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A note on dyadic approximation in Cantor's set

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Abstract

We consider the convergence theory for dyadic approximation in the middle-third Cantor set, K, for approximation functions of the form $\psi_\tau(n) = n^{-\tau}$ ($\tau \geqslant 0$). In particular, we show that for values of τ beyond a certain threshold we have that almost no point in K is dyadically ψ_τ-well approximable with respect to the natural probability measure on K. This refines a previous result in this direction obtained by the first, third, and fourth named authors.

Keywords: Diophantine approximation; Cantor set; Fourier analysis

1. Introduction

Throughout this note, we write K to denote the middle-third Cantor set and denote by μ the natural probability measure on K. We recall that K consists of the real numbers $x \in [0, 1]$ which have a ternary expansion consisting only of 0's and 2's, and that its Hausdorff dimension is

$$\dim_{\mathrm{H}} K = \frac{\log 2}{\log 3} =: \gamma.$$

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The natural measure μ on K is the Hausdorff γ-measure restricted to K, which is a probability measure as $\mathcal{H}^\gamma(K) = 1$. For more information on Hausdorff dimension and Hausdorff measures, we refer the reader to [5].

The study of Diophantine approximation in the Cantor set was suggested by Mahler [13], and has since been an active subject of research — see, for example, [3,4,10,12,14–17]. In [1], the first, third and fourth named authors discussed the problem of approximating elements of K by rationals with denominators that are a power of two: that is, *dyadic rationals*. Our methods realised the dyadic approximation problem as a manifestation of Furstenberg's "times two, times three" phenomenon [6,7].

For $\psi : \mathbb{R} \to [0, \infty)$ and $y \in \mathbb{R}$, define

$$W_2(\psi, y) = \{x \in \mathbb{R} : \|2^n x - y\| < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Here, for $x \in \mathbb{R}$, we write $\|x\|$ to denote the Euclidean distance from x to the nearest integer. In analogy with Khintchine's theorem [11], Velani conjectured that if ψ is monotonic then

$$\mu(W_2(\psi, 0)) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \psi(n) < \infty, \\ 1, & \text{if } \sum_{n=1}^{\infty} \psi(n) = \infty, \end{cases}$$

see [1, Conjecture 1.2]. The two parts of such a dichotomy are commonly referred to as the convergence and divergence theories of metric Diophantine approximation, respectively. The second named author [2] stated the following natural generalisation of Velani's conjecture, dropping the monotonicity condition and introducing an inhomogeneous shift. The latter relates the problem to distribution modulo 1, and also enables one to recast it in terms of shrinking targets [9].

Conjecture 1 *([2, Conjecture 1.2]). If $y \in \mathbb{R}$, then*

$$\mu(W_2(\psi, y)) = \begin{cases} 0, & \textit{if } \sum_{n=1}^{\infty} \psi(n) < \infty, \\ 1, & \textit{if } \sum_{n=1}^{\infty} \psi(n) = \infty. \end{cases}$$

Let us now consider the problem at the level of the exponent. For $\tau \geqslant 0$ and $n \in \mathbb{N}$, define $\psi_\tau(n) = n^{-\tau}$. Plainly $\mu(W_2(\psi_0, y)) = 1$ for any y. By [1, Theorem 1.5], we have

$$\mu(W_2(\psi_\tau, 0)) = 0 \qquad (\tau \geqslant 1/\gamma). \tag{1}$$

It follows from the recent work of the second named author [2] that if $y \in \mathbb{R}$ then

$$\mu(W_2(\psi_\tau, y)) = 1 \qquad (\tau \leqslant 0.01),$$

refining the progress on the divergence side made in [1]. The purpose of this note is to establish the following sharpening and generalisation of (1).

Theorem 2. *Let*

$$\tau > \frac{0.922(1-\gamma)+1}{\gamma(2-\gamma)},$$

and let $y \in \mathbb{R}$. Then $\mu(W_2(\psi_\tau, y)) = 0$.

