Selberg's Central Limit Theorem for families of L-functions

submitted by

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I certify that all material in this dissertation which is not my own work has been identified and that any material that has previously been submitted and approved for the award of a degree by this or any other University has been acknowledged.

Signature: . . . .

Abstract

In this thesis, we present a simple proof of Selberg's Central Limit Theorem for appropriate families of L-functions. As conjectured by Selberg, his central limit theorem can only be proven for the L-functions belonging to the *Selberg Class*.

First, we prove Selberg's central limit theorem for classical automorphic L-functions of degree 2 associated with holomorphic cusp forms. We prove this result in the t aspect.

In Chapter 4, we prove Selberg's central limit theorem for Dirichlet L-functions and quadratic Dirichlet L-functions associated with primitive Dirichlet characters and twisted Hecke-Maass cusp forms respectively. We prove these results in the q-aspect, i.e., instead of integrating we average over Dirichlet characters.

Finally, in Chapter 5, we prove that a sequence of degree 2 automorphic L-functions attached to a sequence of primitive holomorphic cusp forms form a Gaussian process. Also, any two elements from this sequence of L-functions are pair-wise independent. Additionally, we construct a random matrix that generalizes the notion of independence of the families of automorphic L-functions.

Dedication

to

my mentor **Dr Kyle Pratt**

without him, this dissertation would not have been possible.

Thank you for all your guidance and support.

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1. Introduction

The Riemann zeta function (denoted by $\zeta(s)$) encodes information about prime numbers. The study of the behaviour of the Riemann zeta function is an interesting topic in analytic number theory. In 1859, Riemann proposed that every non-trivial zero of $\zeta(s)$ is located at the critical line with $\Re(s) = \frac{1}{2}$. This problem is known as the **Riemann Hypothesis**. It has been open for more than a hundred and fifty years and is one of the most difficult problems in mathematics. There are many applications of the Riemann Hypothesis. In particular, the error term in the prime number theorem is closely related to the position of the zeros. H. V. Koch [Koc01] proved that the Riemann hypothesis implies the "best possible" bound for the error of the prime number theorem. Also, the Riemann hypothesis implies strong bounds on the growth of many other arithmetic functions. For example, the Möbius function μ has Dirichlet series $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$. This relation is valid for every s with real part greater than $\frac{1}{2}$. It is known (see Theorem 14.25 of [Tit86]) that a necessary and sufficient condition for the truth of the Riemann hypothesis is that $M(x) = O(x^{\frac{1}{2}+\epsilon})$, for all $\epsilon > 0$, where $M(x) = \sum_{n \le x} \mu(x)$. The condition would be true if the Möbius sequence $\{\mu(n)\}$ were a random sequence, taking the values $-1, 0,$ and 1, with specified probabilities, those of -1 and 1 being equal.

Hardy and Littlewood [HL16] studied the moments of the Riemann zeta function

$$M_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} dt$$

for $k = 1$. The Lindelöf Hypothesis[1] tells us about the growth rate of the Riemann zeta function, also, it implies that $M_k(T) = O(T^{1+\epsilon})$ for all natural number k. Even though the last decades have seen tremendous progress on this topic, the asymptotic behaviour of $M_k(T)$ still remains unknown. The moment conjecture of the Riemann zeta function states that $M_k(T) \sim C_k T (\log T)^{k^2}$, for all positive real numbers k. Keating and Snaith [KS00a] conjectured a formula for C_k using random matrix theory.

Suppose $X = X(T) \le \log T$ is a parameter tending slowly to infinity with T (to fix ideas one can think of $X(T) = \sqrt{(\log T)}$). Then for typical $t \in [T, 2T]$ (by which

[1] Lindelöf hypothesis, which is implied by the Riemann hypothesis, states that for any $\epsilon > 0$, $\zeta(\frac{1}{2} + it) = O(t^{\epsilon})$ as t tends to infinity.

we mean t lying outside a set of measure $o(T)$) one has

$$\zeta(\sigma+it) \sim \prod_{p\leq X}\left(1-\frac{1}{p^{\sigma+it}}\right)^{-1} \text{ (for } \sigma>1\text{)},$$

In other words, $\zeta(\sigma+it)$ has an almost periodic structure and its value can be usually extracted from knowledge of p^{it} for small primes p.

The value distribution of $\zeta(s)$ on the critical line (for $\Re(s)=\frac{1}{2}$) is different because $\zeta(s)$ does not behave like an almost periodic function and it is hard to determine these values only with the knowledge of p^{it} at small primes. In 1946, Selberg studied the statistical behaviour of the Riemann zeta function. He managed to estimate some integrals involving $\log\zeta(s)$. In the 1940s Selberg made a major contribution on estimating the argument of the Riemann zeta function on the critical line [Sel46b; Sel44]. Selberg [Sel92; Sel46b] established a fundamental theorem that as $t\in[T,2T]$, the quantity $\log\zeta(\frac{1}{2}+it)/\sqrt{\frac{1}{2}\log\log t}$ behaves like a standard complex random variable which means its real and imaginary parts are distributed like independent normal random variables with mean 0 and variance 1. This result is known as Selberg's central limit theorem (SCLT).

For any $\sigma>1$ and $t\in\mathbb{R}$, we have

$$\log|\zeta(\sigma+it)| = \sum_p\sum_{k=1}^{\infty}\frac{1}{k}\frac{\cos(kt\log p)}{p^{k\sigma}}. \tag{1.1}$$

For the proof of (1.1) see Lemma 13.1 of [Har15a]. Considering the Euler product expression for the Riemann zeta function, we have

$$\log\zeta(\sigma+it) = -\sum_p\log\left(1-\frac{1}{p^{\sigma+it}}\right),\ \sigma>1,\, t\in\mathbb{R}.$$

Inserting the Taylor series expression of $\log\left(1-\frac{1}{p^{\sigma+it}}\right)$ and taking the real parts of $\Re(p^{-ikt})=\cos(-kt\log p)=\cos(kt\log p)$, we conclude (1.1). Consider the imaginary part $t\in[T,2T]$, the almost independence arises because of the values $\log p$ are linearly independent over $\mathbb{Q}$ (which is basically a restatement of the uniqueness of prime factorization). So, the terms $\cos(kt\log p)$ vary "almost independently" for distinct primes p. We can consider $\left\{\sum_p\sum_{k=1}^{\infty}\frac{1}{k}\frac{\cos(kt\log p)}{p^{k\sigma}}\right\}_{p\in\mathbb{P}}$ as the sequence of "almost independent" random variable with suitable mean and variance. Then as $T\to\infty$ for $t\in[T,2T]$ the almost random variables-$\left\{\sum_p\sum_{k=1}^{\infty}\frac{1}{k}\frac{\cos(kt\log p)}{p^{k\sigma}}\right\}_{p\in\mathbb{P}}$ converge in distribution to normal $\mathcal{N}(0,\frac{1}{2}\log\log T)$. Hence, we call $\log|\zeta(\sigma+it)|$ has "approximate normal" distribution with mean 0 and variance $\frac{1}{2}\log\log T$.[2]. Now we write the precise definition.

Definition. *If $X(t)$ is approximately normally distributed with mean m and variance ν^2, if, for any fixed positive real number V, as $T\to\infty$, we have*

$$\frac{1}{T}meas\left\{t\in[T,2T]:\frac{X(t)-m}{\nu}\geq v\right\}\sim\frac{1}{\sqrt{2\pi}}\int_v^{\infty}e^{-\frac{x^2}{2}}dx \tag{1.2}$$

[2]Similar argument works for classical automorphic L-functions. In this case we have $L(f,\sigma+it)=\sum_p\sum_{k=1}^{\infty}\frac{1}{k}\frac{\Re(\lambda_f(p))\cos(kt\log p)}{p^{k\sigma}}$.

uniformly for $v \in [-V, V]$.

Now we write the precise statement of Selberg's central limit theorem for the real part of the logarithm of the Riemann zeta function.

Theorem (Selberg)**.** *Let* V *be a fixed positive real number. Then as* $T \to \infty$*, uniformly for all* $v \in [-V, V]$*,*

$$\frac{1}{T} meas\left\{T \leq t \leq 2T : \log|\zeta(\frac{1}{2}+it)| \geq v\sqrt{\frac{1}{2}\log\log T}\right\} \sim \frac{1}{\sqrt{2\pi}}\int_v^\infty e^{-u^2/2}du.$$

Let $N(T)$ denote the number of non-trivial zeros ρ of $\zeta(s)$ inside the critical strip up to height T. Then one can write

$$\begin{aligned} N(T) =& \#\{\rho = \beta + i\gamma : 0 < \gamma \leq T\} \\ =& \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(T^{-1}), \end{aligned}$$

where $S(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+it)$.

To study the statistical behaviour of $\log|\zeta(\frac{1}{2}+it)|$ Selberg observed the behaviour of $S_1(T)$ on the critical line, where $S_1(T)$ is the integrated function given by

$$S_1(T) = \int_0^T S(t)dt.$$

It is known that

$$S_1(T) = \frac{1}{\pi}\int_{\frac{1}{2}}^\infty \log|\zeta(\sigma+it)|d\sigma - \frac{1}{\pi}\int_{\frac{1}{2}}^\infty \log|\zeta(\sigma)|d\sigma.$$

To study the behaviour of $S_1(T)$, Selberg needed a delicate estimate for the non-trivial zeros of the Riemann zeta function. We give an informal overview of Selberg's argument by using T. Tao's [Tao09] survey on Selberg's proof. Selberg's proof has three main steps. In the first step, he established a formula for $\log|\zeta(s)|$. From the Euler product formula and the zeros of the Riemann zeta function, one can write

$$\log|\zeta(s)| = \sum_{p\leq T^\epsilon}\Re(1/p^s) + O\left(\sum_{\rho = s+O(1/\log T)} 1 + \left|\log\frac{|s-\rho|}{1/\log T}\right|\right) + \cdots \qquad (1.3)$$

where ρ denotes the zeros of the Riemann zeta function for $s = \frac{1}{2}+it$ and $t = O(T)$, for $\epsilon > 0$. Notice that (1.3) is a localization of the prime sum to the primes p of size $O(T^{o(1)})$, and the sum of zeros at a distance of $O(1/\log T)$ from s. Note that all the expression in (1.3) can be controlled. The error term (denoted by $\cdots$) has the size $O(1)$ for most values of t, so are a lower order term. The main term contribution is coming from the primes. Let $X_p := \Re(1/p^s) = 1/\sqrt{p}\cos(t\log p)$ for $t \in [T, 2T]$, is a random variable with mean 0 and variance approximately $1/2p$ (if $p \leq T^\epsilon$ for $\epsilon > 0$).

Let us assume that X_p behaves as if they were independent then by the central limit theorem, the sum $\sum_{p\le T^\epsilon} X_p$ behaves like a normal distribution with mean 0 and variance $\sum_{p\le T^\epsilon} 1/2p$. Then by Merten's theorem, one has

$$\sum_{p\le X}\frac{1}{p}=\log\log X+O(1).$$

To establish a formula for (1.3), Selberg started with

$$\int_{\mathbb{R}}\log\left|\zeta\left(s+\frac{iy}{\log T^\epsilon}\right)\right|\psi(y)dy \tag{1.4}$$

where ψ is a bump function with total mass 1. The formula is averaged out for $\log|\zeta(s)|$ in the vertical direction at scale $O(1/\log T^\epsilon)=O(1/\log T)$, where the implied constant depends on ϵ.

One can express (1.4) in two different ways. For the first case consider

$$-\log|\zeta(s)|=\log|s-1|-\sum_\rho \log|s-\rho|+\cdots . \tag{1.5}$$

If one modifies s by $O(1/\log T)$ then the fluctuation of the quantity $\log|s-\rho|$ is not much unless ρ is within $O(1/\log T)$ of s. In that case it will move by about $O\left(1+\log\frac{|s-\rho|}{1/\log T}\right)$. As a result, one has

$$\int_{\mathbb{R}}\log\left|s+\frac{iy}{\log T^\epsilon}-\rho\right|\psi(y)dy\approx\log|s-\rho|$$

where $|\rho-s|\gg 1/\log T$, and

$$\int_{\mathbb{R}}\log\left|s+\frac{iy}{\log T^\epsilon}-\rho\right|\psi(y)dy=\log|s-\rho|+O\left(1+\log\frac{|s-\rho|}{1/\log T}\right).$$

Assuming the imaginary part of s to be large, the quantity $\log|s-1|$ does not move very much by this shift. Inserting these facts to (1.5), once sees that (1.4) (heuristically) equals to

$$\log|\zeta(s)|+\sum_{\rho=s+O(1/\log T)} O\left(1+\log\frac{|s-\rho|}{1/\log T}\right)+\cdots . \tag{1.6}$$

Now we compute (1.4), using

$$\log|\zeta(s)|=\sum_p \Re\frac{1}{p^s}+\cdots . \tag{1.7}$$

Write $s=1/2+it$, one can express (1.4) as

$$\sum_p \Re(1/p^s)\int_{\mathbb{R}} e^{-iy\log p/\log T^\epsilon}\psi(y)dy+\cdots .$$

Let $\hat{\psi}(\xi) := \int_{\mathbb{R}} e^{-iy\xi}\psi(y)dy$ be the Fourier transformation of ψ, then one can write the above equation as $\sum_p \Re(1/p^s) \int_{\mathbb{R}} \hat{\psi}(\log p/\log T^\epsilon) + \cdots$.

Since ψ is a bump function, its Fourier transformation is also a bump function (or a Schwartz function). As a first approximation one can think $\hat{\psi}$ as a smoothed truncation to the region $\{\xi : \xi = O(1)\}$, thus the $\hat{\psi}(\log p/\log T^\epsilon)$ weight is restricting primes to the region $p \leq T^\epsilon$. Thus one can express (1.4) as

$$\sum_{p \leq T^\epsilon} \Re(1/p^s) + \cdots .$$

Comparing this with (1.6), that we have for (1.4), one can obtain (1.3) (formally).

The next step is about the controlling of zeros to take care of the error term of (1.3). Precisely, in this step Selberg showed that $\sum_{\rho = s + O(1/\log T)} 1 + \left|\log \frac{|s-\rho|}{1/\log T}\right|$ is $O(1)$ on the average, for $s = 1/2 + it$ and $t \in [T, 2T]$. In this step, he has used the first moment method. Let $I_\rho = 1 + \left|\log \frac{|s-\rho|}{1/\log T}\right|$ be the random variable for each zero $\rho = s + O(1/\log T)$ and zero otherwise. Thus the target here is to control the expectation of $\sum_\rho I_\rho$. The only relevant zeros are those which have the size of $O(T)$ and we know that there are $O(T \log T)$ zeros of this kind. On the other hand, if one chooses s randomly then, it has a probability $O(1/T \log T)$ of falling within $O(1/\log T)$ of ρ. So, one can expect that each I_ρ have an expected value of $O(1/T \log T)$. By the linearity of expectation one can conclude that $\sum_\rho I_\rho$ has expectation $O(T \log T) \times O(1/T \log T) = O(1)$, and the claim follows.

In the last step, Selberg showed that $\sum_{p \leq T^\epsilon} X_p$ has normal distribution by showing that X_p behaves as if they are jointly independent. To show that X_p has mean 0, one can show that the product $X_{p_1} \cdots X_{p_k}$ have a negligible expectation as long as at least one of the primes occurs at most once. After having this (a similar formula can be computed for the case when all primes occurred twice) one can compute the k-th moment of $(\sum_{p \leq T^\epsilon} X_p)^k$ and can verify that it matches with the answer as predicted in the central limit theorem, which by standard arguments is enough to establish the distribution law [3].

By expanding the product, one can get

$$X_{P_1} \cdots X_{P_k} = \frac{1}{\sqrt{p_1} \cdots \sqrt{p_k}} \cos(t \log p_1) \cdots \cos(t \log p_k).$$

Using the product formula for cosines, the product here can be expressed as a linear combination of cosines $\cos(t\xi)$ where the frequency ξ takes from

$$\xi = \pm \log p_1 \pm \log p_2 \cdots \pm \log p_k.$$

Observe that if each of the p_j's is at most T^ϵ then the numerator and the denominator are at most $T^{k\epsilon}$. Also, ξ is the logarithm of a rational number. By the Fundamental theorem of Arithmetic if one of the primes $p_1, \ldots, p_k$ appears only once then the numerator and the denominator do not cancel. Then ξ can not be

[3] Note that to get close to the normal distribution by a fixed amount of accuracy, it suffices to control a bounded number of moments, which ultimately means that one can treat k as being bounded, $k = O(1)$.

0. Since we know that the denominator is at most $1/T^{k\epsilon}$, the ξ must stay away from 0 at a distant of $1/T^{k\epsilon}$ or more. So, $\cos(t\xi)$ have a wavelength of at most $O(T^{k\epsilon})$, for $t \in [T, 2T]$. If k is fixed and ϵ is smaller than $1/k$ then one can see that the average values of $\cos(t\xi)$ is close to 0 for $t \in [T, 2T]$. Then the product $X_{p_1} \cdots X_{p_k}$ have negligible expectation. Similarly, Selberg computed the expectation of $X_{p_1} \cdots X_{p_k}$ for all primes appear twice at least.

In 2017, Radziwiłł and Soundararajan [RS17] gave a new proof of Selberg's central limit theorem. As the first step of the proof, they have taken away the problem from the critical line[4]. Selberg's original proof is more complicated compared to Radziwiłł and Soundararajan's method because of the interference of the non-trivial zeros of the Riemann zeta function. We have given a simplified version of $\log|\zeta(s)|$. There is a "tail" in (1.3) coming from the zeros of the Riemann zeta function that are more distant from s than $O(1/\log T)$. Also, one has to smooth out the sum in primes p a little bit and allow the implied constant depends on ϵ inside the big-O term. Controlling these zeros (coming from the tail) is making Selberg's argument more complicated. While Radziwiłł and Soundararajan's method is not studying the actual problem on the critical line, just to take away the problem, only a standard zero estimate is needed, not a crucial one like Selberg needed to study the problem on the critical line. This reason makes their proof easier and more elegant.

The rest of the method developed by Radziwiłł and Soundararajan only needs very basic facts about the Riemann zeta function. In the second step, they have introduced an auxiliary series involving the logarithmic derivative of the Riemann zeta function. By restricting the auxiliary series to primes, they have proved that it has approximate Gaussian distribution with mean 0 and variance $\frac{1}{2}\log\log|t|$. Finally, they have taken the help of mollifiers to connect the zeta function with the auxiliary series.

Although Selberg's method is more complicated compared to Radziwiłł and Soundararajan's method to prove SCLT for $\log|\zeta(\frac{1}{2}+it)|$, but for $\arg\zeta(\frac{1}{2}+it)$ Selberg's technique is better. In fact, Radziwiłł and Soundararajan's method can only be used to prove the result for the real part of the logarithm of the Riemann zeta function because the mollification technique can not be applied for $\arg\zeta(\frac{1}{2}+it)$.

Selberg's work provides a good understanding of zeta and L-functions on the critical line. It shows that typical values of $|\zeta(\frac{1}{2}+it)|$ are either very small (say $\frac{1}{A}$ for any A with $\log A = o(\sqrt{\log\log T})$) or large ($> A$ with A as before), and that intermediate values appear only on a set of measure $o(T)$. It has contrast with the fact that $|\zeta(\sigma+it)|$ (with $\sigma > \frac{1}{2}$) is typically of constant size. Some similar results are known for $\zeta(\sigma+it)$ with $\frac{1}{2} < \sigma \leq 1$ but Selberg's result indicates why the problem for $\sigma = \frac{1}{2}$ has an entirely different flavour.

[4] The tail of the Riemann zeta function is defined as $\zeta_n(s) = \sum_{k=n}^{\infty} \frac{1}{k^s}$, for $\Re(s) > 1$. Since Radziwiłł and Soundararajan have proved the result for the real part of $\log|\zeta(s)|$, during the calculations, they have obtained the sums involving $\frac{1}{n^{2\Re(s)}}$. If $\Re(s) = \frac{1}{2}$, because of the tail distribution of the Riemann zeta function, the sums involving these terms will not be controllable. So, for the sake of the calculations, it is necessary to take away the problem from the critical line.

We know that the ζ-function can be considered as an archetype of all L-functions. It is believed that for each L-function there exists a Riemann Hypothesis. So, the behaviour of L-functions in the critical strip encodes valuable information like $\zeta(s)$. The Grand Riemann Hypothesis is a much stronger version of the Riemann Hypothesis. From these facts, we can say that $\log L(s)$ has its own arithmetical significance. Further results were obtained by Selberg on this subject [Sel92; Sel46a]. In these papers, Selberg introduced the properties of a general class of the Dirichlet series, now referred to as the *Selberg class*. Selberg showed that his theory, originally devised for the Riemann zeta-function, carries over to the Selberg class with remarkably few changes.

Keating and Snaith [KS00b] conjectured that the logarithm of the central values of L-functions in families have a normal distribution with suitable mean and variance. In this thesis we prove Selberg's central limit theorem for classical automorphic L-functions and Dirichlet L-functions in t-aspect and q-aspect respectively. In the third chapter, we prove Selberg's theorem for the classical automorphic L-functions (attached with primitive holomorphic cusp form f) given by the Dirichlet series

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}.$$

We prove this result in t-aspect. In the fourth chapter, we prove Selberg's theorem for $GL(3)$ Dirichlet L-functions attached to the Hecke-Maass cusp form f and twisted by the primitive Dirichlet character χ, which is given by the Dirichlet series

$$L(f \otimes \chi, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(1, n)\chi(s)}{n^s}.$$

We give a conjectural proof of this result in the q-aspect using the asymptotic Large Sieve. Further, using a similar method we prove Selberg's theorem for $GL(1)$ Dirichlet L-function attached to primitive Dirichlet character χ[5].

We use the proof technique established by Radziwiłł and Soundararajan [RS17]. But we need a few modifications in their method since we will be working with the L-functions. As we have discussed earlier in the proof given in [RS17], the authors have used the mollification technique to connect the zeta function with the auxiliary series. To establish that connection they evaluated the mean square estimate of the zeta function. We also need such an estimate for our proof but we do not use their method. Since we have the Hecke-eigenvalues of the cusp form (or Maass form) attached to the L-series we require the shifted convolution sum while we are evaluating the integral mean value estimate of the L-function. While working with the q-aspect we need to consider the shifted convolution problem in order to treat the approximate functional equation of the L-function, otherwise, for the mean square estimate, we use a different technique.

Our work is limited to $GL(2)$ and $GL(3)$ L-functions because we don't have much information on the shifted convolution problem for higher degree L-functions.

[5]A detailed information on these families of L-functions has given in the next chapter.

In fact, for $GL(3)$ L-functions, only some hybrid results are possible.

Selberg also introduced the concept of independence of automorphic L-functions attached to distinct cusp forms or Maass forms (for Dirichlet L-function it is associated with distinct Dirichlet characters). Although it was not well defined in Selberg's paper, we get a generalized idea from it. In 2019, Hsu and Wong [HW20] proved this result for families of Dirichlet L-functions associated with the set of distinct primitive Dirichlet characters. We prove this result for families of automorphic L-functions of degree 2 associated with the set of distinct primitive cusp forms. Further, we consider a generalized notion on the independence of the automorphic L-function with the help of the work of Hughes et al [HNY07].

2. Notation and Preliminaries

2.1 Notation

Most of the following notation is standard in Analytic Number Theory.

- For the given functions $f(x)$ and $g(x)$ we write $f(x) = O(g(x))$ or, $f(x) \ll g(x)$ if there exists a constant $c > 0$ such that
$$|f(x)| \leq cg(x)$$
for all x, where c is the implied constant. The notation $f(x) \gg g(x)$ means $g(x) \ll f(x)$.

- For the given functions $f(x)$ and $g(x)$, $f(x) = o(g(x))$ as $x \to x_0$ if
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

- For the given functions $f(x)$ and $g(x)$ we denote $f(x) \sim g(x)$ which means
$$\frac{f(x)}{g(x)} \to 1 \text{ as } x \to \infty.$$

- We write $f(x) \asymp g(x)$ to mean that there exist constants $0 < c < C$ such that
$$cg(x) \leq f(x) \leq Cg(x)$$
for all x of interest. This notation roughly means that the functions $f(x)$ and $g(x)$ have the same size.

- The Von Mangoldt function is the function defined by
$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for prime } p \text{ and integer } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- The Möbius function is denoted as μ given by
$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

- Euler phi function denoted as $\phi(n)$ is the number of non-negative integers less than n that are relatively prime to n. The Euler product formula for the $\phi(n)$ is given by

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

where the product is over the distinct prime numbers dividing n. Note that $\phi^*(q)$ denotes the number of primitive Dirichlet characters modulo q and $\phi^*(q) = 0$ if $2||q$.

- The prime-omega function is denoted as $\Omega(n)$ which denote the number of prime divisors of n counted with multiplicity.

- For $s = \sigma + it$ with $t \neq 0$ we have the Stirling estimate (see 5.113 of [IK04]) asserting that for any fixed σ,

$$\Gamma(\sigma + it) = \sqrt{2\pi}(it)^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}\pi|t|} \left(\frac{|t|}{e}\right)^{it} \left(1 + O\left(\frac{1}{|t|}\right)\right).$$

- As an application of the Stirling's approximation [ET51] it can be shown that for fixed $\delta > 0$ (which is sufficiently small)

$$\frac{\Gamma(\mathfrak{z} + \alpha)}{\Gamma(\mathfrak{z} + \beta)} = \mathfrak{z}^{\alpha - \beta} \left(1 + O\left(\frac{|(\alpha - \beta)(\alpha + \beta - 1)|}{|\mathfrak{z}|}\right)\right),$$

where α and β are arbitrary constants and $|\arg(\mathfrak{z})| \leq \pi - \delta$.

- The exponential function is a mathematical function denoted by $f(x) = \exp(x) = e^x$, with $\exp(0) = 1$.It has the Taylor expansion,

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

It satisfies the following identity

$$\exp(x + y) = \exp(x) \cdot \exp(y) \text{ for } x, y \in \mathbb{C}.$$

Moreover, this function is equal to its own derivative. Note that $e(x) = e^{2\pi i x}$ for all x.

2.2 Basic notion of the L-functions

The main topic of this dissertation is to prove Selberg's central limit theorem for some particular families of L-functions. First, we give a brief overview about the *Selberg Class* then we recall some basic properties of the families of L-functions we will be working on. All the families of L-functions we have considered in the later chapters of this thesis belong to the Selberg Class. Note that for a primitive L-function in the Selberg class (or for a cuspidal automorphic L-function for $GL_n(\mathbb{Q})$),one expects that $\log L(f, \frac{1}{2} + it)$ with $T \leq t \leq 2T$ is distributed like a complex Gaussian with mean 0 and variance $\frac{1}{2} \log\log T$.

2.2.1 Selberg Class

In this section, we define the Selberg Class and discuss some of its properties.

Definition 2.2.1 (Selberg Class)**.** *The Selberg class $\mathcal{S}$ consists of meromorphic functions $L(s)$ satisfying the following properties.*

- **Dirichlet Series:** *It can be expressed as a Dirichlet series*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which is absolutely convergent in the region $\Re(s) > 1$. One can normalize the leading coefficient as $a_n(1) = 1$.

- **Analytic continuation:** *There exists a non-negative integer k, such that $(s-1)^k L(s)$ is an entire function of finite order.*

- **Functional equation:** *There is a number ε with $|\varepsilon| = 1$, and a function $\gamma_L(s)$ of the form*

$$\gamma_L(s) = P(s)Q^s \prod_{j=1}^{w} \Gamma(w_j s + \mu_j)$$

where $Q > 0$, $w_j > 0$, $\Re(\mu_j) \geq 0$ and P is a polynomial whose only zeros in $\sigma > 0$ are at the poles of $L(s)$, such that

$$\xi_L(s) := \gamma_L(s)L(s)$$

is entire, and

$$\xi_L(s) = \varepsilon\bar{\xi}_L(1-s)$$

where $\bar{\xi}_L(s) = \overline{\xi_L(\bar{s})}$ and $\bar{s}$ denotes the complex conjugate of s. The number $2\sum_{j=1}^{w} w_j$ is called the degree of the L-function, and this is conjectured to be an integer. The asymmetric form of the functional equation can be written as

$$L(s) = \varepsilon X_L(s)\bar{L}(1-s)$$

where $X_L(s) = \frac{\bar{\gamma}_L(1-s)}{\gamma_L(s)}$.

- **Euler product:** *There is an Euler product of the form*

$$L(s) = \prod_p L_p(s)$$

where the product is over primes and

$$L_p(s) = \sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}} = \exp\left(\sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}\right)$$

where $b_n \ll n^{\theta}$ with $\theta < \frac{1}{2}$.

- **Ramanujan-Petersson Conjecture:** *For any* $\epsilon > 0$

$$|a_n| = O(n^{\epsilon}). \tag{2.1}$$

The Euler product implies that the coefficients a_n are multiplicative, i.e., $a_{mn} = a_m a_n$ when $(m,n) = 1$.

Generalized Riemann hypothesis (GRH): Let $L(s) \in \mathcal{S}$. If $L(s) = 0$ for $0 < \Re(s) < 1$, then $\Re(s) = \frac{1}{2}$.

The families of L-functions we have considered in the thesis belong to the Selberg class. So, we study some standard results on it.

Selberg's Conjectures: In [Sel92], Selberg made the following conjectures.
Conjecture A (Regularity of distribution). Let $L(s) \in \mathcal{S}$. There exists a constant $c_L \geq 0$, such that as $X \to \infty$,

$$\sum_{p \leq X} \frac{|a_p|^2}{p} = c_L \log\log X + O(1).$$

Conjecture B (Orthonormality). Let $L_1(s), L_2(s) \in \mathcal{S}$ be primitive elements (it means $L_i(s)$ cannot be factored into the product of two or more non-trivial members of $\mathcal{S}$). Then

$$\sum_{p \leq X} \frac{a_{1,p}\bar{a}_{2,p}}{p} = \begin{cases} \log\log X + O(1) & \text{if } L_1(s) = L_2(s) \\ O(1) & \text{otherwise} \end{cases}$$

Hypothesis H (Rudnick-Sarnak). For any fixed $k \geq 2$

$$\sum_{p} \frac{|(\log p) a_{p^k}|^2}{p^2} < \infty.$$

From (2.24) of [RS96] we know that

$$\sum_{n \leq X} \frac{(\log n)\Lambda(n)|a_n|^2}{n} \sim \frac{\log^2 X}{2}.$$

By partial summation, we get

$$\sum_{n \leq X} \frac{\Lambda(n)|a_n|^2}{n \log n} \sim \log\log X. \tag{2.2}$$

If Hypothesis H holds, the contribution on powers of primes can be neglected. This further implies Conjecture A, for any fixed $GL(n)$ L-function.

As we have discussed in the introduction, we will be using very basic results of L-functions to prove our main theorems. We work with Dirichlet L-functions and automorphic L-functions of degrees 2 and 3.

2.2.2 Dirichlet L-functions

In this section we study about the Dirichlet L-functions attached to primitive Dirichlet characters χ.

We start with the definition of Dirichlet characters.

Definition 2.2.2 (Dirichlet Characters)**.** *Let q be a positive integer. A Dirichlet character modulo q is an arithmetic function χ with the following properties:*

- *χ is periodic modulo q i.e., $\chi(n+q) = \chi(n)$ for all $n \in \mathbb{N}$.*
- *χ is completely multiplicative, i.e., $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{N}$ and $\chi(1) = 1$.*
- *$\chi(n) \neq 0$ if and only if $(n, q) = 1$.*

Associated with each character χ, in addition to its modulus q, is a natural number $q^\star$, its conductor. The conductor is the smallest divisor of q such that χ may be written as $\chi = \chi_0 \chi^\star$ where χ_0 is the principal character[1] modulo q and $\chi^\star$ is a character modulo $q^\star$. For some characters the conductor is equal to the modulus. Such characters are called primitive.

Definition 2.2.3 (Dirichlet L-function)**.** *The Dirichlet L-function $L(s, \chi)$ is an L-function of degree 1 given by the Dirichlet series*

$$L(s, \chi) = \sum_n \frac{\chi(n)}{n^s}$$

for $\Re(s) > 1$ with conductor q, gamma factor

$$\gamma(s) = \pi^{-s/2} \Gamma\left(\frac{s+a}{2}\right) \tag{2.3}$$

where $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$. They are called even characters and odd characters respectively.

Then $\xi(f, s)$ has analytic continuation to the entire complex plane and satisfies the functional equation

$$\xi(1-s, \bar{\chi}) = \bar{\varepsilon}(\chi)\xi(s, \chi) \tag{2.4}$$

where $\xi(s, \chi)$ denotes the complete L-function followed by the formula given above, and $\varepsilon(f)$ is the complex number of absolute value 1 called the root number of $L(s, \chi)$.

Since we need to deal with the Dirichlet L-functions it is good to recall a few properties of the Dirichlet characters.

We begin with the orthogonality relations of the Dirichlet characters which say that

[1]The arithmetic function $\chi_0 = \chi_{0,q}$ defined by $\chi_0(n) = 1$ if $(n, q) = 1$ and $\chi_0(n) = 0$ otherwise (i.e., the characteristic function of the integers co-prime with q) is called the principal character modulo q.

$$\sum_{\chi(\bmod q)} \chi(a) = \begin{cases} \phi(q) & \text{if } a \equiv 1 (\mathrm{mod} q) \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

We apply this result in later chapters to average some particular sums over the Dirichlet characters.

The root number of $L(s,\chi)$ is given by $\varepsilon(\chi) = \tau(\chi)/\sqrt{q}$, where

$$\tau(\chi) = \sum_{b(\bmod q)} \chi(b) e\left(\frac{b}{q}\right)$$

is the Gauss sum associated with characters on residue classes modulo q. If χ is non-trivial, then $L(s,\chi)$ is entire, otherwise it has a simple pole at $s = 1$ with residue 1. The approximation of the Gauss sum is given by the next Lemma.

Lemma 2.2.4. *If $\chi(\mathrm{mod} q)$ is primitive, then*

$$|\tau(\chi)| = \sqrt{q}.$$

For the proof of this lemma see Lemma 3.1 of [IK04].

We prove next following lemmas, which we will use in the later chapter.

Lemma 2.2.5 (Euler product expression for $L(s,\chi)$)**.** *If $\Re(s) > 1$, then*

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where (for the primitive Dirichlet characters χ) the finite product is defined to be

$$\lim_{P\to\infty} \prod_{p\le P} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Proof. For any prime p,

$$\left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{k=0}^{\infty} \frac{\chi(p^k)}{p^{ks}}.$$

The above series is absolutely convergent for $\Re(s) > 0$. If we restrict the Euler product of $L(s,\chi)$ upto P and rearrange the term, then we have

$$\prod_{p\le P} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \prod_{p\le P} \sum_{k=0}^{\infty} \frac{\chi(p^k)}{p^{ks}} = \sum_{n=1}^{\infty} \frac{c_P(n)}{n^s},$$

where

$$c_P(n) = \begin{cases} \chi(n) & \text{if all prime factors of } n \text{ are} \le P, \\ 0 & \text{otherwise.} \end{cases}$$

Note that each Dirichlet character are multiplicative and for $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, each pair $(p_i, p_j) = 1$ for $i \neq j$. Then we write

$$\left| L(s,\chi) - \prod_{p \le P} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \right| = \left| \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \right| \le \sum_{\substack{n=1 \\ c_P(n)=0}}^{\infty} \frac{|\chi(n)|}{n^{\Re(s)}}.$$

Since by definition $c_P(n) = \chi(n)$ for $n \le P$, the right hand side of the above equation is

$$\le \sum_{n=P+1}^{\infty} \frac{|\chi(n)|}{n^{\Re(s)}}.$$

If $\Re(s) > 1$, the above expression tends to 0 as $P \to \infty$, concluding the lemma. □

Lemma 2.2.6. *For any $\sigma > 1$ and any $t \in \mathbb{R}$ and primitive Dirichlet characters χ, we have*

$$\log L(\sigma + it, \chi) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k} \frac{\chi(p^k)}{p^{k(\sigma+it)}}$$

and

$$\log |L(\sigma + it, \chi)| = \sum_p \sum_{k=1}^{\infty} \frac{1}{k} \frac{|\chi(p^k)| \cos(kt \log p)}{p^{k\sigma}}.$$

The double series on the right are absolutely convergent.

Proof. From Lemma 2.2.5, we have

$$\log L(\sigma + it, \chi) = -\sum_p \log\left(1 - \frac{\chi(p)}{p^{\sigma+it}}\right), \quad \sigma > 1,\ t \in \mathbb{R}.$$

Inserting the Taylor series expansion of $\log\left(1 - \frac{\chi(p)}{p^{\sigma+it}}\right)$ and taking the real parts and noting that $\Re(p^{-ikt}) = \cos(kt \log p)$, we conclude the lemma. □

Remark 2.2.7. *Note that the proof of Lemma 2.2.5 and Lemma 2.2.6 follows the proof of Lemma 10.2 of [Har15b] and Lemma 13.1 of [Har15a] respectively.*

2.2.3 Classical automorphic L-functions

In a few books classical automorphic L-functions usually refers to the $GL(2)$ L-functions associated with holomorphic or eigenfunctions of the Laplace operator. In this thesis, we deal with the L-functions attached to primitive holomorphic cusp forms and Hecke-Maass cusp forms.

Let $\mathbb{H} = \{x + iy : x \in \mathbb{R}, y > 0\}$. A modular form of weight k and level q for the congruence subgroup

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) | c \equiv 0 \bmod q \right\}$$

is a complex valued function $f : \mathbb{H} \to \mathbb{C}$ such that:

- f is holomorphic;
- $(f|_k\gamma(z)) := (cz+d)^{-k}f(\gamma z) = f(z)$ for each $\gamma \in \Gamma_0(q)$;
- f is holomorphic at all cusps of $\Gamma_0(q)$ which means that the Fourier series at those cusps is a Taylor series (or **q**-series) in **q** $:= e^{2\pi iz}$. The cusps are given by $\gamma(\infty) = \frac{a}{c}$ where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\Gamma_0(q)\backslash \mathrm{SL}(2,\mathbb{Z})$.

Furthermore, f is a cusp form if it is a modular form and if it vanishes at all cusps of $\Gamma_0(q)$. Let $S_k(q)$ denote the vector space of weight k cusp forms on $\Gamma_0(q)$ with trivial character. For the cusp form $f \in S_k(q)$ is not "primitive" if one of the following holds (see 145-146, 173-174, 176-184, 213, 216, 238 of [Kob84]).

- $f \in S_k(q/d)$ for some divisor $d > 1$ of q.
- $f(z) = g(dz)$ and $g \in S_k(q/d)$ for some divisor $d > 1$ of q.

Definition 2.2.8 (Classical automorphic L-functions)**.** *Let f be primitive holomorphic cusp form of weight $k \geq 1$ and level q, with nebentypus[2] ψ. Then f has the Fourier expansion*

$$f(z) = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} \lambda_f(n) e(nz)$$

Then an L-series attached to f is defined as

$$L(f,s) = \sum_n \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} - \frac{\psi(p)}{p^{2s}}\right)^{-1}$$

where $\lambda_f(n)$ is the n^{th} Hecke eigenvalue[3] of f, with conductor q and gamma factor

$$\gamma(f,s) = \pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right) = c_k(2\pi)^{-s}\Gamma\left(s+\frac{k-1}{2}\right)$$

where $c_k = 2^{(3-k)/2}\sqrt{\pi}$ by Legendre duplication formula.

[2] Let $q \geq 1$ be an integer, and ψ a Dirichlet character modulo q (not necessarily primitive). Clearly ψ induces a character of the modular group $\Gamma_0(q)$ by $\psi(g) = \psi(d)$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A modular form of weight k, level q and *nebentypus* (or character) ψ is a holomorphic function f on the upper-half plane $\mathbb{H}$ which satisfies $f|_k\gamma = \psi(\gamma)f$, for all $\gamma \in \Gamma_0(q)$.

[3] For each $n \in \mathbb{N}$, $T_nf(z) := \frac{1}{n}\sum_{ad=n} a^k \sum_{0\leq b<d} f\left(\frac{az+b}{d}\right) = \lambda_f(n)f(z)$, where T_n is the n-th Hecke operator and $\lambda_f(n)$ is the n-th Hecke eigenvalue.

Definition 2.2.8 is taken from section 5.11 of [IK04].

