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When bullwhip increases in the lead time: An eigenvalue analysis of ARMA demand

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ABSTRACT

Problem definition. The impact of lead times on the bullwhip effect produced by the order-up-to (OUT) replenishment policy is studied. *Practical relevance*. Under general auto-regressive moving average (ARMA) demand, we investigate when the OUT policy possesses an always-increasing-in-the-lead-time bullwhip effect and when it does not. *Methodology*. A bullwhip measure based on the difference between the demand and order variance is combined with a novel analysis based on the eigenvalues and impulse response of the ARMA demand process. *Contribution*. We show a positive demand impulse response is a necessary and sufficient condition for an increasing in the lead time bullwhip effect. The ordering of zeros and poles (the eigenvalue ordering) of the z-transform transfer function of the demand process reveals when the demand impulse is positive. To provide further insight, we study ARMA(2,2) demand, which contains six different eigenvalue orderings. Two of these orderings satisfy a sufficient condition (positive demand eigenvalues in a particular order) for a positive impulse response. Two orderings satisfy the inverse of this sufficient condition and do not possess a positive impulse response. The final two orderings do not satisfy the sufficient condition, nor its inverse, but do contain positive impulse responses. *Managerial implications*. Our findings are important as reducing lead-times is often advocated as an improvement action to reduce the bullwhip effect. By identifying the demand characteristics that lead to a bullwhip effect that increases in the lead time we offer prescriptive advice on when, and when not, to invest in lead time reduction.

1. Introduction[1]

Since the important contributions by Lee et al. (1997, 2000), the bullwhip effect has been extensively studied. The bullwhip effect is a term used to describe a supply chain phenomenon where the variance of the outgoing orders is larger than the variance of the incoming demands at each echelon of the supply chain. While little general knowledge has been gained about influence of the lead time on the bullwhip, it is often advocated that the bullwhip effect is an increasing function of the lead time. Indeed, Zhang (2004b) asserts “In general, increasing lead time enhances [the] bullwhip effect regardless of the forecasting methods employed. However, the size of the impact does depend on the forecasting methods”. This advice could motivate companies to reduce lead time by investing in faster production technology or using quicker transportation modes. These actions are often costly and the question of when a lead time reduction yields a bullwhip benefit is the focus of this paper.

1.1. Literature review

We will not present a through literature of the bullwhip effect here; others (Bhattacharya and Bandyopadhyay, 2011), Wang and Disney (2016), and Yang et al. (2021) provide extensive review papers already. Bhattacharya and Bandyopadhyay (2011) provide a review of the literature on the causes of the bullwhip effect, identifying 19 main causes of the bullwhip effect; lead times and lead time variability were identified as important causes. From an operational research perspective, Wang and Disney (2016) found the linear order-up-to (OUT) policy was the most often studied replenishment policy. The OUT policy can

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[1] Early versions of this research were presented at the 9th IFAC/IFIP/IFORS/IISE/INFORMS Conference on Manufacturing Modelling, Management and Control MIM in Berlin, Germany (Gaalman et al., 2019a) , 30th Production and Operations Management Society Annual Conference in Washington USA, (Gaalman et al., 2019b), the 20th International Working Seminar of Production Economics in Innsbruck, Austria (Gaalman et al., 2018), and the 4th World Production and Operations Management Conference in Amsterdam, Holland (Gaalman and Disney, 2012).

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be also be found in many enterprise resource planning (ERP) systems as a standard replenishment policy for controlling high volume products that are ordered and replenished in every period. Yang et al. (2021) conducted a systematic literature review of the behavioural causes of the bullwhip effect. They found the mental model used to make sense of the replenishment decision was the most studied cause of the bullwhip effect in the behavioural bullwhip literature. The anchoring and adjustment heuristic of Sterman (1989), is an important mental model and has long been known to be a cause of judgemental bias which is also closely related to the OUT policy, Riddalls and Bennett (2002).

To focus our literature review on the topics directly relevant to our problem we restrict ourselves to papers that consider: the OUT policy, ARIMA demand processes, and the influence of the lead time on the bullwhip effect. We will introduce references to the technical aspects of our study as required.

When demand is independently and identically distributed, IID, the bullwhip generated by the OUT policy with constant forecasts is independent of the lead-time as always $o_t = d_t$. The first order auto-regressive, AR(1), process is the simplest demand process with auto-correlation (Urban, 2005; Luong, 2007). Zhang (2004b) shows the bullwhip effect in an increasing function of the lead-time under AR(1) demand with positive auto-correlation. First-order auto-regressive moving average, ARMA(1,1), demand processes were studied by Gaalman (2006), Chen and Disney (2007), and Duc et al. (2008). Duc et al. (2008) reveals that bullwhip is increasing in the lead time when the moving average parameter θ is smaller than a positive auto-regressive parameter $\phi > 0$. They also provide a bullwhip expression valid when the lead time approaches infinity, concluding the bullwhip does not always increase in the lead time. Duc et al. (2008) also notes that there is an oscillating bullwhip effect in the lead-time when $\phi < 0$. Chen and Lee (2016) show the bullwhip effect is an increasing function of the lead time under integrated moving average, IMA(0,1,1), demand. The second order ARMA(2,2) process is less frequently considered, although (Gaalman and Disney, 2009) studied the bullwhip produced by a family of the OUT policies reacting to this demand process. Luong and Phien (2007) considered AR(2) demand, noting that "due to the complicated functional form of the bullwhip measure... it is impossible to identify the range of [demand parameter] values ϕ_1 and ϕ_2 in which the bullwhip effect will not always increase when [the lead time] L increases".

Studies that consider higher order ARMA models, or models with seasonal factors, are rare and often conducted via simulation. For example, Bayraktar et al. (2008) use simulation to show that under highly seasonal demand it is necessary to use smaller exponential smoothing parameters to avoid creating excessive bullwhip effects compared to the smoothing parameters recommended for non-seasonal demand. Chen and Lee (2009) study a general demand pattern based on the martingale method of forecast evolution revealing the bullwhip effect was influenced by the correlation in the forecast errors.

Gilbert (2005) studies general auto-regressive, integrated, moving average (ARIMA) demand. He reveals that multi-echelon supply chains with linear OUT policies have the same dynamic behaviour as a single echelon OUT supply chain with a lead time equal to the sum of all the downstream lead-times. Gilbert (2005) also shows how the ARIMA end customer demand changes its structure as it is transferred up the supply chain. Zhang (2004a) considered ARMA end customer demand and noticed an ARMA-in-ARMA-out property that could be predicted with a simple algorithm in OUT policy based supply chains.

General statements (that hold for more than one demand process) about the interaction between the bullwhip effect and the lead-time are rather rare in the literature. Dejonckheere et al. (2003) provide one of the only references that explicitly considers the link between lead times and the bullwhip effect. They show, for all demand processes and for all lead times, the OUT replenishment policy, with exponential smoothing and moving average forecasts, always generates bullwhip. This was extended by Li et al. (2014) who showed Holt's method also has this guaranteed always-increasing-in-the-lead-time bullwhip behaviour, but that damped trend forecasting does not.

1.2. Contribution

While others have revealed the bullwhip/lead time behaviour for *specific ARMA demand* processes, our contribution herein is to determine, *for general ARMA demand*, when the OUT policy with minimum mean squared error (MMSE) forecasting possesses an always increasing in the lead time bullwhip effect and when it does not. We show it is related to positivity of the demand impulse response, which in turn is characterized by the order of the demand eigenvalues. We start by considering the general ARMA(p,q) process, before focusing on the special case of the ARMA(2,2) demand process. We are able to unify all previously known results within a unique theoretical approach based on the order of the eigenvalues. As the demand impulse response is equivalent to the system's autocovariance function, our results provide a framework for managers to predict the bullwhip consequences of altering lead times from the demand process alone.

1.3. Paper structure

The structure of paper is as follows. In Section 2 we review the ARMA(p,q) demand process. In Section 3, we derive the order-up-to policy and propose a new measure for the bullwhip effect. Section 4 highlights the link between the impulse response and the variances required for the bullwhip measure. Section 5 presents our main result, a positive demand impulse response leads to a bullwhip effect that increases in the lead time. Section 6 identifies the necessary and sufficient conditions for a positive ARMA(2,2) impulse response based on the ARMA(2,2) eigenvalues. In Section 7, we study the weekly time series from the so-called M4 dataset and also reflect on how a practicing manager may use our approach, facilitated by an interactive website we have constructed with the R-shiny technology. Section 8 concludes.

2. The ARMA(p,q) demand process

We assume the demand process follows a mean centred ARMA(p,q) process, Box et al. (2008),

$$d_t = \mu_d + \sum_{i=1}^{p} \phi_i \left(d_{t-i} - \mu_d\right) - \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t. \tag{1}$$

Here, d_t is the demand in time period t, μ_d is the mean demand, ϕ_i are p auto-regressive coefficients, θ_j are q moving average coefficients, and ϵ_t is an IID random variable with zero mean and variance σ_ϵ^2. Note, we have made no assumptions about the distribution of the random variable ϵ_t.

2.1. Transfer function of the ARMA(p,q) demand process

Methodologically, we will exploit z-transforms in our analysis. We refer readers to a good control engineering textbook such as (Nise, 2004) or Moudgalaya (2007) for more information on our general methodology and to Disney and Lambrecht (2008) for a review of control theory in a bullwhip/supply chain context. The z-transform requires a linear system to exist; this has both advantages and disadvantages. On the positive side we are able to obtain analytical expressions for the variances of the demand and the orders. However, linear systems dictate we must assume: backlogs rather than lost sales exist, no capacity constraints exist, and that negative demand implies that customers can return unwanted product; negative orders imply finished goods inventory is disassembled into raw materials (or raw materials are returned to suppliers). However, for non-linear systems, a local linear approximation can offer acceptable insights, Lee et al. (2000).

The ARMA(p,q) demand process has the following z-transform transfer function,

$$D[z] = \frac{z^m - \theta_1 z^{m-1} - \cdots - \theta_{m-1} z - \theta_m}{z^m - \phi_1 z^{m-1} - \cdots - \phi_{m-1} z - \phi_m}; \quad m = \max[p, q]. \tag{2}$$

If $q < p$ then $\forall j > q$, $\theta_j = 0$; if $p < q$ then $\forall i > p$, $\phi_i = 0$.

The z-transform representation is useful as it allows us to identify several important characteristics of the demand process.

3. Deriving the inventory cost optimal order-up-to replenishment policy

We consider a discrete time (periodic) inventory system. At the beginning of a time period (at time t), the state of the system is observed; demand and receipts since the last replenishment order was generated are tallied, forecasts for future demands are generated, inventory and work-in-progress (WIP) levels are observed, and replenishment orders are calculated. Notice, the order o_{t-k} is determined during the replenishment period $t-k$; k is the lead-time. When $k=0$, an order is received within the same period it was placed, and its arrival will be observed and accounted for in the next ordering decision made at time $t+1$. With these sequence of events, the following inventory balance equation exists,

$$i_t = i_{t-1} + o_{t-k-1} - d_t. \tag{3}$$

Here, i_t is the inventory level at time t, o_{t-k} is the replenishment order placed k periods ago, and d_t is the demand in period t.

It is desirable that the mean value of the replenishment orders is equal to the mean demand, i.e. $\mu_o = \mu_d$. The mean (target) inventory, μ_i, is not prescribed by (3) and can be set arbitrarily. With per unit, per period, linear and convex inventory holding and backlog costs, h and b, setting the target (mean) inventory $\mu_i = F^{-1}[b/(b+h)]$ to the critical newsvendor fractile[2] minimizes the expected inventory holding and backlog costs (Churchman et al., 1957). The mean inventory value is influenced by the standard deviation of the inventory levels which, in turn, is influenced by the ordering policy.

To find an optimal linear replenishment policy for minimizing the standard deviation of the inventory levels we first notice from (3) that an order placed at time t (at the beginning of time period t) will have its first consequences on the inventory in period $t+k+1$,

$$i_{t+k+1} = i_{t+k} + o_t - d_{t+k+1}. \tag{4}$$

Therefore, the order placed, o_t, must account for the future (and so it must be predicted) inventory and the future (and so it must be predicted) demand. These forecasted demand and inventory variables ($\hat{d}$ and $\hat{i}$) are conditional upon information available at time t (Gaalman and Disney, 2009). Consider how the forecasted inventory builds up over the lead-time and review period,

$$\begin{aligned} \hat{i}_{t+1|t} &= i_t + o_{t-k} - \hat{d}_{t+1|t} \\ \hat{i}_{t+2|t} &= \hat{i}_{t+1|t} + o_{t-k+1} - \hat{d}_{t+2|t} \\ \hat{i}_{t+3|t} &= \hat{i}_{t+2|t} + o_{t-k+2} - \hat{d}_{t+3|t} \\ &\vdots \\ \hat{i}_{t+k+1|t} &= \hat{i}_{t+k|t} + o_t - \hat{d}_{t+k+1|t} \end{aligned} \tag{5}$$

Since the first inventory level that o_t can directly influence is i_{t+k+1}, we set the forecasted inventory level $k+1$ periods ahead to the target mean inventory $\hat{i}_{t+k+1} = \mu_i$. The last equation in (5) then becomes

$$\mu_i = \hat{i}_{t+k|t} + o_t - \hat{d}_{t+k+1|t}. \tag{6}$$

Finally, rearranging (6) and using the relationship $\hat{i}_{t+k|t} = \hat{i}_{t+k-1|t} + o_{t-1} - \hat{d}_{t+k|t}$ from (5) recursively until we eliminate the forecasted inventory $\hat{i}_t$ we obtain,

$$\begin{aligned} o_t &= \mu_i + \hat{d}_{t+k+1|t} - \hat{i}_{t+k|t} \\ o_t &= \mu_i + \hat{d}_{t+k+1|t} + \hat{d}_{t+k|t} - o_{t-1} - \hat{i}_{t+k-1|t} \\ o_t &= \mu_i + \hat{d}_{t+k+1|t} + \hat{d}_{t+k|t} + \hat{d}_{t+k-1|t} - o_{t-1} - o_{t-2} - \hat{i}_{t+k-2|t} \\ &\vdots \end{aligned}$$

$$o_t = \hat{d}_{t+k+1|t} - (i_t - \mu_i) - \sum_{j=1}^{k} \left(o_{t-j} - \hat{d}_{t+j|t} \right). \tag{7}$$

We recognize (7) as the linear OUT policy, Lee et al. (2000). The inventory level plus the open orders (the WIP, the in-transit inventory), $i_t + \sum_{j=1}^{k} o_{t-j}$, is also known as the *inventory position*. Furthermore, $\hat{d}_{t+j|t}$ is a forecast of the demand in period $t+j$ conditional upon the information available at time t. As more accurate forecasts of the demand over the lead time and review period result in smaller inventory variances, lower average inventory levels and small inventory holding and backlog costs, we will adopt the MMSE forecasting method (Box et al., 2008).

