Independence inheritance and Diophantine approximation for systems of linear forms

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Dedicated to Victor Beresnevich and Sanju Velani

Abstract

The classical Khintchine–Groshev theorem is a generalization of Khintchine's theorem on simultaneous Diophantine approximation, from approximation of points in $\mathbb{R}^m$ to approximation of systems of linear forms in $\mathbb{R}^{nm}$. In this paper, we present an inhomogeneous version of the Khintchine–Groshev theorem which does not carry a monotonicity assumption when $nm > 2$. Our results bring the inhomogeneous theory almost in line with the homogeneous theory, where it is known by a result of Beresnevich and Velani (2010) that monotonicity is not required when $nm > 1$. That result resolved a conjecture of Beresnevich, Bernik, Dodson, and Velani (2009), and our work resolves almost every case of the natural inhomogeneous generalization of that conjecture. Regarding the two cases where $nm = 2$, we are able to remove monotonicity by assuming extra divergence of a measure sum, akin to a linear forms version of the Duffin–Schaeffer conjecture. When $nm = 1$ it is known by work of Duffin and Schaeffer (1941) that the monotonicity assumption cannot be dropped.

The key new result is an *independence inheritance phenomenon*; the underlying idea is that the sets involved in the $((n+k)\times m)$-dimensional Khintchine–Groshev theorem ($k \geq 0$) are always k-levels more probabilistically independent than the sets involved the $(n \times m)$-dimensional theorem. Hence, it is shown that Khintchine's theorem itself underpins the Khintchine–Groshev theory.

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Contents

1 Introduction

1.1 The Lebesgue measure theory

For a sequence of balls $\Psi := (B_q)_{q\in\mathbb{N}} \subset \mathbb{R}^m/\mathbb{Z}^m$, and $n \geq 1$, write $\mathbb{I} := [0,1] = \mathbb{R}/\mathbb{Z}$ and let $\mathscr{A}_{n,m}(\Psi)$ denote the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$\mathbf{q}\mathbf{x} + \mathbf{p} \in B_{|\mathbf{q}|}$$

holds for infinitely many pairs $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$, where $|\mathbf{q}| := \max_{1\leq i\leq n} |q_i|$ is the maximum norm. If all the balls have a common center $\mathbf{y}$, and their radius is given by a function $\psi(|\mathbf{q}|)$, then we write $\mathscr{A}^{\mathbf{y}}_{n,m}(\psi)$ instead of $\mathscr{A}_{n,m}(\Psi)$. Unless explicitly specified otherwise, n and m will always be integers such that $n,m \geq 1$. Whenever we refer to balls, we will mean balls with respect to the maximum norm.

Many seminal results in Diophantine approximation have had to do with the sets $\mathscr{A}^{\mathbf{y}}_{n,m}(\psi)$, in particular their metric properties, typically meaning their Lebesgue measure $\left|\mathscr{A}^{\mathbf{y}}_{n,m}(\psi)\right|$ and their Hausdorff measures $\mathscr{H}^f\left(\mathscr{A}^{\mathbf{y}}_{n,m}(\psi)\right)$. The classical Lebesgue theory for these sets is summarized by the following statement.

(Inhomogeneous) Khintchine–Groshev Theorem. *Let $n,m \geq 1$. Then for any $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $\mathbf{y} \in \mathbb{R}^m$, we have*

$$\left|\mathscr{A}^{\mathbf{y}}_{n,m}(\psi)\right| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty\,, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty \quad \ldots \quad \textbf{\textit{and } } \psi \textbf{\textit{ is monotonic.}} \end{cases} \tag{1}$$

Remark (On terminology). The term "homogeneous approximation" refers to the case $\mathbf{y} = \mathbf{0}$, and "inhomogeneous approximation" refers to the general case. "Simultaneous approximation" refers to cases where $n = 1$ and $m > 1$. "Dual approximation" refers to cases where $n > 1$ and $m = 1$. "Approximation of linear forms" is the general case.

Remark (On attribution). When $(n,m)=(1,m)$ the above theorem is known as Khintchine's theorem [24, 25, 1924/1926]. When (n,m) is general, it is known as the Khintchine–Groshev theorem [20, 1938]. An inhomogeneous version of Khintchine's theorem was proved by Szüsz [33, 1958] in dimension $m=1$ and then Schmidt [31, 1964] in higher dimensions. An inhomogeneous version of the Khintchine–Groshev theorem is found as [32, Theorem 12/15 in Chapter 1] in Sprindžuk's book and it only requires monotonicity for $n=1,2$.

The necessity of the emphasized monotonicity assumption in (1) has been one of the motivating questions of modern research in Diophantine approximation, and there have been great strides in the homogeneous setting. Duffin and Schaeffer constructed a counterexample showing that monotonicity cannot be removed when $(n,m)=(1,1)$, and this gave birth to the Duffin–Schaeffer conjecture [16, 1941], a problem which invigorated the field for years until its eventual proof by Koukoulopoulos and Maynard [26, 2020]. Gallagher removed monotonicity for $(1,m)$ when $m\geq 2$ [19, 1965]. As mentioned above, Sprindžuk proved Khintchine–Groshev without monotonicity for (n,m) when $n\geq 3$. Beresnevich and Velani completed the homogeneous story by removing monotonicity when $nm>1$ [11, 2010], thus settling affirmatively a conjecture posed by Beresnevich, Bernik, Dodson, and Velani [5, Conjecture A]. The work of Beresnevich and Velani shows that monotonicity can safely be removed in all homogeneous cases except $(n,m)=(1,1)$, where the Duffin–Schaeffer counterexample had already established that monotonicity could not be removed.

The problem of removing monotonicity in the more general inhomogeneous part of the theory has lagged somewhat, with no further progress having been recorded in the inhomogeneous higher-dimensional linear forms setting since Sprindžuk proved the general theorem without monotonicity for (n,m) where $n\geq 3$. Aside from the earlier work of Sprindžuk, Yu recently removed monotonicity from the general simultaneous inhomogeneous theorem when $n=1$ and $m\geq 3$ [34, 2021]. Since homogeneous approximation is a special case of inhomogeneous approximation, corresponding to $\mathbf{y}=\mathbf{0}$, the Duffin–Schaeffer counterexample already demonstrates that monotonicity cannot be removed in the $(n,m)=(1,1)$ case. In fact, the second author showed that for $(n,m)=(1,1)$ there is no inhomogeneous shift parameter $y\in\mathbb{R}$ for which monotonicity can be removed [29, 2017].

This paper's main contribution to the inhomogeneous theory is to remove monotonicity from the general Khintchine–Groshev theorem whenever $nm>2$. This resolves the natural inhomogeneous generalization of [5, Conjecture A] when $nm>2$, leaving open only the cases $(n,m)=(2,1)$ and $(n,m)=(1,2)$. We formally state the two remaining cases of this inhomogeneous conjecture as Conjecture 1 below, and we make partial progress towards it in Theorem 3.

Theorem 1. *Let $nm>2$. Then for any $\psi:\mathbb{N}\to\mathbb{R}_{\geq 0}$ and $\mathbf{y}\in\mathbb{R}^m$, we have*

$$\left|\mathscr{A}_{n,m}^{\mathbf{y}}(\psi)\right| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty\,, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty\,. \end{cases}$$

Remark. As we have mentioned, the $n\geq 3$ parts of this theorem appear in [32], and the $(1,m)$ cases with $m\geq 3$ appear in [34]. We present a conceptual proof that establishes the result for all $nm>2$.

Theorem 1 is actually a special case of the following more general theorem, which does not require the balls $(B_q)_{q\in\mathbb{N}}$ to be concentric.

Theorem 2. *Let $nm > 2$. For any sequence of balls $\Psi := (B_q)_{q\in\mathbb{N}} \subset \mathbb{R}^m/\mathbb{Z}^m$, we have*

$$\left|\mathscr{A}_{n,m}(\Psi)\right| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}|B_q| < \infty\,, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1}|B_q| = \infty\,. \end{cases}$$

Proof of Theorem 1. This is the special case of Theorem 2 where all of the balls are concentric at the inhomogeneous parameter $\mathbf{y}$. □

We remind the reader that monotonicity is not required to prove the convergence part of the Inhomogeneous Khintchine–Groshev theorem (nor is it required for the convergence part of Theorem 2). The question is really whether one can remove monotonicity from the divergence part. We suspect the truth of the following statement, which is the natural inhomogeneous analogue of [5, Conjecture A] in the cases when $nm = 2$ and comprises the only two cases not covered by Theorem 1 and the work of Duffin and Schaeffer.

Conjecture 1. *Let $nm = 2$. For any $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $\mathbf{y} \in \mathbb{R}^m$, we have*

$$\left|\mathscr{A}_{n,m}^{\mathbf{y}}(\psi)\right| = 1 \qquad \text{if} \qquad \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty.$$

Moreover, in line with the statement of Theorem 2, we actually expect the following more general result to be true when $nm = 2$. Conjecture 1 would follow from this more general result as an immediate corollary.

Conjecture 2. *Let $nm = 2$. For any sequence of balls $\Psi := (B_q)_{q\in\mathbb{N}} \subset \mathbb{R}^m/\mathbb{Z}^m$, we have*

$$\left|\mathscr{A}_{n,m}(\Psi)\right| = 1 \qquad \text{if} \qquad \sum_{q=1}^{\infty} q^{n-1}|B_q| = \infty\,.$$

The challenge in the above conjectures is an issue that causes difficulty in all problems of this type. Without going into detail, it suffices to say that it comes from the fact that any rational point can be expressed in infinitely many ways by changing denominators. Authors contend with this issue in a variety of ways, often under the banner of "overlap estimates" (see, for example, our Lemma 8). Indeed, the Duffin–Schaeffer conjecture arose from an attempt to mitigate this difficulty by requiring that all rational numbers be expressed in their reduced form, and adjusting the divergence condition accordingly. Even after taking this measure, the conjecture stood for almost 80 years. In the meantime, much of the partial progress consisted of proofs of the conjecture under "extra divergence" assumptions on the series [1, 2, 7, 23]. In a similar vein, we have the following theorem, which can be seen as an "extra divergence" version of an inhomogeneous analogue of the Duffin–Schaeffer conjecture for systems of linear forms.

Theorem 3. *Let $\varepsilon > 0$. For any $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $y \in \mathbb{R}$, we have*

$$\left|\mathscr{A}_{2,1}^{y}(\psi)\right| = 1 \qquad \text{if} \qquad \sum_{q=1}^{\infty} \left(\frac{\varphi(q)}{q}\right)^{1+\varepsilon} q\psi(q) = \infty.$$

For any $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $\mathbf{y} \in \mathbb{R}^2$, we have

$$\left|\mathscr{A}_{1,2}^{\mathbf{y}}(\psi)\right| = 1 \qquad \text{if} \qquad \sum_{q=1}^{\infty} \left(\frac{\varphi(q)}{q}\right)^{1+\varepsilon} \psi(q)^2 = \infty.$$

Here, φ denotes the Euler totient function.

Remark. This result improves on [34, Theorem 1.8], where the $(1,2)$ case is proved with an exponent 2 in place of $1+\varepsilon$.

Remark. As with Theorem 1, Theorem 3 is a special case of a more general theorem that does not require concentric balls. It is listed here in Section 5 as Theorem 9.

