystalline site $(X/\mathbf{W})_{\mathrm{crys}}$ and the structure sheaf $\mathcal{O}_{X/\mathbf{W}}$. He considered also the category of crystals of finite presentation, denoted by $\mathrm{Crys}(X/\mathbf{W})$, defined as the category of certain sheaves of $\mathcal{O}_{X/\mathbf{W}}$-modules on $(X/\mathbf{W})_{\mathrm{crys}}$ which verify a rigidity condition. The category of

isocrystals $I_{\mathrm{crys}}(X/\mathbf{W})$ is the category $\mathrm{Crys}(X/\mathbf{W})$ up to isogeny, *i.e.* the category whose objects are exactly those in $\mathrm{Crys}(X/\mathbf{W})$ and whose morphisms are obtained inverting the multiplication by p. Thus we have a natural functor

$$\mathrm{Crys}(X/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(X/\mathbf{W})$$

which is the identity on objects. To distinguish among objects in $\mathrm{Crys}(X/\mathbf{W})$ and in $I_{\mathrm{crys}}(X/\mathbf{W})$ we denote by $K \otimes E$ the image of $E \in \mathrm{Crys}(X/\mathbf{W})$ under the above functor, and we say that E is a lattice for the isocrystal $\mathcal{E}$ if $K \otimes E \cong \mathcal{E}$.

Given a proper and smooth morphism of k-schemes $f \colon X \to S$ and a crystal E on the crystalline site $(X/\mathbf{W})_{\mathrm{crys}}$, there is a morphism of ringed topoi

$$f_{\mathrm{crys}*} \colon ((X/\mathbf{W})_{\mathrm{crys}}^{\sim}, \mathcal{O}_{X/\mathbf{W}}) \longrightarrow ((S/\mathbf{W})_{\mathrm{crys}}^{\sim}, \mathcal{O}_{S/\mathbf{W}})$$

and its derived version $\mathbf{R}f_{\mathrm{crys}*}$. By functoriality the functors $f_{\mathrm{crys}*}$ and $\mathbf{R}f_{\mathrm{crys}*}$ induce corresponding functors in the isogeny categories, so if $\mathcal{E}$ is an isocrystal in $I_{\mathrm{crys}}(X/\mathbf{W})$, then for all $q \geqslant 0$, we get an object $\mathbf{R}^q f_{\mathrm{crys}*}(\mathcal{E})$ in the isogeny category of $\mathcal{O}_{S/\mathbf{W}}$-modules. The main result of the paper is that, if S is smooth, $\mathbf{R}^q f_{\mathrm{crys}*}(\mathcal{E})$ has a richer structure, indeed it is an isocrystal, *i.e.* an object of $I_{\mathrm{crys}}(S/\mathbf{W})$.

Theorem I. *Let $f \colon X \to S$ be a smooth and proper morphism of smooth quasi-compact k-schemes and let $\mathcal{E}$ be an isocrystal in $I_{\mathrm{crys}}(X/\mathbf{W})$. Then, for all $q \geqslant 0$, $\mathbf{R}^q f_{\mathrm{crys}*}(\mathcal{E})$ is an isocrystal in $I_{\mathrm{crys}}(S/\mathbf{W})$.*

The above theorem generalises a result of Morrow, which proved the above theorem for the trivial isocrystal ([Mor19]). Our proof follows the lines of his proof: we explain here the main ideas.

First, using Zariski descent, one reduces to the case in which $S = \mathrm{Spec}\, A$ is affine; now A can be lifted to a p-adically complete flat W-algebra $\mathcal{A}$, such that $\mathcal{A}_n = \mathcal{A}/p^n\mathcal{A}$ is a smooth $W_n \coloneqq W/p^n W$ algebra for all $n \geqslant 1$. Set $\mathbf{W}_n \coloneqq \mathrm{Spec}\, W_n$. Since X is smooth over k, there exists a p-torsion free crystal E on X which is a lattice for $\mathcal{E}$, then one has a Gauss–Manin crystal at one's disposal. Indeed, given a p-torsion free crystal E on $\mathrm{Crys}(X/\mathbf{W}_n)_{\mathrm{crys}}$, one can construct a natural HPD-stratification on the finitely generated $\mathcal{A}$-module $\varprojlim_n (\mathbf{R}^q f_{\mathrm{crys}*}(E))_{\mathrm{Spec}\, \mathcal{A}_n}$ over W. Using the fact that $\mathcal{A}_n$ is W_n-smooth for all $n \in \mathbb{N}^+$, the HPD-stratification on $\varprojlim_n (\mathbf{R}^q f_{\mathrm{crys}*}(E))_{\mathrm{Spec}\, \mathcal{A}_n}$ is equivalent to a crystal $E_{X/\mathcal{A}}^q$ on $(S/\mathbf{W})_{\mathrm{crys}}$ – the Gauss–Manin crystal. Moreover, there is a natural map

$$E_{X/\mathcal{A}}^q \longrightarrow \mathbf{R}^q f_{\mathrm{crys}*}(E)$$

of sheaves on $(S/\mathbf{W})_{\mathrm{crys}}$ which turns out to be an isomorphism after inverting p. This shows that $\mathbf{R}^q f_{\mathrm{crys}*}(E) \otimes K$ (see Definition 2.5) is in $I_{\mathrm{crys}}(S/\mathbf{W})$.

A key ingredient of the above proof is the Berthelot's base change theorem for crystalline cohomology [BO78, Theorem 7.8] which only holds for flat crystals. In Morrow's paper the trivial isocrystal $K \otimes \mathcal{O}_{X/\mathbf{W}}$ admits a lattice, e.g. $\mathcal{O}_{X/\mathbf{W}}$, which is flat, that is, $- \otimes \mathcal{O}_{X/\mathbf{W}}$ is exact in the ringed topos $((X/\mathbf{W})_{\mathrm{crys}}^{\sim}, \mathcal{O}_{X/\mathbf{W}})$. But in general the existence of a flat lattice is not known (see for example [ES15]). This becomes a central theme of this paper: in §2 we develop a crystalline base change theory for crystals that may not be flat; instead of requiring that the base change map is an isomorphism we require that it is an isomorphism after inverting p. The proof follows closely the original proof of Berthelot's base change theorem, namely it uses cohomological descent to reduce the problem to the affine case and then work with the quasi-nilpotent connections and the corresponding de Rham complex. But the argument from

there on has to be changed due to the lack of the flatness condition. We have to use a spectral sequence to find a uniformly large N so that p^N kills both the kernel and the cokernel of the base change map. Shiho also studied in [Shi08a] isocrystals which do not necessarily admit flat lattices, but his results do not fit our situation.

We prove several variants of base change isomorphisms (see Theorem 2.7, Theorem 2.14 and Theorem 2.21). Here we mention the following.

Theorem II. *Consider a cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\ h\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{\ v\ } & S \end{array}$$

of quasi-compact k-schemes with f smooth and proper. Let $E \in \mathrm{Crys}(X/\mathbf{W})$ and assume S is smooth over k. Then for all $n \in \mathbb{N}$ the canonical map

$$v_{\mathrm{crys}}^* \mathbf{R}^n f_{\mathrm{crys}*}(E) \longrightarrow \mathbf{R}^n f'_{\mathrm{crys}*}(h_{\mathrm{crys}}^* E)$$

is an isomorphism of isocrystals in $I_{\mathrm{crys}}(S'/\mathbf{W})$.

A recent result proven by Xu ([Xu19]) deals with a convergent incarnation of Berthelot's conjecture: he proves that the derived pushforward functor preserves convergent isocrystals, in the context of the convergent topos defined by Ogus [Ogu84]. Let $f \colon X \to S$ be a proper and smooth map as above; Xu considers a convergent isocrystal $\mathcal{E} \in I_{\mathrm{conv}}(X/\mathbf{W})$, together with $\mathbf{R}^q f_{\mathrm{conv}*}(\mathcal{E})$; he uses Shiho's base change [Shi08a, Theorem 1.19] to show that $\mathbf{R}^q f_{\mathrm{conv}*}(\mathcal{E})$ is a p-adically convergent isocrystal. Then he develops a strong version of Frobenius descent which allows him to prove that $\mathbf{R}^q f_{\mathrm{conv}*}(\mathcal{E})$ is indeed a convergent isocrystal on S using Dwork's trick. He then proceeds to remove the smoothness hypothesis for the base S. It would be interesting to know if even in our setting one can remove the smoothness hypothesis. In any case, when S is smooth over k, the category of convergent isocrystals is a full subcategory of the category of isocrystals [Ogu84, Theorem 0.7.2]: there is a fully faithful functor $\iota \colon I_{\mathrm{conv}}(S/\mathbf{W}) \to I_{\mathrm{crys}}(S/\mathbf{W})$ (and likewise for $X/\mathbf{W}$). Our result and Xu's result are independent, in the sense that none of the two implies the other. On the other hand they are compatible in the sense that $\iota(\mathbf{R}^q f_{\mathrm{conv}*}\mathcal{E}) \cong \mathbf{R}^q f_{\mathrm{crys}*}(\iota(\mathcal{E}))$ (see Remark 3.2 and the discussion at the end of [Xu19, Section 1.9]).

We remark that if X is a smooth, quasi-compact and connected k-scheme, then the category $I_{\mathrm{crys}}(X/\mathbf{W})$ is a Tannakian category, hence when X has a k-rational point x, one can define the crystalline fundamental group $\pi_1^{\mathrm{crys}}(X/\mathbf{W}, x)$[1]. This group scheme has recently been studied deeply: it has been conjectured by de Jong that for a connected projective variety over an algebraically closed field in characteristic $p > 0$ with trivial étale fundamental group, there are no non-constant isocrystals. The conjecture is still open but several results have been obtained ([Kat18], [ES18], [ES19], [Shi14]). Moreover, we also remark that the pro-unipotent completion of $\pi_1^{\mathrm{crys}}(X/\mathbf{W}, x)$ is considered to be the crystalline realisation of the motivic fundamental group and it has been studied by Shiho in the more general context of log geometry ([Shi00], [Shi02]).

As a consequence of our main result we obtain the Künneth formula for the crystalline fundamental group.

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[1] While the authors were revising this paper, a preprint with a new approach to crystals appeared [Dri18].

Theorem III. *Let k be a perfect field of characteristic $p > 0$, let X and Y be smooth connected k-schemes with Y proper and suppose that $x \in X(k)$, $y \in Y(k)$ are two rational points. Then the canonical morphism between the crystalline fundamental groups*

$$\pi_1^{\mathrm{crys}}(X \times_k Y/\mathbf{W}, (x, y)) \longrightarrow \pi_1^{\mathrm{crys}}(X/\mathbf{W}, x) \times_K \pi_1^{\mathrm{crys}}(Y/\mathbf{W}, y)$$

is an isomorphism.

By the Eckman–Hilton argument we also get the following.

Theorem IV. *Let A be an abelian variety over a perfect field k of positive characteristic. Then $\pi_1^{\mathrm{crys}}(A/\mathbf{W}, 0)$ is an abelian group scheme.*

Analogous results for other fundamental groups have been obtained by Battiston [Bat16] and D'Addezio [DAd21].

The Künneth formula, as in the étale case, is a consequence of the homotopy exact sequence for the crystalline fundamental group, but our argument does not use the homotopy exact sequence. It is an open problem to show the existence of a homotopy exact sequence for the crystalline fundamental group, which has been shown is several other contexts recently ([Zha14], [San15], [LP17], [DS18], ...).

The content of each section is as follows. In §1 we define the crystalline fundamental group; to do so we prove that the category of isocrystals on a smooth, quasi-compact and connected k-scheme is Tannakian. In §2 we prove several generalisations of the base change for crystalline cohomology. We consider a PD-scheme $\mathbf{S} = (\mathbf{S}, I, \gamma)$ over $\mathbf{W}$, requiring that $p \in I$, we let $S = V(I)$, and we consider an S-scheme X. We denote by $g \colon X \longrightarrow \mathbf{S}$ the structure map, and by $g_{X/\mathbf{S}}$ the morphism of topoi $g \circ u_{X/\mathbf{S}} \colon (X/\mathbf{S})^{\sim}_{\mathrm{crys}} \to \mathbf{S}^{\sim}_{\mathrm{Zar}}$. In §2.1 we prove the generalised base change theorem for $g_{X/\mathbf{S}}$ when p is nilpotent in $\mathcal{O}_{\mathbf{S}}$; this includes, as a special case, the classical Berthelot's base change theorem for crystalline cohomology. In §2.2 we consider the case in which $\mathbf{S}$ is affine. In this case we consider the functor $\varprojlim_n \Gamma \circ g_{X/(\mathbf{S}/p^n)*}$; we prove a base change theorem for this functor. In the last part of section §2 we consider a proper and smooth morphism of smooth k-schemes $f \colon X \to S$ as above, and we prove a base change theorem for the functor $f_{\mathrm{crys}*}$. In section §3, we get our main result about Berthelot's conjecture for isocrystals. In §4 we prove the Künneth formula for the crystalline fundamental group.

NOTATION

An ideal I in a ring A is called nil if all elements of I are nilpotent (this is called a locally nilpotent ideal in [Sta19]). We will often use that smooth affine maps have the lifting property for nil ideals (see [Sta19, Tag 07K4]).

1. TANNAKIAN CATEGORIES OF CONNECTIONS AND CRYSTALLINE FUNDAMENTAL GROUP

The goal of this section is to define the crystalline fundamental group. Concretely this means introducing the category of isocrystals and proving that it is a Tannakian category. This is done in essentially four steps.

(1) Reduce the problem to the affine case and compare isocrystals with topologically quasi-nilpotent connections.
(2) Interpret topologically quasi-nilpotent connections as a particular case of connections with respect to a quotient of the sheaf of algebraic differentials.
(3) Show that those connections correspond to differential modules for an associated differential ring.
(4) Study differential rings and differential modules following [Ked12].

This program is done in the reverse order, so that definitions come first.

1.1. **Differential rings.** We start by introducing some general definitions as in [Ked12].

Definition 1.1. A differential ring is a pair (A, Δ_A) where A is a ring and Δ_A is a Lie algebra together with an A-module structure and a Lie algebra homomorphism $\iota \colon \Delta_A \longrightarrow \mathrm{Der}(A/\mathbb{Z}) = \mathrm{Hom}(\Omega_{A/\mathbb{Z}}, A)$ which is A-linear. We moreover ask that the following property holds:

$$(1.1) \qquad [D_1, aD_2] = a[D_1, D_2] + D_1(a)D_2 \text{ for all } a \in A, \ D_1, D_2 \in \Delta_A.$$

Notice that the above equation is automatic if ι is injective. If $X = \mathrm{Spec}\, A$ we sometimes write (X, Δ_A) instead of (A, Δ_A).

If B is a ring, a differential B-algebra is a differential ring (A, Δ_A) such that A is a B-algebra and the map $\iota \colon \Delta_A \longrightarrow \mathrm{Der}(A/\mathbb{Z})$ has image in $\mathrm{Der}(A/B) = \mathrm{Hom}(\Omega_{A/B}, A)$.

A differential (A, Δ_A)-module (or simply differential A-module when Δ_A is clear from the context) is a pair (M, ∇) where M is an A-module and

$$\nabla \colon \Delta_A \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$$

is a morphism of Lie algebras which is A-linear and satisfies the Leibniz rule, *i.e.*

$$(1.2) \qquad D(am) = D(a)m + aD(m) \text{ for all } D \in \Delta_A, \ a \in A, \ m \in M.$$

Above and in what follows we write $D(m)$ instead of $\nabla(D)(m)$.

We denote by $\mathrm{Diff}(A, \Delta_A)$ or simply $\mathrm{Diff}(A)$ the category of differential A-modules which are of finite presentation (as A-modules).

Remark 1.1. Let (A, Δ_A) be a differential ring and (E, ∇_E) and (F, ∇_F) be differential A-modules. Their tensor product is given by the A-module $E \otimes_A F$ and the map

$$\Delta_A \xrightarrow{\;\nabla_{E \otimes_A F}\;} \mathrm{End}_{\mathbb{Z}}(E \otimes_A F)$$
$$D \longmapsto (e \otimes f \mapsto D(e) \otimes f + e \otimes D(f)).$$

Their Hom is instead given by the A-module $\mathrm{Hom}_A(E, F)$ and the map

$$\Delta_A \xrightarrow{\ \nabla_{\mathrm{Hom}_A(E,F)}\ } \mathrm{End}_{\mathbb{Z}}(\mathrm{Hom}_A(E, F))$$
$$D \longmapsto (\phi \mapsto \nabla_F(D) \circ \phi - \phi \circ \nabla_E(D)).$$

See also [Ked12, Def. 1.1.3]. It is easy to see that the category of differential A-modules is symmetric monoidal with unit (A, ∇) where $\nabla \colon \Delta_A \longrightarrow \mathrm{Der}(A/\mathbb{Z}) \subseteq \mathrm{End}_{\mathbb{Z}}(A)$ is the canonical map ι. Moreover the Hom just defined is an internal Hom in the category of differential A-modules, that is if (G, ∇_G) is another differential A-module then the canonical isomorphism

$$\mathrm{Hom}(E \otimes F, G) \longrightarrow \mathrm{Hom}(E, \mathrm{Hom}(F, G))$$

is a map of differential A-modules and preserves the subsets of morphisms of differential A-modules.

If $E \xrightarrow{\phi} F$ is a map of differential A-modules then kernel and cokernels are naturally differential A-modules.

From the discussion above we can conclude that

Proposition 1.2. *If (A, Δ_A) is a differential ring then the category of differential A-modules is symmetric monoidal, abelian and has internal homomorphisms. The same is true for $\mathrm{Diff}(A)$ if A is Noetherian.*

Definition 1.2. Let R be a ring. Let $\mathscr{C}$ be an R-linear category and let R' be an R-algebra. We denote by $\mathscr{C} \otimes_R R'$ the category whose objects are exactly those of $\mathscr{C}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathscr{C} \otimes_R R'}(M, N) := \mathrm{Hom}_{\mathscr{C}}(M, N) \otimes_R R'$$

for any $M, N \in \mathscr{C}$. There is a natural functor $F \colon \mathscr{C} \longrightarrow \mathscr{C} \otimes_R R'$ which is the identity on objects and which is the natural base extension on morphisms. For any object $M \in \mathscr{C}$, in order to emphasize that $F(M)$ is in $\mathscr{C} \otimes_R R'$, we write $M \otimes_R R'$ for $F(M)$.

If $\mathscr{C}$ is symmetric monoidal then also $\mathscr{C} \otimes_R R'$ is symmetric monoidal in a natural way.

Lemma 1.3. *Let R be a ring. Let $\mathscr{C}$ be an R-linear abelian category, and let S be a multiplicative subset of R. Then $\mathscr{C} \otimes_R S^{-1}R$ is also abelian and the natural functor $F \colon \mathscr{C} \longrightarrow \mathscr{C} \otimes_R S^{-1}R$ is exact. Moreover if $\mathscr{C} \longrightarrow \mathscr{D}$ is an R-linear exact functor to an $S^{-1}R$-linear category, then the induced functor $\mathscr{C} \otimes S^{-1}R \longrightarrow \mathscr{D}$ is also exact.*

If $\mathscr{C}$ is symmetric monoidal (with internal homomorphisms) then $F \colon \mathscr{C} \longrightarrow \mathscr{C} \otimes_R S^{-1}R$ is a tensor functor (and preserves internal homomorphisms).

Proof. Set $R' = S^{-1}R$. Since up to isomorphisms every morphism in $\mathscr{C} \otimes_R R'$ comes from $\mathscr{C}$, in order to show the exactness of F it is enough to show that F preserves kernel and cokernel. Let's look at kernel for example. Let $f \colon A \longrightarrow B$ be a morphism in $\mathscr{C}$. Then $\mathrm{Ker}(f)$ is the object in $\mathscr{C}$ which represents the functor that sends any $T \in \mathscr{C}$ to

$$\mathrm{Ker}(\mathrm{Hom}_{\mathscr{C}}(T, A) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(T, B)).$$

By the flatness of $R \longrightarrow R'$ we have the exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(T, \mathrm{Ker}(f)) \otimes R' & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(T, A) \otimes R' & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(T, B) \otimes R' \\
 & & \| & & \| & & \| \\
0 & \longrightarrow & \mathrm{Hom}(T \otimes R', \mathrm{Ker}(f) \otimes R') & \longrightarrow & \mathrm{Hom}(T \otimes R', A \otimes R') & \longrightarrow & \mathrm{Hom}(T \otimes R', B \otimes R')
\end{array}$$

for each $T \otimes_R R' \in \mathcal{C} \otimes_R R'$. Thus $\mathrm{Ker}(f) \otimes_R R'$ represents the kernel of $f \otimes_R R'$.

Now consider an exact linear functor $G \colon \mathcal{C} \longrightarrow \mathcal{D}$ as in the statement and call $G' \colon \mathcal{C} \otimes R' \longrightarrow \mathcal{D}$ the induced functor. Let $A'_\bullet$ be a bounded exact complex in $\mathcal{C} \otimes R'$. In order to show that $G'(A'_\bullet)$ is exact we can multiply each degree map by elements of S. In particular we can assume that all those maps are defined in $\mathcal{C}$ and, multiplying again by elements of S, that they define a complex $A_\bullet$ such that $F(A_\bullet) = A'_\bullet$. Using the exactness of F and G we have

$$F(\mathrm{H}^i(A_\bullet)) \simeq \mathrm{H}^i(A'_\bullet) = 0 \implies 0 = G'F(\mathrm{H}^i(A_\bullet)) = G(\mathrm{H}^i(A_\bullet)) \simeq \mathrm{H}^i(G(A_\bullet)) \simeq \mathrm{H}^i(G'(A'_\bullet)).$$

The last statement follows from a direct check. □

Remark 1.4. Let (A, Δ_A) be a differential ring and S be a multiplicative subset of A. Then $(S^{-1}A, S^{-1}\Delta_A)$ has a natural structure of differential ring. Moreover if (M, ∇) is a differential A-module then $S^{-1}M$ is a differential $S^{-1}A$-module in a natural way.

