Capacity Management for A Leasing System with Different Equipment and Batch Demands

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Capacity Management for A Leasing System with Different Equipment and Batch Demands

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Abstract: This work explores the admission and capacity allocation for a leasing system with two types of equipment and three kinds of batch demands: elementary-specified, premium-specified and unspecified demands. The demands arrive following mutually independent Poisson processes and the rental duration of equipment follows a negative exponential distribution. The lessor can satisfy partially the specified demands with the required type of equipment and satisfy partially the unspecified demands with any type of equipment. The objective is to maximise the expected discounted revenue. We formulate this problem as a Markov Decision Process, prove the anti-multimodularity of the value function and characterise the structure of the optimal policy. We show that the optimal policy has a simple structure and is characterised by state-dependent rationing and priority thresholds. Moreover, a solution algorithm is proposed to calculate the optimal policy. We study the impacts of the system state on the optimal action, and find that the optimal action has limited sensitivity to the system state. Numerical studies are conducted to compare the performance of the optimal policy with that of two heuristic methods and to derive some managerial insights by analysis. We further discuss batch acceptance.

Keywords: leasing system; batch demand; Markov Decision Process; service operations management; capacity rationing

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1 Introduction

The equipment leasing industry has experienced a stable growth in the past decade. It is reported that the equipment and event rental revenue in Europe (including the 28 European Union countries and the European Free Trade Association countries) was €27.7 billion in 2019, and was forecasted to increase by 4.8% in 2021 (ERA, 2020). According to the American Rental Association (ARA), total rental revenues in America could have a 10.2% increase in 2022 to reach $52.7 billion. The equipment rental industry is growing steadily and continues to outpace the overall economy over the next few years (ARA, 2022).

Nowadays, equipment lease involves many industries, e.g., transportation, construction, gardening and landscaping, container leasing and events. For example, in the construction industry, rental companies offer various machines ranging from small machines (e.g., skid steer equipment and generators) to heavy equipment (e.g., telehandlers and hydraulic excavators). In the container leasing industry, different types of containers are provided to satisfy different shipping companies' requirements, e.g., dry containers (20/40/45 ft), refrigerated containers (20/40 ft) and flat racks (20/40 ft). A diversified equipment fleet brings customers a mix of leasing solutions to meet demands. In Particular, even for the same type of equipment there are elementary and premium equipment grades, distinguished by their status. Compared to elementary equipment, premium equipment has some additional functions. For example, in the container leasing industry, elementary refrigerated containers have standard features including power/back-up rechargeable batteries and dehumidification control, while the additional features of premium refrigerated containers include dual voltages, remote monitoring units, remote monitoring plugs, auto vents and water-cooled condensers.

Generally, customers require more than one unit of equipment when they approach leasing companies. For example, contractors rent several types of temporary power equipment such as generators and load banks for site use or during power failures. Some chemical plants require many industrial dehumidifiers to control the humidity of the environment. Transportation companies need many vehicles to carry goods. However, the majority of works on the capacity management of leasing systems assume unit demand (Savin et al., 2005; Yang et al., 2022).

In addition, there are multiple service levels for lessees. Term lease is a common form of equipment lease where lessees sign contracts with the lessors to rent some types of equipment at certain pre-negotiated prices depending on the service levels they choose. Those who select high service levels pay high unit rental rates and are high-value customers.

Another point is that lessors may have multiple kinds of equipment and face different types of demands. Generally, the demands of customers can be divided into two types: specified and unspecified demands. Specified demands have certain requirements regarding equipment type and can only be satisfied by the type of equipment requested while unspecified ones have no requirements regarding equipment type and can be satisfied by any type of equipment. The notion of unspecified demand arises from practical operations. It was pointed out by Steinhardt and Gönsch (2012) that car rental companies may provide different types of cars for customers who require a certain type of car. Another example is from the container leasing industry. Some customers want to rent new containers but can also accept old ones. In other words, they do not have strict requirements for the age of containers and can be satisfied by any type when they arrive. Unspecified demands arise when customers do not have strict requirements on some aspects of target services or products. They provide flexibility for lessors to manage their capacity.

Based on the above-mentioned features of equipment lease, which are batch rental demands, multiple service levels, diversity of rental equipment and multi-type demands, an operational challenge for a leasing company is how to manage its limited capacity efficiently. On the one hand, if the firm accepts each arriving customer, an opportunity cost would be incurred, and potential high-value customers, who could bring a higher profit, may be lost. On the other hand, if the firm keeps equipment idle for potential high-value lessees, it will lose the chances to gain immediate rewards from current customers and have to pay high inventory costs for idle equipment. Moreover, when the firm has different types of equipment and customers have various preferences, this capacity management problem becomes more complicated. If a customer arrives and requires a certain type of equipment which is totally rented out at that moment, whether and how much the firm chooses other equipment to meet the customer's needs is an important problem. Thus, there should be a criterion to guide the firm to decide on efficient and effective admission and allocation actions taking into account different types of equipment and batch demands. This paper addresses this problem.

In this paper, we consider a lessor who has two types of equipment (elementary and premium) constrained by finite capacity and faces customers with stochastic batch demands. That is, each customer requests a certain amount of equipment according to their operational demands. There are three types of batch demands from customers: elementary-specified, premium-specified and unspecified demands. An elementary-specified (premium-specified) batch demand calls for elementary (premium) equipment and can only be satisfied by this type of equipment. An unspecified batch demand has no strict requirements on equipment type. The firm can furnish it with any type of equipment. According to their pre-negotiated prices, customers are divided into several classes. Assume that the arrivals of customers follow mutually independent Poisson processes. When lessees arrive at the firm, they specify their equipment preferences. As a response, the firm selects a number pair that represents the numbers of elementary and premium equipment to be provided. We assume that the lessor can partially meet the demand. The purpose of the lessor is to maximise its total expected discounted revenue over an infinite time horizon with a discount factor. We model this problem as a Markov Decision Process (MDP) and prove the anti-multimodularity of the value function. We show that there exists a series of allocation and priority thresholds guiding the lessor's decisions. The simple structure of the optimal admission and capacity allocation policy allows the policy to be implemented easily. We further investigate the impacts of the system state on the optimal action and derive some managerial insights. We compare the performance of the optimal policy and two heuristic methods that are commonly used in practice. Finally, we explore the case of batch acceptance of demands.

In comparison to the related literature, this paper has the following contributions.

- We first study the optimal admission and capacity allocation policy for a leasing system with two types of equipment and three types of customers who present batch demands. We formulate the problem as an MDP and prove the anti-multimodularity of the optimal value function.
- We prove that partial acceptance is equivalent to dividing a batch demand into multiple single-unit demands and considering them separately.
- We show that the optimal admission and rationing policy has a simple structure and is characterised by a series of state-dependent allocation and priority thresholds. An exact algorithm is designed to calculate the optimal action at each system state by using the anti-multimodularity of the value function.
- We deduce the relationship between the firm's optimal actions at different system states. We show that the optimal action is insensitive to the system state.
- We compare the performance of the optimal policy with that of two heuristic methods to demonstrate the benefit of the optimal policy, and further investigate the batch acceptance case.

The organisation for the remainder of this paper is as follows. Section 2 reviews the relevant literature and identifies the research gap. Section 3 presents the mathematical model. In Section 4, we explore the structural properties of the value function. In Section 5, we characterise the optimal allocation policy and conduct a complementary numerical experiment. In Section 6, we compare the performance of the optimal policy with that of two heuristic methods. In Section 7, we discuss the case with batch acceptance. Conclusions and some future research topics are given in Section 8. All proofs are given in Appendices A–H.

2 Literature Review

This paper addresses the optimal admission and capacity allocation problem faced by a leasing company with two types of equipment, multiple customer classes and batch demands. There are three literature streams which are relevant to this work.

The first literature stream studies the admission control problems for queuing and leasing systems and consists of the following three parts. The first part investigates the admission problem for systems with multiple customer classes and single-unit demand. Miller (1969) showed that the optimal policy for a queuing system with arbitrary customer type has a threshold-type structure. Örmeci et al. (2001) explored the rationing problem for a queuing system facing two customer classes with different service rates. They proved the optimality of a threshold-type policy for this system. Savin et al. (2005) considered a rental system facing two customer classes with different leasing durations. They formulated the capacity rationing problem as a MDP, characterised the optimal policy and developed a heuristic by fluid approximation. Wang et al. (2021) studied the joint decision on production and allocation for a queuing system with different customer classes. Yang et al. (2022) explored the rationing policy for a leasing company with two types of containers. Differently from the above-quoted works, we consider a capacity management problem with batch demands that can be partially met. We prove the anti-multimodularity of the value function and show that the optimal admission and allocation policy still has a threshold structure.

The second part deals with the capacity allocation problem for systems with batch demands. Altman et al. (2001) studied the call admission control problem with batch demands. They established a optimal policy and developed a fluid approximate model for a large-capacity case. Örmeci and Burnetas (2004) considered the admission control problem for a queuing loss system with batch arrivals. An arriving batch included different jobs with the same service rate but different revenues, and the system adopted partial acceptance. They characterised the optimal policy and gave the condition for a class to be preferred. Çil et al. (2007) explored the system with a single server and batch demands where the system takes batch acceptance. There are N classes of jobs and a batch consists of only one class. They showed that the optimal policy is threshold-type when batch size is a constant. Papier and Thonemann (2011) studied a capacity rationing problem for a rental system with classical and premium customers. They adopted the queuing loss system to model this problem and derived the optimal fleet size and the optimal admission policy. In contrast to these authors, we incorporate unspecified demands which can be satisfied by both kinds of equipment.