One computes that $\tau > 1/\gamma - 0.03$ is sufficient. This makes progress towards the convergence part of Velani's conjecture. In [1], it was shown conditionally that

$$\mu(W_2(\psi_\tau, 0)) = \begin{cases} 0, & \text{if } \tau > 1, \\ 1, & \text{if } \tau \leqslant 1, \end{cases} \tag{2}$$

which constitutes a conditional solution to Velani's conjecture at the level of the exponent. Specifically, the appendix of [1] contains empirical data supporting the assertion that

$$D_2(y) + D_3(y) \gg \log y \qquad (y \in \mathbb{N}),$$

where $D_b(y)$ denotes the number of digit changes of y in base b, and (2) was established subject to this hypothesis. We refer the reader to [1, Section 5] for further results of a similar flavour. Theorem 2 is unconditional.

We finish this section by briefly discussing the significance of the exponent $1/\gamma$. By a comparatively simple argument, one can see that if $\tau > 1/\gamma$ and $y \in \mathbb{R}$ then $\mu(W_2(\psi_\tau, y)) = 0$, see the proof of [1, Proposition 1.4]. In [1], we attained the exponent $1/\gamma$ in establishing (1). Thus, as explained in the introduction of that article, dyadic approximation in K behaves very differently to triadic approximation in K, the latter having been thoroughly investigated by Levesley, Salp and Velani [12]. Theorem 2 extends the admissible range for the exponent beyond this threshold.

Notation

For complex-valued functions f and g, we write $f \ll g$ or $f = O(g)$ if $|f| \leqslant C|g|$ pointwise, for some constant $C > 0$.

2. Preliminaries

During our proof of Theorem 2 we will make use of a number of technical results from [1,2,17]. These are detailed below. To this end, let us first recall the following constructive definition of K: let $K_0 := [0,1]$ and let $K_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ be the set obtained by removing the open middle third from K_0. Next, suppose the set K_{n-1} has been defined. Let K_n be the set obtained upon removing the open middle thirds from all the component intervals of K_{n-1}. With the sets K_n constructed in this way, we have

$$K = \bigcap_{n=0}^{\infty} K_n.$$

Note that for each $n \in \mathbb{N}$, the set K_n consists of 2^n closed intervals, each of length 3^{-n}. Let $\mathcal{C}_N$ denote the set of all (left and right) endpoints of the intervals comprising K_N.

The result we use from [1] estimates the μ-measure of a union of balls by counting nearby triadic rationals in $\mathcal{C}_N$ for a sufficiently large $N \in \mathbb{N}$. For $n \in \mathbb{N}$, $\sigma > 0$ and $y \in \mathbb{R}$, denote

$$A_n^y(\sigma) = \{x \in \mathbb{R} : \|2^n x - y\| < \sigma\}.$$

Lemma 3 (*[1, Lemma 2.1], Special Case*)**.** *Let $n, N \in \mathbb{N}$ and $\sigma \in (0,1)$ with $3^{-N} \leqslant \frac{\sigma}{5\cdot 2^n}$, and let $y \in \mathbb{R}$. Then*

$$2^{-(N+1)}\#(\mathcal{C}_N \cap A_n^y(\sigma/5)) \leqslant \mu(A_n^y(\sigma)) \leqslant 2^{-(N-1)}\#(\mathcal{C}_N \cap A_n^y(5\sigma)).$$

The results we use from [2,17] are formulated in terms of the Fourier transform of a measure. Recall that this quantity is defined as follows: given a Borel probability measure ν supported on $[0,1]$, let

$$\hat{\nu}(\xi) = \int e^{-2\pi i \xi x}\, \mathrm{d}\nu(x).$$

Lemma 4 (*[2, Lemma 2.2]*)**.** *Let $N \in \mathbb{N}$, and let $t \in \mathbb{Z} \setminus \{0\}$. Then there exist constants $C_1, C_2 > 0$ independent of N and t such that*

$$\#\left\{0 \leqslant n < N : \left|\hat{\mu}(t2^n)\right| > C_1 N^{-0.078}\right\} \leqslant C_2 N^{0.922}.$$

Lemma 5 (*[17, Theorem 4.1]*)**.** *Let ν be a Borel probability measure on $[0,1]$. Let $\delta \in (0,1)$, let $Q \in \mathbb{N}$, and let $y \in [0,1]$. Then*

$$\nu(\{x \in [0,1] : \|Qx - y\| \leqslant \delta\}) \ll \delta\left(1 + \sum_{\substack{0<|\xi|\leqslant 2Q/\delta \\ Q|\xi}} |\hat{\nu}(\xi)|\right).$$

The statement given in [17, Theorem 4.1] also provides a lower bound for $\nu(\{x \in [0,1] : \|Qx - y\| \leqslant \delta\})$ and applies in arbitrary dimensions, but we will only use this simpler statement.