The condition (1.1) in Definition 1.1 forces the definition of the bracket in $S^{-1}\Delta_A$ as well as in $\mathrm{Der}(S^{-1}A) = \mathrm{Hom}(S^{-1}\Omega_{A/\mathbb{Z}}, S^{-1}A)$.

Also, the Leibniz rule (1.2) in Definition 1.1 forces the definition of the map $\nabla \colon S^{-1}\Delta_A \longrightarrow \mathrm{End}(S^{-1}M)$: this is the unique $S^{-1}A$-linear map such that

$$D(m/s) = D(m)/s - D(s)m/s^2.$$

Indeed everything is well-defined ([Ked12, Rem. 1.1.5]).

Lemma 1.5. *Let R be a ring and let (A, Δ_A) be a differential R-algebra such that Δ_A is a finitely generated A-module and let S be a multiplicative subset of R. Then $\mathrm{Diff}(A)$ is an R-linear category and the functor*

$$\mathrm{Diff}(A) \otimes_R S^{-1}R \longrightarrow \mathrm{Diff}(S^{-1}A)$$

is a fully faithful tensor functor. If A is Noetherian then the above functor is also exact and preserves internal homomorphisms.

Proof. Set $R' = S^{-1}R$ and $A' = S^{-1}A$. The fact that the functor is a tensor functor follows from construction. For the full faithfulness, given two differential A-modules (M, ∇_M) and (N, ∇_N) we want to show that the natural map

$$\phi \colon \mathrm{Hom}_{\mathrm{Diff}(A)}(M, N) \otimes_R R' \longrightarrow \mathrm{Hom}_{\mathrm{Diff}(A')}(M', N')$$

is an isomorphism, where $M' \coloneqq M \otimes_R R'$ and $N' \coloneqq N \otimes_R R'$ are thought of as differential A'-modules. The canonical map

$$\mathrm{Hom}_A(M, N) \otimes_R R' \longrightarrow \mathrm{Hom}_{A'}(M', N')$$

is an isomorphism. Thus we have to show that if $f \in \mathrm{Hom}_{A'}(M', N')$ is a morphism which is compatible with the $\nabla_{*'}$, then it comes from

$$\mathrm{Hom}_{\mathrm{Diff}(A)}(M, N) \otimes_R R' \subseteq \mathrm{Hom}_A(M, N) \otimes_R R' = \mathrm{Hom}_{A'}(M', N').$$

Replacing f by sf for some $s \in S$ we may assume that f comes from $\mathrm{Hom}_A(M, N)$ and we will still use f to denote the lift of f in $\mathrm{Hom}_A(M, N)$. We must show that there exists $s \in S$ such that sf preserves the ∇_*. For $D \in \Delta_A$ and $m \in M$ set

$$g(D, m) = f(D(m)) - D(f(m)) \in N.$$

Since $\nabla_N(D)$ is R-linear we look for an $s \in S$ such that $sg(D, m) = 0$ for all D and m. By hypothesis $g(D, m) = 0$ in $N' = S^{-1}N$. Thus it is enough to notice that, by the Leibniz rule,

$g(D, m)$ is a linear combination of the values of g on generators of Δ_A and M, which are finitely many.

Now assume that A is Noetherian. Then the functor in the statement preserves internal homomorphisms because of how they are constructed and because all modules considered are finitely generated. The exactness follows from Lemma 1.3. $\qquad\square$

Definition 1.3. [Ked12, Def. 1.2.1] A differential ring (A, Δ_A) is called locally simple if for all prime ideals P the differential local ring A_P is simple, *i.e.* A_P contains no proper non zero ideals stable under the action of $(\Delta_A)_P$.

Proposition 1.6. [Ked12, Prop. 1.2.6] *Let A be a locally simple differential ring. If (E, ∇) is a differential A-module of finite presentation then E is locally free as an A-module.*

Theorem 1.7. *Let (A, Δ_A) be a Noetherian locally simple differential ring such that $\mathrm{Spec}\,(A)$ is connected. Then $\mathrm{Diff}(A)$ is a Tannakian category over some subfield $L \subseteq A$. Let k be a field, let A be differential k-algebra and $x \colon \mathrm{Spec}\,k \longrightarrow \mathrm{Spec}\,A$ be a rational point, then $\mathrm{Diff}(A)$ with the fiber functor obtained via x^* is a neutral Tannakian category.*

Proof. By Proposition 1.2 we see that $\mathrm{Diff}(A)$ is an abelian, monoidal and symmetric category with internal homomorphisms. By Proposition 1.6 it is easy to see that $\mathrm{Diff}(A)$ is also rigid and that endomorphisms of the unit are either 0 or isomorphisms, that is $\mathrm{End}_{\mathrm{Diff}(A)}(A) \subseteq A$ is a field. If A is a differential k-algebra and x a k-rational point, then we have that

$$k \subseteq \mathrm{End}_{\mathrm{Diff}(A)}(A) \subseteq k.$$

Therefore $\mathrm{Diff}(A)$, with the fiber functor obtained via x^*, is a neutral Tannakian category.

$\qquad\square$

1.2. **Connections.** We now introduce a natural way of describing differential modules via connections.

Definition 1.4. Let $f \colon Y \longrightarrow S$ be a map of schemes and consider a surjective map of quasi-coherent sheaves $\Omega_{Y/S} \twoheadrightarrow \Omega$ such that the differential $\Omega_{Y/S} \longrightarrow \Omega^2_{Y/S}$ induces $d^1 \colon \Omega \longrightarrow \Omega^2 := \Omega \wedge \Omega$. An Ω-connection on an $\mathcal{O}_Y$-module M is an $f^{-1}\mathcal{O}_S$-linear map

$$\nabla \colon \quad M \longrightarrow M \otimes_{\mathcal{O}_Y} \Omega$$

of sheaves satisfying the Leibniz rule, *i.e.* $\nabla(am) = a\nabla(m) + m \otimes da$ for all sections a, m on $\mathcal{O}_Y$, M respectively over some open.

The connection ∇ induces a map,

$$\nabla^1 \colon M \otimes_{\mathcal{O}_Y} \Omega \to M \otimes_{\mathcal{O}_Y} \Omega^2$$

defined by $\nabla^1(m \otimes \omega) = \nabla(m) \wedge \omega + m \otimes d\omega$ for all sections m, ω on M, Ω respectively over some open, where $\nabla(m) \wedge \omega$ is the image of $\nabla(m) \otimes \omega$ under the canonical map

$$M \otimes_{\mathcal{O}_Y} \Omega \otimes_{\mathcal{O}_Y} \Omega \xrightarrow{\mathrm{id} \otimes \wedge} M \otimes_{\mathcal{O}_Y} \Omega^2.$$

The map ∇^1 is well-defined thanks to [Sta19, Tag 07I0].

The connection ∇ is called integrable if the composition

$$M \xrightarrow{\nabla} M \otimes_{\mathcal{O}_Y} \Omega \xrightarrow{\nabla^1} M \otimes_{\mathcal{O}_Y} \Omega^2$$

is zero.

We denote the category of integrable Ω-connections in finitely presented $\mathcal{O}_Y$-modules by $\mathrm{Conn}(Y/S, \Omega)$.

Lemma 1.8. *Let* $f \colon Y \longrightarrow S$ *be a map of schemes and let* $\mathcal{O}_Y \xrightarrow{d} \Omega^1_{Y/S} \xrightarrow{d^1} \Omega^2_{Y/S}$ *be the canonical differentials. Suppose that* $\phi_1, \phi_2 \in \mathrm{Der}(Y/S)$, *and let* φ_1, φ_2 *be the corresponding maps in* $\mathrm{Hom}_{\mathcal{O}_Y}(\Omega_{Y/S}, \mathcal{O}_Y)$. *We denote* $\varphi_1 \wedge \varphi_2$ *the map* $\Omega^2_{Y/S} \longrightarrow \mathcal{O}_Y$ *sending* $dx \wedge dy$ *to* $\phi_1(x)\phi_2(y) - \phi_2(x)\phi_1(y)$. *Then* $[\phi_1, \phi_2]$ *corresponds to the homomorphism*

$$\varphi_1 \circ d \circ \varphi_2 - \varphi_2 \circ d \circ \varphi_1 - (\varphi_1 \wedge \varphi_2) \circ d^1 \in \mathrm{Hom}_{\mathcal{O}_Y}(\Omega_{Y/S}, \mathcal{O}_Y).$$

Proof. If $\psi \colon \Omega_{Y/S} \longrightarrow \mathcal{O}_Y$ is the map in the statement, it clearly satisfies $\psi \circ d = [\phi_1, \phi_2]$. Thus one has to check that ψ is $\mathcal{O}_Y$ linear. This is a direct computation which we omit. $\square$

Corollary 1.9. *Let* $f \colon Y \longrightarrow S$ *be a map of schemes and* $\Omega_{Y/S} \longrightarrow \Omega$ *a quotient as in Definition 1.4. Then the subsheaf* $\mathcal{H}om_Y(\Omega, \mathcal{O}_Y) \subseteq \mathcal{D}er(Y/S)$ *is a subsheaf of Lie algebras.*

Lemma 1.10. *Let* $f \colon Y := \mathrm{Spec}\, A \longrightarrow S := \mathrm{Spec}\, R$ *be a map of affine schemes and* $\Omega_{Y/S} \longrightarrow \Omega$ *a quotient as in Definition 1.4. Set* $\Delta_A := \mathrm{Hom}_{\mathcal{O}_Y}(\Omega, \mathcal{O}_Y)$. *Assume moreover that* Ω *is locally free of finite type. Then* (A, Δ_A) *is a differential ring over* R. *Moreover the functor*

$$F \colon \ \mathrm{Conn}(Y/S, \Omega) \xrightarrow{\hspace{2.5cm}} \mathrm{Diff}(A)$$
$$(\tilde{M}, \nabla_{\tilde{M}}) \longmapsto (M, \nabla_M)$$

which sends the $\mathcal{O}_Y$*-module* $\tilde{M}$ *to the corresponding* A*-module* M *and the* Ω*-connection* $\nabla_{\tilde{M}}$ *to the map* ∇_M *defined on* $\phi \in \Delta_A$ *as* $\mathrm{H}^0(\tilde{M} \xrightarrow{\nabla_{\tilde{M}}} \tilde{M} \otimes \Omega \xrightarrow{\mathrm{id} \otimes \phi} \tilde{M})$ *is an equivalence of categories.*

Proof. First we prove that the above functor $F \colon (\tilde{M}, \nabla_{\tilde{M}}) \mapsto (M, \nabla_M)$ induces an equivalence between the category of quasi-coherent Ω-connections (not necessarily integrable) and the category of pairs (M, ∇_M), where M is an A-module and ∇_M is an A-linear map $\Delta_A \longrightarrow \mathrm{End}_{\mathbb{Z}}(M)$ satisfying the Leibniz rule (1.2) (not necessarily preserving the Lie bracket).

Full Faithfulness. The faithfulness is clear. Now suppose $\lambda \colon (M, \nabla_M) \longrightarrow (N, \nabla_N)$ is a morphism in the target category. Then we get directly a map $\lambda_{\mathcal{O}} \colon \tilde{M} \longrightarrow \tilde{N}$ between the corresponding $\mathcal{O}_Y$-modules, therefore we only have to check that $\lambda_{\mathcal{O}}$ is compatible with $\nabla_{\tilde{M}}$ and $\nabla_{\tilde{N}}$. We can check the compatibility Zariski locally. We can localize both the Ω-connections and the "not necessarily Lie-bracket preserving differential modules" (Remark 1.4). The functor F is compatible with the localization, thus we are reduced to the case when $\Omega = \mathcal{O}_Y^{\oplus n}$ for some $n \in \mathbb{N}$. Then the map $\nabla_{\tilde{M}}$ (resp. $\nabla_{\tilde{N}}$) becomes a map of the form $\tilde{M} \longrightarrow \prod_{i=1}^n \tilde{M}$ (resp. $\tilde{N} \longrightarrow \prod_{i=1}^n \tilde{N}$). Let p_i^M (resp. p_i^N) be the i-th projection $\prod_{i=1}^n \tilde{M} \longrightarrow \tilde{M}$ (resp. $\prod_{i=1}^n \tilde{N} \longrightarrow \tilde{N}$). Since λ is a map of differential modules, the map $\lambda_{\mathcal{O}}$ is compatible with $p_i^M \circ \nabla_{\tilde{M}}$ and $p_i^M \circ \nabla_{\tilde{N}}$. Therefore, $\lambda_{\mathcal{O}}$ is compatible with $\nabla_{\tilde{M}}$ and $\nabla_{\tilde{N}}$ by the universality of products of modules.

Essential Surjectivity. We cover $\mathrm{Spec}\, A$ by open affines $\mathrm{Spec}\, A_{f_i}$. Suppose Ω is free over each $\mathrm{Spec}\, A_{f_i}$, and suppose the claim holds when Ω is free. Given (M, ∇_M) we get the localizations (M_i, ∇_{M_i}) on each $\mathrm{Spec}\, A_{f_i}$ and the corresponding quasi-coherent connections $(\tilde{M}_i, \nabla_{\tilde{M}_i})$. Note that on $U_{ij} := \mathrm{Spec}\, A_{f_i} \bigcap \mathrm{Spec}\, A_{f_j}$ the sheaf Ω is also free, and by the full faithfulness there is a unique isomorphism

$$(\tilde{M}_i, \nabla_{\tilde{M}_i})|_{U_{ij}} \xrightarrow{\ \simeq\ } (\tilde{M}_j, \nabla_{\tilde{M}_j})|_{U_{ij}}.$$

This allows to glue all $(\tilde{M}_i, \nabla_{\tilde{M}_i})$ together to get $(\tilde{M}, \nabla_{\tilde{M}})$ which corresponds to (M, ∇_M). We are therefore reduced to the case when Ω is free. In this case, we can define $\nabla_{\tilde{M}} \colon \tilde{M} \longrightarrow$

$\tilde{M} \otimes \Omega$ as follows. Choose a basis $s_1, \cdots, s_n$ of Ω and let $f_1, \ldots, f_n$ be its dual basis. We set $\nabla_{\tilde{M}}(m) = \sum_{i=1}^n \nabla_M(f_i)(m) \otimes s_i$ for all $m \in M$.

Now we come back to compare $\mathrm{Conn}(Y/S, \Omega)$ and $\mathrm{Diff}(A)$. To show that the above equivalence induces the equivalence of these two categories we just have to notice the formula in [Kat70, p. 179, last paragraph, 1.0.5] and the fact that $\Delta_A \subseteq \mathrm{Der}(A/R)$ is a sub Lie algebra (Corollary 1.9). $\qquad\square$

1.3. **Crystalline site and crystals.** We recall here the general notion of small crystalline site and crystals on it. This was defined by Berthelot ([Ber74], [BO78]). We use as our main reference for this theory [Sta19, Tag 09PD] and [Sta19, Tag 07GI].

Definition 1.5.
- [Sta19, Tag 07GU] A divided power ring, or a PD-ring, is a triple (A, I, γ) where A is a ring, $I \subset A$ is an ideal, and $\gamma = (\gamma_n)_{n \geqslant 1}$ is a divided power structure on I. A homomorphism of divided power rings $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $\varphi : A \to B$ such that $\varphi(I) \subset J$ and such that $\delta_n(\varphi(x)) = \varphi(\gamma_n(x))$ for all $x \in I$ and $n \geqslant 1$.
- [Sta19, Tag 07GI]. A divided power scheme or a PD-scheme is the natural globalisation of a PD-ring.
- When we want to consider a homomorphism of PD-rings or PD-schemes, we will write it as a morphism of triples. On the other hand if R is a ring an R-PD-ring is a PD-ring (A, I, γ) where A is an R-algebra (and the same for PD-schemes over R).

We fix a prime number p.

Definition 1.6. [Sta19, Tag 07IF] Let $\mathbf{S} = (\mathbf{S}, I, \gamma)$ be a PD-scheme such that $\mathbf{S}$ is a $\mathbb{Z}_{(p)}$-scheme. Let X be an $S = V(I)$-scheme, and we assume moreover that $p \in I$, *i.e.* S is killed by p. An object of the crystalline site $(X/\mathbf{S})_{\mathrm{crys}}$ is given by a triple $(U, \mathbf{T}, \delta)$, where U is a Zariski open of X, $\mathbf{T}$ is an $\mathbf{S}$-scheme, $U \hookrightarrow \mathbf{T}$ is a thickening of $\mathbf{S}$-schemes defined by a nil ideal J and $(\mathbf{T}, J, \delta)$ is a PD-scheme over $(\mathbf{S}, I, \gamma)$. We often denote $(U, \mathbf{T}, \delta)$ simply by $\mathbf{T}$. Morphisms are defined in a natural way, and coverings are defined using the Zariski topology on $\mathbf{T}$. We consider the structure sheaf $\mathcal{O}_{X/\mathbf{S}}$, defined by $\mathcal{O}_{X/\mathbf{S}}(\mathbf{T}) \coloneqq \Gamma(\mathbf{T}, \mathcal{O}_{\mathbf{T}})$.

Remark 1.11. Let the notation be as in Definition 1.6 and set $\mathbf{S}_n \coloneqq \mathbf{S} \times_{\mathrm{Spec}\,(\mathbb{Z})} \mathrm{Spec}\,(\mathbb{Z}/p^n\mathbb{Z})$. Then the crystalline site $(X/\mathbf{S})_{\mathrm{crys}}$ is the direct limit of the sites $(X/\mathbf{S}_n)_{\mathrm{crys}}$.

Remark 1.12. We use [Sta19, Tag 09PD] as the main reference. Here we want to stress the compatibility of Definition 1.6 with more classical references.
   (1) If $\mathbf{S}$ is killed by a power of p, then the site defined in Definition 1.6 is the same as the crystalline site defined in [BO78, p. 5.1], with the hypothesis that $p \in I$.
   (2) When $\mathbf{S} = \mathrm{Spec}\,R$ is the spectrum of a Noetherian ring R which is complete for the I-adic topology, and if $p \in I$, then the crystalline site $(X/\mathbf{S})_{\mathrm{crys}}$ of Definition 1.6 is equivalent to the site $\mathrm{Cris}(X/\hat{S})$ defined in [BO78, p. 7.17] (with $P = I$), where $\hat{S} \coloneqq \mathrm{Spf}\,R$ for the I-adic topology.
   (3) Shiho, in [Shi08a], developed a theory of relative crystalline cohomology for log schemes. He supposes that $I = p$ and (here we are in the simplified case where all the log structures are trivial) he generalised the situation (2) to the case where $\mathbf{S}$ is a p-adic formal scheme separated and topologically of finite type over W.

Definition 1.7. An $\mathcal{O}_{X/\mathbf{S}}$-module E on the site $(X/\mathbf{S})_{\mathrm{crys}}$ is called a crystal if every morphism $\varphi : \mathbf{T} \to \mathbf{T}'$ in $(X/\mathbf{S})_{\mathrm{crys}}$ induces an isomorphism $\varphi^*(E_{\mathbf{T}'}) \to E_{\mathbf{T}}$, where we denote with $E_{\mathbf{T}'}$ (resp. $E_{\mathbf{T}}$) the Zariski sheaf on $\mathbf{T}'$ (resp. on $\mathbf{T}$) induced by E. A crystal is said to be of finite presentation if for every $\mathbf{T} \in (X/\mathbf{S})_{\mathrm{crys}}$ the $\mathcal{O}_{\mathbf{T}}$-module $E_{\mathbf{T}}$ is of finite presentation. The category of crystals of finite presentation on $(X/\mathbf{S})_{\mathrm{crys}}$ is denoted by $\mathrm{Crys}(X/\mathbf{S})$.

For any commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ h\ } & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbf{S}' & \xrightarrow{\ u\ } & \mathbf{S} \end{array}$$

where u is a PD-morphism, we obtain a morphism of ringed topoi $h_{\mathrm{crys}} = (h_{\mathrm{crys}}^*, h_{\mathrm{crys}*})$ ([Sta19, Tag 07KL]). It is known that if E is a crystal in $\mathrm{Crys}(X/\mathbf{S})$, then $h_{\mathrm{crys}}^*(E)$ is a crystal in $\mathrm{Crys}(X'/\mathbf{S}')$ ([Ber74, Corollaire 1.2.4] and Remark 1.11).

Setting 1.8. Let the hypothesis and notation be as in Definition 1.6. Suppose moreover that we have a commutative diagram

(1.3)
$$\begin{array}{ccc} X & \xhookrightarrow{\ i\ } & \mathbf{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\hspace{2cm}} & \mathbf{S} \end{array}$$

in which f is smooth and every scheme is affine: $X = \mathrm{Spec}\, C, \mathbf{X} = \mathrm{Spec}\, P, S = \mathrm{Spec}\, A/I, \mathbf{S} = \mathrm{Spec}\, A$. The map i in the above diagram is a closed immersion defined by an ideal $J \subseteq P$ (in particular $IP \subset J$). Let $\mathbf{D}_{P,\gamma} := \mathrm{Spec}\, D_{P,\gamma}$ be the PD-envelope of $i : X \hookrightarrow \mathbf{X}$ with respect to $(\mathbf{S}, I, \gamma)$ and let D be the p-adic completion of $D_{P,\gamma}$. Set $\mathbf{D} := \mathrm{Spec}\, D, A_n := A/p^n, \mathbf{S}_n := \mathrm{Spec}\, A_n, P_n := P \otimes_A A_n$ and $\mathbf{X}_n := \mathrm{Spec}\, P_n$. Let $\mathbf{D}_{P_n,\gamma} := \mathrm{Spec}\, D_{P_n,\gamma}$ be the PD-envelope of $X \hookrightarrow \mathbf{X}_n$ with respect to $(\mathbf{S}, I, \gamma)$. Thanks to [Sta19, Tag 07KG] we have $D = \varprojlim_{n \in \mathbb{N}} D_{P_n,\gamma}$ as PD-rings.

