From symmetric networks to heteroclinic dynamics and chaos in coupled phase oscillators with higher-order interactions

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Abstract We highlight some results from normal form theory for symmetric bifurcations that give a rational way to organize higher-order interactions between phase oscillators in networks with fully symmetric coupling. For systems near Hopf bifurcation the lowest order (pairwise) interactions correspond to the system of Kuramoto and Sakaguchi. At next asymptotic order one must generically include higher-order interactions of up to four oscillators. We discuss some dynamical consequences of these interactions in terms of heteroclinic attractors, chaos, and chimeras for related systems.

1 Introduction

Network dynamical systems consists of individual dynamical units (nodes) that evolve under mutual interaction. Examples include coupled neural oscillators, flashing fireflies, and power grid networks. Such dynamical systems often give rise to intriguing collective behavior, such as synchronization where nodes eventually behave in unison [1, 2]. Many mathematical descriptions of such network dynamical systems often makes the assumption that nodes interact in a pairwise fashion: The network interactions are determined by the joint state of pairs of nodes, that is, there is an underlying (directed) graph and such that if (j, k) is an edge from node j to node k then the influence of j onto k does not depend on any other nodes. As an example, the interactions in the classical Kuramoto model [3, 4] where the phase $\theta_k \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ of oscillator $k \in \{1, \dots, N\}$ evolves according to

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$$\dot{\theta}_k := \frac{\mathrm{d}}{\mathrm{d}t}\theta_k = \omega_k + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_k), \tag{1}$$

with intrinsic frequency $\omega_k \in \mathbb{R}$ and subject to coupling strength K. In the Kuramoto model, the interactions are all-to-all (i.e., the underlying graph is the complete graph) but pairwise, that is, the influence of node j onto node k is determined by $\sin(\theta_j - \theta_k)$ which does not depend on the state of other nodes. This property allows to generalize the Kuramoto model to arbitrary graphs [5]. Sakaguchi generalized the Kuramoto model by incorporating a phase-shift parameter $\alpha \in \mathbb{T}$ in the interactions function [6].

Recently, the dynamics of networks with nonpairwise interactions—interactions containing nonlinear terms of more than two nodes—have attracted significant attention; cf. [7, 8] for recent reviews as well the other chapters in this book. Such network dynamical systems have been studied in their own right as generalizations of dynamics on graphs to "higher-order" algebraic objects such as simplicial complexes or hypergraphs. Intuitively speaking, a simplicial complex or hypergraph is an object on a number of nodes that may not only contain edges between pairs of nodes but also simplices that are spanned by three or more nodes. For a network dynamical system on a simplex or hypergraph, the interactions along such a simplex corresponds to a nonlinear term in the state variables of the nodes that span it. For example, Skardal and Arenas [9, 10] considered a generalization of the Kuramoto model

$$\begin{aligned}\dot{\theta}_k = \omega_k &+ \frac{K_2}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_k) + \frac{K_3}{N^2}\sum_{j,l=1}^{N}\sin(2\theta_l - \theta_j - \theta_k) \\ &+ \frac{K_4}{N^3}\sum_{j,l,m=1}^{N}\sin(\theta_j + \theta_l - \theta_m - \theta_k),\end{aligned} \tag{2}$$

where K_2 and K_3, K_4 are the coupling strength of the pairwise and nonpairwise interactions, respectively. Here terms such as $\sin(2\theta_l - \theta_j - \theta_k)$ describe the nonadditive joint influence of nodes l, j onto node k. These nonadditive terms can change the properties of the collective dynamics as one may expect [11]: For (2) they lead to a change in the criticality of the synchronization transition [10].

Nonadditively coupled phase oscillator networks—such as (2)—also arise as phase approximations of weakly coupled nonlinear oscillator networks. In other words, they can be *derived* from more general oscillator networks through phase reduction [12, 13]. In this case, the phase dynamics (2) reflect the effective dynamics of the corresponding nonlinear oscillator network [14] and nonadditive terms can reflect the effect of the nonlinearities as the dynamics deviate from the original limit cycle. For example, a globally coupled network of oscillatory nodes close to a Hopf bifurcation has the effective phase dynamics

$$\begin{aligned}\dot{\theta}_k = \omega &+ \sum_{j=1}^{N} g_2(\theta_j - \theta_k) + \sum_{j,l=1}^{N} g_3(\theta_j + \theta_l - 2\theta_k) \\ &+ \sum_{j,l=1}^{N} g_4(2\theta_j - \theta_l - \theta_k) + \sum_{j,l,m=1}^{N} g_5(\theta_j + \theta_l - \theta_m - \theta_k)\end{aligned} \tag{3}$$

up to some order of approximation as shown in [15], where g_2, g_3, g_4, g_5 are 2π-periodic coupling functions. Thus, the dynamics of the phase reduction (3) reflect the effective dynamics of the underlying nonlinear oscillator networks and can reveal the possibility for chaotic phase dynamics [16]. Note that phase dynamics with nonpairwise interaction terms can arise independent of whether the nonlinear oscillator network has pairwise or nonpairwise coupling [14, 17].

In this chapter, we review recent progress on phase reductions in symmetric systems and their effective phase dynamics. We will explicitly also discuss these systems from the perspective of symmetry. First, we will outline the phase reduction of generically coupled symmetric systems close to a Hopf bifurcation [15]; equation (3) yields the resulting phase dynamics to higher order. The phase reduction is based on the calculation of the equivariants of the system. Second, we analyze the phase dynamics (3) and show that due to the inclusion of higher-order terms, chaotic dynamics can arise; see [16]. These dynamics arise in globally coupled networks. Third, we will analyze a variation of (3) that allows to introduce a nontrivial network structure. The resulting equations determine the dynamics of coupled populations of phase oscillator networks, where the coupling within populations and between populations is distinct. We summarize results from a series of papers [18, 19, 20] showing that the network dynamics can not only show localized frequency synchrony (i.e., frequencies are synchronized for some populations but not for others) akin to chimeras [21, 22] but the location of synchrony can also wander around the network through heteroclinic connections. We conclude with some remarks in the final section.

2 Symmetric normal forms and higher-order interactions

An important tool to understand and classify bifurcations of dynamical systems is transformation to a normal form: This a simplest form of nonlinear equation that locally explains the dynamics for all generic cases. In the next subsection we briefly recall relevant ideas from symmetric Hopf bifurcation before applying it to the problem of phase reduction near such a Hopf bifurcation; more details are in [15]. The main result of this section is to show that phase equations (3) with nonpairwise interactions arise as higher-order approximations of the dynamics for symmetric coupled oscillator networks with generic interaction close to a Hopf bifurcation.

