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Optimal Capacity Rationing Policy for a Container Leasing System with Multiple Kinds of Customers and Substitutable Containers

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Abstract. In this paper, we consider a container leasing firm that has elementary and premium containers, which are downward substitutable and for use by elementary contract customers (ECCs), premium contract customers (PCCs), as well as walk-in customers (WICs). ECCs can be satisfied by elementary containers or premium ones at discounted prices while PCCs only accept premium containers. WICs can be satisfied by any type of container at different prices. The objective is to maximise the expected total rental revenue by managing its limited capacity. We formulate this problem as a discrete-time Markov Decision Process and show the submodularity and concavity of the value function. Based on this, we show that the optimal policy can be characterised by a series of rationing thresholds, a series of substitution thresholds and a priority threshold, all of which depend on the system states. We further give conditions under which the optimal policy can be simplified. Numerical experiments are conducted to show the impact of the substitution of two items on the revenue, to compare the performance of the optimal policy with those of the commonly used policies and to investigate the influence of arrival rates on the optimal policy. Last, we extend the basic model to consider different rental durations, ECCs' acceptance behaviour and endogenous prices for WICs.

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Keywords: container leasing • capacity rationing • Markov Decision Process • downward substitution

1. Introduction

According to the United Nations Conference on Trade and Development (UNCTAD 2022), global container port throughput has experienced outstanding growth from 541.76 million TEU in 2010 to 807.33 million TEU in 2019, which shows the increasing demand from shipping companies for containers. Shipping companies either purchase from the trading market to acquire their own fleet or lease from container leasing firms in order to enjoy benefits such as cost saving, quick response to demand changes, and high flexibility (Jiao et al. 2016).

In practice, container leasing firms provide multiple types of containers to meet various customer requirements. Elementary containers, which are equipped with only basic functions, are leased at a relatively low price. Premium containers, which can provide some additional functions and meet some advanced requirements, are leased at a high price. Elementary containers and premium containers are generally differentiated by one dimension in the status of containers, such as 20-foot dry containers that are older versus 20-foot dry containers that are newer or 40-foot refrigerated containers with a typical Partlow recorder versus 40-foot refrigerated containers with an electrical data recorder. Some customers only have basic requirements and prefer to lease elementary containers at low prices, while others have some advanced requirements, which can only be satisfied with premium containers. For example, ice cream and frozen fish suppliers have to transport their goods at extremely low temperatures (down to −35°C) to ensure the quality of the goods, and the transport of tomatoes, potatoes, or bananas requires a multitemperature system during the same voyage. In these cases, the accuracy of the temperature and the nature of the air are extremely important. Thus, refrigerated containers with an electrical recorder are desirable for the transportation of these kinds of goods. Another example is that of high-street fashion suppliers, who require garments-on-hangers containers to transport their clothes, thereby saving time, costs, and money as they can move garments in good condition directly from containers to stores. Generally, the two types of containers are downward substitutable, which means that the requests for elementary containers can be satisfied by premium ones while the requests for premium containers cannot be fulfilled by elementary ones due to the specific functions that elementary containers cannot provide.

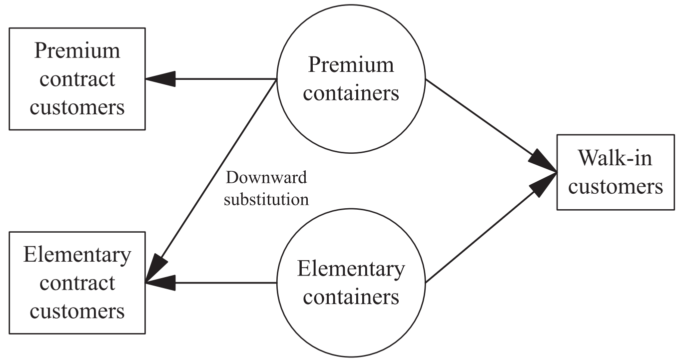
In addition, the majority of shipping companies sign contracts with container leasing companies for a type of container at a specific price. For a container leasing firm, these companies are contract customers. In this paper, we refer to the contract customers who prefer elementary containers as *elementary contract customers (ECCs)* and the contract customers who need premium containers as *premium contract customers (PCCs)*. When contract customers arrive, the leasing firm will get a fixed rental rate if it can provide containers for them and otherwise has to pay a penalty cost for breach of contract. The leasing firm also deals with walk-in customers (WICs) with temporal demands for containers. This type of customer may have no specific requirement for containers, and their needs can be fulfilled by either type of container. When they arrive, the leasing firm can either provide containers at a set price or refuse them with no penalty.

As shown in Figure 1, in this paper, we consider a container leasing firm to be one which owns elementary and premium containers to serve ECCs, PCCs, and WICs. There are several classes of ECCs and PCCs, which are differentiated by their prenegotiated prices. The customers who pay higher prices can enjoy better service, such as more available return depots and a Damage Protection Plan (DPP) with more repair clauses. Suppose that customers arrive at the leasing firm following the Poisson processes and their rental durations follow mutually independent exponential distributions. Each customer requires a single-unit of container upon arrival. Note that "a single-unit of container" here does not necessarily represent a TEU; it may represent 100 TEUs or 500 TEUs. When a customer arrives, the leasing firm first needs to decide whether it accepts this customer and if yes, which type of container to provide.

If the firm allocates the idle containers to customers whenever they arrive, it can immediately gain rewards, but it will lose the chance to serve potential high-value customers and obtain higher future revenue. The firm needs to balance immediate and future revenues. The firm also needs to consider substitution decisions for ECCs when it does not have enough elementary containers. If the firm substitutes, it can ensure good service for ECCs, but its service level for PCCs may be affected; otherwise, it can serve more future PCCs but risks having many unleased premium containers. When a WIC arrives, the firm has three choices: (1) provide an elementary container; (2) provide a premium one; or (3) reject the request.

This problem has not been addressed well in the relevant literature. The existing literature on container leasing focusses on lease duration (Dong and Song 2012), flexible contracts (Liu et al. 2013), and leasing price (Zheng et al. 2016). Jiao et al. (2016) investigated the dynamic pricing problem for leasing firms considering one type of container. However, it is more practical to address the problem of how to manage a firm's capacity when there are different types of substitutable containers and customers with different preferences. Recently, we visited a container leasing company in Tianjin, China. The company has several types of substitutable containers. Since 2021, the demand for refrigerated containers has decreased significantly, but there is high demand for general containers. Possessing limited capacity, the managers told us that they do not have a sensible method to guide them in making appropriate rationing decisions. They have generally adopted a myopic strategy because of its

Figure 1. Operation Pattern of the Leasing Firm



easy implementation. After we told them about the results we obtained in this study, they were very impressed and indicated that they would apply these results in their practice.

We formulated this problem as a dynamic program and showed that the value function is submodular, subconcave, and concave. We then characterised the structure of the optimal policy and showed that the optimal policy has a simple threshold structure (the thresholds depend on the state). We gave conditions under which the optimal policy could be further simplified (the thresholds are independent of the state). Based on these results, we investigated several extensions and obtained the corresponding managerial insights. Numerical experiments were conducted to evaluate the benefit of the optimal policy and the impacts of some parameters on the optimal policy. Some managerial insights were derived from the analysis.

This work is also relevant for general rental businesses (e.g., equipment rental). We can see the old equipment as elementary equipment with low rental rates and the new equipment as premium equipment with high rental rates. Our model is also applicable in this case.

Compared with the existing studies, this work makes the following contributions and findings.

1. We prove that the optimal policy for the capacity allocation problem with substitutable containers and three groups of customers has a simple threshold structure, which is easy to implement.

2. We show that the firm should admit customers if it has enough idle capacity. For ECCs and WICs, which type of container to provide depends on the numbers of both idle elementary containers and premium ones.

3. We give definitions for *preferred class* and *dominated containers* and use them to deduce the conditions under which the optimal policy can be further simplified.

4. Based on real data, we conduct numerical experiments to explore the value of the optimal policy. We find that substitution brings distinct revenue improvement (3%–8%) for the firm (especially when the difference of the rental prices of the two types of containers is small), and the optimal policy outperforms the myopic policy (3%–12% more revenue) and the stochastic rationing policy (3%–8% more revenue).

The remainder of this paper is arranged as follows. In Section 2, we review the related literature to identify the research gaps. In Section 3, we state our model and characterise the optimal rationing policy. In Section 4, the results of several numerical experiments are provided, and some insights are derived. In Section 5, we discuss several extensions. In Section 6, we conclude our study and discuss future topics. All proofs and additional numerical cases are given in the electronic companion (EC).

2. Literature Review

In this paper, we investigate the capacity allocation problem for a container leasing firm with two types of downward substitutable containers and three groups of customers. There are several streams of literature related to this study.

2.1. Admission Control for Rentals and Queuing Systems

An extensive body of literature concerning admission control for rentals and queuing systems exists. Miller (1969) first studied the admission control problem with multiple customer classes. Örmeci et al. (2001) and Savin et al. (2005) examined the systems in which the service durations of two customer classes followed different distributions. Altman et al. (2001) and Örmeci and Burnetas (2004) considered the batch arrival of demands. There are also several works that study pricing in rentals and queuing systems (Gans and Savin 2007, Ahmadi and Shavandi 2015, Jiao et al. 2016, Zhuang et al. 2017). The primary difference between these studies and ours is that they assumed homogeneity of servers or rental items, while we consider two types of substitutable containers.

2.2. Container Leasing

There are some papers on container leasing. Dong and Song (2012) studied the lease duration optimisation problem. Liu et al. (2013) explored leasing contract optimisation for shipping companies. Zheng et al. (2016) studied container rental price measuring with foldable and unfoldable containers using mixed-integer programming. Jiao et al. (2016) investigated dynamic pricing for a container leasing system with customers' hire time and hire quantity preferences by using nonlinear programming. Our work is different from these previous works in that we use a dynamic program to address the problem of capacity rationing with multiple types of customers who arrive dynamically and stochastically.

2.3. Revenue Management with Upgrading and Upselling

The third stream of literature is on revenue management with upgrading and upselling. In the airline industry, upgrades indicate that customers may be offered higher-class seats if the seats they reserved are unavailable (McGill and Van Ryzin 1999). Gallego and Stefanescu (2009) investigated several upselling models and showed the advantages of upgrades in revenue management. Steinhardt and Gönsch (2012) proposed two dynamic program decomposition approaches to solve the revenue management problems with upgrades. They showed that their approaches are tractable for practical problem scales and have better performance than commonly used methods. Gönsch and Steinhardt (2015) developed two reformulations for a dynamic programming model to address an airline network management issue with upgrades. Benigno et al. (2012) investigated a truck rental problem, in which multiple types of trucks were differentiated by their capacities. They designed approximate methods to solve the issue. These works focused on designing heuristic algorithms to accelerate the computing processes. Cui et al. (2017) explored an issue related to pricing for conditional upgrades and constructed a fluid model to obtain the optimal upgrade price. They also investigated the value of offering upgrades. Yilmaz et al. (2017) investigated the performance of standby upgrades in hotel management and identified the conditions under which standby upgrades are profitable for hotels. Çakanyildirim et al. (2020) considered the problem of dynamic pricing for upgrades between the reserve and register time and characterised the optimal price and number of upgrades. We consider the admission and substitution decisions for ECCs simultaneously and explore the structure of optimal policy.