It is well-known that $\varphi(q) \gg q/\log\log q$. Therefore, Theorem 3 immediately implies that in the two cases where $nm = 2$, the divergence part of Theorem 1 holds under the extra divergence condition

$$\sum_{q=1}^{\infty} \frac{q^{n-1}\psi(q)^m}{(\log\log q)^{1+\varepsilon}} = \infty.$$

Readers who are familiar with the Duffin–Schaeffer conjecture and its "extra divergence" precursors will naturally suspect that Theorem 3 is true even with $\varepsilon = 0$. Indeed, Conjecture 1 already predicts something even stronger. We therefore suggest that proving the "$\varepsilon = 0$" case of Theorem 3 may be seen as an interesting intermediate challenge that is likely to be more tractable than full resolutions of the above conjectures.

Interestingly, the guiding principle and key result of this paper (presented in the next subsection) lead us to believe that the "$(n,m) = (2,1)$, $\varepsilon = 0$" case of Theorem 3 would follow from a weak inhomogeneous version of the Duffin–Schaeffer conjecture, stating that

$$\left|\mathscr{A}_{1,1}^{\mathbf{y}}(\psi)\right| = 1 \qquad \text{if} \qquad \sum_{q=1}^{\infty} \frac{\psi(q)\varphi(q)}{q} = \infty,$$

provided it were proved by establishing quasi-independence on average of the necessary sets. Informally, this guiding principle indicates that whenever one can prove something for (n,m), one should expect to be able to do it for $(n+1,m)$ too.

In the present paper, our main motivating objective has been to remove redundant monotonicity assumptions from the classical inhomogeneous Khintchine–Groshev theorem. In another direction, Dani, Laurent, and Nogueira [15, 2015] proved refinements of the classical Khintchine–Groshev theorem where they imposed certain primitivity constraints on their "approximating points". The methods they used relied on the monotonicity of the approximating functions. In addition, their main result [15, Theorem 1.1] is doubly metric, in the sense that it holds for almost every pair $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{nm} \times \mathbb{R}^m$. In future work we hope to investigate whether the approach presented in this paper may be adapted to remove monotonicity from statements given in [15], or to establish a singly metric version of their main result. Any progress in these directions would go some way towards addressing Problems 1 and 2 posed by Laurent in [27, 2016].

1.2 An "independence inheritance" theorem

The results described in this section are the main new tool for establishing Theorems 1 and 2. They also provide an instructive point of view on all statements of Khintchine–Groshev type, and so we find them interesting in their own right. To put it informally, they tell us that statements in the (n,m) case imply statements in the $(n+k,m)$ case, hence, we may take as a guiding principle that

$$\text{“Khintchine”} \qquad \Longrightarrow \qquad \text{“Khintchine–Groshev”}$$

in all its various incarnations. In order to state this principle precisely, some discussion is needed.

The set $\mathscr{A}_{n,m}(\Psi)$ is the limsup set of a sequence of measurable subsets of $\mathbb{I}^{nm}$. As such, its measure is 0 whenever the measures of those subsets have a converging sum — a simple consequence of the First Borel–Cantelli lemma (see, for example, [22, Lemma 1.2]). Indeed, this is why, as we have mentioned, the convergence parts of the theorems in the previous subsection are straightforward to establish.

On the other hand, if the measure sum diverges, it takes much more work to show that $\mathscr{A}_{n,m}(\Psi)$ has full, or even positive, measure. (In fact, it might not have positive measure, as counterexamples in the $(n,m)=(1,1)$ setting have shown.) Invariably, that work has involved showing that the sequence of sets exhibits some sort of "independence," such as is necessary to apply a partial converse of the Borel–Cantelli lemma. In fact, Beresnevich and Velani have shown that, in a sense made precise in [12], a limsup set has positive measure only if there is some independence present.

For us, the relevant notion of independence is that of quasi-independence on average. A sequence $(A_{\mathbf{q}})_{\mathbf{q}\in\mathbb{Z}^n}$ of measurable subsets of a finite measure space (X,μ) is said to be *quasi-independent on average* (or QIA for short) if

$$\limsup_{Q\to\infty} \frac{\left(\sum_{|\mathbf{q}|=1}^{Q} \mu(A_{\mathbf{q}})\right)^2}{\sum_{|\mathbf{q}|,|\mathbf{r}|=1}^{Q} \mu(A_{\mathbf{q}}\cap A_{\mathbf{r}})} > 0. \tag{2}$$

The importance of this concept is made clear by a well-known partial converse to the first Borel–Cantelli lemma, stated here as Lemma 1. It guarantees that if a sequence is quasi-independent on average, then the associated limsup set has positive measure.

The sets we will be working with here are of the following form. Given a ball $B\subset\mathbb{R}^m/\mathbb{Z}^m$ and $\mathbf{q}\in\mathbb{Z}^n$, let $A_{n,m}(\mathbf{q},B)$ denote the set of $\mathbf{x}\in\mathbb{I}^{nm}$ for which there exists a $\mathbf{p}\in\mathbb{Z}^m$ with $\mathbf{q}\mathbf{x}+\mathbf{p}\in B$. When the meaning is clear from context, we may omit the subscript and simply write $A(\mathbf{q},B)$. When $n=1$ we customarily write the non-bold q, i.e. $A(q,B)$. The sets $\mathscr{A}_{n,m}(\Psi)$ appearing in our main results are limsup sets for sequences of sets of this form, so the problem of establishing Theorems 1 and 2 naturally involves studying the independence properties of the sets $A_{n,m}(\mathbf{q},B)$.

As a starting point, we take a naïve point of view, which we will illustrate in the $(2,1)$ case (see Figure 1). The divergence condition $\sum q|B_q|=\infty$ should be understood as the divergence of a measure sum. The summand $q|B_q|$ is, up to a constant, the sum of the measures

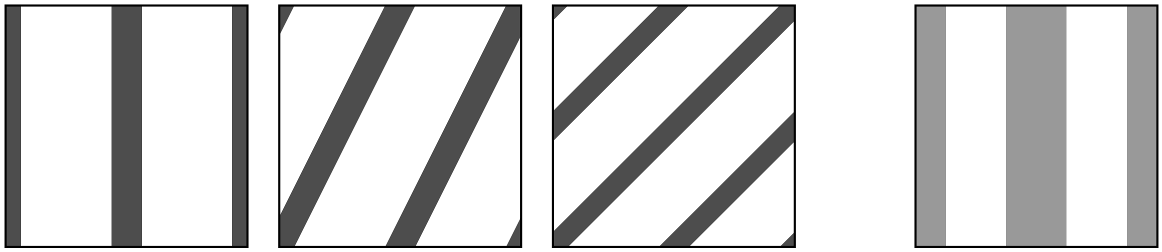

Figure 1: On the left are three of the 16 possible sets $A_{2,1}(\mathbf{q},B_q)$ with $|\mathbf{q}| = q = 2$. On the right is the set $A_{1,1}(\mathbf{q},qB_q)\times[0,1]$. The areas of the 16 possible darker regions sum to the area of the lighter region, up to a constant which depends on the dimensions; in this case, that constant is 8. One may imagine that the lighter region has been fractured into $q = 2$ pieces of equal area, (duplicated 8 times,) and shuffled in order to make the darker regions. Theorem 8 quantifies the extra independence that is gained in this fracturing and shuffling.

of $A_{2,1}(\mathbf{q},B_{|\mathbf{q}|})$ as $\mathbf{q}$ ranges through the sphere of radius q in $\mathbb{Z}^2$. These sets each have measure $|B_q|$, and for each $|\mathbf{q}| = q$, the set $A_{2,1}(\mathbf{q},B_q)$ can be visualized as an array of "stripes" through the unit square, at an angle determined by the angle of $\mathbf{q}$. On the other hand one can interpret the sum $\sum q|B_q|$ as $\sum|qB_q|$, where qB_q denotes the $\times q$-dilation of B_q around its center. This way, one may imagine that the sum is the measure sum corresponding to the sets $A_{1,1}(q,qB_q)$, each of which is a union of intervals. Any metric properties of these sets will be shared trivially by the vertically extended sets $A_{1,1}(q,qB_q)\times[0,1]\subset\mathbb{I}^2$. Notice that this last set, which is a union of thick vertical stripes, has the same measure (up to a constant) as the sum of the measures of the sets $\{A_{2,1}(\mathbf{q},B_q)\}_{|\mathbf{q}|=q}$, which are unions of thin stripes arranged in many different directions. This leads to the naïve suspicion that

the sets $\{A_{2,1}(\mathbf{q},B_{|\mathbf{q}|})\}_{\mathbf{q}\in\mathbb{Z}^2}$ should be at least as independent as the sets $\{A_{1,1}(q,qB_q)\}_{q\in\mathbb{Z}}$,

because it is as if the latter have each been fractured into q pieces which are then shuffled into a disordered arrangement to create the former.

The next result confirms this intuition (and a stronger result, to be discussed, quantifies it). It states that quasi-independence on average for certain sequences of sets in $\mathbb{I}^{nm}$ is inherited by an associated sequence of sets in $\mathbb{I}^{(n+k)m}$ for any integer $k\geq 0$, and this allows us to prove full measure for the corresponding limsup set in $\mathbb{I}^{(n+k)m}$.

Theorem 4 (Independence Inheritance)**.** *Let $n,m\geq 1$. Suppose $(B_q)_{q\in\mathbb{N}}$ is a sequence of balls in $\mathbb{R}^m/\mathbb{Z}^m$. If the sets $\left(A_{n,m}(\mathbf{q},B_{|\mathbf{q}|})\right)_{\mathbf{q}\in\mathbb{Z}^n}$ are quasi-independent on average, then so are the sets $\left(A_{n+k,m}(\mathbf{q},|\mathbf{q}|^{-k/m}B_{|\mathbf{q}|})\right)_{\mathbf{q}\in\mathbb{Z}^{n+k}}$ for every $k\geq 0$. If in addition the sum $\sum_{\mathbf{q}}|B_{|\mathbf{q}|}|$ diverges, then for every $k\geq 0$, the set*

$$\limsup_{|\mathbf{q}|\to\infty} A_{n+k,m}(\mathbf{q},|\mathbf{q}|^{-k/m}B_{|\mathbf{q}|})$$

has full Lebesgue measure.

The philosophy behind applications of Theorem 4 is that Khintchine–Groshev-like statements in the case $(n+k,m)$ are weaker than in the case (n,m). Indeed, by using Theorem 4, one can show that for $m \geq 2$, the m-dimensional Khintchine theorem implies the (n,m)-dimensional Khintchine–Groshev theorem, because the sets involved in Khintchine's theorem are quasi-independent on average in dimensions ≥ 2 [19]. The same implication holds inhomogeneously, as well as inhomogeneously without monotonicity in dimension ≥ 3; that is, the inhomogeneous Khintchine theorem in dimension $m \geq 3$ implies the inhomogeneous Khintchine–Groshev theorem in dimensions (n,m) for any n. The quasi-independence on average of the relevant sets in the cases $(1,m)$ with $m \geq 3$ is established in Proposition 2 as well as in Yu's proof of [34, Theorem 1.8].