The third part uses the mechanism of pricing to control the entrance of customers. Gans and Savin (2007) explored the joint capacity allocation and pricing problem for a rental system with both contract and walk-in customers. Zhuang et al. (2017) extended their work by incorporating Advance Demand Information (ADI). Yildirim and Hasenbein (2010) discussed an admission control and pricing problem with batch arrivals. The service provider first decides the price and then takes the admission action. The value function was proved to show a monotonicity structure. Jiao et al. (2016) investigated the dynamic pricing problem for a container leasing system with consideration of both hire time and quantity preference. The closed-form optimal price was derived and the influence of capacity constraint on different customers was analysed. The above works considered only one type of equipment.

The second literature stream deals with the inventory rationing problems in production/inventory systems. There is a strand of literature that studies inventory rationing problems with a single product. Topkis (1968) considered the inventory control problem in a periodic-review system and derived the optimal ordering and rationing policy. This work was extended by Ha (1997) to investigate make-to-stock systems. Gayon et al. (2009) studied a joint production and rationing problem for a system with several customer classes and ADI. A base-stock production policy and threshold-like allocation policy were derived. Ding et al. (2016) explored inventory systems with multiple classes and set-up costs under continuous review. An algorithm was developed to obtain the optimal rationing and ordering policy. Shen et al. (2019) investigated the joint ordering, expediting and allocation policy for an inventory system with multiple demand classes. Gong et al. (2022) explored the joint inventory control and pricing policy for a firm that has total minimum commitment contracts with its supplier. In the above studies, inventories were used only once. However, industrial equipment is durable, so its return and repeated rental will be considered in this work.

Some works studied inventory rationing problems with multiple products. Smith and Agrawal (2000) investigated an inventory management problem for a retail system with substitution. The inventory level for each item was optimised to maximise the total revenue. Sayah and Irnich (2019) studied a capacity control problem with two types of resources. They presented a method to obtain the optimal policy for both single-unit and multi-unit demands. Feng et al. (2021) explored the inventory replenishment, substitution and allocation policy for a multi-product system with stochastic supply and demand. In contrast to them, we consider three types of batch demands and a leasing system that has an extra return process. We characterise the admission and allocation policy for each batch demand and investigate how the system state influences the optimal action.

The third literature stream related to this study characterises the structure of the optimal policies of queuing or inventory systems by using modularity-related properties. Stidham and Weber (1993) gave an extensive review of previous literature on optimal control problems for queuing systems. Koole (1998) explored several structural properties for queuing systems by using event-based dynamic programming. Altman et al. (2000) investigated the relationship between multimodularity and convexity, and applied the results to analyse the structure of several queuing systems. Altman et al. (2003) discussed the relationship between multimodularity, superconvexity, submodularity and regularity. They studied the structure of the optimal policies in queuing systems with different settings, including admission control, routing, vacancies, service allocation, feedback control etc. In the above works, the authors studied queuing systems and assumed that each customer required one unit of service when they arrive. So the system only needs to provide one server for an accepted customer. In this paper, we study an equipment leasing system with batch demand. The system has to decide how much equipment to provide when a customer arrives. Therefore, the techniques used to prove the multimodularity of the value function in the above works cannot be directly used to show the anti-multimodularity of the value function in this paper. Because of this, we need to find some new properties of the value function (e.g., Lemma 3) and develop some new techniques. In Lemma 3, we derive the relationship between the optimal actions at two states ($(x+2, y)$ and $(x, y+1)$), which is necessary to prove the anti-multimodularity of the operators for unspecified demands and the value function. Moreover, using this new technique, we extend the result of Lemma 3 to show the relationship between the optimal actions at arbitrary different states, which is summarised in Theorems 6-8. We find that the leasing firm's optimal action is insensitive to the system state. Li and Yu (2014) used multimodularity to analyse the optimal policies of three inventory models. Since they did not consider unspecified demands, their proof techniques cannot be applied to our model.

3 Mathematical Model

In this section, we present the notation and mathematical model for the leasing system. We consider a leasing firm (lessor) that has c_1 units of elementary equipment and c_2 units of premium equipment. Generally, the rented equipment will incur usage costs for the leasing firm since the lessor will have to perform some maintenance on equipment after it is returned to keep it in good condition. Premium equipment has a higher usage cost than elementary equipment. In general, this type of cost is a linear function of the amount of rented equipment. We assume that the usage cost is covered by the rental price, which does not affect the analysis and the structure of the optimal policy. Therefore, for elementary and premium equipment, their difference in usage costs is covered by that in the rental prices. Assume that there are n classes of contract customers with different pre-specified prices. The unit rental prices of the elementary and the premium equipment for class-k ($k = 1, 2, ..., n$) customers are r_1^k and r_2^k , respectively. Assume that class-k customers arrive at the leasing firm following the Poisson process with rate λ_k. We assume that customers return one unit of leased equipment at a time. This assumption comes from the practice that many industrial operations are completed stage by stage. The lessees choose to return equipment gradually to maintain their operations (if lessees return all of the rented equipment after all works are finished, they have to spend more on maintenance costs of the equipment). For example, in the container leasing industry, containers are rented by shipping agents and assigned to freighters on different trade routes that have their specific travel times. The rented containers are returned after the trade routes are finished at different times. In addition, some precious equipment has to be returned separately since the lessor will carefully check the degree of damage to them and discuss the compensation (if highly damaged) after they are returned. In addition, when lessees decide to change their operation strategies, they may decrease their amount of leased equipment gradually and return rented equipment one by one. Assume that idle elementary and premium equipment incur unit holding costs h_1 and h_2, respectively.

Each customer has a multi-unit (batch) demand when they arrive at the leasing firm. We consider three types of demands: elementary-specified, premium-specified and unspecified demands. To simplify the statements, we name these three types of demands type-1 demands, type-2 demands and type-3 demands, respectively. Suppose that a class-k customer submits d_{ij}^k units of type-i demand with probability $p_{ij}^k \geq 0$ (where $j = 1, 2, ..., m_k^i$ and m_k^i is the number of optional batch sizes of class-k customers' type-i demands and $\sum_{i=1}^{3} \sum_{j=1}^{m_k^i} p_{ij}^k = 1$) when they arrive at the firm. For an unspecified demand, the firm can provide either elementary or premium equipment to meet it. Then it gives the firm more flexibility to manage its capacity. Because of this, the firm provides a price discount for type-3 demands. Let the unit rental prices of elementary and premium equipment be $\varepsilon_1 r_1^k$ and $\varepsilon_2 r_2^k$ for unspecified demands from class-k customers, respectively ($0 < \varepsilon_1, \varepsilon_2 \leq 1$).

The leasing durations of all demands are stochastic and follow a negative exponential distribution with rate μ*. We assume that the lessor can accept the customers' demands partially. When a specified demand arrives, the lessor partially accepts it with the required type of equipment. For an unspecified demand, the lessor can partially accept it and provide different types of equipment to meet the admitted part. In other words, the leasing firm can split the multi-unit demand into multiple single-unit demands and respond to each of them by one of the following three actions: (1) to provide an elementary equipment†; (2) to provide a premium equipment; or (3) to refuse the demand. We make this assumption since a customer may have leasing contracts with several lessors. A leasing customer may intentionally divide a whole batch demand (if too large) into several parts and rent equipment from different lessors to avoid the risk of a supply shortage. In addition, some customers may own certain items of equipment themselves for emergency use. From the lessor's point of view, in some situations their idle capacity may be insufficient to meet the arriving customer's whole batch demand and can only partially satisfy it. Compared to obtaining no equipment, the arriving customer may be willing to accept some equipment to partly meet the requirement for operational use. In Section 7, we explore the situation where the lessor will either accept the whole batch demands or refuse them.

Let (x, y) be the system state where x and y are non-negative integers which represent the quantities of on-hire elementary equipment and premium equipment, respectively. The system space is $C = \{(x, y) \,|0 \leq x \leq c_1, 0 \leq y \leq c_2\}$. Then an allocation action‡ under the system state (x, y) for a batch

*When leasing durations of different demands follow different distributions, the optimal policy does not have a simple structure. There are no interesting results.

†We briefly use 'an equipment' to denote 'a unit of equipment' in what follows.

‡From the definitions in Puterman (2014), the term 'action' represents the system's allocation behaviour at a certain system state while the term 'policy' represents a set of decision rules and specifies the system's action at any possible state.

demand d_{ij}^k can be denoted by a number pair (a,b), which indicates that the lessor offers a units of elementary equipment, b units of premium equipment, and denies $(d_{ij}^k - a - b)^+$ units of demand. In particular, for elementary-specified demands and premium-specified demands, we have $b = 0$ and $a = 0$, respectively.

The purpose of the lessor is to work out the optimal allocation policy to maximise the total expected discounted revenue, which consists of the immediate leasing reward and the future revenue, with a discount factor α. To simplify the expression, we use 'total revenue' to represent 'total expected discounted revenue' in what follows.

To simplify the analysis and specify the action clearly, we adopt the following convention. *The lessor always chooses to admit if it is indifferent to admitting or refusing a demand, and always provides elementary equipment if it is indifferent to offering elementary or premium equipment.*

When there are more than one allocation actions that maximise the leasing firm's total revenue, this convention provides a criterion to select the unique optimal action without confusion. We formulate the admission and capacity allocation problem for the leasing system with batch-demand customers as a continuous-time MDP and apply the method in Lippman (1975) to uniformise it to an equivalent discrete-time MDP. Define $\sigma = \sum_{k=1}^{n} \lambda_k + \mu(c_1 + c_2)$. The total event rate of the discrete-time MDP is $\Pi = \alpha + \sigma$. For the convenience of analysis, we assume that $\Pi = 1$. Therefore, at a system state (x,y), the next event is one of the following: arrival of a class-k customer with probability λ_k, return of a leased elementary (premium) equipment with probability μx (μy), fictitious return process with probability $\mu(c_1 + c_2 - x - y)$ and transition to the terminal state with probability α.