Zhang (2004b) shows the OUT policy can also be represented by

$$o_t = d_t + s_t - s_{t-1} \tag{8}$$

where the dynamic order-up-to level, $s_t = \sum_{j=1}^{k+1} \hat{d}_{t+j|t}$. s_t is also known as the *base stock level*. Eq. (8) can be obtained from (7) by noticing that

$$o_t = s_t + \mu_i - \sum_{j=1}^{k} o_{t-j}. \tag{9}$$

The difference between o_t and o_{t-1} is given by

$$o_t - o_{t-1} = s_t + \mu_i - \sum_{j=1}^{k} o_{t-j} - \left(s_{t-1} + \mu_i - \sum_{j=1}^{k} o_{t-j-1} \right). \tag{10}$$

Using $o_{t-k-1} = i_t - i_{t-1} + d_t$ from (3), cancelling like terms reveals (8). Lee et al. (1997) highlights that updating the dynamic base stock level, s_t, is an important cause of the bullwhip effect.

3.1. The bullwhip criterion and the impulse response

Bullwhip effect is usually measured as the ratio of σ_o^2, the long run variance of the replenishment orders o_t, divided by σ_d^2, the long run variance of the demand, d_t, Disney and Towill (2003),

$$BI = (\sigma_o^2/\sigma_d^2). \tag{11}$$

Demand must be stationary for these variances to exist. When demand becomes non-stationary, (11) incorrectly suggests $BI = 1$; that is, bullwhip is not present, Gaalman and Disney (2012). An alternative bullwhip criterion, $CB[k]$, provides a better measure that avoids this problem,

$$CB[k] = (\sigma_o^2 - \sigma_d^2)/\sigma_\epsilon^2. \tag{12}$$

There is a clear equivalence between $CB[k]$ and BI; when $CB[k] > 0$, $BI > 1$ and a bullwhip effect exists, when $CB[k] < 0$, $0 \le BI < 1$ and the orders have less variance than the demand. Notice, only σ_o^2 is affected by the lead-time; σ_d^2 is unaffected. Thus, our insights on the bullwhip behaviour remain, regardless of whether (11) or (12) is used to quantify the bullwhip effect.

To facilitate the derivation of our main result (Theorem 1), we elect to use $CB[k]$ here (as also used by Zhang (2005)). The order and demand variances, σ_o^2 and σ_d^2, can be readily obtained by Tsypkin's squared impulse response theorem; the variance of the noise, σ_ϵ^2, is given by assumption. The impulse response is the system's output when the system input is zero $\forall t$ except at $t=0$ when the input is unity.[3] In our case here, the system input is the random noise, ϵ_t, and the output is either the demand d_t, or the orders o_t. The impulse response is also equivalent to the system's autocovariance function and can be easily obtained from the inverse z-transform of the system's transfer function, Nise (2004).

[2] Here $F^{-1}[\cdot]$ is the inverse cumulative distribution function (cdf) of the arbitrary inventory distribution. Linear combinations of normally distributed random variables are also normally distributed; if ϵ_t is normally distributed, $F^{-1}[\cdot]$ can be replaced by the inverse cdf of the normal distribution, $\Phi^{-1}[\cdot]$.

[3] Example impulse responses can be seen in Figs. 3–8.

Lemma 1 (*Tsypkin's Squared Impulse Response Theorem*). *A linear system reacting to an IID white noise input, with variance σ_ϵ^2, has an output x_t, whose long-run variance σ_x^2, is given by the sum of its squared impulse response, $\tilde{x}_t^2$.*

$$\sigma_x^2 = \sigma_\epsilon^2 \sum_{t=0}^{\infty} \tilde{x}_t^2. \tag{13}$$

Proof. All proofs of the Lemmas, Corollaries, and Theorems in this paper are housed in the Appendix. □

4. Impulse responses

Lemma 1 shows that the variances used to measure the bullwhip effect can be obtained from the system's impulse responses. In Section 4.1, we consider the transfer function of the demand and it's impulse response; in Section 4.2 we consider the order impulse response.

4.1. The ARMA(p,q) demand impulse response

The zero-pole form of the rational transfer function (2) is

$$D[z] = \frac{\prod_{i=1}^{m}(z-\lambda_i^{\theta})}{\prod_{i=1}^{m}(z-\lambda_i^{\phi})}. \tag{14}$$

Here λ_i^θ are the zeros and λ_i^ϕ are poles of the transfer function. The zeros are the roots of the numerator of (14) w.r.t. z and are related to the moving average coefficients. The poles are the roots of the denominator of (14) w.r.t. z and are related to the auto-regressive coefficients. The poles and zeros are collectively known as the eigenvalues of the system.

Lemma 2 (*Impulse Response of ARMA(p,q) Demand*). *The impulse response of the ARMA(p,q) demand process, $\tilde{d}_t$, given by*

$$\tilde{d}_{t+1} = \begin{cases} 1, & \text{if } t = 0, \\ \sum_{j=1}^{m} r_j (\lambda_j^{\phi})^t & \text{if } t \geq 1, \end{cases} \tag{15}$$

where,

$$r_j = \frac{\prod_{i=1}^{m}(\lambda_j^{\phi}-\lambda_i^{\theta})}{\prod_{\{i=1, i\neq j\}}^{m}(\lambda_j^{\phi}-\lambda_i^{\phi})}. \quad \square \tag{16}$$

Eq. (15) is fundamental to our analysis and is the basis of all our results. For $t > 0$, the sum of the m power functions, $(\lambda_j^\phi)^t$, is determined by the poles and m coefficients, r_j, which are functions of the poles and zeros. Eq. (15) holds wherever the poles and zeros are located in the complex z-plane. Common poles are allowed and do not lead to fundamentally different insights from our analysis. Furthermore, if the poles are real and positive, the impulse response (12) has a maximum of $m-1$ changes of sign, a property that will be relevant later.

4.2. The impulse response of the orders

Section 4.1 provided the impulse response of the ARMA(p,q) demand. In this section, we derive the impulse response of the replenishment orders.

Lemma 3 (*Impulse Response of the Orders*). *The order impulse response, $\tilde{o}_t$, is*

$$\tilde{o}_t = \begin{cases} \sum_{j=0}^{k+1} \tilde{d}_{t+j}, & \text{if } t = 0, \\ \tilde{d}_{t+k+1}, & \text{if } t > 0. \end{cases} \quad \square \tag{17}$$

5. Demand and order variances

Lemma 1 provides both the long run variance of the demand,

$$\sigma_d^2 = \sigma_\epsilon^2 \sum_{t=0}^{\infty} \tilde{d}_t^2 = \sigma_\epsilon^2 \left(\sum_{t=0}^{k+1} \tilde{d}_t^2 + \sum_{t=k+2}^{\infty} \tilde{d}_t^2 \right), \tag{18}$$

and the long run variance of the orders

$$\sigma_o^2 = \sigma_\epsilon^2 \left(\left(\sum_{j=0}^{k+1} \tilde{d}_j \right)^2 + \sum_{t=k+2}^{\infty} \tilde{d}_t^2 \right). \tag{19}$$

Using (18) and (19), $CB[k] = (\sigma_o^2 - \sigma_d^2)/\sigma_\epsilon^2$ becomes

$$CB[k] = \left(\sum_{j=0}^{k+1} \tilde{d}_j \right)^2 - \sum_{t=0}^{k+1} \tilde{d}_t^2. \tag{20}$$

These relations hold regardless of the demand process. That is, they hold for all types of demand processes, not just ARMA(p,q) demands processes.

Corollary 1 (*Influence of the ARMA Coefficients on the Bullwhip Effect*). *Only the first $k+1$ pairs of ARMA coefficients determine whether a bullwhip effect exists or not.* □

If $k \geq m-1$, all m of the ARMA(p,q) parameter pairs determine whether bullwhip is present.

Theorem 1 (*Necessary-Sufficient Condition for an Increasing Bullwhip Effect*). *$CB[k]$ is always positive and increasing in the lead time $\forall k$ iff $\{\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_{k+1}\} > 0$.* □

Theorem 1, shows that bullwhip is always present and always increasing in the lead-time if, and only if, the demand impulse response is positive for all t; that is, $CB[k]$ is increasing in k iff $\forall t$, $\tilde{d}_t > 0$. Theorem 1 holds for all ARMA(p,q) demand processes, not just for ARMA(2,2) demand processes. There is one important subtlety to consider that we capture in the following Corollary.

Corollary 2 (*Necessary-sufficient Condition for an Increasing Order Variance*). *$CB[k]$ is increasing in the lead time k iff $\{\tilde{d}_2, \ldots, \tilde{d}_{k+1}\} > 0$ and $\tilde{d}_1 > -1$.* □

Notice the subtle difference between Theorem 1 and Corollary 2. Theorem 1 shows when the bullwhip effect is present when $k = 0$ and increases in the lead time k. Corollary 2 shows when bullwhip is not present when $k = 0$ and that the order variance increases in the lead time k.

Corollary 3 (*Bullwhip Lead Time Behaviour Under ARMA(1,1) Demand*). *Stable and invertible ARMA(1,1) demand processes are positive $\forall t$, iff $\phi > 0$ and $\theta < \phi$; for these demand processes, bullwhip increases in the lead time.* □

Corollary 3 simply recovers Proposition 5 in Duc et al. (2008).

Corollary 4 (*Bullwhip Lead Time Behaviour Under MA(q) Demand*). *The bullwhip effect under an invertable[4] MA(q) demand process is increasing in lead-time iff $\forall j$, $\theta_j < 0$.* □

Corollary 4 shows that iff $\forall j$, $\theta_j < 0$, bullwhip is present when $k = 0$ and increases in the lead time until $k = q$, after which bullwhip remains constant.

Corollary 5 (*Bullwhip Lead Time Behaviour Under AR(p) Demand*). *The bullwhip effect under a stable[5] AR(p) demand process is increasing in lead-time if $\forall j$, $\phi_j > 0$.* □

[4] For more information on invertibility see Section 6.1.

[5] For more information on stability, see Section 6.1.

Corollary 5 recovers the result of Luong and Phien (2007) in a simple and direct manner.

Theorem 2 (*Sufficient Condition for a Positive Impulse Response*). *When all the AR eigenvalues are positive, $0 \leq \lambda_j^{\phi} < 1$, if for each λ_j^{ϕ} there are more MA eigenvalues smaller than λ_j^{ϕ} than there AR eigenvalues smaller than λ_j^{ϕ}, then $\forall t$, $\tilde{d}_{t+1} > 0$.* □

For Theorem 2 to hold, the smallest eigenvalues must always be a zero and the largest eigenvalue must always be a pole. Note, to exploit the convolution property, negative poles are not allowed in Theorem 2. However, all poles can be at the origin and negative zeros are allowed. From a control theory perspective, as the λ_j^{ϕ} eigenvalues dominate the λ_j^{θ} eigenvalues, the demand process has low pass frequency filter characteristics, Nise (2004). Low pass filters are an important control theory concept as they are able to filter out high frequencies (noise) and pass on the low frequency (long term trends in demand). Low pass filters are also able to remove resonance (seasonality). The OUT policy strives to reduce inventory fluctuations caused by the demand and as a result the orders have relatively more low frequency harmonics than the demand. For low pass demand processes, this increases the order variance compared with the demand variance.

Corollary 6 (*An Eigenvalue Ordering with a Positive Impulse Response*). *If $\lambda_1^{\phi} > 0 \wedge \lambda_1^{\theta} < \lambda_1^{\phi} < \lambda_2^{\theta} < \lambda_2^{\phi} < \cdots < \lambda_m^{\theta} < \lambda_m^{\phi} < 1$ a positive demand impulse response exists, and bullwhip is always present and increasing in the lead time.*[6] □

Note, $\lambda_1^{\phi} > 0$ allows that $\lambda_1^{\theta} < 0$ in Corollary 6. Further note, if a zero λ_i^{θ} (in an ordering that satisfies Theorem 2) decreases, then $\forall k$ $CB[k]$ increases. As a consequence the ordering given by Corollary 6 has the 'weakest' bullwhip effect and the ordering $\lambda_1^{\phi} > 0 \wedge \lambda_1^{\theta} < \lambda_2^{\theta} < \cdots < \lambda_m^{\theta} < \lambda_1^{\phi} < \lambda_2^{\phi} < \cdots < \lambda_m^{\phi} < 1$ has the 'strongest' bullwhip effect.

The inverse of Theorem 2: If the number of AR eigenvalues smaller than each of the λ_j^{θ} is larger than the number of MA eigenvalues smaller than λ_j^{θ}, then the demand impulse response is not always positive. When the λ_j^{θ} eigenvalues dominate the λ_j^{ϕ} eigenvalues, high-frequency harmonics are present in the demand process.

Corollary 7 (*An Eigenvalue Ordering with a Negative Impulse Response*). *If $0 < \lambda_1^{\phi} < \lambda_1^{\theta} < \lambda_2^{\phi} < \lambda_2^{\theta} < \cdots < \lambda_m^{\phi} < \lambda_m^{\theta} < 1$ then a negative demand impulse exists, and bullwhip is not present and the order variance is always decreasing in the lead time.* □

When Corollary 7 holds, after $\tilde{d}_0 = 1$ and $-1 < \tilde{d}_1 < 0$, all subsequent demands are negative and increasing. This means when the lead time $k = 0$, the order variance is less than the demand variance (i.e. the bullwhip effect is not present), and the order variance is decreasing in the lead time.

In general there are $(2m)!/(m!)^2$ possible eigenvalue orderings. The eigenvalue orderings can be split into 3 subsets: one set that potentially satisfies Theorem 2, one set satisfying the inverse of Theorem 2, and the set of remaining eigenvalue orderings. This last subset also contains orderings which may also have an increasing-in-the-lead-time bullwhip behaviour. For instance, Liu and Bauer (2008), Liu (2011) identify an eigenvalue ordering that has positive impulse responses that are not covered by Theorem 2 or its inverse. Liu's eigenvalue ordering always has a positive pole as the lowest eigenvalue and a positive pole as the largest eigenvalue. The inverse of this ordering was not discussed in Liu and Bauer (2008) or (Liu, 2011); we found this impulse response is negative. The AR(p) demand process, with positive correlation coefficients (Corollary 5), also has a positive demand impulse response that does not conform to Theorem 2 or its inverse.