2.1 Hopf bifurcation with $\mathbf{S}_N$ Symmetry

In the general theory of symmetric (equivariant) dynamical systems [23] we study a system of ordinary differential equations (ODEs)

$$\dot{x} = f(x, \lambda) \tag{4}$$

with $x \in V, \lambda \in \mathbb{R}$, where V is a finite-dimensional space, λ is the bifurcation parameter, and f is a symmetric function.

We say that an invertible $n \times n$ matrix γ is a *symmetry* of (4) if $f(\gamma x, \lambda) = \gamma f(x, \lambda)$ for all $x \in V, \lambda \in \mathbb{R}$. A consequence of this is that if $x(t)$ is a solution to (4), then so is $\gamma x(t)$. For periodic solutions, if $x(t)$ is a a T-periodic solution of (4) then so is $\gamma x(t)$. Uniqueness of solutions to the initial problem for (4) implies that the trajectory of $x(t)$ and $\gamma x(t)$ are either disjoint, in which case we have a new periodic solution, or identical, in which case they differ only by a phase shift, that is, $x(t) = \gamma x(t - t_0)$ for some t_0. In this case we say that the pair (γ, t_0) is a symmetry of the periodic solution $x(t)$. Symmetries of periodic solutions have both a spatial component γ and a temporal component t_0.

Bifurcation Theory investigates how solutions to differential equations can branch as a parameter is varied. Assume that $x = 0$ is an equilibrium of (4) for any λ. The symmetry of f imposes restrictions on the bifurcations that can occur as λ is varied. These can be a steady-state bifurcation, when an eigenvalue of the Jacobian $\mathrm{d}f_\lambda(0)$ of f at $x = 0$ passes through 0 (without loss of generality at $\lambda = 0$) or a Hopf bifurcation, when a pair of complex conjugate eigenvalues of $\mathrm{d}f_\lambda(0)$ crosses the imaginary axis with nonzero speed at $\pm\omega i, \omega \neq 0$ where $i = \sqrt{-1}$.

The problem of N identical and identically interacting smooth (C^∞) dynamical systems on $x_k \in \mathbb{R}^d$ ($d \geq 2$) that simultaneously undergo a Hopf bifurcation is considered in [15]. In such a case the dynamics close to the Hopf bifurcation can be approximated (beyond first order) by a phase oscillator system of the form (3). Specifically, consider the coupled ordinary differential equations

$$\begin{aligned} \dot{x}_1 &= H_\lambda(x_1) + \epsilon h_{\lambda,\epsilon}(x_1; x_2, \dots, x_N) \\ \vdots \;&\;\vdots\;\;\vdots \\ \dot{x}_N &= H_\lambda(x_N) + \epsilon h_{\lambda,\epsilon}(x_N; x_1, \dots, x_{N-1}). \end{aligned} \tag{5}$$

The parameter $\epsilon \in \mathbb{R}$ is such that the system completely decouples for $\epsilon = 0$. We now assume that each system undergoes a Hopf bifurcation of $x = 0$ when $\lambda \in \mathbb{R}$ passes through zero for $\epsilon = 0$. We assume that the uncoupled system for $x \in \mathbb{R}^d$ given by $\dot{x} = H_\lambda(x)$ has a linearly stable fixed point at $x = 0$ for $\lambda < 0$ that undergoes supercritical Hopf bifurcation at $\lambda = 0$, namely $\mathrm{d}H_\lambda(0)$ has a complex pair of eigenvalues $\lambda \pm i\omega$, where $\omega \neq 0$ and all other eigenvalues μ of $\mathrm{d}H_\lambda(0)$ satisfy $Re(\mu) < -r < 0$. Without loss of generality we can assume 0 is an equilibrium in some neighborhood of $(\lambda, \epsilon) = (0, 0)$.

2.2 Normal forms for symmetric Hopf bifurcations with $\mathbf{S}_N$ symmetry

System (5) describes a population of N identical, symmetrically coupled dynamical systems with state $x_k \in \mathbb{R}^d$ ($d \geq 2$) close to a Hopf bifurcation. We assume that the coupling respects the fact that the uncoupled systems can be permuted arbitrarily, i.e., that the system is equivariant under the action of $\mathbf{S}_N$ on $\mathbb{R}^{dN}$ by permutation of coordinates. Since the system is going through a bifurcation, the dynamics can now be reduced to a center manifold using equivariant bifurcation theory [24]: We explain how this can be used as a basis for a phase oscillator description as in [15].

Note that the action of the symmetry group $\mathbf{S}_N$ means that for $\epsilon > 0$ a generic Hopf bifurcation will have centre manifold of dimension either 2 or $2N - 2$. For the uncoupled case $\lambda = \epsilon = 0$ the center manifold will be $2N$ dimensional with each coordinate x_k parametrized by $z_k \in \mathbb{C}$. That is, for $\lambda = \epsilon = 0$ points on the center manifold are given by $(z_1, \ldots, z_N) \in \mathbb{C}^N$. The system on the center manifold is

$$\dot{z}_1 = f_\lambda(z_1) + \epsilon g_\lambda(z_1; z_2, \ldots, z_N) + O(\epsilon^2) \tag{6}$$

etc, where $z \in \mathbb{C}^N$ and we have changed coordinates so that for $z_k = 0$ is an equilibrium that undergoes generic supercritical Hopf bifurcation at $\lambda = 0$. Note that for $N > 1$ this will not be a generic Hopf bifurcation, but still we can assume $f_0(0) = 0$ and $\mathrm{d}f_0(0)$ has a pair of purely imaginary eigenvalues $\pm i\omega$ that pass transversely through the imaginary axis with non-zero speed on changing λ.

The reduced system (6) has symmetries. First, the action of $\gamma \in \mathbf{S}_N$ on $\mathbb{C}^N$ is by permutation of coordinates

$$\gamma(z_1, \ldots, z_N) = (z_{\gamma^{-1}(1)}, \ldots, z_{\gamma^{-1}(N)}), \tag{7}$$

where $(z_1, \ldots, z_N) \in \mathbb{C}^N$ meaning $g_\lambda(z_1; z_2, \ldots, z_N)$ is symmetric under all permutations of the last $N - 1$ arguments. Second, Poincaré–Birkhoff normal form theory [23] means that to all polynomial orders we can assume there is a normal form symmetry given by the action of $\mathbb{T}$ on $\mathbb{C}^N$ so that $\theta \in \mathbb{T}$ acts by

$$\theta(z_1, \ldots, z_N) = e^{i\theta}(z_1, \ldots, z_N). \tag{8}$$

The symmetries (7) and (8) restrict the possible terms that can appear in the normal form; we can characterize these by finding the possible equivariants, one order at a time. Suppose $N \geq 4$. Let $f : \mathbb{C}^N \to \mathbb{C}^N$ be $\mathbf{S}_N \times \mathbb{T}$-equivariant with respect to the action (7), (8) with polynomial components of degree lower or equal than 3. From results in [24, Section 2.1.2] we can write $f = (f_1, f_2, \ldots, f_N)$ where

$$f_1(z_1, z_2, \ldots, z_N) = \sum_{i=-1}^{11} a_i h_i(z_1, z_2, \ldots, z_N) \tag{9}$$

with the other equations obtained by permutation, where the h_i are equivariants listed in [15] and $a_j \in \mathbb{C}$ are constants.