3. Proof of Theorem 2

Set $C > 0$ to be the constant C_1 arising from Lemma 4. Define

$$\beta_1 = 0.078, \qquad \beta_2 = 0.922, \qquad \alpha = \frac{1-\beta_2}{2-\gamma}.$$

Observe that the assumption of the theorem can be rewritten as

$$\tau\gamma > \beta_2 + \alpha = 1 - \alpha(1-\gamma), \tag{3}$$

and that $\tau > \alpha$. Let $N \in \mathbb{N}$ be sufficiently large so that $N^{\tau-\alpha} \geqslant 150$. For $n \in [N, 2N] \cap \mathbb{Z}$, put

$$\sigma_n = n^{-\tau}, \qquad \delta_n = n^{-\alpha}.$$

Write G_N for the set of integers $n \in [N, 2N]$ such that

$$\max\{|\hat{\mu}(t2^n)| : t \in \mathbb{Z},\ 1 \leqslant |t| \leqslant 2/\delta_{2N}\} \leqslant CN^{-\beta_1},$$

and let B_N be its complement in $[N, 2N] \cap \mathbb{Z}$. Applying the union bound, and then Lemma 4 with $2N+1$ in place of N, we have

$$\begin{aligned}\#B_N &\leqslant \sum_{1\leqslant|t|\leqslant 2/\delta_{2N}} \#\{n \in [N,2N]\cap\mathbb{Z} : |\hat{\mu}(t2^n)| > CN^{-\beta_1}\} \\ &\leqslant \sum_{1\leqslant|t|\leqslant 2/\delta_{2N}} \#\{n \in [0,2N+1)\cap\mathbb{Z} : |\hat{\mu}(t2^n)| > C(2N+1)^{-\beta_1}\} \\ &\ll \sum_{1\leqslant|t|\leqslant 2/\delta_{2N}} (2N+1)^{\beta_2} \\ &\ll N^{\beta_2+\alpha}.\end{aligned}$$

Observe that

$$W_2(\psi_\tau, y) = \limsup_{n\to\infty} A_n^y(\sigma_n).$$

By the first Borel–Cantelli lemma [8, Lemma 1.2], it suffices to prove that

$$\sum_{n=1}^{\infty} \mu(A_n^y(\sigma_n)) < \infty. \tag{4}$$

For $n \in B_N$, we use the following estimate, the proof of which follows straightforwardly from the argument in [1, §2.1].

Lemma 6. *Let $y \in \mathbb{R}$. Then*

$$\mu(A_n^y(\sigma_n)) \ll \sigma_n^\gamma \qquad (n \in \mathbb{N}).$$

In the case that $n \in G_N$, we are able to obtain a stronger estimate by transferring data from the coarse scale δ_n to the fine scale σ_n. By Lemma 5, we have

$$\mu(A_n^y(\delta_n)) \ll \delta_n \left(1 + \sum_{1\leqslant|t|\leqslant 2/\delta_n} |\hat{\mu}(t2^n)|\right) \qquad (n \in \mathbb{N}).$$

As $\alpha < \beta_1$, we find that if $n \in G_N$, then

$$\mu(A_n^y(\delta_n)) \ll \delta_n. \tag{5}$$

To pass between the two scales δ_n and σ_n, we require an inhomogeneous analogue of [1, Lemma 2.2]. Its statement and proof are based upon the iterative construction of K, which we now briefly recall, see [1, §2] for further details. For $\mathcal{N} \in \mathbb{N}$, recall that the $\mathcal{N}^{\text{th}}$ level in the construction of the Cantor set, which we denote by $K_\mathcal{N}$, comprises $2^\mathcal{N}$ intervals of length $3^{-\mathcal{N}}$. The left endpoints of these intervals form the set $L_\mathcal{N}$ of rationals $a/3^\mathcal{N}$ such that $a \in [0, 3^\mathcal{N}]$ is an integer whose ternary expansion contains only the digits 0 and 2, and the right endpoints form the set $R_\mathcal{N} = \{1 - x : x \in L_\mathcal{N}\}$. Note that $\mathcal{C}_\mathcal{N} = L_\mathcal{N} \cup R_\mathcal{N}$. The following is an inhomogeneous analogue of [1, Lemma 2.2].