[6] Here $\wedge$ is the logical *and* operator.

For high dimensional ARMA(p,q) demand processes the number of possible eigenvalue orderings grows very large. For example, with $m = 2$ there are 6 orderings, $m = 3$ has 20 orderings, the $m = 7$ has 3432 orderings. Due to its simplicity, in the next section we consider the ARMA(2,2) demand. We are able to find the necessary and sufficient condition for increasing bullwhip with respect to the eigenvalues. This is accompanied by additional insights of the bullwhip characteristics.

6. Bullwhip behaviour over the lead time under ARMA(2,2) demand

While the ARMA(2,2) demand process is of low order, it contains (as special cases) many other ARIMA type demand processes such as IID, AR(1), IMA(0,1,1), MA(1), AR(2), ARMA(2,1), MA(2), and ARMA(1,2). In a study of a European retailer, Ali et al. (2012) found that 75% of 1798 SKU's were ARMA(2,2) demand processes or is special cases. The ARMA(2,2) demand process is given by, Box et al. (2008),

$$d_t = \mu_d + \sum_{i=1}^{2} \phi_i \left(d_{t-i} - \mu_d\right) - \sum_{j=1}^{2} \theta_j \epsilon_{t-j} + \epsilon_t. \tag{21}$$

The z-transform transfer function of the ARMA(2,2) demand process is given by

$$D[z] = \frac{z^2 - \theta_1 z - \theta_2}{z^2 - \phi_1 z - \phi_2}. \tag{22}$$

6.1. Stability and invertibility of the ARMA(2,2) process

When the poles (roots of the denominator of the system transfer function) lie within the unit circle in the complex plane a stable systems exists. Stable systems return to a finite state, after a finite input, in a finite amount of time. When the poles are real, this means $-1 < \lambda_1^{\phi} \leq \lambda_2^{\phi} < 1$. Complex poles are required to be within the unit circle in the complex plane; the stability conditions can be determined directly from the denominator of the transfer function, (22), using Jury's stability test (Jury, 1974) which produces the following (triangular) set of stability conditions,

$$\{1 - \phi_2 > \phi_1 > \phi_2 - 1,\ \phi_2 > -1\}. \tag{23}$$

Box et al. (2008) show that invertible time series allow for the structure of the demand process to be uniquely identified. The zeros must lie inside the unit circle in the complex plane to be invertible. For real zeros this means $-1 < \lambda_1^{\phi} \leq \lambda_2^{\phi} < 1$. When complex zeros are present, applying Jury's criterion to the numerator of the transfer function in (22) reveals the invertibility conditions:

$$\{1 - \theta_2 > \theta_1 > \theta_2 - 1,\ \theta_2 > -1\}. \tag{24}$$

The stability and invertibility conditions provide limits on the allowable ARMA(2,2) parameters.

6.2. The six eigenvalue orderings of ARMA(2,2) demand

The eigenvalues of the ARMA(2,2) demand process can be found by solving for the roots of the numerator and denominator of the transfer function, (22). The MA eigenvalues, the zeros, are the roots of numerator w.r.t. z,

$$\left\{\lambda_1^{\theta} = \frac{1}{2}\left(\theta_1 - \sqrt{\theta_1^2 + 4\theta_2}\right), \lambda_2^{\theta} = \frac{1}{2}\left(\theta_1 + \sqrt{\theta_1^2 + 4\theta_2}\right)\right\}. \tag{25}$$

The AR eigenvalues, the poles, are the roots of denominator w.r.t. z,

$$\left\{\lambda_1^{\phi} = \frac{1}{2}\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right), \lambda_2^{\phi} = \frac{1}{2}\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)\right\}. \tag{26}$$

The poles and zeros can be real, can have common poles or zeros, and can be complex. When $\theta_1^2 + 4\theta_2 < 0$, complex poles exist; when $(\phi_1^2 + 4\phi_2 < 0)$, complex zeros exist.

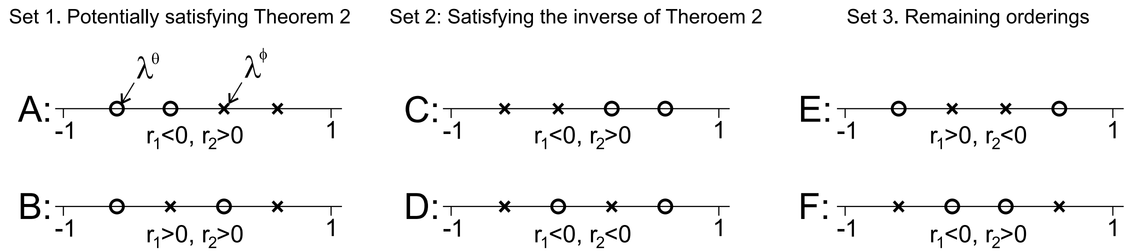

Fig. 1. The six possible real eigenvalue orderings for ARMA(2,2) demand. Circles represent the zeros, crosses represent the poles.

Lemma 2 provides the ARMA(2,2) demand impulse response, $\tilde{d}_t$,

$$\tilde{d}_{t+1} = \begin{cases} 1, & \text{if } t = 0, \\ r_1(\lambda_1^{\phi})^t + r_2(\lambda_2^{\phi})^t, & \text{if } t \geq 1, \end{cases} \tag{27}$$

where

$$r_1 = \frac{(\lambda_1^{\phi} - \lambda_1^{\theta})(\lambda_1^{\phi} - \lambda_2^{\theta})}{(\lambda_1^{\phi} - \lambda_2^{\phi})} \text{ and } r_2 = \frac{(\lambda_2^{\phi} - \lambda_1^{\theta})(\lambda_2^{\phi} - \lambda_2^{\theta})}{(\lambda_2^{\phi} - \lambda_1^{\phi})}. \tag{28}$$

The four eigenvalues of the ARMA(2,2) demand process can be arranged into six different eigenvalue orderings, see Fig. 1. We focus on the case of real poles only as, in a second order system, complex poles result in an impulse response that oscillates between positive and negative values and bullwhip does not always increase in the lead time. However, complex conjugate zeros (where conjugate zeros are projected onto the real axis in the complex plane to determine its eigenvalue ordering) are allowed. Note, in this section we assume the poles can be positive or negative.

6.3. Cases that potentially satisfy Theorem 2

6.3.1. Case A

The eigenvalues order here is $-1 < \text{Re}[\lambda_1^{\theta}] \leq \text{Re}[\lambda_2^{\theta}] < \lambda_1^{\phi} \leq \lambda_2^{\phi} < 1$.[7] It is easy to verify that $r_1 < 0 < r_2$, $\tilde{d}_1 > 0$, and $-r_2/r_1 \geq 1$. This case can exist when complex zeros are present. Three sub-cases are present, depending on the sign of the poles, $\{\lambda_1^{\phi}, \lambda_2^{\phi}\}$:

Case A_1: $0 \leq \lambda_1^{\phi} \leq \lambda_2^{\phi}$. Consider first the case when $0 < \lambda_1^{\phi} < \lambda_2^{\phi}$. Using $r_1 = \tilde{d}_1 - r_2$ in (27) provides

$$\tilde{d}_{t+1} = \tilde{d}_1(\lambda_1^{\phi})^t + r_2((\lambda_2^{\phi})^t - (\lambda_1^{\phi})^t) > 0, \tag{29}$$

which is positive $\forall t \geq 0$ as $\{\tilde{d}_1, r_2, \lambda_1^{\phi}, \lambda_2^{\phi}\} > 0$ and $\lambda_2^{\phi} > \lambda_1^{\phi}$. That $\tilde{d}_{t+1} > 0$ indicates bullwhip is always present and increases in the lead time.

The case of common poles $0 < \lambda_1^{\phi} = \lambda_2^{\phi}$, requires a different approach as $\lambda_1^{\phi} = \lambda_2^{\phi}$ results in a divide by zero in (27). With common poles, the demand impulse evolves via

$$\tilde{d}_{t+1} = \tilde{d}_1(\lambda_1^{\phi})^t + t(\lambda_1^{\phi} - \lambda_1^{\theta})(\lambda_1^{\theta} - \lambda_2^{\theta})(\lambda_1^{\phi})^{t-1}, \tag{30}$$

which is always positive. That is, common poles do not fundamentally alter the solution. The case of common poles at zero, $0 = \lambda_1^{\phi} = \lambda_2^{\phi}$, represents the case of MA(2) demand, as $\forall i,\ \lambda_i^{\phi} = 0$ implies $\forall i,\ \phi_i = 0$. In this situation, Corollary 4 is relevant and an increasing bullwhip in the lead time is present iff $\forall i,\ \theta_i < 0$ as $\tilde{d}_0 = 1,\ \tilde{d}_1 = -\lambda_1^{\theta} - \lambda_2^{\theta} = \theta_1,\ \tilde{d}_2 = \lambda_1^{\theta}\lambda_2^{\theta} = \theta_2$, and $\forall t \geq 2, \tilde{d}_{t+1} = 0$.

Case A_1 is present when $-1 \leq \text{Re}[\lambda_1^{\theta}] \leq \text{Re}[\lambda_2^{\theta}] \leq 1$. That is, case A_1 is present in the whole invertible region regardless of the values of θ_1 and θ_2. Fig. 2 highlights the $\{\theta_1, \theta_2\}$ values required for each of the eigenvalue cases to exist. Case A_1 satisfies the requirements of Theorem 2.

Case A_2: $\lambda_1^{\phi} < 0 < \lambda_2^{\phi}$. The increasing bullwhip condition, $\tilde{d}_{t+1} = r_1(\lambda_1^{\phi})^t + r_2(\lambda_2^{\phi})^t > 0$, can be rearranged into,

$$\left(\frac{\lambda_1^{\phi}}{\lambda_2^{\phi}}\right)^t < -\frac{r_2}{r_1}. \tag{31}$$

As $\lambda_1^{\phi} < 0 < \lambda_2^{\phi}$ and $-r_2/r_1 > 1$, two further sub-cases exist:

- A_{2i}. When $-\lambda_1^{\phi} < \lambda_2^{\phi}$ the LHS of (31) alternates sign with a decaying amplitude strictly less than one, indicating $\tilde{d}_{t+1} > 0$. Using (26) it is clear that $-\lambda_1^{\phi} < \lambda_2^{\phi}$ is equivalent to $\lambda_1^{\phi} + \lambda_2^{\phi} > 0 \implies \phi_1 > 0$. Even though the requirements of Theorem 2 are not met (as one of the AR poles is negative), bullwhip is always present and increases in the lead time for case A_{2i}.
- A_{2ii}. If $-\lambda_1^{\phi} > \lambda_2^{\phi}$, the LHS of (31) will alternate sign with ever increasing amplitude. Initially, when the integer valued t is less than the real valued τ,

$$\tau = \ln(|r_2/r_1|)/\ln(|\lambda_1^{\phi}/\lambda_2^{\phi}|), \tag{32}$$

$(\lambda_1^{\phi}/\lambda_2^{\phi})^t < -r_2/r_1$, indicating the demand impulse $\tilde{d}_{t+1}$ is initially positive and the increasing bullwhip in the lead time criterion holds when the lead time is small. However, the LHS of (31) will alternate with ever increasing amplitude as t increases. When $t \geq \tau$, the increasing bullwhip condition will no longer hold and for even t, $\tilde{d}_{t+1|\text{even } t} < 0$; for odd $t \geq \tau$, $\tilde{d}_{t+1|\text{odd } t} > 0$.

Case A_2 exists whenever $-1 < \text{Re}[\lambda_1^{\theta}] \leq \text{Re}[\lambda_2^{\theta}] < 0$ which is equivalent to the invertible region plus $\theta_1 < 0 \wedge \theta_2 < 0$.

Case A_3: $\lambda_1^{\phi} \leq \lambda_2^{\phi} < 0$. At $t = 0$, $\tilde{d}_0 = 1$ as always, and as relation $(\lambda_1^{\phi}/\lambda_2^{\phi})^0 < (-r_2/r_1)$ holds, $\tilde{d}_1 > 0$. Further insight into the sign of the demand impulse response can be gained from considering the following form of the impulse response,

$$\tilde{d}_{t+1} = (-1)^{t+1}\left(-r_1(-\lambda_2^{\phi})^t\right)\left(\left(\frac{\lambda_1^{\phi}}{\lambda_2^{\phi}}\right)^t - \frac{r_2}{-r_1}\right), \tag{33}$$

which can be obtained from (27). The first factor, $(-1)^{t+1}$ is alternating between positive and negative numbers; for even t, it is negative, for odd t, it is positive. The second factor, $\left(-r_1(-\lambda_2^{\phi})^t\right)$ is always positive. The third factor, $\left((\lambda_1^{\phi}/\lambda_2^{\phi})^t - (r_2/(-r_1))\right)$, is initially negative, up until the threshold τ, then it becomes positive. After $\tilde{d}_0 = 1$, these facts produce two different consequences. Initially, when $t < \tau$, even t result in $\tilde{d}_{t+1|\text{even } t} > 0$ and odd t result in $\tilde{d}_{t+1|\text{odd } t} < 0$. When $t \geq \tau$, even t result in $\tilde{d}_{t+1|\text{even } t} < 0$ and odd t result in $\tilde{d}_{t+1|\text{odd } t} > 0$. In the transition, there may be either two negative demands or two positive demands depending on the value of τ. If $\lceil \tau \rceil$ is odd, two positive demands occur at the transition; if $\lceil \tau \rceil$ is even, two negative demands occur at the transition.

As case A_3 has negative poles, it does not conform to the requirements of Theorem 2. Case A_3 exists whenever $-1 < \text{Re}[\lambda_1^{\theta}] \leq \text{Re}[\lambda_2^{\theta}] < 0$; this is possible in the invertible region plus $\theta_1 < 0 \wedge \theta_2 < 0$, the same area in the $\{\theta_1, \theta_2\}$ parameter plane as Case A_2, see Fig. 2.