Following [15], this means we can write the equation for $\dot{z}_1$ from (6) in Poincaré-Birkhoff normal form [23] as the $\mathbf{S}_N \times \mathbb{T}$-equivariant system

$$\dot{z}_1 = U(z_1) + \epsilon F_1(z_1, \ldots, z_N, \epsilon), \tag{10}$$

where the third order truncated expression for F_1 is given in (35) and the other derivatives $\dot{z}_j$ are obtained by permutation of the indices.

2.3 Perturbations from the uncoupled limit

We assume the Hopf bifurcation of (6) at $\lambda = 0$ has special structure: Following [15] we assume there is an "uncoupled limit" corresponding to $\epsilon = 0$. This extra structure means that

$$\dot{z_1} = U(z_1) := V(z_1)z_1 := \left[\lambda + i\omega + a_1|z_1|^2 + \tau(z_1)\right] z_1, \tag{11}$$

and we write $V(z_1) = V_R(z_1) + iV_I(z_1)$. Note that the uncoupled Hopf is supercritical meaning $a_{1R} < 0$. We seek solutions of (11) of the form

$$z_1(t) = R_1(t)e^{i\phi_1(t)} = R_1(t)e^{i[\Omega t + \psi_1(t)]} \tag{12}$$

for some $R_1(t)$, $\psi_1(t)$ and constant Ω. Substituting this into (11), we require

$$\dot{R}_1 + iR_1\left[\Omega + \dot{\psi}_1\right] = R_1V_R(R_1) + iR_1V_I(R_1)$$

where

$$V_R(R_1) = \lambda + a_{1R}R_1^2 + \tau_R(R_1^2), \qquad V_I(R_1) = \omega + a_{1I}R_1^2 + \tau_I(R_1).$$

From this, it is clear that for small enough $\lambda > 0$ and $\epsilon = 0$ there is a stable periodic orbit at fixed $R_1 = R_* > 0$ such that $V_R(R_*) = 0$, with angular frequency $\Omega = V_I(R_*)$ and arbitrary but fixed phase ψ_1. More precisely, [15] shows that on solving $V_R(R_*) = 0$, we obtain

$$\begin{aligned} R_*^2 &= \frac{\lambda}{-a_{1R}} + O(\lambda^2), \\ \Omega &= V_I(R_*^2) = \omega + a_{1I}R_*^2 + \tau(R_*) = \omega + \frac{a_{1I}}{-a_{1R}}\lambda + O(\lambda^2). \end{aligned} \tag{13}$$

This implies there is a $\lambda_0 > 0$ such that for any $0 < \lambda < \lambda_0$ there is a stable periodic orbit (12) satisfying (13).

2.4 Reduction to phase oscillators

The final stage of the reduction undertaken by [15] is to show that, even though the uncoupled limit cycles for $\lambda > 0$ are weakly stable, the normal form can gives an explicit reduction to coupled phase oscillators as long as $\epsilon = o(\lambda)$. This involves some coordinate changes to ensure that standard results from normally hyperbolic invariant manifolds can be applied, followed by an averaging approximation. Since we will be dealing with multiple timescales here, we will write out the temporal derivatives $\frac{\mathrm{d}}{\mathrm{d}t}$ explicitly in this section.

For $\epsilon = 0$ and any $0 < \lambda < \lambda_0$ there is a stable invariant torus given by

$$(z_1, \ldots, z_N) = (R_* e^{i(\Omega t+\psi_1)}, \ldots, R_* e^{i(\Omega t+\psi_N)}), \tag{14}$$

parametrized by the phases $(\psi_1, \ldots \psi_N) \in \mathbb{T}^N$. This invariant torus is foliated by neutrally stable periodic orbits with period $2\pi/\Omega$ and so for each $0 < \lambda < \lambda_0$, the torus is normally hyperbolic. The theory of normal hyperbolicity [25] implies there is an ϵ_0 such that for $0 < \epsilon < \epsilon_0$ the invariant torus persists and is C^r-smooth for arbitrarily large r. Note that reducing r will restrict the ϵ_0: We will need $r \geq 5$ for the approximation to be valid.

We write $a_k = \alpha_k e^{i\theta_k} = a_{kR} + i a_{kI}$ and $z_k(t) = R_k(t) e^{i(\Omega t+\psi_k(t))} = [R_* + \rho_k(t)] e^{i(\Omega t+\psi_k(t))}$ In particular, we seek solutions such that ρ_k is small and ψ_k varies slowly with t. Re-writing (10), note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_1 + iR_1\left[\Omega + \frac{\mathrm{d}}{\mathrm{d}t}\psi_1\right] = U(R_1) + \epsilon F_1(z_1, \ldots, z_N, 0) e^{-i(\Omega t+\psi_1)} + O(\epsilon^2). \tag{15}$$

Writing U in real and imaginary parts and expanding for small ρ_1, [15] show that $A(\lambda) := U_R'(R_*)/\lambda,\ \ B(\lambda) := V_I'(R_*)/(\lambda^{1/2})$, so that $U(R_1) = \lambda A(\lambda)\rho_1 + iR_1[\Omega + \lambda^{1/2}B(\lambda)\rho_1] + O(\rho_1^2)$. This implies that (15) can be expressed as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_1 + iR_1\left[\Omega + \frac{\mathrm{d}}{\mathrm{d}t}\psi_1\right] &= \lambda A(\lambda)\rho_1 + iR_1[\Omega + \lambda^{1/2}B(\lambda)\rho_1] \\ &\quad + \epsilon F_1(z_1, \ldots, z_N) e^{-i(\Omega t+\psi_1)} + O(\epsilon^2)\end{aligned} \tag{16}$$

Recalling from (13) that $R_*^2 = \lambda/(-a_{1R}) + O(\lambda^2)$, $U(R_*) = U_R(R_*) + iV_I(R_*)R_* = (\lambda + a_{1R}R_*^2 + \tau(R_*))R_*$, $\tau(z) = O(z^4)$, and $\tau'(z) = O(z^3)$ so one can show $A(\lambda) = 1 + 3a_{1R}/(-a_{1R}) + O(\lambda) = -2 + O(\lambda)$ Similarly, one can show $B(\lambda) = 2a_{1I}/\sqrt{-a_{1R}} + O(\lambda)$. In particular, for $\lambda \to 0$ there are finite limits $A(0) = -2$, $B(0) = 2a_{1I}/\sqrt{-a_{1R}}$. By careful expansion of the terms in F_1 and taking real parts of (16) gives the expression (36). The equivalent equation for ψ_1 is obtained by taking imaginary parts of (15) and after cancellation and dividing by R_1, gives (37).