Define the firm's action sets for three types of demands as $S_{1j}^k(x,y) = \left\{ a | x + a \leq c_1, a \leq d_{1j}^k \right\}$, $S_{2j}^k(x,y) = \left\{ b | y + b \leq c_2, b \leq d_{2j}^k \right\}$ and $S_{3j}^k(x,y) = \left\{ (a,b) | x + a \leq c_1, y + b \leq c_2, a + b \leq d_{3j}^k \right\}$. Let $v(x,y)$ be the total revenue under the optimal admission and allocation policy in an infinite time horizon when the system state is (x,y). Thus, we present the optimality equation of the model as

$$v(x,y) = \sum_{k=1}^{n} \lambda_k \sum_{i=1}^{3} \sum_{j=1}^{m_k^i} p_{ij}^k L_{ij}^k[v(x,y)] + T[v(x,y)] + H[v(x,y)],$$

where

$$L_{1j}^k[v(x,y)] = \max_{a \in S_{1j}^k(x,y)} \{v(x+a,y) + a r_1^k\},$$

$$L_{2j}^k[v(x,y)] = \max_{b \in S_{2j}^k(x,y)} \{v(x,y+b) + b r_2^k\},$$

$$L_{3j}^k[v(x,y)] = \max_{(a,b) \in S_{3j}^k(x,y)} \{v(x+a,y+b) + a\varepsilon_1 r_1^k + b\varepsilon_2 r_2^k\},$$

$$T[v(x,y)] = \begin{cases} \mu(c_1+c_2)v(0,0), & x=0, y=0 \\ y\mu v(0,y-1)+\mu(c_1+c_2-y)v(0,y), & x=0, y\geq 1 \\ x\mu v(x-1,0)+\mu(c_1+c_2-x)v(x,0), & x\geq 1, y=0 \\ x\mu v(x-1,y)+y\mu v(x,y-1)+\mu(c_1+c_2-x-y)v(x,y), & x\geq 1, y\geq 1 \end{cases},$$

$$H[v(x,y)] = -h_1(c_1-x)-h_2(c_2-y).$$

The right side of the optimality equation has three parts, with each part representing a value iteration operator. The operator L_{ij}^k represents the system's response to the batch demand d_{ij}^k under the state (x,y). The lessor should select an optimal number pair to maximise the total revenue. The lessor has no choice but to deny the whole batch demand when the state is (c_1,c_2). The operator T reflects the return process and the fictitious return process. The operator H indicates the linear inventory costs. From Puterman (2014), there exists an optimal admission and allocation policy for the system since the action set $S_{ij}^k(x,y)$, state space C and one-period revenue are all finite. Örmeci and Burnetas (2004) investigated a queuing system with parallel servers and random batch demands. In contrast, we investigate a leasing system that supplies two types of equipment and adopts the partial acceptance policy. Yang et al. (2022) examined the optimal rationing policy for a system with two types of containers and multiple customer groups with single-unit demand. We extend their work by further considering batch demands where the lessor can partially meet demands.

From the optimality equation, we define the aggregate value iteration operator as

$$F[v(x,y)] = \sum_{k=1}^{n}\lambda_k\sum_{i=1}^{3}\sum_{j=1}^{m_k^i} p_{ij}^k L_{ij}^k[v(x,y)] + T[v(x,y)] + H[v(x,y)].$$

Let v be a $(c_1+1)\times(c_2+1)$ matrix that collects $v(x,y)$ for each $(x,y)\in C$ and denote an original estimation for v by v^0. Define $v^{j+1}=F[v^j]$ and $F^{l+1}[v^j]=F^l[F[v^j]]$ where $j,l=0,1,2,\ldots$. Then the operator F is convergent and we have $v=\lim_{n\to+\infty}v^n=\lim_{n\to+\infty}F^n[v^0]$ since there is a positive possibility α for the system to be terminated at each system state (Puterman, 2014). In other words, v can be obtained by performing the operator F for an arbitrary v^0 repetitively.

4 Properties of the Value Function

In this section, we prove some properties of the value function that will be used to characterise the structure of the optimal admission and allocation policy.

When the lessor faces a customer and determines the amount of equipment to provide, it has to compare the total revenues resulting from different allocation actions. The total revenue consists of the immediate reward, which is linear with respect to the selected number pair, and the future revenue, which depends on the system state after the allocation action. We can determine the firm's admission action at each system state and further characterise the optimal allocation policy if we understand how the value function $v(x,y)$ changes with respect to (x,y).

To describe the structural properties of the value function, we give some first-order and second-order differences of $v(x,y)$ as follows:

$$\Delta_x v(x,y) = v(x+1,y) - v(x,y),$$
$$\Delta_y v(x,y) = v(x,y+1) - v(x,y),$$

$$\Delta_l v(x,y) = v(x+1,y) - v(x,y+1),$$

$$\Delta_{x,x} v(x,y) = v(x+2,y) - 2v(x+1,y) + v(x,y),$$

$$\Delta_{y,y} v(x,y) = v(x,y+2) - 2v(x,y+1) + v(x,y),$$

$$\Delta_{x,y} v(x,y) = v(x+1,y+1) - v(x+1,y) - v(x,y+1) + v(x,y),$$

$$\Delta_{l,x} v(x,y) = v(x+2,y) - v(x+1,y) - v(x+1,y+1) + v(x,y+1),$$

$$\Delta_{l,y} v(x,y) = v(x+1,y+1) - v(x+1,y) - v(x,y+2) + v(x,y+1).$$

The first-order difference $\Delta_x v(x,y)$ ($\Delta_y v(x,y)$) represents the revenue change caused by renting an additional unit of elementary (premium) equipment. Then $-\Delta_x v(x,y)$ ($-\Delta_y v(x,y)$) is the opportunity cost of renting a unit of elementary (premium) equipment. The diagonal difference $\Delta_l v(x,y)$ is the difference between $\Delta_x v(x,y)$ and $\Delta_y v(x,y)$, which represents the difference between the opportunity costs of renting a unit of premium equipment and a unit of elementary one. The second-order differences indicate how the first-order differences change with respect to x and y.

From now on, we shall show the anti-multimodularity of $v(x,y)$, which is concerned with the above differences. Hajek (1985) gave the definition of the anti-multimodular function as follows.

Definition 1. *(Hajek 1985) Let* $f_0 = (-1,0)$*,* $f_1 = (1,-1)$*,* $f_2 = (0,1)$ *and* $\Gamma = \{f_0, f_1, f_2\}$*. A function* $J(\cdot) : C \subset Z^2 \to R$ *is anti-multimodular in* C *if for any* $e \in C$*,* $u, w \in \Gamma$ *(*$u \neq w$*),*

$$J(e+u) + J(e+w) \leq J(e) + J(e+u+w),$$

where Z *is the integer set.*

Lemma 1 gives necessary and sufficient conditions for a two-dimensional function to be anti-multimodular. Lemma 2 indicates that a two-dimensional anti-multimodular function is concave.

Lemma 1. *Suppose that C is a subset in $Z \times Z$. A function $J(\cdot): C \to R$ is anti-multimodular if and only if J is submodular and subconcave in C. That is, it satisfies*

(1) Submodularity: $\Delta_{x,y}J(x,y) \leq 0$, $\forall (x,y) \in C$;

(2) Subconcavity: $\Delta_{l,x}J(x,y) \leq 0$, $\Delta_{l,y}J(x,y) \geq 0$, $\forall (x,y) \in C$.

Lemma 2. *For $J(\cdot): C \to R$ defined on $C \subseteq Z \times Z$, if $J(\cdot)$ is submodular and subconcave, then it is concave, which satisfies*

$$\Delta_{x,x}J(x,y) \leq 0 \text{ and } \Delta_{y,y}J(x,y) \leq 0, \forall (x,y) \in C.$$

If the value function $v(x,y)$ is anti-multimodular with respect to the system state (x,y), then the opportunity cost of leasing an equipment increases* with the number of items of on-hire equipment. From this we can find a boundary for the system state such that it is optimal to provide equipment if the state is within the boundary and reject the demand otherwise. In addition, the anti-multimodularity of $v(x,y)$ implies that the difference between the opportunity costs of renting a premium and an elementary equipment is monotonic in the system state. Then we can also find a boundary for the system state such that it is more beneficial to provide elementary equipment if the state is within this boundary and to provide premium equipment otherwise. Since the value iteration operator F is convergent, if we prove that F can preserve anti-multimodularity, then $v(x,y)$ is an anti-multimodular function (Porteus, 1982). We need to show that $F[v^0(x,y)]$ is anti-multimodular for any anti-multimodular function $v^0(x,y)$. To this end, we give a technical Lemma as below.

To simplify the statement, for a state (x,y), we denote state $(x+2,y)$ as A_1, $(x+1,y)$ as B_1, $(x+1,y+1)$ as C_1 and $(x,y+1)$ as D_1 in the subsequent analysis.

Lemma 3. *Consider a leasing system W with anti-multimodular value function $v^0(x,y)$. Let (a_1^*,b_1^*) and (a_2^*,b_2^*) (where $a_1^*,b_1^*,a_2^*,b_2^* \geq 0$) be the optimal actions of the system for an unspecified batch d_{3j}^k at states A_1 and D_1, respectively. Then*

(a) When $a_1^* \geq 1$ and $b_1^* = 0$, the possible value for (a_2^*,b_2^*) is $(a_1^*,0)$, $(a_1^*+1,0)$, or $(a_1^*+2,0)$.

(b) When $a_1^* = 0$ and $b_1^* = 0$, the possible value for (a_2^*,b_2^*) is $(0,0)$, $(0,1)$, $(1,0)$, or $(2,0)$.