Fig. 3 illustrates the $\{\phi_1, \phi_2\}$ hyper-plane when the eigenvalue ordering in case A is present for a given set of θ values. It also provides example impulse responses for each sub-case, highlighting the positivity of the impulse response with a sequence of "+ " and "-" beneath each impulse response.

[7] Formally, when there are real zeros, the following order exists $-1 < \lambda_1^{\theta} \leq \lambda_2^{\theta} < \lambda_1^{\phi} \leq \lambda_2^{\phi} < 1$. When complex zeros are present the order is $-1 < \text{Re}[\lambda_1^{\theta}] = \text{Re}[\lambda_2^{\theta}] < \lambda_1^{\phi} \leq \lambda_2^{\phi} < 1$. In order to reduce notational clutter we have combined these two statements.

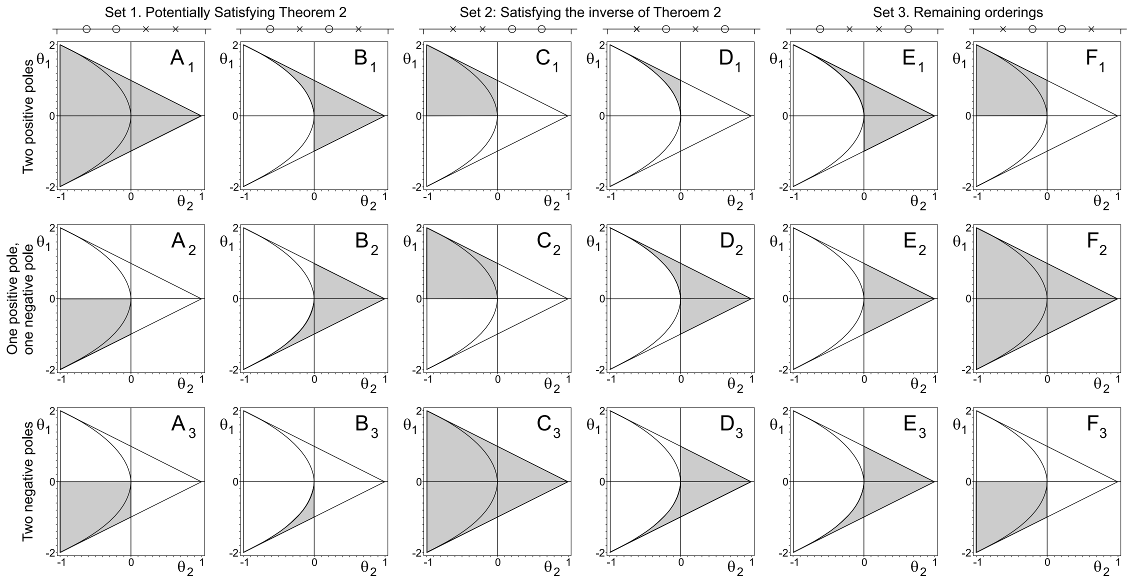

Fig. 2. Case map in the $\{\theta_1, \theta_2\}$ hyper-plane. The grey areas indicates possible θ values for each eigenvalue ordering and their main sub-cases.

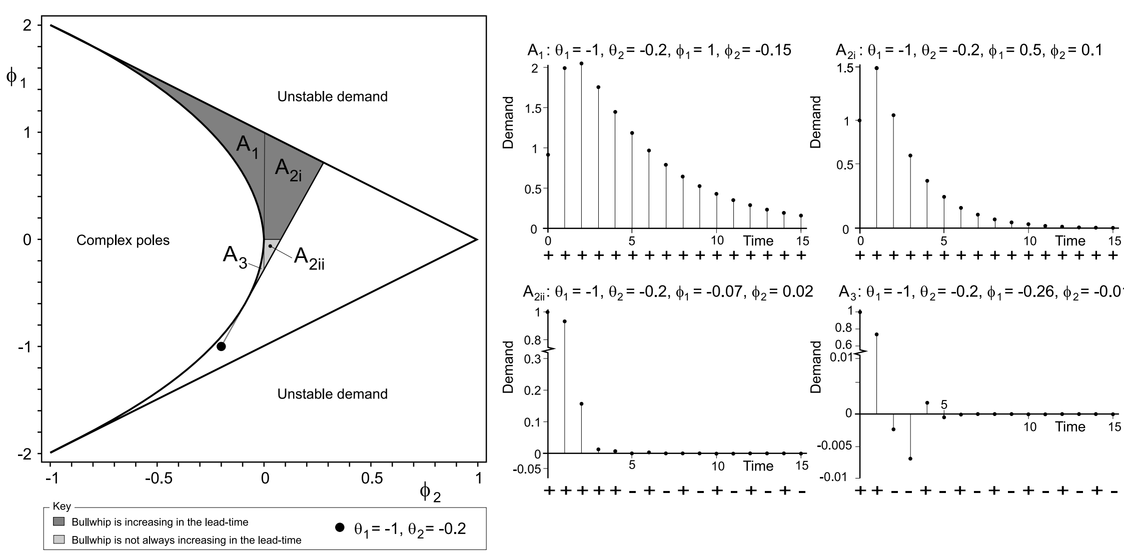

Fig. 3. Areas of increasing bullwhip over the lead-time when $-1 < \mathrm{Re}[\lambda_1^\theta] \le \mathrm{Re}[\lambda_2^\theta] < \lambda_1^\phi \le \lambda_2^\phi < 1$, case A.

6.3.2. Case B

The eigenvalue order is $-1 < \lambda_1^\theta < \lambda_1^\phi < \lambda_2^\theta < \lambda_2^\phi < 1$, $\{r_1, r_2\} > 0$, and by this $\tilde{d}_1 = r_1 + r_2 > 0$. As the zeros enclose a pole, this ordering cannot exist with complex conjugate zeros.

Case B_1: $0 \le \lambda_1^\phi < \lambda_2^\phi$. That $\{r_1, r_2, \lambda_1^\phi, \lambda_2^\phi\} > 0$, implies $\forall t$, $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t > 0$.

Case B_1 exists when $(-1 < \lambda_1^\theta < 1) \wedge (0 < \lambda_2^\theta < 1)$ which is equivalent to the invertible region with $(\theta_1^2 + 4\theta_2 > 0 \wedge \theta_1 > 0) \vee \theta_2 > 0$, where $\vee$ is the logical *or* function. Case B_1 satisfies the requirements of Theorem 2 and Corollary 6.

Case B_2: $\lambda_1^\phi < 0 < \lambda_2^\phi$. The increasing bullwhip criterion, $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t > 0$, can be re-arranged into

$$\left(\frac{\lambda_1^\phi}{\lambda_2^\phi}\right)^t > -\frac{r_2}{r_1}. \tag{34}$$

As $\{r_1, r_2\} > 0$, the RHS of (34) is negative. Thus, all even t have positive impulse responses, $\tilde{d}_{t+1}|_{\text{even } t} > 0$. The positivity of the demand impulse for odd t depends on extra conditions:

Sub-case B_{2i}. If $-\lambda_1^\phi < \lambda_2^\phi$, $(\lambda_1^\phi/\lambda_2^\phi)^t$ alternates between positive and negative numbers that tend towards zero as t increases. This leads to two further sub-sub-cases depending on the sign of $\tilde{d}_2$:

- B_{2ia}. As $-1 < (\lambda_1^\phi/\lambda_2^\phi) < 0$, the minimum $(\lambda_1^\phi/\lambda_2^\phi)^t$ occurs at $t = 1$. If $(\lambda_1^\phi/\lambda_2^\phi)^1 > -r_2/r_1$, $\tilde{d}_2 > 0$, and all subsequent $\tilde{d}_{t+1} > 0$. The requirement that $\tilde{d}_2 > 0$ produces the curve in the $\{\phi_1, \phi_2\}$ hyper-plane, see Fig. 4.
- B_{2ib}. When $(\lambda_1^\phi/\lambda_2^\phi)^1 < -r_2/r_1$, $\tilde{d}_2 < 0$; the impulse response is initially negative for odd t, $\tilde{d}_{t+1}|_{\text{odd } t} < 0$, (and positive for even t, $\tilde{d}_{t+1}|_{\text{even } t} > 0$). As the alternating $(\lambda_1^\phi/\lambda_2^\phi)^t$ has a decreasing amplitude over time, eventually $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$ and all subsequent demands will be positive for both odd and even t. This change in behaviour occurs when $t \ge \tau$.

Sub-case B_{2ii}. When $-\lambda_1^\phi > \lambda_2^\phi$, $(\lambda_1^\phi/\lambda_2^\phi)^t$ alternates with ever-increasing amplitude which will eventually violate the criterion $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$. There are two further sub-cases depending on the positivity of $\tilde{d}_2$:

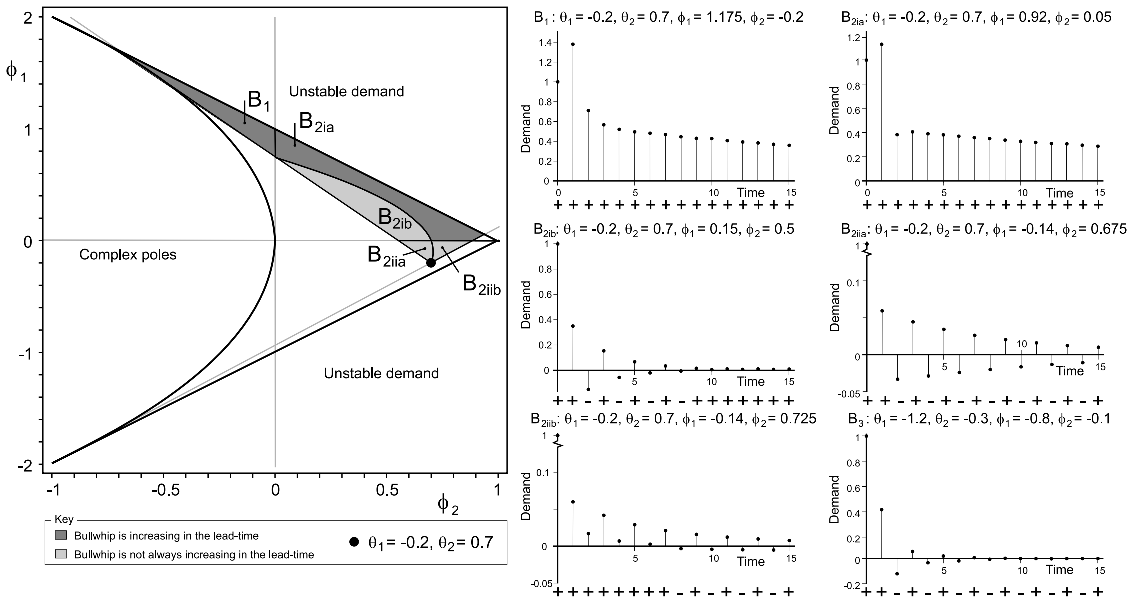

Fig. 4. Areas of increasing bullwhip over the lead-time when $-1 < \lambda_1^\theta < \lambda_1^\phi < \lambda_2^\theta < \lambda_2^\phi < 1$, case B.

- B_{2iia}. If $\tilde{d}_2 < 0$ then for all odd t, $\tilde{d}_{t+1|\text{odd }t} < 0$ and for even t, $\tilde{d}_{t+1|\text{even }t} > 0$, indicating that bullwhip does not always increase in the lead time.
- B_{2iib}. If $\tilde{d}_2 > 0$ then for small (odd and even) $t < \tau$, $\tilde{d}_{t+1} > 0$; when $t \geq \tau$, for odd t, $\tilde{d}_{t+1}|_{\text{odd }t} < 0$ and for even t, $\tilde{d}_{t+1}|_{\text{even }t} > 0$, indicating bullwhip is initially present and increases in the lead time when the lead time is small, but not for a large lead time.

Case B_2 exists when $(-1 < \lambda_1^\theta < 0) \wedge (-1 < \lambda_2^\theta < 1)$ which is equivalent to invertibility region plus $(\theta_1^2 + 4\theta_2 > 0 \wedge \theta_1 < 0) \vee (\theta_2 > 0)$, see Fig. 2. Case B_2 does not satisfy Theorem 2, but sub-case B_{2ia} possesses a positive impulse response, indicating an increasing bullwhip in the lead-time behaviour is present. This again confirms that Theorem 2 is only a sufficient condition for a positive impulse response.

Case B_3: $\lambda_1^\phi < \lambda_2^\phi < 0$. That $\{r_1, r_2\} > 0$, $\{\lambda_1^\phi, \lambda_2^\phi\} < 0$ and $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t$, implies $\tilde{d}_{t+1|\text{even }t} > 0$ for even t and $\tilde{d}_{t+1|\text{odd }t} < 0$ for odd t. This demand impulse response is alternating in sign and the bullwhip effect does not increase in the lead time.

Case B_3 requires that $(-1 < \lambda_1^\theta < \lambda_2^\theta < 0)$; parametrically, this is equivalent to the invertible region plus $(\theta_1^2 + 4\theta_2 > 0 \wedge \theta_2 < 0 \wedge \theta_1 < 0)$. Note, $\{B_1, B_2\}$ and $\{B_2, B_3\}$ can co-exist for a given set of $\{\theta_1, \theta_2\}$, but it is not possible that all three B regions are present for a single set of $\{\theta_1, \theta_2\}$. Fig. 4 highlights the areas within the parameter plane where cases $\{B_1, B_2\}$ exist as well as some of the typical time series responses found in the three different sub-cases.

6.4. Satisfying the inverse of Theorem 2

Here the pole-zero order is the inverse of the order considered in the previous section. This set of eigenvalue orderings never conforms to the eigenvalue ordering requirements of Theorem 2. A positive demand impulse response does not always exist and bullwhip is not always increasing in the lead-time for this class of eigenvalue orderings.

6.4.1. Case C

The eigenvalue order $\lambda_1^\phi \leq \lambda_2^\phi < \text{Re}[\lambda_1^\theta] \leq \text{Re}[\lambda_2^\theta]$ and $r_1 < 0 < r_2$ implies that $-r_1 > r_2$. As a consequence $\tilde{d}_1 = r_1 + r_2 < 0$, and we can conclude immediately that the bullwhip is not always present when $k = 0$ as the order variance is smaller than demand variance for case C.