2.4. Substitution in Demand Management

The fourth stream related to this study is on substitution in demand management. Balakrishnan and Geunes (2000) investigated a production planning problem with multiple types of substitutable products by using integer programming. Hsu et al. (2005) explored multiproduct lot size problems with and without conversion cost, and developed a heuristic method to solve them. Lang (2009) provided a detailed review of this topic. The aim of these works was to propose algorithms to get the solution. Chen et al. (2010) considered a seat capacity control problem with two types of flights and three types of customers in which flights are substitutable for type-three customers. Sayah and Irnich (2019) further investigated a case with batch demands. These two papers explored the structure of the optimal policy. Our study differs from them in the following respects: (a) Our model has an extra return process. We take the return rate of containers into consideration when discussing rationing decisions for customers; (b) The optimal policies in Chen et al. (2010) and Sayah and Irnich (2019) were characterised by several switching curves, while our optimal policy is characterised by certain thresholds. We further explore the conditions under which the thresholds are independent of the system state; (c) We consider not only contract customers but also walk-in ones, which complicates the analyses, especially when the rental prices for walk-in customers are endogenous; (d) They explored a finite-period problem and modelled it as a time-based MDP, in which the stage changes as time moves. We consider a continuous-time problem and model it as an event-based MDP, in which the stage changes as events occur.

Based on the substitutability of some products, many firms have designed flexible or opaque products to manage their demands, and some experts have studied opaque product management. Gallego and Phillips (2004) defined the flexible product as a set of substitutable products. The seller can allocate any type of product in the set to those who buy the flexible product. They showed that flexible products can enhance the seller's profit. Fay and Xie (2008) explored the benefit of selling opaque products to customers in terms of matching demand and capacity. Zhang et al. (2015) studied probabilistic selling in a quality-differentiated market. The aforementioned works considered only static models. Xiao and Chen (2014) studied the issue of opaque selling in a dynamic environment. Huang and Yin (2020) investigated an opaque selling case with high-value and low-value products. The seller dynamically decided whether to sell opaque products, the price for opaque products and the probability of offering high-value products. Different from these works, our aim is to characterise the structure of the dynamic rationing policy according to the currently available capacity, which can be renewed through the return of rented containers.

2.5. Inventory Allocation

This study is also related to the literature on inventory allocation. Topkis (1968) first considered a periodic review system with diverse demand classes. Nahmias and Demmy (1981) extended his model by considering both continuous and periodic review systems. Ha (2000) and De Véricourt et al. (2001) concentrated on the inventory control for make-to-stock production systems, while Liu et al. (2015) considered an inventory allocation problem with stochastic demand processes. In recent years, the problem of inventory allocation has been studied in various fields. Papier (2016) studied an optimal control problem for manufacturing lines considering electricity cost, Sarhangian et al. (2017) investigated the threshold rationing policy for a red blood cell inventory system, and Chen and Thomas (2018) explored inventory allocation in a system with service-level agreements. All of these works studied inventory rationing in production or inventory systems, while we concentrate on a rental case involving a physical return process.

2.6. Assortment Planning

The last stream of relevant literature is on assortment planning. Mahajan and Van Ryzin (2001b) considered a single-period stochastic inventory model with heterogeneous customers, who had substitution behaviour among products. Mahajan and Van Ryzin (2001a) explored inventory competition among companies that provide substitutable products, while Rao et al. (2004) investigated a single-period resource planning problem with multiple downward substitutable products, random demands and set-up costs. Our study differs from the above works because we consider a dynamic resource allocation problem and develop the optimal policy to guide the manager in how to meet different types of customers with two types of containers.

3. Model and Analysis

3.1. Mathematical Model

In this subsection, we present the mathematical model. Consider a leasing firm that has c_1 units of elementary containers and c_2 units of premium containers. The firm faces two types of customers: contract customers and walk-in customers (WICs). Each customer requests a unit of container when arriving at the firm.[1] There are n classes of contract customers who sign contracts with the firm to lease elementary containers at different rates. These kinds of contract customers are called elementary contract customers (ECCs), and the rental rate of class-i ($i = 1, 2, \ldots, n$) ECCs is r_{1i}. There are m classes of contract customers who will lease premium containers and are called premium contract customers (PCCs). The rental rate of class-j ($j = 1, 2, \ldots, m$) PCCs is r_{2j}. Without loss of generality, we assume that $r_{11} \le r_{12} \le \ldots \le r_{1n} \le r_{21} \le r_{22} \le \ldots \le r_{2m}$. When a class-$i$ ECC arrives, the firm has three choices: (1) provide an elementary container, (2) provide a premium container, or (3) refuse the request. If the firm provides an elementary container, it will get a payment r_{1i}. The firm could also choose to upsell a premium container and get a payment βr_{1i}, where β is a constant satisfying $\beta \ge 1$. If the rental demand is refused, the firm has to pay a penalty of π_1 for breach of contract. When a class-j PCC arrives, the firm decides whether to accept it. If the firm accepts the demand, it will provide the customer with a premium container and acquire a payment r_{2j}. Otherwise, the firm has to pay a penalty of π_2. The rental rates of an elementary container and a premium one for WICs are p_1 and p_2 ($r_{1n} \le p_1 < p_2, r_{2m} \le p_2$), respectively. If the firm refuses a WIC, it will not be penalised.

Suppose that class-i ECCs, class-j PCCs, and WICs arrive at the firm following independent Poisson processes with rates λ_{1i} ($1 \le i \le n$), λ_{2j} ($1 \le j \le m$) and λ_3, respectively. The rental durations of all customers are uncertain and follow a mutually independent exponential distribution with a mean of μ^{-1}. We use (x, y) to represent the system state, which means that the firm has rented out x units of elementary containers and y units of premium containers. The system space can then be defined as $C = \{(x, y) \mid 0 \le x \le c_1, 0 \le y \le c_2, x \in N, y \in N\}$ where $N = \{0, 1, 2, \ldots\}$.

In container leasing firms, the holding costs for containers consist of storage costs and maintenance costs. A container leasing firm needs depots/yards for the storage of idle containers, and it pays certain costs to keep the depot operations, including labour costs, depot-leasing costs, and managerial costs. In addition, daily repairs should be undertaken to maintain the quality of containers. In general, leasing companies are responsible for the maintenance of idle containers, and idle premium containers have more holding costs than idle elementary ones since leasing companies have to spend more on the daily maintenance of premium containers. From Drewry (2019), the average rental price per diem for a 20-ft container is about \$0.8. The average holding cost per diem for a 20-ft container is about \$0.4. Reports from Triton and Textainer show that holding costs of containers are significant.[2] In addition, it has been common in the previous literature to incorporate the holding costs of empty containers into the company's total costs (Cheung and Chen 1998, Li et al. 2007, Song and Dong 2012). Therefore, we consider the holding cost of containers in this paper. To simplify the analysis, we assume that the unit holding costs (including storage and maintenance costs) of elementary and premium containers are h_1 and h_2 ($h_1 < h_2$), respectively.

We consider the revenue-maximisation problem along an infinite time horizon with a time discount factor α, which means that a revenue v at time t has the present value $ve^{-\alpha t}$. Given the assumptions of Poisson arrivals of customers and exponential distribution of rental durations, the capacity rationing problem of the system with two types of containers can be formulated as a continuous-time Markov Decision Process (MDP). To simplify the analysis, we use the approach introduced by Lippman (1975) and Serfozo (1979) to transform the model to a discrete-time MDP. The aggregate event rate of the discrete-time MDP is $\alpha + \delta$, where $\delta = \sum_{i=1}^{n} \lambda_{1i} + \sum_{j=1}^{m} \lambda_{2j} + \lambda_3 + \mu(c_1 + c_2)$. Without loss of generality, we assume that $\alpha + \delta = 1$. At any transition epoch, the probabilities of the events are given in Table 1. Let $v(x, y)$ be the expected total discounted revenue[3] from now on with current system state (x, y). Then the problem can be formulated as the dynamic program with Bellman equation as

Table 1. The Event and Probability at a Transition Epoch with System State (x, y)

Event	Arrival of a class-i ECC	Arrival of a class-j PCC	Arrival of a WIC	Return of an elementary container	Return of a premium container	Fictitious return process
Probability	λ_{1i}	λ_{2j}	λ_3	μx	μy	$\mu(c_1 + c_2 - x - y)$

$v(x,y) = T[v(x,y)]$, where

$$T[v(x,y)] = \sum_{i=1}^{n}\lambda_{1i}T_{1i}[v(x,y)] + \sum_{j=1}^{m}\lambda_{2j}T_{2j}[v(x,y)] + \lambda_3 T_3[v(x,y)] + T_4[v(x,y)]$$

$$T_{1i}[v(x,y)] = \begin{cases} \max\{v(x+1,y)+r_{1i}, v(x,y+1) \\ \quad +\beta r_{1i}, v(x,y)-\pi_1\}, & x<c_1, y<c_2 \\ \max\{v(x+1,y)+r_{1i}, v(x,y)-\pi_1\}, & x<c_1, y=c_2 \\ \max\{v(x,y+1)+\beta r_{1i}, v(x,y)-\pi_1\}, & x=c_1, y<c_2 \\ v(x,y)-\pi_1, & x=c_1, y=c_2 \end{cases}$$

$$T_{2j}[v(x,y)] = \begin{cases} \max\{v(x,y+1)+r_{2j}, v(x,y)-\pi_2\}, & y<c_2 \\ v(x,y)-\pi_2, & y=c_2 \end{cases}$$

$$T_3[v(x,y)] = \begin{cases} \max\{v(x+1,y)+p_1, v(x,y+1) \\ \quad +p_2, v(x,y)\}, & x<c_1, y<c_2 \\ \max\{v(x+1,y)+p_1, v(x,y)\}, & x<c_1, y=c_2 \\ \max\{v(x,y+1)+p_2, v(x,y)\}, & x=c_1, y<c_2 \\ v(x,y), & x=c_1, y=c_2 \end{cases}$$

$$T_4[v(x,y)] = \mu x v(x-1,y) + \mu y v(x,y-1) + \mu(c_1+c_2-x-y)v(x,y) - h_1(c_1-x) - h_2(c_2-y).$$

The operator T_{1i} represents the rationing decision for class-i ECCs. The operator T_{2j} represents the rationing decision for class-j PCCs. The operator T_3 represents the rationing decision for WICs. The operator T_4 is related to the return process, fictitious return process and holding costs of the idle containers. Since the state space, action space, and the reward in one period are finite, there exists an optimal rationing policy for the system (Puterman 2014). Miller (1969) considered a case in which a queueing system with identical servers dealt with several groups of customers differentiated by their service rewards. We extend the model in Miller (1969) to a case with two kinds of substitutable items, two groups of contract customers (ECCs and PCCs) and WICs.