In terms of slow time $T = \lambda t$, calculations in [15] show that (36,37) can be written as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}T} r_j &= A(\lambda) r_j + f_j + O(\epsilon) \\
\frac{\mathrm{d}}{\mathrm{d}T} \psi_j &= \epsilon \lambda^{-1} \left[C(\lambda) r_j + h_j \right] + O(\epsilon^2)
\end{aligned}
\tag{17}
$$

for $j = 1, \ldots, N$. Note that f_j and h_j are trigonometric polynomials and A, C, f_j and h_j have finite limits as $\lambda \to 0$. Hence (17) gives a slow timescale for evolution of ψ_j as long as $\epsilon = o(\lambda)$. Defining scaled amplitude variables $\sigma_j := r_j + \frac{f_j(\psi_1, \ldots, \psi_{N-1})}{A(\lambda)}$, system (17) can be expressed as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}T} \sigma_j &= A(\lambda) \sigma_j + O(\epsilon) \\
\frac{\mathrm{d}}{\mathrm{d}T} \psi_j &= \epsilon \lambda^{-1} \left[C(\lambda) \sigma_j + H_j \right] + O(\epsilon^2),
\end{aligned}
\tag{18}
$$

where $H_j = h_j - f_j C(\lambda)/A(\lambda)$. We write $H_j = H_j^0 + \lambda H_j^1 + O(\lambda^2)$, where $H_j^0 = h_j^0 - C(0)/A(0) f_j^0$, $H_j^1 = R_*^2(\lambda)\left[h_j^1 - f_j^1 C(0)/A(0)\right]/\lambda - f_j^0 [C'(0)A(0) - A'(0)C(0)]/A(0)^2$, which is a trigonometric polynomial in $\psi_k - \phi_j$. It can be shown that H_j^0 only involves pairwise coupling while H_j^1 includes coupling of up to four phases (and on $\alpha_2, \ldots, \alpha_{11}$).

After further manipulations [15], the reduced equations for ϕ_j can be written in the form

$$
\frac{\mathrm{d}}{\mathrm{d}t} \phi_j = \Omega + \epsilon \left[H_j^0 + \lambda H_j^1 \right] \tag{19}
$$

where the phase differences $\psi_j - \psi_k = \phi_j - \phi_k$ for all j and k, and the approximation will be close for times $0 < t < \tilde{t}$ with $\tilde{t} = O(\epsilon^{-1}\lambda^{-2})$. For $k = -1, 1, \ldots, 11$ we define β_k and γ_k such that for all θ we have $\beta_k \cos(\gamma_j + \theta) := \alpha_k \sin(\theta_k + \theta) - \frac{C(0)}{A(0)} \alpha_k \cos(\theta_k + \theta)$. Then we can write (19) in the form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \phi_j &= \Omega + \epsilon H_1 \\
&= \tilde{\Omega}(\phi, \epsilon) + \frac{\epsilon}{N} \sum_{k=1}^{N} g_2(\phi_k - \phi_j) + \frac{\epsilon}{N^2} \sum_{k,\ell=1}^{N} g_3(\phi_k + \phi_\ell - 2\phi_j) \\
&\quad + \frac{\epsilon}{N^2} \sum_{k,\ell=1}^{N} g_4(2\phi_k - \phi_\ell - \phi_j) + \frac{\epsilon}{N^3} \sum_{k,\ell,m=1}^{N} g_5(\phi_k + \phi_\ell - \phi_m - \phi_j)
\end{aligned}
\tag{20}
$$

where the various coupling functions have the form

$$\begin{aligned}
\tilde{\Omega}(\phi,\epsilon) &= \Omega + R_*^2\varepsilon\left[\beta_4\cos\gamma_4 + \frac{\beta_5}{N^2}\sum_{j,k}\cos(\gamma_5+\phi_j-\phi_k)\right]\\
g_2(\varphi) &= \beta_{-1}\cos(\gamma_{-1}+\varphi) + R_*^2\left[\beta_2\cos(\gamma_2-\varphi)+\beta_3\cos(\gamma_3+\varphi)\right.\\
&\qquad \left.+\beta_6\cos(\gamma_6+2\varphi)+\beta_8\cos(\gamma_8+\varphi)+\beta_{10}\cos(\gamma_{10}+\varphi)\right]\\
&\qquad -\lambda\frac{C'(0)A(0)-A'(0)C(0)}{A(0)^2}\alpha_{-1}\cos(\theta_{-1}+\varphi)\\
g_3(\varphi) &= R_*^2\left[\beta_7\cos(\gamma_7+\varphi)\right]\\
g_4(\varphi) &= R_*^2\left[\beta_9\cos(\gamma_9+\varphi)\right]\\
g_5(\varphi) &= R_*^2\left[\beta_{11}\cos(\gamma_{11}+\varphi)\right].
\end{aligned} \tag{21}$$

To summarize, we have illustrated how the reduction of [15] demonstrates that, to first order, the generic dynamics of N weakly coupled coupled identical oscillators close to a Hopf bifurcation are approximated by the Kuramoto equations (1) with an additional phase-shift parameter α, i.e., the Kuramoto–Sakaguchi equations [6]. Moreover, at second order in the bifurcation parameter λ we have phase dynamics given by (20), a system very similar to (3): The phase dynamics are determined by

$$\dot{\theta}_k = \tilde{\Omega}(\theta,\epsilon) + \varepsilon\left(F_k^{(2)}(\theta)+F_k^{(3)}(\theta)+F_k^{(4)}(\theta)\right) \tag{22}$$

for $k \in \{1,\dots,N\}$ with

$$F_k^{(2)}(\theta) = \frac{1}{N}\sum_{j=1}^{N} g_2(\theta_j-\theta_k) \tag{23a}$$

$$F_k^{(3)}(\theta) = \frac{1}{N^2}\sum_{j,\ell=1}^{N} g_3(\theta_j+\theta_\ell-2\theta_k) + \frac{1}{N^2}\sum_{j,\ell=1}^{N} g_4(2\theta_j-\theta_\ell-\theta_k) \tag{23b}$$

$$F_k^{(4)}(\theta) = \frac{1}{N^3}\sum_{j,\ell,m=1}^{N} g_5(\theta_j+\theta_\ell-\theta_m-\theta_k) \tag{23c}$$

and coupling functions

$$\begin{aligned}
g_2(\phi) &= \xi_1^0\cos(\phi+\chi_1^0) + \lambda\xi_1^1\cos(\phi+\chi_1^1) + \lambda\xi_2^1\cos(2\phi+\chi_2^1)\\
g_3(\phi) &= \lambda\xi_3^1\cos(\phi+\chi_3^1)\\
g_4(\phi) &= \lambda\xi_4^1\cos(\phi+\chi_4^1)\\
g_5(\phi) &= \lambda\xi_5^1\cos(\phi+\chi_5^1)
\end{aligned} \tag{24}$$

for coefficients ξ_i^j and χ_i^j determined from (21). In particular, this next order includes pairwise, triplet and quadruplet interactions of phases.