$\Lambda(f,s)$ has analytic continuation to the entire complex plane and satisfies the functional equation

$$\Lambda(f,s) = q^{s/2}\gamma(f,s)L(f,s) = \varepsilon(f)\Lambda(\bar{f},1-s) \tag{2.6}$$

where $\Lambda(f,s)$ denotes the complete L-function followed by the formula given above and $\bar{f}$ is an object associated with f (the dual of f) for which $\lambda_{\bar{f}}(n) = \bar{\lambda}_f(n)$ and $\varepsilon(f)$ is the complex number of absolute value 1 called the root number of $L(f,s)$. The dual form $\bar{f}$ satisfies

$$\overline{\lambda}_f(n) = \chi(n)\lambda_f(n) \text{ if } (n,q)=1.$$

Now we recall some standard results on the Hecke-eigenvalues of $L(f,s)$. It is known that $L(f,s)$ satisfies the Ramanujan-Petersson Conjecture by work of Deligne [Del74] for $k \geq 2$ and Deligne-Serre [DS74] for $k=1$.

Ramanujan-Petersson Conjecture: For a modular form $f(z)$ in $SL(2,\mathbb{Z})$ with the weight $k \geq 2$ if its Dirichlet series converges in $\Re(s) > 1$ then the coefficient $\lambda_f(n) = O(n^{\epsilon})$.

Recall that $\lambda_f(n)$ is a real multiplicative function. Then we have [Li10]

$$\lambda_f(n)\lambda_f(m) = \sum_{\substack{d|(m,n)\\(d,q)=1}} \lambda_f\left(\frac{mn}{d^2}\right) \text{ for } m,n \geq 1. \tag{2.7}$$

and

$$\lambda_f(mn) = \sum_{\substack{d|(m,n)\\(d,q)=1}} \mu(d)\lambda_f\left(\frac{m}{d}\right)\lambda_f\left(\frac{n}{d}\right) \text{ for } m,n \geq 1. \tag{2.8}$$

2.2.4 Dirichlet L-functions attached to twisted form

In this section, we define the Dirichlet L-functions associated with a twisted Maass form.

Let $n=3$ and let $\nu = (\nu_1,\nu_2) \in \mathbb{C}^2$. A Maass form for $SL(3,\mathbb{Z})$ of type ν is smooth function $f \in \mathcal{L}^2(SL(3,\mathbb{Z})\backslash\mathfrak{h}^3)$ which satisfies[4]

- $f(\gamma z) = f(z)$, for all $\gamma \in SL(3,\mathbb{Z}), z \in \mathfrak{h}^3$.
- $Df(z) = \lambda_D f(z)$, for all $D \in \mathfrak{D}^3$ with $DI_\nu(z) = \lambda_D \cdot I_\nu(z)$ and

$$I_\nu(z) = \prod_{i,j=1,2} y^{b_{i,j}\nu_j}, \quad b_{i,j} = \begin{cases} ij & \text{if } i+j \leq 3; \\ (2-i)(2-j) & \text{if } i+j \geq 3, \end{cases}$$

and $z = x \cdot y$, where $x, y \in \mathbb{R}_{>0}$.

[4] We define a Maass form as a smooth complex valued cuspidal function on $\mathfrak{h}^3 = GL(3,\mathbb{R})/(O(n,\mathbb{R})) \cdot \mathbb{R}^\times$ which is invariant under the discrete subgroup $SL(3,\mathbb{Z})$ and which is also an eigen-function of every invariant differential operator in $\mathfrak{D}^3$. Let $\mathfrak{gl}(n,\mathbb{R})$ is the additive vector space (over $\mathbb{R}$) of all $n \times n$ matrices with coefficient in $\mathbb{R}$. The differential operator D_α with $\alpha \in \mathfrak{gl}(n,\mathbb{R})$ generate an associative algebra $\mathcal{D}^n$ over $\mathbb{R}$. Then $\mathfrak{D}^n$ is the center of $\mathcal{D}^n$. Recall that λ_D is the Harish-Chandra character.

- $\int_{(SL(3,\mathbb{Z})\cap U)\backslash U} f(uz)du = 0$, for all upper triangular groups U of the form $\left\{\begin{pmatrix} I_{r_1} & & & \\ & I_{r_2} & * & \\ & & \ddots & \\ & & & I_{r_b} \end{pmatrix}\right\}$, with $r_1 + r_2 + \cdots + r_b = 3$. Here I_r denotes the $r \times r$ identity matrix, and $*$ denotes arbitrary real entries (see Definition 5.1.3 of [Gol06]).

Definition 2.2.9 (Twisted Dirichlet L-functions)**.** *Let f be a primitive Hecke-Maass cusp form of type $\nu = (\nu_1, \nu_2)$ for $SL(3,\mathbb{Z})$ with nebentypus[5] ψ. Let $\lambda(m_1, m_2)$ (with $\lambda(1,1) = 1$) be the normalized Fourier coefficients of f. Then f has the Fourier expansion*

$$f(z) = \sqrt{y} \sum_{n \neq 0} \lambda_f(1,n) K_{ir}(2\pi |n| y) e(nx) \tag{2.9}$$

where $K_\nu(\cdot)$ is the K-Bessel function [FL05].

Let χ be a primitive Dirichlet character modulo q. Then the L-function associated with the twisted form $f \otimes \chi$ is given by the Dirichlet series

$$L(f \otimes \chi, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(1,n)\chi(n)}{n^s}$$

in domain $\Re(s) > 1$.

The gamma factor is given by

$$\gamma(f,s) = \prod_{i=1}^{3} \Gamma_{\mathbb{R}}(s - \alpha_i) \tag{2.10}$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$.

Since f is a Hecke-Maass cusp form of type (ν_1, ν_2). Then the Langlands parameters $(\alpha_1, \alpha_2, \alpha_3)$ associated with f are defined as

$$\begin{aligned} \alpha_1 &= -\nu_1 - 2\nu_2 + 1 \\ \alpha_2 &= -\nu_1 + \nu_2 \\ \alpha_3 &= 2\nu_1 + \nu_2 - 1 \end{aligned} \tag{2.11}$$

Then the functional equation of $L(f \otimes \chi, s)$ is given by

$$\Lambda(f \otimes \chi, s) = \varepsilon_\chi \Lambda(\bar{f} \otimes \bar{\chi}, 1 - s) \tag{2.12}$$

where

$$\Lambda(f \otimes \chi, s) = q^{3s/2}\gamma(f,s)L(f \otimes \chi, s)$$

[5]which is an eigenfunction of all the Hecke operators with eigenvalue $\lambda = \frac{1}{4} + r^2$, where $0 \leq r < \frac{1}{2}$.

is the complete L-function. The sign of the L-function ε_χ is given by

$$\varepsilon_\chi = g(\chi)^3/q^{3/2},$$

where $g(\chi)$ is the Gauss sum.

The Ramanujan-Selberg conjecture predicts that $\Re(\alpha_i) = 0$. The bound of the Ramanujan conjecture on $GL(3)$ gives (recently proved by Huang and Zu [HX21]),

$$\lambda(m,n) \ll (mn)^\delta. \tag{2.13}$$

where $\delta = \kappa + \epsilon$ and $\kappa \le 5/14$. Note that Hypothesis H was proved by Rudnick-Sarnak (see Proposition 2.4 of [RS96]) for $GL(3)$ L-functions. We can conclude Conjecture A holds for $GL(3)$ L-functions. From the Rankin-Selberg theory the following average holds.

$$\sum_{n\le x} |\lambda_f(1,n)|^2 \ll x. \tag{2.14}$$

2.2.5 Approximate functional equation

In the proof of the main result, we use approximate functional equation for L-functions.

Lemma 2.2.10. *Let $L(f,s)$ be an L-function belonging to the Selberg Class. Let $G(u)$ be any function which is holomorphic and bounded in the strip $-4 < \Re(u) < 4$, even, and normalized by $G(0) = 1$. Let $X > 0$. Then for s in the strip $0 \le \sigma \le 1$ we have*

$$L(f,s) = \sum_n \frac{\lambda_f(n)}{n^s} V_s\left(\frac{n}{X\sqrt{q}}\right) + \varepsilon(f,s)\sum_n \frac{\bar{\lambda}_f(n)}{n^{1-s}} V_{1-s}\left(\frac{nX}{\sqrt{q}}\right) + R \tag{2.15}$$

where the function $V_s(y)$ is a smooth function given by

$$V_s(y) = \frac{1}{2\pi i}\int_{(3)} y^{-u} G(u) \frac{\gamma(f,s+u)}{\gamma(f,s)}\frac{du}{u}$$

and

$$\varepsilon(f,s) = \varepsilon(f) q(f)^{\frac{1}{2}-s} \frac{\gamma(f,1-s)}{\gamma(f,s)}$$

The last term $R = 0$ if $\Lambda(f,s)$ is entire, otherwise

$$R = (\mathrm{res}_{u=1-s} + \mathrm{res}_{u=-s}) \frac{\Lambda(f,s+u)}{q^{s/2}\gamma(f,s)}\frac{G(u)}{u} X^u.$$

For small y we can shift the contour to the left up to $-1/10 + \varepsilon$. We get the asymptotic expansion

$$V_s(y) = 1 + O(y^{1/10-\varepsilon}).$$

For large y we use the bound $V(y) = O_j(y^{-j})$ which holds for any $j \ge 1$. For the proof of this lemma, see Theorem 5.3 of [IK04].

2.2.6 Miscellaneous

In this section, we state few standard results of L-functions that apply to all degrees of automorphic L-functions.

Lemma 2.2.11. *Let $L(f,s)$ be an L-function. There exist constants $a = a(f)$ and $b = b(f)$ such that*

$$(s(1-s))\Lambda(f,s) = e^{a+bs} \prod_{\rho \neq 0,1} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

where ρ ranges over all zeros of $\Lambda(f,s)$ different from 0,1.

For the proof of Lemma 2.2.11 see Theorem 5.6 of [IK04].

We denote

$$-\frac{L'}{L}(f,s) = \sum_{n \geq 1} \frac{\Lambda_f(s)}{n^s},$$

the expansion of the logarithmic derivative of an L-function in Dirichlet series supported on prime powers. In terms of the local roots $\alpha_i(p)$ of the Euler product we have

$$\Lambda_f(p^k) = \sum_{j=1}^{d} \alpha_j(p)^k \log p.$$

We remark that for Dirichlet characters $\Lambda_\chi(n) = \chi(n)\Lambda(n)$, where $\Lambda(n)$ denotes the Von Mangoldt function. For $GL(2)$ L-functions in the case $k = 1$ we have $\Lambda_f(p) = \lambda_f(p)\log p$ and in the case $m = 2$ we have $\Lambda_f(p^2) = (\lambda_f(p^2) - \psi(p))\log p$ for prime p. Otherwise $|\Lambda_f(n)| \leq 2\Lambda(n)$ holds for every positive integer n where $\Lambda(n)$ denotes von Mangoldt's function. In general $\Lambda_f(p) = \lambda_f(p)\log p$. Note that $\Lambda_{\bar{f}}(n) = \overline{\Lambda_f(n)}$.

We require the following estimates [RS17]. For any $m, n \in \mathbb{N}$, one has

$$\int_T^{2T} \left(\frac{m}{n}\right)^{it} dt = \begin{cases} T & \text{if } m = n; \\ O\left(\min\{T, \frac{1}{|\log(m/n)|}\}\right) & \text{if } m \neq n. \end{cases} \tag{2.16}$$

For $m \neq n$ one further has

$$\frac{1}{|\log(m/n)|} \ll \begin{cases} 1 & \text{if } m \geq 2n, \text{ or } m \leq n/2; \\ \frac{m}{|m-n|} & \text{if } n/2 < m < 2n; \\ \sqrt{mn} & \text{for all } m \neq n. \end{cases} \tag{2.17}$$

Rankin-Selberg L-function: Recall the Rankin-Selberg L-function $L(f \times f, s)$ is defined as

$$L(f \times f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{f \times f}(n)}{n^s} = \zeta^{(q)}(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)^2}{n^s}$$

for $\Re(s) > 1$.

2.3 Shifted Convolution Problem

Let f be a primitive modular form of level q and nebentypus ψ. Let $\lambda_f(n)$ be the Hecke-eigenvalues of f. Then the shifted convolution problem (SCP) consist to estimate the non-trivial bound of the following sum:

$$\sum(f, l_1, l_2, h) = \sum_{l_1 m - l_2 n = h} \lambda_f(n)\bar{\lambda}_f(m)V(m, n) \tag{2.18}$$

where $l_1, l_2 \geq 1$ and V is a compactly supported nice function in $[M, 2M] \times [N, 2N]$. The trivial bound of the sum is

$$\sum(f, l_1, l_2, h) \ll_\epsilon (MN)^\epsilon \max(M, N).$$

Find $\delta > 0$ s.t.

$$\sum(f, l_1, l_2, h) = \text{Main term}(h) + O(M^{1-\delta})$$

when M and N has almost same size. The Main term(h) is the main term and the remaining term is the error term. We can only have a non-zero main term if f is an Eisenstein series [Mic07].

Remark 2.3.1. *If $h = 0$ then the sum is a Rankin-Selberg type partial sum which can be evaluated by shifting the contour. The problem is interesting when $h \neq 0$.*

The non-trivial bound of this type of problem is often used in the sub-convexity problem. In fact, in analytic number theory, the most useful application of SCP is to obtain the sub-convexity bound for the Riemann zeta function and L-functions. But we don't need any deeper knowledge of these types of problems. We only use the non-trivial upper bound of the sum given in (2.18).

From the approximate functional equation of the L-function (see (2.15)), we can obtain the sum

$$\Sigma_V(f; h, l_1, l_2) = \sum_{l_1 m - l_2 n = h} \lambda_f(n)\bar{\lambda}_f(m)\frac{V(\frac{m}{q})V(\frac{n}{q})}{(mn)^{\frac{1}{2}}}.$$

The sum is like a partial sum of Rankin-Selberg type but with an additive shift given by h. As we have stated before that for $h = 0$ the sum is trivial. For $h \neq 0$ the additive shift is non-trivial and one expects some cancellation during the time of averaging the Hecke-eigenvalues $\lambda_f(n), \lambda_f(m)$. Due to Ramanujan-Petersson conjecture we know that $\lambda_f(n) \ll q^\epsilon$ (where $n \sim q$) the non-trivial bound for $\Sigma_V(f; h, l_1, l_2)$ is

$$\Sigma_V(f; h, l_1, l_2) \ll_{\epsilon, f} (qL)^\epsilon L^{3/4} q^{-1/4}, \tag{2.19}$$

where $l_1, l_2 \leq L$.
For the proof of (2.19) see section 4.3.1 and section 4.4 of [Mic07]

2.4 Random Variables and their properties

The main topic of this thesis is to prove Selberg's central limit theorem for L-functions in families for both t-aspect and q-aspect. From the statement of the result stated in the introduction, it is obvious that we need to deal with some basic probabilistic results. Since we are proving the central limit theorem we have to deal with random variables. We prove the main results of this thesis using the method of moments. In this chapter, we recall some standard results on it.

2.4.1 Gaussian random variable

In this section, we go through some basic facts about the Gaussian random variable.

Definition 2.4.1 (Normal random variable)**.** *A continuous random variable Z is said to be a standard normal (standard Gaussian) random variable, denoted as $Z \sim \mathcal{N}(0,1)$, if its probability density function (or PDF) is given by*

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}, \text{ for all } z \in \mathbb{R}.$$

If Z is a standard normal random variable and $Z = \frac{X-\mu}{\sigma}$, then X is a normal random variable with mean μ and variance σ^2 and we write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Definition 2.4.2 (Normal distribution)**.** *A normal (or Gaussian) distribution is a type of continuous probability distribution for a real-valued random variable. The probability density function is given by*

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}.$$

We recall some facts regarding normal random variables. Let X be a normal random variable with mean 0 and variance σ^2. The n-th moment of X satisfies

$$\mathbb{E}[X^n] = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ (1 \cdot 3 \cdots (n-1))\sigma^n & \text{if } n \text{ is even,} \end{cases} \tag{2.20}$$

where $\mathbb{E}[X^n]$ denotes the mean of X^n for positive integer n.

Let X_1 and X_2 be two random variables. Then X_1 and X_2 are independent if their probability distribution

$$P(X_1 < x_1, X_2 < x_2) = P(X_1 < x_1)P(X_2 < x_2),$$

for all x_1, x_2. The co-variance of X_1 and X_2 in terms of expectation is given by

$$\text{Cov}(X_1, X_2) := \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2].$$

Recall that X_1, X_2 is said to be uncorrelated if their co-variance is 0. That means

$$\text{Cov}(X_1, X_2) = 0 \Longleftrightarrow \mathbb{E}[X_1 X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2]$$

The co-variance coefficient defined by

$$\rho(X_1, X_2) := \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}}.$$

If X_1 and X_2 are uncorrelated then the co-variance coefficient $\rho(X_1, X_2)$ is 0. Note that two independent random variables are always uncorrelated, but the converse is not necessarily true. That is, if X_1 and X_2 are uncorrelated, then X_1 and X_2 may or may not be independent.

From the variance of a sum we know that for random variables X_1, X_2 and for real numbers $a, b \in \mathbb{R}$ we write

$$\text{Var}(aX_1 + bX_2) = a^2\text{Var}(X_1) + b^2\text{Var}(X_2) + 2ab\text{Cov}(X_1, X_2)$$

From this equation it is clear that if X_1 and X_2 are uncorrelated then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \tag{2.21}$$

Until now we have only looked at the uni-variate (or one-variable) normal distribution. Now let us recall n-variate (or multivariate) normal distribution.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random vector (or multivariate random variable) in $\mathbb{R}^n$. Let $\mathbb{E}(X_i) = m_i$ and $\text{Var}(X_i) = \sigma_i^2$ for all $i = 1, \ldots, n$. Then $\mathbf{X}$ is called an n-variate normal distribution with probability density function given by

$$f_{\mathbf{X}}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\mathfrak{K})}} e^{-\frac{1}{2}(x-\mathfrak{m})^T \mathfrak{K}^{-1}(x-\mathfrak{m})}$$

where $\mathfrak{m} = (m_1, \ldots, m_n)$, $(x - \mathfrak{m})^T$ denotes the transpose vector $(x - \mathfrak{m})$ and $\mathfrak{K} = (\sigma_{ij})$ is an $n \times n$ symmetric positive definitive matrix of real numbers with $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho(X_i, X_j)\sigma_i\sigma_j$.

We have the following properties for an n-variate normal distribution.

- Let Y be a random variable given by

  $$Y = X_1 + X_2 + \ldots + X_n$$

  then the linearity of expectation tells us that

  $$\mathbb{E}[Y] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]$$

- The variance of Y is given as

  $$\text{Var}(Y) = \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j} \text{Cov}\left(X_i, X_j\right)$$

- If the X_i's are independent then the $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$. In this case one has

  $$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i).$$

Lemma 2.4.3. *Let $(X_i)_{i=1}^n$ be a sequence of normal distributions. Then $(X_i)_{i=1}^n$ is an n-variate normal distribution if and only if any linear combination of X_j's is a normal distribution.*

Suppose, further, that (X_i, X_j) is a bi-variate normal distribution. Then X_i and X_j are independent if and only if they are uncorrelated.

For the proof of Lemma 2.4.3, see Theorem 5.5.33 of [DM88].

Lemma 2.4.4 (Cramér-Wold Theorem)**.** *Let*

$$\bar{X}_n = (X_{n_1}, X_{n_2}) \text{ and } \bar{X} = (X_1, X_2)$$

be random vectors. Then $\bar{X}_n$ converges to $\bar{X}$ in distribution if and only if

$$\sum_{i=1}^{2} a_i X_{n_i} \xrightarrow[n\to\infty]{D} \sum_{i=1}^{2} a_i X_i$$

for each $(a_1, a_2) \in \mathbb{R}^2$, that is, if every linear combination of the coordinates of $\bar{X}_n$ converges in distribution to the correspondent linear combination of coordinates of $\bar{X}$.

For the proof of this lemma see p.383 of [Bil95].

Corollary 2.4.5. *Let*

$$\bar{X}_n(t,\omega) = (X_{n_1}(t,\omega), X_{n_2}(t,\omega)) \text{ and } \bar{X}(t,\omega) = (X_1(t,\omega), X_2(t,\omega))$$

be random vectors, where ω is a random parameter and t is the time parameter. Then $\bar{X}_n(t,\omega)$ parameter wise converges to $\bar{X}(t,\omega)$ in distribution if and only if

$$\sum_{i=1}^{2} a_i X_{n_i}(t) \xrightarrow[t\to\infty]{} \sum_{i=1}^{2} a_i X_i(t) \quad \text{and} \quad \sum_{i=1}^{2} a_i X_{n_i}(\omega) \xrightarrow[n\to\infty]{} \sum_{i=1}^{2} a_i X_i(\omega)$$

for each $(a_1, a_2) \in \mathbb{R}^2$, that is, if every linear combination of the coordinates of $\bar{X}_n(t,\omega)$ parameter wise converges[6] in distribution to the correspondent linear combination of coordinates of $\bar{X}(t,\omega)$.

Remark 2.4.6. *Let (X_1, X_2) are approximate bi-variate normal distribution and converges to (Y_1, Y_2) in distribution, where (Y_1, Y_2) is a bi-variate normal distribution. Then as an application of Lemma 2.4.3 and Corollary 2.4.5, we can deduce that X_1 and X_2 are asymptotically independent if and only if Y_1 and Y_2 are independent.*

In a later chapter, we prove that a finite sequence of L-functions associated with distinct cusp forms (or Maass forms) forms a Gaussian process [Lif12].

Definition 2.4.7 (Gaussian process)**.** *A stochastic process (X_i) is called a Gaussian process if every finite sub-sequence of $(X_i)_{i\in J}$ has a multivariate normal distribution.*

[6]"parameter wise converges" indicates that we fix one parameter (either t or ω) and $\bar{X}_n(t,\omega)$ converges in distribution as the other parameter varies.

2.4.2 Method of moments

We prove our main results by studying the moments of the auxiliary series (which we define in later chapters). We need very basic results on the method of moments.

For some distributions, one can not trace the characteristic function but the moments can be calculated for those distributions. In these cases, one can prove the weak convergence of the distribution by establishing the moment convergence under the condition in which the moments are uniquely determined.

Lemma 2.4.8. *Let* $(\mathfrak{S}, \mathfrak{F}, \mathfrak{P})$ *be a complete probability space. We say that* (X_n) *converges to* X *in distribution if* $\lim_{n\to\infty} \mathfrak{P}(s \in \mathfrak{S} : X_n(s) \leq x) = \mathfrak{P}(s \in \mathfrak{S} : X(s) \leq x)$ *for every* x *such that* $\mathfrak{P}(s \in \mathfrak{S} : X(s) = x) = 0$.

Suppose that the distribution of X *is determined by its moments of all orders, and that* $\lim_{n\to\infty} \mathbb{E}[X_n^r] = \mathbb{E}[X^r]$ *for* $r = 1, 2, \ldots$. *Then* $X_n \Rightarrow X$ *which means* X_n *converges to* X *in distribution.*

For the proof of Lemma 2.4.8 see Theorem 30.2 of [Bil95]. The method of moments plays an important role to determine the central limit theorems and it has several applications in Number Theory as well. For example, see Theorem 30.3 of [Bil95].

3. SCLT for automorphic L-functions in the t-aspect

3.1 Introduction

In this chapter, we present a simple proof of Selberg's central limit theorem for automorphic L-functions of degree 2 associated with primitive holomorphic cusp forms[1] under the assumption of GRH. As we have discussed in the introduction, we follow the method established in [RS17].

As discussed in the introduction, Radziwiłł and Soundararajan's [RS17] method can be extended for higher degree L-functions, but the computations do not follow immediately. To prove SCLT for the families of L-functions with degree $d \geq 3$, one needs to follow the mollification technique, which requires one to prove the second mollified moment of the L-functions. Note that one needs information on the shifted convolution problem to compute the second mollified moment of the L-functions. Due to the limited information on the shifted convolution problem for higher degree L-functions (for degree $d \geq 3$), Radziwiłł and Soundararajan's proof (in t-aspect) can not be extended at the moment. Selberg mentioned that his central limit theorem can be proved for all L-functions belonging to the Selberg class [Sel46a][2].

In this chapter, we have proved SCLT in the t-aspect, for which we have integrated over $t \in [T, 2T]$ for sufficiently large T. Since we are dealing with the L-functions, some modifications are needed in our proof.

Our main result has stated below.

Theorem 3.1.1. *Let V be a fixed positive real number and f is a primitive holomorphic cusp form of weight $k \geq 1$, level q. Then as $T \to \infty$, uniformly for all $v \in [-V, V]$,*

$$\frac{1}{T} meas \left\{ T \leq t \leq 2T : \log |L(f, \frac{1}{2} + it)| \geq v \sqrt{\frac{1}{2} \log\log T} \right\} \sim \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-u^2/2} du,$$

[1] Throughout this chapter, we have considered L-functions as defined in Definition 2.2.8.

[2] Selberg has proved his central limit theorem for the Riemann zeta function and Dirichlet L-functions [Sel46b; Sel46a].

3.2 The Setup

In this section, we write the sketch of the proof of Theorem 3.1.1 for $GL(2)$ L-functions.

Our proof can be broken into four main steps. In the first step, we take away the problem from the critical line. In the next step, we introduce an auxiliary series involving the Hecke-eigenvalues of f. We prove that it has normal distribution by studying its moments. Finally, we use the mollification technique to connect the L-functions with the introduced auxiliary series.

The following proposition proves that $\log|L(f,\frac{1}{2}+it)|$ is typically close to $\log|L(\sigma+it)|$ for suitable σ near $\frac{1}{2}$.

Proposition 3.2.1. *Let T be large and suppose $T\le t\le 2T$. Then for any $\sigma>1/2$ we have*

$$\int_{t-1}^{t+1}\left|\log|L(f,\frac{1}{2}+iy)|-\log|L(f,\sigma+iy)|\right|dy \ll (\sigma-\frac{1}{2})\log T.$$

Remark 3.2.2. *In the first step, we take away the problem from the critical line. If we choose to move $1/\log T$ distance from the $\frac{1}{2}$ line (as $T\to\infty$ the density of zeros of $L(f,s)$ increases), then we might stay very close to $\frac{1}{2}$ line for large enough T. We wish to stay away from the critical line but not too far away. So, we can not choose a large parameter W. That is why we choose $W=o(\sqrt{\log\log T})$. Further, we want to approximate $L(f,s)$ by an Euler product, where the product is going up to X. In order to approximate, we need to consider that X would be going up to a small power of T.*

We fix the parameters

$$W=(\log\log\log T)^4,\ X=T^{1/(\log\log\log T)^2},\ Y=T^{(1/\log\log T)^2},\ \sigma_0=\frac{1}{2}+\frac{W}{\log T},$$

where $T>0$ is sufficiently large so that $W\ge 3$.

From Proposition 3.2.1, we can see that the difference between $\log|L(f,\frac{1}{2}+it)|$ and $\log|L(f,\sigma_0+it)|$ is negligible[3]. So from now on we need to study the behaviour of $\log|L(f,\sigma_0+it)|$ which is a much easier problem since the zeros of the L-functions won't interfere anymore.

As the second step, consider the auxiliary series

$$\mathcal{P}(f,s)=\mathcal{P}(f,s;X)=\sum_{2\le n\le X}\frac{\Lambda_f(n)}{n^s\log n}$$

By computing moments we determine the distribution of $\mathcal{P}(f,s)$.

[3]Using Proposition 3.2.1 we can say that $\log|L(f,\frac{1}{2}+it)|$ and $\log|L(f,\sigma_0+it)|$ differ by at most AW except on a set of measure $O(T/A)$, where $\log A=o(\sqrt{\log\log T})$. If AW is small compared to $\sqrt{\log\log T}$, the difference is negligible and $\log|L(f,s)|$ has almost same distribution at $s=\sigma_0+it$ and $s=\frac{1}{2}+it$.

Proposition 3.2.3. *As t varies in $T \leq t \leq 2T$, the distribution of $\Re(\mathcal{P}(f, \sigma_0 + it))$ is approximately normal*[4] *with mean 0 and variance $\sim \frac{1}{2}\log\log T$. Precisely, let V be a fixed positive real number then as $T \to \infty$, uniformly for all $v \in [-V, V]$,*

$$\frac{1}{T} meas\{T \leq t \leq 2T : \Re(\mathcal{P}(f, \sigma_0 + it)) \geq v\sqrt{\frac{1}{2}\log\log T}\} \sim \frac{1}{\sqrt{2\pi}}\int_v^{\infty} e^{-u^2/2} du.$$

By the definition of the Fourier coefficient of the cusp form $\lambda_f(1) \neq 0$. Then we can define the convolution inverse $(\mu_f(n))$ of the sequence $(\lambda_f(n))$. This is an arithmetic multiplicative function, which satisfies for a prime number p

$$\mu_f(1) = 1,\ \mu_f(p) = -\lambda_f(p),\ \mu_f(p^2) = \lambda_f(p)^2 - \lambda_f(p^2) = \begin{cases} 1 & \text{if } p \nmid q \\ 0 & \text{otherwise} \end{cases}$$

and if $j \geq 3,\ \mu_f(p^j) = 0.$

Now it remains to connect $\Re(\mathcal{P}(f, \sigma_0 + it))$ with $\log|L(f, \sigma_0 + it)|$ for most values of t to prove Theorem 3.1.1. For this, we use the mollification technique. Introducing the Dirichlet polynomial $M(f, s)$ given by

$$M(f, s) = \sum_n \frac{\mu_f(n)a(n)}{n^s}$$

where $a(n)$ is defined by

$$a(n) = \begin{cases} 1 & \text{if } n \text{ is composed only by primes below } X \text{ and has at most } 100\log\log T, \\ & \text{primes below } Y\text{, and at most } 100\log\log\log T \text{ primes between } Y \text{ and } X. \\ 0 & \text{otherwise .} \end{cases}$$

By the definition of $a(n)$ it takes the value 0 except when $n \leq Y^{100\log\log T} X^{100\log\log\log T} < T^{\epsilon}$. It is now evident that $M(f, s)$ is a short Dirichlet polynomial. Our target is to show that $M(f, s)$ can be approximated by $e^{-\mathcal{P}(f,s)}$.

Proposition 3.2.4. *For $T \leq t \leq 2T$*

$$M(f, \sigma_0 + it) = (1 + o(1))\exp(-\mathcal{P}(f, \sigma_0 + it))$$

except perhaps on a subset of measure $o(T)$.

Now it remains to connect the L-functions with the Dirichlet polynomial $M(f, s)$ to prove the theorem. Roughly speaking from the definition of $M(f, s)$ we can see that $L(f, s)$ and $M(f, s)$ are inverse to each other, which we are going to prove as our final step.

Proposition 3.2.5. *For $T \leq t \leq 2T$,*

$$\frac{1}{T}\int_T^{2T} |1 - L(f, \sigma_0 + it)M(f, \sigma_0 + it)|^2\, dt = o(1).$$

[4] see (1.2), introduction.

So that for $T \le t \le 2T$ we have

$$L(f, \sigma_0 + it) M(f, \sigma_0 + it) = 1 + o(1),$$

except perhaps on a set of measure $o(T)$.

Now we are ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1: Recalling Proposition 3.2.5, it says that for all $t \in [T, 2T]$ (outside of a set of measure $o(T)$) we have

$$L(f, \sigma_0 + it) = (1 + o(1)) M(f, \sigma_0 + it)^{-1}.$$

By Proposition 3.2.4, for all $t \in [T, 2T]$ (outside of a set of measure $o(T)$) we know that

$$|L(f, \sigma_0 + it)| = (1 + o(1)) \exp(\Re \mathcal{P}(f, \sigma_0 + it))$$

and by Proposition 3.2.3 we can conclude that $\log |L(f, \sigma_0 + it)|$ is normally distributed with mean 0 and variance $\frac{1}{2} \log \log T$. Finally with the help of Proposition 3.2.1 we deduce that

$$\int_T^{2T} \left| \log |L(f, \frac{1}{2} + it)| - \log |L(f, \sigma_0 + it)| \right| dt \ll T(\sigma_0 - \frac{1}{2}) \log T = WT,$$

So outside of a set of measure $O(T/W) = o(T)$ we have

$$\log |L(f, \frac{1}{2} + it)| = \log |L(f, \sigma_0 + it)| + O(W^2).$$

Since $W^2 = o(\sqrt{\log \log T})$ it follows that similarly like $\log |L(f, \sigma_0 + it)|$,
$\log |L(f, \frac{1}{2} + it)|$ has the normal distribution with mean 0 and variance $\frac{1}{2} \log \log T$, which completes the proof of Theorem 3.1.1. $\square$

3.3 Proof for $GL(2)$ L-functions

In this section, we give the detail of the proof of Theorem 3.1.1 by proving the Propositions.

3.3.1 Proof of Proposition 3.2.1

Let f be a primitive holomorphic cusp form (as defined in Definition 2.2.8) of weight $k \ge 1$ and level q.

Set

$$G(f, s) = q^{s/2} \gamma(f, s) = q^{s/2} c_k (2\pi)^{-s} \Gamma\left(s + \frac{k-1}{2}\right) \tag{3.1}$$

where $\gamma(f, s)$ is the gamma factor of $L(f, s)$ with $c_k = 2^{(3-k)/2} \sqrt{\pi}$. If t is sufficiently large and $y \in [t-1, t+1]$, then by Stirling's formula we will show that

$$\left| \log \frac{G(f, \sigma + iy)}{G(f, 1/2 + iy)} \right| \ll \left(\sigma - \frac{1}{2}\right) \log t.$$

Note that

$$\arg\left(\frac{G(f,\sigma+iy)}{G(f,1/2+iy)}\right) = \arg\left(G(f,\sigma+iy)\right) - \arg\left(G(f,1/2+iy)\right).$$

Now, applying the formula for the argument of the gamma function we have

$$\begin{aligned}\arg\left(G(f,\sigma+iy)\right) =& \arg\left(q^{(\sigma+iy)/2}c_k(2\pi)^{-(\sigma+iy)}\Gamma\left(\sigma+iy+\frac{k-1}{2}\right)\right)\\ =& \arg(q^{(\sigma+iy)/2}) + \arg(c_k(2\pi)^{-(\sigma+iy)}) + \arg\left(\Gamma\left(\sigma+iy+\frac{k-1}{2}\right)\right)\\ =& y/2\log q + y\log 2\pi + (y\log y - y) + \left(\sigma-\frac{1}{2}\right)\frac{\pi}{2} + O\left(\frac{1}{|y|}\right).\end{aligned}$$

Similarly,

$$\arg\left(G(f,1/2+iy)\right) = y/2\log q + y\log 2\pi + (y\log y - y) + O\left(\frac{1}{|y|}\right).$$

Expanding the complex logarithm and putting Stirling's approximation[5] for Gamma function in (3.1) we have,

$$\begin{aligned}&\log\frac{G(f,\sigma+iy)}{G(f,1/2+iy)}\\ =&\log\left|\frac{G(f,\sigma+iy)}{G(f,1/2+iy)}\right| + i\arg\left(\frac{G(f,\sigma+iy)}{G(f,1/2+iy)}\right)\\ =&\log\frac{q^{\frac{\sigma}{2}}c_k(2\pi)^{-\sigma}|\Gamma(\sigma+iy+\frac{k-1}{2})|}{q^{\frac{(1/2)}{2}}c_k(2\pi)^{-1/2}|\Gamma(1/2+iy+\frac{k-1}{2})|} + i\left(\arg\left(G(f,\sigma+iy)\right) - \arg\left(G(f,1/2+iy)\right)\right)\\ =&\log\left(q^{\frac{\sigma-1/2}{2}}(2\pi)^{-(\sigma-1/2)}\frac{\sqrt{2\pi}y^{\sigma-1/2}e^{-\frac{\pi}{2}|y|}}{\sqrt{2\pi}e^{-\frac{\pi}{2}|y|}}\right) + O\left(|y|^{\sigma-\frac{3}{4}}e^{-\frac{\pi}{2}|y|}\right) + O\left(|y|^{-1}e^{-\frac{\pi}{2}|y|}\right)\\ &\qquad + i\left(\left(\sigma-\frac{1}{2}\right)\frac{\pi}{2} + O\left(\frac{1}{|y|}\right)\right)\\ =&\log\left(q^{\frac{\sigma-1/2}{2}}(2\pi)^{-(\sigma-1/2)}y^{(\sigma-\frac{1}{2})}\right) + O\left(\frac{e^{-\frac{\pi}{2}|y|}}{|y|}\right) + i\left(\left(\sigma-\frac{1}{2}\right)\frac{\pi}{2} + O\left(\frac{1}{|y|}\right)\right).\end{aligned}$$

Since, t is large enough and $y \in [t-1, t+1]$, and q is fixed (which means the implicit constant will depend on q), we write[6]

$$\left|\log\frac{G(f,\sigma+iy)}{G(f,1/2+iy)}\right| \ll \left|\log t^{(\sigma-\frac{1}{2})}\right| \ll \left(\sigma-\frac{1}{2}\right)\log t.$$

[5]Consider the Stirling formula given in notation. We can separate the real and the argument part. For the real part we can write $|\Gamma(\sigma+it)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}\left(1+O\left(\frac{1}{|t|}\right)\right)$. The argument can be written as $\arg(\Gamma(\sigma+it)) = t\log t - t + \left(\sigma-\frac{1}{2}\right)\frac{\pi}{2} + O\left(\frac{1}{|t|}\right)$.

[6]Note that the error term coming from the logarithm of the Gamma function is very small. Since we only need the upper-bound, the error will be subsumed by the main term.

Recall the functional equation of the complete L-function (as given in (2.6))

$$\Lambda(f,s) = G(f,s)L(f,s).$$

To prove Proposition 3.2.1 it is enough to prove that

$$\int_{t-1}^{t+1} \left| \log \left| \frac{\Lambda(f,1/2+iy)}{\Lambda(f,\sigma+iy)} \right| \right| dy \ll \left(\sigma - \frac{1}{2}\right) \log T.$$

Recalling Hadamard's factorization formula (see e.g. Lemma 2.2.11), there exist constants $a = a(f)$ and $b = b(f)$ (where $b(f) = -\sum_\rho \Re(1/\rho)$) such that

$$(s(1-s))\Lambda(f,s) = e^{a+bs} \prod_{\rho \neq 0,1} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where ρ ranges over all zeros of $\Lambda(f,s)$ different from 0,1. The product on the left hand side of the above equation is over all non-trivial zeros of $L(f,s)$, all of which lie in the region $0 \leq \Re(\rho) \leq 1$.

Assuming that y is not the ordinate of a zero of $L(f,s)$ we can write

$$\log \left| \frac{\Lambda(f,1/2+iy)}{\Lambda(f,\sigma+iy)} \right| = \sum_\rho \log \left| \frac{(1/2+iy)-\rho}{(\sigma+iy)-\rho} \right|.$$

Suppose $\rho = \beta + i\gamma$ is a non-trivial zero of $L(f,s)$. Integrating over $y \in [t-1, t+1]$ we get

$$\begin{aligned} \int_{t-1}^{t+1} \left| \log \left| \frac{\Lambda(f,\frac{1}{2}+iy)}{\Lambda(f,\sigma+iy)} \right| \right| dy \leq & \sum_\rho \int_{t-1}^{t+1} \left| \log \left| \frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho} \right| \right| dy \\ = & \frac{1}{2} \sum_\rho \int_{t-1}^{t+1} \left| \log \frac{(\beta-\frac{1}{2})^2 + (y-\gamma)^2}{(\beta-\sigma)^2 + (y-\gamma)^2} \right| dy. \end{aligned} \tag{3.2}$$

If $|t-\gamma| \geq 2$ then for any $y \in [t-1, t+1]$ we have

$$\begin{aligned} \left| \log \left| \frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho} \right| \right| = \left| \Re \log \left(1 - \frac{\sigma-\frac{1}{2}}{\sigma+iy-\rho}\right) \right| = & \left| \Re \frac{\sigma-\frac{1}{2}}{\sigma+iy-\rho} \right| + O\left(\frac{(\sigma-\frac{1}{2})^2}{(y-\gamma)^2}\right) \\ = & O\left(\frac{(\sigma-\frac{1}{2})}{(y-\gamma)^2}\right). \end{aligned}$$

So we can write

$$\int_{t-1}^{t+1} \left| \log \left| \frac{(1/2+iy)-\rho}{(\sigma+iy)-\rho} \right| \right| dy \ll \frac{(\sigma-1/2)}{(t-\gamma)^2}.$$

Then contribution of these zeros give

$$\sum_{\substack{\rho \\ |t-\gamma|\geq 2}} \frac{(\sigma - \frac{1}{2})}{(t-\gamma)^2} \ll (\sigma - \frac{1}{2}) \log T.$$

Now consider the range $|t-\gamma| \leq 2$ (which is basically the zeros near t) we have

$$\begin{aligned}
\int_{t-1}^{t+1} \left| \log \left| \frac{1/2 + iy - \rho}{\sigma + iy - \rho} \right| \right| dy =& \frac{1}{2} \int_{t-1}^{t+1} \left| \log \frac{(\beta - \frac{1}{2})^2 + (y-\gamma)^2}{(\beta - \sigma)^2 + (y-\gamma)^2} \right| dy \\
\leq& \frac{1}{2} \int_{-\infty}^{\infty} \left| \log \left| \frac{(\beta - \frac{1}{2})^2 + x^2}{(\beta - \sigma)^2 + x^2} \right| \right| dx \\
=& \pi \left(\sigma - \frac{1}{2} \right).
\end{aligned}$$

So in this case the contribution of zeros is $\ll (\sigma - \frac{1}{2}) \log T$.