(c) When $a_2^* = 0$ and $b_2^* \geq 1$, the possible value for (a_1^*,b_1^*) is $(0,b_2^*-1)$, $(0,b_2^*)$, or $(0,b_2^*+1)$.

(d) When $a_2^* = 0$ and $b_2^* = 0$, the possible value for (a_1^*,b_1^*) is $(0,0)$, or $(0,1)$.

Lemma 3 characterises the relationship between the optimal allocation actions for an unspecified batch demand at states A_1 and D_1 when value function $v^0(x,y)$ is anti-multimodular. This means that the optimal actions at A_1 and D_1 are correlated. Lemma 3 is necessary for us to prove that the operator L_{3j}^k and F preserve anti-multimodularity. Based on Lemma 3, we deduce Lemma 4, which shows that the operator F can preserve anti-multimodularity.

*In this paper, the words 'increase' and 'decrease' are used in the weak sense.

Lemma 4. *If $v^0(x,y)$ is anti-multimodular with respect to (x,y), then $F[v^0(x,y)]$ is also anti-multimodular, i.e., the operator F preserves anti-multimodularity.*

The existing literature on the use of multimodularity in capacity management does not consider batch demand and unspecified demand at the same time. We use a new technique to prove that the optimal actions for an unspecified demand at A_1 and D_1 are correlated (Lemma 3). Based on this, we can prove that the aggregate operator F also preserves anti-multimodularity. Furthermore, this new technique allows us to extend the results in Lemmas 3 and 4 to obtain the relationship between the optimal actions at different states, which is characterised in Theorems 6–8. Thus, this study extends the applications of anti-multimodularity. Since F is convergent, we can choose an arbitrary anti-multimodular $v^0(x,y)$ as the initial estimate of $v(x,y)$ and perform F to get $v(x,y)$. Then we have the following theorem.

Theorem 1. *The value function $v(x,y)$ is anti-multimodular.*

From Lemma 1, Lemma 2 and Theorem 1, we know that the value function is submodular, subconcave and concave. The submodularity and concavity of $v(x,y)$ indicate that the opportunity costs of renting an elementary and premium equipment increase as more equipment is rented out. The subconcavity of $v(x,y)$ means that the difference between the opportunity costs of renting a premium and an elementary equipment decreases in x and increases in y.

In the next section, we show how these properties are used to characterise the structure of the optimal policy.

5 Structure of the Optimal Allocation Policy

In this section, we characterise the structure of the optimal admission and allocation policy, present an illustrative example and investigate the influence of system state on the optimal action.

From Section 3, we know that for specified demands, the lessor just decides the number of items of equipment provided. For unspecified demands, the lessor needs to simultaneously determine the type and the number of equipment items (i.e., number pair). If we work out the optimal actions for unspecified demands, then the optimal admission actions for specified demands can be obtained in a similar but simpler way. In the following, we focus on the optimal actions for unspecified demands.

Although we prove the anti-multimodularity of the value function in Theorem 1, we cannot directly characterise the structure of the optimal policy due to the existence of batch demands. To circumvent such difficulty, we tackle the problem from another point of view. When a batch demand arrives, we view it as multiple single-unit demands. For each single-unit demand, we decide whether it is to be accepted and which type of equipment is provided. By aggregating the allocation actions for each single-unit demand, we can obtain the action for the batch demand. To this end, we propose Algorithm 1.

When an unspecified batch demand d_{3j}^k arrives and the system state is (x,y), we divide the demand into d_{3j}^k single-unit demands and determine the optimal rationing action for each single-unit demand successively by using the following algorithm. Let a and b be the numbers of units of elementary and premium equipment to be provided, respectively. Let d be the number of single-unit demands already handled. They are dynamically updated as the algorithm runs.

Algorithm 1.

- Step 1. *Set* $a = 0$, $b = 0$ *and* $d = 0$, $v(x, c_2 + 1) = v(x+1, c_2+1) = ... = v(c_1, c_2+1) = -M$ *and* $v(c_1+1, y) = v(c_1+1, y+1) = ... = v(c_1+1, c_2) = -M$ *where M is a sufficiently large real number.*

- Step 2. *Calculate* $O_k[v(x+a, y+b)]$ *where* $O_k[v(x+a, y+b)] = \max\{v(x+a+1, y+b) + \varepsilon_1 r_1^k, v(x+a, y+b+1) + \varepsilon_2 r_2^k, v(x+a, y+b)\}$.

- Step 3. *Examine the following three cases in order*

  1. If $O_k[v(x+a, y+b)] = v(x+a+1, y+b) + \varepsilon_1 r_1^k$, *then set* $a = a+1$ *and* $d = d+1$, *go to Step 4.*

  2. If $O_k[v(x+a, y+b)] = v(x+a, y+b+1) + \varepsilon_2 r_2^k$, *then set* $b = b+1$ *and* $d = d+1$, *go to Step 4.*

  3. If $O_k[v(x+a, y+b)] = v(x+a, y+b)$, *then set* $d = d+1$, *go to Step 4.*

- Step 4. *If* $d = d_{3j}^k$, *go to step 5. Otherwise, go to Step 2.*

- Step 5. *Let* $\left(a_{3j}^{'k}(x,y), b_{3j}^{'k}(x,y)\right) = (a, b)$. *Output* $\left(a_{3j}^{'k}(x,y), b_{3j}^{'k}(x,y)\right)$ *and terminate the algorithm.*

Algorithm 1 outputs an allocation action for an unspecified demand. Instead of responding to the whole batch demand directly, Algorithm 1 calculates the optimal actions for each single-unit demand in turn and then aggregates them to obtain a solution. Generally, this solution is not necessarily optimal. However, we can show that it is exactly the optimal allocation action in our problem due to the anti-multimodularity of $v(x,y)$.

Let $\left(a_{3j}^{*k}(x,y), b_{3j}^{*k}(x,y)\right)$ be the optimal allocation action for the batch demand d_{3j}^k at system state (x,y).

Theorem 2. *Algorithm 1 calculates the optimal action, i.e.* $\left(a_{3j}^{'k}(x,y), b_{3j}^{'k}(x,y)\right) = \left(a_{3j}^{*k}(x,y), b_{3j}^{*k}(x,y)\right)$.

Theorem 2 implies that we can obtain the optimal allocation action through Algorithm 1 when a batch demand arrives. It then provides a method to circumvent the difficulty of finding the optimal allocation actions for batch demands. When a batch demand arrives, the algorithm can calculate the optimal action for each single-unit demand and then obtain the aggregative optimal allocation action for the whole batch.

To characterise the structure of the optimal policy, we define some thresholds as follows. The allocation thresholds are

$$R_1^k(y) = \begin{cases} 0, & \Delta_x v(0,y) < -\varepsilon_1 r_1^k \\ t, & \Delta_x v(t,y) < -\varepsilon_1 r_1^k \le \Delta_x v(t-1,y) \\ c_1, & \Delta_x v(c_1-1,y) \ge -\varepsilon_1 r_1^k \end{cases},$$

$$R_2^k(x) = \begin{cases} 0, & \Delta_y v(x,0) < -\varepsilon_2 r_2^k \\ t, & \Delta_y v(x,t) < -\varepsilon_2 r_2^k \le \Delta_y v(x,t-1) \\ c_2, & \Delta_y v(x,c_2-1) \ge -\varepsilon_2 r_2^k \end{cases},$$

$$H_1^k(y) = \begin{cases} 0, & \Delta_x v(0,y) < -r_1^k \\ t, & \Delta_x v(t,y) < -r_1^k \le \Delta_x v(t-1,y) \\ c_1, & \Delta_x v(c_1-1,y) \ge -r_1^k \end{cases},$$

$$H_2^k(x) = \begin{cases} 0, & \Delta_y v(x,0) < -r_2^k \\ t, & \Delta_y v(x,t) < -r_2^k \le \Delta_y v(x,t-1) \\ c_2, & \Delta_y v(x,c_2-1) \ge -r_2^k \end{cases}.$$

The priority thresholds are

$$R_3^k(y) = \begin{cases} 0, & \Delta_l v(0,y) < \varepsilon_2 r_2^k - \varepsilon_1 r_1^k \\ t, & \Delta_l v(t,y) < \varepsilon_2 r_2^k - \varepsilon_1 r_1^k \le \Delta_l v(t-1,y) \\ c_1, & \Delta_l v(c_1-1,y) \ge \varepsilon_2 r_2^k - \varepsilon_1 r_1^k \end{cases},$$

$$R_4^k(x) = \begin{cases} 0, & \Delta_l v(x,0) \ge \varepsilon_2 r_2^k - \varepsilon_1 r_1^k \\ t, & \Delta_l v(x,t) \ge \varepsilon_2 r_2^k - \varepsilon_1 r_1^k > \Delta_l v(x,t-1) \\ c_2, & \Delta_l v(x,c_2-1) < \varepsilon_2 r_2^k - \varepsilon_1 r_1^k \end{cases}.$$

Theorem 3. *At the system state (x,y), the optimal action for class-k customers is as follows.*

(a) For a single-unit elementary-specified demand, it is profitable to provide an elementary equipment if $x < H_1^k(y)$.

(b) For a single-unit premium-specified demand, it is profitable to provide a premium equipment if $y < H_2^k(x)$.

(c) For a single-unit unspecified demand, it is profitable to provide an elementary equipment if $x < R_1^k(y)$; it is profitable to provide a premium equipment if $y < R_2^k(x)$. In addition, it is more profitable to provide an elementary equipment rather than a premium one if $x < R_3^k(y)$; it is more profitable to provide a premium equipment rather than an elementary one if $y < R_4^k(x)$.

Theorem 3 provides us with a criterion to judge which action should be taken for a single-unit demand at any system state. The allocation thresholds $R_1^k(y)$, $R_2^k(x)$, $H_1^k(y)$ and $H_2^k(x)$ are used to evaluate whether it is beneficial to provide an elementary or premium equipment. The priority thresholds, $R_3^k(y)$ and $R_4^k(x)$, are used to determine the offer priority of the two types of equipment.