3.2. The Characteristics of the Optimal Rationing Policy

In this subsection, we analyse the property of the value function $v(x,y)$ and characterise the optimal policy for the system.

Define the first-order and second-order differences as $\Delta_x v(x,y) = v(x+1,y) - v(x,y)$, $\Delta_y v(x,y) = v(x,y+1) - v(x,y)$, $\Delta_{x,x} v(x,y) = v(x+2,y) - 2v(x+1,y) + v(x,y)$, $\Delta_{y,y} v(x,y) = v(x,y+2) - 2v(x,y+1) + v(x,y)$, and $\Delta_{x,y} v(x,y) = v(x+1,y+1) - v(x+1,y) - v(x,y+1) + v(x,y)$. The immediate explanation for $\Delta_x v(x,y)$ ($\Delta_y v(x,y)$) is the system's revenue change from leasing out one more elementary (premium) container at state (x,y). Then $-\Delta_x v(x,y)$ and $-\Delta_y v(x,y)$ are the opportunity costs of leasing an elementary container and a premium one, respectively. The opportunity cost means the potential loss incurred by leasing a container because the firm has less opportunity to serve customers in the future. The difference $\Delta_{x,x} v(x,y)$ reflects how the opportunity cost of leasing out an elementary container changes with respect to x; $\Delta_{y,y} v(x,y)$ reflects how the opportunity cost of leasing out a premium container changes with respect to y; and $\Delta_{x,y} v(x,y)$ reflects how the opportunity cost of leasing out an elementary (premium) container changes with respect to y (x). We then show how these differences help us analyse the firm's decision.

When a customer arrives, the firm needs to decide whether to accept the request for a container. If the firm accepts the customer, it will get immediate payment. If not, it will pay a penalty if the customer is a contract customer. The admission decision depends on the difference between $-\Delta_x v(x,y)$ or $-\Delta_y v(x,y)$ and a certain constant. We give a detailed analysis for operator T_{1i}, which reflects the admission policy for class-i ECCs, to demonstrate how the firm makes a decision (other operators can be analysed similarly). First, the firm judges whether it is profitable to provide an elementary container when a class-i ECC arrives. That is, the firm examines the difference between $v(x+1,y) + r_{1i}$ and $v(x,y) - \pi_1$, which is equivalent to a comparison of $r_{1i} + \pi_1$ with $-\Delta_x v(x,y)$. If $r_{1i} + \pi_1 \geq -\Delta_x v(x,y)$, it is profitable to provide an elementary container rather than to reject the customer. Second, the firm needs to judge whether it is profitable to provide a premium container. That is, the firm has to compare $\beta r_{1i} + \pi_1$ with $-\Delta_y v(x,y)$. If $\beta r_{1i} + \pi_1 \geq -\Delta_y v(x,y)$, then it is profitable to provide a premium container rather than to reject the customer. Finally, if it is beneficial for the firm to provide any type of container for the customer, the firm has to determine which type to provide. Thus, the firm examines the difference between $v(x+1,y) + r_{1i}$ and $v(x,y+1) + \beta r_{1i}$, which is equivalent to a comparison of $v(x+1,y) - v(x,y+1)$ and $(\beta - 1)r_{1i}$. It should provide an elementary container if $v(x+1,y) - v(x,y+1) \geq (\beta - 1)r_{1i}$ and provide a premium one otherwise.

From this analysis, we can see that if we figure out how $-\Delta_x v(x,y)$, $-\Delta_y v(x,y)$, and $v(x+1,y) - v(x,y+1)$ change with respect to (x,y) by proving some properties of $v(x,y)$, it will be easy to characterise the optimal rationing policy. We will next show submodularity, subconcavity, and concavity of $v(x,y)$, which are defined as follows:

1. submodularity: $\Delta_{x,y} v(x,y) \leq 0$;
2. subconcavity: $\Delta_x v(x+1,y) \leq \Delta_x v(x,y+1)$, $\Delta_y v(x,y+1) \leq \Delta_y v(x+1,y)$;
3. concavity: $\Delta_{x,x} v(x,y) \leq 0$, $\Delta_{y,y} v(x,y) \leq 0$.

If we can show that $T[v(x,y)]$ is also submodular, subconcave, and concave under the assumption that $v(x,y)$ is submodular, subconcave, and concave, $v(x,y)$ itself has these properties according to Puterman (2014).

Lemma 1. *$T[v(x,y)]$is submodular, subconcave, and concave with respect to (x, y) if v(x, y) has these properties. Then v(x, y) itself is submodular, subconcave, and concave.*

The submodularity of $v(x, y)$ indicates that the opportunity cost of leasing out an elementary (premium) container increases in y (x).[4] The subconcavity means that the opportunity cost of leasing out an elementary (premium) container increases faster in x (y) than in y (x). The concavity signifies that the opportunity cost of leasing out an elementary (premium) container increases in x (y). We subsequently demonstrate how these properties take effect.

For each customer, the rental rate of a container and the penalty for breach of contract (if it exists) are fixed and independent of (x,y). Consequently, the firm is inclined to keep its capacity for potential high-value customers as x or y becomes larger since the opportunity cost of leasing out a container is increasing.

If an arriving class-j PCC is accepted at state (x_0,y_0), then we have $-\Delta_y v(x_0,y_0) \le \pi_2 + r_{2j}$. From Lemma 1, we have $-\Delta_y v(x_1,y_1) \le \pi_2 + r_{2j}$ for any (x_1,y_1) that $x_1 \le x_0$ and $y_1 \le y_0$. Thus, it is also optimal to accept class-j PCCs at state (x_1,y_1). Until $-\Delta_y v(x,y)$ exceeds $\pi_2 + r_{2j}$, the arriving PCC is rejected. Then the PCCs with lower rental rates should also be refused as well.

Different from PCCs, when ECCs or WICs arrive, the firm needs to decide not only whether to accept them but also which type of container to provide. The analysis of the first problem is similar to that of the preceding discussion. We now discuss the second problem in detail.

Let $g(x,y) = v(x+1,y) - v(x,y+1)$. This is the difference between the revenue of leasing one more elementary container and that of leasing one more premium container at state (x, y).

Lemma 2. *$g(x, y)$ is increasing in y and decreasing in x.*

Lemma 2 means that the difference between the revenue of leasing one more elementary container and that of leasing one more premium container decreases in x and increases in y. This property can help the firm decide which kind of container to provide when an ECC or WIC arrives. When a class-i ECC arrives and the firm determines which kind of container to provide, it needs to compare $v(x+1,y) - v(x,y+1)$ and $(\beta-1)r_{1i}$. If the firm should provide an elementary container at state (x_0,y_0), then we have $g(x_0,y_0) \ge (\beta-1)r_{1i}$. From Lemma 2, we have $g(x_1,y_1) \ge (\beta-1)r_{1i}$ for any (x_1,y_1) that satisfies $x_1 \le x_0$ and $y_1 \ge y_0$. As a result, it is also optimal to provide an elementary container at state (x_1,y_1). Until $g(x, y)$ becomes less than $(\beta-1)r_{1i}$, it is more profitable to provide premium containers to class-i ECCs. When a WIC arrives, the firm should compare $v(x+1,y) - v(x,y+1)$ and $p_2 - p_1$. If the firm provides an elementary container at state (x_2,y_2), then we have $g(x_2,y_2) \ge p_2 - p_1$. Similarly, for any (x_3,y_3) that satisfies $x_3 \le x_2$ and $y_3 \ge y_2$, the leasing firm still chooses an elementary container. It is more profitable to provide premium containers for WICs until $g(x,y) < p_2 - p_1$. From the above analysis, the optimal capacity rationing policy can be characterised in the following Theorem.

Theorem 1. *The optimal capacity rationing policy can be characterised as follows.*

(a) *For class-i ECCs, there exist rationing thresholds $K_{1i}(y)$ and $K_{2i}(x)$ and a substitution threshold $K_{3i}(y)$ such that*

Conditions	$x \ge K_{1i}(y)$ and $y \ge K_{2i}(x)$	$x < K_{1i}(y)$ and $y \ge K_{2i}(x)$	$x < \min\{K_{1i}(y), K_{3i}(y)\}$ and $y < K_{2i}(x)$	$x \ge K_{1i}(y)$ and $y < K_{2i}(x)$	$K_{3i}(y) \le x < K_{1i}(y)$ and $y < K_{2i}(x)$
Policy	Reject	Provide an elementary container		Provide a premium container	

where

$$K_{1i}(y) = \begin{cases} c_1, & -\Delta_x v(c_1-1,y) \le \pi_1 + r_{1i} \\ 0, & -\Delta_x v(0,y) > \pi_1 + r_{1i} \\ k, & -\Delta_x v(k,y) > \pi_1 + r_{1i} \ge -\Delta_x v(k-1,y), \end{cases}$$

$$K_{2i}(x) = \begin{cases} c_2, & -\Delta_y v(x,c_2-1) \le \pi_1 + \beta r_{1i} \\ 0, & -\Delta_y v(x,0) > \pi_1 + \beta r_{1i} \\ k, & -\Delta_y v(x,k) > \pi_1 + \beta r_{1i} \ge -\Delta_y v(x,k-1), \end{cases}$$

$$K_{3i}(y) = \begin{cases} c_1, & v(c_1,y) - v(c_1-1,y+1) \ge (\beta-1)r_{1i} \\ 0, & v(1,y) - v(0,y+1) < (\beta-1)r_{1i} \\ k, & v(k+1,y) - v(k,y+1) < (\beta-1)r_{1i} \\ & \le v(k,y) - v(k-1,y+1); \end{cases}$$

(b) *For class-j PCCs, there exists a rationing threshold $L_j(x)$. A class-j PCC should be provided with a premium container when $y < L_j(x)$ and rejected otherwise, where*

$$L_j(x) = \begin{cases} c_2, & -\Delta_y v(x,c_2-1) \le \pi_2 + r_{2j} \\ 0, & -\Delta_y v(x,0) > \pi_2 + r_{2j} \\ k, & -\Delta_y v(x,k) > \pi_2 + r_{2j} \ge -\Delta_y v(x,k-1); \end{cases}$$

(c) *For WICs, there exist rationing thresholds $R_1(y)$ and $R_2(x)$ and a priority threshold $R_3(y)$ such that*

Conditions	$x \ge R_1(y)$ and $y \ge R_2(x)$	$x < R_1(y)$ and $y \ge R_2(x)$	$x < \min\{R_1(y), R_3(y)\}$ and $y < R_2(x)$	$x \ge R_1(y)$ and $y < R_2(x)$	$R_3(y) \le x < R_1(y)$ and $y < R_2(x)$
Policy	Reject	Provide an elementary container		Provide a premium container	

where

$$R_1(y)=\begin{cases} c_1, & -\Delta_x v(c_1-1,y)\le p_1\\ 0, & -\Delta_x v(0,y)>p_1\\ k, & -\Delta_x v(k,y)>p_1\ge -\Delta_x v(k-1,y),\end{cases}$$

$$R_2(x)=\begin{cases} c_2, & -\Delta_y v(x,c_2-1)\le p_2\\ 0, & -\Delta_y v(x,0)>p_2\\ k, & -\Delta_y v(x,k)>p_2\ge -\Delta_y v(x,k-1),\end{cases}$$

$$R_3(y)=\begin{cases} c_1, & v(c_1,y)-v(c_1-1,y+1)\ge p_2-p_1\\ 0, & v(1,y)-v(0,y+1)<p_2-p_1\\ k, & v(k+1,y)-v(k,y+1)<p_2-p_1\le v(k,y)\\ & \quad -v(k-1,y+1).\end{cases}$$

Theorem 1 characterises the structure of the optimal policy. It shows that the optimal policy still has a simple threshold structure although the firm leases two types of containers, and ECCs and WICs can be satisfied by either type. In addition, the thresholds depend on the system state. The optimal policies for ECCs and WICs are characterised by tables in which each decision is jointly determined by several conditions. The optimal policy for PCCs depends on only one condition, the relation between the number of on-hire premium containers and the corresponding threshold.