Thus in either case

$$\int_{t-1}^{t+1} \left| \log \left| \frac{1/2 + iy - \rho}{\sigma + iy - \rho} \right| \right| dy \ll \frac{(\sigma - \frac{1}{2})}{1 + (t-\gamma)^2}.$$

Inserting this in (3.2) and noting that there are $\log(t+k')$ zeros with $k' \leq |t-\gamma| < k'+1$ (from Theorem 5.38 of [IK04]), we can conclude

$$\begin{aligned}
\int_{t-1}^{t+1} \left| \log \left| \frac{L(f, 1/2 + iy)}{L(f, \sigma + iy)} \right| \right| dy =& \int_{t-1}^{t+1} \left| \log \left| \frac{\Lambda(f, 1/2 + iy)}{\Lambda(f, \sigma + iy)} \right| \right| dy + O\left(\sigma - \frac{1}{2} \right) \log T \\
\ll& \left(\sigma - \frac{1}{2} \right) \log T,
\end{aligned}$$

which completes the proof. □

3.3.2 Proof of Proposition 3.2.3

To prove this proposition we restrict the sum $\mathcal{P}(f, s)$ to primes and compute moments. We know that Gaussian distribution can uniquely be determined by its moments.

For the terms involving higher power of primes i.e. p^k (with $k \geq 3$) we have[7]

$$\left| \sum_{\substack{2 \leq p^k \leq X \\ k \geq 3}} \frac{\Lambda_f(p^k)}{p^{ks}(k \log p)} \right| \leq \sum_{\substack{2 \leq p^k \leq X \\ k \geq 3}} \frac{1}{3p^{k\sigma_0}} = O(1).$$

[7]Note that $\Lambda_f(p^k) = (\alpha_1(p)^k + \alpha_2(p)^k) \log p$ with $|\alpha_1(p)|, |\alpha_2(p)| = 1$, if $(p, q) = 1$ (see Theorem 8.2 of [Del74]).

where $\Re(s) = \sigma_0 > \frac{1}{2}$. As argued in Section 3 of [MN14] (recall that ψ is the nebentypus of $L(f,s)$, see Definition 2.2.8),

$$|\lambda_f(p^2) - \psi(p)|^2 = \left|\frac{\Lambda_f(p^2)}{\log p}\right| \leq \left|\frac{2\Lambda(p^2)}{\log p}\right|^2 = \left|\frac{2\Lambda(p)}{\log p}\right|^2 = 4.$$

By expanding out and using (2.16) and (2.17) contribution of the terms involving prime squares give

$$\int_T^{2T} \left| \sum_{2 \leq p^2 \leq X} \frac{\lambda_f(p^2) - \psi(p)}{p^{2(\sigma_0+it)} \cdot 2} \right|^2 dt \leq \sum_{p_1, p_2 \leq \sqrt{X}} \int_T^{2T} \frac{1}{p_1^{2(\sigma_0+it)} p_2^{2(\sigma_0-it)}} dt \tag{3.3}$$
$$\ll T \sum_{p \leq \sqrt{X}} \frac{1}{p^{4\sigma_0}} + \sum_{\substack{p_1, p_2 \leq \sqrt{X} \\ p_1 \neq p_2}} \frac{1}{p_1^{2\sigma_0} p_2^{2\sigma_0}} \sqrt{p_1 p_2} \ll T.$$

Let $A(t;X) = A(t) = \sum_{2 \leq p \leq X} \frac{\lambda_f(p^2) - \psi(p)}{2p^{2\sigma_0}}$, from (3.3) and Chebyshev's inequality we have

$$meas\{T \leq t \leq 2T : |A(t)| > L\} \leq \frac{1}{L^2} \int_T^{2T} |A(t)|^2 dt \ll T/L^2$$

In other words, we can say that the square of primes in $\mathcal{P}(f,s)$ contribute measure at most $O(T/L^2)$. With a similar explanation of Remark 3.2.2, L can goes up to a small power of T. Then we choose $L = o(\log \log \log T)$.

Now we restrict $\mathcal{P}(f,s)$ to primes. Since we know that $\Lambda_f(p) = \lambda_f(p) \log p$ for all prime p, we can write

$$\mathcal{P}_0(f, \sigma_0 + it) = \mathcal{P}_0(f, \sigma_0 + it, X) = \sum_{p \leq X} \frac{\lambda_f(p)}{p^{\sigma_0+it}}.$$

Let us begin by computing the moments of $\mathcal{P}_0(f, \sigma_0 + it)$ (see Lemma 2.4.8).

Lemma 3.3.1. *Suppose that k and ℓ are non-negative integers with $X^{k+\ell} \ll T$. Then if $k \neq \ell$*

$$\int_T^{2T} \mathcal{P}_0(f, \sigma_0 + it)^k \overline{\mathcal{P}_0(f, \sigma_0 + it)}^\ell dt \ll T.$$

If $k = \ell$, for $\epsilon > 0$ we have

$$\int_T^{2T} |\mathcal{P}_0(f, \sigma_0 + it)|^{2k} dt = k! T (\log \log T)^k + O_k(T (\log \log T)^{k-1+\epsilon}).$$

Proof. Write $\mathcal{P}_0(f,s)^k = \sum_n \frac{a_k(n)\lambda_f(n)}{n^s}$ where

$$a_k(n) = \begin{cases} \frac{k!}{\alpha_1! \cdots \alpha_r!} & \text{if } n = \prod_{j=1}^r {p_j}^{\alpha_j}, p_1 < \ldots < p_r < X, \sum_{j=1}^r \alpha_j = k. \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Therefore (using (2.16)),

$$
\begin{aligned}
&\int_T^{2T} \mathcal{P}_0(\sigma_0+it)^k \overline{\mathcal{P}_0(\sigma_0+it)}^{\ell} dt \\
=&\int_T^{2T} \sum_n \frac{a_k(n)\lambda_f(n)}{n^s} \sum_m \frac{a_\ell(m)\bar{\lambda}_f(m)}{m^{\bar{s}}} dt \\
=&\sum_n \frac{a_k(n)\lambda_f(n)}{n^{\sigma_0}} \sum_m \frac{a_\ell(m)\bar{\lambda}_f(m)}{m^{\sigma_0}} \int_T^{2T} \left(\frac{m}{n}\right)^{it} dt \\
=&T \sum_n \frac{a_k(n)a_\ell(n)\lambda_f(n)\bar{\lambda}_f(n)}{n^{2\sigma_0}} + O\left(\sum_{m\neq n} \frac{a_k(n)a_\ell(m)\lambda_f(n)\bar{\lambda}_f(m)}{(mn)^{\sigma_0}} \frac{1}{\log|m/n|}\right).
\end{aligned}
$$

Notice that if $k \neq \ell$ then $a_k(n)a_\ell(n)$ is 0 by definition. So we don't have to worry about the diagonal term contribution. For the off-diagonal term from (2.17) we can see that the contribution of the denominator is negligible since σ_0 is close to $\frac{1}{2}$ (see Lemma 1 of [Sel46b]). Applying Ramanujan-Petersson conjecture, with $m \neq n$ we have the off-diagonal term contribution given by

$$
\sum_{\substack{m\neq n \\ m\leq X^k \\ n\leq X^\ell}} a_k(n)a_\ell(m)\lambda_f(n)\bar{\lambda}_f(m) \ll X^{k+\ell+\epsilon} \ll T.
$$

We conclude the first part of the Lemma.

For the second part of the lemma (which is the case for $k=\ell$) the diagonal term contributes $T\sum_n \frac{a_k(n)^2|\lambda_f(n)|^2}{n^{2\sigma_0}}$. By the definition for the given positive integers $\alpha_1,\ldots,\alpha_r$ with $\sum_{i=1}^r \alpha_i = k$, the contribution of n of the form $p_1^{\alpha_1},\ldots,p_r^{\alpha_r}$ is given by

$$
\ll T\prod_{i=1}^r \left(\sum_{\substack{p\leq X \\ (p,q)=1}} \frac{|\lambda_f(p)|^2}{p^{2\sigma_0\alpha_i}}\right) \ll T(\log\log T)^{r+\epsilon}.
$$

The terms with n not being square free contributes (where $r \leq k-1$) $O_k((\log\log T)^{k-1+\epsilon})$.

The square free terms n give (see section 3, of [Lü14] and (63) of [MN14]),

$$
\begin{aligned}
k! \sum_{\substack{p_1,\ldots,p_k\leq X \\ \text{all } p_j\text{'s are distinct},(p_j,q)=1}} \frac{|\lambda_f(p_1\cdots p_k)|^2}{(p_1\cdots p_k)^{2\sigma_0}} =&k!\left(\sum_{\substack{p\leq X \\ (p,q)=1}} \frac{|\lambda_f(p)|^2}{p^{2\sigma_0}}\right)^k + O_k\left((\log\log T)^{k-1+\epsilon}\right) \\
=&k!\,(\log\log T)^k + O_k\left((\log\log T)^{k-1+\epsilon}\right).
\end{aligned}
$$

Recalling the definition of X, we conclude the proof. □

Proof of Proposition 3.2.3: Using Lemma 3.3.1 for any odd k,

$$\begin{aligned}\int_T^{2T} \left(\Re\left(\mathcal{P}_0(f,\sigma_0+it)\right)\right)^k dt &= \int_T^{2T} \frac{1}{2^k}\left(\mathcal{P}_0(f,\sigma_0+it)+\overline{\mathcal{P}_0(f,\sigma_0+it)}\right)^k dt\\ &= \frac{1}{2^k}\sum_{\ell=0}^{k}\binom{k}{\ell}\int_T^{2T}\left(\mathcal{P}_0(f,\sigma_0+it)\right)^{\ell}\,\overline{\left(\mathcal{P}_0(f,\sigma_0+it)\right)^{k-\ell}}dt\\ &\ll T,\end{aligned}$$

as it is impossible to have $\ell = k-\ell$ for any odd k. If k is even, then we apply Lemma 3.3.1 with $\ell = k-\ell = k/2$ to obtain,

$$\frac{1}{T}\int_T^{2T} \left(\Re\left(\mathcal{P}_0(f,\sigma_0+it)\right)\right)^k dt = 2^{-k}\binom{k}{k/2}\left(\frac{k}{2}\right)!(\log\log T)^{\frac{k}{2}} + O_k\left((\log\log T)^{\frac{k}{2}-1+\epsilon}\right).$$

The above equation matches with the distribution of the Gaussian random variable (see (2.20)) with mean 0 and variance $\sim \frac{1}{2}\log\log T$, completing the proof. □

3.3.3 Proof of Proposition 3.2.4

To prove this proposition let us first decompose $\mathcal{P}(f,s)$ as $\mathcal{P}_1(f,s)$ and $\mathcal{P}_2(f,s)$ where

$$\begin{aligned}\mathcal{P}_1(f,s) &= \sum_{2\le n\le Y}\frac{\Lambda_f(n)}{n^s\log n},\\ \mathcal{P}_2(f,s) &= \sum_{Y<n\le X}\frac{\Lambda_f(n)}{n^s\log n}.\end{aligned}$$

Set

$$\begin{aligned}\mathcal{M}_1(f,s) &= \sum_{0\le k\le 100\log\log T}\frac{(-1)^k}{k!}\mathcal{P}_1(f,s)^k,\\ \mathcal{M}_2(f,s) &= \sum_{0\le k\le 100\log\log\log T}\frac{(-1)^k}{k!}\mathcal{P}_2(f,s)^k.\end{aligned}$$

Now we state the next lemma.

Lemma 3.3.2. *For $T\le t\le 2T$ we have*

$$\begin{aligned}|\mathcal{P}_1(f,\sigma_0+it)| &\le \log\log T,\\ |\mathcal{P}_2(f,\sigma_0+it)| &\le \log\log\log T.\end{aligned} \tag{3.5}$$

except perhaps for a set of measure $\frac{T}{\log\log\log T}$. We also have

$$\begin{aligned}\mathcal{M}_1(f,\sigma_0+it) &= \exp(-\mathcal{P}_1(f,\sigma_0+it))(1+O(\log T)^{-99}),\\ \mathcal{M}_2(f,\sigma_0+it) &= \exp(-\mathcal{P}_2(f,\sigma_0+it))(1+O(\log\log T)^{-99}).\end{aligned} \tag{3.6}$$

Proof. Starting with the integration, from (2.16) and (2.17) we write,

$$\int_T^{2T} |\mathcal{P}_1(f,\sigma_0+it)|^2 dt$$
$$\ll T \sum_{2\le n_1=n_2\le Y} \frac{\Lambda_f(n_1)\Lambda_f(n_2)}{(n_1n_2)^{\sigma_0}\log n_1 \log n_2} + \sum_{2\le n_1\ne n_2\le Y} \frac{\Lambda_f(n_1)\Lambda_f(n_2)}{(n_1n_2)^{\sigma_0}\log n_1\log n_2}\sqrt{n_1n_2}.$$

Recall that $\Lambda_f(n)$ is supported only on prime powers. For the higher order of prime powers p^k (for $k \ge 3$) the contribution is negligible as argued in the proof of Proposition 3.2.3[8]. The terms involving prime squares contribute

$$\begin{aligned}
&\ll T \sum_{2\le p_1^2=p_2^2\le Y} \frac{|\lambda_f(p_1)-\psi(p)|^2|\lambda_f(p_2)-\psi(p)|^2 \log p_1 \log p_2}{(p_1p_2)^{2\sigma_0} 4\log p_1\log p_2} \\
&\qquad + \sum_{2\le p_1^2\ne p_2^2\le Y} \frac{|\lambda_f(p_1)-\psi(p)|^2|\lambda_f(p_2)-\psi(p)|^2 \log p_1 \log p_2}{(p_1p_2)^{2(\sigma_0-1/2)} 4\log p_1\log p_2} \\
&\ll T \sum_{p_1=p_2\le\sqrt{Y}} \frac{1}{(p_1p_2)^{2\sigma_0}} + \sum_{2\le p_1^2\ne p_2^2\le Y} \frac{1}{(p_1p_2)^{2(\sigma_0-1/2)}} \ll T \sum_{\substack{p\le\sqrt{Y}\\ p_1=p_2=p}} \frac{1}{p^{4\sigma_0}} \ll T.
\end{aligned} \tag{3.7}$$

Note that the denominator in the second term of (3.7) is negligible because σ_0 is close to $\frac{1}{2}$ (see Lemma 1 of [Sel46b]). With a similar argument the terms involving primes contribute

$$\begin{aligned}
&\ll T \sum_{2\le p_1=p_2\le Y} \frac{\lambda_f(p_1)\log p_1 \lambda_f(p_2)\log p_2}{(p_1p_2)^{\sigma_0}\log p_1\log p_2} + \sum_{2\le p_1\ne p_2\le Y} \frac{\lambda_f(p_1)\lambda_f(p_2)\log p_1\log p_2}{(p_1p_2)^{\sigma_0-1/2}\log p_1\log p_2} \\
&\ll T \sum_{2\le p_1=p_2\le Y} \frac{\lambda_f(p_1)\lambda_f(p_2)}{(p_1p_2)^{\sigma_0}} + \sum_{2\le p_1\ne p_2\le Y} \lambda_f(p_1)\lambda_f(p_2) \\
&\ll T \sum_{\substack{2\le p\le Y\\ p_1=p_2=p}} \frac{\lambda_f(p)^2}{p^{2\sigma_0}} \\
&\ll T \log\log T.
\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) we can write

$$\int_T^{2T} |\mathcal{P}_1(f,\sigma_0+it)|^2 dt \ll T\log\log T.$$

[8]Recall that $\Lambda_f(n)$ is the coefficient of the logarithmic derivative of the L-function in Dirichlet series supported on prime powers. Since the argument we have given in the proof of Proposition 3.2.3, we know that the primes with higher power (i.e. p^k with $k \ge 2$) contribute negligible amount so we can write $\frac{1}{T}\int_T^{2T} |\mathcal{P}_1(f,\sigma_0+it)|^2 \approx \sum_{2\le n\le Y} \frac{\Lambda_f(n)^2}{n^{2\sigma_0}(\log n)^2} \approx \sum_{p\le Y} \frac{\lambda_f(p)^2}{p^{2\sigma_0}} \ll \log\log T$. Similarly $\frac{1}{T}\int_T^{2T} |\mathcal{P}_2(f,\sigma_0+it)|^2 \approx \sum_{Y\le p\le X} \frac{\lambda_f(p)^2}{p^{2\sigma_0}} \ll \log\left(\frac{\log X}{\log Y}\right) \ll \log\log\log T$.

Similarly, we have

$$\int_T^{2T} |\mathcal{P}_2(f,\sigma_0+it)|^2 dt \ll T \sum_{Y<p\le X} \frac{\lambda_f(p)^2}{p^{2\sigma_0}} + T \ll T \log\left(\frac{\log X}{\log Y}\right) \ll T \log\log\log T.$$

and the first assertion follows.

Suppose $K \ge 1$ is a real number. If $|z| \le K$ then, using that $k! \ge (k/e)^k$, we write

$$\begin{aligned}\sum_{0\le k\le K} \frac{z^k}{k!} = e^z + O\left(\sum_{k>100K} \frac{K^k}{k!}\right) &= e^z + O\left(\sum_{k>100K} \left(\frac{eK}{k}\right)^k\right) \\ &= e^z + O(e^{-100K}).\end{aligned}$$

Since $|z| \le K$, we may also write the right side above is $e^z(1+O(e^{-99K}))$. Take $z = -\mathcal{P}_1(f,\sigma_0+it)$ and $K = \log\log T$ and (3.6) holds. □

As we have decomposed $\mathcal{P}(f,s)$ similarly, we decompose $M(f,s)$ as $M_1(f,s)$ and $M_2(f,s)$. By the definition of $M(f,s)$ we need to decompose $a(n)$ first.

Set

$$a_1(n) = \begin{cases} 1 & \text{if } n \text{ has at most } 100\log\log T \text{ prime factors with all } p \le Y, \\ 0 & \text{otherwise .}\end{cases}$$

$$a_2(n) = \begin{cases} 1 & \text{if } n \text{ has at most } 100\log\log\log T \text{ prime factors with all } Y < p \le X, \\ 0 & \text{otherwise .}\end{cases}$$

Therefore,

$$\begin{aligned} M(f,s) &= M_1(f,s)M_2(f,s), \\ M_1(f,s) &= \sum_n \frac{\mu_f(n)a_1(n)}{n^s}, \\ M_2(f,s) &= \sum_n \frac{\mu_f(n)a_2(n)}{n^s}. \end{aligned}$$

Lemma 3.3.3. *We have*

$$\int_T^{2T} |\mathcal{M}_1(f,\sigma_0+it) - M_1(f,\sigma_0+it)|^2 dt \ll T(\log T)^{-60},$$

and

$$\int_T^{2T} |\mathcal{M}_2(f,\sigma_0+it) - M_2(f,\sigma_0+it)|^2 dt \ll T(\log\log T)^{-60}.$$

Proof. Expanding $\mathcal{M}_1(f,s)$ into Dirichlet series we write

$$\mathcal{M}_1(f,s) = \sum_n \frac{b(n)\lambda_f(n)}{n^s},$$

where $b(n)$ satisfies the following properties.

1. $|b(n)| \leq 1$ for all n.

2. $b(n) = 0$ unless $n \leq Y^{100 \log\log T}$ has only prime factors below Y.

3. $b(n) = \mu_f(n) a_1(n)$ unless $\Omega(n) > 100 \log\log T$ or, $p \leq Y$ s.t $p^k | n$ with $p^k > Y$.

Set $c(n) = b(n)\lambda_f(n) - \mu_f(n) a_1(n)$, then by (2.16) and (2.17) we have

$$\begin{aligned}
&\int_T^{2T} |\mathcal{M}_1(f, \sigma_0 + it) - M_1(f, \sigma_0 + it)|^2 dt \\
&\quad \ll T \sum_{n_1 = n_2} \frac{|c(n_1)\overline{c(n_2)}|}{(n_1 n_2)^{\sigma_0}} + \sum_{n_1 \neq n_2} \frac{|c(n_1)\overline{c(n_2)}|}{(n_1 n_2)^{\sigma_0}} \sqrt{n_1 n_2} \\
&\quad \ll T (\log T)^{-60}.
\end{aligned}$$

We note that our $a_1(n), a_2(n)$ are exactly the same as given in Radziwiłł and Soundarajan's paper [RS17]. The only difference is that instead of Möbius function we have the convolution inverse $|\mu_f(n)| \leq d(n) \ll n^\epsilon$, where $d(n)$ is the divisor function. Since we know that σ_0 is close to $\frac{1}{2}$, the off-diagonal terms (hence the denominator is negligible due to Lemma 1 of [Sel46b]) with $n_1 \neq n_2$ (by the Ramanujan-Petersson Conjecture) contribute

$$\ll \sum_{n_1 \neq n_2 \leq Y^{100 \log\log T}} (n_1 n_2)^\epsilon \ll T^\epsilon.$$

By recalling property (3) we can say that the diagonal terms (with $n_1 = n_2$) contributes

$$\ll T \sum_{\substack{p|n \implies p \leq Y \\ \Omega(n) > 100 \log\log T}} \frac{1}{n} + T \left(\sum_{\substack{p \leq Y \\ p > Y^k}} \frac{1}{p^k} \right) \left(\sum_{p|n \implies p \leq Y} \frac{1}{n} \right).$$

A small calculation shows that the second term above contribute $\ll T(\log Y)/\sqrt{Y} \ll T(\log T)^{-60}$. For the first term above, note that for $1 < r < 2$ the quantity $r^{\Omega(n) - 100 \log\log T}$ is always non-negative, in fact it is ≥ 1 for those n with $\Omega(n) > 100 \log\log T$. Therefore,

$$\begin{aligned}
T \sum_{\substack{p|n \implies p \leq Y \\ \Omega(n) > 100 \log\log T}} \frac{1}{n} &\ll T r^{-100 \log\log T} \prod_{p \leq Y} \left(1 + \frac{r}{p} + \frac{r^2}{p^2} + \cdots \right) \\
&\ll T (\log T)^{-100 \log r} (\log T)^r.
\end{aligned}$$

Choosing $r = e^{2/3}$, the above estimate becomes $\ll T(\log T)^{-60}$, completing the proof of this lemma. $\square$

Remark 3.3.4. *From the definition of $\mathcal{M}_j(f, s)$ (for $j = 1, 2$) the properties of $b(n)$ can be proved by adopting the argument of Hsu and Wong (see page 696-697 of [HW20]).*

Proof of Proposition 3.2.4: It follows from (3.6) that we have

$$\mathcal{M}_1(f,\sigma_0+it)=\exp(-\mathcal{P}_1(f,\sigma_0+it))(1+O((\log T)^{-99}))$$

and by (3.5) we have

$$(\log T)^{-1}\ll|\mathcal{M}_1(f,\sigma_0+it)|\ll\log T,$$

except on a set of measure $o(T)$, so, we can conclude that

$$\begin{aligned}M_1(f,\sigma_0+it)=&\mathcal{M}_1(f,\sigma_0+it)+O((\log T)^{-25})\\&=\exp(-\mathcal{P}_1(f,\sigma_0+it))(1+O((\log T)^{-20})).\end{aligned}$$

Similarly, except on a set of measure $o(T)$, we have

$$M_2(f,\sigma_0+it)=\exp(-\mathcal{P}_2(f,\sigma_0+it))(1+O((\log\log T)^{-20})).$$

Recalling the decomposition of $M(f,s)$ and $\mathcal{P}(f,s)$, by multiplying these estimates we obtain

$$M(f,\sigma_0+it)=\exp(-\mathcal{P}(f,\sigma_0+it))(1+O((\log\log T)^{-20})),$$

which completes the proof. $\square$

3.3.4 Proof of Proposition 3.2.5

In this section, we prove that the mollifier $M(f,s)$ and the L-function are inverse to each other. As we have followed the method established in [RS17] to prove the other propositions of this chapter, this one is an exception. We need to calculate the second mollified moment of $L(f,s)$. For that, we use the method established by Bernard [Ber15] and Hughes-Young [HY10]. First, we explain why the fourth moment of the Riemann zeta function roughly corresponds to the second moment of the $GL(2)$ L-functions, then we set our parameter to complete the proof. We give brief details of the error term calculations since it is already given in [Ber15; KRZ19].

Expanding the integration as given in Proposition 3.2.5, we get

$$\begin{aligned}&\int_T^{2T}|1-L(f,\sigma_0+it)M(f,\sigma_0+it)|^2dt\\=&\int_T^{2T}|L(f,\sigma_0+it)M(f,\sigma_0+it)|^2dt-2\int_T^{2T}\Re(L(f,\sigma_0+it)M(f,\sigma_0+it))dt+T\\=&S_1-2S_2+S_3.\end{aligned}$$

First we compute S_2. From the approximate functional equation of $L(f,s)$ we write (see (2.15)).

$$L(f,\sigma_0+it)=\sum_n\frac{\lambda_f(n)}{n^{\sigma_0+it}}V_{\sigma_0+it}\left(\frac{n}{q}\right)+\varepsilon(f,\sigma_0+it)\sum_n\frac{\bar{\lambda}_f(n)}{n^{1-\sigma_0-it}}V_{1-\sigma_0-it}(nq) \tag{3.9}$$

where $V_{\sigma_0+it}(y)$ is a smooth function defined by

$$V_{\sigma_0+it}(y)=\frac{1}{2\pi i}\int_{(3)}y^{-s}G(s)\frac{\gamma(f,\sigma_0+it+s)}{\gamma(f,\sigma_0+it)}\frac{ds}{s},$$

$\varepsilon(f)$ is the root number and

$$\varepsilon(f,\sigma_0+it)=\varepsilon(f)\frac{\gamma(f,1-\sigma_0-it)}{\gamma(f,\sigma_0+it)}.$$

For small y we can shift the contour to the left up to $\Re(s)+\sigma_0=-\delta+\epsilon$ (for small $\delta>\epsilon>0$), to get an asymptotic expansion

$$V_{\sigma_0+it}(y)=1+O(y^{\delta-\epsilon}).$$

Then implied constant depends on ϵ. For large y, we use the bound $V(y)=O_j(y^{-j})$, which holds for any $j\geq 1$. We have

$$\begin{aligned}\varepsilon(f,\sigma_0+it)=&\varepsilon(f)\frac{c_k(2\pi)^{-(1-(\sigma_0+it))}\Gamma\left(1-(\sigma_0+it)+\frac{k-1}{2}\right)}{c_k(2\pi)^{-(\sigma_0+it)}\Gamma\left(\sigma_0+it+\frac{k-1}{2}\right)}\\ =&\varepsilon(f)(2\pi)^{2\sigma_0-1}|t|^{1-2\sigma_0}e^{i(\pi/2(1-k)-2t\log\frac{|t|}{2\pi e})}(1+O(|t|^{-1}))\end{aligned}$$

is obtained by Stirling's formula. Since $\sigma_0=\frac{1}{2}+\frac{W}{\log T}$ and k is fixed, for $t\in[T,2T]$ $\varepsilon(f,\sigma_0+it)$ tends to 0 as $T\to\infty$.

Therefore,

$$\begin{aligned}S_2=&\int_T^{2T}\Re(L(f,\sigma_0+it)M(f,\sigma_0+it))dt\\ =&\sum_n\frac{\lambda_f(n)}{n^{\sigma_0}}\sum_{m<T^\epsilon}\frac{a(m)\mu_f(m)}{m^{\sigma_0}}\int_T^{2T}\Re\left(V_{\sigma_0+it}\left(\frac{n}{q}\right)(mn)^{-it}\right)dt\\ &+\sum_n\frac{\lambda_f(n)}{n^{1-\sigma_0}}\sum_{m<T^\epsilon}\frac{a(m)\mu_f(m)}{m^{1-\sigma_0}}\int_T^{2T}\Re\left(V_{1-\sigma_0+it}\left(nq\right)\varepsilon(f,\sigma_0+it)(mn)^{-it}\right)dt\\ =&T+O(T^{1-\epsilon}).\end{aligned}$$

The main term of the above equation is coming from $mn=1$. For $mn\neq 1$, note that by the definition of $a(m)$ if $m\geq T^\epsilon$ the terms in the above equation vanishes. Then we can write

$$\sum\sum_{\substack{n\leq T,m<T^\epsilon\\ mn\neq 1}}\frac{a(m)\lambda_f(n)\mu_f(m)}{(mn)^{\sigma_0}}\int_T^{2T}\Re\left(V_{\sigma_0+it}\left(\frac{n}{q}\right)(mn)^{-it}\right)dt$$

Due the oscillation the integral on the first term of S_2 is bounded by

$$\sum_{\substack{n\leq T,m<T^\epsilon\\ mn\neq 1}}\int_T^{2T}\Re\left(V_{\sigma_0+it}\left(\frac{n}{q}\right)(mn)^{-it}\right)dt\ll 1.$$

Then for $mn \neq 1$, the first term of S_2 is bounded by

$$\ll \sum_{\substack{n \leq T, m < T^{\epsilon} \\ mn \neq 1}} \sum \frac{a(m)\lambda_f(n)\mu_f(m)}{(mn)^{\frac{1}{2}}} \cdot \frac{1}{\log |mn|} \ll \sum_{\substack{n \leq T, m < T^{\epsilon} \\ mn \neq 1}} \sum \frac{1}{(mn)^{\frac{1}{2}}} \ll T^{1/2+\epsilon}.$$

First we bound the following integral using integration by parts. Then we bound the second term of S_2 for $mn \neq 1$. Let

$$f(t) = -2it \log \frac{|t|}{2\pi e}, \quad \text{then} \quad f'(t) = -2it \log \frac{|t|}{2\pi}$$
$$\text{and } dv = f'(t) e^{f(t)} \quad \text{and} \quad u = \frac{(mn)^{-it} V_{1-\sigma_0+it}(nq)}{f'(t)}.$$

Using integration by parts we can bound the following integral by

$$\begin{aligned}
&\int_T^{2T} V_{1-\sigma_0+it}(nq)(mn)^{-it} e^{-2it \log \frac{|t|}{2\pi e}} dt \\
=&\int_T^{2T} \frac{V_{1-\sigma_0+it}(nq)(mn)^{-it}(-2it \log \frac{|t|}{2\pi})}{-2it \log \frac{|t|}{2\pi}} e^{-2it \log \frac{|t|}{2\pi e}} dt \\
\ll &\left| \frac{V_{1-\sigma_0+it}(nq)(mn)^{-it}}{-2i \log \frac{|t|}{2\pi}} e^{-2it \log \frac{|t|}{2\pi e}} \right| \Bigg|_T^{2T} \\
&+ \int_T^{2T} e^{-2it \log \frac{|t|}{2\pi e}} \frac{(-2it \log \frac{|t|}{2\pi}(V'/(mn)^{it} + V'/(mn)^{2it})) + (V \frac{2i}{t} + \log(mn)/(mn)^{it})}{(mn)^{2it}(-2i \log \frac{|t|}{2\pi})^2} dt \\
\ll &\frac{1}{\log 2T} - \frac{1}{\log T} + \log mn \left(1 + \frac{nq}{T^2}\right)^{-A} \\
\ll &\log mn \left(1 + \frac{nq}{T^2}\right)^{-A},
\end{aligned} \tag{3.10}$$

where recalling that $t^j \frac{\partial^j}{\partial t^j} V_{\sigma+it}(y) = \left(1 + \frac{|y|}{t^2}\right)^{-A}$ for any real number $A > 0$, and taking $j = 1$, we bound the integral by

$$\begin{aligned}
&\int_T^{2T} e^{-2it \log \frac{|t|}{2\pi e}} \frac{(-2it \log \frac{|t|}{2\pi}(V'/(mn)^{it} + V'/(mn)^{2it})) + (V \frac{2i}{t} + \log(mn)/(mn)^{it})}{(mn)^{2it}(-2i \log \frac{|t|}{2\pi})^2} dt \\
=&\int_T^{2T} e^{-2it \log \frac{|t|}{2\pi e}} \frac{-2it \log \frac{|t|}{2\pi}(V'_{1-\sigma_0+it}(nq)/(mn)^{it} + V'_{1-\sigma_0+it}(nq)/(mn)^{2it}}{(mn)^{2it}(-2i \log \frac{|t|}{2\pi})^2} dt \\
&\qquad + \int_T^{2T} e^{-2it \log \frac{|t|}{2\pi e}} \frac{(V_{1-\sigma_0+it}(nq) \frac{2i}{t} + \log(mn)/(mn)^{it})}{(mn)^{2it}(-2i \log \frac{|t|}{2\pi})^2} dt \\
\ll &\log mn \left(1 + \frac{nq}{T^2}\right)^{-A}.
\end{aligned}$$

Therefore, for $mn \neq 1$, the integral of the second term of S_2 is

[8]Note that for the sake of simplicity of the notation we have written $V_{1-\sigma_0+it}(nq) = V$ in (3.10).

$$
\begin{aligned}
&\int_T^{2T} \Re\left(V_{1-\sigma_0+it}(nq)(mn)^{-it}\varepsilon(f,\sigma_0+it)\right)dt\\
=&\int_T^{2T} \Re\Bigg(V_{1-\sigma_0+it}(nq)\varepsilon(f)(mn)^{-it}(2\pi)^{2\sigma_0-1}|t|^{1-2\sigma_0}e^{i(\pi/2(1-k)-2t\log\frac{|t|}{2\pi e})}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1+O(|t|^{-1}))\Bigg)dt\\
\ll& T^{-\epsilon'}\int_T^{2T}\Re\left(V_{1-\sigma_0+it}(nq)(mn)^{-it}e^{-2it\log\frac{|t|}{2\pi e}}\right)dt\\
\ll& T^{-\epsilon'}\log mn\left(1+\frac{nq}{T^2}\right)^{-A}\ll T^{-\epsilon'}\log mn.
\end{aligned}
$$

Note that the term $\Re((2\pi)^{2\sigma_0-1}|t|^{1-2\sigma_0}e^{\pi i/2(1-k)})$ in the above integral is bounded by $T^{-\epsilon'}$ because $(1-2\sigma_0=-\epsilon'$ (for some $\epsilon'>0$) and $\Re(e^{\pi i/2(1-k)}))\ll 1$.

From the approximation of (3.10), and with a similar argument (like the error term of the first term of S_2) the error term of second sum of S_2 is bounded by

$$
\begin{aligned}
&\sum\sum_{\substack{n\le T, m<T^\epsilon\\ mn\neq 1}}\frac{a(m)\lambda_f(n)\mu_f(m)}{(mn)^{1-\sigma_0}}\int_T^{2T}\Re\left(V_{1-\sigma_0+it}(nq)\,\varepsilon(f,\sigma_0+it)(mn)^{-it}\right)dt\\
&\ll T^{-\epsilon'}\sum\sum_{\substack{n\le T, m<T^\epsilon\\ mn\neq 1}}\frac{a(m)\lambda_f(n)\mu_f(m)}{(mn)^{1-\sigma_0}}\log mn\\
&\ll T^{-\epsilon'}\sum\sum_{\substack{n\le T, m<T^\epsilon\\ mn\neq 1}}\frac{a(m)\lambda_f(n)\mu_f(m)}{(mn)^{1-\sigma_0-\epsilon''}}\ll\sum\sum_{\substack{n\le T, m<T^\epsilon\\ mn\neq 1}}\frac{1}{(mn)^{1-\sigma_0-\epsilon''}}\ll T^{1-\epsilon},
\end{aligned}
$$

completing the computation of S_2.

In order to prove Proposition 3.2.5, we need the mean value estimate of the L-function. Recall the functional equation of L-function (see (2.6)),

$$
\Lambda(f,s)=\varepsilon(f)\Lambda(\bar{f},1-s),
$$

where $\bar{f}$ is an object associated with f (the dual of f) for which $\lambda_{\bar{f}}(n)=\bar{\lambda}_f(n)$ and $\varepsilon(f)$ is the complex number of absolute value 1 called the root number of $L(f,s)$.

Now we explain that how are the fourth moments of the Riemann zeta function and the second moment of the $GL(2)$ L-functions correspond to each other.

We know that $\zeta^k(s)=\sum_{n=1}^{\infty}\frac{d_k(n)}{n^s}$ and $\lambda_f(n)\ll d(n)$ where $d(n)$ is the divisor function. As argued in [Hea79] we can see that to study the fourth moments of $\zeta(s)$ one needs to study the problem

$$
\sum_{n\le X} d(n)d(n+h)
$$

for suitable ranges of h. Similarly, to study the second moment of $L(f,s)$ we need to study the problem

$$\sum_{n\leq X}\lambda_f(n)\lambda_f(n+h)$$

for suitable ranges of h.

In order to prove this proposition we need compute,

$$\int_T^{2T}|L(f,\sigma_0+it)M(f,\sigma_0+it)|^2. \tag{3.11}$$

As mentioned in Section 2 of [You10], to simplify the upcoming arguments, we smooth the integral in (3.11).

Let $w(t)$ be a smooth function satisfy the following properties:

1. $0\leq w\leq 1$ for all $t\in\mathbb{R}$.
2. w has compact support in $[T/4,2T]$.
3. $w^{(j)}\ll_j T_0^{-j}$ for each $j=0,1,2\ldots$ where $T_0=T/\log T$.

Then we reduce to study the following problem

$$I(\alpha,\beta)=\sum_{h,k}\frac{\mu_f(h)\mu_f(k)a(h)a(k)}{h^{1/2+\alpha}k^{1/2+\beta}}\int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it}w(t)L(\frac{1}{2}+\alpha+it)L(\frac{1}{2}+\beta-it)dt$$

where $\alpha,\beta=\frac{W}{\log T}$.

Lemma 3.3.5. *Let G be any entire function which decays rapidly in vertical strips, even and normalized by $G(0)=1$. Then for α,β equals to $\frac{W}{\log T}$, we have*

$$\begin{aligned}L(f,\frac{1}{2}+\alpha+it)L(f,\frac{1}{2}+\beta-it)=&\sum_{m,n\geq 1}\frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha}n^{1/2+\beta}}\left(\frac{m}{n}\right)^{-it}V_{\alpha,\beta}(mn,t)\\&+X_{\alpha,\beta,t}\sum_{m,n\geq 1}\frac{\lambda_f(m)\lambda_f(n)}{m^{1/2-\alpha}n^{1/2-\beta}}\left(\frac{m}{n}\right)^{-it}V_{-\beta,-\alpha}(mn,t)\end{aligned}$$

where

$$\begin{aligned}g_{\alpha,\beta}(s,t)&=\frac{\gamma(f,\frac{1}{2}+\alpha+s+it)\gamma(f,\frac{1}{2}+\beta+s-it)}{\gamma(f,\frac{1}{2}+\alpha+it)\gamma(f,\frac{1}{2}+\beta-it)},\\V_{\alpha,\beta}(x,t)&=\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}g_{\alpha,\beta}(s,t)x^{-s}ds\end{aligned}$$

and

$$X_{\alpha,\beta,t}=\frac{\gamma(f,\frac{1}{2}-\alpha-it)\gamma(f,\frac{1}{2}-\beta+it)}{\gamma(f,\frac{1}{2}+\alpha+it)\gamma(f,\frac{1}{2}+\beta-it)}.$$

The proof of this lemma is the same as Lemma 1 of [Ber15] or see Theorem 5.3 of [IK04].

