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Second-Price Auctions with Private Entry Costs

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Abstract: We study asymmetric second-price auctions under incomplete information. The bidders have two potentially different, commonly known, valuations for the object and private information about their entry costs. The seller, however, does not benefit from these entry costs. We calculate the equilibrium strategies of the bidders and analyze the optimal design for the seller in this environment in terms of expected entry and the number of potential bidders.

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1. Introduction

In auctions with entry costs, each bidder can enter the auction only if he pays an entry cost. The seller, however, does not benefit from the bidders' entry costs (as opposed to entry fees). This cost of entering is spent regardless of whether a bidder wins and is independent of his bid. It reflects both an opportunity cost of the time of participating and the cost of the effort needed to learn the rules and prepare a strategy (see [1–3]). For large auctions (such as spectrum auctions), it also represents the cost of raising the necessary credit to participate. The usual assumptions on entry costs are that they are common knowledge and identical (for example, see [4–9]. Cao and Tian [10] assume that the entry costs are asymmetric but they are still common knowledge. We instead assume that the bidders' entry costs are private information. This is natural since a bidder should have a much better idea about his own opportunity, learning, and fundraising costs than about such costs of his opponents. Cao et al. [11] assume costs and values are both independently drawn from a joint distribution. They prove existence and find conditions for uniqueness.

Another common assumption in the literature on auctions with entry costs is that bidders' decisions on whether or not to enter the auction are made before they learn their private information. This timing assumption and the assumption that bidders are ex ante symmetric causes the expected profit of each bidder to be zero with all the social surplus going to the seller (as in [5]. We depart also from this timing assumption by having a bidder's entry cost known to the bidder before making his entry decision. Doing so best captures the idea that entry costs are opportunity costs. We note that others have also modeled the timing in this order. Samuelson [7] and Menezes and Monterio [12] consider a model with incomplete information where a player first learns his private value for the object being sold and then decides whether or not to enter the auction. However, contrary to our model, in their symmetric model, all bidders have the same cost of entry, which is common knowledge.

Together, our assumptions on the timing and incomplete information of entry costs allow bidders to earn strictly positive profits and cause the seller's and the social planner's problems to no longer be identical. Finally, we allow ex ante asymmetry among the bidders who have different valuations for the objects. To aid tractability, we do this by assuming that bidders' values are commonly known ex ante and can be, at most, one of two types.

We find that our model has cutoff equilibria, where any bidder with an entry cost higher than the cutoff for his valuation will stay out of the auction and any bidder with an entry cost lower than the cutoff for his valuation will decide to participate in the auction. We show that given these equilibrium strategies, a bidder may wish to have a lower valuation for the object since, surprisingly, his expected payoff may decrease in his valuation. Moreover, the expected payoff of a player may be lower than the expected payoff of his opponents with lower valuations. Cao and Tian [10], who studied a different model with asymmetric entry costs in which bidders have private valuations for the object but participation costs are common knowledge, provided conditions under which there exists only monotonic equilibria, namely equilibrium in which a lower participation cost results in a lower cutoff to participate in an auction.

We analyze the optimal entry (design) in our environment given that the number of bidders (potential entrants) is exogenous. In the special case where bidders are symmetric, that is, they all have the same value for the object, we find that the seller would like to reduce the number of bidders that choose to enter from the equilibrium. While reducing the number that enter, the seller collects more from those that decided to enter. In the case where bidders are asymmetric, that is, they have different values for the object, we show that, independent of the distribution of the bidders' entry costs and the bidders' valuations for the object, the seller always wishes to reduce participation of at least one type of bidders. Sometimes the seller may prefer a less efficient situation where the optimal cutoff of bidders with the high valuation is always smaller than the optimal cutoff of bidders with lower valuation. For example, Gilbert and Klemperer [13] show that an auctioneer may wish to run an inefficient auction to attract weaker bidders to enter the auction. However, we find that if the numbers of bidders of each type are identical, the seller prefers a higher cutoff for the bidders with a high valuation. In the asymmetric case, usually neither entry fees nor reserve prices are sufficient to implement the optimal cutoffs for the seller, and therefore, the seller should find alternative solutions. Only in the symmetric case, these two methods (reserve prices and entry fees) are able to implement the optimal entry in the auction. This result, by the way, is in contrast to the models with common entry costs in which entry fees are useful but reserve price may not be, and thus they are not equivalent tools (see, for example, [5,6]).

Finally, we assume that the number of bidders is endogenous and address the question of what is the optimal number of bidders that maximizes the seller's payoff or the social surplus. We find that the answer to this question is quite ambiguous. We examine three different scenarios. In one, the seller's payoff and the social surplus increase in the number of bidders. In another, both decrease in the number of bidders. Finally, in the last scenario, we find the seller's payoff and the social surplus may react in completely opposite ways whereas an increase in the number of bidders yields an increase in the seller's payoff but a decrease in the social surplus.

It is important to notice that in the symmetric model with private entry costs, the revenue equivalence theorem (see [14,15]) holds whether or not bidders observe how many others have decided to enter before bidding in the auction (see [2,16]) This implies that our results for symmetric second-price auctions will hold, for instance, if the auctions are first-price auctions and bidders are uninformed about who entered the auction before bidding. Indeed, Kaplan and Sela [17] studied a similar model of an all-pay auction with two bidders in which the equilibrium has the same form as in our model. However, for more than two bidders, the equivalence between both models does not hold. Levin and Smith [18] show that for risk-averse bidders, usually, but not always, the seller prefers the first-price auction to the second-price auction when there are entry costs.