Theorem 4. *These thresholds have the following properties.*

(a) $R_1^k(y)$ and $H_1^k(y)$ decrease in y; $R_2^k(x)$ and $H_2^k(x)$ decrease in x. Furthermore, we have $R_1^k(y)-1 \leq R_1^k(y+1)$, $H_1^k(y)-1 \leq H_1^k(y+1)$, $R_2^k(x)-1 \leq R_2^k(x+1)$ and $H_2^k(x)-1 \leq H_2^k(x+1)$.

(b) $R_3^k(y)$ increases in y; $R_4^k(x)$ increases in x.

Theorem 4(a) indicates that the lessor should reserve its capacity for potential high-value lessees when less equipment is available. Moreover, the allocation thresholds have limited sensitivity to the system state. When the system state increases (decreases) by one unit, the corresponding thresholds decrease (increase) by at most one unit. Theorem 4(b) means that the lessor should provide elementary (premium) equipment for customers when there is much premium (elementary) equipment rented out. It is favourable for the lessor to maintain a balance of the numbers of on-hire elementary and premium equipment. Based on the analysis above, we can derive Theorem 5 as below, which completely characterises the optimal allocation policy of the system for three types of demands.

Theorem 5. *At state (x,y) , the system's optimal allocation action is as follows.*

(a) For an elementary-specified batch demand d_{1j}^k, it is optimal to provide $\min\left\{d_{1j}^k, \left(H_1^k(y)-x\right)^+\right\}$ units of elementary equipment.

(b) For a premium-specified batch demand d_{2j}^k, it is optimal to provide $\min\left\{d_{2j}^k, \left(H_2^k(x)-y\right)^+\right\}$ units of premium equipment.

(c) For an unspecified batch demand d_{3j}^k,

(c.1) If $x \geq R_1^k(y)$ and $y \geq R_2^k(x)$, reject the whole batch demand.

(c.2) If $x < R_1^k(y)$ and $y \geq R_2^k(x)$, provide $\min\left\{d_{3j}^k, R_1^k(y)-x\right\}$ units of elementary equipment.

(c.3) If $x \geq R_1^k(y)$ and $y < R_2^k(x)$, provide $\min\left\{d_{3j}^k, R_2^k(x)-y\right\}$ units of premium equipment.

(c.4) If $x < R_1^k(y)$ and $y < R_2^k(x)$, the units of elementary and premium equipment to be provided are calculated by the following algorithm.

Algorithm 2.

Step 1. *Let $a=0$, $b=0$, $d=0$, $a_p=0$ and $b_p=0$.*

Step 2. *Examine the following cases*

1. If $x+a \geq R_1^k(y+b)$ and $y+b \geq R_2^k(x+a)$, then go to Step 4.

2. *If* $x+a<R_1^k(y+b)$ *and* $y+b\geq R_2^k(x+a)$*, set* $a_p=\min\left\{d_{3j}^k-d,R_1^k(y+b)-x-a\right\}$*,* $a=a+a_p$ *and* $d=d+a_p$*. Go to Step 4.*
3. *If* $x+a\geq R_1^k(y+b)$ *and* $y+b<R_2^k(x+a)$*, set* $b_p=\min\left\{d_{3j}^k-d,R_2^k(x+a)-y-b\right\}$*,* $b=b+b_p$ *and* $d=d+b_p$*. Go to Step 4.*
4. *If* $x+a<R_1^k(y+b)$*,* $y+b<R_2^k(x+a)$ *and* $x+a<R_3^k(y+b)$*, set* $a_p=\min\{d_{3j}^k-d,R_1^k(y+b)-x-a,R_3^k(y+b)-x-a\}$*,* $a=a+a_p$ *and* $d=d+a_p$*. Go to Step 3.*
5. *If* $x+a<R_1^k(y+b)$*,* $y+b<R_2^k(x+a)$ *and* $x+a\geq R_3^k(y+b)$*, set* $b_p=\min\{d_{3j}^k-d,R_2^k(x+a)-y-b,R_4^k(x+a)-y-b\}$*,* $b=b+b_p$ *and* $d=d+b_p$*. Go to Step 3.*

Step 3. *If* $d=d_{3j}^k$*, go to Step 4. Otherwise, go to Step 2.*

Step 4. *The units of elementary and premium equipment to be provided are a and b, respectively.*

Remark 1. *Up to now, we have characterised the structure of the optimal policy under the expected discounted cost setting. For the average cost setting, we can get similar results following the arguments in Savin et al. (2005) and Gans and Savin (2007).*

Note that the optimal policy and profit of infinite period MDP can be obtained from the finite period MDP by letting the remaining period run to infinity. Denote the expected total revenue of the leasing firm with no discount by v_M *where M is the number of periods left. Denote the expected average revenue of the leasing firm with infinite period by* $\bar{v}$*. Then we have* $\bar{v}=\lim_{M\to\infty}\frac{1}{M}v_M$*. Since the system state, action space and one-period reward are all limited, there exists an optimal policy for the infinite period MDP under the average cost setting by Puterman (2014).*

Let the event rate be $\sum_{k=1}^{n}\lambda_k+\mu(c_1+c_2)=1$ *and denote the optimality equation by* $v_M(x,y)=\bar{F}[v_{M-1}(x,y)]$ *where* $\bar{F}[v_{M-1}(x,y)]=\sum_{k=1}^{n}\lambda_k\sum_{i=1}^{3}\sum_{j=1}^{m_k^i}p_{ij}^kL_{ij}^k[v_{M-1}(x,y)]+T[v_{M-1}(x,y)]+H[v_{M-1}(x,y)]$*. The operators* L_{ij}^k*, T and H are the same as in Section 3. By letting the discount factor* $\alpha\to 0$*, we can make* $F\to\bar{F}$ *(F is the aggregate value iteration operator defined in Section 3). Since the operator F preserves anti-multimodularity for any given discount factor* $\alpha\in(0,1)$*, the optimal policy under the average cost setting has a similar structure to that under the discounted cost setting.*

Theorem 5 shows that it is easy to derive the optimal action for specified demands. For an unspecified demand, it is also easy to derive the optimal action when lots of elementary or premium equipment are rented out. However, when there are little on-hire elementary and premium equipment, we need to consider which type of equipment to offer. Algorithm 2 is designed to obtain the optimal action for this case. The calculation of the optimal action can be decomposed into several stages. At each stage, the stage-optimal action is calculated according to the system state, allocation thresholds and priority thresholds. The system state is then updated by the stage-optimal action and the calculator goes to the next stage and so forth. Until the demand is entirely satisfied or it is unprofitable to provide any equipment, the algorithm terminates. The optimal action is obtained by summing all stage-optimal actions.

From Theorem 5, we can see that the optimal action is jointly determined by the system state and two types of state-dependent thresholds. It still has a simple structure even if we consider two types of rental items and batch demand. Theorem 5 indicates that, for elementary-specified (premium-specified) demands, the number of units of elementary (premium) equipment to be provided decreases in x (y). For unspecified demands, there are several cases to be considered. To make the optimal action for an unspecified demand be understood easily, a schematic diagram (Figure. 1) is given to clarify Theorem 5(c).

From Figure 1, when there are little idle elementary and premium equipment, it is optimal to reject the whole batch. When there is sufficient idle elementary (premium) equipment but little premium (elementary) equipment, it is optimal to provide elementary (premium) equipment until the number of rented elementary (premium) equipment reaches an upper bound. When there are enough idle elementary and premium equipment, it is optimal to provide two types of equipment. The numbers of two types of equipment to be offered are related to the priority and allocation thresholds.

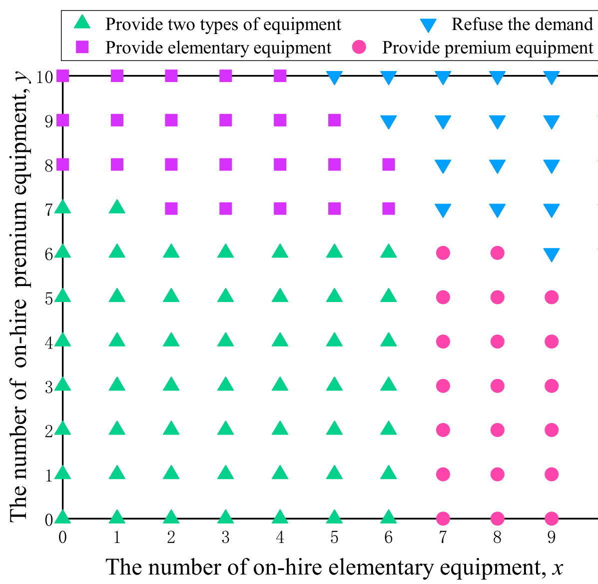

Figure 1. Illustration of Theorem 5(c).