We start with the approximation of $X_{\alpha,\beta,t}$. By Stirling's approximation, we can write

$$\begin{aligned} X_{\alpha,\beta,t} &= \left(\frac{\sqrt{q}}{2\pi}\right)^{-2(\alpha+\beta)} \frac{\Gamma(k/2-\alpha-it)\Gamma(k/2-\beta+it)}{\Gamma(k/2+\alpha+it)\Gamma(k/2+\beta-it)} \\ &= \left(\frac{t\sqrt{q}}{2\pi}\right)^{-2(\alpha+\beta)} \left(1+\frac{i(\alpha^2-\beta^2)}{t}+O\left(\frac{1}{t^2}\right)\right). \end{aligned}$$

and

$$\begin{aligned} g_{\alpha,\beta}(s,t) &= \left(\frac{\sqrt{q}}{2\pi}\right)^{2s} \frac{\Gamma(k/2+\alpha+s+it)\Gamma(k/2+\beta+s-it)}{\Gamma(k/2+\alpha+it)\Gamma(k/2+\beta-it)} \\ &= \left(\frac{t\sqrt{q}}{2\pi}\right)^{2s} \left(1+O\left(\frac{|s|^2}{t}\right)\right). \end{aligned}$$

Additionally, for each integer $j \geq 0$ and for all real number $A > 0$, we have

$$t^j \frac{\partial^j}{\partial t^j} V_{\alpha,\beta}(x,t) \ll A_j \left(1+\frac{|x|}{t^2}\right)^{-A}$$

where $G(s)$ be an even entire function with rapid decay as $|s| \to \infty$.

Note that the Fourier transformation of w we have $\hat{w}(0) = T/2 + O(T_0)$. Suppose that $w(t)$ satisfies the conditions (1),(2) and (3). By Lemma 3.3.5 we have

$$\begin{aligned} I(h,k) &= \int_{-\infty}^{\infty} w(t) \left(\frac{h}{k}\right)^{-it} L(f,\frac{1}{2}+\alpha+it)L(f,\frac{1}{2}+\beta-it)dt \qquad (3.12) \\ &= \sum_{hm=kn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha}n^{1/2+\beta}} \int_{-\infty}^{\infty} w(t)V_{\alpha,\beta}(mn,t)dt \\ &+ \sum_{hm=kn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2-\alpha}n^{1/2-\beta}} \int_{-\infty}^{\infty} w(t)V_{-\beta,-\alpha}(mn,t)X_{\alpha,\beta,t}dt \\ &+ \sum_{hm\neq kn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha}n^{1/2+\beta}} \int_{-\infty}^{\infty} \left(\frac{hm}{kn}\right)^{-it} w(t)V_{\alpha,\beta}(mn,t)dt \\ &+ \sum_{hm\neq kn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2-\alpha}n^{1/2-\beta}} \int_{-\infty}^{\infty} \left(\frac{hm}{kn}\right)^{-it} w(t)V_{-\beta,-\alpha}(mn,t)X_{\alpha,\beta,t}dt \\ &= I_D^{(1)}(h,k) + I_D^{(2)}(h,k) + I_O^{(1)}(h,k) + I_O^{(2)}(h,k). \end{aligned}$$

First, we show the diagonal term calculation then we give a brief overview of the off-diagonal term calculations.

$$I_D(\alpha,\beta) = \sum_{h,k} \frac{\mu_f(h)\mu_f(k)a(h)a(k)}{h^{1/2+\alpha}k^{1/2+\beta}}[I_D^{(1)}(h,k) + I_D^{(2)}(h,k)] \qquad (3.13)$$

In order to simplify, we set

$$\mathcal{S} = \sum_{hm=kn} \frac{\lambda_f(m)\lambda_f(n)\mu_f(h)\mu_f(k)a(h)a(k)}{m^{1/2+\alpha}n^{1/2+\beta}h^{1/2+\alpha}k^{1/2+\beta}}.$$

Choose $h = h_1h_2$, where h_1 is composed only on primes below Y and h_2 is composed only on primes between Y and X. Then $a(h) = a_1(h_1)a_2(h_2)$ with the same notation of Section 3.3.3. Similarly we can write $a(k) = a_1(k_1)a_2(k_2)$ In a similar manner we set $m = m_1m_2$ and $n = n_1n_2$.

We start by dividing the summation

$$\mathcal{S} = \sum_{h_1m_1=k_1n_1} \frac{\lambda_f(m_1)\lambda_f(n_1)\mu_f(h_1)\mu_f(k_1)a(h_1)a(k_1)}{m_1{}^{1/2+\alpha}n_1{}^{1/2+\beta}h_1{}^{1/2+\alpha}k_1{}^{1/2+\beta}} \times \sum_{h_2m_2=k_2n_2} \frac{\lambda_f(m_2)\lambda_f(n_2)\mu_f(h_2)\mu_f(k_2)a(h_2)a(k_2)}{m_2{}^{1/2+\alpha}n_2{}^{1/2+\beta}h_2{}^{1/2+\alpha}k_2{}^{1/2+\beta}}.$$

Consider the first factor of the above equation. If we ignore the fact that h_1 and k_1 must have $100\log\log T$ prime factors, then the resulting sum is simply

$$\sum_{\substack{h_1m_1=k_1n_1 \\ p|h_1k_1m_1n_1 \implies p\le Y}} \frac{\lambda_f(m_1)\lambda_f(n_1)\mu_f(h_1)\mu_f(k_1)}{m_1{}^{1/2+\alpha}n_1{}^{1/2+\beta}h_1{}^{1/2+\alpha}k_1{}^{1/2+\beta}}$$
$$= \prod_{p\le Y} L_p(f\times f, 1+\alpha+\beta)\left(1+\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta}} - \frac{\lambda_f(p)^2}{p^{1+2\alpha}} - \frac{\lambda_f(p)^2}{p^{1+2\beta}} + \psi_0 E_p\right)$$

Let us assume the above relation for now and we will establish this later[9]. Moving on to approximating the first factor by the product above, we incur an error term which is at most

$$\ll \sum_{\substack{h_1m_1=k_1n_1 \\ p|h_1k_1m_1n_1 \implies p\le Y \\ \Omega(h_1)>100\log\log T}} \frac{|\lambda_f(m_1)\lambda_f(n_1)\mu_f(h_1)\mu_f(k_1)|}{m_1{}^{1/2+\alpha}n_1{}^{1/2+\beta}h_1{}^{1/2+\alpha}k_1{}^{1/2+\beta}},$$

where we used symmetry to assume that h_1 has many prime factors. For $\Omega(h_1) \ge 100\log\log T$ the factor $e^{\Omega(h_1)-100\log\log T} \ge 1$ and non-negative. Then the above sum may be bounded by

$$\ll e^{-100\log\log T} \sum_{\substack{h_1m_1=k_1n_1 \\ p|h_1k_1m_1n_1 \implies p\le Y \\ \Omega(h_1)>100\log\log T}} \frac{|\lambda_f(m_1)\lambda_f(n_1)\mu_f(h_1)\mu_f(k_1)|}{m_1{}^{1/2+\alpha}n_1{}^{1/2+\beta}h_1{}^{1/2+\alpha}k_1{}^{1/2+\beta}} e^{\Omega(h_1)}$$
$$\ll e^{-100\log\log T} \sum_{\substack{h_1m_1=k_1n_1 \\ p|h_1k_1m_1n_1 \implies p\le Y \\ \Omega(h_1)>100\log\log T}} \frac{d(m_1)d(n_1)d(h_1)d(k_1)}{m_1{}^{1/2+\alpha}n_1{}^{1/2+\beta}h_1{}^{1/2+\alpha}k_1{}^{1/2+\beta}}, e^{\Omega(h_1)}$$

[9] We have defined E_p later.

where $d(n)$ is the divisor function. Observe that $k_1|h_1m_1$ and $n_1|h_1m_1$. Then $d(k_1) \leq d(h_1m_1)$ and $d(n_1) \leq d(h_1m_1)$. Let $\ell = m_1h_1$, then $m_1|\ell$ and $h_1|\ell$. Therefore, $d(m_1)d(n_1)d(h_1)d(k_1) \ll d^5(\ell)$. Note that $e^{\Omega(h_1)} \leq e^{\Omega(\ell)}$. Then for $\alpha = \beta$ the above equation is bounded by

$$\begin{aligned}
&\ll (\log T)^{-100} \sum_{\substack{\ell \implies p \leq Y \\ \Omega(h_1) > 100 \log\log T}} \frac{d^5(\ell)}{\ell^{2+4\alpha}} e^{\Omega(\ell)} \\
&\ll (\log T)^{-100} \prod_{p \leq Y} \left(1 + \frac{32e}{p}\right) \ll (\log T)^{-13}.
\end{aligned}$$

Similarly, one can obtain that the error term contribution of second factor of $\mathcal{S}$ is bounded by

$$\ll (\log\log T)^{-100} \sum_{\substack{h_2m_2 = k_2n_2 \\ p | h_2k_2m_2n_2 \implies Y < p \leq X \\ \Omega(h_2) > 100 \log\log\log T}} \frac{|\lambda_f(m_2)\lambda_f(n_2)\mu_f(h_2)\mu_f(k_2)|}{{m_2}^{1/2+\alpha}{n_2}^{1/2+\beta}{h_2}^{1/2+\alpha}{k_2}^{1/2+\beta}} e^{\Omega(h_2)} \ll (\log\log T)^{-13}.$$

Then we can write $\mathcal{S}$ as

$$\begin{aligned}
\mathcal{S} = &\left(\sum_{\substack{h_1m_1 = k_1n_1 \\ p | h_1m_1k_1n_1}} \frac{\lambda_f(m_1)\lambda_f(n_1)\mu_f(h_1)\mu_f(k_1)}{{m_1}^{1/2+\alpha}{n_1}^{1/2+\beta}{h_1}^{1/2+\alpha}{k_1}^{1/2+\beta}} + O\left((\log T)^{-13}\right) \right) \\
&\times \left(\sum_{\substack{h_2m_2 = k_2n_2 \\ p | h_2m_2k_2n_2}} \frac{\lambda_f(m_2)\lambda_f(n_2)\mu_f(h_2)\mu_f(k_2)}{{m_2}^{1/2+\alpha}{n_2}^{1/2+\beta}{h_2}^{1/2+\alpha}{k_2}^{1/2+\beta}} + O\left((\log\log T)^{-13}\right) \right).
\end{aligned}$$

Using (2.8), for any prime p such that $p \nmid q$, we have (see (22), (23) of [Ber15])

$$\sum_{\ell \geq 0} \frac{\lambda_f(p^\ell)\lambda_f(p^{(\ell+1)})}{p^{\ell s}} = \lambda_f(p) \left(1 + \frac{1}{p^s}\right)^{-1} \sum_{\ell \geq 0} \frac{\lambda_f(p^\ell)^2}{p^{\ell s}} \tag{3.14}$$

$$\sum_{\ell \geq 0} \frac{\lambda_f(p^\ell)\lambda_f(p^{(\ell+2)})}{p^{\ell s}} = \left(1 + \frac{1}{p^s}\right)^{-1} \left(\lambda_f(p^2) - \frac{1}{p^s}\right) \sum_{\ell \geq 0} \frac{\lambda_f(p^\ell)^2}{p^{\ell s}}. \tag{3.15}$$

Writing the Euler product for the first term of $\mathcal{S}$, we get

$$\prod_{p \leq Y} \left(\sum_{\substack{\ell_1, \ell_2, \ell_3, \ell_4 \geq 0 \\ \ell_1 + \ell_3 = \ell_2 + \ell_4}} \frac{\mu_f(p^{\ell_1})\mu_f(p^{\ell_2})\lambda_f(p^{\ell_3})\lambda_f(p^{\ell_4})}{p^{\ell_1(1/2+\alpha)}p^{\ell_2(1/2+\beta)}p^{\ell_3(1/2+\alpha+s)}p^{\ell_4(1/2+\beta+s)}} \right)$$

By (2.8) the above equation becomes

$$\prod_{\substack{p\le Y\\ p\nmid q}}\Bigg(\left(1+\frac{\lambda_f(p)^2}{p}+\frac{1}{p^2}\right)\sum_{\ell\ge 0}\frac{\lambda_f(p^\ell)^2}{p^{\ell(1+\alpha+\beta+2s)}}$$
$$-\lambda_f(p)\left(\frac{1}{p^{1+\beta+s}}+\frac{1}{p^{1+\alpha+s}}\right)\left(1+\frac{1}{p}\right)\sum_{\ell\ge 0}\frac{\lambda_f(p^\ell)\lambda_f(p^{\ell+1})}{p^{\ell(1+\alpha+\beta+2s)}}$$
$$+\left(\frac{1}{p^{2(1+\beta+s)}}+\frac{1}{p^{2(1+\alpha+s)}}\right)\sum_{\ell\ge 0}\frac{\lambda_f(p^\ell)\lambda_f(p^{\ell+2})}{p^{\ell(1+\alpha+\beta+2s)}}\Bigg)$$
$$\times\prod_{\substack{p\le Y\\ p\mid q}}\left(\left(1+\frac{\lambda_f(p^2)}{p}-\frac{\lambda_f(p^2)}{p^{1+\alpha+s}}-\frac{\lambda_f(p^2)}{p^{1+\beta+s}}\right)\sum_{\ell\ge 0}\frac{\lambda_f(p^\ell)^2}{p^{\ell(1+\alpha+\beta+2s)}}\right). \tag{3.16}$$

For the second summation of $\mathcal{S}$, we get a similar equation like (3.16) for $Y < p \le X$.

By (3.14), (3.15) and (3.16) we write

$$L(f\times f,1+\alpha+\beta+2s)\prod_{p\mid q}\left(\left(1+\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta}}-\frac{\lambda_f(p)^2}{p^{1+2\alpha+s}}-\frac{\lambda_f(p)^2}{p^{1+2\beta+s}}\right)\right)$$
$$\times\prod_{p\nmid q}\Bigg(\left(1+\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta}}+\frac{1}{p^{2(1+\alpha+\beta)}}\right)\left(1-\frac{1}{p^{2(1+\alpha+\beta+2s)}}\right)-\lambda_f(p)^2\left(1-\frac{1}{p^{1+\alpha+\beta+2s}}\right)$$
$$\left(\frac{1}{p^{1+2\beta+s}}+\frac{1}{p^{1+2\alpha+s}}\right)\left(1+\frac{1}{p^{1+\alpha+\beta}}\right)+\left(1-\frac{1}{p^{1+\alpha+\beta+2s}}\right)\left(\lambda_f(p^2)-\frac{1}{p^{1+\alpha+\beta+2s}}\right)$$
$$\left(\frac{1}{p^{2(1+2\beta+s)}}+\frac{1}{p^{2(1+2\alpha+s)}}\right)\Bigg)$$
$$=L(f\times f,1+\alpha+\beta+2s)\prod_{p}\left(1+\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta}}-\frac{\lambda_f(p)^2}{p^{1+2\alpha+s}}-\frac{\lambda_f(p)^2}{p^{1+2\beta+s}}+\psi_0E_p\right)$$
$$\prod_{p>X}\left(\left(1-\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta+2s}}+\psi_0(p)E_p'\right)\left(1+\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta}}-\frac{\lambda_f(p)^2}{p^{1+2\alpha+s}}-\frac{\lambda_f(p)^2}{p^{1+2\beta+s}}+\psi_0E_p\right)^{-1}\right).$$

where ψ_0 is the trivial character modulo q and

$$E_p=\frac{1}{p^2}\Bigg(\frac{1}{p^{2(\alpha+\beta)}}-\frac{1}{p^{2(\alpha+\beta+2s)}}-\frac{\lambda_f(p)^2}{p^s}\left(\frac{1}{p^{2\alpha}}+\frac{1}{p^{2\beta}}\right)\left(\frac{1}{p^{\alpha+\beta}}-\frac{1}{p^{\alpha+\beta+2s}}\right)$$
$$+\frac{\lambda_f(p^2)}{p^{2s}}\left(\frac{1}{p^{4\alpha}}+\frac{1}{p^{4\beta}}\right)\Bigg)+\frac{\lambda_f(p)^2}{p^{3+\alpha+\beta+3s}}\left(\frac{1}{p^{\alpha+\beta}}\left(\frac{1}{p^{2\alpha}}+\frac{1}{p^{2\beta}}\right)-\frac{1}{p^s}\left(\frac{1}{p^{4\alpha}}+\frac{1}{p^{4\beta}}\right)-\frac{1}{p^{2\alpha+2\beta+s}}\right)$$
$$+\frac{1}{p^{4+2(\alpha+\beta+2s)}}\left(\frac{1}{p^{2(2\alpha+s)}}+\frac{1}{p^{2(2\beta+s)}}-\frac{1}{p^{2(\alpha+\beta)}}\right)$$

and

$$E'_p = \frac{2+\lambda_f(p)^2}{p^{2(1+\alpha+\beta+2s)}} - \frac{\lambda_f(p)^2}{p^{3(1+\alpha+\beta+2s)}} + \frac{1}{p^{4(1+\alpha+\beta+2s)}}.$$

The Rankin-Selberg L-function $L(f\times f,s)$ has the Euler product

$$L(f\times f,s) = \prod_p L_p(f\times f,s)$$

where

$$L_p(f\times f,s) = \left(1 - \frac{\lambda_f(p)^2}{p^s} + \psi_0(p)\left(\frac{2+\lambda_f(p)^2}{p^{2s}} - \frac{\lambda_f(p)^2}{p^{3s}} + \frac{1}{p^{4s}}\right)\right),$$

for $\Re(s) > 1$.

Lemma 3.3.6. *For* $\alpha=\beta=\frac{W}{\log T}$ *and* $\Re(s)=\frac{1}{\log T}$,

$$\prod_{p>X}\left(\left(1-\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta+2s}}+\psi_0(p)E'_p\right)\left(1+\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta}}-\frac{\lambda_f(p)^2}{p^{1+2\alpha+s}}-\frac{\lambda_f(p)^2}{p^{1+2\beta+s}}+\psi_0E_p\right)^{-1}\right)=1+o(1).$$

Proof. Note that

$$\begin{aligned}\left(1+\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta}}-\frac{\lambda_f(p)^2}{p^{1+2\alpha+s}}-\frac{\lambda_f(p)^2}{p^{1+2\beta+s}}+\psi_0E_p\right)^{-1} =& 1-\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta}}+\frac{\lambda_f(p)^2}{p^{1+2\alpha+s}}+\frac{\lambda_f(p)^2}{p^{1+2\beta+s}}-\psi_0E_p \\ &+O\left(\frac{1}{p^2}\right) \\ =& 1-\frac{\lambda_f(p)^2}{p^{1+\alpha+\beta}}+\frac{\lambda_f(p)^2}{p^{1+2\alpha+s}}+\frac{\lambda_f(p)^2}{p^{1+2\beta+s}}+O\left(\frac{1}{p^2}\right).\end{aligned}$$

The Sato-Tate conjecture [Bar+11] implies that (see (1.18), (1.19) of [Ran85])

$$\sum_{p\le x}|\lambda_f(p)|^{2\varphi} \sim \theta_\varphi \frac{x}{\log x} \text{ where } \theta_\varphi = \frac{4^\varphi\Gamma\left(\varphi+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\varphi+2)}.$$

Putting $\varphi=1$, we get $\theta_\varphi = 1$.

We have to compute the contribution of $\exp\left(\sum_{p>X}\frac{\lambda_f(p)^2}{p^{1+\alpha}}\right)$. For the dyadic interval $[P,2P]$ we write

$$\sum_{\substack{P<p\le 2P\\ P>X}}\frac{\lambda_f(p)^2}{p^{1+\alpha}} \ll \frac{1}{P^{1+\alpha}}\sum_{\substack{P<p\le 2P\\ P>X}}\lambda_f(p)^2 \ll \frac{1}{P^{1+\alpha}}\sum_{\substack{p\le 2P\\ P>X}}\lambda_f(p)^2 \ll \frac{2P}{P^{1+\alpha}\log(2P)} \ll \frac{1}{P^\alpha\log P}.$$

Adding the dyadic intervals we obtain the bound

$$\sum_{p>X}\frac{\lambda_f(p)^2}{p^{1+\alpha}} \ll \frac{1}{X^\alpha(\log X)}.$$

Recall that $\alpha = \beta = \frac{W}{\log T}$ and $X = T^{1/(\log\log\log T)^2}$ and $W = (\log\log\log T)^4$. By definition, $\alpha > 0$ because W is a small power of T[10]. Therefore, the above series converges for the choice of the parameter α, but the above relation does not hold for general cases, such as $\alpha = 0$.

Then

$$\begin{aligned} X^{\alpha}(\log X) &= \log X \cdot \exp(\alpha \log X) \\ &= \log T \exp\left(\frac{W}{(\log\log\log T)^2}\right) \\ &= \log T \exp\left(\frac{(\log\log\log T)^4}{(\log\log\log T)^2}\right) \\ &= \log T \exp((\log\log\log T)^2). \end{aligned}$$

Hence, $\exp\left(\sum_{p>X} \frac{\lambda_f(p)^2}{p^{1+\alpha}}\right) \ll \exp\left(\frac{1}{X^{\alpha}(\log X)}\right) = 1 + o(1)$ as $T \to \infty$.

For the terms involving p^s, if, $\Re(s) = \frac{1}{\log T}$, then $p^{\frac{1}{\log T}} = \exp\left(\frac{\log p}{\log T}\right)$. Recall that $p \sim X$ and $X = T^{o(1)}$. Hence, $p^{\frac{1}{\log T}} = \exp(o(1))$, completing the proof of the lemma. $\square$

We write

$$\begin{aligned} \left(1 + \frac{\lambda_f(p)^2}{p^{\alpha+\beta}} - \frac{\lambda_f(p)^2}{p^{1+2\alpha+s}} - \frac{\lambda_f(p)^2}{p^{1+2\beta+s}} + \psi_0 E_p\right) &= \frac{L_p(f \times f, \alpha+\beta)}{L_p(f \times f, 1+2\alpha+s) L_p(f \times f, 1+2\beta+s)} \\ &\times \left(1 + L_p(f \times f, 1+2\alpha+s) L_p(f \times f, 1+2\beta+s) \sum_{r=2}^{4} \sum_{\ell} \frac{a_{r,\ell}(p)}{p^{r+Z_{r,\ell}}(\alpha,\beta,s)}\right) \end{aligned}$$

where the sum over ℓ is finite, $Z_{r,\ell}$ are linear forms in α, β, s and $a_{r,\ell}(p)$ are complex numbers with $|a_{r,\ell}(p)| \ll 1$. Then we set

$$A_{\alpha,\beta}(s) = \prod_p \left[1 + \sum_{r,\ell} O\left(\frac{1}{p^{r+Z_{r,\ell}(\Re\alpha,\Re\beta,\Re s)}}\right)\right].$$

Then $A_{\alpha,\beta}(s)$ is an absolutely convergent Euler product in $\{\Re(\alpha+s) > -1\} \cap \{\Re(\beta+s) > -1\} \cap_{r,\ell} \{Z_{r,\ell}(\Re(\alpha), \Re(\beta), \Re(s)) > 1-r\}$.

Therefore,

$$\mathcal{S} \sim \frac{L(f \times f, 1+\alpha+\beta+2s) L(f \times f, 1+\alpha+\beta)}{L(f \times f, 1+2\alpha+s) L(f \times f, 1+2\beta+s)} A_{\alpha,\beta}(s)$$

Lemma 3.3.7. *For $\alpha = \beta$, we have*

$$A_{\alpha,\beta}(0) = 1.$$

[10]The choice of the parameter α comes from the choice of σ_0. As described in Remark 3.2.2, to take away the problem from the critical line, α must be greater than 0 and not equal to 0.

Proof. Recall the definition of $\mathcal{S}$. For $\alpha=\beta$, we write

$$A_{\alpha,\beta}(s)=\sum_{hm=kn}\frac{\mu_f(h)\mu_f(k)\lambda_f(m)\lambda_f(n)}{(hm)^{1/2+\alpha+s}(kn)^{1/2+\beta+s}}=\sum_{h,m}\frac{\mu_f(h)\lambda_f(m)}{(hm)^{(1+2\alpha+2s)}}\sum_{n|hm}\mu_f(hm/n)\lambda_f(n).$$

Since $\big(\mu_f(n)\big)$ is the convolution inverse of $\big(\lambda_f(n)\big)$, we have

$$\sum_{n|d}\mu_f(n/d)\lambda_f(n)=\delta(d).$$

Thus

$$A_{\alpha,\beta}(s)=\mu_f(1)\lambda_f(1)=1.$$

To conclude, we extend this relation to $s=0$ by continuity in the half plane $\Re(s)>0$.

$\square$

The diagonal term Calculations: From Lemma 3.3.5 and the first term of (3.13) we can write

$$\begin{aligned}I_D^{(1)}(\alpha,\beta)&=\sum_{hm=kn}\frac{\lambda_f(m)\lambda_f(n)\mu_f(h)\mu_f(k)a(h)a(k)}{m^{1/2+\alpha}n^{1/2+\beta}h^{1/2+\alpha}k^{1/2+\beta}}\int_{-\infty}^{\infty}w(t)V_{\alpha,\beta}(mn,t)dt\\&=\int_{\mathbb{R}}w(t)\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\sum_{hm=kn}\frac{\lambda_f(m)\lambda_f(n)\mu_f(h)\mu_f(k)a(h)a(k)}{m^{1/2+\alpha+s}n^{1/2+\beta+s}h^{1/2+\alpha}k^{1/2+\beta}}dsdt.\end{aligned}$$

We have $-\frac{L'(f\times f,s)}{L(f\times f,s)}=\sum_{n\geq 1}\frac{\Lambda_f(n)}{n^s}$ with $\Lambda_f(n)\geq 0$, we deduce (see part 5.3 of [IK04])

$$\frac{1}{L(f\times f,\sigma+it)}\ll \ln|t|.$$

By the standard zero-free region of $L(f\times f,s)$, we move the line of integration from $\Re(s)=1$ to $\Re(s)=-(\alpha+\beta)+\frac{1}{\log T}$ crossing a pole at $s=0$ and a pole at $s=-\frac{\alpha+\beta}{2}$. By the choice of our parameter $\alpha=\beta$ and the simplification of $\mathcal{S}$ and $t\in[T,2T]$ and $g_{\alpha,\beta}\ll T^{2s}$, we can bound

$$\begin{aligned}&\int_{\mathbb{R}}w(t)\frac{1}{2\pi i}\int_{1/\left(-(\alpha+\beta)+\frac{1}{\log T}\right)}\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\sum_{hm=kn}\frac{\lambda_f(m)\lambda_f(n)\mu_f(h)\mu_f(k)a(h)a(k)}{m^{1/2+\alpha}n^{1/2+\beta}h^{1/2+\alpha}k^{1/2+\beta}}dsdt\\&=\int_{\mathbb{R}}w(t)\frac{1}{2\pi i}\int_{1/\left(-(\alpha+\beta)+\frac{1}{\log T}\right)}\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\\&\qquad\qquad\frac{L(f\times f,1+\alpha+\beta+2s)L(f\times f,1+\alpha+\beta)}{L(f\times f,1+2\alpha+s)L(f\times f,1+2\beta+s)}A_{\alpha,\beta}(s)dsdt\\&\ll\int_{\mathbb{R}}|w(t)|dtT^{2(-(\alpha+\beta)+1/\log T)}\ll T^{1-o(1)}.\end{aligned}$$

By our choice of parameters $\alpha+\beta \neq 0$. We specialize

$$G(s) = e^{s^2}\frac{(\alpha+\beta)^2-(2s)^2}{(\alpha+\beta)^2}$$

to ensure that $G(-\frac{\alpha+\beta}{2}) = 0$.

Our next step is to check the pole contribution at $s = 0$. Since $G(0) = 1$, by Lemma 3.3.7 we have

$$\begin{aligned}
\int_{\mathbb{R}} w(t)\mathrm{Res}_{s=0}\left[\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\frac{L(f\times f,1+\alpha+\beta+2s)L(f\times f,1+\alpha+\beta)}{L(f\times f,1+2\alpha+s)L(f\times f,1+2\beta+s)}A_{\alpha,\beta}(s)\right]dt \\
+O(T^{1-o(1)})
\end{aligned}$$
$$=\frac{L(f\times f,1+\alpha+\beta)L(f\times f,1+\alpha+\beta)}{L(f\times f,1+2\alpha)L(f\times f,1+2\beta)}\hat{w}(0)+O(T^{1-o(1)}).$$

We have the diagonal term contribution given by

$$I_D^{(1)}(\alpha,\beta) = \hat{w}(0)\frac{L(f\times f,1+\alpha+\beta)L(f\times f,1+\alpha+\beta)}{L(f\times f,1+2\alpha)L(f\times f,1+2\beta)}+O(T^{1-o(1)})$$

The pole $s = 0$ is called the simple pole case which contributes in the main term. For $s=-\frac{\alpha+\beta}{2}$ we have the double pole case. The pole at $s=-\frac{\alpha+\beta}{2}$ gives

$$P_{\alpha,\beta}^{(1)} = \mathrm{Res}_{s=-\frac{\alpha+\beta}{2}}\left[\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\frac{L(f\times f,1+\alpha+\beta+2s)L(f\times f,1+\alpha+\beta)}{L(f\times f,1+2\alpha+s)L(f\times f,1+2\beta+s)}A_{\alpha,\beta}(s)\right]$$

By our choice of G and the fact that $\alpha=\beta$, $P_{\alpha,\beta}^{(1)}$ does not contribute to $I_D^{(1)}(\alpha,\beta)$.

A similar calculation gives

$$\begin{aligned}
I_D^{(2)}(\alpha,\beta) =& -T^{-2(\alpha+\beta)}I_D^{(1)}(\alpha,\beta)+O(T^{1-o(1)}) \\
=& -T^{-2(\alpha+\beta)}\frac{L(f\times f,1+\alpha+\beta)L(f\times f,1+\alpha+\beta)}{L(f\times f,1+2\alpha)L(f\times f,1+2\beta)}\hat{w}(0)+O(T^{1-o(1)}).
\end{aligned}$$

With a similar argument $P_{\alpha,\beta}^{(2)}$ does not contribute in the main term. So the diagonal term contribution gives

$$I_D(\alpha,\beta) = \hat{w}(0)\frac{L(f\times f,1+\alpha+\beta)L(f\times f,1+\alpha+\beta)}{L(f\times f,1+2\alpha)L(f\times f,1+2\beta)}(1-T^{-2(\alpha+\beta)})+O(T^{1-o(1)}).$$

Recall that by the choice of our parameter $\alpha=\beta$, hence the fraction involving Rankin-Selberg L-functions cancel out. As $T\to\infty$, we conclude

$$I_D(\alpha,\beta)\sim\hat{w}(0)+O(T^{1-o(1)}).$$

In the next step we use the smoothing argument (see section 2 of [You10]) to complete the proof of the proposition. By choosing the smooth function w which satisfies (1),(2),(3), and additionally to be an upper bound for the characteristic function of the interval $[3T/2, 2T]$, with support in $[3T/2 - T_0, 2T + T_0]$, we get

$$\int_{3T/2}^{2T} |L(f, \sigma_0 + it)M(f, \sigma_0 + it)|^2 \leq \hat{w}(0) + O(T^{1-o(1)}).$$

Note that $\hat{w}(0) = T/2 + O(T_0)$. Similarly, we get a lower bound. By summing over the dyadic segments we conclude

$$\frac{1}{T}\int_{T}^{2T} |L(f, \sigma_0 + it)M(f, \sigma_0 + it)|^2 = 1 + o(1).$$

Next we briefly explain the off-diagonal term computation.

The off-diagonal term Calculations: In this paragraph we give an overview of the off-diagonal term calculations, since it is already given in [Ber15; KRZ19].

Lemma 3.3.8. *Let $\epsilon > 0$, $0 < \gamma < 1$, $\alpha, \beta = \frac{W}{\log T}$ and $h, k \leq T^{\nu}$ be positive integers. Then for all real number $A > 0$ we have*

$$I_O^{(1)}(h,k) = \sum_{\substack{hm \neq kn \\ mn \ll T^{2+\epsilon} \\ \left|\frac{hm}{kn} - 1\right| \ll T^{\gamma}}} \frac{\lambda_f(m)\lambda_f(n)}{m^{\frac{1}{2}+\alpha} n^{\frac{1}{2}+\beta}} \int_{-\infty}^{\infty} w(t) \left(\frac{hm}{kn}\right)^{-it} V_{\alpha,\beta}(mn,t)dt + O(T^{-A}).$$

For the proof of this lemma see Lemma 3 of [Ber15].

Remark 3.3.9. *Note that the choice of parameters $\alpha, \beta \ll 1/\log T$ in [Ber15]. The proof still holds for our choice of parameters α, β because the properties used in the proof of Lemma 3 and Lemma 4 of [Ber15] is $0 \leq |\Re(\alpha)|, |\Re(\beta)| \leq \frac{1}{2}$, which is satisfied by the choice of our parameters.*

Let us fix an arbitrary smooth function $\rho :]0, \infty[\to \mathbb{R}$ which is compactly supported in $[1, 2]$ and satisfies

$$\sum_{\ell=-\infty}^{\infty} \rho\left(2^{-\ell/2}x\right) = 1$$

To build such a function see section 5 of [Har03]. For each integer ℓ we define

$$\rho_\ell(x) = \rho\left(\frac{x}{A_\ell}\right) \text{ with } A_\ell = 2^{\ell/2}T^{\gamma}.$$

Then by Lemma 4 of [Ber15], one has

$$I_O^{(1)}(h,k) = \sum_{\substack{A_{\ell_1}A_{\ell_1} \ll hkT^{2+\epsilon} \\ A_{\ell_1} \asymp A_{\ell_2} \\ A_{\ell_1}, A_{\ell_2} \gg T^{\gamma}}} \sum_{0 < |h| \ll T^{-\gamma}\sqrt{A_{\ell_1}A_{\ell_2}}} \sum_{hm - kn = r} \lambda_f(m)\lambda_f(n) F_{r;\ell_1,\ell_2}(hm, kn) + O(T^{-A})$$

where

$$F_{r;\ell_1,\ell_2}(x,y) = \frac{h^{\frac{1}{2}+\alpha}k^{\frac{1}{2}+\beta}}{x^{\frac{1}{2}+\alpha}y^{\frac{1}{2}+\beta}} \int_{-\infty}^{\infty} w(t)\left(1+\frac{r}{y}\right)^{-it} V_{\alpha,\beta}\left(\frac{xy}{hk},t\right) dt \times \rho_{\ell_1}(x)\rho_{\ell_2}(y)$$

with positive integers $h, k \leq T^{\nu}$ and $0 < \gamma < 1$. The key ingredient of the proof of this lemma depends on a strong result of shifted convolution sums on average due to Blomer (see Theorem 2 of [Blo05]). A straightforward adaption of Blomer's result and the argument given by Bernard prove the following lemma.

Lemma 3.3.10 (Bernard)**.** *Let ℓ_1, ℓ_2, H and h_1 be positive integers. Let M_1, M_2, P_1, P_2 be real numbers greater than 1. Let $\{g_r\}$ be a family of smooth functions, supported on $[M_1, 2M_1] \times [M_2, 2M_2]$ such that $|g_h^{(ij)}|_\infty \ll_{i,j} (p_1/M_1)^i(P_2/M_2)^j$ for all $i, j \geq 0$. Let $(a(r))$ be sequence of complex numbers such that*

$$a(r) \neq 0 \implies r \leq H, \quad h_1|r \quad \text{and} \left(h_1, \frac{r}{h_1}\right) = 1.$$

If $\ell_1 M_1 \asymp \ell_2 M_2 \asymp A$ and if there exists $\epsilon > 0$ such that

$$H \ll \frac{A}{\max\{P_1, P_2\}} \frac{1}{(\ell_1\ell_2 M_1 M_2 P_1 P_2)^{\epsilon}},$$

then, for all real numbers $\epsilon > 0$, we have

$$\sum_{r=1}^{H} a(r) \sum_{\substack{m_1 m_2 \geq 1 \\ \ell_1 m_1 - \ell_2 m_2 = r}} \lambda_f(m_1)\bar{\lambda}_f(m_2) g_1(m_1, m_2) \ll A^{\frac{1}{2}} h_1^{\theta} ||a||_2 (P_1 + P_2)^{\frac{3}{2}}$$

$$\times \left[\sqrt{P_1 + P_2} + \left(\frac{A}{\max\{P_1, P_2\}}\right)^{\theta} \left(1 + \sqrt{\frac{(h_1, \ell_1\ell_2)H}{h_1, \ell_1\ell_2}}\right)\right] (\ell_1\ell_2 M_1 M_2 P_1 P_2)^{\epsilon}.$$

where θ is the exponent in the Ramanujan-Petersson conjecture.

The next step is to determined the required bound for the test function. Let $\alpha, \beta = \frac{W}{\log T}$ and let $\sigma > 0$ be any positive real number. For all non-negative integers i, j one has

$$x^i y^j \frac{\partial^{i+j} F_{r;\ell_1,\ell_2}}{\partial x^i \partial y^j}(x,y) \ll_{i,j} \left(\frac{a}{A_{\ell_1}}\right)^{\frac{1}{2}+\Re\alpha+\sigma} \left(\frac{b}{A_{\ell_2}}\right)^{\frac{1}{2}+\Re\beta+\sigma} T^{1+2\sigma}(\ln T)^j \quad (3.17)$$

where the implicit constant does not depend on r. The trivial bound for shifted convolution sums can yield

$$\sum_{\ell_1 - m_1\ell_2 m_2 = r} \lambda_f(m_1)\lambda_f(m_2) \ll_\epsilon \min\{M_1, M_2\}(M_1 M_2)^{\epsilon}.$$

This trivial bound together with (3.17) proves the following:

$$I_O^{(1)}(h,k) \ll_\epsilon \min\{h,k\} T^{1+\epsilon}$$

which is not very useful for us. As argued in [Ber15] using Theorem 6.3 of [Ric06] we get

$$I_O^{(1)}(h,k) \ll_\epsilon \min(hk)^{3/4+\theta/2} T^{3/2+2\epsilon}.$$

Instead of using the trivial bound if we use Theorem 1.3 of [Blo04] along with (3.17) then we have

$$I_O^{(1)}(h,k) \ll_\epsilon \min(hk)^{3/4+\theta/2} T^{1/2+2\epsilon}.$$

By Lemma 3.3.10 with $H = T^{-\gamma}\sqrt{A_{\ell_1}A_{\ell_2}}$, $h_1 = 1$ and $a(r) = \begin{cases} 1 & \text{if } r \le H \\ 0 & \text{otherwise.} \end{cases}$ Therefore, from Lemma 3.3.8 we get

$$I_O^{(1)}(h,k) \ll_\epsilon (hk)^{(1+\theta)/2} T^{1/2+\theta+\epsilon}.$$

With a similar argument and from Corollary 5 of [Ber15] one has

$$I_O^{(2)}(h,k) \ll_\epsilon (hk)^{(1+\theta)/2} T^{1/2+\theta+\epsilon}.$$

Then we can trivially bound the off-diagonal term by

$$\begin{aligned} \sum_{h,k\le T^\epsilon} \frac{\mu_f(h)\mu_f(k)a(h)a(k)}{\sqrt{hk}} \left[I_O^{(1)}(h,k) + I_O^{(2)}(h,k)\right] &\ll T^{1/2+3\theta+\epsilon} \sum_{h,k\le T^\epsilon} (hk)^{(1+\theta)/2-1/2} \\ &\ll T^{1-\epsilon} = o(T). \end{aligned}$$

We can use Kim and Sarnak's (see Appendix 2 of [KRS03]) bound, where $\theta = \frac{7}{64}$.

