# ON SECTIONS OF ARITHMETIC FUNDAMENTAL GROUPS OF OPEN $p$-ADIC ANNULI.

Mohamed Saïdi

Abstract. We show the *non-existence* of sections of arithmetic fundamental groups of *open p-adic annuli of small radii*, this implies the *non-existence* of sections of arithmetic fundamental groups of *formal boundaries of formal germs of p-adic curves.*

§**0. Introduction/Statement of the Main Result.** Let $p \geq 2$ be a prime integer, $k$ a $p$-**adic local field** (i.e., $k/\mathbb{Q}_p$ is a finite extension), with ring of integers $\mathcal{O}_k$, uniformiser $\pi$, and residue field $F$. Thus, $F$ is a finite field of characteristic $p$.

Let $X \to \operatorname{Spec} \mathcal{O}_k$ be a flat, proper, relative $\mathcal{O}_k$-curve, with $X$ normal, and $X_k \overset{\text{def}}{=} X \times_{\operatorname{Spec} \mathcal{O}_k} \operatorname{Spec} k$ geometrically connected. Assume $X(F) \neq \emptyset$. Let $x \in X^{\text{cl}}(F)$ be a closed point, $\mathcal{O}_{X,x}$ the local ring at $x$, $\widehat{\mathcal{O}}_{X,x}$ its completion, and $E \overset{\text{def}}{=} \widehat{\mathcal{O}}_{X,x} \otimes_{\mathcal{O}_k} k = \widehat{\mathcal{O}}_{X,x}[\frac{1}{\pi}]$. Write $\mathcal{X} \overset{\text{def}}{=} \operatorname{Spec} E$, which we assume to be geometrically connected. We shall refer to $\mathcal{X}$ as the **formal germ of** $X$ **at** $x$.

Let $\eta$ be a geometric point of $\mathcal{X}$ with values in its generic point. Thus, $\eta$ determines an algebraic closure $\overline{k}$ of $k$, and a geometric point $\bar{\eta}$ of $\mathcal{X}_{\overline{k}} \overset{\text{def}}{=} \mathcal{X} \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. There exists a canonical exact sequence of profinite groups (cf. [Grothendieck], Exposé IX, Théorème 6.1)

$$1 \to \pi_1(\mathcal{X}_{\overline{k}}, \bar{\eta}) \to \pi_1(\mathcal{X}, \eta) \to G_k \to 1. \tag{1}$$

Here, $\pi_1(\mathcal{X}, \eta)$ denotes the arithmetic étale fundamental group of $\mathcal{X}$ with base point $\eta$, $\pi_1(\mathcal{X}_{\overline{k}}, \bar{\eta})$ the étale fundamental group of $\mathcal{X}_{\overline{k}}$ with base point $\bar{\eta}$, and $G_k \overset{\text{def}}{=} \operatorname{Gal}(\overline{k}/k)$ the absolute Galois group of $k$.

The sequence (1) splits if $\mathcal{X}(k) \neq \emptyset$. This is for example the case if the morphism $X \to \operatorname{Spec} \mathcal{O}_k$ is smooth at $x$. If $\mathcal{X}(k) = \emptyset$; for instance if $X$ is stable and regular, and $x$ is an ordinary $F$-rational double point of $X_F \overset{\text{def}}{=} X \times_{\operatorname{Spec} \mathcal{O}_k} F$, the existence of sections $s : G_k \to \pi_1(\mathcal{X}, \eta)$ of the projection $\pi_1(\mathcal{X}, \eta) \twoheadrightarrow G_k$ would provide examples of sections of the projection $\pi_1(X_k, \eta) \twoheadrightarrow G_k$ which are *non-geometric* ($\eta$ induces a geometric point of $X_k$, denoted also $\eta$, via the morphism $\mathcal{X}_k \to X_k$), i.e., which do not arise from rational points. These in turn will provide *counter-examples to the p-adic version of the Grothendieck anabelian section conjecture.* This prompt the following question.