A numerical example is given below to further illustrate Theorem 5. Consider a leasing system with three classes of customers. Each class arrives at the system following the same Poisson process with rate 1. The capacity of the system is $(c_1, c_2) = (50, 50)$. The possible batch sizes and corresponding probabilities for class-1 customers are $d^1_{11} = 12$, $d^1_{21} = 15$, $d^1_{31} = 10$, $d^1_{32} = 15$, $d^1_{33} = 45$, $p^1_{11} = 0.2$, $p^1_{21} = 0.1$, $p^1_{31} = 0.2$, $p^1_{32} = 0.3$ and $p^1_{33} = 0.2$. The possible batch sizes and corresponding probabilities for class-2 customers are $d^2_{11} = 15$, $d^2_{21} = 17$, $d^2_{31} = 15$, $d^2_{32} = 20$, $p^2_{11} = 0.3$, $p^2_{21} = 0.2$, $p^2_{31} = 0.4$ and

$p_{32}^2 = 0.1$. The possible batch sizes and corresponding probabilities for class-3 customers are $d_{11}^3 = 20$, $d_{21}^3 = 10$, $d_{31}^3 = 10$, $d_{32}^3 = 40$, $p_{11}^2 = 0.2$, $p_{21}^2 = 0.4$, $p_{31}^2 = 0.2$ and $p_{32}^2 = 0.2$. The rental prices of elementary and premium equipment for class-1 customers are $r_{11} = 50$ and $r_{12} = 250$, respectively. The rental prices of elementary and premium equipment for class-2 customers are $r_{21} = 90$ and $r_{22} = 110$, respectively. The rental prices of elementary and premium equipment for class-3 customers are $r_{31} = 100$ and $r_{32} = 150$, respectively. The price discounts for unspecified demands are $\varepsilon_1 = 0.8$ and $\varepsilon_2 = 0.85$. The leasing duration follows a negative exponential distribution with rate $\mu = 0.25$. The unit holding costs for elementary and premium equipment are $h_1 = 0.3$ and $h_2 = 0.9$, respectively. The discount factor is $\alpha = 0.5$. Tables 1–3 present the system's optimal allocation action.

Table 1. The optimal allocation action for batch demand $d_{32}^1 = 15$.

(x,y)	(20,40)	(25,40)	(30,40)	(35,40)	(20,45)	(25,45)	(30,45)	(35,45)
$\left(a_{32}^{*1}(x,y), b_{32}^{*1}(x,y)\right)$	(5,10)	(5,10)	(3,10)	(0,10)	(10,5)	(8,5)	(3,5)	(0,5)

Table 2. The optimal allocation action for batch demand $d_{32}^2 = 20$.

(x,y)	(40,20)	(40,25)	(40,30)	(40,35)	(45,20)	(45,25)	(45,30)	(45,35)
$\left(a_{32}^{*2}(x,y), b_{32}^{*2}(x,y)\right)$	(10,10)	(10,10)	(10,6)	(10,1)	(5,15)	(5,11)	(5,6)	(5,1)

Table 3. The optimal allocation action for batch demand $d_{31}^3 = 10$.

(x,y)	(30,20)	(30,25)	(30,30)	(30,35)	(35,20)	(35,25)	(35,30)	(35,35)
$\left(a_{31}^{*3}(x,y), b_{31}^{*3}(x,y)\right)$	(0,10)	(1,9)	(5,5)	(8,2)	(0,10)	(0,10)	(3,7)	(7,3)

We can see from Table 1 that, for given states, it is optimal for the system to allocate all idle premium equipment to class-1 customers since $\varepsilon_2 r_{12}$ is large. The system also offers several units of elementary equipment when x is small but the number of items of elementary equipment provided decreases in x. The system does not supply elementary equipment at the states $(35,40)$ and $(35,45)$ since x has reached (equal to or greater than) the allocation thresholds $R_1^1(40)$ and $R_1^1(45)$, respectively. Table 2 reveals a similar situation to that of Table 1. The system inclines to offer class-2 customers all idle elementary equipment due to the high rental price. The number of items of premium equipment to be provided for class-2 customers decreases in y. Table 3 describes a fairly different condition. Although $\varepsilon_1 r_{31}$ and $\varepsilon_2 r_{32}$ are large enough to ensure that the unspecified demands from class-3 customers would not be rejected, the system still has to make choices between two types of equipment. Even if class-3 customers pay the highest price for elementary equipment, it is not optimal for the system to provide them only with this type. On the contrary, it is beneficial to also offer some premium equipment and reserve some elementary equipment for class-2 customers since the rental rates they pay for premium equipment are relatively low.

Theorem 5 completely characterises the structure of the optimal policy for the leasing system. In what follows, we explore the impacts of the system states on the optimal actions. Define $z = x + y$ to be the number of items of total[*] on-hire equipment. Let (x_0, y_0) be the initial state. The following theorem shows how the optimal action changes as z decreases.

Theorem 6. *For an unspecified batch demand, d^k_{3j}, how the optimal action changes can be characterised as follows:*

(a) As x and y decrease[†], the number of total equipment provided by the firm increases. Moreover, we have

Conditions	Results
If the system offers at least one unit of elementary and one unit of premium equipment at (x_0, y_0)	The increment in the number of items of elementary (premium) equipment provided is smaller than the decrement in x (y)
If the system does not offer elementary or premium equipment at (x_0, y_0)	The increment in the number of items of total equipment provided is smaller than the decrement in z

[*] We use 'total equipment' to represent the summation of two types of equipment.

[†] The words 'decrease', 'increase', 'large' and 'small' are used in the weak sense.

(b) As x decreases, y increases and z decreases, we have

Conditions		Results
General conditions	Additional conditions	
When the system offers at least one unit of elementary equipment at (x_0, y_0)	If the increment in y is small	The decrement in the number of items of premium equipment provided is larger than the increment in y
	If the increment in y is large	The number of items of premium equipment provided decreases to zero
		The number of items of total equipment provided increases and the increment in the number of items of elementary equipment provided is smaller than the decrement in x
When the system does not offer elementary equipment at (x_0, y_0)	If the increment in y is small	The increment of the number of items of total equipment provided is smaller than the decrement in z
	If the increment in y is large	The number of items of premium equipment provided may decrease to zero
		The increment in the number of items of elementary equipment provided is smaller than the decrement in x

(c) As x increases, y decreases and z decreases, we have

Conditions		Results
General conditions	Additional conditions	
When the system offers at least one unit of premium equipment at (x_0, y_0)	If the increment in x is small	The decrement in the number of items of elementary equipment provided is larger than the increment of x
	If the increment in x is large	The number of items of elementary equipment provided decreases to zero
		The number of items of total equipment provided increases and the increment in the number of items of premium equipment provided is smaller than the decrement in y
When the system does not offer premium equipment at (x_0, y_0)	If the increment in x is small	The increment in the number of items of total equipment provided is smaller than the decrement in z
	If the increment in x is large	The number of items of elementary equipment provided may decrease to zero
		The increment in the number of items of premium equipment provided is smaller than the decrement in y

Theorem 6 shows how the system state affects the optimal action for an unspecified demand, which can help us discover the relationship between the optimal actions at different states. Generally, for a given unspecified demand, the system should offer more total equipment when it has more idle total equipment. However, the increment in the number of items of total equipment offered is smaller than that in the number of items of idle total equipment. That is to say, the firm does not have to rent too much equipment out. Which type of equipment to provide depends on the quantities of two types of equipment the system has on hand.

From Theorem 6(a), if a system has enough elementary and premium equipment, it may provide both types. As the number of items of idle elementary and premium equipment increases, the amount of total equipment provided by the system also increases. But the increment in the amount of elementary (premium) provided equipment is less than that in the amount of idle elementary (premium) equipment. If a system does not have enough idle elementary or premium equipment, it can only provide one type. As the firm has more idle elementary and premium equipment, it may provide both types but the amount of total equipment provided has a limited increment.

From Theorem 6(b), as the amount of idle premium equipment decreases but the amount of idle total equipment increases, the system may offer less premium but more total equipment. When the system has less premium equipment on hand, it may only provide elementary equipment to meet current demand and keep premium equipment for potential high-value demands. When the amount of idle elementary equipment decreases and the amount of idle total equipment increases, we can arrive at similar conclusions.

These phenomena can be explained as follows. The system's allocation action depends on the relationship between the opportunity costs of renting the two types of equipment and the rental prices. Low opportunity cost induces the system to provide more equipment for the current customer. High opportunity cost reduces the incentive to rent equipment at the current time. Due to the anti-multimodularity of the value function, the opportunity cost increases with the number of items of on-hire equipment but the increasing rate is limited. The optimal action then does not drastically change when the system state varies.

By similar arguments to those in Theorem 6, we can obtain similar results for specified demands. Theorems 7–8 show how the optimal action for a given specified demand changes with respect to the system state.

Theorem 7. *For an elementary-specified batch demand d_{1j}^k, how the optimal action changes can be characterised as follows:*

(a) As x and y decrease, the number of items of elementary equipment provided increases, but the increment is smaller than the decrement in z ($=x+y$).

(b) As x decreases, y increases and z decreases, the number of items of elementary equipment provided increases, but the increment is smaller than the decrement in x.

(c) As x increases, y decreases and z decreases, if the number of items of elementary equipment provided increases, the increment is smaller than the decrement in z; if the number of items of elementary equipment provided decreases, the decrement is smaller than the increment in x.

Theorem 8. *For a premium-specified batch demand d_{2j}^{k}, how the optimal action changes can be characterised as follows.*

(a) As x and y decrease, the number of items of premium equipment provided increases, but the increment is smaller than the decrement in z (=x+y).

(b) As x increases, y decreases and z decreases, the number of items of premium equipment provided increases, but the increment is smaller than the decrement in y.

(c) As x decreases, y increases and z decreases, if the number of items of premium equipment provided increases, the increment is smaller than the decrement in z; if the number of items of premium equipment provided decreases, the decrement is smaller than the increment in y.

Theorems 7 and 8 show how the system state affects the optimal action for specified demands. The number of items of equipment provided by the system is related to not only the amount of idle equipment required* but also that of other type of equipment. In general, when facing a specified demand, the lessor should provide more equipment when it has more equipment on hand. But it should not allocate too much equipment and should keep its capacity at a moderate level. In particular, when the system has more of elementary and premium equipment on hand, the increment in the amount of equipment provided is smaller than the increment in the amount of idle total equipment. When the system has more of the idle equipment required and less of the idle equipment undesired, the increment in the amount of provided equipment is smaller than that in idle equipment required. When the system has more undesired equipment but less required equipment on hand, the amount of equipment provided may decrease.