The calculation of the diagonal and the off-diagonal term, which completes the proof of this proposition. $\square$

4. SCLT for Dirichlet L-functions in the q-aspect

4.1 Introduction

In this chapter, we prove Selberg's central limit theorem for Dirichlet L-functions in the q-aspect. First, we prove this result for the Dirichlet L-function associated with the primitive Dirichlet characters χ modulo q. Later, we conditionally extend this result for twisted Dirichlet L-functions associated with $SL(3,\mathbb{Z})$ Hecke-Maass cusp form and twisted by the primitive Dirichlet characters χ. Although the key idea of the proof of main theorems of this chapter has been taken from [RS17], since we are proving this result for q-aspect we need the help of asymptotic Large Sieve to compute the mollified second moment for $GL(3)$ L-functions. We give a brief overview of how can one prove the second mollified moment of $L(f\otimes\chi,s)$ using asymptotic Large Sieve. We prove Theorem 4.4.1 conditionally, under the assumption of the moment conjecture for $L(f\otimes\chi,s)$. [1]

Before going into the more details of the proof we describe the notion of "almost normal" distribution for q-aspect (particularly for our setup). For primitive Dirichlet characters χ modulo q, any $\sigma>1$ and $t\in\mathbb{R}$, we have

$$\log|L(\sigma+it,\chi)| = \sum_p\sum_{k=1}^{\infty}\frac{1}{k}\frac{|\chi(p^k)|\cos(kt\log p)}{p^{k\sigma}}.$$

Consider the imaginary part $t\in[-1,1]$, the almost independence arises because of the values $\chi(p^k)$ are linearly independent (orthogonality relation of the primitive Dirichlet characters) over the unit circle for sufficiently large q. So, the terms $|\chi(p^k)|\cos(kt\log p)$ varies "almost independently" for distinct primes p. We can consider $\left\{\sum_p\sum_{k=1}^{\infty}\frac{1}{k}\frac{|\chi(p^k)|\cos(kt\log p)}{p^{k\sigma}}\right\}_{p\in\mathbb{P}}$ as the sequence of "almost independent" random variable with suitable mean and variance. Then as $q\to\infty$ for $t\in[-1,1]$ the almost random variables $\left\{\sum_p\sum_{k=1}^{\infty}\frac{1}{k}\frac{|\chi(p^k)|\cos(kt\log p)}{p^{k\sigma}}\right\}_{p\in\mathbb{P}}$ converge in distribution to normal $\mathcal{N}(0,\frac{1}{2}\log\log q)$. Hence we call $\log|L(\sigma+it,\chi)|$ has "approximate normal" distribution with mean 0 and variance $\frac{1}{2}\log\log q$. Now we write the precise definition.

[1] Throughout this chapter $\sum^{*}_{\chi(\bmod q)}$ denotes the summation over primitive Dirichlet characters.

Definition. *If $X_{\chi,p}$ is approximately normally distributed with mean m and variance ν^2, then for any fixed positive real number V, as $q \to \infty$, we have*

$$\frac{1}{2\phi^*(q)} \sum_{\chi(\bmod q)}^{*} meas \left\{ t \in [-1,1] : \frac{X_{\chi,p} - m}{\nu} \geq v \right\} \sim \frac{1}{\sqrt{2\pi}} \int_v^{\infty} e^{-\frac{x^2}{2}} dx$$

uniformly for $v \in [-V, V]$.

We prove the main result of this chapter into four steps. We give a brief overview of the proof of second mollified moment for $GL(3)$ L-functions under the assumption of moment conjecture. Since we will be working with the q-aspect, we average over the Dirichlet characters and the moduli[2]. Furthermore, notice that we are not proving this result for the hybrid aspect so we fix our t in the range $[-1, 1]$.

4.2 Setup for Dirichlet L-functions

In this section, we prove Selberg's central limit theorem for Dirichlet L-functions as defined in Definition 2.2.3. We need to keep in mind that we will be handling the character sums so the orthogonality relations of the Dirichlet character is a very useful tool here.

Theorem 4.2.1. *Let $\phi^*(q)$ denotes the total number of primitive Dirichlet characters modulo q. Let V be a fixed positive real number. Then as $q \to \infty$, uniformly for all $v \in [-V, V]$*

$$\frac{1}{2\phi^*(q)} \sum_{\chi(\bmod q)}^{*} meas \left\{ -1 \leq t \leq 1 : \log |L(\frac{1}{2} + it, \chi)| \geq v \sqrt{\frac{1}{2} \log \log q} \right\}$$
$$\sim \frac{1}{\sqrt{2\pi}} \int_v^{\infty} e^{-u^2/2} du.$$

We start with the setup of the proof then we will get into the details. In the first step, let us take away the problem from the critical line by proving that for a suitable choice of $\sigma > \frac{1}{2}$, $L(\frac{1}{2} + it, \chi)$ and $L(\sigma + it, \chi)$ are typically close to each other.

Proposition 4.2.2. *Let χ be the primitive Dirichlet character modulo q. Then for any $\sigma > \frac{1}{2}$ we have*

$$\int_{-1}^{1} \left| \log |L(\frac{1}{2} + it, \chi)| - \log |L(\sigma + it, \chi)| \right| dt \ll \left(\sigma - \frac{1}{2} \right) \log q.$$

Remark 4.2.3. *Similarly like the t-aspect in the first step, we take away the problem from the critical line. If we choose to move $1/\log q$ distance from the $\frac{1}{2}$ line (as $q \to \infty$ the density of zeros of $L(s, \chi)$ increases), then we might stay very close*

[2] Note that we have only considered the primitive Dirichlet characters throughout the chapter.

to $\frac{1}{2}$ line for large enough q. We wish to stay away from the critical line but not too far away. So, we can not choose a large parameter W. That is why we choose $W = o(\sqrt{\log\log q})$. Further, we want to approximate $L(s,\chi)$ by an Euler product, where the product is going up to X. In order to approximate, we need to consider that X would be going up to a small power of q.

Fix the parameters

$$W = (\log\log\log q)^4, \quad X = q^{1/(\log\log\log q)^2}, \quad Y = q^{(1/\log\log q)^2}, \quad \sigma_0 = \frac{1}{2} + \frac{W}{\log q}.$$

where q is sufficiently large so $W \geq 3$.

In the next step we consider the auxiliary series given by

$$\mathcal{P}(s,\chi) = \mathcal{P}(s,\chi;X) = \sum_{2\leq n\leq X} \frac{\Lambda(n)\chi(n)}{n^s \log n}$$

where $\Lambda(n)$ is the Von-Mangoldt function. We determine the distribution of the auxiliary series $\mathcal{P}(s,\chi)$ by computing its moments.

Proposition 4.2.4. *For the primitive Dirichlet characters $\chi(\mathrm{mod} q)$, the distribution of $\Re(\mathcal{P}(\sigma_0+it))$ is approximately normal with mean 0 and variance $\frac{1}{2}\log\log q$. Precisely, let V be a fixed positive real number then as $q \to \infty$, uniformly for all $v \in [-V, V]$*

$$\frac{1}{2\phi^*(q)} \sum_{\chi(\bmod q)}^{*} meas\left\{-1 \leq t \leq 1 : \Re(\mathcal{P}(\sigma_0+it,\chi)) \geq v\sqrt{\frac{1}{2}\log\log q}\right\}$$
$$\sim \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-u^2/2} du.$$

It is now obvious that if we can connect the L-function with the auxiliary series $\mathcal{P}(s,\chi)$ that will prove Theorem 4.2.1.

To establish this connection we use the mollification technique. Consider the Dirichlet polynomial $M(s,\chi)$ given by

$$M(s,\chi) = \sum_n \frac{\mu(n)a(n)\chi(n)}{n^s}$$

where $a(n)$ is given by

$$a(n) = \begin{cases} 1 & \text{if } n \text{ is composed only of primes below } X \text{ and has at most } 100\log\log q \\ & \text{primes below } Y \text{ and at most } 100\log\log\log q \text{ primes between } Y \text{ and } X. \\ 0 & \text{otherwise .} \end{cases}$$

By the definition of $a(n)$ it takes the value 0 except when $n \leq Y^{100\log\log q} X^{100\log\log\log q} < q^\epsilon$. It is now evident that $M(s,\chi)$ is a short Dirichlet polynomial. In the next step, we establish the connection between $M(s,\chi)$ and $\mathcal{P}(s,\chi)$.

Proposition 4.2.5. *Given $\epsilon > 0$ and $\delta > 0$, there is some Q so that for all $q > Q$ we have*

$$\frac{1}{2\phi^*(q)} \sideset{}{^*}\sum_{\chi(\bmod q)} meas \left\{ -1 \le t \le 1 : \left| \frac{M(\sigma_0 + it, \chi)}{\exp(-\mathcal{P}(\sigma_0 + it, \chi))} - 1 \right| < \delta \right\} < \epsilon.$$

It remains to prove that the mollifier and the L-function are inverse to each other.

Proposition 4.2.6. *Let $-1 \le t \le 1$,*

$$\frac{1}{\phi^*(q)} \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} |1 - L(\sigma_0 + it, \chi) M(\sigma_0 + it, \chi)|^2 \, dt = o(1). \tag{4.1}$$

Given $\epsilon > 0$ and $\delta > 0$, there is some Q so that for all $q > Q$ we have

$$\frac{1}{2\phi^*(q)} \sideset{}{^*}\sum_{\chi(\bmod q)} meas \{ -1 \le t \le 1 : |L(\sigma_0 + it, \chi) M(\sigma_0 + it, \chi) - 1| < \delta \} < \epsilon.$$

Now we prove our main theorem. We prove the propositions in later sections.

Proof of Theorem 4.2.1: Recalling Proposition 4.2.6, it means that for all $t \in [-1, 1]$ and primitive Dirichlet characters $\chi(\bmod q)$, (for most χ and t) we have

$$L(\sigma_0 + it, \chi) = (1 + o(1)) M(\sigma_0 + it, \chi)^{-1}.$$

By Proposition 4.2.5 (for most χ and t) we know that

$$|L(\sigma_0 + it, \chi)| = (1 + o(1)) \exp(\Re \mathcal{P}(\sigma_0 + it, \chi))$$

and by Proposition 4.2.4 we can conclude that $\log |L(\sigma_0 + it, \chi)|$ is normally distributed with mean 0 and variance $\frac{1}{2} \log \log q$. Finally with the help of Proposition 4.2.2 we deduce that

$$\sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \left| \log |L(\frac{1}{2} + it, \chi)| - \log |L(\sigma_0 + it, \chi)| \right| dt \ll \phi^*(q)(\sigma_0 - \frac{1}{2}) \log q = \phi^*(q) W.$$

So for most χ and t, we have

$$\log |L(\frac{1}{2} + it, \chi)| = \log |L(\sigma_0 + it, \chi)| + O(W^2).$$

Since $W^2 = o(\sqrt{\log \log q})$ it follows that similarly like $\log |L(\sigma_0 + it, \chi)|$, $\log |L(\frac{1}{2} + it, \chi)|$ has the normal distribution with mean 0 and variance $\frac{1}{2} \log \log q$, which completes the proof of Theorem 4.2.1.

□

4.3 Proof of Theorem 4.2.1

In this section, we prove the propositions to complete the proof of Theorem 4.2.1.

4.3.1 Proof of Proposition 4.2.2

Let χ be primitive Dirichlet characters modulo q.

Set

$$G(s,\chi)=\left(\frac{\pi}{q}\right)^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right), \tag{4.2}$$

where

$$a=a(\chi)=\begin{cases}0 & \text{if } \chi(-1)=1\\ 1 & \text{if } \chi(-1)=-1.\end{cases}$$

We will show that

$$\left|\log\frac{G(\sigma+it,\chi)}{G(\frac{1}{2}+it,\chi)}\right|\ll\left(\sigma-\frac{1}{2}\right)\log q.$$

Consider the Taylor expansion of the Gamma function we get

$$\begin{aligned}\Gamma(s+\delta)=&\Gamma(s)+\delta\Gamma'(z)+\frac{\delta^2}{2!}\Gamma''(z)+\cdots\\ =&\Gamma(s)+O(\delta)\end{aligned}$$

where $s=\frac{1}{2}+it$ and $\delta=\frac{W}{\log q}$, with $|t|\le 1$.

Note that

$$\arg\left(\frac{G(\sigma+it,\chi)}{G(\frac{1}{2}+it,\chi)}\right)=\arg\left(G(\sigma+it,\chi)-\arg\left(G(1/2+it,\chi)\right)\right.$$

Now, applying the Taylor expansion for the argument of the gamma function we have

$$\begin{aligned}\arg\left(G(\sigma+it,\chi)\right)=&\arg\left(\left(\frac{\pi}{q}\right)^{-(\sigma+it+a)/2}\Gamma\left(\frac{\sigma+it+a}{2}\right)\right)\\ =&t/2\log q+\arg\left(\Gamma\left(\frac{1/2+it+a}{2}\right)\right)+O(\delta).\end{aligned}$$

Similarly,

$$\arg\left(G(1/2+it,\chi)\right)=t/2\log q+\arg\left(\Gamma\left(\frac{1/2+it+a}{2}\right)\right).$$

Expanding the complex logarithm and putting Taylor expansion for Gamma function in (4.2) we have,

$$
\begin{aligned}
&\log \frac{G(\sigma+it,\chi)}{G(\frac{1}{2}+it,\chi)}\\
=&\log\left|\frac{G(\sigma+it,\chi)}{G(\frac{1}{2}+it,\chi)}\right| + i\arg\left(\frac{G(\sigma+it,\chi)}{G(\frac{1}{2}+it,\chi)}\right)\\
=&\log\frac{\left(\frac{\pi}{q}\right)^{-(\sigma+a)/2}\left|\Gamma\left(\frac{\sigma+it+a}{2}\right)\right|}{\left(\frac{\pi}{q}\right)^{-(1/2+a)/2}\left|\Gamma\left(\frac{1/2+it+a}{2}\right)\right|} + i\Bigg(t/2\log q + \arg\left(\Gamma\left(\frac{1/2+it+a}{2}\right)\right) + O(\delta)\\
&\qquad\qquad - t/2\log q - \arg\left(\Gamma\left(\frac{1/2+it+a}{2}\right)\right)\Bigg)\\
=&\log\left(\frac{q}{\pi}\right)^{(\sigma-1/2)/2} + \log\left|\Gamma\left(\frac{1/2+it+a}{2}\right)\right| - \log\left|\Gamma\left(\frac{1/2+it+a}{2}\right)\right| + O(\delta).
\end{aligned}
$$

Since, q is large enough[3] and $|t|\le 1$, we write

$$
\left|\log\frac{G(\sigma+it,\chi)}{G(\frac{1}{2}+it,\chi)}\right| \ll \left|\log q^{(\sigma-\frac{1}{2})}\right| \ll \left(\sigma-\frac{1}{2}\right)\log q.
$$

Recall the functional equation of the complete L-function

$$
\xi(s,\chi) = G(s,\chi)L(s,\chi).
$$

To prove Proposition 4.2.2 it is enough to prove that

$$
\int_{-1}^{1}\left|\log\left|\frac{\xi(1/2+it,\chi)}{\xi(\sigma+it,\chi)}\right|\right| dt \ll \left(\sigma-\frac{1}{2}\right)\log q.
$$

Recalling Hadamard's factorization formula (see e.g. Lemma 2.2.11), there exist constants $a=a(\chi)$ and $b=b(\chi)$ (where $b(\chi)=-\sum_\rho \Re(1/\rho)$) such that

$$
\xi(s,\chi) = e^{a+bs}\prod_{\rho\in\mathcal{Z}_\chi}\left(1-\frac{s}{\rho}\right)e^{s/\rho},
$$

where $\rho=\beta+i\gamma_\chi\in\mathcal{Z}_\chi$, where $\mathcal{Z}_\chi$ denotes the set of all non-trivial zeros of $L(s,\chi)$ for Dirichlet characters χ modulo q.

Assuming that t is not the ordinate of a zero of $L(s,\chi)$ we can write

$$
\log\left|\frac{\xi(1/2+it,\chi)}{\xi(\sigma+it,\chi)}\right| = \sum_{\rho\in\mathcal{Z}_\chi}\log\left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|.
$$

Suppose $\rho=\beta+i\gamma_\chi\in\mathcal{Z}_\chi$, where $\mathcal{Z}_\chi$ denotes the set of all non-trivial zeros of $L(s,\chi)$ for Dirichlet characters χ modulo q. Integrating over $t\in[-1,1]$ we get

[3]Notice that as $q\to\infty$, $\delta\to 0$

$$\int_{-1}^{1}\left|\log\left|\frac{\xi(1/2+it,\chi)}{\xi(\sigma+it,\chi)}\right|\right|dt \leq \sum_{\rho\in\mathcal{Z}_\chi}\int_{-1}^{1}\left|\log\left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|\right|dt$$
$$=\frac{1}{2}\sum_{\rho\in\mathcal{Z}_\chi}\int_{-1}^{1}\left|\log\frac{(\beta-\frac{1}{2})^2+(t-\gamma_\chi)^2}{(\beta-\sigma)^2+(t-\gamma_\chi)^2}\right|dt. \tag{4.3}$$

If $|t-\gamma_\chi|\geq 2$, then we have

$$\left|\log\left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|\right| = \left|\Re\log\left(1-\frac{\sigma-\frac{1}{2}}{\sigma+it-\rho}\right)\right| = \left|\Re\frac{\sigma-\frac{1}{2}}{\sigma+it-\rho}\right| + O\left(\frac{(\sigma-\frac{1}{2})^2}{(t-\gamma_\chi)^2}\right)$$
$$=O\left(\frac{(\sigma-\frac{1}{2})}{(t-\gamma_\chi)^2}\right)$$

So for $|t|\leq 1$, we can write

$$\int_{-1}^{1}\left|\log\left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|\right|dt \ll \frac{(\sigma-1/2)}{(1\pm\gamma_\chi)^2}.$$

Then contribution of these zeros give

$$\sum_{\gamma_\chi}\frac{(\sigma-\frac{1}{2})}{(1\pm\gamma_\chi)^2} \ll (\sigma-1/2)\log q.$$

Now consider the range $|t-\gamma_\chi|\leq 2$ (which is basically the zeros near the real axis) we have

$$\int_{-1}^{1}\left|\log\left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|\right|dt = \frac{1}{2}\int_{-1}^{1}\left|\log\frac{(\beta-\frac{1}{2})^2+(t-\gamma_\chi)^2}{(\beta-\sigma)^2+(t-\gamma_\chi)^2}\right|dt$$
$$\leq\frac{1}{2}\int_{-\infty}^{\infty}\left|\log\left|\frac{(\beta-\frac{1}{2})^2+x^2}{(\beta-\sigma)^2+x^2}\right|\right|dx$$
$$=\pi\left(\sigma-\frac{1}{2}\right).$$

So in this case the contribution of zeros is $\ll(\sigma-\frac{1}{2})\log q$.

Thus in either case

$$\int_{-1}^{1}\left|\log\left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|\right|dt \ll \frac{(\sigma-\frac{1}{2})}{1+(1\pm\gamma_\chi)^2}.$$

Remark 4.3.1. *Note that the sum $(\sigma-\frac{1}{2})\sum_{\gamma_\chi}\frac{1}{1+(1\pm\gamma_\chi)^2}$ is convergent hence it is finite. In the t-aspect, we have studied the contribution of the non-trivial zeros of the L-functions in two cases (for zeros near t and the zeros far from t). In the q-aspect, we study the problem for the contribution of the zeros near the real axis* *and far from the real axis. Note that the number of zeros in a window of length* 2 *around a given point* t *is given by* $\ll \log q + \log(2+|t|)$ *(see Proposition 5.7 of [IK04]). Similarly, like the* t*-aspect, we compute the contribution of the zeros for each of these small windows, which will lead us to the total contribution, ultimately the desired bound. Also, note that as* q *gets bigger the corresponding* L*-function has more zeros.*

Inserting this in (4.3), from Theorem 5.24 of [IK04] we can conclude

$$\begin{aligned}\int_{-1}^{1}\left|\log\left|\frac{L(1/2+it,\chi)}{L(\sigma+it,\chi)}\right|\right|dt &= \int_{-1}^{1}\left|\log\left|\frac{\xi(1/2+it,\chi)}{\xi(\sigma+it,\chi)}\right|\right|dt + O\left(\sigma-\frac{1}{2}\right)\log q\\ &\ll \left(\sigma-\frac{1}{2}\right)\log q,\end{aligned}$$

which completes the proof. □

4.3.2 Proof of Proposition 4.2.4

Before going into the detail of the proof of Proposition 4.2.4, we prove a general lemma. We use this lemma in later sections as well.

Let $\phi^*(q)$ and $\phi^+(q)$ be the number of primitive and even primitive Dirichlet characters modulo q respectively. Recall that $\phi^+(q) = \frac{\phi^*(q)}{2} + O(1)$ (see section 1 of [CIS12]).

Lemma 4.3.2. *Let* $\phi^*(q)$ *be the number of primitive Dirichlet characters modulo* q*. Recall that* $X = q^{(1/\log\log q)^2}$*. Then for* $\epsilon > 0$ *and complex numbers* $\{a_n\}_{n\in\mathbb{N}}$*, we have*

$$\sideset{}{^*}\sum_{\chi(\bmod q)}\left|\sum_{n\le X}\frac{a_n\chi(n)}{n^{\sigma_0}}\right|^2 = \phi^*(q)\sum_{n\le X}\frac{|a_n|^2}{n^{2\sigma_0}} + O(q^{\epsilon}). \tag{4.4}$$

Proof. Expanding (4.4), we have

$$\sideset{}{^*}\sum_{\chi(\bmod q)}\left|\sum_{n\le X}\frac{a_n\chi(n)}{n^{\sigma_0}}\right|^2 = \sum_{m,n\le X}\frac{a_m\bar{a}_n}{(mn)^{\sigma_0}}\sideset{}{^*}\sum_{\chi(\bmod q)}\chi(m)\bar{\chi}(n). \tag{4.5}$$

Let $\phi^+(q)$ be the number of even primitive Dirichlet characters modulo q. Note that $\sum^+$ denote the summation over the even primitive characters modulo q. Then we have (see section 3 of [IS12]),

$$\sideset{}{^+}\sum_{\chi(\bmod q)}\chi(n) = \frac{1}{2}\sideset{}{^*}\sum_{\chi(\bmod q)}(\chi(m)+\chi(-m)). \tag{4.6}$$

From Lemma 4.1 of [BM11] for $(mn,q)=1$, we can write

$$\sideset{}{^+}\sum_{\chi(\bmod q)}\chi(m)\bar{\chi}(n) = \frac{1}{2}\sum_{\substack{dr=q\\ d|m\pm n}}\mu(r)\phi(d).$$

In order to prove the lemma, we compute equation (4.5) for even and odd primitive Dirichlet characters. We show the computation for even primitive Dirichlet characters, and a similar computation for odd primitive Dirichlet characters follows similarly.

For even primitive Dirichlet characters $\chi(-1) = 1$, inserting the above equation to (4.5), we have

$$\begin{aligned}
&\sum_{m,n\leq X} \frac{a_m \bar{a}_n}{(mn)^{\sigma_0}} \cdot \sum_{\chi(\bmod q)}^{+} \chi(m)\bar{\chi}(n) \\
&= \sum_{m,n\leq X} \frac{a_m \bar{a}_n}{(mn)^{\sigma_0}} \cdot \frac{1}{2} \sum_{\substack{dr=q \\ d|m\pm n}} \mu(r)\phi(d) \\
&= \frac{1}{2} \sum_{dr=q} \mu(r)\phi(d) \sum_{\substack{m\equiv\pm n(\bmod d) \\ m,n\leq X}} \frac{a_m \bar{a}_n}{(mn)^{\sigma_0}}
\end{aligned}$$

We have two cases for the above equation. For the first case, if $d > X$, then $m \equiv \pm n (\bmod d)$, which implies $m = n$. Therefore, the second case arises for $d \leq X$. Then the above equation equals to

$$\frac{1}{2} \sum_{\substack{dr=q \\ d>X}} \mu(r)\phi(d) \sum_{n\leq X} \frac{|a_n|^2}{n^{2\sigma_0}} + \frac{1}{2} \sum_{\substack{dr=q \\ d\leq X}} \mu(r)\phi(d) \sum_{\substack{m\equiv\pm n(\bmod d) \\ m,n\leq X}} \frac{a_m \bar{a}_n}{(mn)^{\sigma_0}}.$$

Observe that in the second term of the above summation, for $d \leq X$, and $m \neq n$ is very small. Then the error term can be bounded by

$$\frac{1}{2} \sum_{\substack{dr=q \\ d\leq X}} \mu(r)\phi(d) \sum_{\substack{m\equiv\pm n(\bmod d) \\ m,n\leq X}} \frac{a_m \bar{a}_n}{(mn)^{\sigma_0}} \ll X^{(o(1))} \ll q^{\epsilon}.$$

Thus, the main term for $m = n$ equals to

$$\sum_{n\leq X} \frac{|a_n|^2}{n^{2\sigma_0}} \left(\frac{1}{2} \sum_{dr=q} \mu(r)\phi(d) + O\left(\sum_{d\leq X} \mu^2(r)\phi(d) \right) \right) = \frac{1}{2}\phi^*(q) \sum_{n\leq X} \frac{|a_n|^2}{n^{2\sigma_0}} + O(q^{\epsilon}),$$

recalling the definition of X we conclude the above bound.

Note that a similar computation can be done for odd primitive characters $\chi(-1) = -1$. In that case, instead of (4.6), we have

$$\sum_{\chi(\bmod q)}^{-} \chi(m) = \frac{1}{2} \sum_{\chi(\bmod q)}^{*} (\chi(m) - \chi(-m)). \qquad (4.7)$$

Combining for the cases of even and odd primitive Dirichlet characters and recalling the definition of X, we conclude the proof.

□

We prove this proposition by restricting the sum to primes and then compute moments. For primes p^k with $k \geq 3$, contributes

$$\left| \sum_{\substack{2 \leq p^k \leq X \\ k \geq 3}} \frac{(\log p)\chi(p^k)}{p^{ks}(k \log p)} \right| \leq \sum_{\substack{2 \leq p^k \leq X \\ k \geq 3}} \frac{1}{3p^{\sigma_0}} = O(1).$$

where $\Re(s) = \sigma_0$. For the prime square contribution we have

$$\sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \left| \sum_{\substack{p^2 \leq X \\ (p,q)=1}} \frac{\chi(p^2)}{p^{2(\sigma_0+it)} \cdot 2} \right|^2 dt = \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \sum_{\substack{p_1, p_2 \leq \sqrt{X} \\ (p_1 p_2, q)=1}} \frac{\chi(p_1^2)\bar{\chi}(p_2^2)}{p_1^{2(\sigma_0+it)} p_2^{2(\sigma_0+it)}} dt. \tag{4.8}$$

First we will show the off-diagonal term treatment for (4.8), then we show the main term computation. As shown in the proof of Lemma 4.3.2, for $d \leq \sqrt{X}$ and $p_1^2 \neq p_2^2$, for the even primitive Dirichlet characters, applying (2.16) and (2.17), the off-diagonal term of (4.8) is bounded by

$$\begin{aligned} \ll \sum_{\substack{p_1 \neq p_2 \\ p_1, p_2 \leq \sqrt{X} \\ (p_1 p_2, q)=1}} \frac{1}{(p_1 p_2)^{2\sigma_0 - \frac{1}{2}}} \sideset{}{^+}\sum_{\chi(\bmod q)} \chi(p_1^2)\bar{\chi}(p_2^2) \ll & \frac{1}{2} \sum_{\substack{dr=q \\ d \leq \sqrt{X}}} \mu(r)\phi(d) \sum_{\substack{p_1^2 \equiv \pm p_2^2 \\ p_1^2, p_2^2 \leq X}} \frac{1}{(p_1 p_2)^{2\sigma_0 - \frac{1}{2}}} \\ \ll & (\sqrt{X})^{(o(1))} \ll q^{\epsilon}. \end{aligned}$$

Applying Lemma 4.3.2 and by (2.16) and (2.17), for even primitive Dirichlet characters (4.8) is bounded by

$$\ll \phi^*(q) \sum_{p \leq \sqrt{X}} \frac{1}{p^{4\sigma_0}} + q^{\epsilon} \ll \phi^*(q).$$

Note that a similar computation can be done for odd primitive Dirichlet characters. Adding their contributions and by the definition of X we conclude

$$\sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \left| \sum_{\substack{p^2 \leq X \\ (p,q)=1}} \frac{\chi(p^2)}{p^{2(\sigma_0+it)} \cdot 2} \right|^2 dt \ll \phi^*(q). \tag{4.9}$$

Let $A(q) = A(q; X) := \sum_{2 \leq p \leq X} \frac{\chi(p^2)}{p^{2\sigma_0}}$, from (4.9) and Chebyshev's inequality we have

$$\frac{1}{2\phi^*(q)} \sideset{}{^*}\sum_{\chi(\bmod q)} meas\{-1 \le t \le 1 : |A(q)| > L\}$$
$$\le \frac{1}{L^2} \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} |A(q)|^2 dt \ll \phi^*(q)/L^2,$$

for any positive real number $L > 1$. In other words we can say that the square of primes in $\mathcal{P}(s,\chi)$ contribute a measure at most $O(\phi^*(q)/L^2)$. With the Same argument given in Remark 4.2.3 we choose $L = o(\log\log\log q)$.

Now it is evident that we can restrict the auxiliary series $\mathcal{P}(s,\chi)$ to primes. Consider the auxiliary series

$$\mathcal{P}_0(\sigma_0 + it, \chi) = \mathcal{P}_0(\sigma_0 + it, \chi; X) = \sum_{p \le X} \frac{\chi(p)}{p^{\sigma_0 + it}}.$$

We study the moments by obtaining a mean value estimate to prove Proposition 4.2.4.

Lemma 4.3.3. *Suppose that k and ℓ are non-negative integers with $X^{k+\ell} \ll q$. Then if $k \neq \ell$,*

$$\sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \mathcal{P}_0(\sigma_0 + it, \chi)^k \overline{\mathcal{P}_0(\sigma_0 + it, \chi)}^{\ell} dt \ll \phi^*(q).$$

If $k = \ell$, for $\epsilon > 0$,

$$\sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} |\mathcal{P}_0(\sigma_0 + it, \chi)|^{2k} dt = k!\phi^*(q)(\log\log Q)^k + O_k(\phi^*(q)(\log\log Q)^{k-1+\epsilon}).$$

Proof. Write $\mathcal{P}_0(s)^k = \sum_n \frac{a_k(n)\chi(n)}{n^s}$ where

$$a_k(n) = \begin{cases} \frac{k!}{\alpha_1! \cdots \alpha_r!} & \text{if } n = \prod_{j=1}^{r} p_j^{\alpha_j}, p_1 < \ldots < p_r < X, \sum_{j=1}^{r} \alpha_j = k. \\ 0 & \text{otherwise}. \end{cases}$$

Therefore,

$$\sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \mathcal{P}_0(\sigma_0 + it, \chi)^k \overline{\mathcal{P}_0(\sigma_0 + it, \chi)}^{\ell} dt$$
$$= \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \left\{ \sum_n \frac{a_k(n)\chi(n)}{n^s} \sum_m \frac{a_\ell(m)\bar{\chi}(m)}{m^{\bar{s}}} \right\} dt$$
$$= \sum_{m,n} \frac{a_k(n)a_\ell(m)}{(mn)^{\sigma_0}} \sideset{}{^*}\sum_{\chi(\bmod q)} \chi(m)\bar{\chi}(n) \int_{-1}^{1} \left(\frac{m}{n}\right)^{it} dt$$

First we compute the sum for even Dirichlet characters. From (4.6), we write

$$\begin{aligned}
&\sum_{m,n} \frac{a_k(n)a_\ell(m)}{(mn)^{\sigma_0}} \sum_{\chi(\bmod q)}^{+} \chi(m)\bar{\chi}(n) \int_{-1}^{1} \left(\frac{m}{n}\right)^{it} dt \\
=&\sum_{m,n} \frac{a_k(n)a_\ell(m)}{(mn)^{\sigma_0}} \cdot \frac{1}{2} \sum_{\substack{dr=q \\ d|m\pm n}} \mu(r)\phi(d) \int_{-1}^{1} \left(\frac{m}{n}\right)^{it} dt \\
=&\frac{1}{2} \sum_{dr=q} \mu(r)\phi(d) \sum_{m\equiv \pm n(\bmod d)} \sum_{m,n} \frac{a_k(n)a_\ell(m)}{(mn)^{\sigma_0}} \int_{-1}^{1} \left(\frac{m}{n}\right)^{it} dt.
\end{aligned}$$

Similarly, like the proof of Lemma 4.3.2 and applying (2.16) and (2.17), we have

$$\frac{1}{2} \sum_{\substack{dr=q \\ d>X^{k+\ell}}} \mu(r)\phi(d) \sum_{n\leq X^{k+\ell}} \frac{a_k(n)a_\ell(n)}{n^{2\sigma_0}} + O\left(\sum_{\substack{dr=q \\ d\leq X^{k+\ell}}} \mu(r)\phi(d) \sum_{\substack{m\equiv \pm n(\bmod d) \\ m\leq X^k, n\leq X^\ell \\ m\neq n}} \frac{a_k(n)a_\ell(m)}{(mn)^{\sigma_0-\frac{1}{2}}} \right)$$

For the diagonal term $a_k(n)a_\ell(n)$ is 0 by definition for $k \neq \ell$. We compute the off-diagonal term of the above equation for $d \leq X^{k+\ell}$ with $m \neq n$. Then the off-diagonal term is bounded by

$$\ll \sum_{\substack{dr=q \\ d\leq X^{k+\ell}}} \mu(r)\phi(d) \sum_{\substack{m\equiv \pm n(\bmod d) \\ m\leq X^k, n\leq X^\ell \\ m\neq n}} \frac{a_k(n)a_\ell(m)}{(mn)^{\sigma_0-\frac{1}{2}}} \ll (X^{k+\ell})^{(o(1))} \ll q^\epsilon.$$

A similar computation can be done for the odd primitive Dirichlet characters. Then we have the off-diagonal term contribution given by

$$\ll \sum_{\chi(\bmod q)}^{*} \sum_{\substack{m\neq n \\ m\leq X^k, n\leq X^\ell}} \frac{a_k(m)a_\ell(n)}{(mn)^{\sigma_0-\frac{1}{2}}} \ll (X^{k+\ell})^{(o(1))} \ll q^\epsilon \ll \phi^*(q).$$

For the off-diagonal term from (2.17) we can see that the contribution of the denominator is negligible since σ_0 is close to $\frac{1}{2}$ (see Lemma 1 of [Sel46b]).

We conclude the first part of the lemma.

For $k = \ell$ the diagonal term contributes $\sum_n \frac{a_k(n)^2}{n^{2\sigma_0}}$. By the definition for the given positive integers $\alpha_1, \ldots, \alpha_r$ with $\sum_{i=1}^{r} \alpha_i = k$ the contribution of the term of n of the form $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is given by

$$\ll \phi^*(q) \prod_{i=1}^{r} \left(\sum_{\substack{p\leq X \\ (p,q)=1}} \frac{1}{p^{2\sigma_0 \alpha_i}} \right) \ll \phi^*(q)(\log\log q)^{r+\epsilon}.$$

The terms with n not being square free contributes (with $r \leq k-1$) $O_k((\log\log q)^{k-1+\epsilon})$. The square free n terms contribute

$$k! \sum_{\substack{p_1,\ldots,p_r \leq X \\ (p_j,q)=1}} \frac{1}{(p_1\cdots p_k)^{2\sigma_0}} = k! \left(\sum_{\substack{p\leq X \\ (p,q)=1}} \frac{1}{p^{2\sigma_0}} \right)^k + O_k((\log\log q)^{k-1+\epsilon})$$
$$= k!(\log\log q)^k + O_k((\log\log q)^{k-1+\epsilon}).$$

Recalling the definition of X we conclude the proof. □

Proof of Proposition 4.2.4: By Lemma 4.3.3 for any odd k we have

$$\begin{aligned}
&\sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} (\Re(\mathcal{P}_0(\sigma_0+it,\chi)))^k \, dt \\
=&\sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \frac{1}{2^k} \left(\mathcal{P}_0(\sigma_0+it,\chi) + \overline{\mathcal{P}_0(\sigma_0+it,\chi)}\right)^k dt \\
=&\frac{1}{2^k}\sum_{\ell=0}^{k} \binom{k}{\ell} \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} (\mathcal{P}_0(\sigma_0+it,\chi))^{\ell}\overline{(\mathcal{P}_0(\sigma_0+it,\chi)}^{k-\ell} dt \\
&\ll \phi^*(q).
\end{aligned}$$

Observe that it is impossible to have $\ell = k-\ell$ for any odd k. For all even k, $\ell = k-\ell = k/2$ and again with the help of Lemma 4.3.3 we obtain,

$$\begin{aligned}
&\frac{1}{\phi^*(q)} \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} (\Re(\mathcal{P}_0(\sigma_0+it,\chi)))^k \, dt \\
=&\frac{1}{2^k}\binom{k}{k/2}\left(\frac{k}{2}\right)!(\log\log q)^{\frac{k}{2}} + O_k\left((\log\log q)^{\frac{k}{2}-1+\epsilon}\right).
\end{aligned}$$

The above equation matches with the Gaussian distribution (see (2.20)) with mean 0 and variance $\frac{1}{2}\log\log q$. □

4.3.3 Proof of Proposition 4.2.5

As we have decomposed $\mathcal{P}(s,\chi)$ in previous chapter, we are going to use the same technique here as well. So we have

$$\mathcal{P}_1(s,\chi) = \sum_{2\leq n\leq Y} \frac{\Lambda(n)\chi(n)}{n^s \log n},$$
$$\mathcal{P}_2(s,\chi) = \sum_{Y<n\leq X} \frac{\Lambda(n)\chi(n)}{n^s \log n}.$$

We further set

$$\mathcal{M}_1(s,\chi) = \sum_{0\le k\le 100\log\log q} \frac{(-1)^k}{k!}\mathcal{P}_1(s,\chi)^k,$$
$$\mathcal{M}_2(s,\chi) = \sum_{0\le k\le 100\log\log\log q} \frac{(-1)^k}{k!}\mathcal{P}_2(s,\chi)^k.$$

Lemma 4.3.4. *For the primitive Dirichlet character χ modulo q,*

$$\begin{aligned} |\mathcal{P}_1(\sigma_0+it,\chi)| &\le \log\log q, \\ |\mathcal{P}_2(\sigma_0+it,\chi)| &\le \log\log\log q. \end{aligned} \tag{4.10}$$

holds for most χ and t. Moreover,

$$\begin{aligned} \mathcal{M}_1(\sigma_0+it,\chi) &= \exp(-\mathcal{P}_1(\sigma_0+it,\chi))(1+O(\log q)^{-99}), \\ \mathcal{M}_2(\sigma_0+it,\chi) &= \exp(-\mathcal{P}_2(\sigma_0+it,\chi))(1+O(\log\log q)^{-99}). \end{aligned} \tag{4.11}$$

Proof. For the first assertion of the lemma[4], with a similar argument of Section 4.3.2 (that the auxiliary series $\mathcal{P}_1$ and $\mathcal{P}_2$ supports only on prime powers) and Lemma 4.3.2[5], we have[6]

[4]As argued in the proof of (3.5), $\sum_{n\le Y}\frac{\Lambda(n)^2}{n^{2\sigma_0}(\log n)^2} \approx \sum_{p\le Y}\frac{1}{p^{2\sigma_0}} \ll \log\log q$. Similarly, $\sum_{Y\le n\le X}\frac{\Lambda(n)^2}{n^{2\sigma_0}(\log n)^2} \approx \sum_{Y\le p\le X}\frac{1}{p^{2\sigma_0}} \ll \log\left(\frac{\log X}{\log Y}\right) \ll \log\log\log q$.

[5]The off-diagonal term treatment is same as (4.8). Note that if we add the contribution of even and odd primitive Dirichlet characters, it will only change the implicit constant which will not make any difference for our result because we are only interested in the upper bound.