**Question A.** *With the above notations, assume $\mathcal{X}(k) = \emptyset$. Does the exact sequence (1) split?*

In this note we investigate the case where $\mathcal{X}$ is a *p-adic open annulus*. Let

$$A \stackrel{\text{def}}{=} \mathcal{O}_k[[S]], \quad B \stackrel{\text{def}}{=} A \otimes_{\mathcal{O}_k} k = A\left[\frac{1}{\pi}\right],$$

$D \stackrel{\text{def}}{=} \operatorname{Spf} A$ is the formal standard open disc, and $\mathcal{D} \stackrel{\text{def}}{=} D_k = \operatorname{Spec} B$ its "generic fibre" which is the standard open disc centred at the point "$S = 0$". Let $\mathbb{A}^1_k = \operatorname{Spec} k[S]$, $Y_k = \mathbb{P}^1_k$ its smooth compactification with function field $k(S)$, and $Y = \mathbb{P}^1_{\mathcal{O}_k}$ the smooth compactification of $\mathbb{A}^1_{\mathcal{O}_k} = \operatorname{Spec} \mathcal{O}_k[S]$. We shall identify $A$ with the completion of the local ring of $Y$ at the closed point "$S = 0$". We have a natural morphism $\mathcal{D} \to \mathbb{P}^1_k$, which induces an identification between the set of closed points of $\mathcal{D}$ and the set

$$\{x \in \mathbb{P}^1_k : |S(x)| < 1\}.$$

For an integer $n \geq 1$, let

$$A_n \stackrel{\text{def}}{=} \frac{\mathcal{O}_k[[S,T]]}{(S^n T - \pi)}, \quad B_n \stackrel{\text{def}}{=} A_n \otimes_{\mathcal{O}_k} k, \text{ and } \mathcal{C}_n \stackrel{\text{def}}{=} \operatorname{Spec} B_n.$$

The natural embedding $\mathcal{C}_n \hookrightarrow \mathcal{D}$ induces an identification between the set of closed points of $\mathcal{C}_n$ and the open annulus

$$\{x \in \mathcal{D} : |\pi|^{\frac{1}{n}} < |S(x)| < 1\}.$$

Further, let $P \stackrel{\text{def}}{=} A_{(\pi)}$ be the localisation of $A$ at the ideal $(\pi)$, and $\widehat{P}$ the completion of $P$, which is a complete discrete valuation ring isomorphic to

$$\mathcal{O}_k[[S]]\{S^{-1}\} \stackrel{\text{def}}{=} \Big\{ \sum_{i=-\infty}^{\infty} a_i S^i : a_i \in \mathcal{O}_k, \lim_{i \to -\infty} |a_i| = 0 \Big\},$$

where $|\ |$ is a normalised absolute value of $\mathcal{O}_k$ (cf. [Bourbaki], §2, 5). Let $L \stackrel{\text{def}}{=} \operatorname{Fr}(\widehat{P})$ be the fraction field of $\widehat{P}$, and $\mathcal{C}_\infty \stackrel{\text{def}}{=} \operatorname{Spec} L$. We shall refer to $\mathcal{C}_\infty$ as a **formal boundary** of the formal germs $\mathcal{D}$, and $\mathcal{C}_i$ for $i \geq 1$. We have natural scheme morphisms

$$\mathcal{C}_\infty \to \cdots \to \mathcal{C}_{n+1} \to \mathcal{C}_n \to \cdots \to \mathcal{C}_1 \to \mathcal{D} \to \mathbb{P}^1_k.$$

Let $\eta$ be a geometric point of $\mathcal{C}_\infty$, which induces a geometric point (denoted also $\eta$) of $\mathcal{C}_n$ for $n \geq 1$. For $i \in \mathbb{N} \cup \{\infty\}$, we have an exact sequence of arithmetic fundamental groups

$$1 \to \pi_1(\mathcal{C}_{i,\overline{k}}, \bar{\eta}) \to \pi_1(\mathcal{C}_i, \eta) \to G_k \to 1, \tag{2}$$

where $\pi_1(\mathcal{C}_i, \eta)$ denotes the arithmetic étale fundamental group of $\mathcal{C}_i$ with base point $\eta$, $\pi_1(\mathcal{C}_{i,\overline{k}}, \bar{\eta})$ the étale fundamental group of $\mathcal{C}_{i,\overline{k}} \stackrel{\text{def}}{=} C_i \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ with base point $\bar{\eta}$; which is induced by $\eta$, and $G_k \stackrel{\text{def}}{=} \operatorname{Gal}(\overline{k}/k)$ the absolute Galois group of $k$. Here $\overline{k}$ is the algebraic closure of $k$ determined by $\eta$.