Case C_1: $0 \leq \lambda_1^\phi \leq \lambda_2^\phi$. The increasing bullwhip criteria is given in (34), as $0 < \lambda_1^\phi/\lambda_2^\phi < 1$ then $0 < (\lambda_1^\phi/\lambda_2^\phi)^t < 1$ is decreasing in t. As $-r_2/r_1 > 0$, when t becomes sufficiently large, $d_t > 0$. There is only one change in the sign of the demand after $t = 1$. If $t < \tau$, there is a negative impulse response, $\tilde{d}_{t+1} < 0$. When $t \geq \tau$, there is a positive impulse response, $\tilde{d}_{t+1} > 0$. If $\tau \leq 1$ then Corollary 2 holds; if $\tau > 1$, then the order variance is initially decreasing in the lead time, before becoming, and remaining, an increasing function of the lead time.

The case of common poles does not fundamentally alter our insights. Common poles at zero result in an MA(2) process and, due to the eigenvalue ordering, all $\theta > 0$ and Corollary 4 does not hold. The C_1 case is present when $0 < \text{Re}[\lambda_1^\theta] \leq \text{Re}[\lambda_2^\theta] < 1$; parametrically, this is equivalent to the invertible region plus $\theta_1 > 0 \wedge \theta_2 < 0$, see Fig. 2.

Case C_2: $\lambda_1^\phi < 0 < \lambda_2^\phi$. Using (34) and that $-r_2/r_1 > 0$, two further sub-cases exist:

Sub-case C_{2i}. If $-\lambda_1^\phi < \lambda_2^\phi$ then $(\lambda_1^\phi/\lambda_2^\phi)^t$ initially alternates between positive and negative numbers with decreasing amplitude that tends to zero as t increases. If $t < \tau$, for even t, $\tilde{d}_{t+1}|_{\text{even }t} < 0$ and for odd t, $\tilde{d}_{t+1}|_{\text{odd }t} > 0$. When $t \geq \tau$, $\tilde{d}_{t+1} > 0$ for both odd and even t. This indicates an increasing in the lead-time bullwhip behaviour is not present with a short lead-time, but will be present when the lead-time is large enough.

Sub-case C_{2ii}. If $-\lambda_1^\phi < \lambda_2^\phi$, $(\lambda_1^\phi/\lambda_2^\phi)^t$ alternates with ever-increasing amplitude between positive and negative numbers. $\tilde{d}_{t+1}$ alternates sign; for odd t, $\tilde{d}_{t+1}|_{\text{odd }t} > 0$ and for even t, $\tilde{d}_{t+1}|_{\text{even }t} < 0$ and an increasing in the lead-time bullwhip does not exist.

Case C_2 is present in the same $\{\theta_1, \theta_2\}$ region as case C_1.

Case C_3: $\lambda_1^\phi < \lambda_2^\phi < 0$. In this case, $0 < -r_2/r_1 < 1$, $(\lambda_1^\phi/\lambda_2^\phi) > 1$ and $(\lambda_1^\phi/\lambda_2^\phi)^t$ is increasing in t. The demand impulse response can be written as

$$\tilde{d}_{t+1} = (-1)^{t+1}(-r_1)(-\lambda_2^\phi)^t((\lambda_1^\phi/\lambda_2^\phi)^t - (-r_2/r_1)). \tag{35}$$

Here, the three multipliers are all positive and the multiplicand is alternating sign in t; with an even t, $\tilde{d}_{t+1|\text{even }t} < 0$ and with an odd t, $\tilde{d}_{t+1|\text{odd }t} > 0$. This odd-even effect means that an always increasing in the lead time bullwhip effect is not present.

Case C_3 is present whenever $-1 < \text{Re}[\lambda_1^\theta] \leq \text{Re}[\lambda_2^\theta] < 1$; C_3 instances can be found over the whole invertible $\{\theta_1, \theta_2\}$ plane, see Fig. 2. Note, that it is possible that all three C cases can be observed for some particular $\{\theta_1, \theta_2\}$ values. Fig. 5 provides an example when case C is present in the $\{\phi_1, \phi_2\}$ hyper-plane, together with some sample impulse responses.

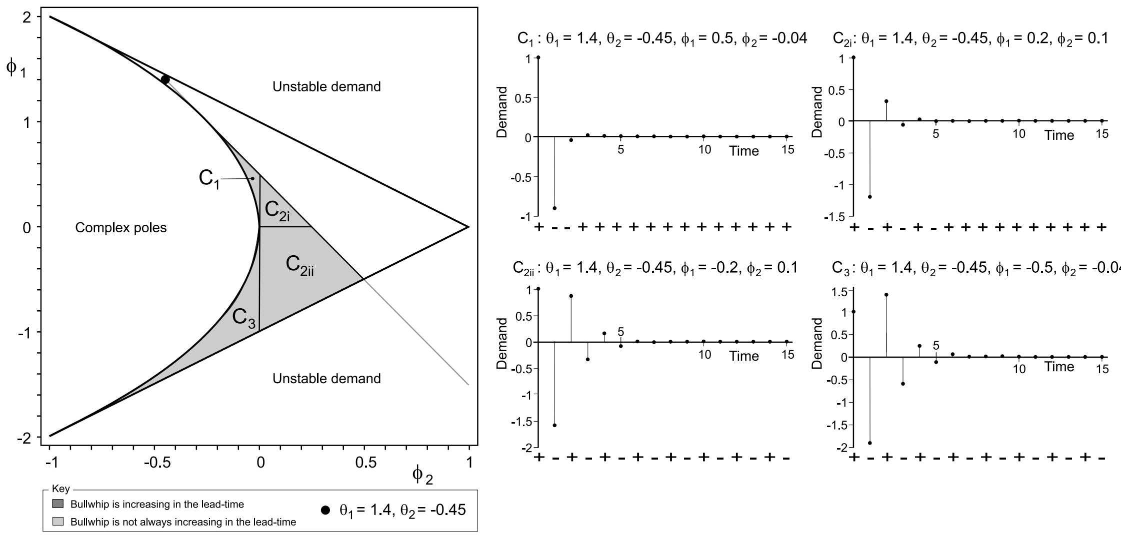

Fig. 5. Areas of increasing bullwhip over the lead-time when $\lambda_1^\phi \le \lambda_2^\phi < \mathrm{Re}[\lambda_1^\theta] \le \mathrm{Re}[\lambda_2^\theta]$, case C.

6.4.2. Case D

The eigenvalue ordering $\lambda_1^\phi < \lambda_1^\theta < \lambda_2^\phi < \lambda_2^\theta$ implies that complex zeros are not possible in case D (as there is a pole between the two zeros). The eigenvalue ordering reveals that $\{r_1, r_2\} < 0$ and $\tilde{d}_1 = r_1 + r_2 < 0$, immediately revealing that bullwhip is not always increasing in the lead-time.

Case D_1: $0 < \lambda_1^\phi < \lambda_2^\phi$. As $\{r_1, r_2\} < 0$ then $\forall t$, $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t < 0$. That is, after $\tilde{d}_0 = 1$, the demand impulse response is always negative. The D_1 case is present when $(0 < \lambda_1^\theta < \lambda_2^\theta < 1)$; this is equivalent to the invertibility region plus $(\theta_1^2 + 4\theta_2 > 0 \wedge (\theta_1 > 0) \wedge \theta_2 < 0)$. Case D_1 conforms to the requirements of Corollary and 7; bullwhip is not present when $k = 0$ and the order variance is decreasing in the lead time.

Case D_2: $\lambda_1^\phi < 0 < \lambda_2^\phi$. A positive impulse response exists if $(\lambda_1^\phi/\lambda_2^\phi)^t < -r_2/r_1$. Here, $-r_2/r_1 < 0$ and $(\lambda_1^\phi/\lambda_2^\phi)^t$ will either alternate between a positive and negative number that decays away to zero (when $-\lambda_1^\phi < \lambda_2^\phi$, sub-case D_{2i}) or alternate with ever-increasing amplitude (when $-\lambda_1^\phi > \lambda_2^\phi$, sub-case D_{2ii}). In both cases, $\tilde{d}_{t+1}|_{\text{even } t} < 0$ when t is even, indicating that the bullwhip is not increasing in the lead-time. The behaviour of $\tilde{d}_{t+1}$ for odd t depends on the sub-case:

Sub-case D_{2i}. When $-\lambda_1^\phi < \lambda_2^\phi$, $(\lambda_1^\phi/\lambda_2^\phi)^t$ is alternating with decreasing amplitude in t. Initially, for small odd t, $(\lambda_1^\phi/\lambda_2^\phi)^t < -r_2/r_1$ implying $\tilde{d}_{t+1}|_{\text{small odd } t} > 0$; for large odd t, the amplitude of $(\lambda_1^\phi/\lambda_2^\phi)^t$ gets smaller and $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$ implying $\tilde{d}_{t+1}|_{\text{large odd } t} < 0$. This means the impulse response is initially alternating, but then it becomes, and remains, negative. Depending on when this change of behaviour occurs, we have two further sub-sub-cases:

- D_{2ia}: If $\tilde{d}_2 > 0$, when $t < \tau$, for even t, $\tilde{d}_{t+1}|_{\text{even } t} < 0$; for odd t, $\tilde{d}_{t+1}|_{\text{odd } t} > 0$. When $t \ge \tau$, $\tilde{d}_{t+1} < 0$ for both odd and even t.
- D_{2ib}: If $\tilde{d}_2 < 0$, all $\tilde{d}_{t+1} < 0$.

Sub-case D_{2ii}. When $-\lambda_1^\phi > \lambda_2^\phi$, $(\lambda_1^\phi/\lambda_2^\phi)^t$ is alternating with ever increasing amplitude that will, for odd t, eventually become more negative than $-r_2/r_1$. Always $\tilde{d}_0 = 1$ and $\tilde{d}_1 < 0$; the sign of $\tilde{d}_2$ determines the fundamental character of the D_{2ii} cases:

- D_{2iia}: If $\tilde{d}_2 > 0$, when t is even, $\tilde{d}_{t+1}|_{\text{even } t} < 0$; when t is odd, $\tilde{d}_{t+1}|_{\text{odd } t} > 0$.
- D_{2iib}: If $\tilde{d}_2 < 0$, when $t < \tau$, after $\tilde{d}_0 = 1$, $\tilde{d}_{t+1} < 0$. When $t \ge \tau$, if t is even, $\tilde{d}_{t+1}|_{\text{even } t} < 0$; if t is odd, $\tilde{d}_{t+1}|_{\text{odd } t} > 0$.

Case D_2 requires $(-1 < \lambda_1^\theta < 1) \wedge (0 < \lambda_2^\theta < 1)$; this is equivalent to the invertibility region plus $(\theta_1^2 + 4\theta_2 > 0 \wedge \theta_1 < 0) \vee \theta_2 > 0$, as illustrated in Fig. 2.

Case D_3: $\lambda_1^\phi < \lambda_2^\phi < 0$. Using $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t$, where $\{r_1, r_2\} < 0$ and $\{(\lambda_1^\phi)^t, (\lambda_2^\phi)^t\}$ alternates around zero, it is clear that $\tilde{d}_{t+1|\text{odd } t} > 0$, and $\tilde{d}_{t+1|\text{even } t} < 0$. Case D_3 occurs when $(-1 < \lambda_1^\theta < 0) \wedge (-1 < \lambda_2^\theta < 1)$, the invertible region plus $(\theta_1^2 + 4\theta_2 > 0 \wedge \theta_1 < 0) \vee \theta_2 > 0$.

Cases D_1 and D_2 can exist together for some specified $\{\theta_1, \theta_2\}$, but neither can exist with D_3, see Fig. 2. Fig. 6 illustrates some case D solutions.

6.5. Other eigenvalue orderings

This set of eigenvalue orderings does not conform to the requirements of Theorem 2 or its inverse. However, we may continue to study this set using Theorem 1.

6.5.1. Case E

The eigenvalue ordering $\lambda_1^\theta < \lambda_1^\phi \le \lambda_2^\phi < \lambda_2^\theta$ implies $r_1 > 0 > r_2$. This case does not exist when complex poles are present as the two zeros are separated by the two poles.

Case E_1: $0 < \lambda_1^\phi \le \lambda_2^\phi$. $\tilde{d}_{t+1}$ may initially be positive. However, as $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t$ and $\lambda_2^\phi > \lambda_1^\phi$, when t becomes sufficiently large, $-r_2(\lambda_2^\phi)^t > r_1(\lambda_1^\phi)^t$ and $\tilde{d}_{t+1}$ turns negative after one change of sign, indicating that bullwhip does not always increase in the lead-time. Depending of when this change of sign occurs we have two sub-cases:

- *Sub-case* E_{1a}. Here $\tilde{d}_1 < 0$, the change of sign has happened immediately and after $\tilde{d}_0 = 1$, all subsequent demands are negative, $\tilde{d}_{1+1} < 0$. This means bullwhip is not present when $k = 0$ and the order variance is decreasing in the lead time.
- *Sub-case* E_{1b}. Here $\tilde{d}_1 > 0$, indicating when $t < \tau$ that $\tilde{d}_{t+1} > 0$; when $t \ge \tau$, $\tilde{d}_{t+1} < 0$.

Case E_1 exists when $(-1 < \lambda_1^\theta < 1) \wedge (0 < \lambda_2^\theta < 1)$ which is equivalent to the invertible region plus $(\theta_1^2 + 4\theta_2 > 0) \wedge (\theta_1 > 0) \vee \theta_2 > 0$, see Fig. 2. **Case** E_2: $\lambda_1^\phi < 0 < \lambda_2^\phi$. As $\lambda_1^\phi < 0$, $r_1(\lambda_1^\phi)^t$ alternates sign, $r_2(\lambda_2^\phi)^t < 0$. $\tilde{d}_{t+1}$ is either always negative or initially alternating sign before becoming negative depending on the relative sizes of $\{r_1, r_2, \lambda_1^\phi, \lambda_2^\phi\}$:

Sub-case E_{2i}. If $-\lambda_1^\phi < \lambda_2^\phi$, there are two sub-sub-cases depending on the positivity of $\tilde{d}_1$. The behaviour of both the sub-sub-cases can ascertained by rearranging (29) into $\left(\lambda_1^\phi/\lambda_2^\phi\right)^t > -r_2/r_1$. Knowing that $-1 < \left(\lambda_1^\phi/\lambda_2^\phi\right)^t \le 1$ is alternating with ever decreasing amplitude leads to the following insights:

- E_{2ia}: As $\tilde{d}_1 < 0$, $-r_2/r_1 > 1$ and after $\tilde{d}_0 = 1$, all $\tilde{d}_{t+1} < 0$.