[6]We know that $\Lambda(n)$ supported only on prime powers. In the proof of Proposition 4.2.4, we have already showed that the terms of $\mathcal{P}(s,\chi)$ involving higher power of primes p^k (with $k\ge 3$), contributes negligible amount.

$$
\begin{aligned}
&\sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} |\mathcal{P}_1(\sigma_0+it,\chi)|^2 dt \\
=&\sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} \left| \sum_{2\le n\le Y} \frac{\Lambda(n)\chi(n)}{n^{\sigma_0+it}\log n} \right|^2 dt \\
=&\sum_{\substack{p,r\le Y\\(p,q)=1}} \frac{1}{(pr)^{\sigma_0}} \sum_{\chi(\bmod q)}^{*} \chi(p)\bar{\chi}(r) \int_{-1}^{1} \left(\frac{p}{r}\right)^{it} dt + \sum_{\substack{p^2,r^2\le Y\\(p,q)=1}} \frac{1}{(pr)^{2\sigma_0}} \sum_{\chi(\bmod q)}^{*} \chi(p^2)\bar{\chi}(r^2) \int_{-1}^{1} \left(\frac{p}{r}\right)^{2it} dt \\
=&2\phi^*(q) \sum_{\substack{p\le Y\\(p,q)=1}} \frac{1}{p^{2\sigma_0}} + O\left(\sum_{\substack{dr=q\\d\le Y}} \mu(r)\phi(d) \sum_{\substack{p\equiv\pm r(\bmod d)\\p,r\le Y}} \frac{1}{(pr)^{\sigma_0-\frac{1}{2}}} \right) \\
&\qquad + 2\phi^*(q) \sum_{\substack{p\le\sqrt{Y}\\(p,q)=1}} \frac{1}{p^{4\sigma_0}} + O\left(\sum_{\substack{dr=q\\d\le\sqrt{Y}}} \mu(r)\phi(d) \sum_{\substack{p\equiv\pm r(\bmod d)\\p,r\le\sqrt{Y}}} \frac{1}{(pr)^{2(\sigma_0-\frac{1}{2})}} \right) \\
\ll&2\phi^*(q) \sum_{\substack{p\le Y\\(p,q)=1}} \frac{1}{p^{2\sigma_0}} + 2\phi^*(q) \\
\ll&2\phi^*(q)\log\log q.
\end{aligned}
$$

Similarly, we have

$$
\sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} |\mathcal{P}_2(\sigma_0+it,\chi)|^2 dt \ll 2\phi^*(q)\log\log\log q.
$$

and the first part of the lemma follows.

Suppose $K\ge 1$ is a real number. If $|z|\le K$ then, using that $k!\ge (k/e)^k$,

$$
\begin{aligned}
\sum_{0\le k\le K} \frac{z^k}{k!} = e^z + O\left(\sum_{k>100K} \frac{K^k}{k!}\right) =&e^z + O\left(\sum_{k>100K} \left(\frac{eK}{k}\right)^k\right) \\
=&e^z + O(e^{-100K}).
\end{aligned}
$$

Since $|z|\le K$, we may also write the right side above is $e^z(1+O(e^{-99K}))$. Taking $z=-\mathcal{P}_1(\sigma_0+it,\chi)$ and $K=\log\log q$, (4.11) holds. $\square$

Now it remains to establish the connection between $M(s,\chi)$ and $\mathcal{P}(s,\chi)$. To do so in a similar way we decompose $M(s,\chi)$ as well. But if we observe the definition of $M(s,\chi)$ we see that we need to decompose $a(n)$ first.

$$a_1(n) = \begin{cases} 1 & \text{if } n \text{ has at most } 100\log\log q \text{ prime factors with all } p \le Y \\ 0 & \text{otherwise }. \end{cases}$$

$$a_2(n) = \begin{cases} 1 & \text{if } n \text{ has at most } 100\log\log\log q \text{ prime factors with all } Y < p \le X \\ 0 & \text{otherwise }. \end{cases}$$

Therefore we have

$$\begin{aligned} M(s,\chi) &= M_1(s,\chi)M_2(s,\chi), \\ M_1(s,\chi) &= \sum_n \frac{\mu(n)a_1(n)\chi(n)}{n^s}, \\ M_2(s,\chi) &= \sum_n \frac{\mu(n)a_2(n)\chi(n)}{n^s}. \end{aligned}$$

Lemma 4.3.5. *For the primitive Dirichlet character χ modulo q, we have*

$$\sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} |\mathcal{M}_1(\sigma_0+it,\chi) - M_1(\sigma_0+it,\chi)|^2 dt \ll \phi^*(q)(\log q)^{-60},$$

$$\sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} |\mathcal{M}_2(\sigma_0+it,\chi) - M_2(\sigma_0+it,\chi)|^2 dt \ll \phi^*(q)(\log\log q)^{-60}.$$

Proof. First we write

$$\mathcal{M}_1(s,\chi) = \sum_n \frac{b(n)\chi(n)}{n^s}$$

where $b(n)$ satisfies the following properties.

1. $|b(n)| \le 1$ for all n.
2. $b(n) = 0$ unless $n \le Y^{100\log\log q}$ has only prime factors below Y.
3. $b(n) = \mu(n)a_1(n)$ unless $\Omega(n) > 100\log\log q$ or, $p \le Y$ s.t $p^k | n$ with $p^k > Y$.

Set $c(n) = (b(n) - \mu(n)a_1(n))$, applying Lemma 4.3.2, we have

$$\begin{aligned} &\sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} |\mathcal{M}_1(\sigma_0+it,\chi) - M_1(\sigma_0+it,\chi)|^2 dt \\ =& \sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} \left| \sum_{n \le Y^{100\log\log q}} \frac{c(n)\chi(n)}{n^{\sigma_0+it}} \right|^2 dt \\ =& 2\phi^*(q) \sum_{n_1=n_2 \le Y^{100\log\log Q}} \frac{c(n_1)c(n_2)}{(n_1n_2)^{\sigma_0}} \\ &+ O\left(\sum_{\substack{dr=q \\ d \le Y^{100\log\log q}}} \mu(r)\phi(d) \sum_{\substack{n_1 \equiv \pm n_2(\bmod d) \\ n_1,n_2 \le Y^{100\log\log q}}} \frac{c(n_1)c(n_2)}{(n_1n_2)^{\sigma_0-\frac{1}{2}}} \right) \\ \ll& \phi^*(q)(\log q)^{-60}. \end{aligned}$$

Note that our $a_1(n), a_2(n), b(n)$ are exactly the same as given in Radziwiłł and Soundarajan's paper [RS17]. The only difference is that now we have a further twisting by primitive Dirichlet characters $|\chi(n)| \le 1$.

The off-diagonal term with $n_1 \neq n_2$ contributes

$$\begin{aligned} &\ll \sum_{\substack{dr=q \\ d\le Y^{100\log\log q}}} \mu(r)\phi(d) \sum_{\substack{n_1\equiv \pm n_2 (\bmod d) \\ n_1,n_2\le Y^{100\log\log q}}} \frac{c(n_1)c(n_2)}{(n_1n_2)^{\sigma_0-\frac{1}{2}}} \\ &\ll \sum_{n_1\neq n_2 \le Y^{100\log\log q}} 1 \ll q^{\epsilon'}, \end{aligned}$$

for $\epsilon' > 0$. The diagonal terms $n_1 = n_2$ contribute (recall the property (3) above),

$$\phi^*(q) \sum_{\substack{p|n \implies p\le Y \\ \Omega(n)>100\log\log q}} \frac{1}{n} + \phi^*(q)\left(\sum_{\substack{p\le Y \\ p^k>Y}} \frac{1}{p^k}\right) + \left(\sum_{p|n \implies p\le Y} \frac{1}{n}\right).$$

A small calculation can show that the second term above is $\ll \phi^*(q)\log Y/\sqrt{Y} \ll \phi^*(q)(\log q)^{-60}$. Note that for any $1 < r < 2$ the quantity $r^{\Omega(n)-100\log\log q}$ is always non-negative and is ≥ 1 with $\Omega(n) > 100\log\log q$. Therefore the first term above become

$$\phi^*(q) \sum_{\substack{p|n \implies p\le Y \\ \Omega(n)>100\log\log q}} \frac{1}{n} \ll \phi^*(q) r^{-100\log\log q} \prod_{p\le Y}\left(1+\frac{r}{p}+\frac{r^2}{p^2}+\cdots\right).$$

Choose $r = e^{2/3}$ (say), the above estimate is $\ll \phi^*(q)(\log q)^{-60}$, completing our proof.

Similarly, the second assertion of the lemma follows. $\square$

Proof of Proposition 4.2.5: It follows from (4.11) that

$$\mathcal{M}_1(\sigma_0+it,\chi) = \exp(-\mathcal{P}_1(\sigma_0+it,\chi))(1+O(\log q)^{-99})$$

and by (4.10) we can write

$$(\log q)^{-1} \ll |\mathcal{M}_1(\sigma_0+it,\chi)| \ll \log q.$$

for most χ and t. Combining these two equations we get

$$\begin{aligned} M_1(\sigma_0+it,\chi) =& \mathcal{M}_1(\sigma_0+it,\chi) + O((\log q)^{-25}) \\ &= \exp(-\mathcal{P}_1(\sigma_0+it,\chi))(1+O(\log q)^{-20}) \end{aligned}$$

Similarly for most χ and t, we have

$$\begin{aligned} M_2(\sigma_0+it,\chi) =& \mathcal{M}_2(\sigma_0+it,\chi) + O((\log\log q)^{-25}) \\ &= \exp(-\mathcal{P}_2(\sigma_0+it,\chi))(1+O(\log\log q)^{-20}). \end{aligned}$$

Recall the decomposition of $\mathcal{M}(s,\chi)$ and $\mathcal{P}(s,\chi)$, by multiplying these estimates we obtain

$$M(\sigma_0+it,\chi) = \exp(-\mathcal{P}(\sigma_0+it,\chi))(1+O(\log\log q)^{-20}),$$

completing the proof of the proposition. $\square$

4.3.4 Proof of Proposition 4.2.6

We prove this proposition using Radziwiłł and Soundararajan's [RS17] method. Expanding (4.1) we get

$$
\begin{aligned}
&\sum_{\chi(\bmod q)}^{*}\int_{-1}^{1}|1-L(\sigma_0+it,\chi)M(\sigma_0+it,\chi)|^2dt\\
=&\sum_{\chi(\bmod q)}^{*}\int_{-1}^{1}|L(\sigma_0+it,\chi)M(\sigma_0+it,\chi)|^2dt-2\int_{-1}^{1}\Re(L(\sigma_0+it,\chi)M(\sigma_0+it,\chi))dt+2\phi^*(q) \qquad (4.12)\\
=&S_1-2S_2+S_3.
\end{aligned}
$$

We compute S_1 and S_2 for even primitive Dirichlet characters, a similar computation can be done for odd primitive Dirichlet characters.

The Dirichlet L-function at $s=\sigma_0+it$ is given by

$$L(\sigma_0+it,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n^{\sigma_0+it}}$$

converges conditionally. We want to truncate the Dirichlet series for $n\leq q^{1+\delta}$, where $\delta>0$ is a small constant. Using partial summation we can write that

$$\sum_{\mathcal{Y}<n\leq\mathcal{X}}\frac{\chi(n)}{n^{\sigma_0+it}}=\frac{\sum_{n\leq\mathcal{X}}\chi(n)}{\mathcal{X}^{\sigma_0+it}}-\frac{\sum_{n\leq\mathcal{Y}}\chi(n)}{\mathcal{Y}^{\sigma_0+it}}+(\sigma_0+it)\int_{\mathcal{Y}}^{\mathcal{X}}\frac{\sum_{n\leq u}\chi(n)}{u^{\sigma_0+it}}du.$$

Using Pólya-Vinogradov [Pól18; Vin18] inequality we have

$$\frac{\sum_{n\leq\mathcal{X}}\chi(n)}{\mathcal{X}^{\sigma_0+it}}\ll\frac{q^{1/2}\log q}{\sqrt{\mathcal{X}}}.$$

The right hand side of the above equation tends to 0 as $\mathcal{X}\to\infty$. Similarly, we have

$$\frac{\sum_{n\leq\mathcal{Y}}\chi(n)}{\mathcal{Y}^{\sigma_0+it}}\ll\frac{q^{1/2}\log q}{\sqrt{\mathcal{Y}}}\ll\frac{\log q}{q^{\delta/2}}\ll q^{-\delta/4}.$$

If $\mathcal{Y}=q^{1+\delta}$, then[7]

$$\int_{\mathcal{Y}}^{\mathcal{X}}\frac{\sum_{n\leq u}\chi(n)}{u^{\sigma_0+it}}du\ll\int_{\mathcal{Y}}^{\mathcal{X}}\frac{q^{1/2}\log q}{u^{3/2}}du\ll\frac{q^{1/2}\log q}{\mathcal{Y}^{1/2}}\ll q^{-\delta/4}.$$

Then we have the approximation,

$$L(\sigma_0+it,\chi)=\sum_{n\leq q^{1+\delta}}\frac{\chi(n)}{n^{\sigma_0+it}}+O(q^{-\delta/4}).$$

[7] Note that $\sigma_0+it=O(1)$

Now for the summation S_2, we write

$$\begin{aligned}&\int_{-1}^{1}\sideset{}{^+}\sum_{\chi(\bmod q)} L(\sigma_0+it)M(\sigma_0+it)dt\\ =&\int_{-1}^{1}\sum_{\substack{n\le q^{1+\delta}\\(n,q)=1}}\frac{1}{n^{\sigma_0+it}}\sum_{\substack{m\le X\\(m,q)=1}}\frac{a(m)}{m^{\sigma_0+it}}\sideset{}{^+}\sum_{\chi(\bmod q)}\chi(mn)\end{aligned}$$

From (3.2) of [IS12], for even primitive Dirichlet characters and $(mn,q)=1$, we have

$$\sideset{}{^+}\sum_{\chi(\bmod q)}\chi(mn)=\sum_{\substack{vw=q\\ mn\equiv 1(\bmod q)}}\mu(v)\phi(w).$$

Therefore,

$$\sum_{vw=q}\mu(v)\phi(w)\sum\sum_{\substack{m\le X,n\le q^{1+\delta}\\(mn,q)=1\\ mn\equiv 1(\bmod q)}}\frac{a(m)}{(mn)^{\sigma_0+it}}.$$

To compute S_2, we have two following cases. For the first case we have $w\le q^{1/3}$,

$$\sum_{\substack{vw=q\\ w\le q^{1/3}}}\phi(w)\sum\sum_{m\le X,n\le q^{1+\delta}}\frac{1}{(mn)^{1/2}}\ll q^{5/6+\delta/2+\epsilon}=o(\phi^*(q)),$$

which is our desired estimate. For the second case we have $w>q^{1/3}$

$$\sum_{\substack{vw=q\\ w>q^{1/3}}}\phi(w)\sum\sum_{m\le X,n\le q^{1+\delta}}\frac{1}{(mn)^{\sigma_0}},$$

We have the main term contribution $\phi^*(q)+O(q^{1/3})$ for $mn=1$. For the off-diagonal term contribution we have $mn\neq 1$. Since we have the condition $mn\equiv 1(\mathrm{mod} w)$, we can write $mn=1+kw$ for $k\ge 1$. Note that $kw\le 1+kw=mn\le q^{1+\delta}X\le q^{1+\delta+\epsilon}$. Then $k\le \frac{q^{1+\delta+\epsilon}}{W}$. Then the off-diagonal term contributes

$$\begin{aligned}\sum_{\substack{vw=q\\ w>q^{1/3}}}\phi(w)\sum_{1\le k\le\frac{q^{1+\delta+\epsilon}}{W}}\frac{1}{(1+kw)^{1/2}}\ll& q^{\epsilon}\sum_{\substack{vw=q\\ w>q^{1/3}}}\frac{\phi(w)}{w^{1/2}}\sum_{1\le k\le\frac{q^{1+\delta+\epsilon}}{W}}\frac{1}{k^{1/2}}\\ \ll& q^{1/2+\delta/2+2\epsilon}\sum_{\substack{vw=q\\ w>q^{1/3}}}\frac{\phi(w)}{w}.\end{aligned}$$

Note that $\sum_{\substack{vw=q\\ w>q^{1/3}}} \frac{\phi(w)}{w} \leq d(q) \leq q^{\epsilon}$, where $d(q)$ is the divisor function. Hence the off-diagonal term contributes $q^{1/2+\delta/2+\epsilon}$.

A similar computation shows that for odd primitive characters we have the main term contribution $\phi^*(q)$ for $mn = 1$. Therefore,

$$S_2 = 2\phi^*(q) + (q^{1/2+\delta/2+\epsilon}).$$

To prove Proposition 4.2.6 we have to show that $S_1 \sim 2\phi^*(q)$. We start with the approximate functional equation of the L-functions. Recall the complete functional equation of L-functions (see (2.4))

$$\xi(1-s,\bar{\chi}) = \bar{\varepsilon}(\chi)\xi(s,\chi)$$

Applying (2.15) we write

$$I = \frac{1}{2\pi i}\int_{(1)} \xi(\sigma_0 + it + s, \chi)\xi(\sigma_0 - it + s, \bar{\chi})\frac{G(s)}{s}ds$$

where $G(s)$ is even and $G(0) = 1$. We move the line integration to $\Re(s) = -1$, picking up the contribution of the simple pole at $s = 0$[8]. The contribution of the pole is given by

$$\xi(\sigma_0 + it, \chi)\xi(\sigma_0 - it, \bar{\chi}) = \left(\frac{q}{\pi}\right)^{\sigma_0} \left|\Gamma\left(\frac{\sigma_0 + it}{2}\right)\right|^2 |L(\sigma_0 + it, \chi)|^2.$$

Since $G(s)$ is an even function, on the new line of integration we change the variable from $s \to -s$ and use the functional equation twice to get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(-1)} \xi(\sigma_0 + it + s, \chi)\xi(\sigma_0 - it + s, \bar{\chi})\frac{G(s)}{s}ds \\
=&-\frac{1}{2\pi i}\int_{(1)} \xi(\sigma_0 + it - s, \chi)\xi(\sigma_0 - it - s, \bar{\chi})\frac{G(s)}{s}ds \\
=&-\frac{1}{2\pi i}\int_{(1)} \xi(1-\sigma_0 + it - s, \chi)\xi(1-\sigma_0 + it + s, \bar{\chi})\frac{G(s)}{s}ds
\end{aligned}$$

Let the above equation be I_1 without the negative sign. Then we can write

$$\left(\frac{q}{\pi}\right)^{\sigma_0} \left|\Gamma\left(\frac{\sigma_0 + it}{2}\right)\right|^2 |L(\sigma_0 + it, \chi)|^2 = I + I_1.$$

We can expand I and I_1, as Dirichlet series, since we are in the region of absolute convergence for the L-functions.

[8]Note that $\Gamma(s/2)$ also has a pole at $s = 0$ but that will cancel by the trivial zero of $L(s,\chi)$ at $s = 0$.

$$
\begin{aligned}
I =&\frac{1}{2\pi i}\int_{(1)}\xi(\sigma_0+it+s,\chi)\xi(\sigma_0-it+s,\bar{\chi})\frac{G(s)}{s}ds\\
=&\sum\sum_{m,n\geq 1}\frac{\chi(m)\bar{\chi}(n)}{(mn)^{\sigma_0}}\left(\frac{m}{n}\right)^{-it}\\
&\qquad\frac{1}{2\pi i}\int_{(1)}\left(\frac{q}{\pi}\right)^{\sigma_0+s}(mn)^{-s}\Gamma\left(\frac{\sigma_0+it+s}{2}\right)\Gamma\left(\frac{\sigma_0-it+s}{2}\right)\frac{G(s)}{s}ds.
\end{aligned}
$$

Similarly, we expand I_1 to get

$$
\begin{aligned}
I_1 =&\frac{1}{2\pi i}\int_{(1)}\xi(\sigma_0+it+s,\chi)\xi(\sigma_0-it+s,\bar{\chi})\frac{G(s)}{s}ds\\
=&\sum\sum_{m,n\geq 1}\frac{\bar{\chi}(m)\chi(n)}{(mn)^{1-\sigma_0}}\left(\frac{m}{n}\right)^{it}\\
&\qquad\frac{1}{2\pi i}\int_{(1)}\left(\frac{q}{\pi}\right)^{1-\sigma_0+s}(mn)^{-s}\Gamma\left(\frac{1-\sigma_0-it+s}{2}\right)\Gamma\left(\frac{1-\sigma_0+it+s}{2}\right)\frac{G(s)}{s}ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{\chi(\bmod q)}^{+}\int_{-1}^{1}|L(\sigma_0+it,\chi)|^2dt =& \sum_{m,n\geq 1}\frac{1}{(mn)^{\sigma_0}}\sum_{\chi(\bmod q)}^{+}\chi(m)\bar{\chi}(n)\int_{-1}^{1}\left(\frac{n}{m}\right)^{it}dtV_{+}\left(\frac{mn}{q/\pi}\right)\\
&+\sum_{m,n\geq 1}\frac{1}{(mn)^{1-\sigma_0}}\sum_{\chi(\bmod q)}^{+}\bar{\chi}(m)\chi(n)\int_{-1}^{1}\left(\frac{m}{n}\right)^{it}dtV_{-}\left(\frac{mn}{q/\pi}\right),
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{aligned}
V_{+}(x) =&\frac{1}{2\pi i}\int_{(1)}x^{-s}\frac{\Gamma\left(\frac{\sigma_0+it+s}{2}\right)\Gamma\left(\frac{\sigma_0-it+s}{2}\right)}{\left|\Gamma\left(\frac{\sigma_0+it}{2}\right)\right|^2}\frac{G(s)}{s}ds.\\
V_{-}(x) =&\frac{1}{2\pi i}\int_{(1)}x^{-s}\frac{\Gamma\left(\frac{1-\sigma_0-it+s}{2}\right)\Gamma\left(\frac{1-\sigma_0+it+s}{2}\right)}{\left|\Gamma\left(\frac{\sigma_0+it}{2}\right)\right|^2}\frac{G(s)}{s}ds.
\end{aligned}
$$

The V_i functions are smooth and has rapid decay as $x\to\infty$. Moving the contour to $-1/4+\epsilon$, applying Lemma 4.3.2, for the even primitive Dirichlet characters, for the first term of (4.13), the pole at $s=0$ contributes

$$
\begin{aligned}
&\text{Res}_{s=0}\Big[\phi^*(q)\sum_{m,n\geq 1}\frac{1}{(mn)^{\sigma_0}}\left(\frac{mn\pi}{q}\right)^{-s}\frac{\Gamma\left(\frac{\sigma_0+it+s}{2}\right)\Gamma\left(\frac{\sigma_0-it+s}{2}\right)}{\left|\Gamma\left(\frac{\sigma_0+it}{2}\right)\right|^2}\frac{G(s)}{s}ds\Big]\\
&\quad=\phi^*(q)\sum_{n\geq 1}\frac{1}{n^{2\sigma_0}}\frac{\Gamma\left(\frac{\sigma_0+it}{2}\right)\Gamma\left(\frac{\sigma_0-it}{2}\right)}{\left|\Gamma\left(\frac{\sigma_0+it}{2}\right)\right|^2}G(0)+O(q^{1/2+\epsilon})\\
&\quad=\phi^*(q)\zeta(2\sigma_0)+O(q^{1/2+\epsilon}).
\end{aligned}
\tag{4.14}
$$

With $G(s) = \frac{(\frac{1}{2}-\sigma_0)^2 - s^2}{(\frac{1}{2}-\sigma_0)^2}$, the pole at $s = \frac{1}{2} - \sigma_0$ contributes

$$\text{Res}_{s=\frac{1}{2}-\sigma_0}\Big[\sum_{m,n\geq 1}\frac{1}{(mn)^{\sigma_0}}\left(\frac{mn\pi}{q}\right)^{-s}\frac{\Gamma\left(\frac{\sigma_0+it+s}{2}\right)\Gamma\left(\frac{\sigma_0-it+s}{2}\right)}{\left|\Gamma\left(\frac{\sigma_0+it}{2}\right)\right|^2}\frac{G(s)}{s}ds\Big] \ll q^{1/2+\epsilon}. \tag{4.15}$$

A similar computation can be done for the odd primitive Dirichlet characters.

Similarly, for the second term of (4.13), we shift the contour to $-1/4+\epsilon$ crossing a pole at $s = 0$ and a pole at $s = -1/2 - \sigma_0$, we get $2\phi^*(q)\zeta(2(1-\sigma_0)) + O(q^{1/2+\epsilon})$.

Now we compute the mollified second moments using Radziwiłł and Soundararajan's mehhod. Using (4.13) we get

$$\begin{aligned}
S_1 &= \sum_{\chi(\bmod q)}^{+}\int_{-1}^{1}|L(\sigma_0+it,\chi)M(\sigma_0+it,\chi)|^2 dt\\
&= \sum_{h,k\geq 1}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{\sigma_0}}\sum_{\chi(\bmod q)}^{+}\chi(h)\bar{\chi}(k)\int_{-1}^{1}\left(\frac{h}{k}\right)^{it}|L(\sigma_0+it,\chi)|^2 dt\\
&= \sum_{h,k\geq 1}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{\sigma_0}}\sum_{\chi(\bmod q)}^{+}\chi(h)\bar{\chi}(k)\int_{-1}^{1}\left(\frac{hm}{kn}\right)^{it}\sum_{m,n\geq 1}\frac{\chi(m)\bar{\chi}(n)}{(mn)^{\sigma_0}}dtV_{+}\left(\frac{mn}{q/\pi}\right)\\
&\quad+\sum_{h,k\geq 1}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{1-\sigma_0}}\sum_{\chi(\bmod q)}^{+}\chi(h)\bar{\chi}(k)\int_{-1}^{1}\left(\frac{hm}{kn}\right)^{it}\sum_{m,n\geq 1}\frac{\chi(m)\bar{\chi}(n)}{(mn)^{1-\sigma_0}}dtV_{-}\left(\frac{mn}{q/\pi}\right)\\
&= \sum_{h,k\geq 1}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{\sigma_0}}\sum_{m,n\geq 1}\frac{1}{(mn)^{\sigma_0}}\sum_{\chi(\bmod q)}^{+}\chi(hm)\bar{\chi}(kn)\int_{-1}^{1}\left(\frac{hm}{kn}\right)^{it}dtV_{+}\left(\frac{mn}{q/\pi}\right)\\
&\quad+\sum_{h,k\geq 1}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{1-\sigma_0}}\sum_{m,n\geq 1}\frac{1}{(mn)^{1-\sigma_0}}\sum_{\chi(\bmod q)}^{+}\chi(hm)\bar{\chi}(kn)\int_{-1}^{1}\left(\frac{hm}{kn}\right)^{it}dtV_{-}\left(\frac{mn}{q/\pi}\right).
\end{aligned}$$

Changing the variable with $m = Nk/(h,k)$ and $n = Nh/(h,k)$, we obtain

$$\sum_{m,n\geq 1}\frac{\chi(m)\bar{\chi}(n)}{(mn)^{\sigma_0}} = \zeta(\sigma_0)\chi(h)\bar{\chi}(k)\left(\frac{(h,k)^2}{hk}\right)^{\sigma_0}.$$

Then by (4.14) and (4.15), the sum S_1 becomes

$$\begin{aligned}
S_1 =& 2\phi^*(q)\zeta(2\sigma_0)\sum_{h,k\geq 1}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{2\sigma_0}}(h,k)^{2\sigma_0}\\
&+2\phi^*(q)\zeta(2(1-\sigma_0))\sum_{h,k\geq 1}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{2(1-\sigma_0)}}(h,k)^{2(1-\sigma_0)}+O(q^{1/2+\epsilon}).
\end{aligned}$$

Let $h = h_1h_2$, where h_1 is composed only of primes below Y, and h_2 is composed of primes between Y and X, and then $a(h) = a(h_1)a(h_2)$, as given in Section 4.3.3. Similarly we write $a(k) = a(k_1)a(k_2)$, we get

$$2\phi^*(q)\zeta(2\sigma_0)\sum_{h,k\geq 1}\frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{2\sigma_0}}(h,k)^{2\sigma_0} \tag{4.16}$$

$$=2\phi^*(q)\zeta(2\sigma_0)\left(\sum_{h_1,k_1}\frac{\mu(h_1)\mu(k_1)a(h_1)a(k_1)}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0}\right)\left(\sum_{h_2,k_2}\frac{\mu(h_2)\mu(k_2)a(h_2)a(k_2)}{(h_2k_2)^{2\sigma_0}}(h_2,k_2)^{2\sigma_0}\right) \tag{4.17}$$

Consider the first factor in (4.16), ignoring the condition that h_1, k_1 must have at most $100\log\log q$ prime factors, the resulting sum gives

$$\sum_{\substack{h_1,k_1\\ p|h_1k_1 \implies p\leq Y}}\frac{\mu(h_1)\mu(k_1)}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0}=\prod_{p\leq Y}\left(1-\frac{1}{p^{2\sigma_0}}\right).$$

Approximating the first factor by the product above, we incur an error term which is at most

$$\ll \sum_{\substack{h_1,k_1\\ p|h_1k_1 \implies p\leq Y\\ \Omega(h_1)>100\log\log q}}\frac{|\mu(h_1)\mu(k_1)|}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0},$$

where we used symmetry to assume that h_1 has many prime factors. Since $e^{\Omega(h_1)-100\log\log q}$ is always non-negative, and is ≥ 1 for those h_1 with $\Omega(h_1)\geq 100\log\log q$, the above may be bounded by

$$\begin{aligned}\ll & e^{-100\log\log q}\sum_{\substack{h_1,k_1\\ p|h_1k_1 \implies p\leq Y}}\frac{|\mu(h_1)\mu(k_1)|}{(h_1k_1)^{2\sigma_0}}(h_1,k_1)^{2\sigma_0}e^{\Omega(h_1)}\\ & \ll (\log q)^{-100}\prod_{p\leq Y}\left(1+\frac{1+2e}{p}\right)\ll(\log q)^{-90}.\end{aligned}$$

Thus the first factor in (4.16) is

$$\prod_{p\leq Y}\left(1-\frac{1}{p^{2\sigma_0}}\right)+O((\log q)^{-90})\sim\prod_{p\leq Y}\left(1-\frac{1}{p^{2\sigma_0}}\right).$$

With a similar computation we can obtain that the second term in (4.16) is

$$\prod_{Y<p\leq X}\left(1-\frac{1}{p^{2\sigma_0}}\right)+O((\log\log q)^{-90})\sim\prod_{Y<p\leq X}\left(1-\frac{1}{p^{2\sigma_0}}\right).$$

Using these above estimates (4.16) becomes

$$\sim 2\phi^*(q)\zeta(2\sigma_0)\prod_{p\leq X}\left(1-\frac{1}{p^{2\sigma_0}}\right)\sim 2\phi^*(q)\prod_{p>X}\left(1-\frac{1}{p^{2\sigma_0}}\right)^{-1}\sim 2\phi^*(q).$$

Recalling the definition of X, W and σ_0, and using the prime number theorem and partial summation we write

$$\sum_{p>X}\frac{1}{p^{2\sigma_0}}\ll\int_X^\infty\frac{1}{t^{2\sigma_0}}\frac{dt}{\log t}\ll\frac{X^{1-2\sigma_0}}{(2\sigma_0-1)\log X}=o(1).$$

In the same way the second term of (4.13) is

$$\begin{aligned}&2\phi^*(q)\zeta(2-2\sigma_0)\left(\sum_{h,k}\frac{\mu(h)\mu(k)a(h)a(k)}{hk}(h,k)^{2-2\sigma_0}\right)\\ \sim&\left(2\phi^*(q)\zeta(2-2\sigma_0)\prod_{p\leq X}\left(1-\frac{2}{p}+\frac{1}{p^{2\sigma_0}}\right)\right)\\ \ll&\phi^*(q)\prod_{p\leq X}\left(1-\frac{1}{p}\right)=o(\phi^*(q)).\end{aligned}$$

Hence, we complete the proof of this proposition. □

4.4 Dirichlet L-function attached to twisted form

In this section, we extend the result of the previous section (under the assumption of GRH) by considering the twisted Dirichlet L-function associated with the $GL(3)$ Hecke-Maass cusp form twisted by the primitive Dirichlet characters χ, as defined in Definition 2.2.9. We prove the following theorem conditionally, under the assumption of the moment conjecture. Note that by the end of this chapter we give a brief overview of how to compute second mollified moment of $L(f\otimes\chi, s)$ under the assumption of the moment conjecture.

We prove our next theorem conditionally for a special choice of the parameter Q. Let $q := q_1q_2$ such that q_1, q_2 are prime numbers. For $q \asymp Q$ and $q_1 \asymp Q_1$ and $q_2 \asymp Q_2$ we may write $Q = Q_1Q_2$ where $Q_1 = Q^{3/4-\delta}$ and $Q_2 = Q^{1/4+\delta}$ for $\delta = 1/100$. Since χ is a primitive Dirichlet character χ modulo all $q \asymp Q$, it can be split as $\chi = \chi_1\chi_2$ with primitive χ_i modulo all $q_i \asymp Q_i$ for $i = 1, 2$. We set

$$N(Q) = \#\{\chi \text{ modulo all } q \asymp Q : \chi \text{ primitive}\}.$$

Theorem 4.4.1. *Let f is a Hecke-Maass cusp form of type $\nu = (\nu_1, \nu_2)$ and $N(Q)$ denote the total number of primitive Dirichlet characters modulo all $q \asymp Q$, for the special choice of parameter Q as defined above. Let V be a fixed positive real number. Then as $Q \to \infty$, uniformly for all $v \in [-V, V]$*

$$\frac{1}{2N(Q)}\sum_{q\asymp Q}\sum_{\chi(\bmod q)}^{*} meas\left\{-1\le t\le 1 : \log|L(f\otimes\chi,\frac{1}{2}+it)|\ge v\sqrt{\frac{1}{2}\log\log Q}\right\}$$
$$\sim\frac{1}{\sqrt{2\pi}}\int_v^{\infty}e^{-u^2/2}du.$$

First, we give the idea of the proof of Theorem 4.4.1. Then we prove Theorem 4.4.1 in four steps. First, we take away the problem from the critical line. Then we introduce an auxiliary series and prove that it has approximate normal distribution with mean 0 and variance $\frac{1}{2}\log\log Q$. Finally, we connect the L-function with the auxiliary series by adapting the mollification technique.

In the first step, we take away the problem from the critical line by counting the zeros of L-functions.

Proposition 4.4.2. *Let χ be primitive Dirichlet characters modulo all $q\asymp Q$, where the choice of parameters q and Q holds the conditions stated above. Then for any $\sigma>\frac{1}{2}$ we have*

$$\int_{-1}^{1}\left|\log|L(f\otimes\chi,\frac{1}{2}+it)|-\log|L(f\otimes\chi,\sigma+it)|\right|dt\ll\left(\sigma-\frac{1}{2}\right)\log Q,$$

where t ranges over $[-1,1]$.

We fix the parameters (with a similar reason given in Remark 4.2.3)

$$W=(\log\log\log Q)^4,\ X=Q^{1/(\log\log\log Q)^2},\ Y=Q^{(1/\log\log Q)^2},\ \sigma_0=\frac{1}{2}+\frac{W}{\log Q},$$

where Q is sufficiently large so that $W\ge 3$.

Next we introduce the auxiliary series given by

$$\mathcal{P}(f\otimes\chi,s)=\mathcal{P}(f\otimes\chi,s;X)=\sum_{2\le n\le X}\frac{\Lambda_f(n)\chi(n)}{n^s\log n}.$$

We compute the moments to determine the distribution of the auxiliary series.

Proposition 4.4.3. *Let us assume Ramanujan-Petersson conjecture. Let χ be primitive Dirichlet characters modulo all $q\asymp Q$, then the distribution of $\Re(\mathcal{P}(f\otimes\chi,s))$ is approximately normal with mean 0 and variance $\sim\frac{1}{2}\log\log Q$. Precisely, let V be a fixed positive real number then as $Q\to\infty$, uniformly for all $v\in[-V,V]$*

$$\frac{1}{2N(Q)}\sum_{q\asymp Q}\sum_{\chi(\bmod q)}^{*} meas\left\{-1\le t\le 1 : \Re(\mathcal{P}(f\otimes\chi,\sigma_0+it))\ge v\sqrt{\frac{1}{2}\log\log Q}\right\}$$
$$\sim\frac{1}{\sqrt{2\pi}}\int_v^{\infty}e^{-u^2/2}du.$$

By the definition of the Fourier coefficient of the Hecke-maass cusp form $\lambda_f(1,1) \neq 0$. Then we define the convolution inverse $(\mu_f(1,n))$ of the sequence $(\lambda_f(1,n))$. This is an arithmetic multiplicative function, for a prime number p which satisfies

$$\mu_f(1,1) = 1,\ \mu_f(1,p) = -\lambda_f(1,p),\ \mu_f(1,p^2) = \lambda_f(p,1),$$
$$\mu_f(1,p^3) = \lambda_f(1,p^3) + \lambda_f(p,p^2) = \begin{cases} -1 & \text{if } p \nmid q \\ 0 & \text{otherwise} \end{cases} \quad \text{and if } j \geq 4, \quad \mu_f(1,p^j) = 0.$$

In the next step, we introduce the mollifier to connect the L-function with the auxiliary series. Let $M(f \otimes \chi, s)$ be a Dirichlet polynomial defined as

$$M(f \otimes \chi, s) = \sum_n \frac{\mu_f(1,n)a(n)\chi(n)}{n^s},$$

where $a(n)$ is given by

$$a(n) = \begin{cases} 1 & \text{if } n \text{ is composed only of primes below } X \text{ and has at most } 100\log\log Q \\ & \text{primes below } Y \text{ and at most } 100\log\log\log Q \text{ primes between } Y \text{ and } X. \\ 0 & \text{otherwise .} \end{cases}$$

Notice that $a(n)$ takes the value 0 except when $n \leq Y^{100\log\log Q}X^{100\log\log\log Q} < Q^{\epsilon}$. So, it can be easily seen that $M(f,s)$ is a short Dirichlet polynomial.

Proposition 4.4.4. *Let χ be primitive Dirichlet characters modulo all $q \asymp Q$. Given $\epsilon > 0$ and $\delta > 0$, there is some R so that for all $q > R$, under the assumption of Ramanujan-Petersson conjecture we have*

$$\frac{1}{2N(Q)} \sum_{q \asymp Q} \sum_{\chi (\bmod q)}^{*} meas\left\{-1 \leq t \leq 1 : \left| \frac{M(f \otimes \chi, \sigma_0 + it)}{\exp(-\mathcal{P}(f \otimes \chi, \sigma_0 + it))} - 1 \right| < \delta \right\} < \epsilon.$$

As the final step, we connect the mollifier with the L-function in order to complete the proof of Theorem 4.4.1.

Proposition 4.4.5. *Let χ be primitive Dirichlet characters modulo all $q \asymp Q$. Given $\epsilon > 0$ and $\delta > 0$, there is some R so that for all $q > R$, under the assumption of Ramanujan-Petersson conjecture we have*

$$\frac{1}{N(Q)} \sum_{q \asymp Q} \sum_{\chi (\bmod q)}^{*} \int_{-1}^{1} |1 - L(f \otimes \chi, \sigma_0 + it)M(f \otimes \chi, \sigma_0 + it)|^2 \, dt = o(1). \quad (4.18)$$

So that for $t \in [-1,1]$,

$$\frac{1}{2N(Q)} \sum_{q \asymp Q} \sum_{\chi (\bmod q)}^{*} meas\left\{-1 \leq t \leq 1 : |L(f \otimes \chi, \sigma_0 + it)M(f \otimes \chi, \sigma_0 + it) - 1| < \delta \right\} < \epsilon.$$

We prove these propositions in the later sections. Let us connect them to write the complete proof for Theorem 4.4.1.

Proof of Theorem 4.4.1: Recalling Proposition 4.4.5, it typically says that for the primitive Dirichlet characters χ modulo all $q \asymp Q$ (for most χ and t), we have

$$L(f \otimes \chi, \sigma_0 + it) = (1 + o(1))M(f \otimes \chi, \sigma_0 + it)^{-1}.$$

By Proposition 4.4.4 (for most χ and t) we know that

$$|L(f \otimes \chi, \sigma_0 + it)| = (1 + o(1)) \exp(\Re\mathcal{P}(f \otimes \chi, \sigma_0 + it))$$

and by Proposition 4.4.3 we can conclude that $\log |L(f \otimes \chi, \sigma_0 + it)|$ is normally distributed with mean 0 and variance $\frac{1}{2} \log \log Q$. Finally, with the help of Proposition 4.4.2 we deduce that

$$\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi (\bmod q)} \int_{-1}^{1} \left| \log |L(f \otimes \chi, \frac{1}{2} + it)| - \log |L(f \otimes \chi, \sigma_0 + it)| \right| dt$$
$$\ll N(Q)(\sigma_0 - \frac{1}{2}) \log Q = N(Q)W.$$

So for most χ and t, we have

$$\log |L(f \otimes \chi, \frac{1}{2} + it)| = \log |L(f \otimes \chi, \sigma_0 + it)| + O(W^2).$$

Since $W^2 = o(\sqrt{\log \log Q})$ it follows that similarly like $\log |L(f \otimes \chi, \sigma_0 + it)|$, $\log |L(f \otimes \chi, \frac{1}{2} + it)|$ has the normal distribution with mean 0 and variance $\frac{1}{2} \log \log Q$, which completes the proof of Theorem 4.4.1. □

4.4.1 Proof of Proposition 4.4.2

In this section, we give the proof of Proposition 4.4.2 and this proof is similar as the prove given in Proposition 4.2.2. Let α_i's are the Langlands parameter as defined in (2.11).