In fact, Theorem 4 follows from a stronger inheritance phenomenon, Theorem 8, which allows us to quantify how much "more independent" the sets in the $(n+k,m)$ case are than the sets in the (n,m) case. We define in Section 4 (Definition 1) a hierachy of the form

$$\text{QIA} = 0\text{-QIA} \implies 1\text{-QIA} \implies 2\text{-QIA} \implies 3\text{-QIA} \implies \cdots$$

where w-QIA stands for "w-weak quasi-independence on average" and 0-QIA coincides with the usual notion defined by (2). Theorem 8 states that the $(n+k,m)$ case inherits an independence that is k steps stronger than the independence enjoyed by the (n,m) case. At this level of detail, we can already say how Theorems 1 and 2 are proved: we establish that the sets involved in the $(1,1)$ case are 2-QIA, the sets in the $(1,2)$ case are 1-QIA, and the sets in the $(1,m)$ case $(m \geq 3)$ are 0-QIA (see Proposition 2). Theorem 8 then gives us 0-QIA for all (n,m) with $nm > 2$.

The full measure statement then follows by a "mixing" argument exploiting the periodicity inherent in the sets $A_{n,m}(\mathbf{q},B)$ (see Lemma 7 and Proposition 1). This argument enables us to overcome another perceived obstacle in our general setting; namely, that the literature lacks a "zero-one" law for inhomogeneous Diophantine approximation. In the homogeneous setting, Beresnevich and Velani had previously shown that the Lebesgue measure of the set $\mathscr{A}^{\mathbf{0}}_{n,m}(\psi)$ is always either zero or one; that is, for homogeneous approximation, these sets satisfy a so-called "zero-one" law [10]. A consequence of this was that in proving the homogeneous Khintchine–Groshev theorem without monotonicity for $nm > 1$ in [11], to obtain the full measure statement, it was enough for Beresnevich and Velani to show that the sets $\mathscr{A}^{\mathbf{0}}_{n,m}(\psi)$ had positive measure subject to the divergence of the appropriate sum. This is a common and effective strategy that has been used countless times in this field. Consequently, zero-one laws are seen as indispensable tools in homogeneous approximation, the most well-known of them being the zero-one laws of Cassels and Gallagher [13, 18]. Since we know of no zero-one law for the general sets $\mathscr{A}^{\mathbf{y}}_{n,m}(\psi)$, that strategy is unavailable to us. However, our Theorem 1 implies an *a posteriori* inhomogeneous zero-one law when $nm > 2$.

1.3 The Hausdorff measure theory

In addition to studying the Lebesgue measure of sets such as $\mathscr{A}^{\mathbf{y}}_{n,m}(\psi)$, much attention in Diophantine approximation is also devoted to studying the more refined Hausdorff measures and Hausdorff dimension of such sets. This is particularly fruitful when the Lebesgue measure of such sets is zero, in which case Hausdorff dimension and Hausdorff measures can often still provide us with a means of differentiating the relative "sizes" of such sets. In classical simultaneous approximation, these types of distinctions were accomplished by the Jarník–Besicovitch theorem and Jarník's theorem. See [8] for a survey discussion.

We conclude this introduction by recording Hausdorff measure counterparts to Theorems 1, 3, and 2. These are, respectively, Theorems 5, 6, and 7. Theorems 5 and 6 can be obtained as a simple consequence of combining the relevant Lebesgue measure statement (in our case, Theorem 1 or 3) with [4, Theorem 5]. The latter statement, appearing below as Theorem AB, is essentially a general "transference principle," proved by the first author and Beresnevich, which allows us to more-or-less immediately read off a Hausdorff measure analogue when given an appropriate Lebesgue measure Khintchine–Groshev type statement. Theorem AB is a consequence of the mass transference principle for systems of linear forms [4, Theorem 1] which, informally speaking, allows us to transfer Lebesgue measure statements for limsup sets determined by neighbourhoods of planes (i.e. systems of linear forms) to Hausdorff measure statements. The mass transference principle for systems of linear forms is just one natural generalization of the original mass transference principle due to Beresnevich and Velani [9, Theorem 2].

Theorem AB ([4]). *Let $\psi:\mathbb{N}\to\mathbb{R}_{\geq 0}$ be an approximating function, let $\mathbf{y}\in\mathbb{I}^m$, and let f and $g: r\to g(r):=r^{-m(n-1)}f(r)$ be dimension functions such that $r^{-nm}f(r)$ is monotonic. Let*

$$\theta:\mathbb{N}\to\mathbb{R}_{\geq 0} \qquad \textit{be defined by} \qquad \theta(q)=q\,g\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}.$$

Then

$$|\mathcal{A}^{\mathbf{y}}_{n,m}(\theta)|=1 \qquad \textit{implies} \qquad \mathcal{H}^f(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi))=\mathcal{H}^f(\mathbb{I}^{nm}).$$

Before presenting our Hausdorff measure results, we recall some definitions. We will say that a function $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ is a *dimension function* if f is continuous, non-decreasing, and $f(r)\to 0$ as $r\to 0$. Given a set $F\subset\mathbb{R}^k$ and a real number $\rho>0$, we will say that a countable collection of balls, say $(B_i:=B(x_i,r_i))_{i\in\mathbb{N}}$, is a *$\rho$-cover* for F if $F\subset\bigcup_{i=1}^{\infty}B_i$ and $r_i<\rho$ for all $i\in\mathbb{N}$. The *ρ-approximate Hausdorff f-measure* of F is defined as

$$\mathcal{H}^f_\rho(F):=\inf\left\{\sum_{i=1}^{\infty}f(r):\{B_i\}_{i\in\mathbb{N}}\text{ is a }\rho\text{-cover for }F\right\}.$$

The *Hausdorff f-measure* of F is then defined to be

$$\mathcal{H}^f(F):=\lim_{\rho\to 0}\mathcal{H}^f_\rho(F)=\sup_{\rho>0}\mathcal{H}^f_\rho(F).$$

In the case that $f(r)=r^s$ for some real $s\geq 0$, $\mathcal{H}^f$ is the perhaps more familiar *Hausdorff s-measure*. In this case we write $\mathcal{H}^s$ in place of $\mathcal{H}^f$. Although we will not require this definition here, for completeness we remark that the *Hausdorff dimension* of F is defined as

$$\dim_{\mathrm{H}}F:=\inf\{s\geq 0:\mathcal{H}^s(F)=0\}.$$

For further details on Hausdorff measures and Hausdorff dimension, we refer the reader to, for example, [17, 28, 30].

Combining Theorem 1 with Theorem AB yields the following Hausdorff measure analogue of Theorem 1. This statement is essentially an inhomogeneous *Hausdorff measure Khintchine–Groshev theorem*.

Theorem 5. *Let $nm > 2$. Let $\mathbf{y} \in \mathbb{I}^m$, and let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be an approximating function. Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-nm} f(r)$ is monotonic. Then,*

$$\mathcal{H}^f(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) < \infty\,, \\ \mathcal{H}^f(\mathbb{I}^{nm}) & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty\,. \end{cases}$$

Proof. Analogously to the Lebesgue measure theory, the convergence part of Theorem 5 follows from a standard covering argument, and the statement is true in this case for any $n, m \geq 1$ with no monotonicity requirements on ψ. We omit the details here but note that the required argument is given very explicitly in [3].

The divergence part of the above statement would follow immediately from Theorem AB provided that we could show that $|\mathcal{A}^{\mathbf{y}}_{n,m}(\theta)| = 1$ for $\theta(q) = q g\left(\frac{\psi(q)}{q}\right)^{\frac{1}{m}}$. Provided that $nm > 2$, it follows from Theorem 1 that $|\mathcal{A}^{\mathbf{y}}_{n,m}(\theta)| = 1$ if

$$\sum_{q=1}^{\infty} q^{n-1}\theta(q)^m = \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty,$$

thus completing the proof. □

Remark. The proof of Theorem 5, both the convergence and divergence parts, is virtually identical to the proof given of Theorem 3.7 in [3]. In all cases except when $n = 1$ or $n = 2$, Theorem 5 was already known to be true without any monotonicity assumptions on ψ, see [4, Theorem 3] (which also appears as [3, Theorem 3.7]). In the case that $n = 1$ or $n = 2$, the conclusion of Theorem 5 was shown to be true in [4, Theorem 3] subject to additional monotonicity assumptions. We refer the reader to [3, 4] for detailed discussion of the precise requirements, as well as for more details on the mass transference principle for linear forms.

In line with how Theorem 1 demonstrates that monotonicity is not required in the classical inhomogeneous Khintchine–Groshev theorem whenever $nm > 2$, Theorem 5 shows that the same is true in the analogous inhomogeneous Hausdorff measure Khintchine–Groshev theorem. Moreover, any further progress towards removing monotonicity from the Lebesgue measure inhomogeneous Khintchine–Groshev theorem in the remaining $(1,2)$ and $(2,1)$ cases can be transferred to analogous progress in the Hausdorff measure setting via Theorem AB. For example, in the same way that we can deduce the divergence part of Theorem 5 from Theorem 1 using Theorem AB, we can deduce the following Hausdorff measure analogue of Theorem 3. We leave the details as an exercise for the interested reader!

Theorem 6. *Let $\varepsilon > 0$ and let $nm = 2$. Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-nm} f(r)$ is monotonic. Then, for any $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$, and any $\mathbf{y} \in \mathbb{I}^m$, we have*

$$\mathcal{H}^f\left(\mathcal{A}^{\mathbf{y}}_{n,m}(\psi)\right) = \mathcal{H}^f(\mathbb{I}^{nm}) \qquad \text{if} \qquad \sum_{q=1}^{\infty} \left(\frac{\varphi(q)}{q}\right)^{1+\varepsilon} q^{n+m-1} g\left(\frac{\psi(q)}{q}\right) = \infty.$$

Remark. Notice that the condition $nm = 2$ only permits the cases $(1,2)$ and $(2,1)$.

Via a more direct application of the mass transference principle for systems of linear forms [4, Theorem 1], it is also possible to prove a more general Hausdorff measure statement analogous to the Lebesgue measure statement given by Theorem 2. More precisely, recall that if $\Psi := (B_q)_{q\in\mathbb{N}}$ is a sequence of balls in $\mathbb{R}^m/\mathbb{Z}^m$ and $n \geq 1$, we denote by $\mathscr{A}_{n,m}(\Psi)$ the set of $\mathbf{x} \in \mathbb{I}^{nm}$ such that

$$\mathbf{q}\mathbf{x} + \mathbf{p} \in B_{|\mathbf{q}|}$$

for infinitely many pairs $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$. By adapting the proof of [4, Theorem 2] (specifically the proof of (11) in [4]), one can apply [4, Theorem 1] directly to obtain the following Hausdorff measure analogue of Theorem 2.

Theorem 7. *Let $nm > 2$. Let f and $g : r \to g(r) = r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-nm} f(r)$ is monotonic. Then,*

$$\mathscr{H}^f(\mathscr{A}_{n,m}(\Psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{|B_q|^{\frac{1}{m}}}{2q}\right) < \infty, \\ \mathscr{H}^f(\mathbb{I}^{nm}) & \text{if } \sum_{q=1}^{\infty} q^{n+m-1} g\left(\frac{|B_q|^{\frac{1}{m}}}{2q}\right) = \infty. \end{cases}$$

Remark. Observe that $\frac{|B_q|^{\frac{1}{m}}}{2q}$ is simply $\frac{r(B_q)}{q}$, where $r(B_q)$ is the radius of the ball B_q.