The policy for ECCs and WICs is relatively complicated since the firm needs to decide not only whether to accept them but also the type of container to provide. When making rationing decisions, the firm should consider at the same time the numbers of on-hire elementary containers and premium containers. The thresholds that decide the optimal policy are also related to both the number of on-hire elementary containers and that of premium ones.

Theorem 1(a) and (c) imply that when an ECC (WIC) arrives, the optimal allocation policy depends on the system state. It is optimal to (a) reject the customer if many elementary and premium containers are rented out, (b) provide an elementary container if there are many idle elementary containers but few available premium containers or (c) provide a premium container if many elementary containers are leased out but there are few on-hire premium containers. There exist several boundaries which are concerned with system parameters and states to separate each decision. The simplicity of the tables in Theorem 1(a) and (c) allow the firm to easily perform the policy according to the system states once the thresholds have been determined. Theorem 1(b) means that when a PCC arrives, it is optimal to provide a premium container if the number of rented premium containers is small and to reject the customer otherwise. To conclude, under the optimal allocation policy, the firm always chooses to keep capacity when there are few containers and to ration capacity when there are enough idle containers. When two types of containers are acceptable for customers, the type of container to be provided depends on the number of the two types of containers that are available.

The thresholds $K_{1i}(y)$, $K_{2i}(x)$, $L_j(x)$, $R_1(y)$ and $R_2(x)$ are rationing thresholds, which are used to determine whether the firm can or cannot benefit from providing a container. $K_{3i}(y)$ is the substitution threshold which indicates whether the firm can benefit from replacing an elementary container with a premium one (upselling for ECCs). $R_3(y)$ is the priority threshold that can be used to determine which type of container has priority in being offered to WICs.

Miller (1969) studied the issue of admission control for a queueing system with identical servers and multiple customer classes, in which the decision maker just needed to decide whether to accept arriving customers. The author showed that the optimal policy has a threshold structure. We generalise these results to our case with two types of downward substitutable items and three groups of customers. Theorem 1 shows that the system still has a simple threshold type policy.

The optimal policy in this work differs in two ways from those of previous studies, in which one type of resource was considered. First, we consider two types of containers. In addition to rationing thresholds, there are substitution and priority thresholds which also affect the firm's optimal allocation policy. Second, the optimal rationing policy depends on the inventory level of two types of containers. When the firm decides whether to provide elementary (premium) containers for customers, it should consider not only the number of idle elementary (premium) containers but also that of idle premium (elementary) ones.

To illustrate Theorem 1, we give a numerical case in Section EC.1. From this experiment, we have an interesting finding that it may be optimal to provide a premium container when an ECC arrives even if there are still some idle elementary containers. This phenomenon is inconsistent with our intuition but can be explained from two aspects.

(i) Idle premium containers have higher holding costs than idle elementary ones. Satisfying ECCs with premium containers rather than elementary ones will decrease the holding cost but not reduce service quality for PCCs and WICs as long as the firm has enough idle premium containers.

(ii) When the firm has many premium containers but few elementary ones on hand, it has enough premium containers for future PCCs. In this situation, the firm should lease premium containers as early as possible. The reason is that leasing premium containers earlier can bring more revenue to the firm. Also, the leased premium containers will be returned earlier.

Corollary 1. *(a) $K_{1i}(y)$ and $R_1(y)$ are nonincreasing in y; $K_{2i}(x)$, $L_j(x)$ and $R_2(x)$ are nonincreasing in x. (b) $K_{3i}(y)$ and $R_3(y)$ are nondecreasing in y.*

Corollary 1(a) implies that the firm is reluctant to provide elementary (premium) containers and would rather reject customers as more premium (elementary) containers are rented out. From the submodularity of the value function, the opportunity cost of leasing an elementary (premium) container increases with the number of on-hire premium (elementary) containers. Therefore, the leasing firm is inclined to keep elementary (premium) containers as there are fewer premium (elementary) containers on hand. Corollary 1(b) means that the firm is willing to provide elementary containers rather than premium ones for ECCs and WICs as there are more on-hire premium containers. From the subconcavity of the value function, as more premium containers are rented out, the opportunity cost of leasing a premium container increases faster than that of an elementary one. Compared with premium containers, the firm is more willing to provide elementary ones. To conclude, under the optimal policy, the firm is unwilling to provide one type of container for customers as the inventory level for the other type of container becomes less. In addition, the firm is willing to provide elementary (premium) containers for customers instead of premium (elementary) ones as more premium (elementary) containers are rented out.

3.3. Simplifying the Optimal Policy

Theorem 1 shows that the optimal policy has a simple structure. However, the thresholds depend on the current system state (x,y). In this subsection, we give conditions under which the thresholds are independent of the system state.

Definition 1. Class-i ECCs are preferred for elementary containers if $K_{1i}(y)=c_1$ for any $0\le y\le c_2$ and for premium containers if $K_{2i}(x)=c_2$ for any $0\le x\le c_1$; class-j PCCs are preferred if $L_j(x)=c_2$ for any $0\le x\le c_1$; WICs are preferred for elementary containers if $R_1(y)=c_1$ for any $0\le y\le c_2$ and for premium containers if $R_2(x)=c_2$ for any $0\le x\le c_1$.

In Gans and Savin (2007), Örmeci and Burnetas (2004), and Savin et al. (2005), the firm had only two options (accept or refuse) when customers arrived. A preferred customer class was always accepted when idle capacity existed. In our work, the firm has three choices (refuse or accept with an elementary container or accept with a premium one) when an ECC or WIC arrives. Then we generalise the concept of *preferred class* in Gans and Savin (2007), Örmeci and Burnetas (2004) and Savin et al. (2005) to present two types of preferences for ECCs and WICs (preferred for elementary containers or premium containers).

From Theorem 1, the conditions for a certain customer class to be preferred can be characterised by determining the upper bounds of $-\Delta_x v(x,y)$ and $-\Delta_y v(x,y)$. To this end, we define

$$\rho_1(t)=t-\frac{\sum_{i=1}^{n}\lambda_{1i}\max(t,r_{1i}+\pi_1)+\lambda_3\max(t,p_1)-h_1}{\sum_{i=1}^{n}\lambda_{1i}+\lambda_3+\mu+\alpha},$$

$$\rho_2(t)=t-\frac{\sum_{i=1}^{n}\lambda_{1i}\max(t,\beta r_{1i}+\pi_1)+\sum_{j=1}^{m}\lambda_{2j}\max(t,r_{2j}+\pi_2)+\lambda_3\max(t,p_2)-h_2}{\sum_{i=1}^{n}\lambda_{1i}+\sum_{j=1}^{m}\lambda_{2j}+\lambda_3+\mu+\alpha}.$$

Lemma 3 (Simplifying Rationing Thresholds).

(a) *$\rho_1(t)$ and $\rho_2(t)$ are increasing in t and the equations $\rho_1(t)=0$ and $\rho_2(t)=0$ have unique solutions b_1^* and b_2^*, respectively. Moreover, $b_1^*\ge\max_{(x,y)\in(C\setminus C_1)}-\Delta_x v(x,y)$ and $b_2^*\ge\max_{(x,y)\in(C\setminus C_2)}-\Delta_y v(x,y)$, where $C_1=\{(c_1,y)\mid 0\le y\le c_2\}$ and $C_2=\{(x,c_2)\mid 0\le x\le c_1\}$.*

(b) *The class-i ECCs are preferred for elementary containers if $r_{1i}\ge b_1^*-\pi_1$ and for premium containers if $r_{1i}\ge(b_2^*-\pi_1)/\beta$; the class-j PCCs are preferred if $r_{2j}\ge b_2^*-\pi_2$; WICs are preferred for elementary containers if $p_1\ge b_1^*$ and for premium containers if $p_2\ge b_2^*$.*

In Lemma 3(a), the functions $\rho_1(t)$ and $\rho_2(t)$ are defined to find the upper bound of $-\Delta_x v(x,y)$ and $-\Delta_y v(x,y)$, respectively. From the submodularity and concavity of the value function $v(x,y)$, we have $\max_{(x,y)\in(C\setminus C_1)}-\Delta_x v(x,y)=-\Delta_x v(c_1-1,c_2)$. The solution of the equation $\rho_1(t)=0$ is large enough to be an upper bound of $-\Delta_x v(c_1-1,c_2)$. Results are similar for the function $\rho_2(t)$. Lemma 3(b) implies that the rationing thresholds $K_{1i}(y)$, $K_{2i}(x)$, $L_j(x)$, $R_1(y)$ and $R_2(x)$ have simpler forms that are independent of system state (x,y). Gans and Savin (2007) gave conditions under which customers are preferred for a leasing system with one kind of rental item. We generalise their results to the case with two kinds of rental items. The expression of b_2^* in our study is more complicated than in theirs since premium containers can be provided for ECCs and WICs. Thus, b_2^* is also affected by the arrival rates of ECCs and WICs.