In addition, in our asymmetric model, there may be a difference among the auctions; in particular, the first-price auction when the bidders are uninformed about who enters may generate lower revenue than the revenue in our model of second-price auctions. This indicates that there is room to study other auction forms in our asymmetric environment. However, other asymmetric auction forms are not necessarily as easily solvable as asymmetric second-price auctions.

The paper is organized as follows: in Section 2, we describe the general environment. We calculate the equilibrium strategies and examine the bidders' behavior in Section 3. Our analysis of the optimal design in our environment is carried out in Section 4. The effects of the number of bidders on these auctions are analyzed in Section 5. Finally, we discuss future extensions in Section 6.

2. The Model

Consider a second-price auction (see [19]) with $n \geq 2$ bidders competing for an indivisible item. The bidder with the highest bid wins the item and pays the second-highest bid with ties broken randomly. If there is no second-highest bid, then the price of the item is zero. The bidders are either type 1 or type 2 with $n_1 \geq 1$ of type 1 and $n_2 \geq 1$ of type 2 (with $n_1 + n_2 = n$). A bidder of type $i \in \{1, 2\}$ has $v_i \geq 0$ valuation for the item, which is common knowledge. Without loss of generality, assume $v_1 \geq v_2$. Additionally, without loss of generality, for each bidder $j \in \{1, \ldots, n\}$, assume the first n_1 bidders are of type 1, and the remaining bidders are of type 2.

For each bidder j, participating in the auction generates a fixed cost c_j, which is private information and drawn independently from the cumulative distribution function F. We assume that F is on the interval $[a, b]$ where $0 \leq a < \min\{v_1, v_2\}$ and that F is continuously differentiable with $F(a) = 0$, all of which is common knowledge. To avoid a trivial solution, assume that $F(v_i) > 0$. The bidders' entry costs are wasted in the sense that the seller does not benefit from these costs. We assume that each bidder knows his entry cost and his value before he makes his decision. This decision made by bidders can be split into two parts: whether to enter or stay out and what to bid if entering. Denote by $d_j(c_j)$ the entry decision (the probability of entering) if one has cost c_j, and let b_j be the bid if one indeed enters .

3. Equilibrium

Once a bidder enters the auction, then we assume he plays his dominant strategy, that is, to bid his value. Given this, the interesting analysis of our model is examining the entry decisions of the bidders. In our model, there frequently are trivial equilibria strategies in which one of the bidders decides to always participate independent of his entry cost, and all the other bidders decide to stay out of the auction. In order to prevent such equilibrium strategies (when $n_1 > 1$ or $n_2 > 1$), we assume that bidders of the same type (same v) follow the same strategy. We say that an equilibrium is type-symmetric if all bidders of the same type follow the same strategy. With a slight reuse of notation, now denote b_i and $d_i(c)$ as the bid and entry decision of bidders of type i.

Proposition 1. *A type-symmetric equilibrium exists and satisfies $b_i = v_i$ and*

$$d_i(c) = \begin{cases} 1 & \text{if } c \leq c_i^*, \\ 0 & \text{if } c > c_i^*, \end{cases}$$

where the equilibrium cutoffs c_i^*, $i = 1, 2$ are given by

$$c_1^* = (v_1 - v_2)(1 - F(c_1^*))^{n_1 - 1} + v_2(1 - F(c_2^*))^{n_2}(1 - F(c_1^*))^{n_1 - 1}, \tag{1}$$

$$c_2^* = v_2(1 - F(c_1^*))^{n_1}(1 - F(c_2^*))^{n_2 - 1}. \tag{2}$$

In the symmetric case where $v_1 = v_2$ and n is the total number of bidders, the symmetric equilibrium is given by $b_i(v) = v$ and

$$d_i(c) = \begin{cases} 1 & \text{if } c \leq c^*, \\ 0 & \text{if } c > c^*, \end{cases}$$

where the equilibrium cutoff $c^* > 0$ is the solution of

$$c^* = v(1 - F(c^*))^{n-1}. \tag{3}$$

The equilibrium described by Proposition 1 is such that any bidder with valuation v_i and an entry cost higher than the equilibrium cutoff c_i^* will stay out of the auction. Any bidder with valuation v_i and an entry cost lower than the equilibrium cutoff c_i^* will participate in the auction.

In the equilibrium, a bidder has a positive payoff only if he is the only entrant. Thus, the payoff of a bidder with valuation v_i and entry cost $c \le c_i^*$ is $c_i^* - c$. Thus, the expected payoff of a bidder with value v_i is

$$\int_0^{c_i^*} (c_i^* - c) dF(c). \tag{4}$$

We note that the Proposition's equilibrium is for $n_1, n_2 \ge 1$. If $n_1 \ge 2, n_2 \ge 2$ and $a = 0$, then any type-symmetric equilibrium must be interior. If $n_1 = 1$ or $n_2 = 1$, the type-symmetric equilibrium can be non-interior with $c_1^* \ge b, c_2^* \le a$ or $c_2^* \ge b, a < c_1^* < b$, and additionally, for $a > v1 - v2$, non-interior with $c_2^* \ge b, c_1^* \le a$ (where a cutoff $c_i > b$ implies that everyone of type i would enter, and a cutoff $c_i < a$ implies that everyone of type i stays out).