2 Lemmas

The first three lemmas are standard facts from measure theory. Recall that the *limsup* of a sequence of measurable sets $(A_q)_{q\in\mathbb{N}}$ in a finite measure space (X,μ) is

$$\begin{aligned} \limsup_{q\to\infty} A_q &:= \{x \in X : x \in A_q \text{ for infinitely many } q \in \mathbb{N}\} \\ &= \bigcap_{Q\geq 1} \bigcup_{q\geq Q} A_q. \end{aligned}$$

Lemma 1 (Divergence Borel–Cantelli Lemma, [22, Lemma 2.3]). *Suppose (X,μ) is a finite measure space and $(A_q)_{q\in\mathbb{N}} \subset X$ is a sequence of measurable subsets such that $\sum \mu(A_q) = \infty$. Then*

$$\mu\left(\limsup_{q\to\infty} A_q\right) \geq \limsup_{Q\to\infty} \frac{\left(\sum_{q=1}^{Q} \mu(A_q)\right)^2}{\sum_{q,r=1}^{Q} \mu(A_q \cap A_r)}.$$

Lemma 2 (Chung–Erdős Lemma, [14, Equation 4]). *Suppose (X,μ) is a probability space and $(A_q)_{q\in\mathbb{N}} \subset X$ is a sequence of measurable subsets. If $\mu\left(\bigcup_{q=1}^{Q} A_q\right) > 0$, then*

$$\mu\left(\bigcup_{q=1}^{Q} A_q\right) \geq \frac{\left(\sum_{q=1}^{Q} \mu(A_q)\right)^2}{\sum_{q,r=1}^{Q} \mu(A_q \cap A_r)}.$$

Lemma 3 ([6, Lemma 6]). *Let (X,d) be a metric space with a finite measure μ such that every open set is μ-measurable. Let A be a Borel subset of X and let $f:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ be an increasing function with $f(x)\to 0$ as $x\to 0$. If for every open set $U\subset X$ we have*

$$\mu(A\cap U)\geq f(\mu(U)),$$

then $\mu(A)=\mu(X)$.

The next lemmas are specific to our problem. Recall that for $\mathbf{q}\in\mathbb{Z}^n$ and a ball $B\subset\mathbb{R}^m/\mathbb{Z}^m$ we define

$$A(\mathbf{q},B)=A_{n,m}(\mathbf{q},B)=\{\mathbf{x}\in\mathbb{I}^{nm}:\mathbf{q}\mathbf{x}+\mathbf{p}\in B \text{ for some } \mathbf{p}\in\mathbb{Z}^m\} \tag{3}$$

and note that it has measure $|B|$. Given two vectors $\mathbf{q}_1,\mathbf{q}_2\in\mathbb{Z}^n$, we will use the notation $\mathbf{q}_1\parallel\mathbf{q}_2$ to denote that $\mathbf{q}_1$ and $\mathbf{q}_2$ are parallel and, similarly, we will write $\mathbf{q}_1\nparallel\mathbf{q}_2$ to indicate that the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ are not parallel.

Lemma 4 ([32, Lemma 9 in Chapter 1]). *Suppose $B_1,B_2\subset\mathbb{R}^m/\mathbb{Z}^m$ are balls, and $\mathbf{q}_1,\mathbf{q}_2\in\mathbb{Z}^n$ with $\mathbf{q}_1\nparallel\mathbf{q}_2$. Then*

$$|A(\mathbf{q}_1,B_1)\cap A(\mathbf{q}_2,B_2)|=|A(\mathbf{q}_1,B_1)||A(\mathbf{q}_2,B_2)|.$$

Lemma 5. *Suppose $B_1,B_2\subset\mathbb{R}^m/\mathbb{Z}^m$ are balls. Let $n,n'\geq 1$ be integers and suppose that $\mathbf{q}_1\parallel\mathbf{q}_2\in\mathbb{Z}^n$ and $\mathbf{r}_1\parallel\mathbf{r}_2\in\mathbb{Z}^{n'}$ are such that $|\mathbf{q}_i|=|\mathbf{r}_i|$ and $\mathbf{q}_1\cdot\mathbf{q}_2=\mathbf{r}_1\cdot\mathbf{r}_2$ (that is, the pairs either both point in the same direction, or both point in opposite directions). Then*

$$\left|A_{n,m}(\mathbf{q}_1,B_1)\cap A_{n,m}(\mathbf{q}_2,B_2)\right|=\left|A_{n',m}(\mathbf{r}_1,B_1)\cap A_{n',m}(\mathbf{r}_2,B_2)\right|.$$

Remark. Throughout this work, we have allowed $\mathbf{q}$ to take values in $\mathbb{Z}^n$. As it turns out, all of our arguments work if we restrict $\mathbf{q}$ to lie in the positive orthant $\mathbb{Z}^n_{\geq 0}$. Under that restriction, the question (seen for example in the above lemma) of whether $\mathbf{q}_1$ and $\mathbf{q}_2$ point in the same direction becomes irrelevant.

Proof of Lemma 5. We will show that

$$\left|A_{n,m}(\mathbf{q}_1,B_1)\cap A_{n,m}(\mathbf{q}_2,B_2)\right|=\left|A_{1,m}(|\mathbf{q}_1|,B_1)\cap A_{1,m}(\pm|\mathbf{q}_2|,B_2)\right|, \tag{4}$$

where the $\pm$ is determined according to whether $\mathbf{q}_1\cdot\mathbf{q}_2$ is positive or negative. To see this, let $\mathbf{q}\in\mathbb{Z}^n$ and $k_1,k_2\in\mathbb{Z}$ be such that $\mathbf{q}_1=k_1\mathbf{q}$ and $\mathbf{q}_2=k_2\mathbf{q}$. By possibly changing the sign on $\mathbf{q}$ we may ensure that k_1 is positive. Now $A_{n,m}(\mathbf{q}_1,B_1)\cap A_{n,m}(\mathbf{q}_2,B_2)$ is the preimage of $A_{1,m}(k_1,B_1)\cap A_{1,m}(k_2,B_2)$ under the projection $\mathbf{x}\mapsto\mathbf{q}\mathbf{x}\pmod 1$. This mapping is measure-preserving (see [32, Lemma 8 in Chapter 1]), therefore

$$|A(\mathbf{q}_1,B_1)\cap A(\mathbf{q}_2,B_2)|=|A(k_1,B_1)\cap A(k_2,B_2)|.$$

But notice that $A(|\mathbf{q}_1|,B_1)\cap A(\mathrm{sgn}(k_2)|\mathbf{q}_2|,B_2)$ is the preimage of $A(k_1,B_1)\cap A(k_2,B_2)$ under the "$\times|\mathbf{q}|\pmod 1$" map, which is also measure-preserving, so we have

$$|A(k_1,B_1)\cap A(k_2,B_2)|=|A(|\mathbf{q}_1|,B_1)\cap A(\mathrm{sgn}(k_2)|\mathbf{q}_2|,B_2)|.$$

Combining this with the previous observation, we have (4), where the $\pm$ is determined by $\mathrm{sgn}(k_2)$, which is itself determined by whether $\mathbf{q}_1$ and $\mathbf{q}_2$ point in the same or opposite direction. Since the same argument could have been carried out with $\mathbf{r}_1$ and $\mathbf{r}_2$, we are done. $\square$

Lemma 6. *Suppose $B_1, B_2 \subset \mathbb{R}^m/\mathbb{Z}^m$ are balls, and q_1, q_2 are integers with $1 \le q_1 \le q_2$. Then, for every $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{Z}^n$ with $\mathbf{q}_1 \parallel \mathbf{q}_2$ and $|\mathbf{q}_i| = q_i$,*

$$\left|A(\mathbf{q}_1, q_1^{-1/m}B_1) \cap A(\mathbf{q}_2, q_2^{-1/m}B_2)\right| \le \frac{1}{q_1}\left|A(\mathbf{q}_1, B_1) \cap A(\mathbf{q}_2, (q_1/q_2)^{1/m}B_2)\right|.$$

Proof. Lemma 5 implies that it is enough to prove this for $n = 1$, that is, we must prove that

$$\left|A(q_1, q_1^{-1/m}B_1) \cap A(\pm q_2, q_2^{-1/m}B_2)\right| \le \frac{1}{q_1}\left|A(q_1, B_1) \cap A(\pm q_2, (q_1/q_2)^{1/m}B_2)\right|,$$

where the $\pm$ is determined by whether $\mathbf{q}_1$ and $\mathbf{q}_2$ point in the same or opposite direction. Seeing $A(q_1,B_1) \cap A(\pm q_2, (q_1/q_2)^{1/m}B_2)$ as a subset of $[0,1]^m$, we contract it by $q_1^{-1/m}$ and see that

$$\left|A(q_1,B_1) \cap A(\pm q_2, (q_1/q_2)^{1/m}B_2)\right| = q_1\left|q_1^{-1/m}\left(A(q_1,B_1) \cap A(\pm q_2, (q_1/q_2)^{1/m}B_2)\right)\right|.$$

Notice that

$$q_1^{-1/m}\left(A(q_1,B_1) \cap A(\pm q_2, (q_1/q_2)^{1/m}B_2)\right) = \underbrace{\left[q_1^{-1/m}(A(q_1,B_1))\right]}_{C} \cap \underbrace{\left[q_1^{-1/m}\left(A(\pm q_2, (q_1/q_2)^{1/m}B_2)\right)\right]}_{D}.$$

Since B_1 and B_2 are balls in $\mathbb{R}^m/\mathbb{Z}^m$, they have diameter ≤ 1, and so each of the square bracketed sets on the right-hand side (labelled C and D) is a union of essentially disjoint balls of volume $q_1^{-1}|B_1|$ and $q_2^{-1}|B_2|$ respectively. We may see $C \cap D$ as an intersection of several balls in $(q_1^{-1/m}\mathbb{R})^m/(q_1^{-1/m}\mathbb{Z})^m$ (a shrunken torus) such that no point is contained in more than two of them. The sets $A(q_1, q_1^{-1/m}B_1)$ and $A(\pm q_2, q_2^{-1/m}B_2)$ are obtained from C and D by translating the disjoint balls so that their centers lie at their corresponding points in $\mathbb{R}^m/\mathbb{Z}^m$. Since those corresponding points can be obtained via a scaling of the metric by $q_1^{1/m} \ge 1$, the pairwise distances between centers have necessarily increased. Moreover, since the balls making up C and D are only translated to their new positions (not scaled), the measures of the pairwise intersections cannot have increased. Therefore, we have

$$\left|A(q_1,B_1) \cap A(\pm q_2, (q_1/q_2)^{1/m}B_2)\right| \ge q_1\left|A(q_1, q_1^{-1/m}B_1) \cap A(\pm q_2, q_2^{-1/m}B_2)\right|,$$

proving the lemma. □

Lemma 7. *For any $n, m \ge 1$ there exists a constant $C := C_m > 0$ such that for every open set $U \subset \mathbb{I}^{nm}$ the following holds: for all $\mathbf{q} \in \mathbb{Z}^n$ of sufficiently large norm ($|\mathbf{q}| \ge Q_U$),*

$$|A(\mathbf{q}, B) \cap U| \ge C|A(\mathbf{q}, B)||U|$$

holds for every ball $B \subset \mathbb{R}^m/\mathbb{Z}^m$. (Note that the subscripts are added to the constants here to indicate their dependencies, e.g. C_m indicates that C depends on m.)

Proof. First, find a finite union V of disjoint balls contained in U such that $|V| \ge |U|/2$. (In principle, we could get as close to the measure of U as we want.) We may even, and indeed will, assume all the balls in V have the same radius, $r > 0$.