1.3.1. *Crystals and connections over complete PD-envelopes.* We denote by Ω_D the p-adic completion of the module of PD-differentials $\Omega_{D_{P,\gamma}/A,\bar\gamma}$ (see [Sta19, Tag 07HQ]). Notice that Ω_D is a finite projective D-module: indeed

$$\Omega_{D_{P,\gamma}/A,\bar\gamma} \simeq \Omega_{P/A} \otimes_P D_{P,\gamma}$$

(see [Sta19, Tag 07HW]) and, when we take the p-adic completion, the left hand side, by definition, becomes Ω_D, while the right hand side is isomorphic to $\Omega_{P/A} \otimes_P D$ because $\Omega_{P/A}$ is a finite projective P-module. Therefore

$$\Omega_D \simeq \Omega_{P/A} \otimes_P D$$

which is a finite projective D-module. We denote by $\Omega_{\mathbf{D}}$ the sheaf on $\mathbf{D}$ associated to Ω_D.

Remark 1.13. Thanks to [Sta19, Tag 07KG] we have

$$\Omega_D \otimes_A A_n \simeq \Omega_{P_n/A} \otimes_{P_n} D_{P_n,\gamma} \simeq \Omega_{D_{P_n,\gamma}/A,\bar\gamma}$$

for n large. This allows us to construct a map

$$\Omega_{D/A} \longrightarrow \Omega_D$$

which is split surjective. Indeed, the section

$$\Omega_D \simeq \Omega_{P/A} \otimes_P D \to \Omega_{D/A}$$

is given by the extension of scalars of the natural map $\Omega_{P/A} \longrightarrow \Omega_{D/A}$ along the map $P \longrightarrow D$.

Definition 1.9. In the situation of **Setting** 1.8, we denote by $\mathrm{Conn}(X/\mathbf{S}, i, f)$ the full subcategory of the category $\mathrm{Conn}(\mathbf{D}/\mathbf{S}, \Omega_{\mathbf{D}})$ consisting of integrable $\Omega_{\mathbf{D}}$-connections $(\tilde{M}, \nabla_{\tilde{M}})$, where M is a finitely presented p-adically complete D-module.

Remark 1.14.

(1) If $\mathbf{D}$ is Noetherian, then $\mathrm{Conn}(X/\mathbf{S}, i, f) = \mathrm{Conn}(\mathbf{D}/\mathbf{S}, \Omega_{\mathbf{D}})$ because in this case any finitely presented D-module is p-adically complete.

(2) If M is a p-adically complete D-module, the module $M \otimes_D \Omega_D$ is p-adically complete because Ω_D is a finite projective D-module. In particular the connections defined above agree with the pairs considered in [Sta19, Tag 07J7].

(3) If the diagram in (1.3) is Cartesian, then the PD-structure γ extends to $X \hookrightarrow \mathbf{X}$ ([Sta19, 07H1]), and $\mathbf{D}_{P,\gamma} = \mathbf{X}$. Indeed, since the diagram is cartesian, $IP = J$ and (P, IP) verifies the universal property of the PD-envelope. With these hypothesis we get that $\Omega_{D_{P,\gamma}/A,\bar{\gamma}} = \Omega_{P/A}$ (see [Sta19, Tag 07HW]). Therefore the p-adic completions are isomorphic

$$\widehat{\Omega_{D/A}} \simeq \widehat{\Omega_{P/A}} \simeq \Omega_D.$$

Moreover

$$\mathrm{Hom}(\Omega_D, D) = \mathrm{Der}(D/A);$$

indeed a map from $\Omega_{D/A}$ to a p-adically complete module factors through Ω_D. We remark that any derivation in $\mathrm{Der}(D/A)$ is $\mathbb{Z}$-linear, hence it is automatically p-adically continuous.

(4) If we have another commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\;\;i'\;\;} & \mathbf{X}' \\
\downarrow & & \downarrow{\scriptstyle f'} \\
S' & \xrightarrow{\hspace{1.5cm}} & \mathbf{S}'
\end{array}
$$

mapping to the original one, there is an induced map $D' \longrightarrow D$ which yields a map $\Omega_D \otimes_D D' \longrightarrow \Omega_{D'}$. Via this map we obtain a functor $\mathrm{Conn}(X/\mathbf{S}, i, f) \longrightarrow \mathrm{Conn}(X'/\mathbf{S}', i', f')$.

1.3.2. *Topologically quasi-nilpotent connections.*

Definition 1.10. In the situation of **Setting** 1.8 where (1.3) is cartesian, a connection $(\tilde{M}, \nabla_{\tilde{M}}) \in \mathrm{Conn}(X/\mathbf{S}, i, f)$ is called *topologically quasi-nilpotent* if for all $n \geqslant 1$ its reduction modulo p^n is quasi-nilpotent in the sense of [BO78, Definition 4.10, Remark 4.11].

We denote by $\mathrm{QNCf}(X/\mathbf{S}, i, f)$ the full subcategory of $\mathrm{Conn}(X/\mathbf{S}, i, f)$ consisting of topologically quasi-nilpotent connections.

Theorem 1.15. *Let $X, \mathbf{X}, \mathbf{S}$ be as in Definition 1.10. Then there is a fully faithful additive tensor functor*

$$\mathrm{Crys}(X/\mathbf{S}) \longrightarrow \mathrm{Conn}(X/\mathbf{S}, i, f)$$

whose essential image is $\mathrm{QNCf}(X/\mathbf{S}, i, f)$. *Moreover, the above functor is functorial with respect to the diagram* (1.3).

Proof. Given $E \in \mathrm{Crys}(X/\mathbf{S})$, we take its restriction $E_n \in \mathrm{Crys}(X/\mathbf{S}_n)$, obtaining $(\tilde{M}_n, \nabla_{\tilde{M}_n}) \in \mathrm{Conn}(X/\mathbf{S}_n, i_n, f_n)$ by [BO78, Theorem 6.6]. Here the P_n-module M_n is $\mathrm{H}^0(E_{\mathbf{X}_n})$. There are transition maps $\phi_n \colon M_{n+1} \longrightarrow M_n$ which are horizontal, that is they preserve the connections. Since E is a crystal, we have $M_{n+1}/p^n M_{n+1} \simeq M_n$.

The limit $M \coloneqq \varprojlim_{n \in \mathbb{N}^+} M_n$ is a D-module since $D/p^n D = P_n$ and $M/p^n M = M_n$ by [Sta19, 09B8]. Moreover, M also comes with a connection. This association defines the functor $\mathrm{Crys}(X/\mathbf{S}) \longrightarrow \mathrm{Conn}(X/\mathbf{S}, i, f)$, which is easily seen to be linear and to preserve the tensor product.

The full faithfulness and the claim about the essential image follow from the corresponding statements in the p^n-torsion case (see e.g. [BO78, Corollary 6.8] or [Ber74, Théorème 1.6.5, p. 247]).

$\square$

Remark 1.16.

(1) The naturality of the functor in Theorem 1.15 indicates that the pullback of a topologically quasi-nilpotent connection is topologically quasi-nilpotent.
(2) Directly from the definition one sees that $(\tilde{M}, \nabla_{\tilde{M}}) \in \mathrm{Conn}(X/\mathbf{S}, i, f)$ belongs to $\mathrm{QNCf}(X/\mathbf{S}, i, f)$ if and only if its pullback to $X/\mathbf{S}_n$ belongs to $\mathrm{QNCf}(X/\mathbf{S}_n, i_n, f_n)$ (see Remark 1.11 for the notation) for some $n \in \mathbb{N}$.

Lemma 1.17. *Suppose that we are in the situation of Definition 1.10. Then*

$$\mathrm{QNCf}(X/\mathbf{S}, i, f) \subseteq \mathrm{Conn}(X/\mathbf{S}, i, f)$$

is a full subcategory closed under taking subobjects, quotients, tensor products and internal homomorphisms.

Proof. Directly from Definition 1.10 it is clear that subobjects and quotient objects of topologically quasi-nilpotent connections are topologically quasi-nilpotent. We still have to show that if $(\tilde{E}, \nabla_{\tilde{E}})$ and $(\tilde{F}, \nabla_{\tilde{F}})$ are topologically quasi-nilpotent connections, then their tensor product and their Hom are topologically quasi-nilpotent. This follows by checking the following relations for all $D \in \mathrm{Der}(\mathbf{D}/\mathbf{S})$: $[\nabla_{E \otimes F}(D)]^n(e \otimes f)$ is

$$D^n(e \otimes f) = D^n(e) \otimes f + \cdots + \binom{n}{r} D^{n-r+1}(e) \otimes D^r(f) + \cdots + e \otimes D^n(f)$$

and $[\nabla_{\mathrm{Hom}(E,F)}(D)]^n(\phi)$ is

$$\nabla_F(D)^n \circ \phi + \cdots + (-1)^r \binom{n}{r} \nabla_F(D)^{n-r} \circ \phi \circ \nabla_E(D)^r + \cdots + (-1)^n \phi \circ \nabla_E(D)^n. \quad \square$$

1.3.3. *The situation when* (1.3) *is cartesian.*

Lemma 1.18. *Let* $(\mathbf{S} = \mathrm{Spec}\, A, I, \gamma)$ *be an affine PD-scheme over* $\mathbb{Z}_{(p)}$ *such that* $p \in I$. *As above set* $S = \mathrm{Spec}\, A/I$, $A_n = A/p^n$ *and* $\mathbf{S}_n = \mathrm{Spec}\, A_n$ *for all* $n \in \mathbb{N}$.

The closed embedding $S \hookrightarrow \mathbf{S}_1$ *is a locally nilpotent thickening, that is* I/pA *is a nil ideal in* A/p. *In particular, if the ideal* I *is finitely generated, then the closed embedding* $S \hookrightarrow \mathbf{S}_1$ *is a nilpotent thickening.*

Proof. The ideal I has a PD-structure and therefore $p!\gamma_p(x) = x^p$ for all $x \in I$, so that $x^p \in pA$ as required. $\qquad\square$

Remark 1.19. Let $\mathbf{S} = (\mathbf{S}, I, \gamma)$ be a PD-scheme as in Lemma 1.18. Suppose moreover that I is finitely generated. Let $g \colon X = \operatorname{Spec} C \longrightarrow S = V(I)$ be a smooth map. Under the assumptions of Lemma 1.18, we can build up a diagram (1.3) out of the given map $g \colon X \longrightarrow S$ and the closed immersion $S \hookrightarrow \mathbf{S}$ such that it is a cartesian diagram. Indeed, by [Sta19, 07M8] we can lift g to a smooth affine map $f \colon \mathbf{X} = \operatorname{Spec} P \longrightarrow \mathbf{S} = \operatorname{Spec} A$ not necessarily uniquely along $S \hookrightarrow \mathbf{S}$. Note that by [Ill05, Theorem 8.5.9] the lifts of g along $S \hookrightarrow \mathbf{S}_1$ and $\mathbf{S}_n \hookrightarrow \mathbf{S}_{n+1}$ are unique. Thanks to Remark 1.14 (3) and the uniqueness of the lift to $\mathbf{S}_n$ for all n, the spectrum $\mathbf{D}$ of the p-adically completed PD-envelope D, which is the p-adic completion of P, does not depend on the lift $f \colon \mathbf{X} \longrightarrow \mathbf{S}$ we chose for g.

Definition 1.11. Let $\mathbf{S} = (\mathbf{S}, I, \gamma)$ be a PD-scheme as in Lemma 1.18 and we assume that I is finitely generated. Let $g \colon X \longrightarrow S = V(I)$ be a smooth map. We construct a cartesian diagram as in Remark 1.19. As observed in Remark 1.19, the category $\operatorname{Conn}(X/\mathbf{S}, i, f)$ does not depend on the choice of f and i such that (1.3) is cartesian, so, in this case, we will just write $\operatorname{Conn}(X/\mathbf{S})$ instead of $\operatorname{Conn}(X/\mathbf{S}, i, f)$. Thanks to Theorem 1.15 the full subcategory $\operatorname{QNCf}(X/\mathbf{S}, i, f)$ does not depend on the choice of such f and i either, thus we will write $\operatorname{QNCf}(X/\mathbf{S})$ instead of $\operatorname{QNCf}(X/\mathbf{S}, i, f)$ when the conditions of Lemma 1.18 are met.

Lemma 1.20. *Let $(\mathbf{S} = \operatorname{Spec} A, I, \gamma)$ be as in in Lemma 1.18, and let $g \colon X \longrightarrow S = V(I)$ be a smooth map. If $\mathbf{S}$ is Noetherian, then we have*

$$\operatorname{Conn}(X/\mathbf{S}) = \operatorname{Conn}(\mathbf{D}/\mathbf{S}, \Omega_{\mathbf{D}}).$$

Therefore, the category $\operatorname{Conn}(X/\mathbf{S})$ is an abelian, symmetric monoidal category with internal homomorphisms.

Proof. If $\mathbf{S}$ is Noetherian, then D is Noetherian and p-adically complete, so every finitely presented D-module is p-adically complete. The last claim follows from Lemma 1.10 and Proposition 1.2. $\qquad\square$
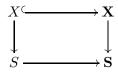
Lemma 1.21. *Let $(\mathbf{S} = \operatorname{Spec} A, I, \gamma)$ be as in in Lemma 1.18, and let $g \colon X = \operatorname{Spec} C \longrightarrow S = V(I)$ be a smooth map. Suppose moreover that $\mathbf{S} = \operatorname{Spec} A$, where A is a complete DVR of mixed characteristic $(0, p)$ with perfect residue field k and fraction field K.*

*If X is connected, then the rings D (see **Setting** 1.8) and $D \otimes_A K$ are regular domains and $(D \otimes_A K, \operatorname{Der}(D/A) \otimes_A K)$ is a locally simple differential ring.*

Proof. We lift, as in Remark 1.19, the smooth map $g \colon X = \operatorname{Spec} C \longrightarrow S = V(I)$ to a smooth map $f \colon \mathbf{X} = \operatorname{Spec} P \longrightarrow \mathbf{S} = \operatorname{Spec} A$.

We first show that D is a regular domain. Thanks to [Sta19, 07QW] the ring P is excellent, so it is a G-ring. According to [Sta19, 0AH2] the completion $P \longrightarrow D$ is a regular map (*i.e.* has geometrically regular fibers). Taking into account [Sta19, 031E] and the fact that $A \longrightarrow P$ is regular by construction, we can conclude that D is a regular ring.

In order to conclude that D is also a domain, it is enough to show that $\mathbf{D} = \operatorname{Spec} D$ is connected. Since the ideal I is finitely generated, by Lemma 1.18 the maps $S \hookrightarrow \mathbf{S}_n$ are nilpotent thickenings as well as the maps $X \hookrightarrow \mathbf{X}_n = \operatorname{Spec} D/p^n$ because the diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathbf{X} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathbf{S} \end{array}$$

is cartesian. Therefore $\mathbf{X}_n = \operatorname{Spec} D/p^n$ is connected for all $n \in \mathbb{N}$, because X is connected by hypothesis.

In particular if $a \in D$ is an idempotent element, then

$$a_n := a \mod p^n D$$

is either 0 or 1 in D/p^n. As X is non-empty, none of those (D/p^n)'s is a zero ring, so $a_n \in D/p^n$ has to be 0 for all n or 1 for all n. Thus $a = 0$ or 1 in D, which implies that $\mathbf{D}$ is connected.

From the fact that D is a regular domain we deduce that its localization $D \otimes_A K$ is a regular domain as well.

Thus it remains to prove that $(D \otimes_A K, \operatorname{Der}(D/A) \otimes_A K)$ is a locally simple differential ring. By [Ogu84, Lemma 1.19] and its proof we see that for any closed point $x \in \mathbf{D} \times_\mathbf{S} \operatorname{Spec} K$ the map

$$m_x/m_x^2 \longrightarrow \Omega_D \otimes k(x)$$

is injective, where m_x and $k(x)$ are the maximal ideal and the residue field of x respectively. Applying $\operatorname{Hom}_{k(x)}(-, k(x))$ and recalling that Ω_D is locally free we obtain a surjective map

$$\operatorname{Hom}_D(\Omega_D, D) \otimes_D k(x) \longrightarrow \operatorname{Hom}_{k(x)}(m_x/m_x^2, k(x)).$$

Since $\operatorname{Hom}(\Omega_D, D) = \operatorname{Der}(D/A)$ the result follows from [Ked12, Proposition 1.2.3]. $\qquad \square$

Theorem 1.22. *Let* $(\mathbf{S} = \operatorname{Spec} A, I, \gamma)$ *be an affine PD-scheme over* $\mathbb{Z}_{(p)}$ *such that* $p \in I$ *and let* $g \colon X = \operatorname{Spec} C \longrightarrow S = \operatorname{Spec}(A/I)$ *be a smooth map. Suppose moreover that* $\mathbf{S} = \operatorname{Spec} A$, *where* A *is a complete DVR of mixed characteristic* $(0, p)$ *with perfect residue field* k *and fraction field* K. *If* X *is connected, then we have a diagram of Tannakian categories*

$$
\begin{array}{ccc}
\operatorname{QNCf}(X/\mathbf{S}) \otimes_A K \longrightarrow \operatorname{Conn}(\mathbf{D}/\mathbf{S}, \Omega_\mathbf{D}) \otimes_A K \longrightarrow \operatorname{Conn}(\mathbf{D} \otimes_A K/K, \Omega_\mathbf{D} \otimes_A K) \\
\Big\downarrow \wr \qquad\qquad\qquad\qquad \Big\downarrow \wr \\
\operatorname{Diff}(D, \operatorname{Der}(D/A)) \otimes_A K \longrightarrow \operatorname{Diff}(D \otimes_A K, \operatorname{Der}(D/A) \otimes_A K)
\end{array}
$$

where all the functors are fully faithful tensor exact functors.

Proof. The two vertical equivalences come from Lemma 1.10 since $\operatorname{Der}(D/A) = \operatorname{Hom}_D(\Omega_D, D)$ and that Ω_D is locally free. Notice that $\mathbf{D} = \operatorname{Spec} D$ is Noetherian because D is a completion of an affine smooth A-algebra. In particular the horizontal arrows on the right are fully faithful, exact, tensorial and preserve internal homomorphisms thanks to Lemma 1.5. The left horizontal arrow is fully faithful, exact, tensorial and preserves internal homomorphisms by Lemma 1.17.

By Theorem 1.7 and Lemma 1.21 we can conclude that $\operatorname{Diff}(D \otimes_A K, \operatorname{Der}(D/A) \otimes K)$ is a Tannakian category. From this it easily follows that for all other categories there exists a fiber functor and the endomorphisms of the trivial object form a field. The rigidity of those categories also follows. Indeed we must check that for all objects M, N in one of those categories the natural arrow

$$M^\vee \otimes N \longrightarrow \underline{\operatorname{Hom}}(M, N)$$

where $\underline{\operatorname{Hom}}(M, N)$ denotes the internal Hom, is an isomorphism. Because all functors preserves internal homomorphisms and tensor product, this morphism become an isomorphism in $\operatorname{Diff}(D \otimes_A K, \operatorname{Der}(D/A) \otimes K)$ and, because all functors are fully faithful, this morphism has to be an isomorphism. $\qquad \square$

1.4. **Crystalline fundamental group.** In this section we consider the following situation. Let k be a perfect field of characteristic $p > 0$, and let W be the ring of Witt vectors of k. Set $\mathbf{W} \coloneqq \operatorname{Spec} W$. We denote by γ the canonical PD-structure on pW, K the fraction field of W. Set $W_n \coloneqq W/p^n W$ and $\mathbf{W}_n \coloneqq \operatorname{Spec} W_n$. We denote by γ_n the induced PD-structure on pW_n. The base PD-scheme $(\mathbf{S}, I, \gamma)$ is $(\mathbf{W}, pW, \gamma)$, and $S = \operatorname{Spec} k$.

Definition 1.12. Let X be a scheme over k. We denote by $I_{\mathrm{crys}}(X/\mathbf{W})$ the category of finitely presented isocrystals. This is the category $\mathrm{Crys}(X/\mathbf{W})$ up to isogeny, *i.e.* the category whose objects are exactly those in $\mathrm{Crys}(X/\mathbf{W})$ and whose morphisms are obtained inverting the multiplication by p. Thus we have a natural functor

$$\mathrm{Crys}(X/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(X/\mathbf{W})$$

which is the identity on objects. To distinguish objects in $\mathrm{Crys}(X/\mathbf{W})$ from those in $I_{\mathrm{crys}}(X/\mathbf{W})$ we denote by $K \otimes E$ the image of $E \in \mathrm{Crys}(X/\mathbf{W})$ under the above functor, and we say that E is a lattice for the isocrystal $\mathcal{E}$ if $K \otimes E \cong \mathcal{E}$.

The main result of the section is the following

Theorem 1.23. *If X is a smooth, quasi-compact and connected k-scheme, then the category $I_{\mathrm{crys}}(X/\mathbf{W})$ is a Tannakian category over a field extending K.*

If Y is another smooth, quasi-compact and connected k-scheme with a map $Y \longrightarrow X$, then the pullback $I_{\mathrm{crys}}(X/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(Y/\mathbf{W})$ is an exact tensor functor. Moreover $I_{\mathrm{crys}}(\operatorname{Spec} k/\mathbf{W}) = \mathsf{Vect}(K)$ and, if $x \colon \operatorname{Spec} k \longrightarrow X$ is a rational point, then $I_{\mathrm{crys}}(X/\mathbf{W})$ is a neutral K-Tannakian category via $x^* \colon I_{\mathrm{crys}}(X/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(\operatorname{Spec} k/\mathbf{W}) = \mathsf{Vect}(K)$.