The following example shows that type-symmetric equilibrium is not necessarily unique and the difference among the equilibrium points is meaningful. (Note that for simplicity of exposition, in our examples, we will write the equilibrium cutoff equations, (1) and (2), assuming there is an interior solution and then see if this is indeed the case.)

Example 1. *Consider an auction where $n_1 = 2$, $n_2 = 1$, $v_1 = 2.25$, $v_2 = 2$ and F is a uniform distribution on $[0, 1]$.*

By (1) and (2), the equilibrium interior cutoffs are given by:

$$\begin{aligned} c_1^* &= (2.25 - 2)(1 - c_1^*) + 2(1 - c_2^*)(1 - c_1^*), \\ c_2^* &= 2(1 - c_1^*)^2. \end{aligned}$$

There are two solutions to this system of equations: 1. $c_1^* = 0.34255$ and $c_2^* = 0.8644$; 2. $c_1^* = 0.62993$ and $c_2^* = 0.2739$. Note that in the first solution, the bidders exhibit paradoxical behavior in the following sense. The equilibrium cutoff of the bidder with the low valuation v_2 is higher than the equilibrium cutoff of the bidders with the higher valuation v_1. This results in the expected payoff of the bidder with the low valuation v_2 being larger than the expected payoff of his opponents with the higher valuations v_1. We note that when $v_1 > 2v_2$, such a paradoxical equilibrium cannot exist. In a similar vein, there always exists an equilibrium where the cutoff c_1 is larger than the cutoff c_2, but as the example shows, it is not always unique.

Corollary 1. *A bidder with a relatively high valuation and low entry cost may decide to stay out of the auction, whereas a bidder with a relatively low valuation and high entry cost may decide to participate in the auction.*

The intuition for why this is possible is that a bidder's willingness to enter depends upon his expected surplus of being in the auction. This surplus depends upon not only the bidder's valuation but who else decides to enter the auction. Hence, if high bidders are less likely to enter the auction, then it is indeed possible for low-value bidders to be more willing to enter since they are more likely to be alone and reap all the profits.

We note that this result (or any) is not dependent on two bidders having the same value. In a three-type model where values are $v_1 = 2.25$, $v_2 = 2.24$, $v_3 = 2$, there are still two possible interior equilibria: $c_1^* = 0.34828$, $c_2^* = 0.8762$, $c_3^* = 0.327774$ and $c_1^* = 0.635516$, $c_2^* = 0.629634$, $c_3^* = 0.269985$..

Another paradoxical behavior in the asymmetric auctions is illustrated in the following example.

Example 2. *Consider an auction where $n_1 = n_2 = 1$ and F is a uniform distribution on $[1/2, 1]$. Let $v_1 > v_2$ and v_1, v_2 be in $(1/2, 1)$.*

The only potential interior equilibrium is given by

$$c_1^* = \frac{2v_1(v_1 + v_2 - 1)}{4(v_1)^2 - 1} \text{ and } c_2^* = \frac{4(v_1)^2 - (v_1 + v_2)}{4(v_1)^2 - 1}.$$

It is indeed interior when $v_2 < (4(v_1)^2 - 1)/2v_1 + 1 - v_1$ which is possible when $v_1 > 0.78$. Note that in this case $c_1^* > c_2^*$. It is clear that c_2^* decreases with v_2 and c_1^* increases with v_2. Furthermore, one can show that c_2^* increases with v_1, and c_1^* decreases with v_1. We note that by (4), a type-1 bidder's expected payoff is increasing in cutoff c_1^* leading to the following corollary.

Corollary 2. *The expected payoff of a player may decrease in his valuation and increase in the valuations of his opponents.*

We gain intuition for why this may happen by comparing our model with private entry costs to the model with commonly known entry costs. For example, consider the case of two bidders with valuations $v_1 > v_2$ and entry costs of c_1, c_2. The valuations as well as the entry costs are common knowledge. Then the auction reduces to the following 2×2 game:

		Bidder 2	
		In	*Out*
Bidder 1	*In*	$(v_1 - v_2 - c_1, -c_2)$	$(v_1 - c_1, 0)$
	Out	$(0, v_2 - c_2)$	$(0, 0)$

This game has two pure-strategy equilibrium points and one mixed-strategy equilibrium. The pure-strategy equilibrium strategies are identical to those in our model in which, independent of costs, one bidder decides to enter and the other bidder decides to stay out of the auction. In the mixed-strategy equilibrium the probability that player 1 will participate is $p_1 = \frac{v_2 - c_2}{v_2}$, and the probability that player 2 will participate is $p_2 = \frac{v_1 - c_1}{v_2}$. Note that the probabilities that both bidders will participate in the auction increase in their opponents' valuation, and the probability that bidder 2 will participate even decreases in his own valuation. Hence, we can say that the probabilities of participation in both models have almost the same properties. However, while the expected payoff of the bidders in this 2×2 game is zero and, hence, independent on the bidders' valuations, in our model, the expected payoff of each bidder is positive and may decrease in his valuation as we can see in Example 2.

4. The Optimal Entry

We assume that the number of bidders is exogenous such that the seller cannot determine the number of bidders and the bidders use type-symmetric equilibria. However, the seller can change the entry decision of the players (the equilibrium cutoff) by imposing entry fees, reserve prices, bid caps, or other methods. In this section, we examine what the optimal level of entry is in this environment. We do so under two conditions: (i) the seller must treat all bidders equally and (ii) the equilibrium is a type-symmetric equilibrium.