Now let $W \subset \mathbb{I}^{nm}$ be any ball of radius r. It will be enough to show that

$$|A(\mathbf{q},B)\cap W| \geq C'|A(\mathbf{q},B)||W| \tag{5}$$

for all $|\mathbf{q}| \geq Q_r$, where $C' > 0$ is some absolute constant which may depend on n and m, and Q_r only depends on r. Importantly for us, Q_r does not depend on B and so, given (5), one can deduce the lemma with $C = C'/2$.

Let us write $\mathbf{x} \in \mathbb{I}^{nm}$ as $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m)$ where $\mathbf{x}_j$ are column vectors. Then for any $\mathbf{q} \in \mathbb{Z}^n$ we have that $\mathbf{q}\mathbf{x} = (\mathbf{q}\cdot\mathbf{x}_1, \ldots, \mathbf{q}\cdot\mathbf{x}_m) \in \mathbb{I}^m$, and the condition that

$$\mathbf{q}\mathbf{x} + \mathbf{p} \in B \qquad \text{for} \qquad \mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m$$

is equivalent to $\mathbf{q}\cdot\mathbf{x}_j + p_j \in B_j$ for each j, where B_j is the projection of B to the jth component. Therefore, we have

$$A_{n,m}(\mathbf{q},B)\cap W = \left(\prod_{j=1}^{m} A_{n,1}(\mathbf{q},B_j)\right)\cap W = \prod_{j=1}^{m} A_{n,1}(\mathbf{q},B_j)\cap W_j, \tag{6}$$

where W_j is the projection of W to the copy of $\mathbb{I}^n$ corresponding to the jth column of $\mathbb{I}^{nm}$.

Suppose for the moment that $|\mathbf{q}|$ is achieved in the first coordinate of $\mathbf{q}$. For any $\mathbf{z} \in \mathbb{I}^{n-1}$ (representing the last $n-1$ coordinates), let

$$S_{\mathbf{z}} = \big(A_{n,1}(\mathbf{q},B_j)\cap W_j\big)_{\mathbf{z}} = A_{n,1}(\mathbf{q},B_j)_{\mathbf{z}} \cap (W_j)_{\mathbf{z}}$$

be the cross-section through $\mathbf{z}$ parallel to the first coordinate. Then

$$\begin{aligned}\big|A_{n,1}(\mathbf{q},B_j)\cap W_j\big| &= \int_{\mathbb{I}^{n-1}} |S_{\mathbf{z}}|\, d\mathbf{z} \\ &= \int_{Y_j} |S_{\mathbf{z}}|\, d\mathbf{z}\end{aligned}$$

where Y_j is the projection of W_j to the last $n-1$ coordinates. Meanwhile, $(W_j)_{\mathbf{z}}$ is an interval of length $2r$ and $A_{n,1}(\mathbf{q},B_j)_{\mathbf{z}}$ is a union of $|\mathbf{q}|$ disjoint intervals of length $|B_j|/|\mathbf{q}| = |B|^{1/m}/|\mathbf{q}|$ with centers spaced $1/|\mathbf{q}|$ apart, in $(\mathbb{I}^{nm})_{\mathbf{z}} \cong \mathbb{R}/\mathbb{Z}$. Therefore, $(W_j)_{\mathbf{z}}$ fully contains at least $2r|\mathbf{q}| - 2$ of the intervals constituting $A_{n,1}(\mathbf{q},B_j)_{\mathbf{z}}$, so we have

$$|S_{\mathbf{z}}| \geq (2r|\mathbf{q}| - 2)\frac{|B_j|}{|\mathbf{q}|},$$

which exceeds $r|B_j|$ as soon as $|\mathbf{q}| \geq 2/r$. In that case, we have

$$\begin{aligned}\big|A_{n,1}(\mathbf{q},B_j)\cap W_j\big| &= \int_{Y_j} |S_{\mathbf{z}}|\, d\mathbf{z} \\ &\geq r|B_j| \int_{Y_j} d\mathbf{z} \\ &= r|B_j||Y_j| \\ &= 2^{n-1} r^n |B_j|.\end{aligned}$$

Note that the argument in this paragraph did not depend on the supposition that $|\mathbf{q}|$ was achieved in the first of the n coordinates.

Now, by (6) and recalling that $W \subset \mathbb{I}^{nm}$ was an arbitrary ball of radius r, we have

$$|A(\mathbf{q},B)\cap W| \geq 2^{nm-m} r^{nm} \prod_{j=1}^{m} |B_j| \geq \frac{1}{2^m}|B||W|$$

for all $|\mathbf{q}| \geq 2/r$. Since $|A(\mathbf{q},B)| = |B|$, we are done. □

Lemma 8 (Overlap estimates). *Let $m \geq 1$. Suppose B_1 and B_2 are balls in $\mathbb{R}^m/\mathbb{Z}^m$ and let $r,q \in \mathbb{Z}\setminus\{0\}$. Then*

$$\left|A_{1,m}(r,B_1)\cap A_{1,m}(q,B_2)\right| \leq 2^m \left(|B_1||B_2| + |B_2||q|^{-m}\gcd(r,q)^m\right).$$

Proof. Since $A(q,B) = A(-q,-B)$, it is enough to prove this lemma for $r,q \geq 1$.

Suppose the radii of B_1 and B_2 are ψ_1 and ψ_2, respectively. Let

$$\delta = 2\min\left\{\frac{\psi_1}{r},\frac{\psi_2}{q}\right\} \qquad \text{and} \qquad \Delta = 2\max\left\{\frac{\psi_1}{r},\frac{\psi_2}{q}\right\}.$$

Then

$$|A(r,B_1)\cap A(q,B_2)| \leq \delta^m N$$

where

$$N = \#\left\{(\mathbf{a},\mathbf{b}) \in \mathbb{Z}^m \times \mathbb{Z}^m : |\mathbf{a}| \leq q, |\mathbf{b}| \leq r, r\mathbf{a} - q\mathbf{b} \in B\right\},$$

and B is a ball of diameter Δrq whose center depends on B_1, B_2, r, q, but we do not need to specify it here. Essentially, N gives us an upper bound on the number of regions of intersection we can possibly see between $A(r,B_1)$ and $A(q,B_2)$, and δ^m is a trivial upper bound for the measure of one of these regions of intersection.

If $\Delta qr \geq \gcd(q,r)$ then B can contain at most

$$\left(\frac{\Delta qr}{\gcd(q,r)} + 1\right)^m$$

integer points of the form $r\mathbf{a} - q\mathbf{b}$. Furthermore, an integer point can be realized in the form $r\mathbf{a} - q\mathbf{b}$ in at most $\gcd(q,r)^m$ different ways, for if

$$r\mathbf{a} - q\mathbf{b} = r\mathbf{a}' - q\mathbf{b}',$$

then this implies

$$\frac{\mathbf{a}-\mathbf{a}'}{q} = \frac{\mathbf{b}-\mathbf{b}'}{r},$$

and there are $\gcd(q,r)^m$ rational points of this form in $\mathbb{I}^m$. So, in this case, we can bound

$$\begin{aligned}|A(r,B_1)\cap A(q,B_2)| &\leq \left(\frac{2\Delta qr}{\gcd(q,r)}\right)^m \gcd(q,r)^m \delta^m \\ &= 2^m \Delta^m \delta^m q^m r^m \\ &= 8^m \psi_1^m \psi_2^m \\ &= 2^m |B_1||B_2|.\end{aligned}$$

On the other hand, if $\Delta qr < \gcd(q,r)$, then B contains at most one integer point of the form $r\mathbf{a} - q\mathbf{b}$ which (if it exists) can be realized in that form in at most $\gcd(q,r)^m$ different ways. Therefore, in this case, $N \le \gcd(q,r)^m$ and we have

$$|A(r,B_1) \cap A(q,B_2)| \le \delta^m \gcd(q,r)^m \le 2^m \frac{\psi_2^m}{q^m} \gcd(q,r)^m = |B_2| q^{-m} \gcd(q,r)^m.$$

Combining the two possible cases, $\Delta qr < \gcd(q,r)$ and $\Delta qr \ge \gcd(q,r)$, the proof of the lemma is complete. □

3 Proof of Theorem 4

The following definition is the way of quantifying "weak quasi-independence on average" that we described in Section 1.2.

Definition 1. Given a sequence of balls $\Psi := (B_q)_{q\in\mathbb{N}}$ in $\mathbb{R}^m/\mathbb{Z}^m$ and $w \in \mathbb{Z}$, we will say that the sets $\left(A_{n,m}(\mathbf{q}, B_{|\mathbf{q}|})\right)_{\mathbf{q}\in\mathbb{Z}^n}$ are *w-weakly quasi-independent on average* (w-QIA, for short) if we have

$$\limsup_{Q\to\infty} \left(\sum_{|\mathbf{q}|=1}^{Q} |B_{|\mathbf{q}|}|\right)^2 \left(\sum_{1\le|\mathbf{q}_1|\le|\mathbf{q}_2|\le Q} \left(\frac{\gcd(|\mathbf{q}_1|,|\mathbf{q}_2|)}{|\mathbf{q}_1|}\right)^w \left| A\left(\mathbf{q}_1, B_{|\mathbf{q}_1|}\right) \cap A\left(\mathbf{q}_2, \left(\frac{|\mathbf{q}_1|}{|\mathbf{q}_2|}\right)^{\frac{w}{m}} B_{|\mathbf{q}_2|}\right)\right|\right)^{-1} > 0. \tag{7}$$

For tidiness, we use the notation

$$\Pi(\mathbf{q}_1,\mathbf{q}_2,\Psi,w) = \Pi_{n,m}(\mathbf{q}_1,\mathbf{q}_2,\Psi,w) := \left| A\left(\mathbf{q}_1, B_{|\mathbf{q}_1|}\right) \cap A\left(\mathbf{q}_2, \left(\frac{|\mathbf{q}_1|}{|\mathbf{q}_2|}\right)^{\frac{w}{m}} B_{|\mathbf{q}_2|}\right)\right|$$

when $|\mathbf{q}_1| \le |\mathbf{q}_2|$, and for integers $q, r \ge 1$, we write

$$\Gamma(q,r) = \frac{\gcd(q,r)}{\min(q,r)}.$$

This way, the expression (7) becomes

$$\limsup_{Q\to\infty} \left(\sum_{|\mathbf{q}|=1}^{Q} |B_{|\mathbf{q}|}|\right)^2 \left(\sum_{1\le|\mathbf{q}_1|\le|\mathbf{q}_2|\le Q} \Gamma(|\mathbf{q}_1|,|\mathbf{q}_2|)^w \Pi(\mathbf{q}_1,\mathbf{q}_2,\Psi,w)\right)^{-1} > 0. \tag{8}$$

Note that w-QIA implies w'-QIA whenever $w \le w'$ and that 0-QIA coincides with QIA.

The independence inheritance claimed in Theorem 4 follows directly from a stronger quantitative independence inheritance theorem which we now state and prove.

Theorem 8 (Strong Independence Inheritance)**.** *Let $n, m \ge 1$ and let $w \ge 0$. Suppose $(B_q)_{q\in\mathbb{N}}$ is a sequence of balls in $\mathbb{R}^m/\mathbb{Z}^m$. If the sets*

$$\left(A_{n,m}(\mathbf{q}, B_{|\mathbf{q}|})\right)_{\mathbf{q}\in\mathbb{Z}^n}$$

are w-QIA, then the sets

$$\left(A_{n+k,m}(\mathbf{q}, |\mathbf{q}|^{-k/m} B_{|\mathbf{q}|})\right)_{\mathbf{q}\in\mathbb{Z}^{n+k}}$$

are $\max(w-k,0)$-QIA for every integer $k \ge 0$.