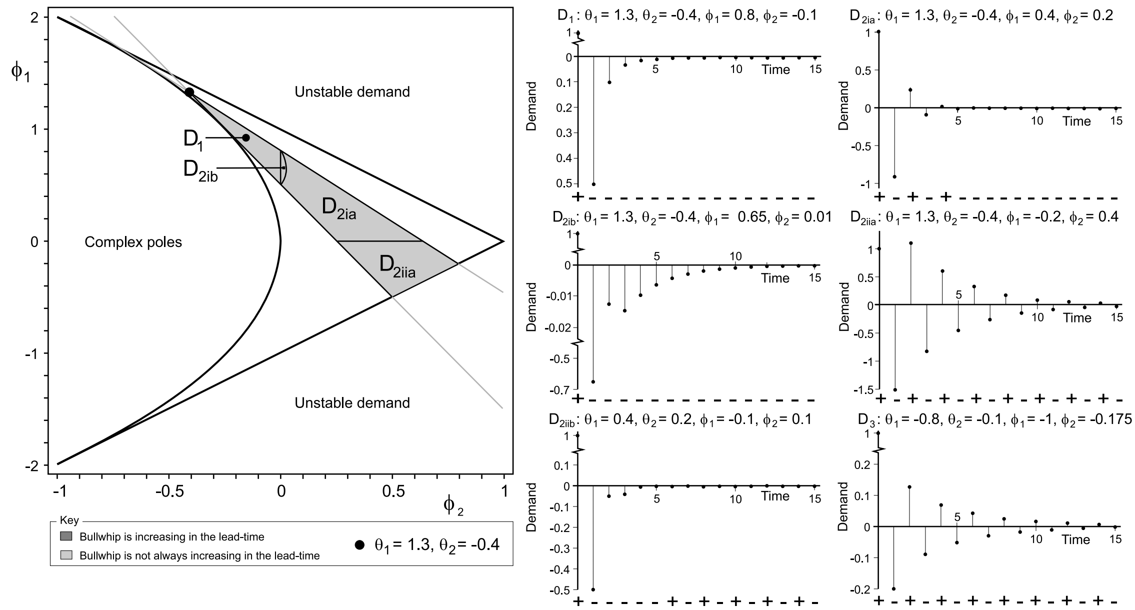

Fig. 6. Areas of increasing bullwhip over the lead-time when $\lambda_1^\phi < \lambda_1^\theta < \lambda_2^\phi < \lambda_2^\theta$, case D.

- E_{2ib}: As $\tilde{d}_1 > 0$, $0 < -r_2/r_1 < 1$ and when $t < \tau$, $\tilde{d}_{t+1}$ is initially alternating between a positive ($\tilde{d}_{t+1|\text{even }t} > 0$) and negative number ($\tilde{d}_{t+1|\text{odd }t} < 0$). When $t \geq \tau$, $\tilde{d}_{t+1}$ will become negative, $\tilde{d}_{t+1} < 0$.

Sub-case E_{2ii}. When $-\lambda_1^\phi > \lambda_2^\phi$, $(\lambda_1^\phi/\lambda_2^\phi)^t$, is alternating in t with ever increasing amplitude. The bullwhip criterion remains as $\left(\lambda_1^\phi/\lambda_2^\phi\right)^t > -r_2/r_1$, and two important sub-sub-cases are present:

- E_{2iia}: When $\tilde{d}_1 < 0$, $-r_2/r_1 > 1$ and after $t = 0$, $\tilde{d}_{t+1}$ is initially negative when $t < \tau$, but when $t \geq \tau$, it alternates sign; for odd t, $\tilde{d}_{t+1|\text{odd }t} < 0$, and for even t, $\tilde{d}_{t+1|\text{even }t} > 0$.
- E_{2iib}: When $\tilde{d}_1 > 0$, $0 < -r_2/r_1 < 1$ and the demand impulse response is alternating; for odd t, $\tilde{d}_{t+1|\text{odd }t} < 0$, and for even t, $\tilde{d}_{t+1|\text{even }t} > 0$.

Case E_2 exists when $-1 < \lambda_1^\theta < 0 < \lambda_2^\theta < 1$ which is equivalent to the invertible region plus $\theta_2 > 0$.

Case E_3: $\lambda_1^\phi \leq \lambda_2^\phi < 0$. Using $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t > 0$, as $\lambda_1^\phi/\lambda_2^\phi > 1$, $(\lambda_1^\phi/\lambda_2^\phi)^t$ increases in t and $-r_2/r_1 > 0$. When t is even, the increasing bullwhip criterion becomes $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$; when t is odd the increasing bullwhip criterion becomes $(\lambda_1^\phi/\lambda_2^\phi)^t < -r_2/r_1$. Depending on the positivity of $\tilde{d}_1$ there are two sub-cases:

- *Sub-case* E_{3a}: That $\tilde{d}_1 < 0$ implies $-r_2/r_1 > 1$. This means initially when $t < \tau$, $(\lambda_1^\phi/\lambda_2^\phi)^t < -r_2/r_1$, and for even t, $\tilde{d}_{t+1|\text{even }t} < 0$; for odd t, $\tilde{d}_{t+1|\text{odd }t} > 0$. Later, when $t \geq \tau$ $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$, and for even t, $\tilde{d}_{t+1|\text{even }t} > 0$; for odd t, $\tilde{d}_{t+1|\text{odd }t} < 0$. During the transition, there will be either two consecutive positive demands or two consecutive negative demands. If $\lceil\tau\rceil$ is odd, two negative demands occur at the transition; if $\lceil\tau\rceil$ is even, two positive demands occur at the transition.
- *Sub-case* E_{3b}: $\tilde{d}_1 > 0$ means that $-r_2/r_1 < 1$ and $(\lambda_1^\phi/\lambda_2^\phi)^t > 1$ is increasing in t. Together with the increasing bullwhip conditions highlighted above means that $\tilde{d}_{t+1|\text{even }t} > 0$ and $\tilde{d}_{t+1|\text{odd }t} < 0$.

Bullwhip is not always increasing in the lead time in case E_3. Case E_3 exists when $(-1 < \lambda_1^\theta < 0) \wedge (-1 < \lambda_2^\theta < 1)$ which is equivalent to the invertibility region plus $(\theta_1^2 + 4\theta_2 > 0 \wedge \theta_1 < 0) \vee \theta_2 > 0$, see Fig. 2. It is not possible to illustrate all possible subsets of our 4-D parameter space on a single 2-D map. Hence, case E_3 is not shown on the parameter hyper-plane in Fig. 7.

6.5.2. Case F

The eigenvalue order $\lambda_1^\phi < \text{Re}[\lambda_1^\theta] \leq \text{Re}[\lambda_2^\theta] < \lambda_2^\phi$ implies $r_1 < 0 < r_2$, but the relative size of r_1 and r_2 is unknown. Complex conjugate poles can be present.

Case F_1: $0 < \lambda_1^\phi \leq \lambda_2^\phi$. From $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t$ and that $\lambda_2^\phi \geq \lambda_1^\phi$, we can see that as t increases, eventually $r_2(\lambda_2^\phi)^t > -r_1(\lambda_1^\phi)^t$ and $\lim_{t\to\infty}\tilde{d}_{t+1} = 0^+$. From (31), as $0 < (\lambda_1^\phi/\lambda_2^\phi) < 1$, then $(\lambda_1^\phi/\lambda_2^\phi)^t$ is decreasing t. Furthermore $-r_2/r_1 > 0$, thus the positivity of $\tilde{d}_1$ is sufficient to reveal the behaviour of the F_1 cases.

Sub-case F_{1a}. Here, $-r_2/r_1 < 1$, $\tilde{d}_1 = \phi_1 - \theta_1 < 0$, and bullwhip is not present when $k = 0$. However, after $t \geq \tau$, $\tilde{d}_{t+1}$ becomes, and remains, positive. After $t = 1$, there is only one change of sign. If $\tau < 1$, $\forall t \geq 1$, $\tilde{d}_t > 0$ and Corollary 2 holds.

Sub-case F_{1b}. If $\tilde{d}_1 = \phi_1 - \theta_1 > 0$ then $-r_2/r_1 > 1$ and $\forall t\ \tilde{d}_{t+1} > 0$, indicating that an increasing bullwhip effect in the lead time is present. The F_{1b} case illustrates again that Theorem 2 is only a sufficient condition for increasing bullwhip in the lead-time. The positive impulse responses identified by Liu and Bauer (2008) correspond to case F_{1b}.

Case F_1 exists when $(0 < \text{Re}[\lambda_1^\theta] \leq \text{Re}[\lambda_2^\theta] < 1)$ which is equivalent to the invertibility region plus $\theta_1 > 0 \wedge \theta_2 < 0$, see Fig. 2.

Case F_2: $\lambda_1^\phi < 0 < \lambda_2^\phi$. Consider the increasing bullwhip criterion, $(\lambda_1^\phi/\lambda_2^\phi)^t < (-r_2/r_1)$, in (31). As $\lambda_1^\phi/\lambda_2^\phi < 0$ and $-r_2/r_1 > 0$, further sub-cases exist:

Sub-case F_{2i}. Here $-\lambda_1^\phi/\lambda_2^\phi < 1$, $(\lambda_1^\phi/\lambda_2^\phi)^0 = 1$ and $(\lambda_1^\phi/\lambda_2^\phi)^t$ alternates with decreasing amplitude in t. This leads to two sub-sub-cases:

- F_{2ia}. Here $\tilde{d}_1 < 0$, $-r_2/r_1 < 1$. Initially, when $t < \tau$, the demand alternates in sign with $\tilde{d}_{t+1|\text{odd }t} > 0$ and $\tilde{d}_{t+1|\text{even }t} < 0$, before becoming always positive when $t \geq \tau$.
- F_{2ib}. Here $\tilde{d}_1 > 0$, $-r_2/r_1 > 1$. The demand impulse is always positive, $\tilde{d}_{t+1} > 0$.

Sub-case F_{2ii}. Here $-\lambda_1^\phi/\lambda_2^\phi > 1$, $(\lambda_1^\phi/\lambda_2^\phi)^0 = 1$ and $(\lambda_1^\phi/\lambda_2^\phi)^t$ alternates with ever increasing amplitude in t. This leads to two sub-sub-cases:

- F_{2iia}. Here $\tilde{d}_1 < 0$, $-r_2/r_1 < 1$. The demand is forever alternating sign with $\tilde{d}_{t+1|\text{odd }t} > 0$ and $\tilde{d}_{t+1|\text{even }t} < 0$.
- F_{2iib}. Here $\tilde{d}_1 > 0$, $-r_2/r_1 > 1$, and while $(\lambda_1^\phi/\lambda_2^\phi)^t$ alternates sign with ever increasing amplitude in t, initially, when $t < \tau$, $(\lambda_1^\phi/\lambda_2^\phi)^t < (-r_2/r_1)$ and the demand impulse is positive, $\tilde{d}_{t+1} > 0$. While for all odd t, $(\lambda_1^\phi/\lambda_2^\phi)^t < -r_2/r_1$, as t gets larger, eventually

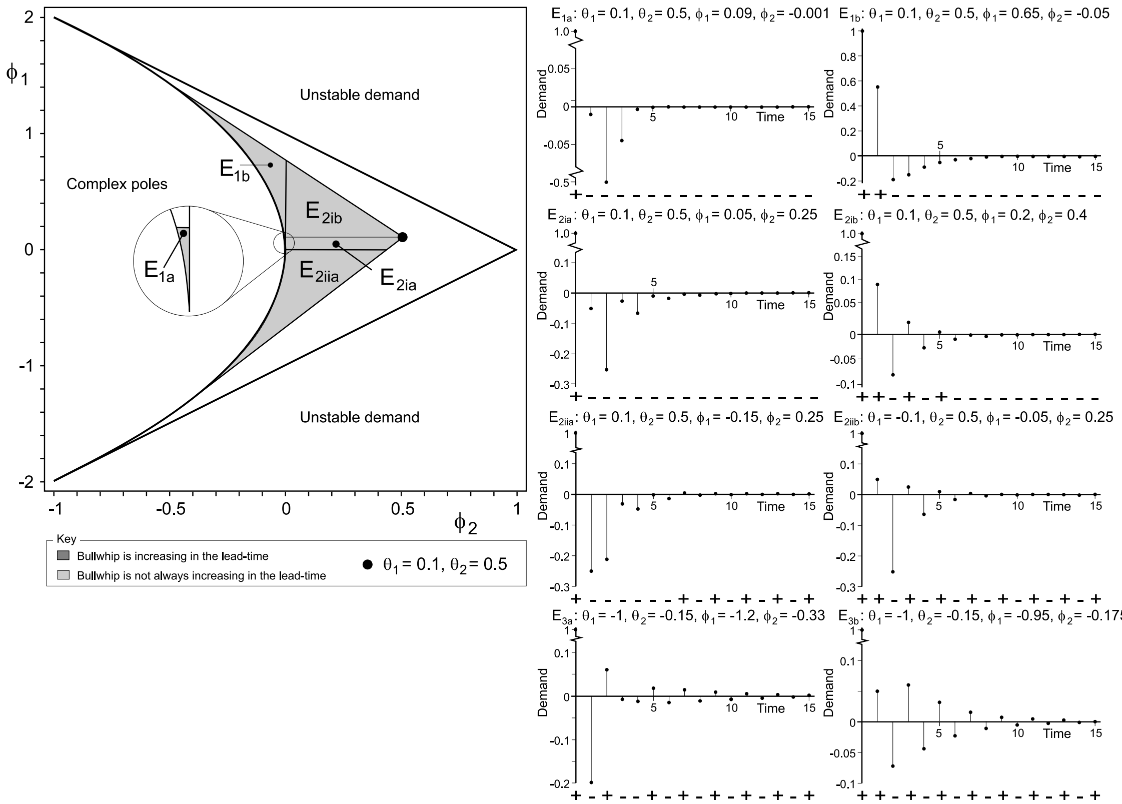

Fig. 7. Areas of increasing bullwhip over the lead-time when $\lambda_1^\theta < \lambda_1^\phi \le \lambda_2^\phi < \lambda_2^\theta$, case E.

for even t $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$, indicating, when $t \ge \tau$, the demand impulse will alternate with $\tilde{d}_{t+1|\text{even } t} < 0$ and $\tilde{d}_{t+1|\text{odd } t} > 0$.