Definition 1.13. Let X be a smooth, quasi-compact and connected k-scheme with a rational point $x \in X(k)$. We define $\pi_1^{\mathrm{crys}}(X/\mathbf{W}, x)$ as the Tannaka dual of the neutral Tannakian category $I_{\mathrm{crys}}(X/\mathbf{W})$ endowed with the fiber functor x^* (see Theorem 1.23).

Remark 1.24. The prounipotent completion of the group scheme defined in Definition 1.13 has been defined and studied by Shiho in [Shi00] and [Shi02] (in the more general situation of log schemes).
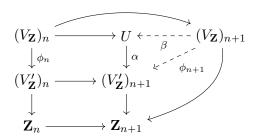
Lemma 1.25. *Let R be a complete Noetherian ring with respect to an ideal $I \subseteq R$, and set $Z \coloneqq \operatorname{Spec} R/I$, $\mathbf{Z} \coloneqq \operatorname{Spec} R$. Consider also a smooth affine map $V \longrightarrow Z$. We denote by $(-)_n$ the base change to $\mathbf{Z}_n = \operatorname{Spec}(R/I^n)$. Then:*

(1) There exists a smooth affine map $\widetilde{V} = \operatorname{Spec} \widetilde{D} \longrightarrow \mathbf{Z}$ lifting $V \longrightarrow Z$.

(2) There exists an affine and flat map $V_{\mathbf{Z}} = \operatorname{Spec} D \longrightarrow \mathbf{Z}$ lifting $V \longrightarrow Z$ such that D is an I-adically complete ring. We can choose as D the I-adic completion of an R-algebra $\widetilde{D}$ as in (1). Moreover, $V_{\mathbf{Z}}$ is a Noetherian scheme and all $(V_{\mathbf{Z}})_n \longrightarrow \mathbf{Z}_n$ are smooth.

(3) If $V_{\mathbf{Z}} \longrightarrow \mathbf{Z}$ and $V'_{\mathbf{Z}} \longrightarrow \mathbf{Z}$ are two lifts as in (2) then there exists a (not necessarily unique) $\mathbf{Z}$-isomorphism $V_{\mathbf{Z}} \longrightarrow V'_{\mathbf{Z}}$ lifting $\operatorname{id}_V \colon V \longrightarrow V$.

Proof. (1) This is [Sta19, Tag 07M8].

(2) Let D be the I-adic completion of $\widetilde{D}$ and set $V_{\mathbf{Z}} \coloneqq \operatorname{Spec} D$. By [Sta19, Tag 05GH] and [Sta19, Tag 0912] the ring D is ID-adically complete, Noetherian, $D/ID = \widetilde{D}/I\widetilde{D}$ and $\widetilde{D} \longrightarrow D$ is flat, so that $V_{\mathbf{Z}} \longrightarrow \mathbf{Z}$ is flat as well.

(3) It is enough to find a system of compatible $\mathbf{Z}_n$-maps $\phi_n \colon (V_{\mathbf{Z}})_n \longrightarrow (V'_{\mathbf{Z}})_n$ with $\phi_0 = \mathrm{id}_V$ (and thus automatically isomorphisms). Consider the diagram

$$
\begin{array}{ccc}
(V_{\mathbf{Z}})_n \longrightarrow U \xleftarrow{\ \ \beta\ \ } (V_{\mathbf{Z}})_{n+1} \\
\Big\downarrow{\scriptstyle \phi_n} \qquad \Big\downarrow{\scriptstyle \alpha} \qquad \quad {\scriptstyle \phi_{n+1}} \\
(V'_{\mathbf{Z}})_n \longrightarrow (V'_{\mathbf{Z}})_{n+1} \\
\Big\downarrow \qquad\qquad \Big\downarrow \\
\mathbf{Z}_n \longrightarrow \mathbf{Z}_{n+1}
\end{array}
$$

where $\alpha \colon U \longrightarrow (V'_{\mathbf{Z}})_{n+1}$ is any flat lift of ϕ_n, which exists by (2) because ϕ_n is an isomorphism and thus it is smooth. Since $\mathbf{Z}_n$ is affine, by [Ill05, Theorem 8.5.9, pp. 213-214] we can find the dashed $\mathbf{Z}_{n+1}$-isomorphism $\beta \colon (V'_{\mathbf{Z}})_{n+1} \longrightarrow U$ making the above diagram commutative. The choice $\phi_{n+1} = \alpha \circ \beta$ yields the desired lifting of ϕ_n. $\qquad\qquad\square$

Lemma 1.26. *Let X be a smooth affine scheme over k. Then:*
   (1) *There exists a smooth affine map $\widetilde{X} = \operatorname{Spec} \widetilde{B} \longrightarrow \mathbf{W}$ lifting $X \longrightarrow \operatorname{Spec} k$.*
   (2) *There exists a flat and affine $\mathbf{W}$-scheme $X_{\mathbf{W}} = \operatorname{Spec} B \longrightarrow \mathbf{W}$ lifting $X \longrightarrow \operatorname{Spec} k$ and such that B is p-adically complete. We can choose as B the p-adic completion of a W-algebra $\widetilde{B}$ as in (1). Moreover, $X_{\mathbf{W}}$ is a Noetherian scheme and all maps $(X_{\mathbf{W}})_n = \operatorname{Spec} B/p^n B \longrightarrow \mathbf{W}_n$ are smooth.*
   (3) *If $f \colon Y \longrightarrow X$ is a smooth affine map over k and $X_{\mathbf{W}}, Y_{\mathbf{W}} \longrightarrow \mathbf{W}$ are the complete lifts of X, Y as in (2) respectively then there exists a flat map $f_{\mathbf{W}} \colon Y_{\mathbf{W}} \longrightarrow X_{\mathbf{W}}$ lifting $f \colon Y \longrightarrow X$. Moreover, all $(f_{\mathbf{W}})_n \colon (Y_{\mathbf{W}})_n \longrightarrow (X_{\mathbf{W}})_n$ are smooth.*
   (4) *If $X_{\mathbf{W}} \longrightarrow \mathbf{W}$ and $X'_{\mathbf{W}} \longrightarrow \mathbf{W}$ are two lifts as in (2) then there exists a $\mathbf{W}$-isomorphism $X_{\mathbf{W}} \longrightarrow X'_{\mathbf{W}}$ lifting $\mathrm{id}_X \colon X \longrightarrow X$.*
   (5) *If $f_{\mathbf{W}}, f'_{\mathbf{W}} \colon Y_{\mathbf{W}} \longrightarrow X_{\mathbf{W}}$ are two lifts as in (3) then there exists an automorphism σ of $Y_{\mathbf{W}}$ fitting in the diagram*

$$
\begin{array}{ccccccc}
Y & \xrightarrow{\ \mathrm{id}_Y\ } & Y & \longrightarrow & X & \longrightarrow & \operatorname{Spec} k \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
Y_{\mathbf{W}} & \dashrightarrow_{\ \sigma\ } & Y_{\mathbf{W}} & \xrightarrow{\ f_{\mathbf{W}}\ } & X_{\mathbf{W}} & \longrightarrow & \mathbf{W} \\
& & & \underset{f'_{\mathbf{W}}}{\searrow\nearrow} & & &
\end{array}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad .$

Proof. If we apply Lemma 1.25 with $R = W$, $I = pW$ and $V = X$, so that $Z = \operatorname{Spec} k$ and $\mathbf{Z} = \mathbf{W}$, we obtain (1), (2) and (4).

Now consider the situation of (3) and (5) and set $X_{\mathbf{W}} = \operatorname{Spec} R$. We apply Lemma 1.25 with $I = pR$ and $V = Y$, so that $Z = X$ and $\mathbf{Z} = X_{\mathbf{W}}$. Lemma 1.25 (3) directly implies case (5). From Lemma 1.25 (2) we obtain a lift $V_{\mathbf{Z}} \longrightarrow \mathbf{Z} = X_{\mathbf{W}}$ of $Y \longrightarrow X$ and, using (4), we find a $\mathbf{W}$-isomorphism ϕ from $Y_{\mathbf{W}} \longrightarrow \mathbf{W}$ to $V_{\mathbf{Z}} \longrightarrow X_{\mathbf{W}} \longrightarrow \mathbf{W}$ which lifts $\mathrm{id}_Y \colon Y \longrightarrow Y$: the composition $Y_{\mathbf{W}} \xrightarrow{\ \phi\ } V_{\mathbf{Z}} \longrightarrow X_{\mathbf{W}}$ is the desired map $f_{\mathbf{W}}$ in (3). $\qquad\square$

Proposition 1.27. *If X is a smooth and quasi-compact k-scheme then $\operatorname{Crys}(X/\mathbf{W})$ is a symmetric monoidal, abelian category with internal homomorphisms.*

Proof of Theorem 1.23 and Proposition 1.27. Firstly note that the category $\mathrm{Crys}(X/\mathbf{W})$ is a symmetric monoidal additive $\mathbf{W}$-linear category. It also admits cokernels as the pullback functor of sheaves of modules is right exact and cokernels of maps of finitely presented modules are still finitely presented ([Sta19, Tag 0519]).

Now we consider the existence of kernels and the internal homomorphisms. Let $\{U_i\}_{i\in I}$ be a finite Zariski covering of X such that each U_i is an affine non-empty scheme. Taking into account Lemma 1.26, for each U_i we can choose a smooth lift $\mathbf{U}_i = \mathrm{Spec}\, A_i \longrightarrow \mathbf{W}$ of $U_i \longrightarrow \mathrm{Spec}\, k$. Set $(U_i)_\mathbf{W}$ for the spectrum of the p-adic completion of A_i. By Lemma 1.17 and Theorem 1.15 we see that each $\mathrm{Crys}(U_i/\mathbf{W})$ admits kernels and internal homomorphisms.

It is straightforward that $\mathrm{Crys}(-/\mathbf{W})$ is a stack on the small Zariski site of X. If U_{ij} is a non-empty affine open inside $U_i \cap U_j$, then by Lemma 1.26 (3) there is a flat $\mathbf{W}$-lift $(U_{ij})_\mathbf{W} \longrightarrow (U_i)_\mathbf{W}$ (note that this is not an open immersion!), whose flatness implies that kernels and internal homomorphisms are preserved at the level of topologically quasi-nilpotent connections by the pullback. We can glue kernels and internal homomorphisms in $\mathrm{Crys}(X/\mathbf{W})$ using the universal property defining them.

Thus we can conclude that $\mathrm{Crys}(X/\mathbf{W})$ and, by Lemma 1.3, $I_{\mathrm{crys}}(X/\mathbf{W})$ are abelian categories, because the canonical map from the coimage to the image is an isomorphism (as it is an isomorphism when restricted to each U_i). Moreover by construction and again by Lemma 1.3 restriction to an open is exact, tensorial and preserves internal homomorphisms for both $\mathrm{Crys}(-/\mathbf{W})$ and $I_{\mathrm{crys}}(-/\mathbf{W})$. This ends the proof of Proposition 1.27.

We now deal with the proof of Theorem 1.23. In particular we assume that X is connected. In particular X and all U_i are integral schemes. Since we are in the situation of Remark 1.19, the category $I_{\mathrm{crys}}(U_i/\mathbf{W})$ is Tannakian by Theorem 1.22.

It is easy to check that $I_{\mathrm{crys}}(-/\mathbf{W})$ is a prestack in the small Zariski site of X, that is morphisms between isocrystals form a Zariski sheaf. In particular $I_{\mathrm{crys}}(X/\mathbf{W})$ is rigid because all $I_{\mathrm{crys}}(U_i/\mathbf{W})$'s are Tannakian.

Next we will show that the ring of endomorphisms of the trivial object $\mathcal{O}_{X/\mathbf{W}} \otimes_W K \in I_{\mathrm{crys}}(X/\mathbf{W})$ is a field. Let ϕ be a non zero endomorphism of $\mathcal{O}_{X/\mathbf{W}} \otimes_W K$. We must show that ϕ is invertible. Since $I_{\mathrm{crys}}(-/\mathbf{W})$ is a prestack we must show that its restriction ϕ_i over U_i is invertible. As $I_{\mathrm{crys}}(U_i/\mathbf{W})$ is Tannakian, it is enough to show that $\phi_i \neq 0$. By contradiction assume that $\phi_i = 0$. The functor $I_{\mathrm{crys}}(U_j/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(U_{ij}/\mathbf{W})$ is exact, K-linear and tensorial, so it is faithful by [Del90, p. 2.10]. Since $(\phi_i)_{|U_{ij}} = 0$, we have $\phi_j = 0$ for all j by the connectedness of X. But this would imply that $\phi = 0$.
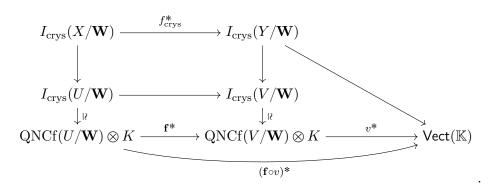
Hence the endomorphisms of $\mathcal{O}_{X/\mathbf{W}} \otimes_W K$ form a field. Let's denote it by L. A fiber functor for $I_{\mathrm{crys}}(X/\mathbf{W})$ is obtained composing a fiber functor of $I_{\mathrm{crys}}(U_i/\mathbf{W})$ with the tensor exact functor $I_{\mathrm{crys}}(X/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(U_i/\mathbf{W})$.

In conclusion $I_{\mathrm{crys}}(X/\mathbf{W})$ is a Tannakian category over L (see [Del90, p. 1.9]).

Let now $f\colon Y \longrightarrow X$ be a map as in the statement of Theorem 1.23 and denote by $f^*_{\mathrm{crys}}\colon I_{\mathrm{crys}}(X/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(Y/\mathbf{W})$ the pullback. We know that f^*_{crys} is a tensor functor and we must show that it is exact.

Let $U \subseteq X$ and $V \subseteq Y$ be non-empty affine open subsets such that $f(V) \subseteq U$. Let $\mathbf{f}\colon V_\mathbf{W} \longrightarrow U_\mathbf{W}$ be a lift of $V \longrightarrow U$ as in Lemma 1.26, (3) and $v\colon \mathrm{Spec}\,\mathbb{K} \longrightarrow V_\mathbf{W} \times_\mathbf{W} K$ be

a geometric point. Using Theorem 1.15 we have a commutative diagram

$$
\begin{array}{ccc}
I_{\mathrm{crys}}(X/\mathbf{W}) & \xrightarrow{\;f^*_{\mathrm{crys}}\;} & I_{\mathrm{crys}}(Y/\mathbf{W}) \\
\downarrow & & \downarrow \\
I_{\mathrm{crys}}(U/\mathbf{W}) & \longrightarrow & I_{\mathrm{crys}}(V/\mathbf{W}) \\
\downarrow\wr & & \downarrow\wr \\
\mathrm{QNCf}(U/\mathbf{W})\otimes K & \xrightarrow{\;\mathbf{f}^*\;} & \mathrm{QNCf}(V/\mathbf{W})\otimes K & \xrightarrow{\;v^*\;} & \mathsf{Vect}(\mathbb{K})
\end{array}
$$

with the bottom arrow $(\mathbf{f}\circ v)^*$.

Notice that $v^*\colon \mathrm{QNCf}(V/\mathbf{W})\otimes K \longrightarrow \mathsf{Vect}(\mathbb{K})$ is the composition

$$\mathrm{QNCf}(V/\mathbf{W})\otimes K \longrightarrow \mathrm{Conn}(V_{\mathbf{W}}\times_{\mathbf{W}} K/K, \Omega_{V_{\mathbf{W}}}\otimes_W K) \longrightarrow \mathsf{Vect}(\mathbb{K})$$
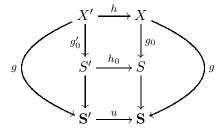
and it is a fiber functor by construction (or we can check it directly because modules in the middle category are locally free). The same happens to U and $(\mathbf{f}\circ v)^*$. In particular those arrows and therefore also $I_{\mathrm{crys}}(X/\mathbf{W}) \longrightarrow \mathsf{Vect}(\mathbb{K}), I_{\mathrm{crys}}(Y/\mathbf{W}) \longrightarrow \mathsf{Vect}(\mathbb{K})$ are exact and faithful. From this it follows that $I_{\mathrm{crys}}(X/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(Y/\mathbf{W})$ is exact.

Let's conclude computing $I_{\mathrm{crys}}(X/\mathbf{W})$ for $X = \mathrm{Spec}\,k$. We have $X_{\mathbf{W}} = \mathbf{W}$ and, in particular, $\Omega_{X_{\mathbf{W}}} = \Omega_{\mathbf{W}/\mathbf{W}} = 0$. In particular $\mathrm{QNCf}(X/\mathbf{W}) = \mathrm{Conn}(X/\mathbf{W})$ is just the category of finitely generated $\mathbf{W}$-modules. Tensoring by K one exactly gets $\mathsf{Vect}(K)$. $\square$

2. Base change theorems for crystalline cohomology

In this section we generalise in various ways the classical base change theorem for crystalline cohomology proven in [Ber74, V, Proposition 3.5.2], [BO78, Theorem 7.8]. Let k be a perfect field of characteristic $p > 0$, and let W be the ring of Witt vectors of k. Set $\mathbf{W} \coloneqq \mathrm{Spec}\,W$. We denote by γ the canonical PD-structure on pW, K the fraction field of W. Set $W_n \coloneqq W/p^nW$ and $\mathbf{W}_n \coloneqq \mathrm{Spec}\,W_n$. We denote by γ_n the induced PD-structure on pW_n

Setting 2.1. Let $\mathbf{S} = (\mathbf{S}, I, \gamma)$ be a PD-scheme such that $\mathbf{S}$ is a $\mathbf{W}$-scheme and $p \in I$. Denote by S the zero locus $V(I)$ of I inside $\mathbf{S}$, which is a k-scheme because $p \in I$. Let X be an S-scheme and denote by $g\colon X \longrightarrow \mathbf{S}$ the structure map. Consider a commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\;h\;} & X \\
g'_0\downarrow & & \downarrow g_0 \\
S' & \xrightarrow{\;h_0\;} & S \\
\downarrow & & \downarrow \\
\mathbf{S}' & \xrightarrow{\;u\;} & \mathbf{S}
\end{array}
$$

with g on both the left and right outer arrows.

where $\mathbf{S}' = (\mathbf{S}', I', \gamma')$ is a PD-scheme, $S' = V(I')$, X' is a scheme, u is a PD-morphism and the top square is cartesian. We assume moreover that all schemes are quasi-compact and g_0 is smooth, quasi-compact and quasi-separated. We consider a crystal of finite presentation $E \in \mathrm{Crys}(X/\mathbf{S})$.

We define
$$\Gamma((X/\mathbf{S})_{\mathrm{crys}}, -)\colon \mathsf{Mod}(\mathcal{O}_{X/\mathbf{S}}) \longrightarrow \mathsf{Mod}(\mathrm{H}^0(\mathcal{O}_\mathbf{S}))$$
as the functor of global sections ([BO78, p. 5.5]). It is easy to see that
$$\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E) = \varprojlim_n \Gamma((X/\mathbf{S}_n)_{\mathrm{crys}}, E_{|(X/\mathbf{S}_n)_{\mathrm{crys}}}) \in \mathsf{Mod}(\widehat{\mathrm{H}^0(\mathcal{O}_\mathbf{S})})$$
where E is a sheaf of $\mathcal{O}_{X/\mathbf{S}}$-modules on $(X/\mathbf{S})_{\mathrm{crys}}$, $\mathbf{S}_n \coloneqq \mathbf{S} \times_\mathbf{W} \mathbf{W}_n$ and $\widehat{\mathrm{H}^0(\mathcal{O}_\mathbf{S})}$ is the p-adic completion of $\mathrm{H}^0(\mathcal{O}_\mathbf{S})$.

There is a canonical projection from the crystalline ringed topos to the Zariski ringed topos [Sta19, Tag 07IL]
$$u_{X/\mathbf{S}}\colon ((X/\mathbf{S})^\sim_{\mathrm{crys}}, \mathcal{O}_{X/\mathbf{S}}) \to (X^\sim_{\mathrm{Zar}}, g^{-1}\mathcal{O}_\mathbf{S})$$
where $g^{-1}\mathcal{O}_\mathbf{S}$ is the pullback of $\mathcal{O}_\mathbf{S}$ along g. Concretely, we have

(1) For $F \in (X/\mathbf{S})^\sim_{\mathrm{crys}}$ and $j\colon U \hookrightarrow X$ an open,
$$(u_{X/\mathbf{S}*}(F))(U) = \Gamma((U/\mathbf{S})_{\mathrm{crys}}, F);$$

(2) For $G \in X^\sim_{\mathrm{Zar}}$ and $(U, T, \delta) \in (X/\mathbf{S})_{\mathrm{crys}}$,
$$(u_{X/\mathbf{S}}{}^*(G))(U, T, \delta) = G(U).$$

By composition we get a morphism of topoi
$$g_{X/\mathbf{S}} \coloneqq g \circ u_{X/\mathbf{S}}\colon ((X/\mathbf{S})^\sim_{\mathrm{crys}}, \mathcal{O}_{X/\mathbf{S}}) \to (\mathbf{S}^\sim_{\mathrm{Zar}}, \mathcal{O}_\mathbf{S}).$$
Notice that
$$\Gamma((X/\mathbf{S})_{\mathrm{crys}}, -) = \Gamma \circ g_{X/\mathbf{S}*}(-)$$
where $\Gamma\colon \mathsf{Mod}(\mathcal{O}_\mathbf{S}) \longrightarrow \mathsf{Mod}(\mathrm{H}^0(\mathcal{O}_\mathbf{S}))$ is the functor of global sections.