Under these two conditions and the additional assumption that any bidder considered in the mechanism must pay the entry cost, we show in Appendix A that the second-price auction with the optimal level of entry set is also the optimal mechanism.

The seller's expected surplus in the auction is the total surplus minus the bidders' surplus. The total surplus for the seller and the bidders together must equal the chance that at least one bidder with value v_1 enters times v_1 plus the chance that no bidder with value v_1 enters and at least one bidder with value v_2 enters time v_2 minus the expected cost of entry for both types of bidders. The chance that at least one bidder with value v_1 enters is $1-(1-F(c_1^*))^{n_1}$, while the chance that no one with value v_1 enters and at least one bidder with value v_2 enters is $(1-F(c_1^*))^{n_1}(1-(1-F(c_2^*))^{n_2})$. The expected entry cost of a bidder with value v_i is $\int_0^{c_i^*} cdF$. Hence, we can write the seller's expected surplus as:

$$\begin{aligned}\pi_s(c_1^*,c_2^*) &= (1-(1-F(c_1^*))^{n_1})v_1+(1-F(c_1^*))^{n_1}(1-(1-F(c_2^*))^{n_2})v_2 \\ &\quad -n_1\int_0^{c_1^*} cdF - n_2\int_0^{c_2^*} cdF - n_1\int_0^{c_1^*}(c_1^*-c)dF - n_2\int_0^{c_2^*}(c_2^*-c)dF \\ &= (1-(1-F(c_1^*))^{n_1})v_1+(1-F(c_1^*))^{n_1}(1-(1-F(c_2^*))^{n_2})v_2 \\ &\quad -n_2c_2^*F(c_2^*)-n_1c_1^*F(c_1^*)\end{aligned} \tag{5}$$

Consider now that the seller could influence the equilibrium cutoff. By using the above expression, we show that the seller always wishes to decrease the equilibrium cutoff of at least one type of bidders, namely, he wishes to reduce the participation of these bidders.

Proposition 2. *For at least one type v_i, the optimal cutoff c_i^{op} for the revenue-maximizing seller is smaller than the equilibrium cutoff c_i^*, that is, either $c_1^{op} < c_1^*$ or $c_2^{op} < c_2^*$. However, for either type, it is possible that the optimal cutoff is larger (or smaller) than the equilibrium cutoff*

Proof. For the first part, see Appendix A. For the second part, we can further examine Example 1. Recall that in Example 1, $n_1 = 2, n_2 = 1$, $v_1 = 2.25$, $v_2 = 2$ and F is a uniform distribution on $[0,1]$. It can be shown that the optimal cutoffs are $c_1^{op} = 0.452$, $c_2^{op} = 0.3$. Recall that there are two equilibria: $c_1^* = 0.343$ and $c_2^* = 0.864$; $c_1^* = 0.62993$ and $c_2^* = 0.2739$. This shows that the optimal cutoff c_i^{op} can be either smaller or larger than the equilibrium cutoff c_i^*. □

The optimal cutoff from the above proposition is optimal given the limitation of symmetry (all bidders should have identical strategies). Any mechanism that induces behavior according to this optimal cutoff is an optimal mechanism (as shown in Appendix A).

A consequence of Proposition 2 is that the seller may wish to either decrease or increase the equilibrium cutoff of either type of bidders. However, if the number of bidders from each type is identical, the seller will always prefer the participation of the bidders with the higher type.

Proposition 3. *If $n_1 = n_2$, then the optimal cutoff of the bidders with the high valuation c_1^{op} is always larger than the optimal cutoff c_2^{op} of the bidders with the lower valuation.*

Proof. See the Appendix A. □

On the other hand, if the number of bidders of each type is not identical, that is, $n_1 \neq n_2$, then the seller does not necessarily prefer participation of bidders with the higher type, as we can see in the following example.

Example 3. *Consider an auction where $n_1 = 5$, $n_2 = 2$, $v_1 = 1.5$, $v_2 = 1.4$, and F is a uniform distribution on $[0.8, 1]$.*

By (A1) and (A3), the optimal cutoffs are $c_1^{op} = 0.699$ and $c_2^{op} = 0.979$. That is, the seller would prefer to get the bidders with the lower valuation v_2 to almost always enter while leaving the bidders with the higher valuation v_1 to partially stay out (30% of the time).

Finally, we can investigate the symmetric case by setting $v_1 = v_2$ and obtain:

Proposition 4. *When $v_1 = v_2$, the optimal cutoff c^{op} for the revenue-maximizing seller is strictly positive and always smaller than the equilibrium cutoff c^*.*

Proof. See Appendix A. □

The relevant question now is: how can the seller implement the optimal entry? Can he implement the optimal entry by entry fees or reserve prices? As we show below, the implementation of the optimal entry is quite simple in the symmetric case and much more complicated in the asymmetric case.

Let us begin by examining the symmetric case, where $v_1 = v_2$. Here, we find that the optimal critical entry cost can be obtained by imposing an entry fee or, alternatively, a reserve price.