The solution for $\rho_1(t)=0$ is not less than the maximum opportunity cost of leasing an elementary container.[5] From the definition of $\rho_1(t)$, b_1^* satisfies

$$b_1^*\left[\sum_{i=1}^{n}\lambda_{1i}+\lambda_3+\mu+\alpha\right]=\sum_{i=1}^{n}\lambda_{1i}\max(b_1^*,r_{1i}+\pi_1)+\lambda_3\max(b_1^*,p_1)-h_1. \tag{1}$$

On the one hand, the left of (1) is the maximum opportunity cost of leasing an elementary container times the sum of the arrival rate of ECCs, the arrival rate of

WICs, the return rate of a container and the discount rate. The sum of the arrival rate of ECCs and that of WICs represents the total arrival of customers who require elementary containers. The return rate of a container and the discount rate are also involved since they are concerned with the opportunity cost of leasing an elementary container. The left of (1) can then be seen as a type of "adjusted total opportunity cost." On the other hand, $\max\{b_1^*, r_{1i}+\pi_1\}$ ($\max\{b_1^*, p_1\}$) is the maximum opportunity cost of leasing an elementary container after a class-i ECC (WIC) arrives, which indicates that the operator T_{1i} (T_3) should be performed and the value function is iterated. The right of (1) is the sum of the arrival rates of customers who require elementary containers times the corresponding opportunity cost and then minus the unit holding cost of elementary containers. We can see it as a type of "iterated total opportunity cost." Therefore, we can conclude that the maximum opportunity cost always ensures that the adjusted total opportunity cost is equal to the iterated total opportunity cost. The definition of $\rho_2(t)$ can be explained in a similar way.

Lemma 3 gives the sufficient condition for a customer class to be preferred. We can judge whether a customer class is preferred by comparing the rental price and an upper bound calculated from Lemma 3. If the immediate benefit brought by a customer class is larger than b_1^* (b_2^*), it is profitable for the firm to provide elementary (premium) containers unless there is no idle capacity. Moreover, if a customer class is preferred, then any customer classes that pay higher rental prices are also preferred. Note that $r_{1n}+\pi_1$ is the immediate benefit of providing an elementary container for a class-i ECC and p_1 is the immediate benefit of providing an elementary container for a WIC. Among all customers who accept elementary containers, class-n ECCs (WICs) are the highest-valued customers for the elementary containers if $r_{1i}+\pi_1 > p_1$ ($r_{1n}+\pi_1 \le p_1$). It is obvious that $\max\{r_{1n}+\pi_1, p_1\} > b_1^*$ since $\rho_1(\max\{r_{1n}+\pi_1, p_1\}) > 0$, which indicates that the highest-valued customers are always preferred. The firm should always satisfy the demand from the highest-valued customers unless there are no idle containers. We can also find that $b_1^* < 0$ when h_1 is very large. In this situation, all customers are preferred for elementary containers due to the unaffordable holding cost. Similarly, all customers are preferred for premium containers when h_2 is very large. In these conditions, the firm is glad to rent all of the idle containers out. The optimal policy is a myopic-type policy, in which the firm provides elementary (premium) containers for ECCs (PCCs) and WICs until there is no idle capacity. In addition, when a customer class pays high rental rates for both elementary containers and premium ones, it may be *preferred for elementary containers* and *preferred for premium containers* at the same time. In this case, which type of container should be provided for them depends on the difference between the rental price of elementary containers and that of premium ones. We will discuss this topic in the next theorem.

Definition 2. Class-i ECCs are elementary container dominated if $K_{3i}(y)=c_1$ for any $0 \le y \le c_2-1$ and are premium container dominated if $K_{3i}(y)=0$ for any $0 \le y \le c_2-1$; WICs are elementary container dominated if $R_3(y)=c_1$ for any $0 \le y \le c_2-1$ and are premium container dominated if $R_3(y)=0$ for any $0 \le y \le c_2-1$.

Elementary (premium) container domination means that it is always more profitable to provide elementary (premium) containers than to offer premium (elementary) ones when the firm has two types of containers on hand. From Theorem 1, it will be helpful for us to characterise the conditions under which customers are *elementary container dominated* or *premium container dominated* if we can figure out the upper and lower bounds of $v(x+1,y)-v(x,y+1)$.

To this end, let $\sigma_{1i}(t)=\max[t,(\beta-1)r_{1i}, \beta r_{1i}+\pi_1-b_3^*, b_2^* - r_{1i}-\pi_1]$, $\sigma_{2j}(t)=\max(t, r_{2j}+\pi_2-b_3^*)$, $\sigma_3(t)=\max(t, p_2-p_1, p_2-b_3^*, b_2^*-p_1)$, $\omega_{1i}(t)=\min[t,(\beta-1)r_{1i}, -b_1^*+\beta r_{1i}+\pi_1, b_4^*-\pi_1-r_{1i}]$, $\omega_{2j}(t)=\min(t,-b_1^*+r_{2j}+\pi_2)$, $\omega_3(t)=\min(t, p_2-p_1, -b_1^*+p_2, b_4^*-p_1)$, $b_3^*=\min_{(x,y)\in(C\backslash C_1)}-\Delta_x v(x,y)$, $b_4^*=\min_{(x,y)\in(C\backslash C_2)}-\Delta_y v(x,y)$. We then define

$$\rho_3(t)=t-\frac{\sum_{i=1}^m \lambda_{1i}\sigma_{1i}(t)+\sum_{j=1}^n \lambda_{2j}\sigma_{2j}(t)+\lambda_3\sigma_3(t)+h_1-h_2}{\sum_{i=1}^m \lambda_{1i}+\sum_{j=1}^n \lambda_{2j}+\lambda_3+\mu+\alpha},$$

$$\rho_4(t)=t-\frac{\sum_{i=1}^m \lambda_{1i}\omega_{1i}(t)+\sum_{j=1}^n \lambda_{2j}\omega_{2j}(t)+\lambda_3\omega_3(t)+h_1-h_2}{\sum_{i=1}^m \lambda_{1i}+\sum_{j=1}^n \lambda_{2j}+\lambda_3+\mu+\alpha}.$$

Lemma 4 (Simplifying Substitution and Priority Thresholds).

(a) *$\rho_3(t)$ and $\rho_4(t)$ are increasing in t and the equations $\rho_3(t)=0$ and $\rho_4(t)=0$ have unique solutions b_5^* and b_6^*, respectively. Then $b_5^* \ge \max_{(x,y)\in(C\backslash C_3)} g(x,y)$ and $b_6^* \le \min_{(x,y)\in(C\backslash C_3)} g(x,y)$, where $C_3=\{(x,y) \mid x=c_1, 0\le y\le c_2 \text{ or } y=c_2, 0\le x\le c_1\}$.*

(b) *Class-i ECCs are elementary container dominated if $b_6^* \ge (\beta-1)r_{1i}$ and premium container dominated if $b_5^* < (\beta-1)r_{1i}$; WICs are elementary container dominated if $b_6^* \ge p_2-p_1$ and premium container dominated if $b_5^* < p_2-p_1$.*

In Lemma 4, we give upper and lower bounds for $v(x+1,y)-v(x,y+1)$ by solving the equations $\rho_3(t)=0$ and $\rho_4(t)=0$, respectively. The functions $\rho_3(t)$ and $\rho_4(t)$ have similar properties to $\rho_1(t)$. From Lemma 4, for WICs or a certain class of ECCs, if the difference between the rental price of premium containers and that of elementary ones is sufficiently large (small), it is always more profitable to provide them with premium (elementary) containers rather than elementary (premium) ones. Therefore, the firm should either provide them with premium (elementary) containers or refuse them. To conclude, if a customer class is *elementary container dominated* (*premium container dominated*), then it is better to provide the class with elementary (premium) containers instead of premium (elementary) ones once idle elementary (premium) containers exist.

Summarizing Lemmas 3 and 4, we get Theorem 2, which completely characterises the simplified optimal rationing policy.

Theorem 2. *For any $(x,y) \in (C \setminus C_3)$, the optimal rationing policy is characterised as follows:*

(a) *For class-i ECCs, the optimal policy is*

	$r_{1i} \geq b_1^* - \pi_1$ or $r_{1i} \geq (b_2^* - \pi_1)/\beta$	
Conditions	$b_6^* \geq (\beta - 1) r_{1i}$	$b_5^* < (\beta - 1) r_{1i}$
Policy	Provide elementary containers	Provide premium containers

(b) *For class-j PCCs, the optimal policy is to provide premium containers if $r_{2j} \geq b_2^* - \pi_2$.*

(c) *For WICs, the optimal policy is*

	$p_1 \geq b_1^*$ or $r_2 \geq b_2^*$	
Conditions	$b_6^* \geq p_2 - p_1$	$b_5^* < p_2 - p_1$
Policy	Provide elementary containers	Provide premium containers

Theorem 2(a) and (c) imply that for ECCs (WICs) preferred for elementary or premium containers, the firm should provide them with elementary containers if the difference between the rental rates of premium and elementary containers for them is very small and with premium containers if the difference is very large. Theorem 2(b) means that the firm always provides premium containers for PCCs if their rental rate is sufficiently large. Therefore, the policy can be characterised as a simpler format and be more easily implemented.

4. Benefits of the Optimal Policy

In this subsection, we carry out numerical studies to show the value of the model and the optimal policy based on the data from Drewry (2019) and a real leasing firm located in Tianjin, China. The detailed data used in this part are presented in Section EC.2. We also conduct numerical experiments to investigate the influence of customers' arrival rates on the optimal policy, which are presented in Section EC.3.

4.1. Benefits of Substitution

This work assumes that the firm has two types of containers and premium containers have a substitution effect. If ECCs can only be satisfied by elementary containers and WICs can only be satisfied by one type of container, then the system can be reduced to the system in Miller (1969). To demonstrate the value of this work, we explore the benefits of substitution. Let $v_1(x,y)$ be the expected total discounted revenue without substitution. Then the dynamic program model of the system without substitution is $v_1(x,y) = T^1[v_1(x,y)]$, where

$$T^1[v_1(x,y)] = \sum_{i=1}^{n} \lambda_{1i} T_{1i}^1[v_1(x,y)] + \sum_{j=1}^{m} \lambda_{2j} T_{2j}[v_1(x,y)] + \lambda_3 T_3[v_1(x,y)] + T_4[v_1(x,y)]$$

$$T_{1i}^1[v_1(x,y)] = \begin{cases} \max\{v_1(x+1,y) \\ \quad +r_{1i}, v_1(x,y) - \pi_1\}, & x < c_1 \\ v_1(x,y) - \pi_1, & x = c_1. \end{cases}$$

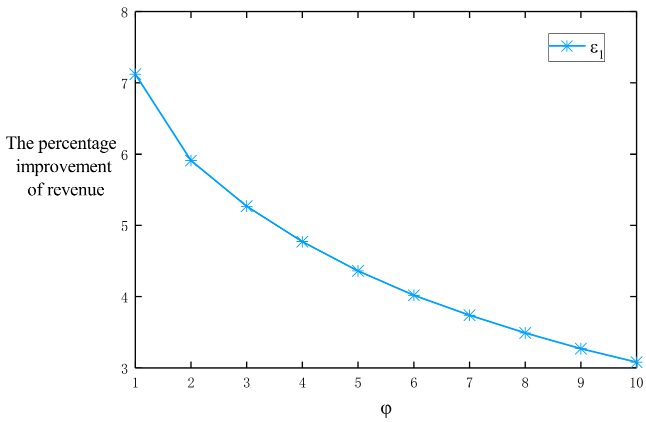
Other than that the firm does not provide premium containers for ECCs, other settings of this system are the same as those of the system in Section 3. Without loss of generality, we suppose that both systems begin at state (0, 0). Let $\varepsilon_1 = (v(0,0) - v_1(0,0))/(v_1(0,0)) \times 100$. We use the relative improvement percentage of expected rental revenue to measure the value of our system. To explore the benefits of substitution, we further consider the difference between the rental prices of the two types of containers for WICs. To this end, we keep p_1 as a constant and change p_2. Let $p_2 = \varphi p_1$, where $\varphi = 1, 2, \ldots, 10$. Other parameters are provided in Section EC.2.