Proof. Let $\Psi_0 := (B_q)_{q\in\mathbb{N}}$ be the given sequence of balls in $\mathbb{R}^m/\mathbb{Z}^m$, and for each integer $k \ge 0$ denote $\Psi_k := (q^{-k/m}B_q)_{q\in\mathbb{N}}$.

It suffices to prove the theorem with $k = 1$ since all other cases can be deduced from this by induction. Furthermore, since 0-QIA implies 1-QIA, we may take $w \ge 1$. For $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{Z}^{n+1}$ with $|\mathbf{q}_i| = q_i$ and $q_1 \le q_2$, we are concerned with overlaps of the form

$$\Pi_{n+1,m}(\mathbf{q}_1,\mathbf{q}_2,\Psi_1,w-1) := \left| A_{n+1,m}(\mathbf{q}_1, q_1^{-1/m}B_{q_1}) \cap A_{n+1,m}\left(\mathbf{q}_2, \left(\frac{q_1}{q_2}\right)^{\frac{w-1}{m}} q_2^{-1/m}B_{q_2}\right)\right|. \tag{9}$$

If $\mathbf{q}_1 \nparallel \mathbf{q}_2$ then

$$\begin{aligned}\Pi_{n+1,m}(\mathbf{q}_1,\mathbf{q}_2,\Psi_1,w-1) &= \left|A_{n+1,m}(\mathbf{q}_1, q_1^{-1/m}B_{q_1})\right| \left|A_{n+1,m}\left(\mathbf{q}_2, \left(\frac{q_1}{q_2}\right)^{\frac{w-1}{m}} q_2^{-1/m}B_{q_2}\right)\right| \\ &\le \left|A_{n+1,m}(\mathbf{q}_1, q_1^{-1/m}B_{q_1})\right| \left|A_{n+1,m}(\mathbf{q}_2, q_2^{-1/m}B_{q_2})\right|\end{aligned}$$

by Lemma 4. In particular, we have

$$\sum_{1\le q_1\le q_2\le Q} \sum_{\substack{\mathbf{q}_1\nparallel\mathbf{q}_2\in\mathbb{Z}^{n+1}\\ |\mathbf{q}_i|=q_i}} \Gamma(q_1,q_2)^{w-1}\Pi_{n+1,m}(\mathbf{q}_1,\mathbf{q}_2,\Psi_1,w-1) \le \left(\sum_{\substack{\mathbf{q}\in\mathbb{Z}^{n+1}\\ |\mathbf{q}|\le Q}} |A_{n+1,m}(\mathbf{q},|\mathbf{q}|^{-1/m}B_{|\mathbf{q}|})|\right)^2. \tag{10}$$

On the other hand, for $\mathbf{q}_1 \parallel \mathbf{q}_2$, we have as a consequence of Lemma 5 that

$$\begin{aligned}&\sum_{1\le q_1\le q_2\le Q} \sum_{\substack{\mathbf{q}_1\parallel\mathbf{q}_2\in\mathbb{Z}^{n+1}\\ |\mathbf{q}_i|=q_i}} \Gamma(q_1,q_2)^{w-1}\Pi_{n+1,m}(\mathbf{q}_1,\mathbf{q}_2,\Psi_1,w-1) \\ &\qquad \ll \sum_{1\le q_1\le q_2\le Q} \sum_{\substack{\mathbf{r}_1\parallel\mathbf{r}_2\in\mathbb{Z}^{n}\\ |\mathbf{r}_i|=q_i}} \gcd(q_1,q_2)\Gamma(q_1,q_2)^{w-1}\Pi_{n,m}(\mathbf{r}_1,\mathbf{r}_2,\Psi_1,w-1),\end{aligned}$$

noting that in $\mathbb{Z}^d$ the number of parallel pairs with norms q_1, q_2 is comparable to $\gcd(q_1,q_2)^{d-1}$. Then by Lemma 6 we see that we may follow this with

$$\ll \sum_{1\le q_1\le q_2\le Q} \sum_{\substack{\mathbf{r}_1\parallel\mathbf{r}_2\in\mathbb{Z}^{n}\\ |\mathbf{r}_i|=q_i}} \Gamma(q_1,q_2)^{w}\Pi_{n,m}(\mathbf{r}_1,\mathbf{r}_2,\Psi_0,w). \tag{11}$$

By our assumption that the sets $\left(A_{n,m}(\mathbf{q},B_{|\mathbf{q}|})\right)_{\mathbf{q}\in\mathbb{Z}^n}$ are w-QIA, there are infinitely many values of $Q \in \mathbb{N}$ for which this last sum is

$$\ll \left(\sum_{\substack{\mathbf{r}\in\mathbb{Z}^n\\ |\mathbf{r}|\le Q}} |A(\mathbf{r},B_{|\mathbf{r}|})|\right)^2.$$

Since

$$\sum_{\substack{\mathbf{q}\in\mathbb{Z}^{n+1}\\ |\mathbf{q}|\le Q}} |A_{n+1,m}(\mathbf{q},|\mathbf{q}|^{-1/m}B_{|\mathbf{q}|})| \asymp \sum_{q=1}^{Q} q^{n-1}|B_q| \asymp \sum_{\substack{\mathbf{r}\in\mathbb{Z}^n\\ |\mathbf{r}|\le Q}} |A_{n,m}(\mathbf{r},B_{|\mathbf{r}|})|,$$

we have shown that there are infinitely many values of $Q \in \mathbb{N}$ for which

$$\sum_{1 \le q_1 \le q_2 \le Q} \sum_{\substack{\mathbf{q}_1 \| \mathbf{q}_2 \in \mathbb{Z}^{n+1} \\ |\mathbf{q}_i| = q_i}} \Gamma(q_1, q_2)^{w-1} \Pi_{n+1,m}(\mathbf{q}_1, \mathbf{q}_2, \Psi_1, w-1) \ll \left(\sum_{\substack{\mathbf{q} \in \mathbb{Z}^{n+1} \\ |\mathbf{q}| \le Q}} |A_{n+1,m}(\mathbf{q}, |\mathbf{q}|^{-1/m} B_{|\mathbf{q}|})| \right)^2 .$$

Combining this with (10), we see that there is some $C > 1$ and infinitely many values of $Q \in \mathbb{N}$ for which

$$\sum_{1 \le |\mathbf{q}_1| \le |\mathbf{q}_2| \le Q} \Gamma(q_1, q_2)^{w-1} \Pi_{n+1,m}(\mathbf{q}_1, \mathbf{q}_2, \Psi_1, w-1) \le C \left(\sum_{|\mathbf{q}| \le Q} \left| A_{n+1,m}(\mathbf{q}, |\mathbf{q}|^{-1/m} B_{|\mathbf{q}|}) \right| \right)^2 .$$

That is, the sets $\left(A_{n+1,m}(\mathbf{q}, |\mathbf{q}|^{-1/m} B_{|\mathbf{q}|})\right)_{\mathbf{q} \in \mathbb{Z}^{n+1}}$ are $(w-1)$-QIA as the theorem claims. The general statement follows by induction. □

To obtain the full measure part of Theorem 4, we require the following proposition.

Proposition 1 (Full measure). *Let $n, m \ge 1$ and suppose $(B_q)_{q \in \mathbb{N}}$ is a sequence of balls in $\mathbb{R}^m/\mathbb{Z}^m$. If the sets $\left(A(\mathbf{q}, B_{|\mathbf{q}|})\right)_{\mathbf{q} \in \mathbb{Z}^n}$ are 0-QIA and the sum $\sum_{\mathbf{q} \in \mathbb{Z}^n} |B_{|\mathbf{q}|}|$ diverges, then the set*

$$\limsup_{|\mathbf{q}| \to \infty} A_{n,m}(\mathbf{q}, B_{|\mathbf{q}|})$$

has full Lebesgue measure.

Proof. Let $U \subset \mathbb{I}^{nm}$ be an open set. Then for infinitely many $Q \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{|\mathbf{q}_1|, |\mathbf{q}_2| \le Q} |A(\mathbf{q}_1, B_{|\mathbf{q}_1|}) \cap A(\mathbf{q}_2, B_{|\mathbf{q}_2|}) \cap U| &\le C_1 \left(\sum_{|\mathbf{q}| \le Q} \left| A(\mathbf{q}, B_{|\mathbf{q}|}) \right| \right)^2 \\ &\overset{\text{Lemma 7}}{\le} \frac{C_2}{|U|^2} \left(\sum_{|\mathbf{q}| \le Q} \left| A(\mathbf{q}, B_{|\mathbf{q}|}) \cap U \right| \right)^2 , \end{aligned}$$

where $C_1 > 0$ is a universal constant coming from the 0-QIA assumption and $C_2 > 0$ is a constant which may depend on m. Now, as a consequence of Lemma 1, we see that

$$\left| \limsup_{|\mathbf{q}| \to \infty} A_{n,m}(\mathbf{q}, B_{|\mathbf{q}|}) \cap U \right| \ge \frac{|U|^2}{C_2} .$$

The theorem now follows by Lemma 3. □

Proof of Theorem 4. The independence statement follows directly from Theorem 8. The full measure statement follows from Proposition 1. □

4 Proof of Theorem 2

Proposition 2 (Base case for Theorem 2)**.** *Suppose $(B_q)_{q\in\mathbb{N}}$ is a sequence of balls in $\mathbb{R}^m/\mathbb{Z}^m$ such that $\sum |B_q|$ diverges.*

- *If $m \geq 3$, then the sets $(A_{1,m}(q,B_q))_{q\in\mathbb{N}}$ are quasi-independent on average.*
- *If $m \geq 2$, then the weaker estimate*

$$\sum_{1\leq r\leq q\leq Q} \frac{\gcd(q,r)}{r}\left|A_{1,m}(r,B_r)\cap A_{1,m}(q,(r/q)^{1/m}B_q)\right| \ll \left(\sum_{q=1}^{Q}|A_{1,m}(q,B_q)|\right)^2,$$

  holds.
- *If $m \geq 1$, then the still weaker estimate*

$$\sum_{1\leq r\leq q\leq Q} \left(\frac{\gcd(q,r)}{r}\right)^2\left|A_{1,m}(r,B_r)\cap A_{1,m}(q,(r/q)^{2/m}B_q)\right| \ll \left(\sum_{q=1}^{Q}|A_{1,m}(q,B_q)|\right)^2,$$

  holds.

Remark. In the language introduced in Definition 1, this proposition says that the sets are $\max\{(3-m),0\}$-QIA.