Our main result in this note is the following.

**Theorem A.** *We use notations as above. There exists an integer $N \geq 1$, such that for every integer $n \geq N$, the projection $\pi_1(\mathcal{C}_n, \eta) \twoheadrightarrow G_k$* **doesn't split***.*

The author ignores, for the time being, if the projection $\pi_1(\mathcal{C}_1, \eta) \twoheadrightarrow G_k$ splits or not.

As a corollary of Theorem A, we obtain the following.

**Theorem B.** *The projection $\pi_1(\mathcal{C}_\infty, \eta) \twoheadrightarrow G_k$* **doesn't split***.*

One of the consequences of Theorems A, and B, is that one can not produce examples of sections of hyperbolic curves over $p$-adic local fields, which arise from sections of arithmetic fundamental groups of boundaries of formal fibres, or open annuli with small radii. Those sections would be non-geometric, hence would provide counter-examples to the $p$-adic version of the Grothendieck anabelian section conjecture.

Finally we observe the following. For $i \in \mathbb{N} \cup \{\infty\}$, let $\pi_1(\mathcal{C}_{i,\overline{k}}, \overline{\eta})^{\text{ab}}$ be the maximal abelian quotient of $\pi_1(\mathcal{C}_{i,\overline{k}}, \overline{\eta})$, and consider the push-out diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathcal{C}_{i,\overline{k}}, \overline{\eta}) & \longrightarrow & \pi_1(\mathcal{C}_i, \eta) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(\mathcal{C}_{i,\overline{k}}, \overline{\eta})^{\text{ab}} & \longrightarrow & \pi_1(\mathcal{C}_i, \eta)^{(\text{ab})} & \longrightarrow & G_k & \longrightarrow & 1 \end{array} \tag{3}$$

Thus, $\pi_1(\mathcal{C}_i, \eta)^{(\text{ab})}$ is the geometrically abelian quotient of $\pi_1(\mathcal{C}_i, \eta)$.

**Proposition C.** *The projection $\pi_1(\mathcal{C}_i, \eta)^{(\text{ab})} \twoheadrightarrow G_k$* **splits***, $\forall i \in \mathbb{N}$.*

The author ignores, for the time being, if the projection $\pi_1(\mathcal{C}_\infty, \eta)^{(\text{ab})} \twoheadrightarrow G_k$ splits or not.

§**1. Proof of Theorem A.** In this section we shall prove Theorem A. We use the notations in §0. We argue by contradiction, and **assume** that the projection $\pi_1(\mathcal{C}_n, \eta) \twoheadrightarrow G_k$ **splits**, $\forall n \geq 1$.

**Proposition 1.1.** *There exists a relative curve $X \to \operatorname{Spec} \mathcal{O}_k$ with the following properties.*

*(i) The morphism $X \to \operatorname{Spec} \mathcal{O}_k$ is flat, proper,* **stable***, and $X_k \overset{\text{def}}{=} X \times_{\operatorname{Spec} \mathcal{O}_k} \operatorname{Spec} k$ is geometrically connected.*

*(ii) $X$ is* **regular***.*

*(iii) The set of singular points $X_F^{\text{sing}}$ of the special fibre $X_F \overset{\text{def}}{=} X \times_{\operatorname{Spec} \mathcal{O}_k} \operatorname{Spec} F$ of $X$ consists of $F$-***rational ordinary double points***, $U \overset{\text{def}}{=} X_F \setminus X_F^{\text{sing}}$ is $F$-smooth, and $U(F) = \emptyset$ holds.*