From Figure 2, we find that ε_1 is positive. This means that the downward substitution considered in this study can bring much revenue improvement to the firm (between 3% and 8%, with an average of 4.50%). Moreover, the smaller φ is, the larger is the revenue improvement brought by substitution. This phenomenon can be explained as follows. When φ is small, the rental price difference of the two types of containers for WICs is small. The firm has more flexibility to use both types of containers to satisfy WICs. Thus, the firm can satisfy more ECCs in our system. As φ grows, the difference between the rental prices of the two types of containers for WICs is large, and the firm will prefer to use premium containers to satisfy WICs in both systems, which decreases the revenue improvement brought by substitution.

4.2. Optimal Policy vs. Myopic Policy

In this part, we compare the revenues brought by the optimal policy and the myopic policy (a commonly used heuristic policy which is also used by the firm we recently visited). The myopic policy can be characterised as follows. The firm always provides ECCs and WICs with elementary containers when it has elementary containers on hand, provides them with premium containers when it has idle premium containers but does not have idle elementary containers and

Figure 2. (Color online) The Benefits of Substitution



refuses them only when there is no idle container. The firm always provides PCCs with premium containers when there are idle premium containers and otherwise refuses them.

Let the expected total discounted revenue of the firm under the myopic policy be $v_2(x, y)$ when the system state is (x, y). We still use the relative improvement percentage of the total revenue to measure the value of the optimal policy. Let $\varepsilon_2 = (v(0,0) - v_2(0,0))/(v_2(0,0)) \times 100$. We shall see how customers' arrival rates affect the performance of the optimal policy. Set $\lambda_{11} = 0, 10, \ldots, 120$, 130.[6] Other parameters are provided in Section EC.2. Under this setting, the revenue, $v_2(0,0)$, brought by the myopic policy can be regarded as the firm's real revenue. From Table 2, we find that the total revenues of the firm under the optimal and myopic policies first increase and then decrease in λ_{11}. The myopic policy brings the maximum revenue when $\lambda_{11} = 50$, while the optimal policy results in the maximum revenue when $\lambda_{11} = 60$. This indicates that the optimal policy can help the firm to better utilise its capacity and serve more customers. We also find that the optimal policy brings more revenue for the firm (between 3% and 12%, with an average of 6.48%) than the myopic policy does. Moreover, ε_2 decreases in λ_{11} when $\lambda_{11} \leq 50$ and increases in λ_{11} when $\lambda_{11} \geq 50$, which means that the optimal policy performs better when the arrival rate is either small or large.

To explain this, we introduce two symbols: $\varpi = (\sum_{i=1}^{n} \lambda_{1i} + \sum_{j=1}^{m} \lambda_{2j} + \lambda_3)/\mu$ (which represents the number of customers that the firm deals with in an average leasing duration) and $\tau = \varpi/(c_1 + c_2)$ (which represents the matching degree of the supplies and demands). When τ is close to 1, supply and demand are matched. When τ is much larger or smaller than 1, supply and demand are mismatched. We have $\tau = 1.02$ when $\lambda_{11} = 50$. When $\tau < 1.02$, the firm's capacity is adequate. Supply and demand are mismatched. Each customer is reasonably valuable and their demand should be met in an appropriate way. In this case, the optimal policy helps the firm make the most profitable decision for each customer, and thus it performs much better than the myopic policy. As τ increases, the capacity will be insufficient and the value of customers will decrease; then the revenue improvement brought by the optimal policy decreases. When $\tau = 1.02$, supply and demand match well, and then the myopic policy also performs well. The optimal policy brings the least revenue improvement. When $\tau > 1.02$, there is a shortage of containers compared with customers, in which case containers become more valuable. Supply and demand do not match well. The firm should make use of each container to meet customers properly and earn more profit. In this case, the optimal policy helps the firm make better rationing decisions than the myopic policy does. As τ increases, the value of each container increases and the revenue improvement brought by the optimal policy also increases.

4.3. Optimal Policy vs. Stochastic Rationing Policy

In this subsection, we compare the performance of the optimal policy and a stochastic rationing policy. We consulted some container leasing companies and found that their commonly used capacity rationing policy is very similar to the stochastic rationing policy we will characterise. Under the stochastic rationing policy, the firm randomly provides idle elementary and premium containers for ECCs and WICs. Each idle container has the same probability of being

Table 2. Optimal Policy vs. Myopic Policy

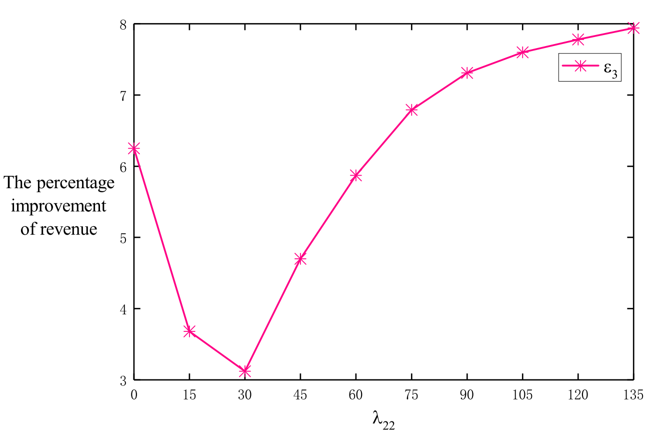
λ_{11}	0	10	20	30	40	50	60
$v_2(0,0)$/million dollars	99.498	112.470	126.160	139.570	149.282	153.064	152.547
$v(0, 0)$/million dollars	107.872	120.665	133.423	145.552	154.288	157.909	158.033
ε_2	8.42	7.29	5.76	4.29	3.35	3.17	3.60
λ_{11}	70	80	90	100	110	120	130
$v_2(0,0)$/million dollars	149.758	145.818	141.272	136.390	131.315	126.128	120.875
$v(0, 0)$/million dollars	156.334	153.675	150.453	146.884	143.062	139.068	134.934
ε_2	4.39	5.39	6.50	7.69	8.95	10.26	11.63

selected. The firm always provides premium containers for PCCs unless it does not have premium containers on hand.

Let the expected total discounted revenue under the stochastic rationing policy be $v_3(x,y)$. We use the relative improvement percentage of the total revenue to measure the value of the optimal policy. Let $\varepsilon_3 = (v(0,0) - v_3(0,0))/(v_3(0,0)) \times 100$. We investigate how ε_3 changes with respect to the arrival rate of class-2 PCCs. Let $\lambda_{22} = 0, 15, 30, \ldots, 120, 135$. Other parameters are also set in Section EC.2.

From Figure 3, we find that the optimal policy brings 3%–8% (average 6.10%) more revenue than the stochastic rationing policy does. We observe that ε_3 decreases in λ_{22} when $\lambda_{22} \le 30$ ($\tau \le 0.90$) and increases in λ_{22} when $\lambda_{22} \ge 30$ ($\tau \ge 0.90$). This means that when supply and demand are mismatched, our policy performs much better than the stochastic rationing policy. The reason is as follows. When $\lambda_{22} \le 30$, the firm has adequate capacity to meet customers' demands. Under the optimal policy, the firm will use more premium containers to satisfy ECCs and WICs to earn more income and pay fewer holding costs than under the stochastic rationing policy. As λ_{22} increases, more premium containers should be allocated to PCCs, while fewer premium containers should be provided for ECCs and WICs. There is then less difference in the performances of the two policies. Thus, the optimal policy brings less revenue improvement. When $\lambda_{22} \ge 30$ and increases, the firm cannot meet all customers' demands. PCCs can only be satisfied by premium containers. However, the stochastic rationing policy still allocates some premium containers to ECCs and WICs. This will decrease the service level for PCCs and reduce revenue. Therefore, the optimal policy performs much better than the stochastic rationing policy. When $\lambda_{22} = 30$, the optimal policy brings the least revenue improvement.

Figure 3. (Color online) Optimal Policy vs. Stochastic Rationing Policy



5. Extension

5.1. Different Expected Rental Durations

In this subsection, we relax the assumption of identical expected rental duration for elementary and premium containers and consider different expected rental durations for them. We give a counter example to show that the value function is not necessarily subconcave. Therefore, the optimal policy does not have a threshold-type structure. The counter example and the corresponding explanations are presented in Section EC.4.

5.2. ECCs' Acceptance Behaviours

In this subsection, we consider ECCs' acceptance behaviours and investigate the following two cases: (a) ECCs accept substitution at a constant probability p ($0 < p < 1$); and (b) the acceptance probability is a function of the rental price of premium containers determined by the firm.

5.2.1. Constant Acceptance Probability. In this case, ECCs accept the premium containers offered by the firm at a constant acceptance probability p and refuse the substitution at the probability $1-p$. If ECCs accept the substitution, they pay a rental rate (βr_{1i} for class-i ECCs) to get a premium container. If they refuse the substitution, they will receive a penalty π_1 from the firm. The other setting is the same as in Section 3. Let $u(x, y)$ be the expected total discounted revenue of the firm when the state is (x, y). Then the optimality equation is given by $u(x,y) = T^{cp}[u(x,y)]$ where $S_i(x,y) = p(u(x,y+1) + \beta r_{1i}) + (1-p)(u(x,y) - \pi_1)$, $T^{cp}[u(x,y)] = \sum_{i=1}^{n} \lambda_{1i} T_{1i}^{cp}[u(x,y)] + \sum_{j=1}^{m} \lambda_{2j} T_{2j}[u(x,y)] + \lambda_3 T_3[u(x,y)] + T_4[u(x,y)]$,

$$T_{1i}^{cp}[u(x,y)] = \begin{cases} \max\{u(x+1,y) + r_{1i}, S_i(x,y), \\ \quad u(x,y) - \pi_1\}, & x < c_1, y < c_2 \\ \max\{u(x+1,y) + r_{1i}, u(x,y) - \pi_1\}, & x < c_1, y = c_2 \\ \max\{S_i(x,y), u(x,y) - \pi_1\}, & x = c_1, y < c_2 \\ u(x,y) - \pi_1, & x = c_1, y = c_2. \end{cases}$$

The operator T_{1i}^{cp} represents the firm's rationing decision for class-i ECCs. The operators T_{2j}, T_3 and T_4 are the same as those in Section 3. We also give a counter example to show that the value function is not necessarily subconcave. The counter example is given in Section EC.5.1.