Lemma 2.1. *Assume that p is nilpotent in $\mathcal{O}_\mathbf{S}$ and $\mathbf{S}$ is separated. Let $E \in \mathrm{Crys}(X/\mathbf{S})$. Then $\mathbf{R}g_{X/\mathbf{S}*}(E)$ is quasi-isomorphic to a bounded complex of quasi-coherent $\mathcal{O}_\mathbf{S}$-modules. If $\mathbf{S}$ is affine, then this is quasi-isomorphic to the complex of quasi-coherent $\mathcal{O}_\mathbf{S}$-modules associated with any complex of $\mathrm{H}^0(\mathcal{O}_\mathbf{S})$-modules representing $\mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E)$.*

Proof. The complex $\mathbf{R}g_{X/\mathbf{S}*}(E)$ is cohomologically bounded and has quasi-coherent cohomology thanks to [BO78, Theorem 7.6]. By a standard argument it is quasi-isomorphic to a bounded complex of quasi-coherent sheaves $B^\bullet$ [BN93, Corollary 5.5]. If $\mathbf{S}$ is affine the degenerate spectral sequence (see [Wei94, p. 5.7.9])
$$E_2^{pq} = \mathrm{H}^p((\mathbf{R}^q\Gamma)(B^\bullet)) \Rightarrow \mathbf{R}^{p+q}\Gamma(B^\bullet)$$
tells us that $\Gamma(B^\bullet) \simeq \mathbf{R}\Gamma(B^\bullet) \simeq \mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E)$ as desired. $\qquad\square$

Remark 2.2.

(a) If p is nilpotent in $\mathcal{O}_\mathbf{S}$ we define a map
$$(2.1) \qquad\qquad \mathbf{L}u^*\mathbf{R}g_{X/\mathbf{S}*}(E) \to \mathbf{R}g'_{X'/\mathbf{S}'*}h^*_{\mathrm{crys}}(E)$$

in $D(\mathbf{S}^\sim_{\mathrm{Zar}})$ as follows. Applying adjunction to the canonical map $\mathbf{L}h^*_{\mathrm{crys}}(E) \to h^*_{\mathrm{crys}}(E)$ we obtain a map
$$E \to \mathbf{R}h_{\mathrm{crys}*}(h^*_{\mathrm{crys}}(E)).$$

Applying $\mathbf{R}g_{X/\mathbf{S}*}$ and using $g_{X/\mathbf{S}} \circ h_{\mathrm{crys}*} = u_* \circ g'_{X'/\mathbf{S}'}$ (see [Sta19, Tag 07MH]) we get
$$\mathbf{R}g_{X/\mathbf{S}*}(E) \to \mathbf{R}u_*\mathbf{R}g'_{X'/\mathbf{S}'*}(h^*_{\mathrm{crys}}(E)).$$

The map (2.1) is obtained applying adjunction again, which can be done because $\mathbf{R}g_{X/\mathbf{S}*}(E)$ is bounded above thanks to Lemma 2.1.

(b) If $\mathbf{S}$ and $\mathbf{S}'$ are affine, but p is not necessarily nilpotent in $\mathcal{O}_{\mathbf{S}}$, we can still define a map

$$(2.2) \qquad \mathbf{L}u^*\mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E) \to \mathbf{R}\Gamma((X'/\mathbf{S}')_{\mathrm{crys}}, h^*_{\mathrm{crys}}E)$$

in $D^-(\mathrm{H}^0(\mathcal{O}_{\mathbf{S}'}))$. The construction is the same and it is possible since $\mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E)$ is bounded above as we will prove in Corollary 2.12.

Definition 2.2. Let $\mathscr{A}$ be an abelian category, p a given prime and $N \in \mathbb{N}$. A map of objects of $\mathscr{A}$ is a p^N-isogeny if its kernel and its cokernel are killed by p^N, it is an isogeny if it is a p^r-isogeny for some $r \in \mathbb{N}$.

Definition 2.3. Let $\mathscr{C}$ be a W-linear category and $E \in \mathscr{C}$. Given $n \in \mathbb{N}$ we say that E is W_n-flat if p^n kills E and, for all $0 \leqslant j \leqslant n$, the quotient $E/p^j E$ exists and the map

$$E/p^j E \xrightarrow{p^{n-j}} E$$

is injective. We say that E is W-flat or p-torsion free if $E \xrightarrow{p} E$ is injective in $\mathscr{C}$.

Remark 2.3. If $\mathscr{C} = \mathsf{Mod}(W)$ then the notion of flatness just introduced and the classical one agrees. This is an easy consequence of testing flatness on ideals.

Lemma 2.4. *In the hypothesis of* **Setting** *2.1, if* $E \in \mathrm{Crys}(X/\mathbf{S})$ *is p-torsion free, then* $E_n = E_{|(X/\mathbf{S}_n)_{\mathrm{crys}}} \in \mathrm{Crys}(X/\mathbf{S}_n)$ *is W_n-flat.*

Proof. Indeed, since $\mathrm{Crys}(X/\mathbf{S})$ satisfies Zariski descent, we can assume that X and $\mathbf{S}$ are affine. We apply Theorem 1.15 twice. The crystal E corresponds to a module M with an integrable connection over $\mathrm{Spec}\,B$, where B is a p-adically complete W-algebra and $\mathrm{Spec}\,B$ is a lift of X. The B-module M is p-torsion free, thus W-flat, so its restriction $M_n \coloneqq M \otimes_W W_n$ is W_n-flat. Therefore, the crystal E_n, which corresponds to M_n, is also W_n-flat. $\square$

Remark 2.5. Let $E \in \mathrm{Crys}(X/\mathbf{S})$ be p-torsion free. It is not true that the map $E \xrightarrow{p} E$ is injective in the ringed topos $((X/\mathbf{S})^{\sim}_{\mathrm{crys}}, \mathcal{O}_{X/\mathbf{S}})$. For example, we can look at the trivial crystal $\mathcal{O}_{\mathrm{Spec}\,k/\mathbf{W}}$ on $(\mathrm{Spec}\,k/\mathbf{W})_{\mathrm{crys}}$: the map $W_1 \xrightarrow{p} W_1$ at the thickening $\mathrm{Spec}\,k \hookrightarrow \mathbf{W}_1$ is not injective.

Lemma 2.6. *In the hypothesis of* **Setting** *2.1, if* $\mathbf{S}$ *is flat over* $\mathbf{W}$, *and if* $E \in \mathrm{Crys}(X/\mathbf{S})$ *is a flat crystal* [BO78, p. 7.10], *then E is p-torsion free in* $\mathrm{Crys}(X/\mathbf{S})$.

Proof. Indeed, to see this we may assume X and $\mathbf{S}$ are affine. Then by Theorem 1.15, E corresponds to a flat module M equipped with an integrable connection over the flat $\mathbf{S}$-lift $\mathrm{Spec}\,B$ of X, where B is a p-adically complete W-algebra. Since $\mathbf{S}$ is flat over $\mathbf{W}$, M is W-flat, hence it is p-torsion free in $\mathrm{QNCf}(X/\mathbf{S})$. Thus E is p-torsion free in $\mathrm{Crys}(X/\mathbf{S})$ as well by Theorem 1.15. $\square$

2.1. **The case of a base killed by a power of p.** The next theorem deals with the situation in Remark 2.2 (a) and the map in (2.1).

Theorem 2.7. *In the situation of* **Setting** *2.1, assume moreover that p is nilpotent in $\mathcal{O}_{\mathbf{S}}$. The following hold.*

(a) *There exists $r \in \mathbb{N}$, which depends only on $X \xrightarrow{g_0} S$, such that for all open U of $\mathbf{S}$ and $i > r$ we have*

$$\mathbf{R}^i(g_{|g^{-1}(U)})_{g^{-1}(U)/U*}(E_{|(g^{-1}(U)/U)_{\mathrm{crys}}}) = 0.$$

(b) *The map* (2.1) *is an isomorphism if u is flat or E is a flat crystal* [BO78, p. 7.10].

(c) *The map* (2.1) *is an isomorphism if E is W_n-flat, $\mathbf{S}$ is a flat $\mathbf{W}_n$-scheme and if there exists a map of schemes $u_0 \colon \mathbf{Z} \longrightarrow \mathbf{W}_n$ such that u is the base change of u_0 along $\mathbf{S} \longrightarrow \mathbf{W}_n$.*

(d) *Suppose that S is smooth of finite type over k. Let $E_{\mathbf{W}} \in \mathrm{Crys}(X/\mathbf{W})$ and set $E = (E_{\mathbf{W}})_{|(X/\mathbf{S})_{\mathrm{crys}}}$. Then there exists $N \colon \mathbb{Z} \longrightarrow \mathbb{N}$, independent of the closed immersion $S \hookrightarrow \mathbf{S}$, such that the i-th cohomology of the map* (2.1) *is a p^{N_i}-isogeny in the ringed topos $(\mathbf{S}'^{\sim}_{\mathrm{Zar}}, \mathcal{O}_{\mathbf{S}'})$.*

Before giving the proof of this theorem we prove some preliminary results.

Lemma 2.8. *Let $\pi \colon \mathscr{A} \longrightarrow \mathscr{B}$ be a left exact functor between abelian categories. Assume that $\mathscr{A}$ has enough injectives and that there exists $n_0 > 0$ such that $\mathbf{R}^n\pi = 0$ for all $n \geqslant n_0$, so that, by* [Sta19, Tag 07K7], *there is a functor $\mathbf{R}\pi \colon D(\mathscr{A}) \longrightarrow D(\mathscr{B})$. Let also $\alpha \colon C \longrightarrow D$ be a map in $D(\mathscr{A})$ and $N \colon \mathbb{Z} \longrightarrow \mathbb{N}$ a function such that $\mathrm{H}^i(\mathbf{R}\pi(\alpha))$ is a p^{N_i}-isogeny and $N_i = 0$ for $i \gg 0$.*

Then there exists $N' \colon \mathbb{Z} \longrightarrow \mathbb{N}$, which depends only on N and n_0, such that $\mathrm{H}^i(\mathbf{R}\pi(\alpha))$ is a $p^{N'_i}$-isogeny and $N'_i = 0$ for $i \gg 0$.

Proof. Applying $\mathbf{R}\pi$ to the exact triangle of the cone of α and taking cohomology we get a long exact sequence

$$\cdots \to \mathbf{R}^i\pi C \to \mathbf{R}^i\pi D \to \mathbf{R}^i\pi(\mathrm{Cone}(\alpha)) \to \mathbf{R}^{i+1}\pi C \to \cdots.$$

From this we are reduced to show that if $G \in D(\mathscr{A})$ satisfies that $\mathrm{H}^i(G)$ is killed by p^{N_i}, then we can find N' as in the statement such that $\mathbf{R}^i\pi G$ is killed by $p^{N'_i}$ and $N'_i = 0$ for all $i \gg 0$.

We consider the truncation

$$\tau_{\geqslant n}(G) := \cdots \to 0 \to (G^n/\mathrm{Im}(d^{n-1})) \xrightarrow{d^n} G^{n+1} \xrightarrow{d^{n+1}} G^{n+2} \to \cdots.$$

By [Sta19, Tag 08J5], we have an exact triangle

$$\mathrm{H}^n(G)[-n] \longrightarrow \tau_{\geqslant n}(G) \to \tau_{\geqslant n+1}(G) \longrightarrow \mathrm{H}^n(G)[-n+1]$$

hence the exact triangle

$$\mathbf{R}\pi(\mathrm{H}^n(G)[-n]) \longrightarrow \mathbf{R}\pi(\tau_{\geqslant n}(G)) \to \mathbf{R}\pi(\tau_{\geqslant n+1}(G)) \longrightarrow \mathbf{R}\pi(\mathrm{H}^n(G)[-n+1]).$$

We show that there exists $f \colon \mathbb{Z} \longrightarrow \mathbb{N}$ such that the multiplication by p^{f_n} induces 0 on all cohomologies of $\mathbf{R}\pi(\tau_{\geqslant n}(G))$ and $f_n = 0$ for $n \gg 0$.

For $n \in \mathbb{N}$ satisfying $N_m = 0$ for $m \geqslant n$ we can set $f_n = 0$. Indeed in this case $\tau_{\geqslant n}G$ (and therefore also $\mathbf{R}\pi(\tau_{\geqslant n}G)$) is acyclic by assumption.

Moreover $\mathrm{H}^n(G)$ and, by linearity, all $\mathbf{R}\pi(\mathrm{H}^n(G)[u])$ ($u \in \mathbb{Z}$) are killed by p^{N_n} in the derived category. We can therefore define $f \colon \mathbb{Z} \longrightarrow \mathbb{N}$ working by reverse induction on $\mathbb{Z}$.

Next we show that

$$\mathbf{R}^i\pi(G) \longrightarrow \mathbf{R}^i\pi(\tau_{\geqslant n}G)$$

is an isomorphism for $n < i - n_0$ so that the function $N'_i = f_{i-n_0-1}$ satisfies the requests in the statement.

By [Sta19, Tag 07K7] we can assume that G is made by right acyclic objects for π. For all $n \in \mathbb{N}$ we have an exact sequence of complexes

$$0 \longrightarrow \sigma_{\geqslant n+1}G \longrightarrow \tau_{\geqslant n}G \longrightarrow (G_n/\operatorname{Im}(d_{n-1}))[-n] \longrightarrow 0$$

where $\sigma_{\geqslant n+1}G$ denotes truncation. Since $\mathbf{R}^q\pi = 0$ for $q \geqslant n_0$, we can conclude that $\mathbf{R}^i\pi(\sigma_{\geqslant n+1}G) \longrightarrow \mathbf{R}^i\pi(\tau_{\geqslant n}G)$ is an isomorphism for $n \leqslant i - n_0$. Since $\mathbf{R}\pi(G) = \pi(G)$ and $\mathbf{R}\pi(\sigma_{\geqslant n+1}G) = \pi(\sigma_{\geqslant n+1}G)$ we can also conclude that

$$\mathbf{R}^i\pi(\sigma_{\geqslant n+1}G) \longrightarrow \mathbf{R}^i\pi(G)$$

is an isomorphism for $n + 1 < i$. $\qquad\square$

Lemma 2.9. *Let $\mathscr{A}$ be an abelian category, $l, N \in \mathbb{N}$ and*

$$E_2^{uv} \Rightarrow H^{u+v}$$

be a convergent spectral sequence in $\mathscr{A}$.

If $E_2^{uv} = 0$ for $v > 0$ or $u < 0$ or $u > l$, then there is an associated map

$$\omega_n \colon H^n \longrightarrow E_\infty^{n0} \longrightarrow E_2^{n0}$$

and, if p^N kills all E_2^{uv} for $v \neq 0$, this map is a $p^{N(l+1)}$-isogeny.

If $E_2^{uv} = 0$ for $u > 0$ or $v < 0$ or $v > l$, then there is an associated map

$$\omega_n \colon E_2^{0n} \longrightarrow E_\infty^{0n} \longrightarrow H^n$$

and, if p^N kills all E_2^{uv} for $u \neq 0$, this map is a $p^{N(l+1)}$-isogeny.

Proof. We consider only the first case because the second one is analogous. By convergence there is a filtration

$$0 = F^tH^n \subseteq \cdots \subseteq F^{u+1}H^n \subseteq F^uH^n \subseteq \cdots \subseteq F^sH^n = H^n$$

for some $s < t$ such that

$$E_\infty^{u,n-u} \simeq (F^uH^n)/(F^{u+1}H^n).$$

The vanishing in the hypothesis tells us that $F^uH^n = F^{u+1}H^n$ if $u < 0$ or $u > l$ or $n > u$. Thus we can choose $t = l + 1$ and $s = \max(0, n)$ in the above filtration. In particular, $F^nH^n = H^n$ for all n.

Since $E_2^{uv} = 0$ for $v > 0$ all differentials landing in $(u, 0)$ are zero in all pages. It follows that $E_\infty^{u0} \subseteq E_2^{u0}$. Moreover there is a map

$$\omega_n \colon H^n = F^nH^n \longrightarrow (F^nH^n)/(F^{n+1}H^n) \simeq E_\infty^{n0} \hookrightarrow E_2^{n0}.$$

Assume now that p^N kills all the modules E_2^{uv} for $v \neq 0$. It follows that p^N kills all modules E_r^{uv} for $v \neq 0$ and $r \geqslant 2$. In particular E_{r+1}^{n0} is the kernel of a map from E_r^{n0} to an object killed by p^N. Moreover the differentials at page $l + 1$ must be 0, so that $E_{l+1}^{uv} = E_\infty^{uv}$. From this it follows that

$$\operatorname{Coker}\omega_n = E_2^{n0}/E_\infty^{n0}$$

is killed by $p^{N(l+1)}$.

It remains to look at $\operatorname{Ker}(\omega_n) = F^{n+1}H^n$. But this object has a filtration of lenght l of subobjects whose partial cokernels are killed by p^N. It follows that it must be killed by $p^{N(l+1)}$. $\qquad\square$

Lemma 2.10. *Let B be a smooth W-algebra, let $\hat{B}$ the p-adic completion of B and let $\Omega_{\hat{B}}$ be the p-adic completion of the module of algebraic differentials $\Omega_{\hat{B}/W}$ (As in Remark 1.13, $\Omega_{\hat{B}}$ is a quotient of $\Omega_{\hat{B}/W}$). Let $(M, \nabla) \in \mathrm{Conn}(\hat{B}/W, \Omega_{\hat{B}})$. Then there exist $l, a, b \in \mathbb{N}$ and maps of $\hat{B}$-modules $\alpha\colon \hat{B}^l \longrightarrow M$, $\beta\colon M \longrightarrow \hat{B}^l$ satisfying $p^a(\alpha\beta - p^b\mathrm{id}_M) = 0$. In particular if $F\colon \mathsf{Mod}(\hat{B}) \longrightarrow \mathscr{C}$ is any linear functor with values in a linear category and $F(\hat{B}) = 0$ then p^{a+b} kills $F(M)$.*

Proof. The last claim follows by linearity applying the functor F to the given expression and using that $F(\hat{B})$ and therefore $F(\beta)$ are zero.

Applying Proposition 1.6, Lemma 1.10 and Lemma 1.21 we can conclude that $M[1/p]$ is a finitely generated projective $\hat{B}[1/p]$-module. In particular there exist maps $\alpha\colon \hat{B}[1/p]^l \longrightarrow M[1/p]$ and $\beta\colon M[1/p] \longrightarrow \hat{B}[1/p]^l$ such that $\alpha\beta = \mathrm{id}$. Multiplying α and β by a power of p we can find $b \in \mathbb{N}$ and $\alpha\colon \hat{B}^l \longrightarrow M$ and $\beta\colon M \longrightarrow \hat{B}^l$ such that $\alpha\beta = p^b\mathrm{id}$ in $M[1/p]$. In particular there also exists $a \in \mathbb{N}$ such that $p^a(\alpha\beta - p^b\mathrm{id}_M) = 0$ as required. $\square$

Proof of Theorem 2.7. We follow the proof of [BO78, Theorem 7.8], in particular the proofs of (a) and (b) are essentially the same as the one given in the above reference.

We may assume that $\mathbf{S}'$ and $\mathbf{S}$ are affine. We want to reduce to the case where X is also affine by using cohomological descent as in [Ber74, Proposition 3.5.2] and [BO78, Theorem 7.8]. If $U \subseteq \mathbf{S}$ is any open subset then we have

$$\mathbf{R}(g|_{g^{-1}(U)})_{g^{-1}(U)/U_*}(E|_{(g^{-1}(U)/U)_{\mathrm{crys}}}) = \mathbf{R}g_{X/\mathbf{S}_*}(E)|_U.$$

Thus in (a) we may assume $U = \mathbf{S}$.

We take a finite affine covering $\{U_i\}_{i=0,\cdots,n}$ of X. From the covering we obtain the topos $(X^\bullet/\mathbf{S})_{\mathrm{crys}}^\sim$ as in [Ber74, p. 335, p. 344], and the morphism of topoi

$$\pi\colon (X^\bullet/\mathbf{S})_{\mathrm{crys}}^\sim \to (X/\mathbf{S})_{\mathrm{crys}}^\sim.$$

Similarly, we have the topos $(X'^\bullet/\mathbf{S}')_{\mathrm{crys}}^\sim$ and the corresponding morphism of topoi $\pi'\colon (X'^\bullet/\mathbf{S}')_{\mathrm{crys}}^\sim \to (X'/\mathbf{S}')_{\mathrm{crys}}^\sim$. Thus we have a diagram of topoi

Then cohomological descent implies that there are canonical isomorphisms [Ber74, V, Proposition 3.4.8]

$$E \xrightarrow{\cong} \mathbf{R}\pi_*(\pi^*E) \qquad \text{and} \qquad h_{\mathrm{crys}}^*E \xrightarrow{\cong} \mathbf{R}\pi'_*(\pi'^* h_{\mathrm{crys}}^*(E)).$$

Applying $\mathbf{R}g_{X/\mathbf{S}*}$ to the first above isomorphism, $\mathbf{R}g'_{X'/\mathbf{S}'*}$ to the second, we obtain the following commutative diagram
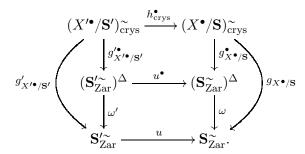
(2.3)
$$\begin{array}{ccc} \mathbf{L}u^*\mathbf{R}g_{X/\mathbf{S}*}(E) & \xrightarrow{\cong} & \mathbf{L}u^*\mathbf{R}g_{X\bullet/\mathbf{S}*}(\pi^*(E)) \\ \downarrow & & \downarrow \\ \mathbf{R}g'_{X'/\mathbf{S}'*}h^*_{\text{crys}}(E) & \xrightarrow{\cong} & \mathbf{R}g'_{X'\bullet/\mathbf{S}'*}h^{\bullet*}_{\text{crys}}(\pi^*(E)). \end{array}$$

The vertical map on the right is obtained via adjunctions as in **Setting** 2.1, using that $\mathbf{R}g_{X\bullet/\mathbf{S}}(\pi^*(E))$ is bounded being isomorphic to $\mathbf{R}g_{X/\mathbf{S}*}(\mathbf{R}\pi_*(\pi^*E)) = \mathbf{R}g_{X/\mathbf{S}*}(E)$ which is bounded by [BO78, Theorem 7.6]. This means that we can work with $X^\bullet$ and $X'^\bullet$, instead of X and X' respectively.