Note, under AR(2) demand with positive correlation coefficients and two zeros at the origin, there will be one negative pole and one positive pole. In this situation sub-case F_{1ii} does not exist, and Corollary 5 is equivalent to sub-case F_{2i}. Case F_2 is present when $(-1 < \text{Re}[\lambda_1^\theta] \le \text{Re}[\lambda_2^\theta] < 1)$; that is, F_2 solutions can exist in the whole invertible region.

Case F_3: $\lambda_1^\phi \le \lambda_2^\phi < 0$. As $\lambda_1^\phi/\lambda_2^\phi > 1$, $(\lambda_1^\phi/\lambda_2^\phi)^t > 1$ and increases in t. The finite $-r_2/r_1 > 0$. Using $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t > 0$, for even t, we require $(\lambda_1^\phi/\lambda_2^\phi)^t < -r_2/r_1$; for odd t, we require $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$. There are two sub-cases depending on the sign of $\tilde{d}_1$:

Sub-case F_{3a}. As $\tilde{d}_1 < 0$, $-r_2/r_1 < 1$ and the demand impulse is alternating sign; for even t, $\tilde{d}_{t+1|\text{even } t} < 0$; for odd t, $\tilde{d}_{t+1|\text{odd } t} > 0$.

Sub-case F_{3b}. As $\tilde{d}_1 > 0$, $-r_2/r_1 > 1$, and initially, when $t < \tau$, for even t, $\tilde{d}_{t+1|\text{even } t} > 0$ and for odd t, $\tilde{d}_{t+1|\text{odd } t} < 0$. Later, when $t \ge \tau$, $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$, and for even t, $\tilde{d}_{t+1|\text{even } t} < 0$; for odd t, $\tilde{d}_{t+1|\text{odd } t} > 0$. During the transition, there will be either two consecutive positive demands or two consecutive negative demands. If $\lceil\tau\rceil$ is odd, two positive demands occur at the transition; if $\lceil\tau\rceil$ is even, two negative demands occur at the transition".

In Case F_3, bullwhip is not always increasing in the lead time. Case F_3 occurs when $(-1 < \text{Re}[\lambda_1^\theta] \le \text{Re}[\lambda_2^\theta] < 0)$. This is equivalent to the invertible region plus $\lambda_1 > 0 \wedge \lambda_2 < 0$. Fig. 8 illustrates some instances of case F.

To summarize our ARMA(2,2) investigation, in some low pass filter settings (where Theorem 2 holds, cases A and B), two positive poles (sub case 1) or a single negative pole (sub case 2) can lead to a positive impulse response and an increasing in the lead time bullwhip effect. However, two negative poles (sub case 3) always lead to an alternating impulse response, and bullwhip does not always increase in the lead-time. When the inverse of Theorem 2 holds (cases C and D), bullwhip does not always increase in the lead time. Indeed, case D_1 possessed an always decreasing in the lead time bullwhip effect. There are also positive impulses responses when neither Theorem 2, nor its inverse, hold (case F). However, a maximum of one pole can be negative for an increasing in the lead time bullwhip behaviour to exist.

7. Verification via simulation of bullwhip response to real demand data

In order to gain some insight on the practical implications of our research we investigated the bullwhip lead-time behaviour when the demand is drawn from the M4 dataset. The M-competitions are a series forecasting competitions organized by Spyros Makridakis. In the forth M-competition, M4, a dataset of 100,000 time series were used to judge the accuracy of different forecasting mechanisms, Makridakis et al. (2020). The M4 dataset is available in the R software package via the *M4comp2018* package. We considered only time series with a weekly period as most practical supply chain planning operations we know operate on a weekly schedule. We also truncated this subset so as to consider only the last 104 (2 years) periods in each time series (or length of the time series if the original series contained less than 104 periods of data). This was to avoid over-fitting of high order ARMA models and also reflects industrial practice of only having access to only the most recent demand data. Using the *auto.arima* and *ndiffs* functions in R we identified the ARIMA model for each time series. Of the 359 weekly time series, 70 time series were identified as stationary, stable, and invertible (that is, they had no integrative terms and no poles or zeros outside the unit circle) and were amenable to our analysis.

For each of the 70 time series we: identified the ARMA coefficients, plotted the demand impulse response function with the identified ARMA coefficients, calculated the poles and zeros, and simulated the OUT polices bullwhip lead time behaviour (up to lead time k = 30). In all cases, the simulated bullwhip lead time behaviour was as predicted by the impulse response, confirming the robustness and accuracy of our analysis.

Fig. 9, summaries our results. Of the 70 time series:

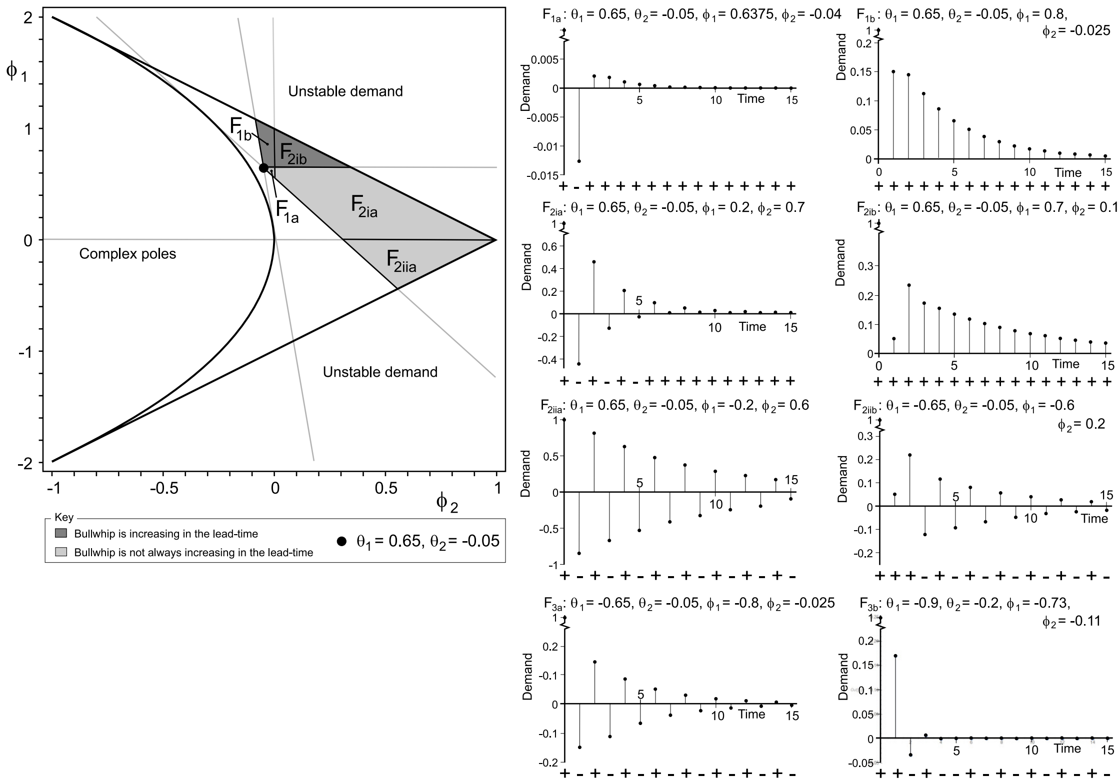

Fig. 8. Areas of increasing bullwhip over the lead-time when $\lambda_1^{\phi} < \mathrm{Re}[\lambda_1^{\theta}] \le \mathrm{Re}[\lambda_2^{\theta}] < \lambda_2^{\phi}$, case F.

- One time series was identified as IID; for all lead-times the bullwhip ratio, $BI = 1$, an obvious result as $o_t = d_t$ under IID demand with MMSE forecasts.
- Six time series had an MA(1) structure. All six MA(1) processes had $\theta > 0$, and a constant bullwhip effect over the lead time with $BI < 1$, confirming Corollary 4 that requires a $\theta < 0$ for a bullwhip effect to exist.
- There were 26 AR(1) time series, all with $\phi > 0$. Corollary 5 and Lee et al. (1997) showed that the bullwhip effect is increasing in the lead time under AR(1) demand with positive correlation.
- Fourteen ARMA(1,1) time series all conformed to Corollary 3 and Duc et al. (2008), with $\phi > 0$ and $\theta < \phi$. The bullwhip effect always increases in the lead time for these time series.
- Six ARMA(1,2) time series were present. Two of the ARMA(1,2) belonged to case A_1 and had complex zeros. Three time series was from the B_1 case that satisfies Theorem 2 and Corollary 6. One time series was from the E_{1b} case. All six cases had a bullwhip effect that increased in the lead time.
- Three AR(2) demand processes: the two increasing bullwhip in the lead time were from the A_1 and F_{2ib} subcases, the multi-periods oscillation case had complex poles.
- Three ARMA(2,1) and two ARMA(2,2) demand processes all had complex poles that produced multi-periods oscillations.

All the above time series were subsets of the ARMA(2,2) demand process and our analysis in Section 6 applies directly. The final row of Fig. 9 contained higher order ARMA processes that we do not yet fully understand:

- One AR(3) demand with two complex poles and increasing bullwhip in the lead-time.
- One AR(4) demand with four complex poles and a multi-period oscillation.
- An ARMA(3,1), ARMA(3,2) and ARMA(2,3) all with complex poles, exhibiting a multi-period oscillation in the impulse response.
- Two ARMA(1,3) and two ARMA(1,4) demands which satisfied Theorem 2 and had an increasing bullwhip effect in the lead time.

All time series were simulated; in all cases the theoretical bullwhip-lead time behaviour concurred with a simulation of the OUT policy with the identified demand parameters reacting to the M4 time series. This provides additional support that our theoretical results can be used to predict the OUT policies bullwhip-lead time behaviour from a demand time series alone.

8. Managerial insights

The practicing manager, having observed an ARMA process structure in demand, may want to consider lead time reduction. Depending on the demand process observed, there may or may not be a bullwhip benefit from reducing the lead time. If there is a benefit, the cost of reducing the lead time may be offset against reduced capacity costs, (Hosoda and Disney, 2012); if bullwhip does not increase in the lead time, perhaps different (cheaper, more ecologically friendly) transport modes or production technology (with longer lead-times) can be used to access other financial or environmental benefits while still reducing capacity costs. The impulse response-eigenvalue analysis that we introduced here can be used to answer these questions.

We have created a R-shiny web app (https://www.bullwhip.co.uk/#shiny) where users can explore the bullwhip lead time behaviour of ARMA demand (up until ARMA(12,12)). The shiny app also includes some real demand patterns we have collected and identified their ARMA coefficients. The practicing manager is also able to upload their own demand data to the shiny app. Using the auto.arima function in R, the shiny will automatically identify the ARMA coefficients and

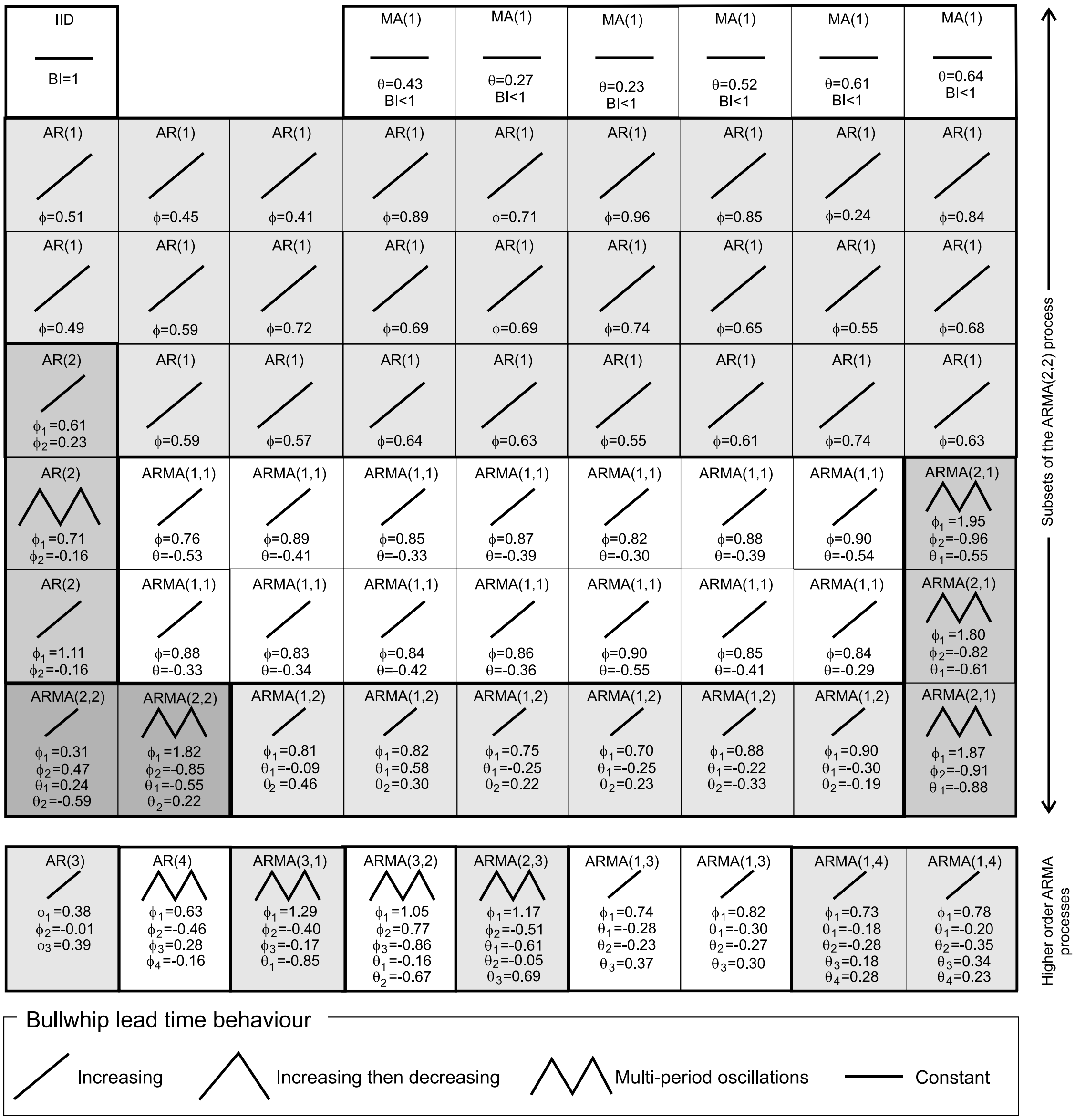

Fig. 9. The bullwhip lead time behaviours observed in the stationary, stable, and invertible weekly time series in the M4 dataset.

compare the simulated bullwhip lead time performance based on the actual demand to the theoretical performance based on the ARMA model identified. Fig. 10 illustrates the workflow when using the shiny apps to analyse the bullwhip lead time behaviour from the demand time series.