*(iv) $X(k) = \emptyset$ holds.*

*Proof.* First, assume $p \neq 2$. Let $\widetilde{C} \overset{\text{def}}{=} \mathbb{P}^1_F$ with function field $k(\widetilde{C})$. Thus, $\operatorname{Card}(\widetilde{C}(F)) = \operatorname{Card} F + 1$ is even. Arrange the set $\widetilde{C}(F)$ in pairs of $F$-rational points: $\widetilde{C}(F) = \{(x_i, y_i)\}_{1 \leq i \leq \frac{\operatorname{Card} F + 1}{2}}$. One can identify in $\widetilde{C}$ the points $x_i$ and $y_i$; $1 \leq i \leq \frac{\operatorname{Card} F + 1}{2}$, to construct a stable proper $F$-curve $C$ which is geometrically connected and geometrically reduced, with normalisation $\widetilde{C} \to C$. Moreover, the set of singular points $C^{\text{sing}} = \{c_i\}_{1 \leq i \leq \frac{\operatorname{Card} F + 1}{2}}$ consists of $F$-rational ordinary double points, and the pre-image of $c_i$ in $\widetilde{C}$ consists of the two $F$-rational points $\{x_i, y_i\}$. In particular, $C(F) = C^{\text{sing}} = \{c_i\}_{1\leq i\leq \frac{\operatorname{Card} F+1}{2}}$. More precisely, for $1 \leq i \leq \frac{\operatorname{Card} F+1}{2}$, let $\widetilde{\mathcal{O}}_i \overset{\text{def}}{=} \mathcal{O}_{\widetilde{C},x_i} \cap \mathcal{O}_{\widetilde{C},y_i} \subset k(\widetilde{C})$, $\mathcal{N}_{x_i} \overset{\text{def}}{=} \mathfrak{m}_{x_i} \cap \widetilde{\mathcal{O}}_i$, and $\mathcal{N}_{y_i} \overset{\text{def}}{=} \mathfrak{m}_{y_i} \cap \widetilde{\mathcal{O}}_i$, where $\mathfrak{m}_{x_i}$ (resp. $\mathfrak{m}_{y_i}$) is the maximal ideal of $\mathcal{O}_{\widetilde{C},x_i}$ (resp. $\mathcal{O}_{\widetilde{C},y_i}$). Define $\mathcal{O}_{c_i} \overset{\text{def}}{=} F + \mathcal{N}_{x_i}\mathcal{N}_{y_i} \subset \widetilde{\mathcal{O}}_i$. Then $\mathcal{O}_{c_i}$ is a local ring (with maximal ideal $\mathcal{N}_{x_i}\mathcal{N}_{y_i}$, and residue field $F$) whose integral closure is $\mathcal{O}_{c_i}$ (cf. [Aubry-Lezzi], Proposition 3.1, Theorem 3.4, and the references therein for the properties of $\mathcal{O}_{c_i}$, as well as the existence of $C$ with the required properties).

In case $p = 2$. Consider the affine $F$-curve $\operatorname{Spec}(\frac{F[s,t]}{(st)})$, and $\tilde{C}$ its smooth compactification. Thus, $\tilde{C}$ consists of two $F$-smooth irreducible components $\tilde{C}_1 = \mathbb{P}^1_F$, and $\tilde{C}_2 = \mathbb{P}^1_F$, which intersect at the $F$-rational ordinary double point $c = (s,t) \in \operatorname{Spec}(\frac{F[s,t]}{(st)})$. On each irreducible component $\tilde{C}_i$ of $\tilde{C}$; $1 \leq i \leq 2$, the set of $F$-rational points of $\tilde{C}_i \setminus \{c\}$ is non-empty and comes into pairs of rational points $\{(x_{i,j}, y_{i,j})\}_{1\leq j\leq \frac{\operatorname{Card} F}{2}}$. As above we can identify each of those pairs of $F$-rational points $(x_{i,j}, y_{i,j})$ into an $F$-rational ordinary double point $c_{i,j}$ to construct a reducible and geometrically connected stable curve $F$-curve $C$ such that the set of singular points $C^{\text{sing}}$ consists of $F$-rational ordinary double points, a double point $c_{i,j}$ lies on a unique irreducible component of $C$, and $C^{\text{sing}} = C(F)$ (the local ring at $c_i$ is defined as above; the case $p \neq 2$. See. [Rosenlicht], §4, for a discussion of this procedure and the existence of such a curve $C$ in the case of reducible curves).