In the following, we give the conditions under which the value function $u(x, y)$ has submodularity, subconcavity, and concavity. Using similar arguments as those used to prove Lemma 1, we can show that the operators T_{2j}, T_3, and T_4 can preserve the submodularity, subconcavity, and concavity of the value function. The following Lemma shows that when λ_{1i} ($i = 1, 2, \ldots, n$) is small enough, T^{cp} can also preserve the submodularity, subconcavity, and concavity.

Lemma 5. *There exists a vector $\boldsymbol{l} = (l_1, l_2, \ldots, l_n)$ such that the value function $u(x, y)$ is submodular, subconcave, and concave when $\lambda_{1i} \le l_i$ for each $i = 1, 2, \ldots, n$. The value of $\boldsymbol{l}$ depends on system parameters.*

From Lemma 5, the firm's optimal rationing policy can be characterised under some conditions. Since the optimal rationing policy for PCCs and WICs has a similar formation to those in Section 3, we only give the optimal policy for ECCs in the following theorem.

Theorem 3. *When λ_{1i}, $\forall i = 1, 2, \ldots, n$ is small enough, the value function is submodular, subconcave, and concave. And for class-i ECCs, there exist rationing thresholds $K^{cp}_{1i}(y)$ and $K^{cp}_{2i}(x)$ and a substitution threshold $K^{cp}_{3i}(y)$ such that*

Conditions	$x \ge K^{cp}_{1i}(y)$ and $y \ge K^{cp}_{2i}(x)$	$x < K^{cp}_{1i}(y)$ and $y \ge K^{cp}_{2i}(x)$	$x < \min\{K^{cp}_{1i}(y), K^{cp}_{3i}(y)\}$ and $y < K^{cp}_{2i}(x)$	$x \ge K^{cp}_{1i}(y)$ and $y < K^{cp}_{2i}(x)$	$K^{cp}_{3i}(y) \le x < K^{cp}_{1i}(y)$ and $y < K^{cp}_{2i}(x)$
Rationing policy	Reject	Provide an elementary container		Provide a premium container	

where

$$K^{cp}_{1i}(y) = \begin{cases} c_1, & -\Delta_x u(c_1 - 1, y) \le \pi_1 + r_{1i} \\ 0, & -\Delta_x u(0, y) > \pi_1 + r_{1i} \\ k, & -\Delta_x u(k, y) > \pi_1 + r_{1i} \ge -\Delta_x u(k-1, y), \end{cases}$$

$$K^{cp}_{2i}(x) = \begin{cases} c_2, & -\Delta_y u(x, c_2 - 1) \le \pi_1 + \beta r_{1i} \\ 0, & -\Delta_y u(x, 0) > \pi_1 + \beta r_{1i} \\ k, & -\Delta_y u(x, k) > \pi_1 + \beta r_{1i} \ge -\Delta_y u(x, k-1), \end{cases}$$

$$K^{cp}_{3i}(y) = \begin{cases} c_1, & (1-p)\Delta_x u(c_1 - 1, y) + p[u(c_1, y) - u(c_1 - 1, y+1)] \\ & \quad \ge (\beta p - 1) r_{1i} - (1-p)\pi_1 \\ 0, & (1-p)\Delta_x u(0, y) + p[u(1, y) - u(0, y+1)] \\ & \quad < (\beta p - 1) r_{1i} - (1-p)\pi_1 \\ k, & k = \min\{x \mid (1-p)\Delta_x u(x, y) + p[u(x+1, y) \\ & \quad - u(x, y+1)] < (\beta p - 1) r_{1i} - (1-p)\pi_1\}. \end{cases}$$

From Theorem 3, we can see that the optimal policy still has a simple threshold structure under certain conditions. Due to the existence of the probability that ECCs may refuse premium containers offered by the firm, the operators for ECCs are more complicated. Therefore, the substitution threshold $K^{cp}_{3i}(y)$ is more complicated than in the original model. Although ECCs may refuse the substitution, it is still optimal to provide them with premium containers when the firm has few elementary containers but many premium ones on hand.

5.2.2. Rental Price-Related Acceptance Probability.

In this part, we consider the case in which ECCs' acceptance probability depends on the rental price of premium containers for ECCs.

Suppose that the firm chooses a price from a limited set $Q_i = \{q_i^1, q_i^2, \ldots, q_i^{J_i}\}$ (where J_i is the number of alternative prices for class-i ECCs) when it offers premium containers to class-i ECCs. If the firm chooses a price $q_i^{j_i}$ $(1 \le j_i \le J_i)$, class-i ECCs' acceptance probability is $f_i(q_i^{j_i})$ and the probability of refusing the firm's offer is $1 - f_i(q_i^{j_i})$. If class-i ECCs accept, they should pay the price $q_i^{j_i}$ to get premium containers; if they refuse, the firm pays a penalty π_1. Without loss of generality, we assume $q_i^1 \le q_i^2 \le \ldots \le q_i^{J_i} (\forall i = 1, 2, \ldots, n)$ and that ECCs are price-sensitive such that $f_i(q_i^1) \ge f_i(q_i^2) \ge \ldots \ge f_i(q_i^{J_i}) = 0$.[7] Let the expected total discounted revenue of the firm under state (x, y) be $v(x,y)$. The optimality equation of this MDP is given by $v(x,y) = T^{vp}[v(x,y)]$ where

$$T^{vp}[v(x,y)] = \sum_{i=1}^{n} \lambda_{1i} T^{vp}_{1i}[v(x,y)] + \sum_{j=1}^{m} \lambda_{2j} T_{2j}[v(x,y)] + \lambda_3 T_3[v(x,y)] + T_4[v(x,y)]$$

$$T^{vp}_{1i}[v(x,y)] = \begin{cases} \max\{v(x+1, y) + r_{1i}, R_i(x,y), \\ \quad v(x,y) - \pi_1\}, & x < c_1, y < c_2 \\ \max\{v(x+1, y) \\ \quad + r_{1i}, v(x,y) - \pi_1\}, & x < c_1, y = c_2 \\ \max\{R_i(x,y), v(x,y) - \pi_1\}, & x = c_1, y < c_2 \\ v(x,y) - \pi_1, & x = c_1, y = c_2 \end{cases}$$

$$R_i(x,y) = \max_{q_i \in Q_i}\{f_i(q_i)[v(x, y+1) + q_i] + [1 - f_i(q_i)][v(x,y) - \pi_1]\}.$$

The operator T^{vp}_{1i} reflects the firm's decisions for a class-i ECC, which consists of directly providing an elementary container at the prenegotiated price, offering a premium container at a certain price and refusing. Other operators are the same as those in Section 3.

Define a new function as $R'_i(x,y) = \max_{q_i \in Q_i}\{f_i(q_i)[v(x, y+1) - v(x,y) + q_i + \pi_1]\}$. By calculation, we have $R_i(x,y) = R'_i(x,y) + v(x,y) - \pi_1$. The function $R'_i(x,y)$ is the additional revenue function, which represents the immediate expected reward brought by the substitution decision for a class-i ECC. Define $q_i(x,y) = \arg\max_{q_i \in Q_i}\{f_i(q_i)[v(x, y+1) - v(x,y) + q_i + \pi_1]\}$ as the firm's pricing function. When the firm makes decisions, it first calculates the optimal price that maximises $R'_i(x,y)$ and then compares the benefits of providing an elementary container at a prenegotiated price, providing a premium one at its optimal price and refusing. However, in this situation, the subconcavity of the value function may not hold. This is shown by the counter example in Section EC.5.2.

Note that the only difference between this model and the model in Section 3 is that the operator T_{1i} is replaced by T^3_{1i}. We can show the submodularity, subconcavity, and concavity of the value function under the assumption that λ_{1i} $(i = 1, 2, \ldots, n)$ is small.

Lemma 6. *There exists a vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$ such that the value function $v(x, y)$ is submodular, subconcave, and concave when $\lambda_{1i} \le \eta_i$ $(i = 1, 2, \ldots, n)$. The value of $\boldsymbol{\eta}$ depends on the system parameters.*

Lemma 6 shows that the value function is submodular, subconcave, and concave under certain conditions. To explore the firm's optimal policy, we also need to discuss the properties of $R'_i(x, y)$ and $q_i(x, y)$. To this end, define the functions $h^i(a) = \max_{q_i \in Q_i}\{f_i(q_i)(a + q_i + \pi_1)\}$ and $q^i(a) = \arg\max_{q_i \in Q_i}\{f_i(q_i)(a + q_i + \pi_1)\}$. We then have the following lemma.

Lemma 7. *The function $h^i(a)$ increases in a; the function $q^i(a)$ decreases in a.*

From Lemma 7, we can see that the additional revenue function $R'_i(x, y)$ decreases with the opportunity cost of leasing out a premium container and the pricing function $q_i(x, y)$ increases with the opportunity cost. Combining these properties and Lemma 6, we can characterise the optimal policy as follows.

Theorem 4. *When λ_{1i} is sufficiently small, the value function is submodular, subconcave, and concave, and the optimal policy for class-i ECCs can be characterised as follows.*

(a) *For the optimal price of premium containers for class-i ECCs, we have $q_i(x, y) \le q_i(x + 1, y) \le q_i(x, y + 1)$.*

(b) *When the firm has no container on hand, it will reject the arriving customer. When the firm only has idle elementary containers, there is a threshold $K^{vp}_{1i}(y)$ such that it provides an elementary container if $x < K^{vp}_{1i}(y)$ and otherwise refuses the customer. When the firm only has idle premium containers, it provides a premium container at the optimal price. When the firm has two types of containers on hand, there exists a threshold $K^{vp}_{3i}(y)$ such that the firm provides an elementary container when $x < K^{vp}_{3i}(y)$ and otherwise provides a premium one at its optimal price, where*

$$K^{vp}_{1i}(y) = \begin{cases} c_1, & -\Delta_x v(c_1 - 1, y) \le \pi_1 + r_{1i} \\ 0, & -\Delta_x v(0, y) > \pi_1 + r_{1i} \\ k, & -\Delta_x v(k, y) > \pi_1 + r_{1i} \ge -\Delta_x v(k - 1, y), \end{cases}$$

$$K^{vp}_{3i}(y) = \begin{cases} c_1, & \Delta_x v(c_1 - 1, y) - R'_{1i}(c_1 - 1, y) + r_{1i} + \pi_1 \ge 0 \\ 0, & \Delta_x v(0, y) - R'_{1i}(0, y) + r_{1i} + \pi_1 < 0 \\ k, & k = \min\{x \mid \Delta_x v(x, y) - R'_{1i}(x, y) + r_{1i} + \pi_1 < 0\}. \end{cases}$$