6 The Benefit of the Optimal Policy

In this section, to show the benefit of the optimal policy, we compare the performance of the optimal policy with that of two heuristic methods, which are commonly used in practice.

The first heuristic method is the *elementary-prior myopic policy* which is characterised as follows. The leasing firm always tries to satisfy the specified demand with the required equipment. For an

*For elementary-specified (premium-specified) demands, elementary (premium) equipment is the required equipment while premium (elementary) equipment is the undesired equipment.

unspecified demand, the leasing firm first tries to use elementary equipment to satisfy the demand. If there is insufficient idle elementary equipment, then the firm uses premium equipment to satisfy the demand. If the demand cannot be completely satisfied by idle equipment, the remainder of the demand is refused.

The second heuristic method is the *premium-prior myopic policy*. Under this policy, the leasing firm uses the same principle to deal with specified demands as used under the elementary-prior myopic policy. But for unspecified demands, the firm uses first idle premium equipment and then idle elementary equipment to satisfy them. If the total amount of idle equipment is insufficient to meet the batch demand, the remainder of the demand is refused.

Let $v_1(x,y)$ and $v_2(x,y)$ be the total revenue of the firm at state (x,y) under elementary-prior and premium-prior myopic policies, respectively. Define

$$\rho_1 = \frac{v(0,0)-v_1(0,0)}{v(0,0)},\ \rho_2 = \frac{v(0,0)-v_2(0,0)}{v(0,0)}.$$

Then ρ_1 and ρ_2 can measure the revenue improvement brought by the optimal policy compared to the above two myopic policies. To explore the influence of the difference between equipment rental prices for different classes of customers on the revenue improvement, we change the rental prices for a class of customer and keep other parameters as constants in the following numerical experiments.

Suppose that the leasing firm owns $c_1 = c_2 = 10$ units of elementary and premium equipment, and faces three classes of customers with arrival rates $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 2$, respectively. Other parameters are set as follows: $\mu = 0.8$, $\alpha = 0.5$, $h_1 = 0.5$, $h_2 = 0.7$, $d^1_{11} = 12$, $d^1_{21} = 5$, $d^1_{31} = 10$, $d^2_{11} = 8$, $d^2_{21} = 12$, $d^2_{31} = 9$, $d^3_{11} = d^3_{21} = d^3_{31} = 10$, $p^1_{11} = 0.3$, $p^1_{21} = 0.4$, $p^1_{31} = 0.3$, $p^2_{11} = 0.3$, $p^2_{21} = 0.5$, $p^2_{31} = 0.2$, $p^3_{11} = 0.3$, $p^3_{21} = 0.6$, $p^3_{31} = 0.1$, $r^1_1 = 70$, $r^1_2 = 200$, $r^2_1 = 90$, $r^2_2 = 290$, $\varepsilon_1 = \varepsilon_2 = 0.8$. Define $\Delta_1 = r^3_1 - r^2_1$ as the difference between the rental prices of elementary equipment for class-3 and class-2 customers. Define $\Delta_2 = r^3_2 - r^2_2$ as the difference between the rental prices of premium equipment for class-3 and class-2 customers. Tables 4 and 5 summarise how the differences between rental prices influence ρ_1 and ρ_2, respectively.

Table 4. The influence of the rental price differences on ρ_1

ρ_1 (Δ_1 \ Δ_2)	0	50	100	150	200
0	0.017	0.026	0.041	0.058	0.074
50	0.021	0.028	0.042	0.059	0.074
100	0.030	0.035	0.048	0.062	0.077
150	0.052	0.055	0.065	0.076	0.091
200	0.080	0.082	0.090	0.098	0.109

From Tables 4 and 5, we find that the optimal policy performs much better than the myopic ones

Table 5. The influence of the rental price differences on ρ_2

ρ_2 \ Δ_2 / Δ_1	0	50	100	150	200
0	0.023	0.037	0.054	0.072	0.090
50	0.033	0.042	0.058	0.075	0.092
100	0.046	0.053	0.066	0.082	0.098
150	0.071	0.076	0.087	0.098	0.113
200	0.102	0.105	0.113	0.122	0.134

(the optimal policy brings 1.7%–13.4% revenue improvement on the myopic methods). In addition, as Δ_1 (Δ_2) increases, the revenue improvement also increases. This means that, as the differences in rental prices increase, the optimal policy performs better. The reason for this is that, under myopic policies, the firm always uses idle equipment to meet as many as possible of the arriving demands. Under the optimal policy, the firm may keep some equipment for customers with high prices to earn more profits. Another phenomenon is that ρ_2 is always larger than ρ_1. This means that it is better to use elementary than premium equipment to satisfy unspecified demands. The reason is that premium-specified demands arrive more frequently than elementary-specified ones do.

7 Batch Acceptance

In Section 3, we assumed that the leasing firm can accept the demand partially. In this section, we consider the case that the firm adopts a batch acceptance policy. That is, when a batch demand arrives, the firm either refuses it or accepts it as a whole. The return process is the same as before*. We consider a special case where the batch demands from all customers have the same size d. Let $u(x,y)$ be the firm's total revenue when the current system state is (x,y). Then the optimality equation is given as follows.

$$u(x,y) = \sum_{k=1}^{n} \lambda_k \sum_{i=1}^{3} p_i^k L_{di}^k[u(x,y)] + T[u(x,y)] + H[u(x,y)],$$

where

$$L_{d1}^k[u(x,y)] = \max\{u(x+d,y) + dr_1^k, u(x,y)\},$$

$$L_{d2}^k[u(x,y)] = \max\{u(x,y+d) + dr_2^k, u(x,y)\},$$

$$L_{d3}^k[u(x,y)] = \max\{u(x+d,y) + d\varepsilon_1 r_1^k, u(x,y+d) + d\varepsilon_2 r_2^k, u(x,y)\}.$$

The operator L_{di}^k represents the firm's allocation policy for class-k customers' type-i demands. The

*The analysis for the case with batch acceptance and batch return is similar to that of the case with single-unit demand. Therefore, we omit it.

operators T and H are the same as before. To characterise the properties of the value function, define d-jump differences of $u(x,y)$ as:

$$\Delta_x^d u(x,y) = u(x+d,y) - u(x,y),$$

$$\Delta_y^d u(x,y) = u(x,y+d) - u(x,y),$$

$$\Delta_l^d u(x,y) = u(x+d,y) - u(x,y+d),$$

$$\Delta_{x,x}^d u(x,y) = u(x+2d,y) - 2u(x+d,y) + u(x,y),$$

$$\Delta_{y,y}^d u(x,y) = u(x,y+2d) - 2u(x,y+d) + u(x,y),$$

$$\Delta_{x,y}^d u(x,y) = u(x+d,y+d) - u(x+d,y) - u(x,y+d) + u(x,y),$$

$$\Delta_{l,x}^d u(x,y) = u(x+2d,y) - u(x+d,y) - u(x+d,y+d) + u(x,y+d),$$

$$\Delta_{l,y}^d u(x,y) = u(x+d,y+d) - u(x+d,y) - u(x,y+2d) + u(x,y+d).$$

To keep the threshold structure of the optimal policy in this situation, we need to establish some properties for the value function $u(x,y)$. In particular, following the definition of Q-jump convexity in Yang et al. (2014) and Yang et al. (2017), we define a two-dimensional d-jump anti-multimodular function as follows.

Definition 2. *Let $f_0 = (-d,0)$, $f_1 = (d,-d)$, $f_2 = (0,d)$ and $\Gamma = \{f_0, f_1, f_2\}$. A function $J(\cdot) : C \subset Z^2 \to R$ is d-jump anti-multimodular in C if for any $e \in C$, $u,w \in \Gamma$ ($u \neq w$),*

$$J(e+u) + J(e+w) \leq J(e) + J(e+u+w),$$

where Z is the integer set.

Lemma 5. *A function $J(\cdot) : C \to R$ is d-jump anti-multimodular if and only if J is d-jump submodular, d-jump subconcave and d-jump concave in C. That is, it satisfies*

(1) d-jump submodularity: $\Delta_{x,y}^d J(x,y) \leq 0$, $\forall (x,y) \in C$;

(2) d-jump subconcavity: $\Delta_{l,x}^d J(x,y) \leq 0$, $\Delta_{l,y}^d J(x,y) \geq 0$, $\forall (x,y) \in C$;

(3) d-jump concavity: $\Delta_{x,x}^d J(x,y) \leq 0$, $\Delta_{y,y}^d J(x,y) \leq 0$, $\forall (x,y) \in C$.

We can characterise the optimal allocation policy if we can show the d-jump anti-multimodularity of $u(x,y)$. The following theorem shows that $u(x,y)$ is d-jump anti-multimodular when the expected rental duration μ^{-1} is sufficiently large.

Theorem 9. *There exists μ_m such that $u(x,y)$ is d-jump anti-multimodular when $\mu \leq \mu_m$. The value of μ_m depends on the system parameters.*

The d-jump anti-multimodularity of the value function $u(x,y)$ indicates that the d-jump difference $\Delta_x^d u(x,y)$ is not greater than $\Delta_x^d u(x,y+d)$, and $\Delta_x^d u(x,y+d)$ is not greater than $\Delta_x^d u(x+d,y)$; the d-jump difference $\Delta_y^d u(x,y)$ is not greater than $\Delta_y^d u(x+d,y)$, and $\Delta_y^d u(x+d,y)$ is not greater than $\Delta_y^d u(x,y+d)$.