3 Coupled phase oscillators networks with nonpairwise interactions

In this section, we recall some results from [16] and related literature that explores the phase equations (22) with higher-order interactions. For concreteness, we set $\tilde{\Omega}(\theta, \epsilon) = \omega$ and fix $\lambda = \varepsilon = 1$. That is, we consider (3) with the coupling functions

$$
\begin{aligned}
g_2(\phi) &= \xi_1 \cos(\phi + \chi_1) + \xi_2 \cos(2\phi + \chi_2) \\
g_3(\phi) &= \xi_3 \cos(\phi + \chi_3) \\
g_4(\phi) &= \xi_4 \cos(\phi + \chi_4) \\
g_5(\phi) &= \xi_5 \cos(\phi + \chi_5)
\end{aligned} \tag{25}
$$

such that for general N the function g_2 determines pairwise, g_3, g_4 triplet and g_5 quadruplet interaction.

3.1 Symmetric phase oscillator networks

The symmetries of the phase equations (3) have consequences for the dynamics. Here the phase equations "inherit" symmetries from the generically coupled system (5): First, the phase equations are symmetric with respect to the rotation by a common angle. As a consequence, we may assume—without loss of generality—that the phase of the first oscillator θ_1 is always equal to zero by going into a co-rotating reference frame that moves with oscillator $k = 1$. Second, the $\mathbf{S}_N$-symmetry acts by permuting oscillators. By using the permutational symmetry, we may assume that the phases are in ascending order. Note that these properties are due to the symmetry alone, independent of whether the phase oscillators are subject to pairwise or nonpairwise interactions; cf. [26].

Because of the symmetries, we do not need to consider the dynamics of (3) on the entire phase space $\mathbb{T}^N$ but we can restrict the analysis to a smaller but still representative subset. Specifically, define the *canonical invariant region* (CIR) [26] as the set of phases

$$C = \left\{ \theta \in \mathbb{T}^N \mid 0 = \theta_1 < \theta_2 < \cdots < \theta_N < 2\pi \right\}. \tag{26}$$

The CIR is a $(N-1)$-simplex whose boundary consists of *cluster configurations* where the phases of two or more oscillators are equal. The intersection of all cluster configurations is the fully synchronized phase configuration

$$\mathrm{S} := \left\{ (\theta_1, \ldots, \theta_N) \in \mathbb{T}^N \mid \theta_k = \theta_{k+1} \right\} \tag{27a}$$

where the phases of all oscillators are equal. At the centroid of the CIR is the *splay phase configuration*

$$\mathrm{D} := \left\{ (\theta_1, \dots, \theta_N) \in \mathbb{T}^N \;\middle|\; \theta_{k+1} = \theta_k + \frac{2\pi}{N} \right\}, \tag{27b}$$

where the oscillator phases are uniformly distributed on the circle. As fixed point subsets of symmetries—e.g., S is invariant under ay permutation of the oscillator indices—the cluster configurations are also dynamically invariant.

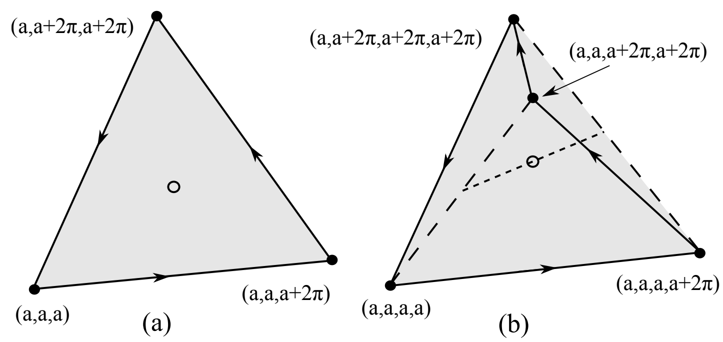


Fig. 1 Structure of the canonical invariant region C for $N = 3$ and $N = 4$ (see [26]). Panels (a,b) show C as an orthogonal projection of into $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The edges of C for (a) and the faces of C for (b) are points where two oscillators have the same phase. The filled circles represent the fully synchronous phase configuration S; the open circle represents the splay phase configuration D in C. In (b) the solid lines correspond to 3:1 cluster configurations where three oscillators have the same phase and one is distinct while the long-dashed lines correspond to 2:2 cluster configurations of two clusters of two oscillators. The short-dashed lines are points $(a, b, a+\pi, b+\pi)$. For any N there is a residual $\mathbb{Z}/N\mathbb{Z}$ symmetry that "rotates" the canonical invariant region (the direction of rotation is indicated by the arrows in (b)). Overall $(N-1)!$ symmetric copies of C pack a generating region for the torus. [Reprinted with permission from [16].]

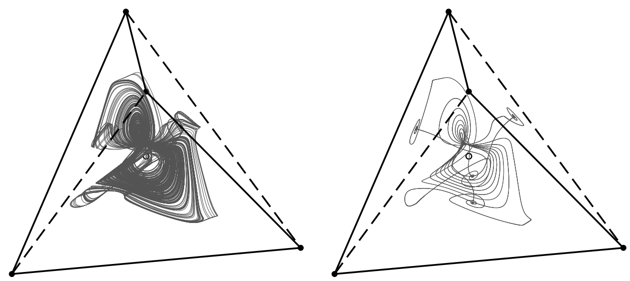
The CIR for $N = 3$ and $N = 4$ is illustrated in Figure 1. For $N = 3$ the CIR is a two-dimensional simplex and we cannot expect any chaotic dynamics [27]. For $N = 4$ the CIR is three-dimensional which does not preclude chaotic dynamics. If the coupling is pairwise with a single harmonic as in the Kuramoto–Sakaguchi model, there is additional degeneracy that prevent chaotic attractors to emerge [28]. If the coupling is pairwise but one allows for higher harmonics in the coupling function (cf. [29]) one may observe chaotic dynamics for a pairwise coupling function with four harmonics [30]. But no further examples of coupling functions with fewer harmonics are known for fully symmetric systems with pairwise interactions.

3.2 Chaos in globally coupled phase oscillator networks with higher-order interactions

The phase dynamics of (3) with nonpairwise coupling mediated by the functions (25) can give rise to chaotic dynamics. Following [16] we fix Fourier coefficients

$$\xi = (-0.3, 0.3, 0.02, 0.8, 0.02) \tag{28}$$

while varying the phase shifts χ. Calculating the maximal Lyapunov exponent λ_{max} reveals a region in parameter space where $\lambda_{max} > 0$ and chaotic attractors appear in the canonical invariant region. Figure 2(a) shows a solution $\theta(t)$ in C for $\chi = (0.154, 0.318, 0, 1.74, 0)$. While the attracting set lies in the interior of C, the trajectories on the chaotic attractor approach come close to its boundary that consist where oscillators are clustered. Indeed, a small variation shows periodic dynamics that appear to be close to a heteroclinic network: Figure 2(b) shows a stable periodic orbit close to such a heteroclinic network for parameters $\chi = (0.2, 0.316, 0, 1.73, 0)$.