Proof of Proposition 2. First, suppose $m \geq 3$. By Lemma 8,

$$\sum_{q,r=1}^{Q}\left|A(r,B_r)\cap A(q,B_q)\right| \ll \sum_{q,r=1}^{Q}\left(|B_r||B_q| + |B_q|q^{-m}\gcd(q,r)^m\right).$$

The second sum on the right-hand side is

$$\begin{aligned}\sum_{q,r=1}^{Q}|B_q|q^{-m}\gcd(q,r)^m &\ll \sum_{q=1}^{Q}\sum_{r=1}^{q}|B_q|q^{-m}\gcd(q,r)^m\\ &= \sum_{q=1}^{Q}q^{-m}|B_q|\underbrace{\sum_{r=1}^{q}\gcd(q,r)^m}.\end{aligned}$$

The indicated gcd sum is $\ll q^m$ if $m \geq 3$. After all,

$$\begin{aligned}\sum_{r=1}^{q}\gcd(q,r)^m &\leq \sum_{d|q}d^m(q/d)\\ &= \sum_{d|q}(q/d)^m d\\ &\leq q^m\underbrace{\sum_{i=1}^{q}i^{-m+1}},\end{aligned}$$

and the indicated sum is bounded by an absolute constant for $m \ge 3$. Therefore, we have

$$\sum_{q,r=1}^{Q} \left|A(r,B_r)\cap A(q,B_q)\right| \ll \sum_{q,r=1}^{Q} |B_r||B_q| + \sum_{q=1}^{Q}|B_q| \ll \left(\sum_{q=1}^{Q}|B_q|\right)^2,$$

since $\sum |B_q|$ diverges.

Now let us suppose only that $m \ge 2$. Then a nearly identical calculation can be done. By Lemma 8, we have

$$\sum_{1\le r\le q\le Q} \frac{\gcd(q,r)}{r}\left|A(r,B_r)\cap A(q,(r/q)^{1/m}B_q)\right| \ll \sum_{1\le r\le q\le Q}\left(|B_r||B_q| + \frac{r}{q}|B_q|q^{-m}r^{-1}\gcd(q,r)^{m+1}\right).$$

This time the second sum on the right-hand side is

$$\sum_{1\le r\le q\le Q} |B_q|q^{-m-1}\gcd(q,r)^{m+1} \ll \sum_{q=1}^{Q} q^{-m-1}|B_q| \underbrace{\sum_{r=1}^{q}\gcd(q,r)^{m+1}}$$

and (by the same argument as above) the gcd sum is $\ll q^{m+1}$ since we have assumed $m \ge 2$ and so $m+1 \ge 3$. Therefore, we have

$$\sum_{1\le r\le q\le Q} \frac{\gcd(q,r)}{r}\left|A(r,B_r)\cap A(q,(r/q)^{1/m}B_q)\right| \ll \sum_{q,r=1}^{Q}|B_q||B_r| + \sum_{q=1}^{Q}|B_q| \ll \left(\sum_{q=1}^{Q}|B_q|\right)^2,$$

as in the previous paragraph.

Finally, suppose only that $m \ge 1$. Again by Lemma 8, we have

$$\sum_{1\le r\le q\le Q} \left(\frac{\gcd(q,r)}{r}\right)^2\left|A(r,B_r)\cap A(q,(r/q)^{2/m}B_q)\right| \ll \sum_{1\le r\le q\le Q}\left(|B_r||B_q| + \left(\frac{r}{q}\right)^2|B_q|q^{-m}r^{-2}\gcd(q,r)^{m+2}\right).$$

This time the second sum on the right-hand side is

$$\sum_{1\le r\le q\le Q} |B_q|q^{-m-2}\gcd(q,r)^{m+2} \ll \sum_{q=1}^{Q} q^{-m-2}|B_q| \underbrace{\sum_{r=1}^{q}\gcd(q,r)^{m+2}}$$

and the gcd sum is $\ll q^{m+2}$ since we have assumed $m \ge 1$. Therefore, we have

$$\sum_{1\le r\le q\le Q} \left(\frac{\gcd(q,r)}{r}\right)^2\left|A(r,B_r)\cap A(q,(r/q)^{2/m}B_q)\right| \ll \sum_{q,r=1}^{Q}|B_q||B_r| + \sum_{q=1}^{Q}|B_q| \ll \left(\sum_{q=1}^{Q}|B_q|\right)^2,$$

as before. This finishes the proof. □

Proof of Theorem 2. Let $nm > 2$. The convergence part is easily proved by an application of the First Borel–Cantelli lemma, so let us focus on the divergence part.

First, notice that we lose no generality by choosing some $c > 0$ and assuming that

$$(\forall q \in \mathbb{N}) \qquad q^{n-1}|B_q| \le c. \tag{12}$$

After all, for any q where it does not hold, we can shrink the ball until $q^{n-1}|B_q| = c$ and work with the possibly-shrunk sequence of balls instead. The divergence condition will still hold for the smaller balls. Let us therefore assume that (12) holds with a fixed $c < 1$.

Now we may obtain a new sequence of balls $\widehat{B}_q = q^{(n-1)/m} B_q \subset \mathbb{R}^m/\mathbb{Z}^m$. Notice that the sum $\sum |\widehat{B}_q| = \sum q^{n-1}|B_q|$ diverges by assumption, so Proposition 2 guarantees that the sets

$$A_{1,m}(q, \widehat{B}_q) = A_{1,m}(q, q^{(n-1)/m} B_q)$$

are $\max\{(3-m), 0\}$-QIA, in the sense of Definition 1. Therefore, by Theorem 8 the sets

$$A_{n,m}(\mathbf{q}, |\mathbf{q}|^{-(n-1)/m} \widehat{B}_{|\mathbf{q}|}) = A_{n,m}(\mathbf{q}, B_{|\mathbf{q}|})$$

inherit $\max\{(3-m)-(n-1), 0\}$-QIA, which, since $4-(m+n) \le 0$, is 0-QIA. Now, by Proposition 1,

$$\limsup_{|\mathbf{q}| \to \infty} A_{n,m}(\mathbf{q}, B_{|\mathbf{q}|})$$

has full measure, and this proves the theorem. □

5 Proof of Theorem 3

The following theorem is a more general version of Theorem 3 that does not require the balls to be concentric.

Theorem 9. *Suppose $nm = 2$ and $\varepsilon > 0$. For any sequence of balls $\Psi := (B_q)_{q \in \mathbb{N}} \subset \mathbb{R}^m/\mathbb{Z}^m$, we have*

$$\left|\mathscr{A}_{n,m}(\Psi)\right| = 1 \qquad \text{if} \qquad \sum_{q=1}^{\infty} q^{n-1} \left(\frac{\varphi(q)}{q}\right)^{1+\varepsilon} |B_q| = \infty.$$

Remark. The above theorem only has two cases: $(1,2)$ and $(2,1)$. The corresponding divergence conditions are

$$\sum_{q=1}^{\infty} \left(\frac{\varphi(q)}{q}\right)^{1+\varepsilon} |B_q| = \infty \qquad \text{and} \qquad \sum_{q=1}^{\infty} \left(\frac{\varphi(q)}{q}\right)^{1+\varepsilon} q|B_q| = \infty.$$

Remark. Following the arguments outlined in Section 1.3, one could obtain a Hausdorff measure analogue of Theorem 9 akin to statements seen in Section 1.3. We leave the details to the reader.

Proof of Theorem 9 for $(n,m)=(2,1)$. Suppose $(B_q)_{q\in\mathbb{N}}$ is a sequence of balls such that for some $\varepsilon>0$ the series

$$\sum_{q=1}^{\infty}\left(\frac{\varphi(q)}{q}\right)^{1+\varepsilon} q|B_q| \asymp \sum_{|\mathbf{q}|=1}^{\infty}\left(\frac{\varphi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{1+\varepsilon}|A(\mathbf{q},B_{|\mathbf{q}|})|=\infty.$$

By Lemma 5, we have

$$\begin{aligned}\sum_{\substack{1\le|\mathbf{q}_1|\le|\mathbf{q}_2|\le Q\\ \mathbf{q}_1\|\mathbf{q}_2}}\left|A(\mathbf{q}_1,B_{|\mathbf{q}_1|})\cap A(\mathbf{q}_2,B_{|\mathbf{q}_2|})\right| &= \sum_{\substack{1\le|\mathbf{q}_1|\le|\mathbf{q}_2|\le Q\\ \mathbf{q}_1\|\mathbf{q}_2}}\left|A_{1,1}(|\mathbf{q}_1|,B_{|\mathbf{q}_1|})\cap A_{1,1}(\pm|\mathbf{q}_2|,B_{|\mathbf{q}_2|})\right| \\ &\ll \sum_{1\le q_1\le q_2\le Q}\gcd(q_1,q_2)\left|A(q_1,B_{q_1})\cap A(q_2,B_{q_2})\right|, \quad (13)\end{aligned}$$

and by Lemma 6, we have

$$\left|A(q_1,B_{q_1})\cap A(q_2,B_{q_2})\right|\le\frac{1}{q_1}\left|A(q_1,q_1B_{q_1})\cap A(q_2,(q_1/q_2)q_2B_{q_2})\right|. \quad (14)$$

Combining (13) and (14) brings us to

$$\sum_{\substack{1\le|\mathbf{q}_1|\le|\mathbf{q}_2|\le Q\\ \mathbf{q}_1\|\mathbf{q}_2}}\left|A(\mathbf{q}_1,B_{|\mathbf{q}_1|})\cap A(\mathbf{q}_2,B_{|\mathbf{q}_2|})\right|\ll\sum_{1\le q_1\le q_2\le Q}\frac{\gcd(q_1,q_2)}{q_1}\left|A(q_1,q_1B_{q_1})\cap A(q_2,(q_1/q_2)q_2B_{q_2})\right|,$$

and, by Lemma 8, we have

$$\left|A(q_1,q_1B_{q_1})\cap A(q_2,(q_1/q_2)q_2B_{q_2})\right|\ll q_1^2|B_{q_1}||B_{q_2}|+q_1|B_{q_2}|q_2^{-1}\gcd(q_1,q_2).$$

Hence,

$$\begin{aligned}\sum_{1\le q_1\le q_2\le Q}&\frac{\gcd(q_1,q_2)}{q_1}\left|A(q_1,q_1B_{q_1})\cap A(q_2,(q_1/q_2)q_2B_{q_2})\right| \\ &\ll\left(\sum_{q=1}^{Q}q|B_q|\right)^2+\sum_{1\le q_1\le q_2\le Q}\frac{\gcd(q_1,q_2)^2}{q_2}|B_{q_2}| \\ &=\left(\sum_{q=1}^{Q}q|B_q|\right)^2+\sum_{q_2=1}^{Q}|B_{q_2}|q_2^{-1}\sum_{q_1=1}^{q_2}\gcd(q_1,q_2)^2.\end{aligned}$$

Now,

$$\sum_{r=1}^{q}\gcd(r,q)^2=\sum_{d|q}d^2\varphi\left(\frac{q}{d}\right)=\sum_{d|q}\left(\frac{q}{d}\right)^2\varphi(d)=q^2\sum_{d|q}\frac{\varphi(d)}{d^2}\le q^2\sum_{d|q}\frac{1}{d}=q\sum_{d|q}d\ll\frac{q^3}{\varphi(q)}, \quad (15)$$

by [21, Theorem 329]. Therefore, we have

$$\begin{aligned}\sum_{\substack{1\le|\mathbf{q}_1|\le|\mathbf{q}_2|\le Q\\ \mathbf{q}_1\|\mathbf{q}_2}}\left|A(\mathbf{q}_1,B_{|\mathbf{q}_1|})\cap A(\mathbf{q}_2,B_{|\mathbf{q}_2|})\right| &\ll\left(\sum_{q=1}^{Q}q|B_q|\right)^2+\sum_{q=1}^{Q}|B_q|\frac{q^2}{\varphi(q)} \\ &\ll\left(\sum_{|\mathbf{q}|=1}^{Q}|A(\mathbf{q},B_{|\mathbf{q}|})|\right)^2+\sum_{|\mathbf{q}|=1}^{Q}|A(\mathbf{q},B_{|\mathbf{q}|})|\frac{|\mathbf{q}|}{\varphi(|\mathbf{q}|)}, \quad (16)\end{aligned}$$

an estimate which can be extended to the nonparallel pairs $\mathbf{q}_1 \nparallel \mathbf{q}_2$, since they are genuinely pairwise independent.