5.3. Endogenous Rental Prices for WICs

In this subsection, we consider the case in which the rental prices for WICs are endogenous decision variables and WICs are price-sensitive. In this situation, when a WIC arrives, the firm first decides the optimal rental prices of the two types of containers and then specifies the container type and its corresponding price. Assume that the prices of elementary and premium containers are selected from the sets $P_1 = \{p^1_1, p^2_1, \ldots, p^{L_1}_1\}$ and $P_2 = \{p^1_2, p^2_2, \ldots, p^{L_2}_2\}$, respectively. If a WIC is provided with a price p^l_1 as the rental price of elementary containers, they accept the offer with probability $t_1(p^l_1)$. If the customer is provided with a price p^l_2 as the rental price of premium containers, they accept the offer with probability $t_2(p^l_2)$. Without loss of generality, we assume that $0 \le p^1_1 \le p^2_1 \le \ldots \le p^{L_1}_1$, $0 \le p^1_2 \le p^2_2 \le \ldots \le p^{L_2}_2$, $t_1(p^l_1) \ge t_1(p^2_1) \ge \ldots \ge t_1(p^{L_1}_1) = 0$ and $t_2(p^l_2) \ge t_2(p^2_2) \ge \ldots \ge t_2(p^{L_2}_2) = 0$. Let the expected total discounted revenue of the firm under state (x, y) be $w(x, y)$. Then the optimality equation of the model is $w(x, y) = T^{ew}[w(x, y)]$, where $D_1(x, y) = \max_{p_1 \in P_1}\{t_1(p_1)(w(x + 1, y) - w(x, y) + p_1)\}$, $D_2(x, y) = \max_{p_2 \in P_2}\{t_2(p_2)\,(w(x, y + 1) - w(x, y) + p_2)\}$,

$$T^{ew}[w(x, y)] = \sum_{i=1}^{n} \lambda_{1i} T_{1i}[w(x, y)] + \sum_{j=1}^{m} \lambda_{2j} T_{2j}[w(x, y)] + \lambda_3 T^{ew}_3[w(x, y)] + T_4[w(x, y)],$$

$$T^{ew}_3[w(x, y)] = \begin{cases} \max\{D_1(x, y) + w(x, y), D_2(x, y) + w(x, y), w(x, y)\}, & x < c_1, y < c_2 \\ \max\{D_1(x, y) + w(x, y), w(x, y)\}, & x < c_1, y = c_2 \\ \max\{D_2(x, y) + w(x, y), w(x, y)\}, & x = c_1, y < c_2 \\ w(x, y), & x = c_1, y = c_2. \end{cases}$$

The operator T^{ew}_3 represents the firm's pricing decision for WICs. $D_1(x, y)$ and $D_2(x, y)$ are the additional revenue functions under optimal pricing decisions for elementary containers and premium ones, respectively. Define $p_1(x, y) = \arg\max_{p_1 \in P_1}\{t_1(p_1)[w(x + 1, y) - w(x, y) + p_1]\}$ and $p_2(x, y) = \arg\max_{p_2 \in P_2}\{t_2(p_2)[w(x, y + 1) - w(x, y) + p_2]\}$ as the firm's optimal pricing functions for elementary and premium containers, respectively. When a WIC arrives, if the firm has both types of containers on hand, it calculates the optimal prices of the two types of containers and then decides which type of container to provide at the optimal price. When the firm has one type of container on hand, the firm calculates the optimal price of this kind of container and announce the price to the WIC. When the firm has no idle container, the firm refuses the WIC. Under this situation, the value function $w(x, y)$ is not necessarily subconcave. A counter example is presented in Section EC.6.

In what follows, we identify conditions under which the subconcavity of the value function holds. Define the functions $d_1(a) = \max_{p_1 \in P_1}\{t_1(p_1)(-a + p_1)\}$ and $d_2(a) = \max_{p_2 \in P_2}\{t_2(p_2)(-a + p_2)\}$. From the similarity of $d_1(a)$, $d_2(a)$ and $h^i(a)$ in Section 5.2, we know that $d_1(a)$ and $d_2(a)$ decrease in a by similar arguments to those in Lemma 7. Then we obtain that the function

$D_1(x,y)$ ($D_2(x,y)$) decreases with the opportunity cost of leasing an elementary (premium) container.

To study the properties of $d_1(a)$ and $d_2(a)$, we make the following assumption.

Assumption 1. $(d_2(a_2)-d_2(a_1))/(a_2-a_1)=(d_1(a_2)-d_1(a_1))/(a_2-a_1)=\gamma$ *for any* a_1 *and* a_2 *where* γ *is a constant.*

If a is the opportunity cost of leasing an elementary (premium) container, $d_1(a)$ ($d_2(a)$) is by definition the expected revenue of leasing an elementary (premium) container to a WIC. In Assumption 1, $(d_2(a_2)-d_2(a_1))/(a_2-a_1)$ and $(d_1(a_2)-d_1(a_1))/(a_2-a_1)$ are the revenue-cost ratios of leasing an elementary and premium container to WICs, respectively. Assumption 1 indicates that the revenue-cost ratios (which do not change with the system state) of leasing the two types of containers to WICs are the same. Then the two types of containers have the same importance in terms of serving WICs.

Lemma 8. *Under Assumption 1, we have* $-1\le\gamma\le 0$.

Lemma 9. *Under Assumption 1, the value function* $w(x,y)$ *is submodular, subconcave, and concave.*

Based on Lemma 9, we can characterise the structure of the optimal policy. Since the optimal rationing policy for ECCs and PCCs in this situation has a similar structure to those in Section 3, we only give the firm's optimal pricing policy for WICs.

Theorem 5. *The optimal pricing policy for WICs can be characterised as follows.*

(a) *For the optimal prices for elementary and premium containers, we have* $p_1(x,y)\le p_1(x,y+1)\le p_1(x+1,y)$ *and* $p_2(x,y)\le p_2(x+1,y)\le p_2(x,y+1)$.

(b) *When the firm has no container on hand, it refuses the arriving customer. When the firm has only elementary (premium) containers on hand, it provides an elementary (premium) container for the customer at its optimal price. When the firm has both types of containers on hand, a pricing threshold* $R_3^{ew}(y)$ *exists such that the firm provides an elementary container at its optimal price if* $x<R_3^{ew}(y)$ *and otherwise provides a premium container at its optimal price, where*

$$R_3^{ew}(y)=\begin{cases} c_1, & D_1(c_1-1,y)-D_2(c_1-1,y)\ge 0\\ 0, & D_1(0,y)-D_2(0,y)<0\\ k, & D_1(k,y)-D_2(k,y)<0\le D_1(k-1,y)\\ & \quad -D_2(k-1,y).\end{cases}$$

6. Conclusion

In this paper, we studied the problem of capacity rationing for a firm with two types of containers dealing with two groups of contract customers (ECCs and PCCs) and WICs. ECCs and WICs can be satisfied by either elementary or premium containers, while PCCs can only be satisfied by premium ones. We formulated the problem as an event-based MDP problem. Under the assumptions that the customers' rental durations follow the same distribution, ECCs accept the substitution with certainty and the rental prices are exogenous, we analysed the properties of the value function, derived the optimal rationing policy and characterised its structure. Specifically, we proved that there exist three types of thresholds that can provide a guideline for the firm. Rationing thresholds are used to determine whether it is profitable to provide elementary or premium containers for customers according to the current system state. Substitution thresholds determine whether it is profitable to substitute elementary containers with premium ones when an ECC arrives. A priority threshold can be applied to determine whether elementary containers have priority in being offered to WICs. By combining the three types of thresholds, the firm can make the optimal decision at any system state. Furthermore, we provided the conditions under which the optimal policy can be further simplified and is easier to implement.

To show the effectiveness of our model and the optimal policy, we conducted several numerical experiments based on the data from Drewry (2019) as well as real data from a container leasing company in China. We first showed that the optimal policy brings 3%–8% (average 4.50%) more revenue than when the firm separately manages the different types of containers. We then compared the performances of the optimal policy with those of the two commonly used policies, namely myopic policy and stochastic rationing policy. We found that the optimal policy can bring 3%–12% (average 6.48%) more revenue than the myopic policy, and 3%–8% (average 6.10%) more revenue than the stochastic rationing policy.

Moreover, we discussed some extensions considering different rental durations, ECCs' acceptance behaviours for premium containers and endogenous prices for WICs. In the case of different rental durations, we gave a counter example to show that the value function is not necessarily subconcave. In the case of ECCs' acceptance behaviours for premium containers, we showed that the subconcavity of the value function does not generally hold and gave conditions under which the optimal policy has a simple threshold-type structure. In the case of endogenous prices for WICs, we gave conditions under which the value function is submodular, subconcave, and concave. We then characterised how the optimal prices change with the system state, and the optimal rationing policy still has a threshold-type structure.

Based on this work, there are several topics that can be studied further.

- We consider two types of rental items and three groups of customers. In reality, a container leasing firm may have more than two kinds of items, and customers may have more complex preference behaviours. It is an interesting topic to generalise the results to a system with multi-items and customers with more complex preference behaviours.

• In this study, we assume that each customer upon arrival has one unit of demand. In reality, customers may require different units of containers. Thus, one future research direction could be to generalise the results to a case in which customers request more than one unit of containers.

• In this paper, we consider one firm's rental problem at a single location. In reality, containers can be rented from one location but returned to another one. Extending this study to the case of multiple-locations needs to be studied further.

• Only one leasing firm is discussed in this study. In reality, there is more than one firm in the container leasing market, and they compete for customers. Generalizing the results of this study to a competitive environment is an interesting and challenging topic for further study.

• In this paper, we consider a short-term lease in the container leasing industry, and then we assume stationary parameter settings. In practice, shipping companies can also choose a long-term lease in which they lease a whole fleet of containers for a couple of years (up to 8–10 years). Hence, it would also be interesting to generalise this study to a case with fluctuating parameters.

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Endnotes

[1] To simplify the description, we use "a container" to represent "a unit of container" in the discussion.

[2] Triton (2020) showed that the operating lease revenue and the direct operating cost in the fourth quarter of 2019 were $321.626 million and $23.718 million, respectively. From Textainer (2019), container expense can be influenced by storage cost: "Direct container expense—owned fleet, increased $1.1 million compared with the second quarter of 2019, primarily due to an increase in storage costs, partially offset by reductions in other direct costs. Direct container expense—owned fleets, decreased $2.3 million compared with the third quarter of 2018 from a reduction in repositioning expense, partially offset by higher storage costs."

[3] Throughout this paper, we use "revenue" to represent the firm's rental incomes deducting holding costs of containers and potential penalties for breach of contracts.

[4] In the discussion, the words "increase" and "decrease" indicate "non-strictly increase" and "non-strictly decrease," respectively, unless we use "strictly increase" and "strictly decrease."

[5] To better understand the definition of $\rho_1(t)$, we view the solution as the maximum opportunity cost in the explanations.

[6] We observe a similar phenomenon when the arrival of ECCs is kept at a constant rate and the arrival rate of PCCs changes.

[7] The assumption of $f_i(q_i^{J_i}) = 0$ means that providing a premium container for class-i ECCs at the price $q_i^{J_i}$ is equivalent to rejecting them.

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