(a) Chaotic attractor

(b) Periodic orbit near heteroclinic network

Fig. 2 Heteroclinic networks organize chaotic behavior in C for networks of $N = 4$ oscillators; line styles on the boundary of C are as in Fig. 1. The right panel shows a trajectory with positive maximal Lyapunov exponents for phase shift parameters $\chi = (0.154, 0.318, 0, 1.74, 0)$ that comes close to the boundary of C. For nearby parameter values $\chi = (0.2, 0.316, 0, 1.73, 0)$ there is an attracting periodic orbit close to a heteroclinic network involving two saddle equilibria, one a saddle-focus, on the boundary of C. [Reprinted with permission from [16].]

Since the equilibria on the boundary include a saddle-focus, the chaotic dynamics appear to arise through a nonstandard Shilnikov saddle-focus scenario [31]. Indeed, modulo the residual $\mathbb{Z}_N$ symmetry on C the heteroclinic network on the boundary of C involves two equilibria. Grines and Osipov [32] took this observation as a starting point to determine what homoclinic and heteroclinic trajectories are possible in (3) for $N = 4$ oscillators. More specifically, the symmetries of the system restrict the saddle connections that are possible between equilibria that lie on the boundary of C such as those in Figure 2(b).

While $N = 4$ is the smallest number of oscillators for which chaos can arise in the phase equations, chaotic dynamics also arise in networks with $N > 4$ phase oscillators. In [16] we gave explicit parameter values for which $\lambda_{max} > 0$ but a detailed analysis of these larger phase oscillator networks is still an outstanding problem.

4 Chimeras and other creatures for multiple populations

The dynamics of a globally coupled network (3) of N identical oscillators with non-pairwise interactions is constrained by the symmetries of the system. Since the system is $\mathbf{S}_N$-equivariant, the asymptotic average frequencies $\Omega_k(\theta(0)) := \lim_{T\to\infty} \theta(T)/T$ for any initial condition $\theta(0) \in \mathbb{T}^N$ are identical[1]: We have $\Omega_k = \Omega_j$ for all $k, j \in \{1, \dots, N\}$ independent of the initial condition and the oscillators are *frequency synchronized* [21]. This restriction breaks down if the $\mathbf{S}_N$ symmetry is broken. In this section we discuss the dynamics of a generalization of (29) where the phase θ_k evolves according to

$$\begin{aligned}\dot{\theta}_k &= \omega + \sum_{j=1}^{N} a_2^{(jk)} g_2(\theta_j - \theta_k) + \sum_{j,l=1}^{N} a_3^{(jlk)} g_3(\theta_j + \theta_l - 2\theta_k)\\ &\quad + \sum_{j,l=1}^{N} a_4^{(jlk)} g_4(2\theta_j - \theta_l - \theta_k) + \sum_{j,l,m=1}^{N} a_5^{(jlmk)} g_5(\theta_j + \theta_l - \theta_m - \theta_k)\end{aligned} \tag{29}$$

where $a_2^{(jk)} \in \mathbb{R}$ and $a_3^{(jlk)}, a_4^{(jlk)}, a_5^{(jlmk)} \in \mathbb{R}$ are the coupling strength of pairwise and nonpairwise interactions. For nonhomogeneous choice of these coupling coefficients, the system (29) can describe coupled populations of phase oscillators that allow for frequency synchrony to be localized in one or more populations.

4.1 Frequency synchrony in coupled oscillator populations

Suppose that the N oscillators are grouped into M populations (assuming $N = MQ$) indexed by $\sigma \in \{1, \dots, M\}$. The first Q oscillators belong to population $\sigma = 1$, oscillators $k \in \{Q+1, \dots, 2Q\}$ to population $\sigma = 2$ etc., and we write $k = (\sigma, q)$ if oscillator k (in the linear ordering as above) corresponds to oscillator q in population σ and $\theta_{\sigma,q}$ denote its phase and $\Omega_{\sigma,q}$ the asymptotic average frequency. We now consider networks with coupling coefficients $a_2^{(jk)} = K_\mathrm{p}^{(\sigma)}/Q$ if k, j belong to population σ and $a_2^{(jk)} = 0$ otherwise so that interactions within populations are pairwise and $a_3^{(jlk)} = a_4^{(jlk)} = 0$, and $a_5^{(jlmk)} = K_\mathrm{np}^{(\sigma\tau)}/Q^3$ if and only if oscillators m, k belong to population σ and oscillators j, l to population τ and $a_5^{(jlmk)} = 0$ otherwise determine the nonpairwise interactions. With this choice of coefficients the M populations are globally and identically coupled through pairwise interactions while the nonpairwise interactions mediate the coupling between distinct populations.

The specific form of network coupling induces symmetries: The dynamical system is $(\mathbf{S}_Q \times \mathbb{T})^M$-equivariant where, for each population, $\mathbf{S}_Q$ acts by permuting the oscillators and $\mathbb{T}$ acts by shifting all oscillators of the given population by a constant.

[1] Here we assume that the limit exists; for a generalization to frequency intervals see [22].

Note that there is one phase-shift symmetry for each population. For population σ, write $\theta_\sigma = (\theta_{\sigma,1}, \dots, \theta_{\sigma,q})$ to denote the state of the population. Recall that S and D, as defined in (27), denote the synchronized and splay configurations in a network consisting of a single population. For the network of interacting populations, write

$$\theta_1 \cdots \theta_{\sigma-1} \mathrm{S} \theta_{\sigma+1} \cdots \theta_M = \left\{ \theta \in \mathbb{T}^N \;\middle|\; \theta_\sigma \in \mathrm{S} \right\} \tag{30a}$$

$$\theta_1 \cdots \theta_{\sigma-1} \mathrm{D} \theta_{\sigma+1} \cdots \theta_M = \left\{ \theta \in \mathbb{T}^N \;\middle|\; \theta_\sigma \in \mathrm{D} \right\} \tag{30b}$$

to indicate that population σ is fully phase synchronized or in splay phase. Because of the symmetry these sets are dynamically invariant. We extend this notation to intersections of the sets (30), so that $\mathrm{S} \cdots \mathrm{S}$ (M times) denotes cluster states where all populations are fully phase synchronized and $\mathrm{D} \cdots \mathrm{D}$ (M times) the set where all populations are in splay phase.