9. Conclusions

Bullwhip may be measured with either a ratio of variances ($BI = (\sigma_o^2/\sigma_d^2)$) or a difference of variances ($CB[k] = (\sigma_o^2 - \sigma_d^2)/\sigma_\epsilon^2$). The bullwhip metric based on the difference, $CB[k]$, useful when large order and demand variances are present, allowed us to concisely derive the conditions under which a bullwhip effect is always present and always increasing in the lead time. Building upon Tsypkin's squared impulse response theorem, we derived our core contribution, Theorem 1, which showed the positivity of the demand impulse response determines whether the bullwhip effect is present and increasing in the lead-time. Theorem 1 is both necessary and sufficient. We showed how the impulse response could be expressed as a function of the eigenvalues, $\{\lambda^\phi, \lambda^\theta\}$, of the demand process rather than the AR and MA parameters, $\{\phi, \theta\}$, directly. This is an important contribution as it proved to be efficient as only the order of the eigenvalues determines a lead-time/bullwhip relationship, not the specific value of the eigenvalues or the demand parameters.

Theorem 2 identified a class of easy-to-identify eigenvalue orderings for which the general demand processes behaves as a low pass filter that is sufficient to describe when the bullwhip is an increasing function of the lead-time. Using Theorem 2 we found three different sets of eigenvalue orderings exist:

- a set where Theorem 2 is potentially satisfied and a bullwhip effect that increases over the lead-time is possible,
- an set where the inverse of Theorem 2 and bullwhip is not always increasing in the lead-time,
- and a third set where neither Theorem 2 or its inverse holds but which does include a bullwhip effect that increases over the lead-time.

While our main results (Theorems 1 and 2) hold for general ARMA(p,q) demand, we furthered our investigation by studying all the possible eigenvalue orderings of the ARMA(2,2) demand process with our unique impulse response/eigenvalue approach. The ARMA(2,2)

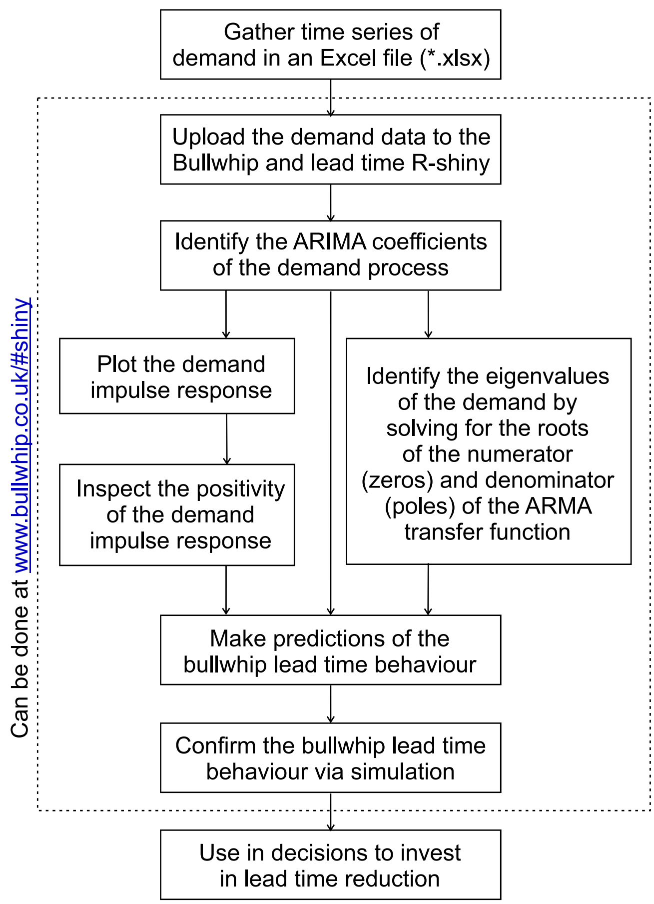

Fig. 10. Procedure for analysing bullwhip lead time behaviour from a time series of demand.

includes a number of well-known sub-ARMA cases and is also practically relevant. For all possible stable and invertible ARMA(2,2) demand processes, we fully characterized the bullwhip/lead-time behaviour. This numerical example illustrated the ARMA(2,2) demand process contained a rich set of dynamic responses. Low pass demand processes, where bullwhip was an increasing function of the lead-time with positive poles, were found as predicted by Theorem 2. Other demand processes with an increasing bullwhip in the lead-time were dominated by high frequency harmonics caused had a negative pole, confirming that Theorem 2 is sufficient, but not a necessary condition. Furthermore, demand processes that did not conform to Theorem 2, or its inverse, also had a bullwhip effect that increased in the lead time. The impulse response and bullwhip properties for higher order ARMA demand processes is generally complex and we do not yet fully understand their behaviour.

9.1. Further work

The characteristics of both the demand process and the OUT policy are important factors that determines whether a bullwhip effect is increasing in the lead-time or not. Our study has shown that an increasing bullwhip in the lead-time only happens in a small subset of the possible demand processes with a specific eigenvalue ordering. It would be interesting to better understand the propensity of real demand processes belonging to each of the eigenvalue subsets, perhaps extending our analysis conducted for Fig. 9. Studying how the lead-time influences the bullwhip behaviour under higher order ARMA demand processes, with other inventory replenishment policies (such as the (s, S) policy), and other forecasting techniques are also worthy areas for future research. The impact of stochastic lead times on the bullwhip effect might also be interesting to study, especially given the recent supply chain congestion issues.

Data availability

We have used the publically available dataset from the M4 competition that is available in R.

Appendix. Proof of all Lemmas, Corollaries, and Theorems

A.1. Proof of Lemma 1: Tsypkin's squared impulse response theorem

Tsypkin (1964, pp 183–192) and Boute et al. (2022) provide a proof of this relation.

A.2. Proof of Lemma 2 : Impulse response of ARMA(p,q) demand

Following Moudgalaya (2007), we can re-write (14) using polynomial long division as

$$D[z] = 1 + \frac{\prod_{i=1}^{m}(z-\lambda_i^{\theta}) - \prod_{i=1}^{m}(z-\lambda_i^{\phi})}{\prod_{i=1}^{m}(z-\lambda_i^{\phi})}. \tag{36}$$

Applying partial fraction expansion to (36) provides

$$D[z] = 1 + \sum_{j=1}^{m} \frac{r_j}{z-\lambda_j^{\phi}}. \tag{37}$$

Finally, taking the inverse z-transform of (37) gives (15).

A.3. Proof of Lemma 3 : Impulse response of the orders

When the demand is an impulse response, the forecasted demand $\hat{d}_{t+j|t} = \tilde{d}_{t+j}$ for $t \geqslant 0$ and $\hat{d}_{t+j|t} = 0$ otherwise. Substituting these relations into (8) produces (17).

A.4. Proof of Corollary 1 : Influence of the ARMA coefficients on the bullwhip effect

Eq. (12) shows that bullwhip exists iff $CB[k] > 0$; (20) shows only the demand impulses until time $t = k+1$ influence the difference between the order and demand variances. The demand impulse is given by

$$\tilde{d}_t = \begin{cases} 1, & \text{if } t = 0, \\ \sum_{j=1}^{t-1} \phi_j \tilde{d}_{t-j} + (\phi_t - \theta_t) & \text{if } 0 < t \leq m, \\ \sum_{j=1}^{m} \phi_j \tilde{d}_{t-j} & \text{if } t > m. \end{cases} \tag{38}$$

After the initial demand impulse of unity at time $t = 0$, the demand impulse evolves by adding, in each time period, the next pair of $\{\phi, \theta\}$ until all m pairs are present. At, and after, $t = m$ all the parameter pairs of $\{\phi, \theta\}$ are present in the demand impulse.

Eq. (38) shows at time t, $\tilde{d}_t$ is influenced only by $\{\phi_j, \theta_j\}$ with $j \leq t$. Therefore, the question whether bullwhip exists or not is fully determined by the first $k+1$ pairs of ARMA coefficients.

A.5. Proof of Theorem 1 : Necessary-sufficient condition for an increasing bullwhip effect

$CB[k]$ is positive and increasing in k if $CB[0] > 0$ and $\forall k$, $CB[k] - CB[k-1] > 0$. Note always, $\tilde{d}_0 = 1$. $CB[0] = \left(\sum_{j=0}^{1} \tilde{d}_j\right)^2 - \sum_{t=0}^{1} \tilde{d}_t^2 = 2\tilde{d}_0\tilde{d}_1$ is positive if additionally $\tilde{d}_1 > 0$. $CB[1] - CB[0] = 2(\tilde{d}_0 + \tilde{d}_1)\tilde{d}_2$ is positive if additionally $\tilde{d}_2 > 0$. $CB[2] - CB[1] = 2(\tilde{d}_0 + \tilde{d}_1 + \tilde{d}_2)\tilde{d}_3$ is positive if additionally $\tilde{d}_3 > 0$. This process can be continued $\forall k$, indicating that bullwhip is always present and increasing in the lead-time iff the demand impulse response is positive for all t.

A.6. Proof of Corollary 2 : Necessary-sufficient condition for an increasing order variance

If $CB[0] \geq 0$, Theorem 1 holds and that the order variance is increasing in the lead time follows naturally. However, if $CB[0] < 0$ then $\tilde{d}_1 < 0$ as $\tilde{d}_0 = 1$ and $CB[0] = 2\tilde{d}_0\tilde{d}_1$. This implies the order variance is smaller than the demand variance when $k = 0$. The order variance increases when the lead time k increases to unity if $CB[1] - CB[0] = 2(\tilde{d}_0 + \tilde{d}_1)\tilde{d}_2 > 0$. This implies $\tilde{d}_2 > 0$ and $\tilde{d}_1 > -1$. The order variance increases when the lead time k increases from unity to two if $CB[2] - CB[1] = 2(\tilde{d}_0 + \tilde{d}_1 + \tilde{d}_2)\tilde{d}_3 > 0$. This implies that additionally $\tilde{d}_3 > 0$. This last step can be continued $\forall k \geq 3$, indicating that the order variance is increasing the lead-time iff $\{\tilde{d}_2, \ldots, \tilde{d}_{k+1}\} > 0$ and $\tilde{d}_1 > -1$.

A.7. Proof of Corollary 3 : Bullwhip lead time behaviour under ARMA(1,1) demand

The proof of Corollary 3 follows by simple inspection of the demand impulse response. Eq. (15) shows after $\tilde{d}_0 = 1$, the demand impulse response evolves via $\tilde{d}_{t+1} = (\lambda_1^\phi - \lambda_1^\theta)(\lambda_1^\phi)^t = \phi^t(\phi - \theta)$. The last equality is a consequence of $\lambda_1^\phi = \phi$ and $\lambda_1^\theta = \theta$ under ARMA(1,1) demand.

A.8. Proof of Corollary 4 : Bullwhip lead time behaviour under MA(q) demand

The proof of Corollary 4 follows by adapting (36) for MA(q) demand:

$$D[z] = 1 + \frac{\prod_{i=1}^{q}(z - \lambda_i^\theta) - z^q}{z^q} = \frac{\prod_{i=1}^{q}(z - \lambda_i^\theta)}{z^q} = 1 - \sum_{i=1}^{q} \theta_i z^{-i}. \quad (39)$$

Taking the inverse z-transform of (39) reveals the demand impulse response; after $\tilde{d}_0 = 1$, evolves via $\tilde{d}_t = -\theta_t$ until $\tilde{d}_q = -\theta_q$, after which $\forall t > q$, $\tilde{d}_t = 0$.

A.9. Proof of Corollary 5 : Bullwhip lead time behaviour under AR(p) demand

The proof of Corollary 5 follows by simple inspection of the demand impulse response which after $\tilde{d}_0 = 1$ evolves via $\tilde{d}_t = \sum_{i=1}^{t} \phi_j d_{t-j}$, with $\phi_j = 0$ when $j > p$.

A.10. Proof of Theorem 2 : Sufficient condition for a positive impulse response

We use the pole-zero transfer function of $\tilde{d}_t$ given in (14) and the convolution theorem in our proof. Theorem 1 showed a positive impulse response of $D[z]$ led to a bullwhip effect that always increases in the lead time. Because of the restrictions placed on the λ_j^ϕ eigenvalues in relation to the λ_j^θ eigenvalues in Theorem 2, $\lambda_j^\phi > \lambda_j^\theta$ and $\lambda_j^\phi > 0$. Noting

$$D[z] = \prod_{j=1}^{m} D_j[z]; \text{ where } D_j[z] = \left(\frac{z - \lambda_j^\theta}{z - \lambda_j^\phi} \right), \quad (40)$$

and $\lambda_j^\phi > \lambda_j^\theta$ and $\lambda_j^\phi > 0$, it is then clear that each $D_j[z]$ has a positive impulse response as

$$\tilde{d}_{j,t} = Z^{-1}[D_j[z]] = \begin{cases} 1 \text{ if } t = 0, \\ (\lambda_j^\phi)^{t-1}(\lambda_j^\phi - \lambda_j^\theta) \text{ if } t \geq 1. \end{cases} \quad (41)$$

Multiplication of the factors $D_j[z]$ in (40) is equivalent to convolution of (41) in the time domain. As convolution involves addition and multiplication operations, any combination of addition and multiplication of positive terms produces a positive outcome. $D[z]$ has a positive impulse response because of these properties.

A.11. Proof of Corollary 6: An eigenvalue ordering with a positive impulse response

The eigenvalue ordering means $\forall j,\ r_j > 0$ and $\forall i,\ \lambda_i^\phi > 0$. It is then easy to see from (15) that $\tilde{d}_{t+1} > 0$ when Corollary 6 holds.

A.12. Proof of Corollary 7: An eigenvalue ordering with a negative impulse response

The eigenvalue ordering means $\forall j,\ r_j < 0$ and $\forall i,\ \lambda_i^\phi > 0$. It is then easy to see from (15) that $\tilde{d}_{t+1} < 0$ when Corollary 7 holds.

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