Now the stable $F$-curve $C$ can be deformed to a semi-stable $\mathcal{O}_k$-curve $X \to \operatorname{Spec}\mathcal{O}_k$ with special fibre $X_F = C$ satisfying (i) and (ii) (cf. [Talpo-Vistoli], Proposition 7.10, Corollary 7.11 and its proof). By our construction (iii) holds also. If $x \in X(k)$, then $X$ specialises in a point $\bar{x} \in C(F)$ which is a regular point of $C$ and lies on a unique irreducible component of $C$ (cf. [Liu], Corollary 9.1.32). Thus, (iv) follows from (iii). $\square$

Let $X \to \operatorname{Spec}\mathcal{O}_K$ be a regular, proper, flat, and stable $\mathcal{O}_k$-curve as in Proposition 1.1. Let $y \in X_F(F)$ be an $F$-rational point, which is an ordinary double point and a regular point of $X$ (cf. Proposition 1.1 (ii) and (iii)). We fix an isomorphism $\rho : \widehat{\mathcal{O}}_{X,y} \xrightarrow{\sim} R[[S,T]]/(ST - \pi)$, and identify $\mathcal{X} \overset{\text{def}}{=} \operatorname{Spec}(\widehat{\mathcal{O}}_{X,y} \otimes_{\mathcal{O}_k} k)$ with $\mathcal{C}_1$ via the isomorphism $\rho_k : \widehat{\mathcal{O}}_{X,y} \otimes_{\mathcal{O}_k} k \xrightarrow{\sim} \frac{R[[S,T]]}{(ST-\pi)} \otimes_{\mathcal{O}_k} k$ induced by $\rho$. Thus, we have scheme morphisms

$$\mathcal{C}_\infty \to \cdots \to \mathcal{C}_{n+1} \to \mathcal{C}_n \to \cdots \to \mathcal{C}_1 \to X_k.$$

For $n \geq 1$, write

$$\pi_1(X_k \setminus \mathcal{S}_n, \eta) \overset{\text{def}}{=} \varprojlim_{S_n \subset X_k \setminus \mathcal{C}_n} \pi_1(X_k \setminus S_n, \eta),$$

where the projective limit is over all finite sets of closed points $S_n \subset X_k \setminus \mathcal{C}_n$, and $\pi_1(X_k \setminus S_n, \eta)$ is the arithmetic fundamental group of the affine curve $X_k \setminus S_n$ with base point $\eta$. (Here, we identify the set of closed points of $\mathcal{C}_n$; $n \geq 1$, with a subset of the set of closed points of $X_k$ which specialise in $y$.) There is a natural projection $\pi_1(X_k \setminus \mathcal{S}_n, \eta) \twoheadrightarrow G_k$, and we have a commutative diagram

$$\begin{array}{ccc} \pi_1(\mathcal{C}_n, \eta) & \longrightarrow & G_k \\ \downarrow & & \| \\ \pi_1(X_k \setminus \mathcal{S}_n, \eta) & \longrightarrow & G_k \end{array} \tag{4}$$

where the left vertical map is induced by the morphism $\mathcal{C}_n \to X_k$.

Further, we have a natural map

$$\varprojlim_{n\geq 1} \pi_1(\mathcal{C}_n, \eta) \to \varprojlim_{n\geq 1} \pi_1(X_k \setminus \mathcal{S}_n, \eta),$$

and $\varprojlim_{n\geq 1} \pi_1(X_k \setminus \mathcal{S}_n, \eta)$ is naturally identified with the absolute Galois group $G_{k(X)} \overset{\text{def}}{=} \text{Gal}\,(k(X)^{\text{sep}}/k(X))$, where $k(X)^{\text{sep}}$ is the separable closure of the function field $k(X)$ of $X$ determined by the geometric point $\eta$.

**Lemma 1.2.** *The projection $G_{k(X)} \twoheadrightarrow G_k$* **splits***.*

*Proof.* First, our assumption that the projection $\pi_1(\mathcal{C}_n, \eta) \twoheadrightarrow G_k$ **splits**, implies that the projection $\pi_1(X_k \setminus \mathcal{S}_n, \eta) \to G_k$ splits, $\forall n \geq 1$ (cf. diagram (4)).