These invariant sets can display frequency synchrony that is *localized* in a specific part of the network: The oscillators within one populations are frequency synchronized while oscillators in different populations are not. This is a characterizing feature of a *weak chimera* [21, 22]. To see this take $K_{\mathrm{np}}^{(\sigma\tau)} = 0$, that is, there is no coupling between different populations. If population σ is phase synchronized, that is, $\theta_\sigma(0) = (\theta_{\sigma,1}(0), \dots, \theta_{\sigma,Q}(0)) \in \mathrm{S}$ we have

$$\Omega_{\sigma,k}(\theta_\sigma(0)) = \omega + K_{\mathrm{p}}^{(\sigma)} g_2(0). \tag{31}$$

Similarly, if population σ is phase synchronized, that is, $\theta_\sigma(0) \in \mathrm{D}$ we have

$$\Omega_{\sigma,k}(\theta_\sigma(0)) = \omega + \sum_{j=1}^{Q} \frac{K_{\mathrm{p}}^{(\sigma)}}{Q} g_2\left(\frac{2\pi j}{Q}\right). \tag{32}$$

Since these two values are distinct for a generic pairwise coupling function g_2, we have that any set of the form $\mathrm{DS} \cdots \mathrm{S}$ has populations with distinct frequency. Moreover, this property is preserved for sufficiently small $\left|K_{\mathrm{np}}^{(\sigma,\tau)}\right| > 0$.

4.2 Heteroclinic cycles and networks

While much attention has focused on localized frequency and chimeras to be attractors in network dynamical systems [33], the nonpairwise interactions also allow for heteroclinic dynamics that connect different localized frequency synchrony patterns. For us, a heteroclinic cycle consists of a finite number of normally hyperbolic invariant sets ξ_s, $s \in \{1, \dots, S\}$, together with trajectories $[\xi_s \to \xi_{s+1}]$ (indices are taken modulo S) that lie in the intersection of the unstable manifold of ξ_s and the stable manifold of ξ_{s+1}; cf. [34, 35]. Trajectories close to a heteroclinic cycle show "switching dynamics": The trajectory will spend time close to one of the invariant sets ξ_s before a rapid transition to the next set.

For small networks that consist of $M = 3$ populations of $Q \in \{2, 3\}$ we can explicitly give conditions for the existence of robust heteroclinic cycles that are asymptotically stable. Here we outline the results for $Q = 2$ oscillators and refer to [18, 19, 20] for more detailed results.

Theorem Consider $M = 3$ populations of $Q = 2$ oscillators with coupling functions $g_2(\vartheta) = \sin(\vartheta + \alpha_2) + r\sin(2(\vartheta + \alpha_2))$ and $g_4(\vartheta) = \sin(\vartheta + \alpha_4)$ and nonpairwise coupling parameters $K_{\mathrm{np}}^{(\sigma\tau)} = -K$ if $\tau = \sigma - 1$, $K_{\mathrm{np}}^{(\sigma\tau)} = K$ if $\tau = \sigma + 1$, and $K_{\mathrm{np}}^{(\sigma\tau)} = 0$ if $\tau = \sigma$ and . Then there exists an open set of parameter values K, r, α_2, α_4 such that the coupled phase oscillator network (29) with higher-order interactions has an asymptotically stable robust heteroclinic cycle. □

The main ideas of the proof is as follows. First, note that because of the $\mathbf{S}_Q^M$ symmetry we can reduce the 6-dimensional dynamics to a system of 3 phase difference variables $\psi_\sigma = \theta_{\sigma,2} - \theta_{\sigma,1}$ for each population $\sigma \in \{1, 2, 3\}$. In the reduced coordinates invariant sets of the form SSS, DSS, . . . are equilibrium points. Second, we can linearize the equations close to these equilibria. This allows to write down conditions that ensure that the equilibria have the right (local) stability properties. For example, we can impose that DSS is stable in the invariant subspaces DSθ_3 and θ_1SS but unstable in the invariant subspace Dθ_2S. Moreover, we want that DDS is stable in Dθ_2S and DDθ_3 but unstable in θ_1DS. The stability conditions for the other equilibria are similar. Third, we have to ensure that there are heteroclinic connections between the equilibria: There is a connection [DSS → DDS] if there are no other equilibria in the one-dimensional invariant set Dθ_2S. This condition—as well as conditions for the other heteroclinic connections—can be explicitly expressed in terms of the coupling parameters. Fourth, we have that the resulting heteroclinic cycle is in the class of quasi-simple heteroclinic cycles; see [36]. This allows to write down explicit conditions for the stability of the resulting cycle [20]. Heteroclinic structures organize the dynamics even if these structures are broken by perturbations: Typically, periodic or chaotic dynamics appear that closely mimic the switching dynamics of the cycle.

For a larger number of populations, such heteroclinic cycles can be part of larger networks of heteroclinic connections. Existence of a heteroclinic network in $M = 4$ coupled populations of $Q = 2$ oscillators each is proved in [20]. This network consists of two cycles of the form discussed above with the difference that from the equilibrium SDSS there are two distinct heteroclinic connections [SDSS → SDDS] and [SDSS → SDSD] resulting in a network that contains two distinct heteroclinic cycles. In other words, the second population can desynchronize either the third or the fourth population. If weak noise is added to the system nearby trajectories exhibit dynamics that can follow either of the two cycles in the network. As quasi-simple heteroclinic cycles—one can calculate their stability properties explicitly.

5 Outlook

In this chapter, we have reviewed results from [15, 16] and related literature [18, 19, 20] that discuss how nonpairwise interactions in phase oscillator networks arise naturally in phase reductions and their consequences for the phase dynamics. The framework of symmetric Hopf bifurcation theory helps organize and understand the importance these nonpairwise interactions of the phase dynamics in a rigorous manner. We have discussed the dynamics of the resulting phase oscillator networks and a generalization thereof that allows for a more general network structure other than global and identical coupling.

One of the more puzzling aspects of higher order interactions in phase oscillator networks is that it seems to be hard to characterize the dynamical restrictions imposed by having only pairwise interactions. With a few exceptions (e.g., the scenarios for cluster state stabilities considered in [15]), pairwise coupled systems are remarkably rich in their dynamics. This may be the reason why higher order interactions have only recently become of interest. In another approach, Komarov and Pikovsky [37] consider a phase oscillator system of the form

$$\dot{\phi}_k = \Omega + \omega + S(\phi_k)F \tag{33}$$

where F depends on the mean fields. They show that the second order phase dynamics are given by

$$\dot{\theta}_k = \omega + \varepsilon\left(F_k^{(2)}(\theta) + F_k^{(3)}(\theta)\right) \tag{34}$$

with $F_k^{(2)}(\theta) = \frac{1}{N}\sum_{j=1}^{N} g_2(\theta_j - \theta_k)$, $F_k^{(3)}(\theta) = \frac{1}{N^2}\sum_{j,\ell=1}^{N} g_3(\theta_j + \theta_\ell - 2\theta_k)$ and the interactions between the phases are given by $g_2(\phi) = \xi_1\cos(\phi + \chi_1) + \xi_2\cos(2\phi + \chi_2)$, $g_3(\phi) = \xi_3\cos(\phi + \chi_3)$. This is a special case of (29) where the coupling functions g_4, g_5 are zero. Similarly, the phase oscillator network (2) considered by Skardal and Arenas [9, 10] is a special case of (29) as mentioned above.