When the seller imposes an entry fee, e, the symmetric equilibrium is given by $b_i(v) = v$ and

$$d_i(c) = \begin{cases} 1 & \text{if } c_i \leq c^e, \\ 0 & \text{if } c_i > c^e \end{cases}$$

where the equilibrium cutoff c^e is the solution of

$$c^e + e = v(1 - F(c^e))^{n-1}. \tag{6}$$

Now, if we set $e = \frac{F(c^{op})}{F'(c^{op})}$, then the solution of (6) yields the optimal cutoff: From the proof of Proposition 4 (see Appendix A), the optimal entry cost is such that $F(c^{op}) > 0$. In addition, $F'(c^{op}) > 0$ since if $F'(c^{op}) = 0$ then $\frac{d\pi_s}{dc^o}(c^{op}) < 0$, which would be a contradiction given our continuity assumptions that implies $\frac{d\pi_s}{dc^o}(c^{op}) = 0$. It can be easily verified that setting the optimal reserve price is an equivalent operation to setting an entry fee in the symmetric setup, both of which yields the optimal entry in the auction. This result is in contrast to the models with common entry costs in which a reserve price and an entry fee are not equivalent tools. For example, Levin and Smith [5] show that in common value auctions, the seller should discourage entry by charging a positive entry fee but no reserve price.

In contrast to symmetric auctions, in asymmetric auctions, the seller's aim to obtain the optimal entry costs (c_1^{op}, c_2^{op}) is not simple, since, in these auctions, using tools such as entry fees or reserve prices are not always sufficient for the seller to reach this goal. Moreover, these tools are not equivalent as they are in the symmetric auctions.

Example 4. *Consider an auction where $n_1 = n_2 = 1$, $v_1 = 1$, $v_2 = 0.5$, and F is a uniform distribution on $[0, 1]$.*

The unique equilibrium is $c_1^* = 1, c_2^* = 0$. The optimal cutoffs obtained by the solution of Equations (A1) and (A3) are: $c_1^{op} = \frac{7}{15}, c_2^{op} = \frac{2}{15}$.

Therefore, the aim of the seller is to decrease the equilibrium cutoff of the bidder with the high valuation while increasing the equilibrium cutoff of the bidder with the low valuation.

If the seller imposes an entry fee e, the equilibrium cutoffs c_i^e $i = 1, 2$ are given by

$$c_1^e + e = (v_1 - v_2) + v_2(1 - F(c_2^e)) = 0.5 + 0.5(1 - F(c_2^e)),$$

$$c_2^e + e = v_2(1 - F(c_1^e)) = 0.5(1 - F(c_1^e)).$$

If the entry fees are restricted to be the same, the solution of these equations yields $c_1 = 1 - 2/3e$ and $c_2 = -2/3e$. Then, the seller cannot induce an interior solution with a uniform e. Any positive e will result in $c_2 = 0$ and $c_1 = 1 - e$, and any negative e will result in $c_1 = 1$ and $c_2 = -e$. The negative e would not be profitable since it will only cause additional unuseful entry. With a positive e, the seller's profit will be $e(1 - e)$, which reaches its maximum at $e = 1/2$ with a profit of $1/4$.

On the other hand, if the seller imposes a reserve price r (less than v_2), then the equilibrium cutoffs c_i^r $i = 1, 2$ are given by

$$c_1^r = (v_1 - v_2) + (v_2 - r)(1 - F(c_2^r)) = 0.5 + (0.5 - r)(1 - c_2^r),$$

$$c_2^r = (v_2 - r)(1 - F(c_1^r)) = (0.5 - r)(1 - c_1^r), \text{ for every } 0 < r < 0.5.$$

By using a reserve price r, the equilibrium cutoff of the bidder with the high valuation decreases, and the equilibrium cutoff of the bidder with the low valuation increases.

The solution to the above equations yields

$$\begin{aligned} c_1 &= \frac{3 - 4r^2}{3 + 4r - 4r^2}, \\ c_2 &= \frac{2r - 4r^2}{3 + 4r - 4r^2}, \end{aligned}$$

and the seller's profit is

$$\pi_s(r) = \frac{4r(3 - 2r^2 - 4r^3)}{(3 - 4(r - 1)r)^2}.$$

The maximum profit achievable by using a reservation price is 0.252366, which, although close, is still different than the optimal profit 0.26666. However, it is better than the best achievable with uniform entry fees, which is 0.25.

Unlike in Levin and Smith [5], the seller's payoff is not equivalent to the social surplus. This social surplus π_{ss} is the chance that someone enters and gets the object minus the expected costs of all the entrants. Thus, the social surplus is given by:

$$\begin{aligned} \pi_{ss}(c_1^*, c_2^*) &= (1 - (1 - F(c_1^*))^{n_1})v_1 + \\ & (1 - F(c_1^*))^{n_1}(1 - (1 - F(c_2^*))^{n_2})v_2 - n_1 \int_0^{c_1^*} c dF - n_2 \int_0^{c_2^*} c dF. \end{aligned} \quad (7)$$

It can be easily verified that the first-order conditions of the social surplus are identical to the equilibrium conditions. Therefore, in order to maximize the social surplus, we should not use tools such as entry fees or reserve prices; rather, we should just let the bidders compete without any interference (this is under the restriction that the seller must run an auction and cannot treat bidders differently).