Let $(H_i)_{i\in I}$ be a projective system of quotients $G_{k(X)} \twoheadrightarrow H_i$, where $H_i$ sits in an exact sequence $1 \to F_i \to H_i \to G_k \to 1$ with $F_i$ finite, and $G_{k(X)} = \varprojlim_{i\in I} H_i$. [More precisely, write $G_{k(X)}$ as a projective limit of finite groups $\{\tilde{H}_i\}_{i\in I}$. Then $\tilde{H}_i$ fits in an exact sequence $1 \to F_i \to \tilde{H}_i \to G_i \to 1$, where $G_i$ is a quotient of $G_k$, and $F_i$ a quotient of $\text{Gal}(k(X)^{\text{sep}}/k(X)\bar{k})$. Let $1 \to F_i \to H_i \to G_k \to 1$ be the pull-back of the group extension $1 \to F_i \to \tilde{H}_i \twoheadrightarrow G_i \to 1$ by $G_k \twoheadrightarrow G_i$. Then $G_{k(X)} = \varprojlim_{i\in I} H_i$]. The set $\text{Sect}(G_k, G_{k(X)})$ of group-theoretic sections of the projection $G_{k(X)} \twoheadrightarrow G_k$ is naturally identified with the projective limit $\varprojlim_{i\in I} \text{Sect}(G_k, H_i)$ of the sets $\text{Sect}(G_k, H_i)$ of group-theoretic sections of the projection $H_i \twoheadrightarrow G_k$. For each $i \in I$, the set $\text{Sect}(G_k, H_i)$ is non-empty. Indeed, $H_i$ (being a quotient of $G_{k(X)}$) is a quotient of $\pi_1(X_k \setminus \mathcal{S}_n, \eta)$ for some $n \geq 1$, this quotient $\pi_1(X_k \setminus \mathcal{S}_n, \eta) \twoheadrightarrow H_i$ commutes with the projections onto $G_k$, and we know the projection $\pi_1(X_k \setminus \mathcal{S}_n, \eta) \to G_k$ splits. Hence the projection $H_i \twoheadrightarrow G_k$ splits. Moreover, the set $\text{Sect}(G_k, H_i)$ is, up to conjugation by the elements of $F_i$, a torsor under the group $H^1(G_k, F_i)$ which is finite since $k$ is a $p$-adic local field (cf. [Neukirch-Schmidt-Winberg], (7.1.8) Theorem (iii)). Thus, $\text{Sect}(G_k, H_i)$ is a nonempty finite set. Hence the set $\text{Sect}(G_k, G_{k(X)})$ is nonempty being the projective limit of nonempty finite sets. This finishes the proof of Lemma 1.1. (See also the proof of Proposition 1.5 in [Saïdi] for similar arguments in a slightly different context.) $\square$

Let $s : G_k \to G_{k(X)}$ be a section of the projection $G_{k(X)} \twoheadrightarrow G_k$ (cf. Lemma 1.2).

**Lemma 1.3.** *The section $s$ is* **geometric***, i.e., $s(G_k) \subset D_x$, where $D_x \subset G_{k(X)}$ is a decomposition group associated to a (unique) rational point $x \in X(k)$. In particular, $X(k) \neq \emptyset$.*

*Proof.* This follows from [Koenigsmann], Proposition 2.4 (2). $\square$

Now the conclusion of Lemma 1.3 that $X(k) \neq \emptyset$ contradicts the assertion (iv) in Proposition 1.1 that $X(k) = \emptyset$. This is a contradiction. Thus, our assumption that the projection $\pi_1(\mathcal{C}_n, \eta) \twoheadrightarrow G_k$ **splits**, $\forall n \geq 1$, can not hold. Let $N \geq 1$ be such that the projection $\pi_1(\mathcal{C}_N, \eta) \twoheadrightarrow G_k$ doesn't splits. Then the projection $\pi_1(\mathcal{C}_n, \eta) \twoheadrightarrow G_k$ doesn't splits, $\forall n \geq N$ as required. Indeed, this follows from the fact that for $n \geq N$ we have a natural homomorphism $\pi_1(\mathcal{C}_n, \eta) \to \pi_1(\mathcal{C}_N, \eta)$ which commutes with the projections onto $G_k$. Hence if the projection $\pi_1(\mathcal{C}_n, \eta) \twoheadrightarrow G_k$ splits then the projection $\pi_1(\mathcal{C}_N, \eta) \twoheadrightarrow G_k$.