Now let Δ be the opposite category of the category whose objects are subsets of $I := \{0, 1, 2, \ldots, n\}$ and whose morphisms are the inclusions of subsets. As in [Ber74, pp. V, 3.4.3] we obtain the commutative diagram

$$\begin{array}{ccc} (X'^\bullet/\mathbf{S}')^\sim_{\text{crys}} & \xrightarrow{h^\bullet_{\text{crys}}} & (X^\bullet/\mathbf{S})^\sim_{\text{crys}} \\ & & \\ g'_{X'\bullet/\mathbf{S}'} \downarrow g'^\bullet_{X'\bullet/\mathbf{S}'} & & \downarrow g^\bullet_{X\bullet/\mathbf{S}} \quad g_{X\bullet/\mathbf{S}} \\ (\mathbf{S}'^\sim_{\widetilde{\text{Zar}}})^\Delta & \xrightarrow{u^\bullet} & (\mathbf{S}^\sim_{\widetilde{\text{Zar}}})^\Delta \\ \omega' \downarrow & & \downarrow \omega \\ \mathbf{S}'^\sim_{\widetilde{\text{Zar}}} & \xrightarrow{u} & \mathbf{S}^\sim_{\widetilde{\text{Zar}}}. \end{array}$$

We know that $\mathbf{R}g^\bullet_{X\bullet/\mathbf{S}}(\pi^*(E))$ has bounded cohomologies by [Ber74, pp. 340, 320]. Then by [Ber74, p. V. 3.4.9], one has the isomorphism

(2.4)
$$\mathbf{L}u^*(\mathbf{R}\omega_*(\mathbf{R}g^\bullet_{X\bullet/\mathbf{S}_*}(\pi^*(E)))) \xrightarrow{\cong} \mathbf{R}\omega'_*(\mathbf{L}u^{\bullet*}(\mathbf{R}g^\bullet_{X\bullet/\mathbf{S}_*}(\pi^*(E)))).$$

Note that by [Ber74, Prop. V. 3.4.9, i), p. 340] we have $\mathbf{R}^i\omega_*(-) = \mathbf{R}^i\omega'_*(-) = 0$ for all $i \geqslant n+1$ or $i < 0$, so by [Sta19, Tag 07K7] $\mathbf{R}\omega$ and $\mathbf{R}\omega'$ make sense. The right vertical arrow in (2.3) is the composition of (2.4) with the map obtained by applying $\mathbf{R}\omega'_*(-)$ to

(2.5)
$$\mathbf{L}u^{\bullet*}\mathbf{R}g^\bullet_{X\bullet/\mathbf{S}_*}(\pi^*(E)) \to \mathbf{R}g'^\bullet_{X'\bullet/\mathbf{S}'_*}(h^{\bullet*}_{\text{crys}}(\pi^*(E))).$$

Therefore, in (a), (b) and (c) we can replace $\mathbf{S}^\sim_{\widetilde{\text{Zar}}}$ and $\mathbf{S}'^\sim_{\widetilde{\text{Zar}}}$ by $(\mathbf{S}^\sim_{\widetilde{\text{Zar}}})^\Delta$ and $(\mathbf{S}'^\sim_{\widetilde{\text{Zar}}})^\Delta$ respectively. When (a) is proved, we can conclude that both $\mathbf{L}u^{\bullet*}\mathbf{R}g^\bullet_{X\bullet/\mathbf{S}_*}(\pi^*(E))$ and $\mathbf{R}g'^\bullet_{X'\bullet/\mathbf{S}'_*}(h^{\bullet*}_{\text{crys}}(\pi^*(E)))$ have cohomologies bounded from above with a bound i_0 depending only on g_0. Thus in (d) we can also replace $\mathbf{S}^\sim_{\widetilde{\text{Zar}}}$ and $\mathbf{S}'^\sim_{\widetilde{\text{Zar}}}$ by $(\mathbf{S}^\sim_{\widetilde{\text{Zar}}})^\Delta$ and $(\mathbf{S}'^\sim_{\widetilde{\text{Zar}}})^\Delta$ respectively, because we can reset the N obtained for $(\mathbf{S}^\sim_{\widetilde{\text{Zar}}})^\Delta$ and $(\mathbf{S}'^\sim_{\widetilde{\text{Zar}}})^\Delta$ to

$$N'_i := \begin{cases} N_i & \text{if } i \leqslant i_0 \\ 0 & \text{if } i > i_0 \end{cases}$$

so that the conditions of Lemma 2.8 are satisfied.

By [Ber74, Prop. V.3.4.4] and [Ber74, Prop. V.3.4.5] we see that $\mathbf{L}u^{\bullet*}$ and $\mathbf{R}g^\bullet_{X\bullet/\mathbf{S}_*}$ are computed componentwise. An intersection of open affine subsets of X may not be affine, but

it is separated. Thus one can first reduce the problem to the case when X is separated and, after, to the case when X is affine.

Now let $\mathbf{S} = \operatorname{Spec} A$ and $\mathbf{S}' = \operatorname{Spec} A'$. Since $g_0 \colon X \longrightarrow S$ is smooth and X, S are affine, there is a smooth affine lift $\tilde{g}_0 \colon \operatorname{Spec} B = \mathbf{X} \longrightarrow \mathbf{S}$ by [Sta19, Tag 07M8], and by pulling back along $u \colon \mathbf{S}' \longrightarrow \mathbf{S}$ we get a lift of g_0' to $\tilde{g}_0' \colon \operatorname{Spec} B' = \mathbf{X}' \longrightarrow \mathbf{S}'$. The comparison theorem (e.g. [Sta19, Tag 07LG]) tells us that there is a commutative diagram

(2.6)
$$
\begin{array}{ccc}
\mathbf{L}u^* \mathbf{R} g_{X/\mathbf{S}_*}(E) & \longrightarrow & \mathbf{R} g_{X'/\mathbf{S}'_*} h^*_{\mathrm{crys}}(E) \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \cong} \\
\mathbf{L}u^* \tilde{g}_{0*}(\mathcal{M} \otimes_{\mathcal{O}_{\mathbf{X}}} \Omega^\bullet_{\mathbf{X}/\mathbf{S}}) & \xrightarrow{\phi} & \tilde{g}'_{0*}(\mathcal{M}' \otimes_{\mathcal{O}_{\mathbf{X}'}} \Omega^\bullet_{\mathbf{X}'/\mathbf{S}'})
\end{array}
$$

where $\mathcal{M} \otimes_{\mathcal{O}_{\mathbf{X}}} \Omega^\bullet_{\mathbf{X}/\mathbf{S}}$ and $\mathcal{M}' \otimes_{\mathcal{O}_{\mathbf{X}'}} \Omega^\bullet_{\mathbf{X}'/\mathbf{S}'}$ are the de Rham complex associated to the topologically quasi-nilpotent connections corresponding to the crystal E and $h^*_{\mathrm{crys}} E$ respectively via the map in Theorem 1.15.

Proof of (a). We see from the comparison theorem [Sta19, Tag 07LG] that $\mathbf{R} g_{X/\mathbf{S}_*}(E)$ has bounded cohomologies whose bound depends only on the relative dimension of $g_0 \colon X \longrightarrow S$, so the proof of (a) is finished. $\square$

Proof of (b). Replacing the affine schemes $\mathbf{X}, \mathbf{S}, \mathbf{X}', \mathbf{S}'$ by the rings B, A, B', A' and the quasi-coherent sheaves $\mathcal{M}, \mathcal{M}'$ by modules M, M' respectively we obtain the map

$$
\phi \colon \mathbf{L}u^*(M \otimes_B \Omega^\bullet_{B/A}) \longrightarrow M' \otimes_{B'} \Omega^\bullet_{B'/A'}
$$

which is the ring version of the map ϕ in (2.6) (which we still call ϕ). The functoriality in Theorem 1.15 tells us that $M' = M \otimes_B B'$. Since $\Omega^\bullet_{B'/A'} = \Omega^\bullet_{B/A} \otimes_A A'$ we see that the target of ϕ is just $u^*(M \otimes_B \Omega^\bullet_{B/A})$. If we denote by $Q^\bullet$ the bounded complex of A-modules $M \otimes_B \Omega^\bullet_{B/A}$, it follows that the map ϕ we are considering is the canonical map

(2.7)
$$
\mathbf{L}u^*(Q^\bullet) \longrightarrow u^* Q^\bullet.
$$

When u is flat the map (2.7) is a quasi-isomorphism. The same holds if E is flat because in this case $Q^\bullet$ is a complex of flat A-modules. $\square$

Proof of (c). We proceed as in (b) and get the map (2.7). Assume that E is W_n-flat and that there is a map of rings $W_n \longrightarrow R$ such that $A' = A \otimes_{W_n} R$ as an A-algebra, then the module M is W_n-flat in $\mathsf{Mod}(B)$ and therefore it is flat as W_n-module. Therefore the complex $Q^\bullet$ is a complex of flat W_n-modules. Using the flatness of A and Q^i over W_n one can easily check that

$$
\mathbf{L}u^*(Q^\bullet) = Q^\bullet \overset{\mathbf{L}}{\otimes}_A A' \simeq Q^\bullet \overset{\mathbf{L}}{\otimes}_A (A \otimes_{W_n} R) \simeq Q^\bullet \overset{\mathbf{L}}{\otimes}_{W_n} R \simeq Q^\bullet \otimes_{W_n} R.
$$

Thus (2.7) is an isomorphism. $\square$

Proof of (d). We proceed as in (b) and get the map (2.7). We consider the converging cohomological spectral sequences [Wei94, Proposition 5.7.6, with the convention on Dual Definition 5.2.3]

$$
E_2^{xy} = \mathrm{H}^x(\mathbf{L}^y u^*(Q^\bullet)) \Rightarrow \mathbf{L}^{x+y} u^*(Q^\bullet) = H^{x+y}
$$

where $\mathbf{L}^y u^*(Q^\bullet)$ is the complex obtained by applying $\mathbf{L}^y u^*$ on each terms of $Q^\bullet$. This sequence is obtained from the double complex made by the projective resolutions of the modules in $Q^\bullet$.

It is a fourth quadrant spectral sequence, *i.e.* $E_2^{xy} = 0$ when $y > 0$ or $x < 0$ or $x > m$ (where m is the relative dimension of g_0).

By Lemma 2.9, from the spectral sequence, we obtain a map

(2.8) $$H^i = \mathbf{L}^i u^*(Q^\bullet) \longrightarrow \mathrm{H}^i(u^* Q^\bullet) = E_2^{i0}$$

which is the i-th cohomology of the map we are considering.

By Lemma 1.26 we have the following diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X_{\mathbf{W}} \coloneqq \operatorname{Spec} \widetilde{B} \\
\downarrow & & \downarrow \\
S & \longrightarrow & S_{\mathbf{W}} \coloneqq \operatorname{Spec} \widetilde{A} \\
\downarrow & & \downarrow \\
\operatorname{Spec} k & \longrightarrow & \mathbf{W}
\end{array}
$$

where $\widetilde{A}$ is the p-adically complete flat lift of the smooth k-algebra A/I to W, and $\widetilde{B}$ is the p-adically complete flat lift of the smooth A/I-algebra B/I to $\widetilde{A}$. Since $\widetilde{A}$ is p-adically formally smooth over W and $A \longrightarrow A/I$ is a quotient of p-adically discrete W-algebras defined by a nil ideal, we can choose a W-map $\widetilde{A} \longrightarrow A$. In the same way, we can choose an $\widetilde{A}$-map $\widetilde{B} \longrightarrow B$.

If $(\hat{M}, \hat{\nabla}) \in \operatorname{QNCf}(X/\mathbf{W})$ is the quasi-nilpotent connection corresponding to $E_{\mathbf{W}} \in \operatorname{Crys}(X/\mathbf{W})$ via Theorem 1.15, then $\hat{M} \otimes_{\widetilde{B}} B \simeq M$ by the crystalline nature of $E_{\mathbf{W}}$. Applying Lemma 2.10 to $\hat{M}$ and the functors

$$F_{yt}(-) \coloneqq \mathbf{L}^y u^*(- \otimes_{\widetilde{B}} \Omega^t_{B/A}) \qquad (y \in \mathbb{Z}_{\neq 0}, t \in \mathbb{N})$$

we find $N' \in \mathbb{N}$ such that $p^{N'}$ kills $F_{yt}(\hat{M})$ hence also $\mathrm{H}^x(F_{y\bullet}(\hat{M})) = E_2^{xy}$ ($y \neq 0$). Notice that $N' \in \mathbb{N}$ depends only on the $\widetilde{B}$-module $\hat{M}$ and the $\widetilde{B}$-module $\hat{M}$ depends only on $E_{\mathbf{W}} \in \operatorname{Crys}(X/\mathbf{W})$.

Set $N_i \coloneqq (m+1)N'$ for all $i \in \mathbb{Z}$, where m is the relative dimension of g_0. By Lemma 2.9, the i-th cohomology of (2.1) is a p^{N_i}-isogeny. $\qquad \square$

The proof of the theorem is done. $\qquad \square$

Remark 2.11. Note that in the proof of Theorem 2.7 (a), (d), the bound r and the function $N : \mathbb{Z} \longrightarrow \mathbb{N}$ depend not only on the relative dimension of g_0, but also on the number of opens in the affine covering $\{U_i\}_{i=0,\cdots,n}$ of X and the affine coverings of the arbitrary intersections of $\{U_i\}_{i=0,\cdots,n}$. Indeed this was used during the reduction of X to the affine case (see the two paragraphs after (2.5)). Since this is a choice on X which is part of the map $g_0 : X \longrightarrow S$, we didn't specify it.

2.2. **The case of an affine base.** In this section we treat the case in which the base $\mathbf{S}$ is affine. The first result is a corollary of the base change theorem proven in the previous subsection (Theorem 2.7).

Corollary 2.12. *In the situation of* **Setting** *2.1 assume that* $\mathbf{S} = \operatorname{Spec} A$ *is affine and set* $A_n = A/p^n$, $\mathbf{S}_n = \operatorname{Spec} A_n$ *and* $E_n = E_{|(X/\mathbf{S}_n)_{\mathrm{crys}}}$. *Then the following hold.*

(a) *There exists* $r \in \mathbb{N}$, *which depends only on* $X \xrightarrow{g_0} S$ *such that for all* $i \geqslant r$ *we have* $\mathbf{R}^i\Gamma((X/\mathbf{S}_n)_{\mathrm{crys}}, E_n) = 0$. *Moreover, we have*

(2.9) $$\mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E) \simeq \mathbf{R}\varprojlim_n \mathbf{R}\Gamma((X/\mathbf{S}_n)_{\mathrm{crys}}, E_n)$$

is quasi-isomorphic to a bounded complex.

(b) If **S** is flat over **W** and E is p-torsion free then the system $\{\mathbf{R}\Gamma((X/\mathbf{S}_n)_{\mathrm{crys}}, E_n)\}$ is quasi-consistent in the sense of [BO78, B.4] and, if moreover **S** is Noetherian, then
$$\mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E) \overset{\mathbf{L}}{\otimes}_A A_n \simeq \mathbf{R}\Gamma((X/\mathbf{S}_n)_{\mathrm{crys}}, E_n).$$

(c) If **S** is flat over **W** and Noetherian, E is p-torsion free, X/S is proper and $S = S_1$ ($I = (p)$), then $\mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E)$ is quasi-isomorphic to a bounded complex of finitely generated $\hat{A}$-modules, where $\hat{A}$ is the p-adic completion of A, and
$$\mathrm{H}^i((X/\mathbf{S})_{\mathrm{crys}}, E) \simeq \varprojlim_n \mathrm{H}^i((X/\mathbf{S}_n)_{\mathrm{crys}}, E_n).$$

Moreover, the projective system on the right hand side satisfies the Mittag-Leffler condition, and is made by finitely generated $\hat{A}$-modules.

Remark 2.13. The proof of Corollary 2.12 is the same as the proof of [ES19, Proposition 5.3 1)] and [Shi08a, **claim** in pp. 10–11] .

Proof of Corollary 2.12. Firstly, notice that we can replace A by its p-adic completion thanks to [Sta19, 05GG].

(a) The isomorphism (2.9) follows from [Sta19, Tag 07MV]. By Theorem 2.7 (a) and [BO78, Remark B.1.6] we also get the boundness.

(b) The quasi-consistency follows from Lemma 2.4 and Theorem 2.7 (c) because the maps $\mathbf{S}_{n-1} \longrightarrow \mathbf{S}_n$ are base changes of the maps $\operatorname{Spec} W_{n-1} \longrightarrow \operatorname{Spec} W_n$. From the quasi-consistency and [BO78, Proposition B.5, 3)] we obtain the last isomorphism.

(c) Assume that **S** flat over **W**, E is p-torsion free, X/S is proper and $S = S_1$ ($I = (p)$). Since $\mathbf{R}\Gamma((X/\mathbf{S}_1)_{\mathrm{crys}}, E_1)$ has finitely generated cohomologies and all the $\mathbf{R}\Gamma((X/\mathbf{S}_n)_{\mathrm{crys}}, E_n)$ are uniformly cohomologically bounded thanks to [BO78, p.7.7], the result follows from [BO78, Lemma B.6 and Proposition B.7]. Here we use that a bounded complex with finitely generated cohomology is quasi-isomorphic to a bounded complex of finitely generated modules. $\square$

Always in **Setting** 2.1, we consider now the situation in Remark 2.2 (b). We analyse under which condition the map in (2.2) is an isomorphism (or an isogeny).

Theorem 2.14. *Let the notation and hypothesis be as in* **Setting** *2.1. Assume moreover that* **S** *is Noetherian and* **W** *-flat. Let* $\mathbf{S}' := \operatorname{Spec} A'$, $\mathbf{S} := \operatorname{Spec} A$, *where* A *and* A' *are* p*-adically complete rings. Suppose that one of the following is true:* p *is nilpotent in* $\mathcal{O}_{\mathbf{S}'}$ *or* X/S *is proper and* $S = \operatorname{Spec} A/p$ *(i.e.* $I = (p)$*). Then the following hold.*

(a) *Let* $E_{\mathbf{W}} \in \operatorname{Crys}(X/\mathbf{W})$ *and set* $E = (E_{\mathbf{W}})_{|(X/\mathbf{S})_{\mathrm{crys}}}$. *Assume that* S *is smooth over* k. *Then there exists* $N \colon \mathbb{Z} \longrightarrow \mathbb{N}$, *depending only on* $E_{\mathbf{W}}$ *and* g_0, *such that the* i*-th cohomology of the map (2.2) is a* p^{N_i}*-isogeny isomorphism.*

(b) *The map (2.2) is an isomorphism if* $E \in \operatorname{Crys}(X/\mathbf{S})$ *is a flat crystal* [BO78, p. 7.10].

(c) *The map (2.2) is an isomorphism if* $E \in \operatorname{Crys}(X/\mathbf{S})$ *is* p*-torsion free and all* $u_n \colon \mathbf{S}'_n \longrightarrow \mathbf{S}_n$ *are either flat or the base change of a map to* $\mathbf{W}_n$.

Before proving this theorem, we consider two remarks.

Remark 2.15. If M is a flat W-module, that is it is p-torsion free, then so is its p-adic completion. Indeed let $m_n \in M$ be a collection of elements such that $m_{n+1} - m_n = p^n x_n \in p^n M$ and $pm_n = p^n y_n \in p^n M$. Then $m_n = p^{n-1} y_n$ and
$$p^n y_{n+1} - p^{n-1} y_n = p^n x_n \implies y_n \in pM \implies m_n \in p^n M.$$

Remark 2.16. [ES18, after Remark 2.5] If X is a smooth and quasi-compact k-scheme and $E_{\mathbf{W}} \in \mathrm{Crys}(X/\mathbf{W})$, then there exists a p-torsion free $E'_{\mathbf{W}} \in \mathrm{Crys}(X/\mathbf{W})$ and an isogeny $E_{\mathbf{W}} \longrightarrow E'_{\mathbf{W}}$. Indeed one can check locally, using Proposition 1.27 and Theorem 1.15, that the sequence $E_{\mathbf{W}}[p^n]$ stabilizes to a subobject $E_{\mathbf{W}}[p^\infty]$ which is killed by a power of p. Thus $E'_{\mathbf{W}} = E_{\mathbf{W}}/(E_{\mathbf{W}}[p^\infty])$ meet the requirements.

Proof of Theorem 2.14. By Remark 2.16 we can assume that E is p-torsion free in (a). If E is a flat crystal, then E is p-torsion free by Lemma 2.6.

Now, for $n \in \mathbb{N}$, let $u_n = u \times \mathbf{W}_n \colon \mathbf{S}'_n \longrightarrow \mathbf{S}_n$ and consider the base change map

$$(2.10) \qquad \mathbf{L}u_n^* \mathbf{R}\Gamma((X/\mathbf{S}_n)_{\mathrm{crys}}, E_n) \longrightarrow \mathbf{R}\Gamma((X'/\mathbf{S}'_n)_{\mathrm{crys}}, h_{\mathrm{crys}}^* E_n).$$

Firstly we would like to prove that the $\mathbf{R}\varprojlim$ of (2.10) yields the map (2.2).