Let χ be primitive Dirichlet characters modulo q for all $q \asymp Q$.

Set

$$G(f \otimes \chi, s) = q^{3s/2} \gamma(f \otimes \chi, s) = q^{3s/2} \prod_{i=1}^{3} \Gamma_{\mathbb{R}}(s - \alpha_i)$$

where the α_i's are the Langlands parameter as defined in (2.11).

We show that

$$\left| \log \frac{G(f \otimes \chi, \sigma + it)}{G(f \otimes \chi, 1/2 + it)} \right| \ll \left(\sigma - \frac{1}{2} \right) \log q.$$

Consider the Taylor expansion of the Gamma function we get

$$\begin{aligned}\Gamma(s+\delta) =&\Gamma(s)+\delta\Gamma'(z)+\frac{\delta^2}{2!}\Gamma''(z)+\cdots\\ =&\Gamma(s)+O(\delta)\end{aligned}$$

where $s=\frac{1}{2}+it$ and $\delta=\frac{W}{\log q}$, with $|t|\le 1$.

Note that

$$\arg\left(\frac{G(f\otimes\chi,\sigma+it)}{G(f\otimes\chi,1/2+it)}\right)=\arg\left(G(f\otimes\chi,\sigma+it)\right)-\arg\left(G(f\otimes\chi,1/2+it)\right).$$

Since f is a Hecke-Maass cusp form of type (ν_1,ν_2) then by Ramanujan-Selberg conjecture we know that $\Re(\alpha_i)=0$.

Now, applying the Taylor expansion we have

$$\begin{aligned}\arg\left(G(f\otimes\chi,\sigma+it)\right)=&\arg\left(q^{3(\sigma+it)/2}\gamma(f\otimes\chi,(\sigma+it))\right)\\ =&\arg\left(q^{3(\sigma+it)/2}\prod_{i=1}^{3}\Gamma_{\mathbb{R}}(\sigma+it-\alpha_i)\right)\\ =&3t/2\log q+\sum_{i=1}^{3}\arg\left(\Gamma_{\mathbb{R}}(\frac{1}{2}+it-\alpha_i)\right)+O(\delta).\end{aligned}$$

Similarly,

$$\arg\left(G(f\otimes\chi,1/2+it))\right)=3t/2\log q+\sum_{i=1}^{3}\arg\left(\Gamma_{\mathbb{R}}(\frac{1}{2}+it-\alpha_i)\right).$$

Expanding the complex logarithm and putting Taylor expansion for Gamma function we have,

$$\begin{aligned}&\log\frac{G(f\otimes\chi,\sigma+it)}{G(f\otimes\chi,1/2+it)}\\ =&\log\left|\frac{G(f\otimes\chi,\sigma+it)}{G(f\otimes\chi,1/2+it)}\right|+i\arg\left(\frac{G(f\otimes\chi,\sigma+it)}{G(f\otimes\chi,1/2+it)}\right)\\ =&\log\left(\frac{q}{\pi}\right)^{3(\sigma-1/2)/2}+\sum_{i=1}^{3}\log\left|\Gamma_{\mathbb{R}}(\frac{1}{2}+it-\alpha_i)\right|-\sum_{i=1}^{3}\log\left|\Gamma_{\mathbb{R}}(\frac{1}{2}+it-\alpha_i)\right|+O(\delta)\\ &+i\left(3t/2\log q+\sum_{i=1}^{3}\arg\left(\Gamma_{\mathbb{R}}(\frac{1}{2}+it-\alpha_i)\right)+O(\delta)-3t/2\log q+\sum_{i=1}^{3}\arg\left(\Gamma_{\mathbb{R}}(\frac{1}{2}+it-\alpha_i)\right)\right).\end{aligned}$$

Since, q is large enough and $|t|\le 1$, we write

$$\left|\log\frac{G(f\otimes\chi,\sigma+it)}{G(f\otimes\chi,1/2+it)}\right|\ll\left|\log q^{(\sigma-\frac{1}{2})}\right|\ll\left(\sigma-\frac{1}{2}\right)\log q.$$

Recall the functional equation of the complete L-function

$$\Lambda(f\otimes\chi,s)=G(f\otimes\chi,s)L(f\otimes\chi,s).$$

To prove Proposition 4.4.2 it is enough to prove that

$$\int_{-1}^{1}\left|\log\left|\frac{\Lambda(f\otimes\chi,1/2+it)}{\Lambda(f\otimes\chi,\sigma+it)}\right|\right|dt\ll\left(\sigma-\frac{1}{2}\right)\log Q.$$

Recalling Hadamard's factorization formula (see e.g. Lemma 2.2.11), there exist constants $a=a(f\otimes\chi)$ and $b=b(f\otimes\chi)$ (where $b(f\otimes\chi)=-\sum_{\rho}\Re(1/\rho)$) such that

$$(s(s-1))\Lambda(f\otimes\chi,s)=e^{a+bs}\prod_{\rho\in\mathcal{Z}_{f\otimes\chi}}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where $\rho=\beta+i\gamma\in\mathcal{Z}_{f\otimes\chi}$, where $\mathcal{Z}_{f\otimes\chi}$ denotes the set of all non-trivial zeros of $L(f\otimes\chi,s)$ for Dirichlet characters χ modulo all $q\asymp Q$[9].

Assuming that t is not the ordinate of a zero of $L(f\otimes\chi,s)$ we can write

$$\log\left|\frac{\Lambda(f\otimes\chi,1/2+it)}{\Lambda(f\otimes\chi,\sigma+it)}\right|=\sum_{\rho\in\mathcal{Z}_{f\otimes\chi}}\log\left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|.$$

Suppose $\rho=\beta+i\gamma\in\mathcal{Z}_{f\otimes\chi}$, where $\mathcal{Z}_{f\otimes\chi}$ denotes the set of all non-trivial zeros of $L(f\otimes\chi,s)$ for Dirichlet characters χ modulo all $q\asymp Q$. Integrating over $|t|\leq 1$ we get

$$\begin{aligned}\int_{-1}^{1}\left|\log\left|\frac{\Lambda(f\otimes\chi,1/2+it)}{\Lambda(f\otimes\chi,\sigma+it)}\right|\right|dt&\leq\sum_{\rho\in\mathcal{Z}_{f\otimes\chi}}\int_{-1}^{1}\left|\log\left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|\right|dt\\&=\frac{1}{2}\sum_{\rho\in\mathcal{Z}_{f\otimes\chi}}\int_{-1}^{1}\left|\log\frac{(\beta-\frac{1}{2})^2+(t-\gamma)^2}{(\beta-\sigma)^2+(t-\gamma)^2}\right|dt.\end{aligned}\tag{4.19}$$

If $|t-\gamma|\geq 2$ then with $|t|\leq 1$, we have

$$\begin{aligned}\left|\log\left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|\right|=\left|\Re\log\left(1-\frac{\sigma-\frac{1}{2}}{\sigma+it-\rho}\right)\right|&=\left|\Re\frac{\sigma-\frac{1}{2}}{\sigma+it-\rho}\right|+O\left(\frac{(\sigma-\frac{1}{2})^2}{(t-\gamma)^2}\right)\\&=O\left(\frac{(\sigma-\frac{1}{2})}{(t-\gamma)^2}\right)\end{aligned}$$

So we can write

$$\int_{-1}^{1}\left|\log\left|\frac{(1/2+it)-\rho}{(\sigma+it)-\rho}\right|\right|dt\ll\frac{(\sigma-1/2)}{(t-\gamma)^2}.$$

[9]For simplicity we denote $\gamma_{f\otimes\chi}$- as γ.

Then contribution of these zeros give

$$\sum_{\substack{\rho \in \mathcal{Z}_{f\otimes\chi} \\ |t-\gamma|\geq 2}} \frac{(\sigma - \frac{1}{2})}{(t-\gamma)^2} \ll \log Q.$$

Now consider the range $|t-\gamma| \leq 2$ (which is basically the zeros near t) we have

$$\begin{aligned} \int_{-1}^{1} \left|\log \left|\frac{\frac{1}{2}+it-\rho}{\sigma+it-\rho}\right|\right| dt =& \frac{1}{2}\int_{t-1}^{t+1} \left|\log \frac{(\beta-\frac{1}{2})^2+(t-\gamma)^2}{(\beta-\sigma)^2+(t-\gamma)^2}\right| dt \\ \leq& \frac{1}{2}\int_{-\infty}^{\infty} \left|\log \left|\frac{(\beta-\frac{1}{2})^2+x^2}{(\beta-\sigma)^2+x^2}\right|\right| dx \\ =& \pi\left(\sigma - \frac{1}{2}\right). \end{aligned}$$

So in this case the contribution of zeros is $\ll (\sigma - \frac{1}{2})\log Q$.

Thus in either case

$$\int_{-1}^{1} \left|\log \left|\frac{1/2+it-\rho}{\sigma+it-\rho}\right|\right| dt \ll \frac{(\sigma-\frac{1}{2})}{1+(t-\gamma)^2}.$$

Inserting this in (4.19), from Theorem 5.8 of [IK04] we can conclude

$$\begin{aligned} \int_{-1}^{1} \left|\log \left|\frac{L(f\otimes\chi,1/2+it)}{L(f\otimes\chi,\sigma+it)}\right|\right| dt =& \int_{-1}^{1} \left|\log \left|\frac{\Lambda(f\otimes\chi,1/2+it)}{\Lambda(f\otimes\chi,\sigma+it)}\right|\right| dt + O\left(\sigma-\frac{1}{2}\right)\log Q \\ \ll& \left(\sigma - \frac{1}{2}\right)\log Q \end{aligned}$$

which completes the proof. □

4.4.2 Proof of Proposition 4.4.3

In this section, we study the moments of the auxiliary series $\mathcal{P}(f\otimes\chi,s)$ to prove that it has normal distribution with mean 0 and variance $\frac{1}{2}\log\log Q$.

Similarly like Lemma 4.3.2, we prove the following lemma by averaging over the Dirichlet characters and moduli.

Lemma 4.4.6. *Let $N(Q)$ be the number of primitive Dirichlet characters for all modulo $q \asymp Q$. Recall that $X = Q^{(1/\log\log Q)^2}$. Then for $\epsilon > 0$ and complex numbers $\{a_n\}_{n\in\mathbb{N}}$, we have*

$$\sum_{q\asymp Q} \sideset{}{^*}\sum_{\chi (\bmod q)} \left|\sum_{n\leq X}\frac{a_n\chi(n)}{n^{\sigma_0}}\right|^2 = N(Q)\sum_{n\leq X}\frac{|a_n|^2}{n^{2\sigma_0}} + O(Q^{1+\epsilon}). \tag{4.20}$$

Proof. Expanding (4.20), we have

$$\sum_{q\asymp Q}\sideset{}{^*}\sum_{\chi(\bmod q)}\left|\sum_{n\leq X}\frac{a_n\chi(n)}{n^{\sigma_0}}\right|^2=\sum_{m,n\leq X}\frac{a_m\bar{a}_n}{(mn)^{\sigma_0}}\sum_{q\asymp Q}\sideset{}{^*}\sum_{\chi(\bmod q)}\chi(m)\bar{\chi}(n). \tag{4.21}$$

Let $\phi^+(q)$ be the number of even primitive Dirichlet characters modulo q. Note that $\sum^+$ denote the summation over the even primitive characters modulo q. From Lemma 4.1 of [BM11] for $(mn,q)=1$, we can write

$$\sideset{}{^+}\sum_{\chi(\bmod q)}\chi(m)\bar{\chi}(n)=\frac{1}{2}\sum_{\substack{dr=q\\ d|m\pm n}}\mu(r)\phi(d).$$

In order to prove the lemma, we compute equation (4.21) for even and odd primitive Dirichlet characters. We show the computation for even primitive Dirichlet characters, and a similar computation for odd primitive Dirichlet characters follows similarly.

For even primitive Dirichlet characters $\chi(-1)=1$, inserting the above equation to (4.21), we have

$$\begin{aligned}
&\sum_{m,n\leq X}\frac{a_m\bar{a}_n}{(mn)^{\sigma_0}}\cdot\sideset{}{^+}\sum_{\chi(\bmod q)}\chi(m)\bar{\chi}(n)\\
=&\sum_{m,n\leq X}\frac{a_m\bar{a}_n}{(mn)^{\sigma_0}}\cdot\frac{1}{2}\sum_{\substack{dr=q\\ d|m\pm n}}\mu(r)\phi(d)\\
=&\frac{1}{2}\sum_{dr=q}\mu(r)\phi(d)\sum_{\substack{m\equiv\pm n(\bmod d)\\ m,n\leq X}}\frac{a_m\bar{a}_n}{(mn)^{\sigma_0}}
\end{aligned}$$

We have two cases for the above equation. For the first case, if $d>X$, then $m\equiv\pm n(\bmod d)$, which implies $m=n$. Therefore, the second case arises for $d\leq X$. Then the above equation equals to

$$\frac{1}{2}\sum_{\substack{dr=q\\ d>X}}\mu(r)\phi(d)\sum_{n\leq X}\frac{|a_n|^2}{n^{2\sigma_0}}+\frac{1}{2}\sum_{\substack{dr=q\\ d\leq X}}\mu(r)\phi(d)\sum_{\substack{m\equiv\pm n(\bmod d)\\ m,n\leq X}}\frac{a_m\bar{a}_n}{(mn)^{\sigma_0}}.$$

Observe that in the second term of the above summation, for $d\leq X$, and $m\neq n$ is very small. Then the error term can be bounded by

$$\frac{1}{2}\sum_{\substack{dr=q\\ d\leq X}}\mu(r)\phi(d)\sum_{\substack{m\equiv\pm n(\bmod d)\\ m,n\leq X}}\frac{a_m\bar{a}_n}{(mn)^{\sigma_0}}\ll X^{(o(1))}\ll q^{\epsilon}.$$

Thus, the main term for $m=n$ equals to

$$\sum_{n\leq X}\frac{|a_n|^2}{n^{2\sigma_0}}\left(\frac{1}{2}\sum_{dr=q}\mu(r)\phi(d)+O\left(\sum_{d\leq X}\mu^2(r)\phi(d)\right)\right)=\frac{1}{2}\phi^*(q)\sum_{n\leq X}\frac{|a_n|^2}{n^{2\sigma_0}}+O(q^{\epsilon}),$$

recalling the definition of X we conclude the above bound.

Note that a similar computation can be done for odd primitive characters $\chi(-1) = -1$. Now averaging over the moduli we get

$$\sum_{q \asymp Q} \left(\frac{1}{2} \phi^*(q) \sum_{n \leq X} \frac{|a_n|^2}{n^{2\sigma_0}} + O(q^{\epsilon}) \right)$$

The summation $\sum_{q \asymp Q} \phi^*(q)$ is the number of primitive Dirichlet characters modulo all $q \asymp Q$, which is $N(Q)$ by definition. Hence,

$$\sum_{q \asymp Q} \sum_{\chi (\bmod q)}^{+} \left| \sum_{n \leq X} \frac{a_n \chi(n)}{n^{\sigma_0}} \right|^2 = \frac{1}{2} N(Q) \sum_{n \leq X} \frac{|a_n|^2}{n^{2\sigma_0}} + O(Q^{1+\epsilon}).$$

Note that a similar computation can be done for odd primitive characters $\chi(-1) = -1$. In that case instead of (4.6), we have (4.7).

Adding the cases of even and odd primitive Dirichlet characters and recalling the definition of X, we conclude the proof.

$\square$

We prove this proposition by restricting the sum to primes and then compute moments. For primes p^k with $k \geq 3$ contributes

$$\left| \sum_{\substack{2 \leq p^k \leq X \\ k \geq 3}} \frac{\Lambda_f(p^k)\chi(p^k)}{p^{ks}(k \log p)} \right| \ll \sum_{\substack{2 \leq p^k \leq X \\ k \geq 3}} \frac{1}{3p^{k\sigma_0}} = O(1).$$

where $\Re(s) = \sigma_0$. For the prime square contribution we have

$$\begin{aligned}
&\sum_{q \asymp Q} \sum_{\chi (\bmod q)}^{*} \int_{-1}^{1} \left| \sum_{\substack{p^2 \leq X \\ (p,q)=1}} \frac{\Lambda_f(p^2)\chi(p^2)}{p^{2(\sigma_0+it)} \cdot 2} \right|^2 dt \\
=&\frac{1}{4} \sum_{q \asymp Q} \sum_{\chi (\bmod q)}^{*} \int_{-1}^{1} \sum_{\substack{p_1, p_2 \leq \sqrt{X} \\ (p,q)=1}} \frac{\Lambda_f(p_1^2)\Lambda_f(p_2^2)\chi(p_1^2)\chi(p_2^2)}{p_1^{2(\sigma_0+it)} p_2^{2(\sigma_0+it)} \log p_1 \log p_2} dt.
\end{aligned}$$

Note that $\Lambda_f(p^2) = \sum_{i=1}^{3} \gamma_i(p)^2 \log p$ where $\gamma_i(p)$ are the complex roots of the quadratic equation $\mathfrak{X} - \lambda_f(1,p)\mathfrak{X}^2 + \lambda_f(p,1)\mathfrak{X} - 1 = 0$. By assuming Ramanujan-Petersson (see (2.1)) conjecture we can say that $|\sum_{i=1}^{3} \gamma_i(p)^2|^2 \ll p^{4\epsilon}$.

First we will show the off-diagonal term treatment, then we show the main term computation. As shown in the proof of Lemma 4.4.6, for $d \leq \sqrt{X}$, $p_1^2 \equiv \pm p_2^2 (\bmod d)$ implies that $p_1^2 \neq p_2^2$. Then for the even primitive Dirichlet characters, applying (2.16) and (2.17), the off-diagonal term of the above equation is bounded by

$$\begin{aligned}
&\ll \sum_{\substack{p_1 \neq p_2 \\ p_1, p_2 \leq \sqrt{X} \\ (p_1 p_2, q)=1}} \frac{1}{(p_1 p_2)^{2\sigma_0 - \frac{1}{2}}} \sum_{q \asymp Q} \sum_{\chi(\bmod q)}^{+} \chi(p_1^2) \bar{\chi}(p_2^2) \\
&\ll \frac{1}{2} \sum_{q \asymp Q} \sum_{\substack{dr=q \\ d \leq \sqrt{X}}} \mu(r) \phi(d) \sum_{\substack{p_1^2 \equiv \pm p_2^2 \\ p_1^2, p_2^2 \leq X}} \frac{1}{(p_1 p_2)^{2\sigma_0 - \frac{1}{2}}} \ll Q^{1+\epsilon}.
\end{aligned}$$

Applying Lemma 4.4.6 and by (2.16) and (2.17), the above integration is bounded by

$$\ll N(Q) \sum_{p \leq \sqrt{X}} \frac{1}{p^{4\sigma_0}} + Q^{1+\epsilon} \ll N(Q). \tag{4.22}$$

Let $A(Q;X) = A(Q) = \sum_{2 \leq p \leq \sqrt{X}} \frac{\sum_{i=1}^{3} \gamma_i(p)^2}{p^{2\sigma_0}}$, from (4.22) and Chebyshev's inequality we have

$$\begin{aligned}
&\frac{1}{2N(Q)} \sum_{q \asymp Q} \sum_{\chi(\bmod q)}^{*} meas\{-1 \leq t \leq 1 : |A(Q)| > L\} \\
&\leq \frac{1}{L^2} \sum_{q \asymp Q} \sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} |A(Q)|^2 dt \ll N(Q)/L^2
\end{aligned}$$

for any positive real number $L > 1$. In other words we can say that the square of primes in $\mathcal{P}(f \otimes \chi, s)$ contribute a measure at most $O(N(Q)/L^2)$. With the same argument given in Remark 4.2.3, we choose $L = o(\log \log \log Q)$.

Now we restrict the sum $\mathcal{P}(f \otimes \chi, s)$ to primes and define

$$\mathcal{P}_0(f \otimes \chi, s) = \mathcal{P}_0(f \otimes \chi, s; X) = \sum_{p \leq X} \frac{\lambda_f(1,p)\chi(p)}{p^{\sigma_0 + it}}.$$

Next we study the moments of $\mathcal{P}_0(f \otimes \chi, s)$.

Lemma 4.4.7. *Suppose k and ℓ are non-negative integers with $X^{k+\ell} \ll Q$. If $k \neq \ell$*

$$\sum_{q \asymp Q} \sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} \mathcal{P}_0(f \otimes \chi, \sigma_0 + it)^k \overline{\mathcal{P}_0(f \otimes \chi, \sigma_0 + it)}^{\ell} dt \ll N(Q).$$

If $k = \ell$, for $\epsilon > 0$, we have

$$\sum_{q \asymp Q} \sum_{\chi(\bmod q)}^{*} \int_{-1}^{1} |\mathcal{P}_0(f \otimes \chi, \sigma_0 + it)|^{2k} dt = k! 2N(Q)(\log \log Q)^k + O_k(N(Q)(\log \log Q)^{k-1+\epsilon}).$$

Proof. Write

$$\mathcal{P}_0(f\otimes\chi,s)^k=\sum_n \frac{a_k(n)\lambda_f(1,n)\chi(n)}{n^s}$$

where

$$a_k(n)=\begin{cases}\frac{k!}{\alpha_1!\cdots\alpha r!} & \text{if } n=\prod_{j=1}^r p_j^{\alpha_j}, p_1<\ldots<p_r<X, \sum_{j=1}^r \alpha_j=k.\\ 0 & \text{otherwise .}\end{cases}$$

Let m,n are positive integers, then we write

$$\Delta(m,n)=\sum_{q\asymp Q}\sideset{}{^*}\sum_{\chi(\bmod q)}\chi(m)\bar{\chi}(n).$$

Therefore,

$$\begin{aligned}&\sum_{q\asymp Q}\sideset{}{^*}\sum_{\chi(\bmod q)}\int_{-1}^{1}\mathcal{P}_0(f\otimes\chi,\sigma_0+it)^k\overline{\mathcal{P}_0(f\otimes\chi,\sigma_0+it)}^{\ell}dt\\ =&\sum_{q\asymp Q}\sideset{}{^*}\sum_{\chi(\bmod q)}\int_{-1}^{1}\left\{\sum_n\frac{a_k(n)\lambda_f(1,n)\chi(n)}{n^s}\cdot\sum_m\frac{a_\ell(m)\bar{\lambda}_f(1,m)\bar{\chi}(m)}{m^{\bar{s}}}\right\}dt.\\ &\sum_{m,n}\frac{a_k(n)a_\ell(m)\lambda_f(1,n)\bar{\lambda}_f(1,m)}{(mn)^{\sigma_0}}\Delta(m,n)\int_{-1}^{1}\left(\frac{m}{n}\right)^{it}dt\\ =&2\sum_n\frac{a_k(n)a_\ell(n)\lambda_f(1,n)\bar{\lambda}_f(1,n)}{n^{2\sigma_0}}\Delta(n,n)+\sum_{m,n}\frac{a_k(n)a_\ell(m)\lambda_f(1,n)\bar{\lambda}_f(1,m)}{(mn)^{\sigma_0-\frac{1}{2}}}\Delta(m,n)\end{aligned}$$

First we compute the sum for even Dirichlet characters. From (4.6) and applying (2.16) and (2.17) we write

$$\begin{aligned}&\sum_{m,n}\frac{a_k(n)a_\ell(m)\lambda_f(1,n)\bar{\lambda}_f(1,m)}{(mn)^{\sigma_0}}\sum_{q\asymp Q}\sideset{}{^+}\sum_{\chi(\bmod q)}\chi(m)\bar{\chi}(n)\int_{-1}^{1}\left(\frac{m}{n}\right)^{it}dt\\ =&\sum_{m,n}\frac{a_k(n)a_\ell(m)\lambda_f(1,n)\bar{\lambda}_f(1,m)}{(mn)^{\sigma_0}}\cdot\frac{1}{2}\sum_{q\asymp Q}\sideset{}{^+}\sum_{\substack{dr=q\\ d\mid m\pm n}}\mu(r)\phi(d)\int_{-1}^{1}\left(\frac{m}{n}\right)^{it}dt\\ =&\frac{1}{2}\sum_{q\asymp Q}\sideset{}{^+}\sum_{\substack{dr=q\\ d\mid m\pm n}}\mu(r)\phi(d)\sum_{m,n}\frac{a_k(n)a_\ell(m)\lambda_f(1,n)\bar{\lambda}_f(1,m)}{(mn)^{\sigma_0}}\int_{-1}^{1}\left(\frac{m}{n}\right)^{it}dt.\end{aligned}$$

Similarly, like the proof of Lemma 4.4.6 and applying (2.16) and (2.17), we have

$$\frac{1}{2}\sum_{q\asymp Q}\sideset{}{^+}\sum_{\substack{dr=q\\ d>X^{k+\ell}}}\mu(r)\phi(d)\sum_{n\le X^{k+\ell}}\frac{a_k(n)a_\ell(n)|\lambda_f(1,n)|^2}{n^{2\sigma_0}}$$

$$+O\left(\frac{1}{2}\sum_{q\asymp Q}\sideset{}{^+}\sum_{\substack{dr=q\\ d\le X^{k+\ell}}}\mu(r)\phi(d)\sum_{\substack{m\equiv n(\bmod d)\\ m\le X^k\\ n\le X^\ell}}\frac{a_k(n)a_\ell(m)\lambda_f(1,n)\bar{\lambda}_f(1,m)}{(mn)^{\sigma_0-\frac{1}{2}}}\right).$$

For the diagonal term $a_k(n)a_\ell(m)$ is 0 by definition for $k\neq\ell$. The off-diagonal term of the above equation is is bounded by

$$\frac{1}{2}\sum_{q\asymp Q}\sideset{}{^+}\sum_{\substack{dr=q\\ d\le X^{k+\ell}}}\mu(r)\phi(d)\sum_{\substack{m\equiv\pm n(\bmod d)\\ m\le X^k\\ n\le X^\ell}}\frac{a_k(n)a_\ell(m)\lambda_f(1,n)\bar{\lambda}_f(1,m)}{(mn)^{\sigma_0-\frac{1}{2}}}\ll Q^{1+\epsilon}\ll N(Q).$$

A similar computation can be done for the odd primitive Dirichlet characters. Then the off-diagonal term is bounded by

$$\ll\sum_{q\asymp Q}\sideset{}{^*}\sum_{\chi(\bmod q)}\sum_{\substack{m\neq n\\ m\le X^k\\ n\le X^\ell}}\frac{a_k(n)a_\ell(m)\lambda_f(1,n)\bar{\lambda}_f(1,m)}{(mn)^{\sigma_0-\frac{1}{2}}}\ll N(Q).$$

We conclude the first part of the lemma.

For the second part, for $k=\ell$ the diagonal term contributes $\sum_n\frac{a_k(n)^2|\lambda_f(1,n)|^2|\chi(n)|^2}{n^{2\sigma_0}}=\sum_n\frac{a_k(n)^2|\lambda_f(1,n)|^2}{n^{2\sigma_0}}$. By the definition for the given positive integers $\alpha_1,\dots,\alpha_r$ with $\sum_{i=1}^r\alpha_i=k$ the contribution of the term of n of the form $p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ is given by

$$\ll N(Q)\prod_{i=1}^r\left(\sum_{\substack{p\le X\\ (p,q)=1}}\frac{|\lambda_f(1,p)|^2}{p^{2\sigma_0\alpha_i}}\right)\ll N(Q)(\log\log Q)^{r+\epsilon}$$

The terms with n not being square free contributes (with $r\le k-1$) $O_k((\log\log Q)^{k-1+\epsilon})$. The square-free n terms give (by Proposition 2.4 of [RS96] and noting that $n=p^k$

(for $k \geq 2$) contributes $O(1)$ in (2.2))

$$\begin{aligned}
&k! \sum_{\substack{p_1,\ldots,p_k \leq X \\ \text{all } p_j\text{'s are distinct, } (p_j,q)=1}} \frac{|\lambda_f(1, p_1\cdots p_k)|^2}{(p_1\cdots p_k)^{2\sigma_0}} \\
=&k! \left(\sum_{\substack{p\leq X \\ (p,q)=1}} \frac{|\lambda_f(1,p)|^2}{p^{2\sigma_0}} \right)^k + O_k((\log\log Q)^{k-1+\epsilon}) \\
=&k!(\log\log Q)^k + O_k((\log\log Q)^{k-1+\epsilon}).
\end{aligned}$$

Recalling the definition of X completes the proof. □

Proof of Proposition 4.4.3: By Lemma 4.4.7 for any odd k we have

$$\begin{aligned}
&\sum_{q\asymp Q} \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} (\Re(\mathcal{P}_0(f\otimes\chi, \sigma_0+it)))^k \, dt \\
=&\sum_{q\asymp Q} \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \frac{1}{2^k} \left(\mathcal{P}_0(f\otimes\chi, \sigma_0+it) + \overline{\mathcal{P}_0(f\otimes\chi, \sigma_0+it)}\right)^k dt \\
=&\frac{1}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{q\asymp Q} \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} (\mathcal{P}_0(f\otimes\chi,\sigma_0+it))^\ell \overline{(\mathcal{P}_0(f\otimes\chi,\sigma_0+it)}^{k-\ell} dt \\
&\ll N(Q).
\end{aligned}$$

Observe that it is impossible to have $\ell = k-\ell$ for any odd k, for all even k, we have $\ell = k - \ell = k/2$ and again with the help of Lemma 4.4.7 we obtain,

$$\begin{aligned}
&\frac{1}{2N(Q)} \sum_{q\asymp Q} \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} (\Re(\mathcal{P}_0(f\otimes\chi, \sigma_0+it)))^k \, dt \\
=&\frac{1}{2^k} \binom{k}{k/2} \left(\frac{k}{2}\right)! (\log\log Q)^{\frac{k}{2}} + O_k\left((\log\log Q)^{\frac{k}{2}-1+\epsilon}\right).
\end{aligned}$$

The above equation matches with the Gaussian distribution (see (2.20)) with mean 0 and variance $\frac{1}{2}\log\log Q$. □

4.4.3 Proof of Proposition 4.4.4

Similarly like Proposition 3.2.4 and 4.2.5 we decompose $\mathcal{P}(f\otimes\chi, s)$ and $M(f\otimes\chi, s)$. Starting with the decomposition of $\mathcal{P}(f\otimes\chi, s)$ into $\mathcal{P}_1(f\otimes\chi, s)$ and $\mathcal{P}_2(f\otimes\chi, s)$ we have

$$\begin{aligned}
\mathcal{P}_1(f\otimes\chi, s) &= \sum_{2\leq n\leq Y} \frac{\Lambda_f(n)\chi(n)}{n^s \log n}, \\
\mathcal{P}_2(f\otimes\chi, s) &= \sum_{Y<n\leq X} \frac{\Lambda_f(n)\chi(n)}{n^s \log n}.
\end{aligned}$$

Set

$$\mathcal{M}_1(f\otimes\chi,s)=\sum_{0\le k\le 100\log\log Q}\frac{(-1)^k}{k!}\mathcal{P}_1(f\otimes\chi,s)^k,$$
$$\mathcal{M}_2(f\otimes\chi,s)=\sum_{0\le k\le 100\log\log\log Q}\frac{(-1)^k}{k!}\mathcal{P}_2(f\otimes\chi,s)^k.$$

In the next lemma we establish the connection between $\mathcal{P}_j(f\otimes\chi,s)$ and $\mathcal{M}_j(f\otimes\chi,s)$ for $j=1,2$.

Lemma 4.4.8. *Let χ be a primitive Dirichlet character modulo q with $q\asymp Q$, we have*

$$|\mathcal{P}_1(f\otimes\chi,s)|\le\log\log Q, \qquad |\mathcal{P}_2(f\otimes\chi,s)|\le\log\log\log Q, \tag{4.23}$$

is true for most χ and t. Moreover,

$$\mathcal{M}_1(f\otimes\chi,s)=\exp(\mathcal{P}_1(f\otimes\chi,s))(1+O(\log Q)^{-99}), \qquad \mathcal{M}_2(f\otimes\chi,s)=\exp(\mathcal{P}_2(f\otimes\chi,s))(1+O(\log\log Q)^{-99}). \tag{4.24}$$

Proof. With a similar argument of Section 4.4.2 (that the auxiliary series $\mathcal{P}_1$ and $\mathcal{P}_2$ supports only on prime powers) and from Lemma 4.4.6, we have[10] (by Proposition 2.4 of [RS96] and noting that $n=p^k$ (for $k\ge 2$) contributes $O(1)$ in (2.2))

[10]Recall that $\Lambda_f(n)$ is the coefficient of the logarithmic derivative of the L-function in Dirichlet series supported on prime powers. Since the argument we have given in the proof of Proposition 4.4.3, we know that the primes with higher power (i.e. p^k with $k\ge 2$) contribute negligible amount so we can write $\sum_{n\le Y}\left|\frac{\Lambda_f(n)}{n^{\sigma_0+it}}\right|^2\approx\sum_{n\le Y}\frac{\lambda_f(1,p)^2}{p^{2\sigma_0}}\ll\log\log Q$. Similarly $\sum_{Y\le n\le X}\left|\frac{\Lambda_f(n)}{n^{\sigma_0+it}}\right|^2\approx\sum_{Y\le n\le X}\frac{\lambda_f(1,p)^2}{p^{2\sigma_0}}\ll\log\left(\frac{\log X}{\log Y}\right)\ll\log\log\log Q$.

$$
\begin{aligned}
&\sum_{q\asymp Q}\ \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} |\mathcal{P}_1(f\otimes\chi,\sigma_0+it)|^2 dt \\
=&\sum_{q\asymp Q}\ \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} \left| \sum_{2\le n\le Y} \frac{\Lambda_f(n)\chi(n)}{n^{\sigma_0+it}\log n} \right|^2 dt \\
=&\sum_{\substack{p,r\le Y\\(p,q)=1}} \frac{\lambda_f(1,p)\lambda_f(1,r)}{(pr)^{\sigma_0}}\Delta(p,r)\int_{-1}^{1}\left(\frac{p}{r}\right)^{it}dt+\sum_{\substack{p^2,r^2\le Y\\(p,q)=1}} \frac{\lambda_f(1,p^2)\lambda_f(1,r^2)}{(pr)^{2\sigma_0}}\Delta(p^2,r^2)\int_{-1}^{1}\left(\frac{p}{r}\right)^{2it}dt \\
=&2N(Q)\sum_{\substack{p\le Y\\(p,q)=1}} \frac{|\lambda_f(1,p)|^2}{p^{2\sigma_0}}+O\left(\sum_{q\asymp Q}\sum_{\substack{dr=q\\d\le Y}}\mu(r)\phi(d)\sum_{\substack{p\equiv\pm r(\bmod d)\\p,r\le Y}}\frac{\lambda_f(1,p)\bar{\lambda}_f(1,r)}{(pr)^{\sigma_0-\frac{1}{2}}}\right) \\
&\quad +2N(Q)\sum_{\substack{p\le\sqrt{Y}\\(p,q)=1}} \frac{|\lambda_f(1,p)|^2}{p^{4\sigma_0}}+O\left(\sum_{q\asymp Q}\sum_{\substack{dr=q\\d\le\sqrt{Y}}}\mu(r)\phi(d)\sum_{\substack{p\equiv\pm r(\bmod d)\\p,r\le\sqrt{Y}}}\frac{\lambda_f(1,p)\bar{\lambda}_f(1,r)}{(pr)^{2(\sigma_0-\frac{1}{2})}}\right) \\
\ll&2N(Q)\sum_{\substack{p\le Y\\(p,q)=1}} \frac{\lambda_f(1,p)^2}{p^{2\sigma_0}}+2N(Q) \\
\ll&N(Q)\log\log Q.
\end{aligned}
$$

Similarly, we have

$$
\sum_{q\asymp Q}\ \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} |\mathcal{P}_2(f\otimes\chi,\sigma_0+it)|^2 dt \ll N(Q)\log\log\log Q,
$$

completing the proof of (4.23).

Suppose $K\ge 1$ is a real number. If $|z|\le K$ then, using that $k!\ge(k/e)^k$, we write

$$
\begin{aligned}
\sum_{0\le k\le K}\frac{z^k}{k!}=e^z+O\left(\sum_{k>100K}\frac{K^k}{k!}\right)=&e^z+O\left(\sum_{k>100K}\left(\frac{eK}{k}\right)^k\right) \\
=&e^z+O(e^{-100K}).
\end{aligned}
$$

Since $|z|\le K$, we may also write the right side above is $e^z(1+O(e^{-99K}))$. Take $z=-\mathcal{P}_1(f\otimes\chi,\sigma_0+it)$ and $K=\log\log Q$, (4.24) holds. $\square$

In order to complete the proof of Proposition 4.4.4 next we establish the connection between $M_j(f\otimes\chi,s)$ and $\mathcal{M}_j(f\otimes\chi,s)$ for $j=1,2$.

We establish this connection by decomposing $M(f\otimes\chi,s)$.

$$
\begin{aligned}
M(f \otimes \chi, s) &= M_1(f \otimes \chi, s) M_2(f \otimes \chi, s), \\
M_1(f \otimes \chi, s) &= \sum_n \frac{\mu_f(1,n) a_1(n) \chi(n)}{n^s}, \\
M_2(f \otimes \chi, s) &= \sum_n \frac{\mu_f(1,n) a_2(n) \chi(n)}{n^s},
\end{aligned}
$$

where

$$
a_1(n) = \begin{cases} 1 & \text{if } n \text{ has at most } 100 \log\log Q \text{ prime factors with all } p \le Y \\ 0 & \text{otherwise .} \end{cases}
$$

$$
a_2(n) = \begin{cases} 1 & \text{if } n \text{ has at most } 100 \log\log\log Q \text{ prime factors with all } Y < p \le X \\ 0 & \text{otherwise .} \end{cases}
$$

Lemma 4.4.9. *Let χ be a primitive Dirichlet character modulo q with $q \asymp Q$ then under the assumption of Ramanujan-Petersson conjecture we have*

$$
\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi (\bmod q)} \int_{-1}^{1} |\mathcal{M}_1(f \otimes \chi, \sigma_0 + it) - M_1(f \otimes \chi, \sigma_0 + it)|^2 \, dt \ll N(Q) (\log Q)^{-60},
$$

$$
\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi (\bmod q)} \int_{-1}^{1} |\mathcal{M}_2(f \otimes \chi, \sigma_0 + it) - M_2(f \otimes \chi, \sigma_0 + it)|^2 \, dt \ll N(Q) (\log\log Q)^{-60}.
$$

Proof. We write

$$
\mathcal{M}_1(f \otimes \chi, s) = \sum_n \frac{b(n) \lambda_f(1,n) \chi(n)}{n^s}
$$

where $b(n)$ satisfies the following properties:

1. $|b(n)| \le 1$ for all n.
2. $b(n) = 0$ unless $n \le Y^{100 \log\log Q}$ has only prime factors below Y.
3. $b(n) = \mu_f(1,n) a_1(n)$ unless $\Omega(n) > 100 \log\log Q$ or, $p \le Y$ s.t $p^k | n$ with $p^k > Y$.

Set $c(n) = b(n) \lambda_f(1,n) - \mu_f(1,n) a_1(n)$, we have

$$
\begin{aligned}
&\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi (\bmod q)} \int_{-1}^{1} |\mathcal{M}_1(f \otimes \chi, \sigma_0 + it) - M(f \otimes \chi, \sigma_0 + it)|^2 dt \\
= &\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi (\bmod q)} \int_{-1}^{1} \left| \sum_{n \le Y^{100 \log\log Q}} \frac{c(n) \chi(n)}{n^{\sigma_0 + it}} \right|^2 dt \\
\ll &N(Q) (\log Q)^{-60}.
\end{aligned}
$$

We note that our $a_1(n), a_2(n)$ are exactly the same as Radziwiłł and Soundararajan's method. Instead of Möbius function we have the convolution inverse of $\lambda_f(1,n)$. We have a further twist by $|\lambda_f(1,n)|$ and the primitive Dirichlet character $|\chi(n)| \leq 1$. The Ramanujan-Petersson conjecture (see (2.1)) asserts that $|\lambda_f(1,n)| \leq d_3(n) \ll n^\epsilon$ for $\epsilon > 0$, where $d_m(n)$ denotes the number of representations of n as the product of m natural numbers. Then by definition $|\mu_f(1,n)| \leq d_3(n) \ll n^\epsilon$.