Lemma 7. *Fix an absolute constant $c > 0$. Let $n, \mathcal{N}, \mathcal{M} \in \mathbb{N}$ and $\sigma, \delta \in \mathbb{R}$ be such that $\mathcal{N} \geqslant \mathcal{M}$ and*

$$0 < \sigma < \delta \leqslant 1, \qquad 3^{-\mathcal{N}} \geqslant \frac{c\sigma}{2^n}, \qquad \frac{\sigma}{2^n} \leqslant 3^{-\mathcal{M}} \leqslant \frac{\delta}{2^n},$$

and let $y \in \mathbb{R}$. Then

$$\#(\mathcal{C}_\mathcal{N} \cap A_n^y(\sigma)) \ll \#(\mathcal{C}_\mathcal{M} \cap A_n^y(2\delta)).$$

Proof. We imitate the proof of [1, Lemma 2.2]. By symmetry, it suffices to prove that

$$\#(L_\mathcal{N} \cap A_n^y(\sigma)) \ll \#(L_\mathcal{M} \cap A_n^y(2\delta)). \tag{6}$$

Suppose $x \in L_\mathcal{N} \cap A_n^y(\sigma)$. Then $x = a/3^\mathcal{N}$ for some integer $a \in [0, 3^\mathcal{N})$ whose ternary expansion contains only the digits 0 and 2. Further, there exists an integer $b \in [0, 2^n]$ such that

$$\left| x - \frac{b+y}{2^n} \right| < \frac{\sigma}{2^n}.$$

Therefore $\#(L_{\mathcal{N}} \cap A_n^y(\sigma))$ is bounded above by the number of integer solutions (a, b) to the inequality

$$\left| \frac{a}{3^{\mathcal{N}}} - \frac{b+y}{2^n} \right| < \frac{\sigma}{2^n}$$

such that $a \in [0, 3^{\mathcal{N}})$, $b \in [0, 2^n]$, and each ternary digit of a is 0 or 2.

We write

$$a = 3^{\mathcal{N}-\mathcal{M}} a_1 + a_2, \qquad a_1, a_2 \in \mathbb{Z}, \qquad 0 \leqslant a_1 < 3^{\mathcal{M}}, \qquad 0 \leqslant a_2 < 3^{\mathcal{N}-\mathcal{M}}.$$

This reveals that $\#(L_{\mathcal{N}} \cap A_n^y(\sigma))$ is bounded above by the number of integer solutions (a_1, a_2, b) to

$$\left| \frac{3^{\mathcal{N}-\mathcal{M}} a_1 + a_2}{3^{\mathcal{N}}} - \frac{b+y}{2^n} \right| < \frac{\sigma}{2^n} \tag{7}$$

such that

$$0 \leqslant a_1 < 3^{\mathcal{M}}, \qquad 0 \leqslant a_2 < 3^{\mathcal{N}-\mathcal{M}}, \qquad 0 \leqslant b \leqslant 2^n,$$

and the ternary digits of a_1, a_2 are all 0 or 2. As

$$\left| \frac{a_1}{3^{\mathcal{M}}} - \frac{b+y}{2^n} \right| \leqslant \left| \frac{a_1}{3^{\mathcal{M}}} + \frac{a_2}{3^{\mathcal{N}}} - \frac{b+y}{2^n} \right| + \frac{a_2}{3^{\mathcal{N}}} < \frac{\sigma}{2^n} + \frac{1}{3^{\mathcal{M}}} \leqslant \frac{2}{3^{\mathcal{M}}}, \tag{8}$$

we must have $a_1/3^{\mathcal{M}} \in A_n^y(2\delta)$ for any such solution.