If, for some $a \in \mathbb{N}$, $p^a = 0$ in $\mathcal{O}_{\mathbf{S}'}$, the map $u \colon \mathbf{S}' \longrightarrow \mathbf{S}$ factors through u_n for $n \geqslant a$ and therefore

$$\mathbf{L}u^* \mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E) \simeq \mathbf{L}u_n^* \mathbf{R}\Gamma((X/\mathbf{S}_n)_{\mathrm{crys}}, E_n).$$

So what remains is the case where p is not nilpotent in $\mathcal{O}_{\mathbf{S}'}$ and X/S is proper and $S = \mathrm{Spec}\, A/p$ (*i.e.* $I = (p)$). By Corollary 2.12 (c) we have that $\mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E)$ is quasi-isomorphic to a complex $P^\bullet$ of A-modules which is bounded above and it is made by finite free A-modules. In this case, by Corollary 2.12 (b), $\mathbf{R}\Gamma((X/\mathbf{S}_n)_{\mathrm{crys}}, E_n) \simeq P_n^\bullet := P^\bullet \otimes_A A_n$ and

$$\mathbf{L}u_n^* P_n^\bullet \simeq P_n^\bullet \otimes_{A_n} A'_n.$$

This is a complex of flasque projective systems in the sense of [BO78, Remark B.1.4]. In particular by [BO78, Remark B.1.6] we have

$$\mathbf{R}\varprojlim \mathbf{L}u_n^* P_n^\bullet \simeq \varprojlim[(P^\bullet \otimes_A A') \otimes_{A'} A'_n] \simeq P^\bullet \otimes_A A'.$$

The last isomorphism holds because $P^\bullet \otimes A'$ is a complex of finite free A'-modules which are therefore complete. Since $P^\bullet \otimes_A A' \simeq \mathbf{L}u^* \mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E)$ we get the result.

(a) Applying Theorem 2.7 (d) we know that there exists $N \colon \mathbb{Z} \longrightarrow \mathbb{N}$, which depends only on $E_{\mathbf{W}}$ and g_0 (thus not on n), such that the i-th cohomology of (2.10) is a p^{N_i}-isogeny. Letting n vary we can consider (2.10) as a map of complexes in $D^-(\mathbb{N}, A'_*)$ whose i-th cohomology is a p^{N_i}-isogeny. By Theorem 2.7 (a) we can suppose $N_i = 0$ for $i \gg 0$, and by [BO78, Remark B.1.6] we have that $\mathbf{R}^i \varprojlim = 0$ for $i \geqslant 2$. Now applying $\mathbf{R}^i \varprojlim$ to (2.10) we get our result by Lemma 2.8.

(b) We consider, as in (a), the map in (2.10). Applying Theorem 2.7 (b) we get that the map (2.10) is a quasi-isomorphism. Again applying $\mathbf{R}^i \varprojlim$ to (2.10) yields the quasi-isomorphism (2.2).

(c) The proof is exactly as in (b), using Theorem 2.7 (c). $\qquad \square$

Remark 2.17. A result along the same lines is proven in [Shi08a, Theorem 1.19] and [ES19, Proposition 5.3].

2.3. **Pullback in the crystalline site revisited.** Suppose that we are in **Siuation** 2.1. In what follows we collect some properties of pullback of sheaves in the crystalline topoi, following the discussion in [Ber74, Chapter III, Section 2.2, p. 196]. We denote by

$$h_{\mathrm{crys}}^{-1} \colon (X/\mathbf{S})_{\mathrm{crys}}^\sim \longrightarrow (X'/\mathbf{S}')_{\mathrm{crys}}^\sim$$

the pullback in the morphism of topoi h_{crys} (not ringed topoi) induced by the morphism $h \colon X' \to X$. We instead denote by h_{crys}^* the pullback of $\mathcal{O}_{X/\mathbf{S}}$-modules.

Definition 2.4. Given $\mathbf{T}' \in (X'/\mathbf{S}')_{\mathrm{crys}}$ and $\mathbf{T} \in (X/\mathbf{S})_{\mathrm{crys}}$ a h-PD-morphism $\mathbf{T}' \longrightarrow \mathbf{T}$ is a PD-morphism $v \colon \mathbf{T}' \longrightarrow \mathbf{T}$ which is compatible with h and u.

For $\mathbf{T}' \in (X'/\mathbf{S}')_{\mathrm{crys}}$ we define the category

$$I_h^{\mathbf{T}'} = \{h\text{-PD-morphisms } \mathbf{T}' \longrightarrow \mathbf{T} \text{ with } \mathbf{T} \in (X/\mathbf{S})_{\mathrm{crys}}\}.$$

Given $x' \in \mathbf{T}'$ we also define the category

$$I_h^{x',\mathbf{T}'} = \{h\text{-PD-morphisms } \mathbf{V} \longrightarrow \mathbf{T} \text{ with } \mathbf{T} \in (X/\mathbf{S})_{\mathrm{crys}} \text{ and } x' \in \mathbf{V} \subseteq \mathbf{T}' \text{ open}\}.$$

Lemma 2.18. *Let F be a sheaf on $(X/\mathbf{S})_{\mathrm{crys}}$, $\mathbf{T}' \in (X'/\mathbf{S}')_{\mathrm{crys}}$ and $x' \in \mathbf{T}'$. Then*

(1) $h_{\mathrm{crys}}^{-1}(F)$ is the sheafification of the presheaf

$$\mathbf{T}' \mapsto \operatorname*{colim}_{(q \colon \mathbf{T}' \longrightarrow \mathbf{T}) \in I_h^{\mathbf{T}'}} F(\mathbf{T}).$$

(2) for $q \colon \mathbf{V} \longrightarrow \mathbf{T}$ in $I_h^{x',\mathbf{T}'}$ there is a canonical map

$$q^{-1}(F_{\mathbf{T}}) \longrightarrow h_{\mathrm{crys}}^{-1}(F)_{\mathbf{V}}.$$

(3) the set $I_h^{\mathbf{T}',x'}$ is filtered; moreover taking stalks at $x' \in \mathbf{V} \subseteq \mathbf{T}$ of the maps in (2) and passing to the limit we obtain an isomorphism

$$\operatorname*{colim}_{(q \colon \mathbf{V} \longrightarrow \mathbf{T}) \in I_h^{x',\mathbf{T}'}} q^{-1}(F_{\mathbf{T}})_{x'} \longrightarrow (h_{\mathrm{crys}}^{-1}(F)_{\mathbf{T}'})_{x'}.$$

Proof. Point (1) is [Ber74, Chapter III, Section 2, eq (2.2.10)], while (2) is an easy consequence of (1). The proof that $I_h^{x',\mathbf{T}'}$ is filtered is given in the first paragraph of [Ber74, Chapter III, p. 199]. As in [Ber74, Chapter III, eq (2.2.11), p. 199], taking a double limit in (1) we have

$$(h_{\mathrm{crys}}^{-1}(F)_{\mathbf{T}'})_{x'} = \operatorname*{colim}_{(q \colon \mathbf{V} \longrightarrow \mathbf{T}) \in I_h^{x',\mathbf{T}'}} F(\mathbf{T}).$$

By definition of $I_h^{x',\mathbf{T}'}$ it is easy to rewrite the above equation as

$$(h_{\mathrm{crys}}^{-1}(F)_{\mathbf{T}'})_{x'} = \operatorname*{colim}_{(q \colon \mathbf{V} \longrightarrow \mathbf{T}) \in I_h^{x',\mathbf{T}'}} (F_{\mathbf{T}})_{q(x')} = \operatorname*{colim}_{(q \colon \mathbf{V} \longrightarrow \mathbf{T}) \in I_h^{x',\mathbf{T}'}} q^{-1}(F_{\mathbf{T}})_{x'}.$$

$\square$

Lemma 2.19. *Let F be a sheaf of $\mathcal{O}_{X/\mathbf{S}}$-modules on $(X/\mathbf{S})_{\mathrm{crys}}$, $\mathbf{T}' \in (X'/\mathbf{S}')_{\mathrm{crys}}$ and $x' \in \mathbf{T}'$. Then*

(1) for $q \colon \mathbf{V} \longrightarrow \mathbf{T}$ in $I_h^{x',\mathbf{T}'}$ there is a canonical map

$$q^*(F_{\mathbf{T}}) \longrightarrow h_{\mathrm{crys}}^*(F)_{\mathbf{V}}.$$

(2) taking stalks at $x' \in \mathbf{V} \subseteq \mathbf{T}$ of the maps in (2) and passing to the limit we obtain an isomorphism

$$\operatorname*{colim}_{(q \colon \mathbf{V} \longrightarrow \mathbf{T}) \in I_h^{x',\mathbf{T}'}} q^*(F_{\mathbf{T}})_{x'} \longrightarrow (h_{\mathrm{crys}}^*(F)_{\mathbf{T}'})_{x'}.$$

Proof. By definition we have

$$h_{\mathrm{crys}}^*(F) = h_{\mathrm{crys}}^{-1}(F) \otimes_{h_{\mathrm{crys}}^{-1}\mathcal{O}_{X/\mathbf{S}}} \mathcal{O}_{X'/\mathbf{S}'}.$$

Properties (1) and (2) follows from Lemma 2.18, taking into account that tensor products commute with filtered colimits. $\square$

Lemma 2.20. *Let $F^\bullet$ be a complex of sheaves of $\mathcal{O}_{X/\mathbf{S}}$-modules on $(X/\mathbf{S})_{\mathrm{crys}}$, $\mathbf{T}' \in (X'/\mathbf{S}')_{\mathrm{crys}}$ and $x' \in \mathbf{T}'$. Then*

(1) for $q\colon \mathbf{V} \longrightarrow \mathbf{T}$ in $I_h^{x',\mathbf{T}'}$ there is a canonical map of complexes

$$q^*(F_{\mathbf{T}}^\bullet) \longrightarrow h_{\mathrm{crys}}^*(F^\bullet)_{\mathbf{V}};$$

(2) for all $j \geqslant 0$, taking j-th cohomology, stalks at x' and passing to the limit we obtain an isomorphism

$$\operatorname*{colim}_{(q\colon \mathbf{V} \longrightarrow \mathbf{T}) \in I_h^{x',\mathbf{T}'}} \mathrm{H}^j(q^*(F_{\mathbf{T}}^\bullet))_{x'} \longrightarrow (\mathrm{H}^j(h_{\mathrm{crys}}^* F^\bullet)_{\mathbf{T}'})_{x'};$$

(3) if $F^\bullet$ is bounded from above then we have a canonical isomorphism

$$\operatorname*{colim}_{(q\colon \mathbf{V} \longrightarrow \mathbf{T}) \in I_h^{x',\mathbf{T}'}} (\mathbf{L}^j q^*(F_{\mathbf{T}}^\bullet))_{x'} \longrightarrow ((\mathbf{L}^j h_{\mathrm{crys}}^* F^\bullet)_{\mathbf{T}'})_{x'}.$$

Proof. (1), (2) follows literally from Lemma 2.19 (1) and (2) respectively. If $F^\bullet$ is bounded, then by [BO78, p. 7.7-7.8] we can replace $F^\bullet$ by a complex of flat $\mathcal{O}_{X/\mathbf{S}}$-modules. By [Ber74, Chapter III, Cor 3.5.2, p. 211] we have that $F_{\mathbf{T}}$ is a complex of flat $\mathcal{O}_{\mathbf{T}}$-modules for any $\mathbf{T} \in (X/\mathbf{S})_{\mathrm{crys}}$. We can therefore replace $\mathbf{L}^j$ in (3) by H^j, but this is just (2). $\qquad\square$

2.4. Crystalline base change.

Definition 2.5. Let $f\colon X \to S$ be a morphism of k-schemes. There is a morphism of ringed topoi [Sta19, Tag 07IK]

$$f_{\mathrm{crys}}\colon ((X/\mathbf{W})_{\mathrm{crys}}^\sim, \mathcal{O}_{X/\mathbf{W}}) \longrightarrow ((S/\mathbf{W})_{\mathrm{crys}}^\sim, \mathcal{O}_{S/\mathbf{W}}).$$

For a sheaf of $\mathcal{O}_{X/\mathbf{W}}$-modules E on $(X/\mathbf{W})_{\mathrm{crys}}$ we consider the higher direct images $\mathbf{R}^n f_{\mathrm{crys}*} E$ and also $K \otimes \mathbf{R}^n f_{\mathrm{crys}*} E$, which belong to $K \otimes \mathsf{Mod}(\mathcal{O}_{S/\mathbf{W}})$.

Theorem 2.21. *Consider a cartesian diagram*

$$
\begin{array}{ccc}
X' & \stackrel{h}{\longrightarrow} & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
S' & \stackrel{v}{\longrightarrow} & S
\end{array}
$$

of quasi-compact k-schemes with f smooth and quasi-compact. Let $E \in \mathrm{Crys}(X/\mathbf{W})$ and assume that E is flat (resp. S is smooth over k). Then there is a natural map in $D((S'/\mathbf{W})_{\mathrm{crys}})$

(2.11) $$\mathbf{L}v_{\mathrm{crys}}^* \mathbf{R}f_{\mathrm{crys}_*}(E) \longrightarrow \mathbf{R}f'_{\mathrm{crys}_*} h_{\mathrm{crys}}^*(E)$$

which is an isomorphism (resp. induces isogeny on coholomogy).

Proof. The definition of the map in the statement is also given in the proof of [Ber74, Chapter V, Theorem 3.5.1, p. 342]. Applying adjunction to the canonical map $\mathbf{L}h_{\mathrm{crys}}^*(E) \to h_{\mathrm{crys}}^*(E)$ we obtain a map

$$E \to \mathbf{R}h_{\mathrm{crys}*}(h_{\mathrm{crys}}^*(E)).$$

Applying $\mathbf{R}f_{\mathrm{crys}*}$ we get

$$\mathbf{R}f_{\mathrm{crys}*}(E) \to \mathbf{R}v_{\mathrm{crys}*} \mathbf{R}f'_{\mathrm{crys}*}(h_{\mathrm{crys}}^*(E)).$$

The map (2.11) is obtained applying adjunction again, which is possible because $\mathbf{R}f_{\mathrm{crys}*}(E)$ is bounded: if $(U, \mathbf{T}, \delta) \in (S/W)_{\mathrm{crys}}$ then ([Sta19, Tag 07MJ], [Ber74, Corollaire V, 3.2.3, p. 328])

$$(\mathbf{R}f_{\mathrm{crys}*}(E))_{\mathbf{T}} \simeq \mathbf{R}(f_{|f^{-1}(U)})_{f^{-1}(U)/\mathbf{T}*}(E_{|(f^{-1}(U)/\mathbf{T})_{\mathrm{crys}}}),$$

which is bounded uniformly thanks to Theorem 2.7 (a).

The case when E is flat is essentially contained in [Ber74, Chapter V, Theorem 3.5.1, p. 342], but we include the proof for completeness.

Let's fix $(U', \mathbf{T}', \delta) \in (S'/\mathbf{W})_{\mathrm{crys}}$ and $x' \in \mathbf{T}'$. It is enough to check that the map

$$(2.12) \qquad (\mathbf{L}^j v_{\mathrm{crys}}^*(\mathbf{R}f_{\mathrm{crys}*}(E))_{\mathbf{T}'})_{x'} \longrightarrow (\mathbf{R}^j f'_{\mathrm{crys}*}(h_{\mathrm{crys}}^*(E))_{\mathbf{T}'})_{x'}$$

is a quasi-isomorphism (resp. p^{N_j}-isogeny for some N_j) for all $\mathbf{T}'$ and x'. We follow notation from §2.3, for instance recall that $I_v^{x', \mathbf{T}'}$ is the filtered category of v-PD-morphisms $\mathbf{V} \longrightarrow \mathbf{T}$ where $x \in \mathbf{V} \subseteq \mathbf{T}'$ is an open and $\mathbf{T} \in (S/\mathbf{W})_{\mathrm{crys}}$.
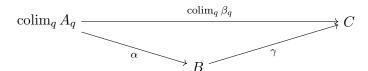
Let $q: \mathbf{V} \longrightarrow \mathbf{T}$ be an object of $I_v^{x', \mathbf{T}'}$. By Lemma 2.20 we have maps

$$\mathbf{L}q^*(\mathbf{R}f_{\mathrm{crys}*}(E)_{\mathbf{T}}) \xrightarrow{\quad c_q \quad} \mathbf{R}f'_{\mathrm{crys}*}(h_{\mathrm{crys}}^*(E))_{\mathbf{V}}.$$

with maps a_q to $\mathbf{L}v_{\mathrm{crys}}^*(\mathbf{R}f_{\mathrm{crys}*}(E))_{\mathbf{V}}$ and b_q from it.

By [Sta19, Tag 07MJ] the map c_q is the map considered in Theorem 2.7 (c) (resp. (d)). Therefore, c_q is a quasi-isomorphism (resp. we find $N: \mathbb{Z} \longrightarrow \mathbb{N}$ depending only on f and E, such that the map $\mathrm{H}^j(c_q)$ is an p^{N_j}-isogeny).

Now, on the diagram above, we take j-th cohomology and the stalk at x'. The map b_q becomes the map (2.12). This map and, in particular, its source and target do not depend on $q \in I_v^{x', \mathbf{T}'}$. Let's call it $B \longrightarrow C$. Passing to the colimit for $q \in I_v^{x', \mathbf{T}'}$ (at the level of complexes) we get the diagram of the form

$$\mathrm{colim}_q A_q \xrightarrow{\quad \mathrm{colim}_q \beta_q \quad} C$$

with map α to B and map γ from B to C.

with the map α an isomorphism by Lemma 2.20. If c_q is a quasi-isomorphism, then so is $\mathrm{colim}_q \beta_q$, hence so is γ. This finishes the proof in the case when E is flat.

Let's now focus on the "resp." case. Taking the limit of the exact sequence

$$0 \longrightarrow K_q \longrightarrow A_q \xrightarrow{\beta_q} C \longrightarrow D_q \longrightarrow 0$$

we obtain that

$$\mathrm{Ker}(\gamma) \simeq \mathrm{colim}_q K_q \text{ and } \mathrm{Coker}(\gamma) \simeq \mathrm{colim}_q D_q.$$

Because all the β_q are p^{N_j}-isogenies, p^{N_j} kills all K_q, D_q and therefore $\mathrm{Ker}(\gamma)$ and $\mathrm{Coker}(\gamma)$, as required. $\qquad\square$

3. HIGHER PUSH-FORWARD OF ISOCRYSTALS

This section is dedicated to the proof of Theorem I.

Theorem 3.1. *Let $f\colon X \to S = \operatorname{Spec} A$ be a smooth and proper morphism between smooth k-schemes, and let $\mathcal{A}$ be a p-adically complete flat lift of A over W and $E \in \operatorname{Crys}(X/\mathbf{W})$ be a p-torsion free crystal. Then for each $n \in \mathbb{N}$ there is a crystal $E_{X/\mathcal{A}}^n$ in $\operatorname{Crys}(S/\mathbf{W})$ with a morphism of sheaves $\eta_n\colon E_{X/\mathcal{A}}^n \longrightarrow \mathbf{R}^n f_{\mathrm{crys}_*}(E)$ on the crystalline site $(S/\mathbf{W})_{\mathrm{crys}}$ which induces the isomorphism*

$$\varprojlim_e (E_{X/\mathcal{A}}^n)_{\operatorname{Spec}(\mathcal{A}/p^e)} \simeq \varprojlim_e (\mathbf{R}^n f_{\mathrm{crys}_*}(E))_{\operatorname{Spec}(\mathcal{A}/p^e)}.$$

Moreover,

$$\eta_n \otimes K\colon E_{X/\mathcal{A}}^n \otimes K \longrightarrow \mathbf{R}^n f_{\mathrm{crys}_*}(E) \otimes K$$

is an isomorphism and $E_{X/\mathcal{A}}^n$ corresponds, via Theorem 1.15, to the $\mathcal{A}$-module

$$\mathrm{H}^n((X/\mathbf{S})_{\mathrm{crys}}, E_{|(X/\mathbf{S})_{\mathrm{crys}}})$$

equipped with a topologically quasi-nilpotent connection.

Proof of Theorem I as a consequence of Theorem 3.1. By Remark 2.16 we can assume $\mathcal{E} = E \otimes K$, where $E \in \operatorname{Crys}(X/\mathbf{W})$ is p-torsion free. By Theorem 3.1 the statement is true when S is affine. By descent for isocrystals ([Ogu90, Lemma 0.7.5]), we can conclude that an $\mathcal{O}_{X/\mathbf{W}}$-module on $(X/\mathbf{W})_{\mathrm{crys}}$ in the isogeny category is an isocrystal if and only if it is Zariski locally so. This finishes the proof. $\square$

Proof of Theorem 3.1. Set $\mathcal{A}_e = \mathcal{A}/p^e$, $\mathbf{S}_e = \operatorname{Spec}\mathcal{A}_e$,

$$E_{\mathbf{S}} = E_{|(X/\mathbf{S})_{\mathrm{crys}}} \text{ and } H_n = \mathrm{H}^n((X/\mathbf{S})_{\mathrm{crys}}, E_{|(X/\mathbf{S})_{\mathrm{crys}}}).$$

We construct the crystal $E_{X/\mathcal{A}}^n$ in $\operatorname{Crys}(S/W)$ with the morphism $\eta_n\colon E_{X/\mathcal{A}}^n \longrightarrow \mathbf{R}^n f_{\mathrm{crys}_*}(E)$. Let $\mathbf{D}(e)$ be the p-adic completion of the PD-envelope of S inside $\mathbf{S} \times_{\mathbf{W}} \mathbf{S} \cdots \times_{\mathbf{W}} \mathbf{S}$ (the fiber product over $\mathbf{W}$ of e copies of $\mathbf{S}$). Since S is smooth, the projections

$$p_i : \mathbf{D}(e) \longrightarrow \mathbf{S}$$

are flat ([BO78, p. 3.32], [Sta19, Tag 0912]). By Theorem 2.14 (c) we get canonical isomorphisms

$$p_i^* \mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E_{\mathbf{S}}) \longrightarrow \mathbf{R}\Gamma((X/\mathbf{D}(e))_{\mathrm{crys}}, E_{|(X/\mathbf{D}(e))_{\mathrm{crys}}}).$$

Taking cohomology we also get canonical isomorphisms

$$p_i^* H_n \longrightarrow \mathrm{H}^n((X/\mathbf{D}(e))_{\mathrm{crys}}, E_{|(X/\mathbf{D}(e))_{\mathrm{crys}}}).$$

This defines an HPD-stratification on the $\mathcal{A}$-module H_n, which is finitely generated by Corollary 2.12. Similarly to [BO78, p. 6.6], this HPD-stratification defines a crystal $E_{X/\mathcal{A}}^n \in \operatorname{Crys}(S/\mathbf{W})$. Let's recall here its construction.