5. The Optimal Number of Bidders

Thus far, we assumed that the number of potential bidders is exogenous. Suppose that the seller can determine the number of bidders. We also assume that the bidders that are excluded will not pay entry costs. We note that this is consistent with our vision that the seller cannot reduce the entry costs of the bidders. Here, we picture that the seller is simply able to send a message that the auction is closed at an early enough stage. Usually, in auctions under incomplete information, the optimal number of bidders is infinity. In our model, the optimal number of bidders is more complex. In order to demonstrate this point, it is sufficient to consider the simpler case of symmetric auctions. The following example consists of three cases and shows that an increase in the number of potential bidders has an ambiguous effect both on the seller's expected payoff and on the social surplus.

Example 5. *Consider an auction where* $v_1 = v_2 = 1$.

By (3), (5) and (7), the equilibrium cutoff, the seller's payoff and the social surplus, respectively, are given for three different cases as follows:

Case 1: The bidders' entry costs are distributed according to a uniform distribution on $[0, 1]$.

Number of Bidders	Equilibrium Cutoff	Seller's Payoff	Social Surplus
2	0.5	0.25	0.5
3	0.381966	0.326238	0.545085
4	0.317672	0.379581	0.581412
5	0.275508	0.420873	0.610635
10	0.175699	0.546468	0.700819
1000	0.00524	0.9673	0.9810

In this case, an increase in the number of potential bidders yields an increase in both the seller's payoff and the social surplus.

Case 2: The bidders' entry costs are distributed according to a uniform distribution on $[0.5, 0.75]$.

Number of Bidders	Equilibrium Cutoff	Seller's Payoff	Social Surplus
2	0.6	0.16	0.2
3	0.5625	0.15625	0.17969
4	0.5457	0.15501	0.17171
5	0.53608	0.15443	0.16745
10	0.51764	0.15368	0.1599
1000	0.50017	0.15335	0.15340

In this case, an increase in the number of potential bidders yields a decrease in both the seller's payoff and social surplus.

Case 3: The bidders' entry costs are distributed according to a uniform distribution on $[0.5, 1]$.

Number of Bidders	Equilibrium Cutoff	Seller's Payoff	Social Surplus
2	0.66667	0.11111	0.16667
3	0.60961	0.12311	0.15915
4	0.58244	0.12947	0.15665
5	0.56626	0.13355	0.1555
10	0.5337	0.1426	0.15396
1000	0.5003	0.1533	0.1534

In this case, an increase in the number of bidders yields an increase in the seller's payoff but a decrease in the social surplus. In all three cases, as the number of potential entrants increase, the seller captures more of the social surplus.

Bolton and Farrell [20] use games of entry with private costs of entry to compare centralized to decentralized decision making. Centralization has the manager (seller) limit the potential entrants by assignment. They find that depending upon the parameters either centralized decision making is superior by avoiding coordination costs or decentralized decision making is superior by providing entry-cost advantages. Our examples show that this trade-off exists for auctions with entry: while a seller could save on coordination costs by a centralized assignment of participants, the seller would potentially lose out due to higher entry costs. In case one, decentralization is superior while in case two, centralization is superior. In case three, centralization is superior for a seller concerned only with revenue, while decentralization is superior for a seller concerned with efficiency (such as a government).

In Example 5, we showed that the seller's payoff in equilibrium may either increase or decrease with the number of bidders. The following example shows that the seller's payoff with the optimal cutoff (seller's optimal payoff) may also decrease with the number of bidders. (While we will not show it here, the seller's optimal payoff may also increase with the number of bidders.)

Example 6. *Consider an auction where $v_1 = v_2 = 1$ and F is a uniform distribution function on* $[0.5, 0.75]$.

By (4) and (A4), the seller's optimal cutoff and the seller's optimal payoff are as follows:

Number of Bidders	Seller's Optimal Cutoff	Seller's Optimal Payoff
2	0.58333	0.16667
3	0.55481	0.15915
4	0.54122	0.15665
5	0.53313	0.15550
10	0.51685	0.15396
1000	0.500173	0.1534

Note that the seller's optimal cutoff as well as the seller's optimal payoff decrease with the number of bidders. Thus, the optimal number of bidders for the seller in this example is 2.

6. Discussion

We study environments with one area of private information (costs) and one of complete information (values). Our environments have the advantage that the introduction of asymmetry does not obstruct solvability. Cao et al. [11] studied environments with two areas of private information: costs and values. They showed that there always exists an equilibrium in this general setting with two-dimensional types of ex ante heterogeneous bidders. When bidders are ex ante homogeneous, there is a unique symmetric equilibrium, but asymmetric equilibria may also exist. They further examined conditions that guarantee uniqueness when there are two bidders. Their results are consistent with the existence of the asymmetric equilibria present in our model.

We also wish to bring up a motivating example suggested by a referee. There is a physical used-car auction held downtown in the morning of a workday. The two types of bidders are consumers and dealers. Consumers (type 1) value the car more than dealers with resale motives (type 2). On the morning of the auction, both the consumer and the dealer check their calendar, and figure out whether it is worth it to drive downtown to attend. The *iid* assumption on entry costs is realistic given that everyone is busy and people live all over the place. There can be two types of equilibria: consumer-prevalent (consumers go almost regardless of cost, so dealers only go if not too busy) and dealer-prevalent (dealers go almost regardless of cost, so consumers only go if not too busy). The dealer-prevalent equilibrium is somewhat counter-intuitive in that a consumer with the same opportunity cost as a dealer may not go even though the consumer values the car more.