Given a_1, the inequality (8) forces $b/2^n$ to lie in an interval of length $4/3^{\mathcal{M}}$, and so there are at most $O(1)$ possibilities for b. Next, suppose we are given a_1 and b. Then, by (7), the integer a_2 is forced to lie in the interval of length $3^{\mathcal{N}}\sigma 2^{1-n}$ centred at $3^{\mathcal{N}}(b+y)2^{-n} - 3^{\mathcal{N}-\mathcal{M}} a_1$. Consequently, as $3^{-\mathcal{N}} \geqslant c\sigma/2^n$, there are at most $O(1)$ solutions a_2 to (7). Finally, since $a_1/3^{\mathcal{M}} \in L_{\mathcal{M}} \cap A_n^y(2\delta)$, we conclude that there are $O(\#(L_{\mathcal{M}} \cap A_n^y(2\delta)))$ solutions in total. This confirms (6) and completes the proof of the lemma. □

Let $n \in [N, 2N] \cap \mathbb{Z}$. Let $\mathcal{N}, \mathcal{M}$ be positive integers such that

$$\frac{\sigma_n}{15 \cdot 2^n} < 3^{-\mathcal{N}} \leqslant \frac{\sigma_n}{5 \cdot 2^n} \quad \text{and} \quad \frac{\delta_n}{30 \cdot 2^n} < 3^{-\mathcal{M}} \leqslant \frac{\delta_n}{10 \cdot 2^n}.$$

We apply Lemma 3 with $\sigma = \sigma_n$ and $\mathcal{N}$ in place of N therein, giving

$$\mu(A_n^y(\sigma_n)) \ll 2^{-\mathcal{N}} \#(\mathcal{C}_{\mathcal{N}} \cap A_n^y(5\sigma_n)).$$

As $\delta_n/\sigma_n = n^{\tau-\alpha} \geqslant N^{\tau-\alpha} \geqslant 150$, we may apply Lemma 7 with $\sigma = 5\sigma_n$ and $\delta = \delta_n/10$, giving

$$\#(\mathcal{C}_{\mathcal{N}} \cap A_n^y(5\sigma_n)) \ll \#(\mathcal{C}_{\mathcal{M}} \cap A_n^y(\delta_n/5)).$$

Next we apply Lemma 3 again, now with $\sigma = \delta_n$ and $\mathcal{M}$ in place of N therein, giving

$$\#(\mathcal{C}_{\mathcal{M}} \cap A_n^y(\delta_n/5)) \ll 2^{\mathcal{M}} \mu(A_n^y(\delta_n)).$$

Note that we have

$$2^{-\mathcal{N}} \ll (\sigma_n/2^n)^\gamma \quad \text{and} \quad 2^{-\mathcal{M}} \gg (\delta_n/2^n)^\gamma,$$

and that, combined with the above, these inequalities furnish

$$\mu(A_n^y(\sigma_n)) \ll \frac{(\sigma_n/2^n)^\gamma}{(\delta_n/2^n)^\gamma} \mu(A_n^y(\delta_n)).$$

Thus, by (5), for $n \in G_N$ we have

$$\mu(A_n^y(\sigma_n)) \ll \delta_n^{1-\gamma} \sigma_n^{\gamma}.$$

Hence, by Lemma 6 and our earlier observation that $\#B_N \ll N^{\beta_2+\alpha}$, we have

$$\begin{aligned} \sum_{n=N}^{2N} \mu(A_n^y(\sigma_n)) &\ll \sum_{n=N}^{2N} \delta_n^{1-\gamma} \sigma_n^{\gamma} + \sum_{n \in B_N} \sigma_n^{\gamma} \\ &\ll \sum_{n=N}^{2N} \frac{1}{n^{\tau\gamma+\alpha(1-\gamma)}} + N^{\beta_2+\alpha-\tau\gamma}. \end{aligned}$$

In view of (3), and noting that we can write

$$\begin{aligned} \sum_{n=1}^{\infty} \mu(A_n^y(\sigma_n)) &\leqslant \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}} \mu(A_n^y(\sigma_n)) \\ &\ll \sum_{k=0}^{\infty} \left(\sum_{n=2^k}^{2^{k+1}} \frac{1}{n^{\tau\gamma+\alpha(1-\gamma)}} + 2^{k(\beta_2+\alpha-\tau\gamma)} \right), \end{aligned}$$

we finally have (4), which completes the proof of Theorem 2.

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