While phase oscillators with nonpairwise interactions can be analyzed in their own right, it is instructive to remember that such interaction terms arise in phase reductions as discussed here. The nonpairwise interactions capture the nonlinearities of the (unreduced) nonlinear oscillator system and their interactions. Thus, phase oscillator networks with nonpairwise interactions can capture some properties of their dynamics. It seems natural to assume that it is especially when one moves away from the weakly coupled limit that higher-order interactions will become decisive: For example, the discontinuous synchronization transitions in [38] appear in a strongly-coupled oscillator network, while [39] also consider effects that can be viewed as associated with higher-order interactions.

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Truncated expressions for phase and amplitude dynamics

For completeness, the expression for the cubic truncated Hopf normal form from [15] is

$$\begin{aligned}F_1 = \Bigg[& a_{-1}\frac{1}{N}\sum_j z_j + a_2\frac{z_1^2}{N}\sum_j \bar{z}_j + a_3\frac{|z_1|^2}{N}\sum_j z_j \\ & + a_4\frac{z_1}{N}\sum_j |z_j|^2 + a_5\frac{z_1}{N^2}\sum_{j,k} z_j\bar{z}_k + a_6\frac{\bar{z}_1}{N}\sum_j z_j^2 \\ & + a_7\frac{\bar{z}_1}{N^2}\sum_{j,k} z_j z_k + a_8\frac{1}{N}\sum_j |z_j|^2 z_j + a_9\frac{1}{N^2}\sum_{j,k} z_j^2\bar{z}_k \\ & + a_{10}\frac{1}{N^2}\sum_{j,k} z_j|z_k|^2 + a_{11}\frac{1}{N^3}\sum_{j,k,\ell} z_j z_k\bar{z}_\ell \Bigg] + \tilde{F}_1 + O(\epsilon).\end{aligned} \tag{35}$$

where the $\epsilon = 0$ error term is $\tilde{F}_1 = O(|z|^5)$, $\sum_i$ denotes $\sum_{i=1}^N$, $\sum_{i,j}$ denotes $\sum_{i=1}^N\sum_{j=1}^N$ and $\sum_{i,j,k}$ denotes $\sum_{i=1}^N\sum_{j=1}^N\sum_{k=1}^N$. This can be recovered from [24].

The radial dynamics for phase reduction is

$$\begin{aligned}\dot{\rho}_1(t) = \lambda A(\lambda)\rho_1 + \epsilon \Big[& \alpha_{-1}\textstyle\sum_j' R_j\cos(\theta_{-1} + \psi_j - \psi_1) \\ & +\alpha_2\textstyle\sum_j' R_1^2 R_j\cos(\theta_2 + \psi_1 - \psi_j) \\ & +\alpha_3\textstyle\sum_j' R_1^2 R_j\cos(\theta_3 + \psi_j - \psi_1) \\ & +\alpha_4\textstyle\sum_j' R_1 R_j^2\cos\theta_4 \\ & +\alpha_5\textstyle\sum_{j,k}' R_1 R_j R_k\cos(\theta_5 + \psi_j - \psi_k) \\ & +\alpha_6\textstyle\sum_j' R_1 R_j^2\cos(\theta_6 + 2\psi_j - 2\psi_1) \\ & +\alpha_7\textstyle\sum_{i,j}' R_1 R_i R_j\cos[\theta_7 + (\psi_i - \psi_1) + (\psi_j - \psi_1)] \\ & +\alpha_8\textstyle\sum_j' R_j^3\cos(\theta_8 + \psi_j - \psi_1) + \\ & +\alpha_9\textstyle\sum_{j,k}' R_j^2 R_k\cos(\theta_9 + 2\psi_j - \psi_k - \psi_1) \\ & +\alpha_{10}\textstyle\sum_{j,k}' R_j R_k^2\cos(\theta_{10} + \psi_j - \psi_1) \\ & +\alpha_{11}\textstyle\sum_{i,j,k}' R_i R_j R_k\cos(\theta_{11} + \psi_i + \psi_j - \psi_k - \psi_1)\Big] \\ & +O(\rho^2, \epsilon^2)\end{aligned} \tag{36}$$

where $\rho^2 = \max_j(\rho_j^2)$ and $\sum_j' a_j := \frac{1}{N}\sum_{j=1}^N a_j$, $\sum_{j,k}' a_{j,k} := \frac{1}{N^2}\sum_{j,k=1}^N a_{j,k}$, etc are the normalized sums. Similarly the phase dynamics are given by

$$
\begin{aligned}
\dot{\psi}_1(t) = \lambda^{1/2}B(\lambda)\rho_1 + \epsilon\big[&\alpha_{-1}\textstyle\sum_j'(R_j/R_1)\sin(\theta_{-1}+\psi_1-\psi_j)\\
&+\alpha_2\textstyle\sum_j' R_1R_j\sin(\theta_2+\psi_1-\psi_j)\\
&+\alpha_3\textstyle\sum_j' R_1R_j\sin(\theta_3+\psi_j-\psi_1)\\
&+\alpha_4\textstyle\sum_j' R_j^2\sin\theta_4\\
&+\alpha_5\textstyle\sum_{j,k}' R_jR_k\sin(\theta_5+\psi_j-\psi_k)\\
&+\alpha_6\textstyle\sum_j' R_j^2\sin(\theta_6+2(\psi_j-\psi_1))\\
&+\alpha_7\textstyle\sum_{i,j}' R_iR_j\sin[\theta_7+(\psi_i-\psi_1)+(\psi_j-\psi_1)]\\
&+\alpha_8\textstyle\sum_j'(R_j^3/R_1)\sin(\theta_8+\psi_j-\psi_1)\\
&+\alpha_9\textstyle\sum_{j,k}'(R_j^2R_k/R_1)\sin(\theta_9+2\psi_j-\psi_k-\psi_1)\\
&+\alpha_{10}\textstyle\sum_{j,k}'(R_jR_k^2/R_1)\sin(\theta_{10}+\psi_j-\psi_1)\\
&+\alpha_{11}\textstyle\sum_{i,j,k}'(R_iR_jR_k/R_1)\sin(\theta_{11}+\psi_i+\psi_j-\psi_k-\psi_1)\big]\\
&+\tfrac{1}{R_1}O(\rho^2,\epsilon^2)
\end{aligned}
\tag{37}
$$