In the realm of market entry games, our model fits in the class of one-shot games. There is another class of multi-period games [20,21]. While these capture the ability of firms to enter if a market is left empty, they lack the potential sunk cost associated with such actions (as one would expect if entry decisions need to be made several periods in advance). Such a multi-period model (entry if the market is left empty) cannot be used to study auctions with sunk entry costs since they will allow non-winning participants to avoid the entry cost. For example, the seller could run a multi-period mechanism that imitates a Dutch auction, where each period he offers the item to a random potential entrant and in case of non-entry slightly lowers his price. This scenario is not reasonable if, for instance, the costs represent raising the necessary credit to participate. One extreme of the multi-period models is to keep the time element of the entry, but have entry decisions made completely in advance. In such a model, earlier entry costs more, but provides higher expected profits. These have been studied as all-pay auctions by Kaplan et al. [22] and Kaplan et al. [23]. There, the coordination failure is costly only to the loser. Such a scenario best fits patent races and markets with strong network externalities. This paper presents another extreme that de-emphasizes the time element of entry and keeps the dual cost of coordination failure. This fits cases where entry decisions must be made far in advance, but there is not an overwhelming advantage to the first entrant. All such models have their place and complement each other, with this paper's model being the most tractable.

Finally, while we talk about entry costs in auctions, our results can also be applied to the Bertrand price competition with entry costs. They add to this literature, particularly in regards to the number of potential entrants. Lang and Rosenthal [24] show that if the number of entrants is unknown at the time of bidding, but there is symmetry and complete information about values and costs, then the total welfare (which equals the seller's surplus) decreases with the number of potential entrants. Elberfeld and Wolfstetter [25] show that when there is complete information about the symmetric entry costs and the number of entrants, but incomplete information about the values, then total welfare decreases in the number of potential entrants. Thomas [26] on the other hand, shows that with complete information about entry costs and about the number of entrants, but with asymmetric entry costs, the total welfare increases in the number of potential entrants. This built upon Samuelson [7], who has complete information about entry costs, but incomplete information about values. Our results in this paper show that all the situations described above are possible in one model. The Bertrand setting can also make use of our optimal design analysis. Here, the optimal design for the seller is the same as an optimal design for a regulatory agency with the consumer interest as the objective. This may shed light on which policies may work best in price competitions such as the Bertrand competition.

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Appendix A

Appendix A.1. Proof of Proposition 1

The standard result that bidding your value in a second-price auction is the weakly dominant strategy (see Vickrey, 1961) holds in our asymmetric environment. (We will ignore other equilibria that are not trembling-hand perfect and therefore would not withstand some uncertainty in the values.)

Given the equilibrium bidding behavior, a bidder with a low valuation v_2 will profit only when he is in the auction alone. The probability of this is $(1-F(c_1^*))^{n_1}(1-F(c_2^*))^{n_2-1}$, which implies Equation (2). On the other hand, a bidder with a high valuation v_1 will profit $v_1 - v_2$ when there are no other v_1-value bidders and will profit an *additional* v_2 when there are no v_2-value bidders as well. These happen with probabilities $(1-F(c_1^*))^{n_1-1}$ and $(1-F(c_2^*))^{n_2}(1-F(c_1^*))^{n_1-1}$, respectively, which implies Equation (1).

Another way to derive Equation (1) is to see that a bidder with value v_1 profits v_1 when alone in the auction, which occurs with probability $(1-F(c_2^*))^{n_2}(1-F(c_1^*))^{n_1-1}$ and profits $v_1 - v_2$ when not alone, but there are only bidders with value v_2 in the auction, which occurs with probability $(1-F(c_1^*))^{n_1-1}(1-(1-F(c_2^*))^{n_2})$, which then simplifies to the expression.

The existence of the equilibrium is derived by Brower's fixed-point theorem. The RHS of Equations (1) and (2) form a bounded function from $[0, v_1] \times [0, v_2]$ to $[0, v_1] \times [0, v_2]$ that is continuous since F is continuous. Therefore, a fixed point must exist (note that if the cutoff c_i^* of the fixed point is above b, then it would imply that everyone with value v_i enters. Likewise, if cutoff c_i^* of the fixed point is below a, then it would imply that everyone with value v_i stays out).

In the following we show that if n_1, $n_2 \geq 2$, and $a = 0$, then any fixed point is interior; that is, $F(c_1^*), F(c_1^*)$ are from $(0,1)$. (We also assume that $F'(a) > 0$.) The RHS of Equations (1) and (2) are decreasing in c_1^* and c_2^*. If $F(c_1^*) = 0$, then the RHS of (1) is greater than or equal to $v_1 - v_2 > 0$—a contradiction. If $F(c_1^*) = 1$, then the RHS of (1) is zero—also a contradiction. Hence, $0 < F(c_1^*) < 1$. A similar argument shows that $0 < F(c_2^*) < 1$ as well.

The symmetric case can be shown in a similar, but simpler manner.

Appendix A.2. Proof That the Optimal Cutoff Is the Optimal Mechanism for the Seller

Here we show that an auction with an equilibrium with an optimal cutoff is the optimal mechanism for the seller. Since our participation costs are wasted, any bidder that agrees to participate in the mechanism must incur his costs. This eliminates the possibility of a mechanism that queries the bidders about their costs before they are incurred. We also restrict mechanisms to being symmetric in regards to the individual bidders and assume that the equilibrium of such mechanisms are also symmetric. (If this restriction were lifted it may be indeed possible for the optimal mechanism to be asymmetric.)