For each object $\chi = (U, \mathbf{T}, \delta) \in (S/\mathbf{W})_{\mathrm{crys}}$ with $\mathbf{T}$ affine we get, thanks to [Sta19, Tag 07K4] and the smoothness of S, a commutative diagram

We set $(E_{X/\mathcal{A}}^n)_\mathbf{T}$ to be the quasi-coherent sheaf on $\mathbf{T}$ associated to $\alpha_\chi^* H_n$. The structure of the HPD-stratification allows us to define the transition morphisms and to prove the functoriality of the correspondence $\chi = (U, \mathbf{T}, \delta) \mapsto (E_{X/\mathcal{A}}^n)_\mathbf{T}$.

By Theorem 2.14 (a) with $S' = U$ and $\mathbf{S}' = \mathbf{T}$ there exists $N \colon \mathbb{Z} \longrightarrow \mathbb{N}$, depending only on E and f, such that the i-th cohomology of

$$(3.1) \qquad \gamma_\chi \colon \mathbf{L}\alpha_\chi^* \mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E_\mathbf{S}) \longrightarrow \mathbf{R}\Gamma((f^{-1}(U)/\mathbf{T})_{\mathrm{crys}}, E_{|(f^{-1}(U)/\mathbf{T})_{\mathrm{crys}}})$$

is a p^{N_i}-isogeny.

Notice that ([Sta19, Tag 07MJ], [Ber74, Corollaire V.3.2.3, p. 318])

$$(\mathbf{R}f_{\mathrm{crys}*}(E))_\mathbf{T} \simeq \mathbf{R}f_{f^{-1}(U)/\mathbf{T}*}(E_{|(f^{-1}(U)/\mathbf{T})_{\mathrm{crys}}})$$

is quasi-isomorphic to the complex of $\mathcal{O}_\mathbf{T}$-modules associated to any complex of $\mathrm{H}^0(\mathcal{O}_\mathbf{T})$-modules representing the right hand side of (3.1).

Moreover there is a canonical map

$$\iota_\chi \colon \alpha_\chi^* H_n \longrightarrow \mathrm{H}^n(\mathbf{L}\alpha_\chi^* \mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E_\mathbf{S})).$$

Putting everything together we get a canonical morphism

$$(\eta_n)_\mathbf{T} \colon (E_{X/\mathcal{A}}^n)_\mathbf{T} \longrightarrow (\mathbf{R}^n f_{\mathrm{crys}*}(E))_\mathbf{T}.$$

If $\chi = (S, \mathbf{S}_e, \delta_e)$ and $\alpha_\chi \colon \mathbf{S}_e \longrightarrow \mathbf{S}$ is the obvious closed immersion, then, by Theorem 2.14 (c), the map γ_χ is a quasi-isomorphism and $(\eta_n)_{\mathbf{S}_e}$ becomes the map of quasi-coherent sheaves on $\mathbf{S}_e$ associated to the map

$$H_n \otimes \mathcal{A}_e \longrightarrow \mathrm{H}^n((X/\mathbf{S}_e)_{\mathrm{crys}}, E_{|(X/\mathbf{S}_e)_{\mathrm{crys}}}).$$

By Corollary 2.12 the projective limit of the above maps is an isomorphism as required. The limit H_n, which corresponds to $E_{X/\mathcal{A}}^n$ via Theorem 1.25, is therefore the module with the topologically quasi-nilpotent connection in the statement.

It remains to show that $\eta_n \otimes K$ is an isomorphism. It is enough to show that there exists a $\overline{N} \in \mathbb{N}$ such that for all $\chi = (U, \mathbf{T}, \delta) \in (S/W)_{\mathrm{crys}}$ the map $(\eta_n)_\mathbf{T}$ is a $p^{\overline{N}}$-isogeny. Since γ_χ is a p^{N_n}-isogeny, we have to prove the analogous statement for ι_χ.

Set $M := \mathbf{R}\Gamma((X/\mathbf{S})_{\mathrm{crys}}, E_\mathbf{S})$. By [Wei94, Proposition 5.7.6, with the convention on Dual Definition 5.2.3] there is a convergent spectral sequence

$$E_2^{uv} = \mathbf{L}^u \alpha_\chi^*(\mathrm{H}^v(M)) \Rightarrow \mathbf{L}^{u+v}\alpha_\chi^*(M) = H^{u+v}.$$

Since M is bounded there exists $l \in \mathbb{N}$ such that $E_2^{uv} = 0$ for $v < 0$ or $v > l$. Moreover $E_2^{uv} = 0$ if $u > 0$. By Lemma 2.9 we obtain a map

$$E_2^{0n} = \alpha_\chi^*(\mathrm{H}^n(M)) \longrightarrow \mathbf{L}^n \alpha_\chi^*(M) = H^n$$

which coincides with the map ι_χ.

Since $\mathrm{H}^v(M) = H_v$ is endowed with a topologically quasi-nilpotent connection on $\mathcal{A}$, by Lemma 2.10 there exists $N_v \in \mathbb{N}$, depending only on H_v, such that $\mathbf{L}^q \alpha_\chi^*(\mathrm{H}^v(M))$ is killed by p^{N_v} for any $q \neq 0$. Since $\mathbf{L}^q \alpha_\chi^*(\mathrm{H}^v(M)) = 0$ for all $v < 0$ or $v > l$, we can choose N large, so that it kills $\mathbf{L}^q \alpha_\chi^*(\mathrm{H}^v(M))$ for all $q \neq 0$ and v. Thus Lemma 2.9 tells us that ι_χ is a $p^{N(l+1)}$-isogeny. $\qquad\square$

Remark 3.2. We want to compare [Xu19, Theorem 1.9] and Theorem I and, in particular, show how they are compatible. Assume the common settings for those results, that is, let $f\colon X \longrightarrow S$ be a smooth and proper morphism of smooth k-schemes and $\mathcal{E} \in I_{\mathrm{conv}}(X/\mathbf{W})$, where $I_{\mathrm{conv}}(X/\mathbf{W})$ denotes the category of convergent isocrystals.

By [Ogu84, Theorem 0.7.2], there is a fully faithful functor $\iota\colon I_{\mathrm{conv}}(X/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(X/\mathbf{W})$ and similarly for S. Moreover, $\mathbf{R}^i f_{\mathrm{conv}*}(\mathcal{E}) \in I_{\mathrm{conv}}(S/\mathbf{W})$ by [Xu19, Theorem 1.9] and $\mathbf{R}^i f_{\mathrm{crys}*}(\iota(\mathcal{E})) \in I_{\mathrm{crys}}(S/\mathbf{W})$ by Theorem I. We claim that there is a canonical isomorphism

$$\iota(\mathbf{R}^i f_{\mathrm{conv}*}(\mathcal{E})) \simeq \mathbf{R}^i f_{\mathrm{crys}*}(\iota(\mathcal{E})) \text{ in } I_{\mathrm{crys}}(S/\mathbf{W}).$$

By descent for isocrystals ([Ogu90, Lemma 0.7.5]) we can assume that S is affine and, by 2.16, choose a p-torsion free crystal E such that $\iota(\mathcal{E}) \simeq E \otimes K$. We use the notations from Theorem 3.1 and freely refer to its proof. In particular we consider the schemes $\mathbf{D}(e)$ with projections $p_i\colon \mathbf{D}(e) \longrightarrow \mathbf{S}$ and the module H_n with stratification defined at the beginning of the proof.

Since all $\mathbf{D}(e) \longrightarrow \mathbf{W}$ are flat, the associated formal schemes $\mathbf{P}(e)$ belong to the convergent site of $S/\mathbf{W}$. We use the description of $\iota\colon I_{\mathrm{conv}}(S/\mathbf{W}) \longrightarrow I_{\mathrm{crys}}(S/\mathbf{W})$ given in [Xu19, Section 3.20]. Applying [Xu19, Theorem 3.22] (or [Shi08a, Theorem 2.36]) to $X/\mathbf{P}(e)$ (be aware that the $g_{X/\mathbf{P}(e),\mathrm{crys}*}$ in the reference is what we denoted by $g_{X/\mathbf{D}(e)*}$) we see that $H_n \otimes K$ is the module with stratification inducing $\iota(\mathbf{R}^i f_{\mathrm{conv}*}(\mathcal{E}))$ (see also the proof of [Xu19, Lemma 4.10]). This shows the claim.

Proof of Theorem II as a consequence of Theorem 3.1. By Theorem 2.21 there is an isogeny

$$\mathrm{H}^n(\mathbf{L}v_{\mathrm{crys}}^* \mathbf{R}f_{\mathrm{crys}_*}(E)) \longrightarrow \mathbf{R}^n f'_{\mathrm{crys}_*} h_{\mathrm{crys}}^*(E).$$

Set $M := \mathbf{R}f_{\mathrm{crys}_*}(E)$. There is a canonical map

$$\phi\colon v_{\mathrm{crys}}^* \mathbf{R}^n f_{\mathrm{crys}*}(E) = v_{\mathrm{crys}}^*(\mathrm{H}^n(M)) \longrightarrow \mathrm{H}^n(\mathbf{L}v_{\mathrm{crys}}^* M).$$

We have to prove that it is a p^{N_n}-isogeny with an $N_n \in \mathbb{N}$ depending only on E and f. We are going to show that there exists $N_n \in \mathbb{N}$, depending only on E and f, such that for all $\chi = (U', \mathbf{T}', \delta') \in (S'/\mathbf{W})_{\mathrm{crys}}$ the $\mathcal{O}_{\mathbf{T}'}$-linear map

$$\phi_\chi\colon v_{\mathrm{crys}}^*(\mathrm{H}^n(M))_{\mathbf{T}'} \longrightarrow \mathrm{H}^n(\mathbf{L}v_{\mathrm{crys}}^* M)_{\mathbf{T}'}$$

is a p^{N_n}-isogeny. To show this it is enough to show that for each $x' \in \mathbf{T}'$ the map on stalks

$$\phi_{\chi,x'}\colon (v_{\mathrm{crys}}^*(\mathrm{H}^n(M))_{\mathbf{T}'})_{x'} \longrightarrow \mathrm{H}^n((\mathbf{L}v_{\mathrm{crys}}^* M)_{\mathbf{T}'})_{x'}$$

is a p^{N_n}-isogeny. Now we use notation from §2.3. Recall that $I_v^{x',\mathbf{T}'}$ is the filtered category of v-PD-morphisms $\mathbf{V} \xrightarrow{u} \mathbf{T}$ where $x \in \mathbf{V} \subseteq \mathbf{T}'$ is an open and $\mathbf{T} \in (S/\mathbf{W})_{\mathrm{crys}}$. Then we have a commutative diagram

$$
\begin{array}{ccc}
u^*(\mathrm{H}^n(M)_{\mathbf{T}}) & \longrightarrow & \mathrm{H}^n(\mathbf{L}u^* M_{\mathbf{T}}) \\
\downarrow & & \downarrow \\
v_{\mathrm{crys}}^*(\mathrm{H}^n(M))_{\mathbf{V}} & \longrightarrow & \mathrm{H}^n(\mathbf{L}v_{\mathrm{crys}}^* M)_{\mathbf{V}}.
\end{array}
$$

If we take the stalk at x' in the above diagram, then the bottom horizontal map is exactly $\phi_{\chi,x'}$. Moreover, if we take the colimit of the vertical arrows over all $u \in I_v^{x',\mathbf{T}'}$, then the vertical arrows are isomorphisms by Lemma 2.20. Thus it is enough for us to show that the top horizontal arrow is a p^{N_n}-isogeny with N_n depending only on E and f.

Now consider $(U', \mathbf{T}', \delta') \in (S'/\mathbf{W})_{\mathrm{crys}}$, $(S, \mathbf{T}, \delta) \in (S/\mathbf{W})_{\mathrm{crys}}$ and a commutative diagram

$$
\begin{array}{ccc}
f'^{-1}(U') & \xrightarrow{\ h\ } & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
U' & \xrightarrow{\ v\ } & S \\
\downarrow & & \downarrow \\
\mathbf{T}' & \xrightarrow{\ u\ } & \mathbf{T}
\end{array}
$$

where v, u form a PD-map. We have to show that the map

$$
u^* \operatorname{H}^n(M_{\mathbf{T}}) \longrightarrow \operatorname{H}^n(\mathbf{L}u^*(M_{\mathbf{T}}))
$$

is a p^{N_n}-isogeny for some $N_n \in \mathbb{N}$ depending only on E and f. Notice that by Theorem 2.7 (a) the complex $M_{\mathbf{T}} \simeq \mathbf{R}f_{X/\mathbf{T}*}(E_{|(X/\mathbf{T})_{\mathrm{crys}}})$ is bounded with a bound depending only on f. By [Wei94, Proposition 5.7.6, with the convention on Dual Definition 5.2.3] there is a convergent spectral sequence

$$
E_2^{ab} = \mathbf{L}^a u^*(\operatorname{H}^b(M_{\mathbf{T}})) \Rightarrow \mathbf{L}^{a+b} u^*(M_{\mathbf{T}}) = H^{a+b}.
$$

The upper bound of $M_{\mathbf{T}}$ provides a number $l \in \mathbb{N}$ depending only on f (so independent of the choice of $\mathbf{T}$), such that $E_2^{ab} = 0$ for $b < 0$ or $b > l$. Moreover $E_2^{ab} = 0$ if $a > 0$. By Lemma 2.9 we obtain a map

$$
E_2^{0n} = u^*(\operatorname{H}^n(M_{\mathbf{T}})) \longrightarrow \mathbf{L}^n u^*(M_{\mathbf{T}}) = H^n
$$

which coincides with the map ϕ_χ.

By Lemma 2.9 we must show that there exists N_n, which depends only on E and f, such that $\mathbf{L}^a u^*(\operatorname{H}^b(M_{\mathbf{T}}))$ is killed by p^{N_n} for $a \neq 0$. We can assume that $\mathbf{T}$ and S are affine. By Remark 2.16 and Theorem 3.1 there exists a crystal $H \in \operatorname{Crys}(S/\mathbf{W})$ which is isogenous to $\operatorname{H}^b(M)$. Thus it is enough to look at $\mathbf{L}^a u^* H_{\mathbf{T}}$. By Theorem 1.15 H corresponds to some $(P, \nabla) \in \operatorname{QNCf}(S/\mathbf{W})$. Let $S_{\mathbf{W}} = \operatorname{Spec}\mathcal{A} \longrightarrow \mathbf{W}$ be a lift of S as in Lemma 1.26 (2), so that P is an $\mathcal{A}$-module. The smoothness of $S_{\mathbf{W}_n}$ over $\mathbf{W}_n$ for all $n \in \mathbb{N}$ and [Sta19, Tag 07K4] imply the existence of a map $\mathbf{T} \longrightarrow S_{\mathbf{W}}$ lifting the identity map of S along $S \subseteq S_{\mathbf{W}}$. In particular $P \otimes \mathcal{O}_{\mathbf{T}} \simeq H_{\mathbf{T}}$. Applying Lemma 2.10 to P and $F = \mathbf{L}^a u^*(- \otimes \mathcal{O}_{\mathbf{T}})$ we find the $N_n \in \mathbb{N}$ depending only on E and f such that p^{N_n} kills $F(P) = \mathbf{L}^a u^* H_{\mathbf{T}}$ for $a \neq 0$. $\square$

4. The Künneth Formula

In this last section we prove Theorems III and IV.

Proof of Theorem III. Consider the following diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1^{\mathrm{crys}}(Y/\mathbf{W}, y) & \longrightarrow & \pi_1^{\mathrm{crys}}(X \times_k Y/\mathbf{W}, (x,y)) & \longrightarrow & \pi_1^{\mathrm{crys}}(X/\mathbf{W}, x) & \longrightarrow & 1 \\
& & \| & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1^{\mathrm{crys}}(Y/\mathbf{W}, y) & \longrightarrow & \pi_1^{\mathrm{crys}}(X/\mathbf{W}, x) \times_k \pi_1^{\mathrm{crys}}(Y/\mathbf{W}, y) & \longrightarrow & \pi_1^{\mathrm{crys}}(X/\mathbf{W}, x) & \longrightarrow & 1.
\end{array}
$$

It is enough to show that the top sequence is exact. Consider the diagram

$$
\text{(4.1)} \qquad
\begin{array}{ccc}
Y & \xrightarrow{\ x\ } & X \times_k Y \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p_1} \\
\operatorname{Spec} k & \xrightarrow{\ u\ } & X.
\end{array}
$$

Since x is a section of the projection, it gives a closed embedding on fundamental group schemes, while the projection yields a surjection on fundamental group schemes. We are going to apply [EHS08, Theorem A.1 (iii)] to prove the exactness in the middle. So we have to check:

(a) If $\mathcal{E} \in I_{\mathrm{crys}}(X \times_k Y/\mathbf{W})$, then $x^*_{\mathrm{crys}}\mathcal{E}$ is a trivial object in $I_{\mathrm{crys}}(Y/\mathbf{W})$ if and only if there exists $\mathcal{F} \in I_{\mathrm{crys}}(X/\mathbf{W})$ such that $p^*_{1\mathrm{crys}}\mathcal{F} \simeq \mathcal{E}$.

(b) We have to check that for any isocrystal $\mathcal{E} \in I_{\mathrm{crys}}(X \times_k Y/\mathbf{W})$, the maximal trivial subobject of $x^*_{\mathrm{crys}}\mathcal{E}$ comes from a subobject $\mathcal{F} \subseteq \mathcal{E}$, where $\mathcal{F}$ is defined over $X/\mathbf{W}$.

(c) If $\mathcal{G} \in I_{\mathrm{crys}}(Y/\mathbf{W})$, then there exists $\mathcal{E} \in I_{\mathrm{crys}}(X \times_k Y/\mathbf{W})$ such that $\mathcal{G}$ is a subobject of $x^*_{\mathrm{crys}}\mathcal{E}$.

Condition (c) follows because x is a section of the projection $X \times_k Y \xrightarrow{p_2} Y$. Also the "if" part of (a) is obvious from (4.1), and the "only if" part is a consequence of (b). Thus let's focus on (b).

Since $p_{1\mathrm{crys}_*}$ and $p^*_{1\mathrm{crys}}$ are a pair of adjoint functors between the category of sheaves of $\mathcal{O}$-modules on $(X \times_k Y/\mathbf{W})_{\mathrm{crys}}$ and that on $(X/\mathbf{W})_{\mathrm{crys}}$, and thanks to Theorem I, the induced pair of functors between the isogeny categories $I_{\mathrm{crys}}(X \times Y/\mathbf{W})$ and $I_{\mathrm{crys}}(X/\mathbf{W})$ are also adjoint to each other. The map p_1 induces a map on fundamental group schemes

$$\pi^{\mathrm{crys}}_1(p_1)\colon \pi^{\mathrm{crys}}_1(X \times_k Y/\mathbf{W}, (x,y)) \longrightarrow \pi^{\mathrm{crys}}_1(X/\mathbf{W}, x)$$

which is surjective because p_1 has a section. It follows that $p_{1\mathrm{crys}_*}$ on isocrystals corresponds to taking invariants by the kernel of $\pi^{\mathrm{crys}}_1(p_1)$. In particular the map

$$\mathcal{F} \coloneqq p^*_{1\mathrm{crys}}p_{1\mathrm{crys}_*}\mathcal{E} \longrightarrow \mathcal{E}$$

is injective.

The same argument applied to $I_{\mathrm{crys}}(Y/\mathbf{W})$ and $I_{\mathrm{crys}}(\mathrm{Spec}\,k/\mathbf{W})$ shows that

$$g^*_{\mathrm{crys}}g_{\mathrm{crys}_*}x^*_{\mathrm{crys}}\mathcal{E} \longrightarrow x^*_{\mathrm{crys}}\mathcal{E}$$

is injective and $g^*_{\mathrm{crys}}g_{\mathrm{crys}_*}x^*_{\mathrm{crys}}\mathcal{E}$ is the maximal trivial subobject of $x^*_{\mathrm{crys}}\mathcal{E}$.

Using the base change isomorphism in Theorem II in (4.1), we can conclude that applying x^*_{crys} to $\mathcal{F} \longrightarrow \mathcal{E}$ we get the map $g^*_{\mathrm{crys}}g_{\mathrm{crys}_*}x^*_{\mathrm{crys}}\mathcal{E} \longrightarrow x^*_{\mathrm{crys}}\mathcal{E}$ as required. □

Proof of Theorem IV. By the binary operation on $\pi^{\mathrm{crys}}_1(A/\mathbf{W}, 0)$ induced by the addition of the abelian variety A, $\pi^{\mathrm{crys}}_1(A/\mathbf{W}, 0)$ becomes group object in the category of affine group schemes over K. Then, by the calculation given in [EH62, Theorem 5.4.2], $\pi^{\mathrm{crys}}_1(A/\mathbf{W}, 0)$ is an abelian group scheme. □

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