This finishes the proof of Theorem A. $\square$

**§2. Proof of Theorem B.** Next, we explain how Theorem B can be derived from Theorem A. We have, $\forall n \geq 1$, a commutative diagram

$$\begin{array}{ccc} \pi_1(\mathcal{C}_\infty, \eta) & \longrightarrow & G_k \\ \big\downarrow & & \big\| \\ \pi_1(\mathcal{C}_n, \eta) & \longrightarrow & G_k \end{array}$$

where the horizontal maps are the natural projections, and the left vertical map is induced by the morphism $\mathcal{C}_\infty \to \mathcal{C}_n$.

Now assume that the projection $\pi_1(\mathcal{C}_\infty, \eta) \twoheadrightarrow G_k$ splits. Then the projection $\pi_1(\mathcal{C}_n, \eta) \twoheadrightarrow G_k$ splits, $\forall n \geq 1$, by the above diagram. But this contradicts Theorem A.

This finishes the proof of Theorem B. $\square$

**§3. Proof of Proposition C.** Let $n \geq 1$ be an integer, and $\ell_1$, $\ell_2$, distinct prime integers such that $\ell_1 \geq 2n$, and $\ell_2 \geq 2n$. Let $\mathcal{O}_1$, and $\mathcal{O}_2$, be totally ramified extensions of $\mathcal{O}_k$ of degree $\ell_1$, and $\ell_2$, with fraction fields $L_1 = \mathrm{Fr}(\mathcal{O}_1)$, and $L_2 = \mathrm{Fr}(\mathcal{O}_2)$; respectively. Thus, the extensions $L_1/k$ and $L_2/k$, are disjoint and $\mathcal{C}_n(L_i) \neq \emptyset$, for $i \in \{1, 2\}$. A restriction-corestriction argument shows that the class $[\pi_1(\mathcal{C}_n, \eta)^{(\mathrm{ab})}]$ of the group extension $\pi_1(\mathcal{C}_n, \eta)^{(\mathrm{ab})}$ in $H^2(G_k, \pi_1(\mathcal{C}_{n,\overline{k}}, \bar{\eta})^{\mathrm{ab}})$ is trivial. Thus the group extension $\pi_1(\mathcal{C}_n, \eta)^{(\mathrm{ab})}$ splits.

This finishes the proof of Proposition C. $\square$

References.


[Aubry-Lezzi], Aubry, Y., Lezzi, A., On the Maximum Number of Rational Points on Singular Curves over Finite Fields. Mosc. Math. J. (2015), Volume 15, Number 4, Pages 615-627, arXiv:1501.03676.

[Bourbaki] N. Bourbaki, Algèbre Commutative, Chapitre 9, Masson, 1983.

[Grothendieck] Grothendieck, A., Revêtements étales et groupe fondamental, Lecture Notes in Math. 224, Springer, Heidelberg, 1971.

[Koenigsmann] Koenigsmann, J., On the section conjecture in anabelian geometry. J. Reine Angew. Math. 588 (2005), 221-235.

[Liu] Liu, Q., Algebraic geometry and arithmetic curves, Oxford graduate texts in mathematics 6. Oxford University Press, 2002.

[Neukirch-Schmidt-Winberg] Neukirch, J., Schmidt, A., Winberg, K., Cohomology of number fields, first edition, Springer, Grundlehren der mathematischen Wissenschaften Bd. 323, 2000.

[Rosenlicht] Rosenlicht, M., Equivalence relations on algebraic curves, Annals of Mathematics, Vol. 56, No. 1 (Jul., 1952), pp. 169-191.

[Saïdi] Saïdi, M., On the existence of non-geometric sections of arithmetic fundamental groups, Mathematische Zeitschrift 277, no. 1-2 (2014), 361-372.

[Talpo-Vistoli], Talpo, M., Vistoli, A., Deformation theory from the point of view of fibered categories, Handbook of Moduli (Volume III), Editors: Gavril Farkas and Ian Morrison, publisher Int. Press, Somerville, MA series Adv. Lect. Math. (ALM), volume 26, pages 281-397, arXiv:1006.0497.



Mohamed Saïdi
College of Engineering, Mathematics, and Physical Sciences

University of Exeter
Harrison Building
North Park Road
EXETER EX4 4QF
United Kingdom
M.Saidi@exeter.ac.uk