Let $U \subset \mathbb{I}^2$ be an open set. Then by a combination of (16) and Lemma 7, for $Q \in \mathbb{N}$ sufficiently large, we have

$$\begin{aligned}\sum_{1\le|\mathbf{q}_1|\le|\mathbf{q}_2|\le Q}\left|A(\mathbf{q}_1,B_{|\mathbf{q}_1|})\cap A(\mathbf{q}_2,B_{|\mathbf{q}_2|})\cap U\right| &\ll \left(\sum_{|\mathbf{q}|=1}^{Q}|A(\mathbf{q},B_{|\mathbf{q}|})|\right)^2+\sum_{|\mathbf{q}|=1}^{Q}|A(\mathbf{q},B_{|\mathbf{q}|})|\frac{|\mathbf{q}|}{\varphi(|\mathbf{q}|)}\\ &\ll \frac{1}{|U|^2}\left(\sum_{|\mathbf{q}|=1}^{Q}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\right)^2+\frac{1}{|U|}\sum_{|\mathbf{q}|=1}^{Q}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\frac{|\mathbf{q}|}{\varphi(|\mathbf{q}|)}\\ &\ll \frac{1}{|U|^2}\left[\left(\sum_{|\mathbf{q}|=1}^{Q}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\right)^2+\sum_{|\mathbf{q}|=1}^{Q}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\frac{|\mathbf{q}|}{\varphi(|\mathbf{q}|)}\right]. \end{aligned} \tag{17}$$

The implicit constant does not depend on U.

Following a strategy from the proof of [34, Theorem 1.8], we let

$$D_\ell=\left\{q\in\mathbb{N}:2^\ell\le\frac{q}{\varphi(q)}<2^{\ell+1}\right\}.$$

If there exists $\ell\ge 0$ for which

$$\sum_{|\mathbf{q}|\in D_\ell}\left(\frac{\varphi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{1+\varepsilon}|A(\mathbf{q},B_{|\mathbf{q}|})|=\infty,$$

then we can restrict our attention to D_ℓ. In this case, estimate (16) would immediately lead to 0-QIA, and we would then be done by Proposition 1. So assume that there is no such ℓ, and put

$$\Sigma_\ell:=\sum_{|\mathbf{q}|\in D_\ell}\left(\frac{\varphi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{1+\varepsilon}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\quad\text{and}\quad \Sigma_{\ell,Q}:=\sum_{\substack{|\mathbf{q}|\in D_\ell\\1\le|\mathbf{q}|\le Q}}\left(\frac{\varphi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{1+\varepsilon}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|.$$

Notice that $\Sigma_\ell<\infty$ for every $\ell\ge 0$, by assumption, and that the Σ_ℓs form a divergent series, i.e. $\sum_{\ell=1}^{\infty}(\Sigma_\ell)=\infty$. This latter statement can be seen by taking the divergence hypothesis in the statement of the theorem together with Lemma 7. Then, by Lemma 2 and the symmetry inherent in the sum $\sum_{\substack{1\le|\mathbf{r}|,|\mathbf{q}|\le Q\\|\mathbf{r}|,|\mathbf{q}|\in D_\ell}}\left|A(\mathbf{q},B_{|\mathbf{q}|})\cap A(\mathbf{r},B_{|\mathbf{r}|})\cap U\right|$, we have

$$\begin{aligned}\left|\bigcup_{\substack{|\mathbf{q}|\in D_\ell\\|\mathbf{q}|\le Q}}A(\mathbf{q},B_{|\mathbf{q}|})\cap U\right| &\ge \frac{\left(\sum_{\substack{|\mathbf{q}|\in D_\ell\\|\mathbf{q}|\le Q}}\left|A(\mathbf{q},B_{|\mathbf{q}|})\cap U\right|\right)^2}{\sum_{\substack{1\le|\mathbf{r}|,|\mathbf{q}|\le Q\\|\mathbf{r}|,|\mathbf{q}|\in D_\ell}}\left|A(\mathbf{q},B_{|\mathbf{q}|})\cap A(\mathbf{r},B_{|\mathbf{r}|})\cap U\right|}\\ &\gg \frac{\left(\sum_{\substack{|\mathbf{q}|\in D_\ell\\|\mathbf{q}|\le Q}}\left|A(\mathbf{q},B_{|\mathbf{q}|})\cap U\right|\right)^2}{\sum_{\substack{1\le|\mathbf{r}|\le|\mathbf{q}|\le Q\\|\mathbf{r}|,|\mathbf{q}|\in D_\ell}}\left|A(\mathbf{q},B_{|\mathbf{q}|})\cap A(\mathbf{r},B_{|\mathbf{r}|})\cap U\right|}\end{aligned}$$

for every ℓ and Q for which the union on the left-hand side has positive measure. This is guaranteed to be the case for infinitely many ℓ, since the measure sum diverges. Then, for $Q \geq Q_\ell$ where Q_ℓ is sufficiently large that the estimates in (17) take effect, we have

$$\begin{aligned}
\left|\bigcup_{\substack{|\mathbf{q}|\in D_\ell\\|\mathbf{q}|\leq Q}} A(\mathbf{q},B_{|\mathbf{q}|})\cap U\right| &\gg |U|^2\left(\frac{\left(\sum_{\substack{|\mathbf{q}|\in D_\ell\\|\mathbf{q}|\leq Q}}\left|A(\mathbf{q},B_{|\mathbf{q}|})\cap U\right|\right)^2}{\left(\sum_{\substack{|\mathbf{q}|\in D_\ell\\1\leq|\mathbf{q}|\leq Q}}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\right)^2+\sum_{\substack{|\mathbf{q}|\in D_\ell\\1\leq|\mathbf{q}|\leq Q}}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\frac{|\mathbf{q}|}{\varphi(|\mathbf{q}|)}}\right)\\
&= |U|^2\left(1+\frac{\sum_{\substack{|\mathbf{q}|\in D_\ell\\1\leq|\mathbf{q}|\leq Q}}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\frac{|\mathbf{q}|}{\varphi(|\mathbf{q}|)}}{\left(\sum_{\substack{|\mathbf{q}|\in D_\ell\\1\leq|\mathbf{q}|\leq Q}}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\right)^2}\right)^{-1} \qquad (18)
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\sum_{\substack{|\mathbf{q}|\in D_\ell\\1\leq|\mathbf{q}|\leq Q}}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\frac{|\mathbf{q}|}{\varphi(|\mathbf{q}|)}}{\left(\sum_{\substack{|\mathbf{q}|\in D_\ell\\1\leq|\mathbf{q}|\leq Q}}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\right)^2} &= \frac{\sum_{\substack{|\mathbf{q}|\in D_\ell\\1\leq|\mathbf{q}|\leq Q}}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\left(\frac{\varphi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{1+\varepsilon}\left(\frac{|\mathbf{q}|}{\varphi(|\mathbf{q}|)}\right)^{2+\varepsilon}}{\left(\sum_{\substack{|\mathbf{q}|\in D_\ell\\1\leq|\mathbf{q}|\leq Q}}|A(\mathbf{q},B_{|\mathbf{q}|})\cap U|\left(\frac{\varphi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{1+\varepsilon}\left(\frac{|\mathbf{q}|}{\varphi(|\mathbf{q}|)}\right)^{1+\varepsilon}\right)^2}\\
&\leq \frac{(2^{\ell+1})^{2+\varepsilon}}{2^{2\ell(1+\varepsilon)}\Sigma_{\ell,Q}}.
\end{aligned}$$

Putting this into (18), we find that

$$\left|\bigcup_{\substack{|\mathbf{q}|\in D_\ell\\|\mathbf{q}|\leq Q}} A(\mathbf{q},B_{|\mathbf{q}|})\cap U\right| \gg |U|^2\left(\frac{1}{1+\frac{2^{2+\varepsilon}}{2^{\varepsilon\ell}\Sigma_{\ell,Q}}}\right).$$

Now, the fact that the Σ_ℓ form a divergent series implies that there are ℓ and corresponding Q for which $2^{\varepsilon\ell}\Sigma_{\ell,Q}$ is arbitrarily large. In particular, there are infinitely many $\ell \in \mathbb{N}$ (and corresponding $Q \in \mathbb{N}$) for which the above string of inequalities gives

$$\underbrace{\left|\bigcup_{\substack{|\mathbf{q}|\in D_\ell\\|\mathbf{q}|\leq Q}} A(\mathbf{q},B_{|\mathbf{q}|})\cap U\right|}_{C_\ell} \geq \frac{|U|^2}{2C},$$

where C is the implicit constant in the above estimates. Since the sets C_ℓ all have measure at least $|U|^2/2C$, their associated limsup set must have at least that measure. Furthermore, since

$$\limsup_{\ell\to\infty} C_\ell \subset \limsup_{|\mathbf{q}|\to\infty} A(\mathbf{q},B_{|\mathbf{q}|})\cap U,$$

this implies that

$$\left|\limsup_{|\mathbf{q}|\to\infty} A(\mathbf{q},B_{|\mathbf{q}|})\cap U\right| \geq \frac{|U|^2}{2C}.$$

The theorem now follows by Lemma 3. □

Proof of Theorem 9 for $(n,m)=(1,2)$. Suppose $(B_q)_{q\in\mathbb{N}}$ is a sequence of balls such that for some $\varepsilon>0$ the series

$$\sum_{q=1}^{\infty}\left(\frac{\varphi(q)}{q}\right)^{1+\varepsilon}|B_q| \asymp \sum_{|\mathbf{q}|=1}^{\infty}\left(\frac{\varphi(|\mathbf{q}|)}{|\mathbf{q}|}\right)^{1+\varepsilon}|A(\mathbf{q},B_{|\mathbf{q}|})| = \infty.$$

Lemma 8 gives

$$\begin{aligned}
\sum_{1\leq r\leq q\leq Q}\left|A_{1,2}(r,B_r)\cap A_{1,2}(q,B_q)\right| &\ll \sum_{1\leq r\leq q\leq Q}\left(|B_r||B_q| + |B_q|q^{-2}\gcd(r,q)^2\right)\\
&\ll \left(\sum_{1\leq q\leq Q}|B_q|\right)^2 + \sum_{1\leq q\leq Q}|B_q|q^{-2}\sum_{r=1}^{q}\gcd(r,q)^2\\
&\overset{(15)}{\ll} \left(\sum_{1\leq q\leq Q}|B_q|\right)^2 + \sum_{1\leq q\leq Q}|B_q|\frac{q}{\varphi(q)}\\
&\ll \left(\sum_{q=1}^{Q}|A(q,B_q)|\right)^2 + \sum_{q=1}^{Q}|A(q,B_q)|\frac{q}{\varphi(q)}.
\end{aligned}$$

Now the rest of the proof follows the proof for the $(2,1)$ case verbatim, starting at (16) and replacing every instance of $\mathbf{q}$ with q. □

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