For $\tau = 0,1,2...,d-1$, let $[\tau]_d = \{\omega \in N | \omega = \kappa d + \tau, \kappa = 0,1,2,...\}$ which is the set of integers that have the remainder of τ when divided by d. Let $C_1 = \{x | 0 \leq x \leq c_1\}$ and $C_2 = \{y | 0 \leq y \leq c_2\}$. Then we have $\Delta_x^d u(x_1,y) \geq \Delta_x^d u(x_2,y)$ and $\Delta_l^d u(x_1,y) \geq \Delta_l^d u(x_2,y)$ if $x_1,x_2 \in [\tau_0]_d \cap C_1$ with $x_1 \leq x_2$, and $\Delta_y^d u(x,y_1) \geq \Delta_y^d u(x,y_2)$ and $\Delta_l^d u(x,y_1) \leq \Delta_l^d u(x,y_2)$ if $y_1,y_2 \in [\tau_0]_d \cap C_2$ with $y_1 \leq y_2$. Based on these properties, we can characterise the optimal policy.

For a state (x,y), let τ_x and τ_y be the remainders when x and y are divided by d, respectively. Then x (y) is included in the set $[\tau_x]_d$ ($[\tau_y]_d$). Let $\iota_x^{1a} = \max\{\iota | \iota \in [\tau_x]_d \cap C_1\}$ and $\iota_y^{2a} = \max\{\iota | \iota \in [\tau_y]_d \cap C_2\}$, and $\iota_x^{1i} = \min\{\iota | \iota \in [\tau_x]_d \cap C_1\}$ and $\iota_y^{2i} = \min\{\iota | \iota \in [\tau_y]_d \cap C_2\}$.

Define the thresholds as

$$H_{\tau_x}^{k1}(y) = \begin{cases} \iota_x^{1i}, & \Delta_x^d v(\iota_x^{1i},y) < -dr_1^k \\ t \in [\tau_x]_d, & \Delta_x^d v(t,y) < -dr_1^k \leq \Delta_x^d v(t-d,y) \\ \iota_x^{1a}, & \Delta_x^d v(\iota_x^{1a}-d,y) \geq -dr_1^k \end{cases},$$

$$H_{\tau_y}^{k2}(x) = \begin{cases} \iota_y^{2i}, & \Delta_y^d v(x,\iota_y^{2i}) < -dr_2^k \\ t \in [\tau_y]_d, & \Delta_y^d v(x,t) < -dr_2^k \leq \Delta_y^d v(x,t-d) \\ \iota_y^{2a}, & \Delta_y^d v(x,\iota_y^{2a}-d) \geq -dr_2^k \end{cases},$$

$$R_{\tau_x}^{k1}(y) = \begin{cases} \iota_x^{1i}, & \Delta_x^d v(\iota_x^{1i},y) < -d\varepsilon_1 r_1^k \\ t \in [\tau_x]_d, & \Delta_x^d v(t,y) < -d\varepsilon_1 r_1^k \leq \Delta_x^d v(t-d,y) \\ \iota_x^{1a}, & \Delta_x^d v(\iota_x^{1a}-d,y) \geq -d\varepsilon_1 r_1^k \end{cases},$$

$$R_{\tau_y}^{k2}(x) = \begin{cases} \iota_y^{2i}, & \Delta_y^d v(x,\iota_y^{2i}) < -d\varepsilon_2 r_2^k \\ t \in [\tau_y]_d, & \Delta_y^d v(x,t) < -d\varepsilon_2 r_2^k \leq \Delta_y^d v(x,t-d) \\ \iota_y^{2a}, & \Delta_y^d v(x,\iota_y^{2a}-d) \geq -d\varepsilon_2 r_2^k \end{cases},$$

$$R_{\tau_x}^{k3}(y) = \begin{cases} \iota_x^{1i}, & \Delta_l^d v(\iota_x^{1i},y) < d\varepsilon_2 r_2^k - d\varepsilon_1 r_1^k \\ t \in [\tau_x]_d, & \Delta_l^d v(t,y) < d\varepsilon_2 r_2^k - d\varepsilon_1 r_1^k \leq \Delta_l^d v(t-d,y) \\ \iota_x^{1a}, & \Delta_l^d v(\iota_x^{1a}-d,y) \geq d\varepsilon_2 r_2^k - d\varepsilon_1 r_1^k \end{cases}.$$

Theorem 10. *When $\mu \leq \mu_m$, the system's optimal action at state (x,y) is as follows:*

(a) For an elementary-specified batch demand from class-k customers, it is optimal to provide d units of elementary equipment if $x < H_{\tau_x}^{k1}(y)$ and to refuse the demand otherwise.

(b) For a premium-specified batch demand from class-k customers, it is optimal to provide d units of premium equipment if $y < H_{\tau_y}^{k2}(x)$ and to refuse the demand otherwise.

(c) For an unspecified batch demand from class-k customers, it is optimal (1) to provide d units of elementary equipment if $x < R_{\tau_x}^{k1}(y)$ and $y \geq R_{\tau_y}^{k2}(x)$, or $x < R_{\tau_x}^{k1}(y)$, $y < R_{\tau_y}^{k2}(x)$ and $x < R_{\tau_x}^{k3}(y)$; (2) to provide d units of premium equipment if $x \geq R_{\tau_x}^{k1}(y)$ and $y < R_{\tau_y}^{k2}(x)$, or $x < R_{\tau_x}^{k1}(y)$, $y < R_{\tau_y}^{k2}(x)$ and $x \geq R_{\tau_x}^{k3}(y)$; and (3) to refuse the demand if $x \geq R_{\tau_x}^{k1}(y)$ and $y \geq R_{\tau_y}^{k2}(x)$.

Since the value function is not generally anti-multimodular, the thresholds in this section are slightly different from those seen in Section 5. For example, in the original model, the value of threshold $H_1^k(y)$ only depends on y. In the batch acceptance model, the value of threshold $H_{\tau_x}^{k1}(y)$ depends on y as well as the remainder of d dividing x. In addition, to find the optimal action, we need to get the thresholds, $H_0^{k1}(y)$, $H_1^{k1}(y)$,..., $H_{d-1}^{k1}(y)$.

Theorem 10 characterises the system's optimal allocation policy in the case with batch acceptance. The optimal policy still has a simple structure. From Theorem 10(a) and (b), we find that the optimal policy depends on not only x and y themselves, but also the remainders of d dividing x and y. For an elementary-specified (premium-specified) demand and given remainders of d dividing x and y, the system will offer d units of elementary (premium) equipment if it has much idle elementary (premium) equipment and refuse the demand otherwise. From Theorem 10(c), we find that for an unspecified demand and given remainders of d dividing x and y, the system will offer d units of elementary (premium) equipment if it has much idle elementary (premium) equipment and little idle premium (elementary) equipment; the system will refuse the demand if it has little idle elementary and premium equipment.

Theorem 11. *The thresholds have the following properties.*

(a) $H_{\tau_x}^{k1}(y) - d \leq H_{\tau_x}^{k1}(y+d) \leq H_{\tau_x}^{k1}(y)$ *and* $H_{\tau_y}^{k2}(x) - d \leq H_{\tau_y}^{k2}(x+d) \leq H_{\tau_y}^{k2}(x)$;

(b) $R_{\tau_x}^{k1}(y) - d \leq R_{\tau_x}^{k1}(y+d) \leq R_{\tau_x}^{k1}(y)$, $R_{\tau_y}^{k2}(x) - d \leq R_{\tau_y}^{k2}(x+d) \leq R_{\tau_y}^{k2}(x)$ *and* $R_{\tau_x}^{k3}(y+d) \geq R_{\tau_x}^{k3}(y)$.

From Theorem 11, when y increases by d units, the thresholds $H_{\tau_x}^{k1}(y)$ and $R_{\tau_x}^{k1}(y)$ will decrease, but their decrements are not greater than d. When x increases by d units, the thresholds $H_{\tau_y}^{k2}(x)$ and $R_{\tau_y}^{k2}(x)$ will decrease but their decrements are not greater than d. The system is more willing to refuse the current demand when the number of idle elementary or premium equipment decreases by d. The threshold $R_{\tau_x}^{k3}(y)$ increases when y increases by d units, which indicates that the system is more willing to provide elementary equipment for unspecified demands.

8 Conclusions

In this work, we explore the capacity allocation problem faced by a leasing system with two types of equipment and three types of batch demands: elementary-specified, premium-specified and unspecified demands. For specified demands, the lessor can admit them partially with only the required type of equipment. For unspecified demands, the system can admit them partially and satisfy the accepted part with any type of equipment. The lessor's aim is to maximise the total revenue by working out the optimal amounts of elementary and premium equipment to provide. We formulate the problem as an MDP and show that the value function is anti-multimodular. We further show that it is equivalent to dividing the batch demand into multiple single-unit demands and taking the action for each single-unit demand. Based on this, we develop an algorithm to compute the optimal number of units of equipment provided for customers and characterise the optimal policy. We show that the optimal allocation policy can be characterised by state-dependent rationing and priority thresholds, which serves as an effective method for the lessor's decision-making in practice. A numerical experiment is carried out to illustrate the optimal allocation policy and to derive several managerial insights. We show how the system state affects the optimal action by comparing the firm's optimal actions at different system states. We further find that the optimal action is insensitive to the system state. We demonstrate the benefit of the optimal policy by comparing the performance of the optimal policy with that of two heuristic methods that are commonly used in practice. Finally, we explore the case of batch acceptance.

Based on this study, there are several research topics that can be investigated further.

In addition to the difference in demand volumes and rental prices, lessees may also have differences in leasing durations. It would be an interesting topic to consider the system with diverse customer classes differentiated not only by demand volumes and rental prices but also by leasing durations.

In practice, a container leasing company has many depots in different places. Containers are assigned dynamically among these depots to meet rental demands. It is challenging to examine the capacity allocation problem considering the assignment of containers among multiple depots.

Another research direction is to take the competition from the trading market into consideration. That is, there are two or more leasing firms in the market and customers can rent equipment from them. If customers cannot get equipment from their favourite firm, they may switch to others.

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