If in equilibrium of the mechanism m, a bidder with cost c_1 enters and receives expected payoff $m(c_1)$, then a bidder with cost $c_2 < c_1$ would also have to enter in equilibrium since he can always imitate a bidder with cost c_1. This, as before, leads to cutoff strategies. Since at the cutoff, anyone above the cutoff can not pretend to have cost c^o, the expected profits of the cutoff must be zero, $m(c^o) - c^o = 0$. Additionally, since any bidder with cost below c^o can always receive payoff $m(c^o)$ and the bidder with cutoff c^o can always pretend to have lower costs, the payoff to entering must be $m(c^o)$. Thus, the expected profits of a bidder entering with costs c must be $m(c^o) - c = c^o - c$. Therefore, the expected profits of each bidder is the same as shown for the auction and likewise for the seller's profits.

Appendix A.3. Proof of Proposition 2 (First Part)

The derivative of the seller's surplus with respect to the cutoff of bidders with value v_1 is:

$$\begin{aligned}\frac{d\pi_s}{dc_1^o} \quad = \quad & n_1\Big[(v_1 - v_2)(1-F(c_1^o))^{n_1-1}+ \\ & v_2(1-F(c_1^o))^{n_1-1}(1-F(c_2^0))^{n_2} - c_1^o\Big]F'(c_1^o) - n_1F(c_1^o). \qquad \text{(A1)}\end{aligned}$$

Substituting the equilibrium cutoff (1) in (A1) yields

$$\frac{d\pi_s}{dc_1^o}(c_1^*, c_2^*) = -n_1 F(c_1^*) < 0. \tag{A2}$$

Since the RHS of (A1) is decreasing in both c_1^o and c_2^o, we obtain for any $c_1 \geq c_1^*$ and $c_2 \geq c_2^*$ the expression $\frac{d\pi_s}{dc_1^o}(c_1, c_2) < 0$.

Likewise,

$$\frac{d\pi_s}{dc_2^o} = n_2\Big[v_2(1 - F(c_1^o))^{n_1}(1 - F(c_2^0))^{n_2-1} - c_2^o\Big]F'(c_2^o) - n_2 F(c_2^o). \tag{A3}$$

Substituting the equilibrium entry cost (2) in (A2) yields

$$\frac{d\pi_s}{dc_2^o}(c_1^*, c_2^*) = -n_2 F(c_2^*) < 0.$$

Since the RHS of (A3) is decreasing in both c_1^o and c_2^o, then we obtain that for any $c_1 \geq c_1^*$ and $c_2 \geq c_2^*$, we have $\frac{d\pi_s}{dc_2^o}(c_1, c_2) < 0$. Thus, $\pi_s(c_1, c_2) < \pi_s(c_1^*, c_2^*)$ for all $c_1 \geq c_1^*, c_2 \geq c_2^*$ with at least one strict inequality. This implies that it is better to decrease at least one of the equilibrium cutoffs c_1^*, c_2^*, and particularly, these equilibrium cutoffs are not optimal.

Appendix A.4. Proof of Proposition 3

Assume that $c_1^{op} < c_2^{op}$. Let us compare the sellers' profits using the optimal cutoffs to the sellers' profits reversing the optimal cutoffs, i.e., using c_2^{op} for the cutoff for a bidder with value v_1 and vice versa. The advantage of the optimal cutoffs to this new set of cutoffs is $\pi_s(c_1^{op}, c_2^{op}) - \pi_s(c_2^{op}, c_1^{op})$. Since $n_1 = n_2$, we have $\pi_s(c_1^{op}, c_2^{op}) - \pi_s(c_2^{op}, c_1^{op}) = -(v_1 - v_2)((1 - F(c_1^{op}))^{n_1} - (1 - F(c_2^{op}))^{n_1}) < 0$. This is a contradiction to the optimality of the cutoffs and therefore the optimal cutoffs must satisfy $c_1^{op} > c_2^{op}$.

Appendix A.5. Proof of Proposition 4

The derivative of the seller's profit with respect to c^o yields:

$$\frac{d\pi_s}{dc^o}(c^{op}) = n\Big[v(1 - F(c^{op}))^{n-1} - c^{op}\Big]F'(c^{op}) - nF(c^{op}). \tag{A4}$$

Substituting the equilibrium entry cost c^* (3) in (A4) yields that

$$\frac{d\pi_s}{dc^o}(c^*) = -nF(c^*) < 0.$$

Furthermore, $v(1 - F(c^{op}))^{n-1} - c^{op}$ is decreasing in c^{op}. Additionally, $-nF(c^{op})$ is decreasing. Therefore, for any $c > c^*$, the $\frac{d\pi_s}{dc^o}(c) < 0$ as well. Thus, the optimal critical entry cost c^{op} is always smaller than the equilibrium critical entry cost c^*.

Note, the optimal critical entry cost is strictly positive (with strictly positive entry $F(c^{op}) > 0$). We can see this since the profits for no entry is zero. Thus, we need to only show that there is a possibility for the seller to make a profit. Our assumption that $F(v) > 0$ and continuity of F imply that there exists a c' such that $v(1 - F(c'))^{n-1} - c' > 0$ and $F(c') > 0$. If the seller set an additional entry fee $e = v(1 - F(c'))^{n-1} - c'$, all bidders with $c < c'$ will enter. Hence, the seller would make profit of at least $F(c') \cdot e > 0$.

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