Under the assumption of The Ramanujan-Petersson conjecture (see (2.1)) the off-diagonal terms with $n_1 \neq n_2$ contribute

$$\begin{aligned}
&\ll \sum_{q \asymp Q} \sum_{\substack{dr=q \\ d \leq Y^{100 \log\log Q}}} \mu(r)\phi(d) \sum_{\substack{n_1 \equiv \pm n_2 (\bmod d) \\ n_1, n_2 \leq Y^{100 \log\log Q}}} \frac{c(n_1)c(n_2)}{(n_1 n_2)^{\sigma_0 - \frac{1}{2}}} \\
&\ll \sum_{q \asymp Q} Y^{100 \log\log Q} \sum_{n_1 \neq n_2 \leq Y^{100 \log\log Q}} (n_1 n_2)^\epsilon \ll Q^{1+\epsilon'}.
\end{aligned}$$

As argued in the proof of Lemma 4.3.5 and under the assumption of The Ramanujan-Petersson conjecture (see (2.1)) the diagonal terms $n_1 = n_2$ contribute $\ll N(Q)(\log Q)^{-60}$, which completes the proof of the lemma. □

Proof of Proposition 4.4.4: It follows from (4.24) that for most χ and t,

$$\mathcal{M}_1(f \otimes \chi, \sigma_0 + it) = \exp(-\mathcal{P}_1(f \otimes \chi, \sigma_0 + it))(1 + O(\log Q)^{-99})$$

and by (4.23) (for most χ and t) we can write

$$(\log Q)^{-1} \ll |\mathcal{M}_1(f \otimes \chi, \sigma_0 + it)| \ll \log Q.$$

Combining these two equations we get

$$\begin{aligned}
M_1(f \otimes \chi, \sigma_0 + it) =& \mathcal{M}_1(f \otimes \chi, \sigma_0 + it) + O((\log Q)^{-25}) \\
=& \exp(-\mathcal{P}_1(f \otimes \chi, \sigma_0 + it))(1 + O(\log Q)^{-20})
\end{aligned}$$

Similarly, for most χ and t, we have

$$\begin{aligned}
M_2(f \otimes \chi, \sigma_0 + it) =& \mathcal{M}_2(f \otimes \chi, \sigma_0 + it) + O((\log\log Q)^{-25}) \\
=& \exp(-\mathcal{P}_2(f \otimes \chi, \sigma_0 + it))(1 + O(\log\log Q)^{-20}).
\end{aligned}$$

Recall the decomposition of $\mathcal{M}(f \otimes \chi, s)$ and $\mathcal{P}(f \otimes \chi, s)$, by multiplying these estimates we obtain

$$M(f \otimes \chi, \sigma_0 + it) = \exp(-\mathcal{P}(f \otimes \chi, \sigma_0 + it))(1 + O(\log\log Q)^{-20}),$$

completing the proof of the proposition. □

4.4.4 Proof of Proposition 4.4.5

In this chapter, we have used the method established in [RS17] to prove Theorem 4.4.1. But to prove the last step of Proposition 4.4.5 we do not follow the same method. Since we are working with the q-aspect we give a proof strategy of this proposition using the argument given in [CIS13; CIS12].

Expanding (4.18) we get

$$\begin{aligned}
&\sum_{q\asymp Q}\ \sideset{}{^*}\sum_{\chi(\bmod q)}\int_{-1}^{1}\left|1-L(f\otimes\chi,\sigma_0+it)M(f\otimes\chi,\sigma_0+it)\right|^2 dt\\
=&\sum_{q\asymp Q}\ \sideset{}{^*}\sum_{\chi(\bmod q)}\int_{-1}^{1}\left|L(f\otimes\chi,\sigma_0+it)M(f\otimes\chi,\sigma_0+it)\right|^2 dt\\
&\qquad -2\sum_{q\asymp Q}\ \sideset{}{^*}\sum_{\chi(\bmod q)}\int_{-1}^{1}\Re(L(f\otimes\chi,\sigma_0+it)M(f\otimes\chi,\sigma_0+it))dt+2N(Q)\\
=&S_1-2S_2+S_3 \qquad (4.25)
\end{aligned}$$

We compute S_1 and S_2 for even primitive Dirichlet characters, a similar computation can be done for odd primitive Dirichlet characters.

The Dirichlet L-function at $s=\sigma_0+it$,

$$L(f\otimes\chi,\sigma_0+it)=\sum_{n\geq 1}\frac{\lambda_f(1,n)\chi(n)}{n^{\sigma_0+it}}$$

converges conditionally. From the approximate functional equation of L-functions (see (2.15)) we write

$$\begin{aligned}
&\sideset{}{^+}\sum_{\chi(\bmod q)} L(f\otimes\chi,\sigma_0)M(f\otimes\chi,\sigma_0)\\
=&\sideset{}{^+}\sum_{\chi(\bmod q)}\sum_{n\geq 1}\frac{\lambda_f(1,n)\chi(n)}{n^{\sigma_0}}V\left(\frac{n}{q^{3/2}}\right)\sum_{m\leq q^{\epsilon}}\frac{\mu_f(1,m)a(m)\chi(m)}{m^{\sigma_0}}\\
&+\sideset{}{^+}\sum_{\chi(\bmod q)}\frac{g(\chi)^3}{q^{3/2}}\sum_{n\geq 1}\frac{\bar{\lambda}_f(1,n)\bar{\chi}(n)}{n^{1-\sigma_0}}\bar{V}\left(\frac{n}{q^{3/2}}\right)\sum_{m\leq q^{\epsilon}}\frac{\bar{\mu}_f(1,m)a(m)\bar{\chi}(m)}{m^{1-\sigma_0}} \qquad (4.26)
\end{aligned}$$

For the first summation of the above equation we have

$$\begin{aligned}
&\sideset{}{^+}\sum_{\chi(\bmod q)}\sum_{\leq q^{3/2}}\frac{\lambda_f(1,n)\chi(n)}{n^{\sigma_0}}V\left(\frac{n}{q^{3/2}}\right)\sum_{m\leq q^{\epsilon}}\frac{\mu_f(1,m)a(m)\chi(m)}{m^{\sigma_0}}\\
=&\sum_{n\leq q^{3/2},m\leq q^{\epsilon}}\frac{\lambda_f(1,n)\mu_f(1,m)a(m)}{(mn)^{\sigma_0}}V\left(\frac{n}{q^{3/2}}\right)\sideset{}{^+}\sum_{\chi(\bmod q)}\chi(n)\bar{\chi}(m)\\
=&\sum_{n\leq q^{3/2},m\leq q^{\epsilon}}\frac{\lambda_f(1,n)\mu_f(1,m)a(m)}{(mn)^{\sigma_0}}V\left(\frac{n}{q^{3/2}}\right)\sum_{\substack{vw=q\\ mn\equiv\pm1(\bmod w)}}\sum\mu(v)\phi(w)\\
=&\sum\sum\nolimits_{vw=q}\mu(v)\phi(w)\sum_{mn\equiv\pm1(\bmod w)}\sum\frac{\lambda_f(1,n)\mu_f(1,m)a(m)}{(mn)^{\sigma_0}}V\left(\frac{n}{q^{3/2}}\right)
\end{aligned}$$

Assuming The Ramanujan-Petersson conjecture (see (2.1)), let $k = mn$ which means $k \leq q^{3/2+\epsilon}$ and $k = \pm 1 + \ell w$. Therefore the above equation is approximately

$$\approx \sum\sum_{vw=q} \mu^2(v)\phi(w) \sum_{\ell \leq \frac{q^{3/2}}{w}} \frac{1}{\ell^{1/2} w^{1/2}} \ll \sum\sum_{vw=q} \mu^2(v)\phi(w) \frac{1}{w^{1/2}} \left(\frac{q^{3/2}}{w}\right)^{1/2}$$

$$\ll q^{3/2+\epsilon} \sum\sum_{vw=q} \mu^2(v) \frac{\phi(w)}{w} \ll q^{3/4+\epsilon}.$$

By averaging over Q we get the main term of size $N(Q)$ and error term of size $Q^{2-\frac{1}{4}+\epsilon} = Q^{7/4+\epsilon}$. A similar computation can be done for odd primitive Dirichlet characters.

Now we study the following

$$\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi(\bmod q)} g(\chi)^3 \chi(n)\bar{\chi}(m)$$

Assume that $q = q_1 q_2$ where q_1, q_2 are primes. As $q \asymp Q$, we set $q_i \asymp Q_i$ (for $i = 1, 2$), where $Q = Q_1 Q_2$. Since χ is a primitive Dirichlet character modulo q then it splits as $\chi = \chi_1 \chi_2$ with χ_i primitive modulo q_i (for $i = 1, 2$).

We consider the Gauss sum of the product $\chi_1\chi_2$ i.e., (for detailed computation see Section 3 of [MS15])

$$g(\chi_1\chi_2) = \sum_{a(\bmod q_1 q_2)} \chi_1(a)\chi_2(a) e_{q_1 q_2}(a)$$

Each a in the above sum can be written uniquely as $a = a_1 q_2 \bar{q}_2 + a_2 q_1 \bar{q}_1$ with $a_i (\bmod q_i)$ (for $i = 1, 2$). Consequently we get

$$\begin{aligned} g(\chi_1\chi_2) =& \sum_{a_1(\bmod q_1)} \sum_{a_2(\bmod q_2)} \chi_1(a_1)\chi_2(a_2) e_{q_1}(a_1\bar{q}_2) e_{q_2}(a_2\bar{q}_1) \\ =& \chi_1(q_2)\chi_2(q_1) g(\chi_1) g(\chi_2). \end{aligned}$$

with the above equation we consider the sum

$$\sideset{}{^*}\sum_{\chi_i(\bmod q_i)} g^3_{\chi_i} \chi_r \bar{\chi}_i(mn),$$

where $(r, q_i) = 1$ for $i = 1, 2$. Opening the Gauss sum we get

$$\begin{aligned} &\sideset{}{^*}\sum_{\chi_i(\bmod q_i)} \left(\sum_{a_i(\bmod q_i)} \chi_i(a) e_{q_i}(a) \right)^3 \chi_i(rabc)\bar{\chi}_i(mn) \\ =& \phi^*(q_i) \sideset{}{^*}\sum_{a,b(\bmod q_i)} e_{q_i}(a + b + mn\overline{rab}) - c_{q_i}(1)^3. \end{aligned}$$

Here $c_{q_i}(1)$ stands for the Ramanujan sum modulo q_i, and since we are taking q_i prime we have $c_{q_i}(1) = -1$. We set

$$K_{q_i}(u) = \sum\nolimits^*_{a,b(\bmod q_i)} e_{q_i}(a + b + u\overline{ab}).$$

The above sum is a hyper-Kloosterman sum. Square-root cancellation in such sums was proved by Deligne [Del77]:

$$K_{q_i}(u) \ll q_i,$$

for any integer u.

Then by Lemma 1 of [MS15] for q_i prime and $q_i \nmid rmn$, we have

$$\sum\nolimits^*_{\chi_i(\bmod q_i)} g(\chi_i)^3 \chi_i(r)\bar{\chi}_i(mn) = \phi^*(q_i)K_{q_i}(mn\bar{r}) + 1$$

Following the computation of Section 5 of [MS15] and using the error term estimate for the first term of (4.26), we conclude that the error term of the second sum of (4.26) is bounded by

$$\begin{aligned}
&\ll (Q_1 + Q_2) \sum_{q \asymp Q} \sum_{n=1}^{\infty} \frac{\lambda_f(1,n)}{n^{1-\sigma_0}} \sum_{m \le q^{\epsilon}} \frac{\lambda_f(1,m)a(m)}{m^{1-\sigma_0}} \left| V_{\sigma_0+it}\left(\frac{mn}{q^{3/2}}\right) \right| \\
&\ll Q^{1/2+7/4-1/2+\epsilon} \ll Q^{7/4+\epsilon}
\end{aligned}$$

This term is satisfactory for our purpose because of our choice of parameter $(Q_1+Q_2) \ll Q^{1/2}$. Similarly like the first summation, the main term for the second sum of (4.26) contributes $N(Q)$.

Since $t \in [-1, 1]$, by (2.16) and (2.17) we can say that integrating over t will not have any significant contribution in the error term. For the main term with $mn = 1$, $\int_{-1}^{1} \left(\frac{m}{n}\right)^{it} dt = 2$. Then,

$$S_2 = 2N(Q) + (Q^{7/4+\epsilon}).$$

To prove Proposition 4.4.5 we need to show that the first term of (4.25) is $\sim 2N(Q)$. The second mollified moment of $GL(3)$ L-functions is an interesting problem in Analytic Number Theory. We give a brief overview of the proof technique of the following conjecture, using the methods established in [CIS13; CIS12][11].

[11] Note that we only give a brief overview of how can one prove Conjecture 4.4.10. The proof technique of the conjectures of this section is similar to the given references but do not follow immediately, that's why we state them as conjectures not theorems.

Conjecture 4.4.10. *Let f be a Hecke-Maass cusp form of type $\nu = (\nu_1, \nu_2)$ and $N(Q)$ denote the total number of primitive Dirichlet character modulo all $q \asymp Q$.*

$$\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi(\bmod q)} \int_{-1}^{1} |L(f \otimes \chi, \sigma_0 + it) M(f \otimes \chi, \sigma_0 + it)|^2 dt \sim 2N(Q).$$

Let $\alpha = \beta = \frac{W}{\log Q}$. Then we consider the mollified integral

$$I_{f \otimes \chi}(\alpha, \beta) = \int_{-\infty}^{\infty} L(f \otimes \chi, 1/2 + \alpha + it) L(\bar{f} \otimes \bar{\chi}, 1/2 + \beta - it) |M(f \otimes \chi, \frac{1}{2} + it)|^2 w(t) dt \tag{4.27}$$

We assume that $w(t)$ is smooth, $w(t) \geq 0$ with

$$\hat{w}(1) = \int_{-\infty}^{\infty} w(t) dt > 0$$

and

$$(1 + |t|)^j w^{(j)}(t) \ll (1 + |t|)^{-A}$$

for any $j \geq 0$ and any $A \geq 0$, the implied constant depending on j and A. Since $t \in [-1, 1]$ the smoothing factor in (4.27) can be easily replaced by the sharp cut $|t| \leq 1$ by exploiting the positivity features. In this case $\hat{w}(t) = 2$, while in general $\hat{w}(t) \asymp 1$.

Opening up the mollifier we write

$$|M(f \otimes \chi, \frac{1}{2} + it)|^2 = \sum_{h,k \leq T^{\epsilon}} \frac{\mu_f(1,h) \bar{\mu}_f(1,k) a(h) a(k)}{h^{1/2+\alpha} k^{1/2+\beta}} \chi(h) \bar{\chi}(k) \left(\frac{h}{k}\right)^{-it}$$

Applying (4.27) we get

$$I_{f \otimes \chi}(\alpha, \beta) = \sum_{h,k \leq T^{\epsilon}} \frac{\mu_f(1,h) \bar{\mu}_f(1,k) a(h) a(k)}{h^{1/2+\alpha} k^{1/2+\beta}} I_{f \otimes \chi}(\alpha, \beta; h, k)$$

where

$$I_{f \otimes \chi}(\alpha, \beta; h, k) = \chi(h) \bar{\chi}(k) \int_{-\infty}^{\infty} w(t) \left(\frac{h}{k}\right)^{-it} L(f \otimes \chi, 1/2 + \alpha + it) L(\bar{f} \otimes \bar{\chi}, 1/2 + \beta - it) dt.$$

Using the proof technique of [CIS13; CIS12] we will reach to a similar version of Corollary 1 of [CIS12] but we have an extra twist by the mollifer so the expression would be similar like Theorem 2 of [CIS13]. Also we have to take care of the terms $a(h)a(k)$ coming from the mollifier.

Conjecture 4.4.11. *Let f be a Hecke-Maass cusp form of type $\nu = (\nu_1, \nu_2)$ and $N(Q)$ denote the total number of primitive Dirichlet character modulo all $q \asymp Q$.*

$$\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi(\bmod q)} I_{f \otimes \chi}(\alpha, \beta) dt \sim \hat{w}(1) N(Q) = 2N(Q).$$

Now we give a brief overview of how to prove Conjecture 4.4.11 to complete the explanation of the proof of Proposition 4.4.5.

Lemma 4.4.12 (Soundararajan [Sou09])**.** *Let $h > 0$ and $\Delta > 0$ be given. Let $\chi_{[-h,h]}$ denote the characteristic function of the interval $[-h, h]$. There exist even analytic functions $F_-(u)$, and $F_+(u)$ satisfying the following properties.*

1. *$F_-(u) \le \chi_{[-h,h]}(u) \le F_+(u)$ for real u.*
2. *We have*
$$\int_{-\infty}^{\infty} |F_\pm(u) - \chi_{[-h,h]}(u)| du \le \frac{1}{\Delta}.$$
3. *$\hat{F}_\pm(x) = 0$ for $|x| \ge \Delta$ where $\hat{F}_\pm(x) = \int_{-\infty}^{\infty} F_\pm(u) e^{-2\pi i x u} du$ denotes the Fourier transform. Also,*
$$\hat{F}_\pm(x) = \frac{\sin(2\pi h x)}{\pi x} + O\left(\frac{1}{\Delta}\right).$$

Such functions were constructed by Selberg (see [Sel89]), using Beurling's approximation to the signum function. We use Beurling-Selberg function to change the integral. Set

$$\chi_{[-1,1]}(x) := \begin{cases} 1 & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise .} \end{cases}$$

Then by the second condition of Lemma 4.4.12 we can write

$$\int_{-\infty}^{\infty} \left(\chi_{[-1,1]}(x) - F_\pm(x)\right) dx \le \frac{1}{\Delta}.$$

Note that $\hat{F}_\pm(y)$ is supported in $[-\Delta, \Delta]$. In order to prove Conjecture 4.4.11 one can choose $w(t)$ to be the Beurling-Selberg function in (4.27). In that case $F_\pm(y)$ is supported in $[-1, 1]$ and $|F_\pm(s)| \ll \frac{e^{2\pi\Delta|t|}}{\Delta|s|^2}$.

Conjecture 4.4.13. *Let $F_\pm(t)$ be a Beurling-Selberg function. Then*

$$\sum_{h,k \le T^\epsilon} \frac{\mu_f(1,h)\bar{\mu}_f(1,k)a(h)a(k)}{h^{1/2+\alpha}k^{1/2+\beta}} \chi(h)\bar{\chi}(k) \times$$
$$\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi (\bmod q)} \int_{-\infty}^{\infty} F_\pm(t) \exp\left(-it \log\left(\frac{h}{k}\right)\right) L(f \otimes \chi, 1/2 + \alpha + it) L(f \otimes \chi, 1/2 + \beta - it) dt \sim N(Q).$$

Remark 4.4.14. *One can start with*

$$\int_{-\infty}^{\infty} F_\pm(t) |L(f \otimes \chi, \sigma_0 + it) M(f \otimes \chi, \sigma_0 + it)|^2 dt$$
$$= \sum_{h,k \le T^\epsilon} \frac{\mu_f(1,h)\bar{\mu}_f(1,k)a(h)a(k)}{h^{1/2+\alpha}k^{1/2+\beta}} \chi(h)\bar{\chi}(k)$$
$$\int_{-\infty}^{\infty} F_\pm(t) \left(\frac{h}{k}\right)^{-it} L(f \otimes \chi, 1/2 + \alpha + it) L(f \otimes \chi, 1/2 + \beta - it) dt.$$

Then one may have the Fourier transform

$$
\begin{aligned}
&\int_{-\infty}^{\infty} F_{\pm}(t) \left(\frac{h}{k}\right)^{-it} L(f\otimes\chi, 1/2+\alpha+it)L(f\otimes\chi, 1/2+\beta-it)dt \\
=&\int_{-\infty}^{\infty} F_{\pm}(t) \exp\left(-it\log\left(\frac{h}{k}\right)\right) L(f\otimes\chi, 1/2+\alpha+it)L(f\otimes\chi, 1/2+\beta-it)dt.
\end{aligned}
$$

Note that we do not want h and k to be far away from each other. So we set

$$
\left|\log\left(\frac{h}{k}\right)\right| \le \Delta \implies e^{-\Delta} \le \left(\frac{h}{k}\right) \le e^{\Delta},
$$

where $\Delta = o(\log T)^{1/2}$.

From Conjecture 4.4.13, one can deduce Conjecture 4.4.11.

5. Independence of automorphic L-functions

5.1 Overview of the Problem

After the derivation of the central limit theorem, Selberg [Sel92] mentioned his orthogonality conjecture, following from which he further remarked that the primitive L-functions belonging to the Selberg class are statistically independent. Although, Selberg did not give any precise description of independence. In the paper [HW20], P. Hsu and P. Wong, proved the same result for the multiple Dirichlet L-functions associated with the primitive Dirichlet characters. In this chapter by following the method established in Chapter 3, we prove the following theorems.

Theorem 5.1.1. *Let f_1 and f_2 be two distinct primitive holomorphic cusp forms. For sufficiently large T and $t \in [T, 2T]$, the random vector*

$$\left(\log\left|L\left(f_1, \frac{1}{2}+it\right)\right|, \log\left|L\left(f_2, \frac{1}{2}+it\right)\right|\right)$$

is approximately a bi-variate normal distribution with mean vector 0_2 and covariance matrix $\frac{1}{2}(\log\log T)I_2$.
More precisely, let V be a fixed real number. Then as $T \to \infty$, uniformly for all $v_1, v_2 \in [-V, V]$

$$\frac{1}{T} meas\left\{t \in [T, 2T] : \frac{\log|L(f_i, \frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} \geq v_i : i = 1, 2\right\}$$
$$\sim \frac{1}{\sqrt{2\pi}} \int_{v_1}^{\infty} e^{-x_1^2/2} dx_1 \int_{v_2}^{\infty} e^{-x_2^2/2} dx_2.$$

Consequently, $\log\left|L\left(f_1, \frac{1}{2}+it\right)\right|$ and $\log\left|L\left(f_2, \frac{1}{2}+it\right)\right|$ are asymptotically independent.

As a more generalized version of this theorem, we further prove that the real part of logarithm of the automorphic L-functions form a Gaussian process. Since we intend to prove Theorem 5.1.1 by the method established in Chapter 3, we need to prove our next theorem by studying the joint distribution of these L-functions.

Theorem 5.1.2. *Let $(f_j)_{j=1}^{n}$ be a sequence of distinct primitive holomorphic cusp forms. Then for all large T and $t \in [T, 2T]$ the random vector*

$$\left(\log\left|L\left(f_1, \frac{1}{2}+it\right)\right|, \ldots, \log\left|L\left(f_j, \frac{1}{2}+it\right)\right|\right)$$

is approximately n-variate normal distribution with mean vector 0_n and co-variance matrix $\frac{1}{2}(\log\log T)I_n$, where 0_n is the zero vector and I_n is the $n\times n$ identity matrix. More precisely, let V be a fixed real number. Then as $T\to\infty$, uniformly for all $v_1,\ldots,v_n\in[-V,V]$

$$\frac{1}{T}meas\left\{t\in[T,2T]:\frac{\log|L(f_i,\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}}\geq v_i: i=1,2,\ldots,n\right\}$$
$$\sim\frac{1}{\sqrt{2\pi}}\int_{v_1}^{\infty}e^{-x_1{}^2/2}dx_1\cdots\int_{v_n}^{\infty}e^{-x_n{}^2/2}dx_n.$$

Consequently, the random variables $\log|L(f_j,\frac{1}{2}+it)|$'s (for $j=1,\ldots,n$) are asymptotically independent and $\log|L(f_j,\frac{1}{2}+it)|_{f_j\in J}$ forms a Gaussian process (see Definition 2.4.7) for any totally ordered set J of distinct primitive holomorphic cusp forms.

Let $X_1,X_2,\ldots,X_n$ be a n-variate normal distribution with mean vector 0_n and co-variance matrix approximately ν^2I_n. Each term $X_j=\log|L(f_j,\frac{1}{2}+it)|$ (for $j=1,\ldots,n$). As argued in the Definition (see introduction), with a similar argument each entry X_j (for $j=1,\ldots,n$) of the random vector varies “almost independently” for distinct primes p. We also know that each X_j (for $j=1,\ldots,n$) is “approximately normally” distributed. Every linear combination of X_j (for $j=1,\ldots,n$), of its component is “approximately normally” distributed.

Definition. *If $X_1,X_2,\ldots,X_n$ is an approximately n-variate normal distribution with mean 0_n and variance ν^2I_n, for any fixed positive real number V, as $T\to\infty$, we have*

$$\frac{1}{T}meas\left\{t\in[T,2T]:\frac{X_i}{\nu}\geq v_i: i=1,2,\ldots,n\right\}$$
$$\sim\frac{1}{\sqrt{2\pi}}\int_{v_1}^{\infty}e^{-x_1{}^2/2}dx_1\cdots\int_{v_n}^{\infty}e^{-x_n{}^2/2}dx_n,$$

uniformly for all $v_i\in[-V,V]$, where $i=1,2,\ldots,n$.

We prove these theorems in the next section. Moreover, we propose a more generalized problem on the statistical independence of the families of L-functions.

5.2 Gaussian Process for automorphic L-functions

In this section, we prove Theorem 5.1.1 and 5.1.2. For the sake of the calculations we start with the proof of Theorem 5.1.1, the computation for the next theorem is rather complicated.

5.2.1 Proof of Theorem 5.1.1

Theorem 5.2.1. *Let f_1 and f_2 be distinct primitive holomorphic cusp forms. Let V be a fixed real number. As $T\to\infty$, for $a_1,a_2\in\mathbb{R}$, (with $a_1,a_2\neq 0$) for all*

$v \in [-V, V]$,

$$\frac{1}{T} meas \left\{ t \in [T, 2T] : \log |\mathcal{L}_{a_1,a_2}\left(\frac{1}{2}+it\right)| \geq v\sqrt{\frac{{a_1}^2+{a_2}^2}{2}\log\log T} \right\}$$
$$\sim \frac{1}{\sqrt{2\pi}}\int_v^\infty e^{-\frac{x^2}{2}}dx,$$

where

$$\mathcal{L}_{a_1,a_2}(s) = \mathcal{L}(f_1, f_2, s; a_1, a_2) := |L(f_1, s)|^{a_1}|L(f_2, s)|^{a_2}.$$

The proof of this theorem is similar to the proof of Theorem 3.1.1. As in that proof we define the auxiliary series[1],

$$\mathcal{P}_{a_1,a_2,0}(s) = \mathcal{P}_0(f_1, f_2, s; X) := \sum_{p\leq X}\frac{a_1\lambda_{f_1}(p) + a_2\lambda_{f_2}(p)}{p^s},$$

where $\lambda_{f_1}, \lambda_{f_2}$ are the Fourier coefficients of the primitive holomorphic cusp forms f_1, f_2 respectively. Similarly to the proof of Theorem 3.1.1, we need moment calculation for $\mathcal{P}_{a_1,a_2,0}(s)$.

Lemma 5.2.2. *Let f_1, f_2 be distinct primitive holomorphic cusp forms. Suppose, k, ℓ are non-negative integers with $X^{k+\ell} \ll T$. Then for any real numbers a_1, a_2 (with $a_1, a_2 \neq 0$), we have,*

$$\int_T^{2T}\mathcal{P}_{a_1,a_2,0}(\sigma_0+it)^k\overline{\mathcal{P}_{a_1,a_2,0}(\sigma_0+it)^\ell}dt \ll_{|a_1|,|a_2|} T,$$

for $k \neq \ell$, and

$$\int_T^{2T}|\mathcal{P}_{a_1,a_2,0}(\sigma_0+it)|^{2k}dt = k!T((a_1^2+a_2^2)\log\log T)^k + O_k(T(\log\log T)^{k-1+\epsilon}).$$

Proof. Set

$$\psi(p) = \psi_{a_1,a_2}(p) := a_1\lambda_{f_1}(p) + a_2\lambda_{f_2}(p)$$

and

$$\Psi_k(n) := \prod_{j=1}^r \psi(p_j)^{\alpha_j}$$

where $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ with $\alpha_1 + \cdots + \alpha_r = k$, we can write

$$\mathcal{P}_{a_1,a_2,0}(\sigma_0+it)^k = \sum_n \frac{a_k(n)\Psi_k(n)}{n^{\sigma_0+it}},$$

where $a_k(n)$ is same as defined in (3.4). Therefore using (2.16) and (2.17) we write,

[1] For the convenience of the reader we write $\mathcal{P}_{a_1,a_2,0}(s)$ instead of $\mathcal{P}_{a_1,a_2,0}(f_1, f_2, s)$.

$$\begin{aligned}
&\int_T^{2T} \mathcal{P}_{a_1,a_2,0}(\sigma_0+it)^k \overline{\mathcal{P}_{a_1,a_2,0}(\sigma_0+it)^\ell} dt \\
=&\int_T^{2T} \sum_n \frac{a_k(n)\Psi_k(n)}{n^s} \sum_m \frac{a_\ell(m)\bar{\Psi}_\ell(m)}{m^{\bar{s}}} dt \\
=&T\sum_n \frac{a_k(n)a_\ell(n)\Psi_k(n)\overline{\Psi_\ell(n)}}{n^{2\sigma_0}} + O\left(\sum_{n\neq m} \frac{a_k(n)a_\ell(m)|\Psi_k(n)\overline{\Psi_\ell(m)}|}{(nm)^{\sigma_0-\frac{1}{2}}}\right).
\end{aligned} \tag{5.1}$$

Using Lemma 1 of [Sel46b] since σ_0 is close to $\frac{1}{2}$, we can see the the denominator in the error term of (5.1) is negligible. From the definition of $\Psi_k(n)$, for $n=\prod_{j=1}^r p_j^{\alpha_j}$, by the Ramanujan-Petersson conjecture (due to Deligne [Del74]) we know that $\lambda_{f_j}(n) \ll n^\epsilon$ for $\epsilon>0$ (and $j=1,2$). Then we write[2]

$$|\Psi_k(n)| = \prod_{j=1}^r |\psi(p_j)|^{\alpha_j} \leq \prod_{j=1}^r (|a_1\lambda_{f_1}(p_j)| + |a_2\lambda_{f_2}(p_j)|)^{\alpha_j} \ll X^{k\epsilon}(|a_1|+|a_2|)^k.$$

Thus, the big-O term is at most

$$X^{(k+\ell)\epsilon}(|a_1|+|a_2|)^{k+\ell} \sum_{\substack{n\neq m\\ m\leq X^k\\ n\leq X^\ell}} a_k(n)a_\ell(m) \ll_{|a_1|,|a_2|} X^{(k+\ell)(1+\epsilon)} \ll_{|a_1|,|a_2|} T.$$

By the definition $a_k(n)a_\ell(n)=0$ if $k\neq\ell$, we conclude the first assertion of Lemma 5.2.2. It remains to prove the lemma for $k=\ell$.
For $n=\prod_{j=1}^r p_j^{\alpha_j}$, we write

$$\Psi_k(n)\overline{\Psi_k(n)} = \prod_{j=1}^r \left(a_1^2|\lambda_{f_1}(p_j)|^2 + a_2^2|\lambda_{f_2}(p_j)|^2 + a_1a_2\lambda_f(p_j) + a_1a_2\bar{\lambda}_f(p_j)\right)^{\alpha_j},$$

where $\lambda_f := \lambda_{f_1}\bar{\lambda}_{f_2}$. From the above equation and the fact that $\sum_j \alpha_j = k$, for the second part of the lemma we have

$$\left|\sum_{n \text{ non square free}} \frac{a_k(n)a_k(n)\Psi_k(n)\overline{\Psi_k(n)}}{n^{2\sigma_0}}\right| \leq T^\epsilon(|a_1|+|a_2|)^{2k} \sum_{n \text{ non square free}} \frac{a_k(n)^2\lambda_f(n)^2}{n^{2\sigma_0}}.$$

Therefore as argued in the proof of Lemma 3.3.1 the non-square free term contributes a quantity of order $O((\log\log T)^{k-1+\epsilon})$. For n square free we can express $\Psi(n)\bar{\Psi}(n)$ as

[2] Note that each $p_j<X$ from which we can write $n\leq X^k$ and $m\leq X^\ell$. Then $\prod_{j=1}^r(|a_1\lambda_{f_1}(p_j)|+|a_2\lambda_{f_2}(p_j)|)^{\alpha_j} \leq \prod_{j=1}^r(|a_1|+|a_2|)^{\alpha_j}(|\lambda_{f_1}(p_j)|+|\lambda_{f_2}(p_j)|)^{\alpha_j}$. Then for each $j=1,\ldots,r$, $(|\lambda_{f_1}(p_j)|+|\lambda_{f_2}(p_j))^{\alpha_j}| < |\lambda_{f_1}(p_j)|^{\alpha_j}\cdot|\lambda_{f_2}(p_j)|^{\alpha_j}$. Then $\prod_{j=1}^r |\lambda_{f_1}(p_j)|^{\alpha_j}\cdot|\lambda_{f_2}(p_j)|^{\alpha_j} \ll X^{k\epsilon}$, for $\epsilon>0$.

$$\sum \left(\sum_{i=0}^{\beta} {a_1}^{2i} {a_2}^{2(\beta-i)} |\lambda_{f_1}(p_1 \cdots p_i)|^2 |\lambda_{f_2}(p_1 \cdots p_{\beta-i})|^2 \right) (a_1 a_2)^{k-\beta} \lambda_f(m) \bar{\lambda}_f(m')$$

where the sum is over $\beta + \Omega(m) + \Omega(m') = k$ such that $0 \leq \beta \leq k$ and $n = n'mm'$ where n has the prime composition of $p_1, \ldots, p_k$.

For $i = 0$ and $i = \beta$, the above equation can be written as

$$\begin{aligned} &\sum {a_1}^{2\beta} |\lambda_{f_1}(n')|^2 (a_1 a_2)^{k-\beta} \lambda_f(m) \bar{\lambda}_f(m') + \sum {a_2}^{2\beta} |\lambda_{f_2}(n')|^2 (a_1 a_2)^{k-\beta} \lambda_f(m) \bar{\lambda}_f(m') \\ &+ \sum \left(\sum_{i=1}^{\beta-1} {a_1}^{2i} {a_2}^{2(\beta-i)} |\lambda_{f_1}(p_1 \cdots p_i)|^2 |\lambda_{f_2}(p_1 \cdots p_{\beta-i})|^2 \right) (a_1 a_2)^{k-\beta} \lambda_f(m) \bar{\lambda}_f(m') \end{aligned} \tag{5.2}$$

We can express the first term of (5.2) as

$$\sum_{0 \leq \beta \leq k} {a_1}^{2\beta} (a_1 a_2)^{k-\beta} \sum_n \frac{a_k(n)^2 |\lambda_{f_1}(n')|^2 \lambda_f(m) \bar{\lambda}_f(m')}{n^{2\sigma_0}}$$

where the inner sum runs over $n = n'mm'$ (with n', m, m' pair-wise co-prime) and $\beta + \Omega(m) + \Omega(m') = k$. Since $\lambda_{f_1}(n)$ is real multiplicative function we have

$$k! \sum_{0 \leq \gamma \leq k-\beta} \frac{k!}{\beta! \gamma! (k-\beta-\gamma)!} \cdot \sum_{\substack{p_1, \ldots, p_k \leq X \\ \text{all } p_j\text{'s are distinct} \\ (p_j, q)=1}} \frac{|\lambda_{f_1}(p_1 \cdots p_\beta)|^2 \lambda_f(p_{\beta+1} \cdots p_{\beta\gamma})) \bar{\lambda}_f(p_{\beta\gamma+1} \cdots p_k)}{(p_1 \cdots p_k)^{2\sigma_0}}$$

which is

$$k! \sum_{0 \leq \gamma \leq k-\beta} \frac{k!}{\beta! \gamma! (k-\beta-\gamma)!} \left(\sum_{\substack{p \leq X \\ (p,q=1)}} \frac{|\lambda_{f_1}(p)|^2}{p^{2\sigma_0}} \right)^{\beta} \left(\sum_{\substack{p \leq X \\ (p,q=1)}} \frac{\lambda_f(p)}{p^{2\sigma_0}} \right)^{\gamma} \left(\sum_{\substack{p \leq X \\ (p,q=1)}} \frac{\bar{\lambda}_f(p)}{p^{2\sigma_0}} \right)^{k-\beta-\gamma}. \tag{5.3}$$

Similarly, for the second term of (5.2) gives

$$k! \sum_{0 \leq \gamma \leq k-\beta} \frac{k!}{\beta! \gamma! (k-\beta-\gamma)!} \left(\sum_{\substack{p \leq X \\ (p,q=1)}} \frac{|\lambda_{f_2}(p)|^2}{p^{2\sigma_0}} \right)^{\beta} \left(\sum_{\substack{p \leq X \\ (p,q=1)}} \frac{\lambda_f(p)}{p^{2\sigma_0}} \right)^{\gamma} \left(\sum_{\substack{p \leq X \\ (p,q=1)}} \frac{\bar{\lambda}_f(p)}{p^{2\sigma_0}} \right)^{k-\beta-\gamma}. \tag{5.4}$$

With a similar computation we get contributions from the third term of (5.2). Note that the sum in (5.1) has its main contribution for $\beta = k$. Taking $\beta = k$ in (5.3) and (5.4) and adding up the contribution coming from the third term of (5.2), from section 3 of [Lü14] and (63) of [MN14] we conclude

$$\begin{aligned}
&k!\left((a_1^2\log\log T)^k + (a_2^2\log\log T)^k\right) + k!\left(\sum_{i=1}^{k-1}\binom{k-1}{i}a_1^i a_2^{(k-1)-i}(\log\log T)^{k-1}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + O_k((\log\log T)^{k-1+\epsilon})\\
=&k!\left((a_1^2+a_2^2)\log\log T\right)^k - k!\left(\sum_{i=1}^{k-1}\binom{k-1}{i}a_1^i a_2^{(k-1)-i}(\log\log T)^{k-1}\right)\\
&\qquad\qquad + k!\left(\sum_{i=1}^{k-1}\binom{k-1}{i}a_1^i a_2^{(k-1)-i}(\log\log T)^{k-1}\right) + O_k((\log\log T)^{k-1+\epsilon})\\
=&k!\left((a_1^2+a_2^2)\log\log T\right)^k + O_k((\log\log T)^{k-1+\epsilon})
\end{aligned}$$

completing the proof. □

Let $X_j = \log|L(f_j, \sigma_0 + it)|$ for $j = 1, 2$. We are now ready to prove Theorem 5.2.1.

Proof of Theorem 5.2.1: Observe that $\log|\mathcal{L}_{a_1,a_2(s)}| = a_1X_1 + a_2X_2$ (where $X_j = \log|L(f_j, \frac{1}{2} + it)|$ for $j = 1, 2$). Like a similar argument of Proposition 3.2.1 we have

$$\int_{t-1}^{t+1}\left|\log|\mathcal{L}_{a_1,a_2(\frac{1}{2}+iy)}| - |\mathcal{L}_{a_1,a_2(\sigma_0+iy)}|\right| dy \ll \left(\sigma_0 - \frac{1}{2}\right)\log T. \tag{5.5}$$

However, we can study the function $\log|\mathcal{L}_{a_1,a_2}(\sigma_0 + it)|$ away from the critical line $\Re(s) = \frac{1}{2}$.

Define $\mathcal{P}_{a_1,a_2}(s) := a_1\mathcal{P}(f_1, s) + a_2\mathcal{P}(f_2, s)$ where $\mathcal{P}(f_j, s)$'s (for $j = 1, 2$) are defined as the auxiliary series $\mathcal{P}(f, s)$. Similarly, the contribution of the higher order terms of the primes is at most $O(1)$. For the terms involving p^2 as shown in the proof of Proposition 3.2.3, it contributes $O(T/L^2)$, for any real number $L > 1$. Now by Lemma 5.2.2 for $X^k \ll T$ and odd k,

$$\begin{aligned}
&\int_T^{2T}(\Re(\mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)))^k dt\\
=&\frac{1}{2^k}\sum_{\ell=0}^{k}\binom{k}{\ell}\int_T^{2T}\mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)^\ell\,\overline{\mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)^{k-\ell}}dt \ll T.
\end{aligned}$$

Also, by Lemma 5.2.2, for even k,

$$\begin{aligned}&\int_T^{2T} (\Re(\mathcal{P}_{a_1,a_2,0}(\sigma_0+it)))^k dt\\ =&2^{-k}\binom{k}{k/2}\left(\frac{k}{2}\right)!\left(\frac{a_1^2+a_2^2}{2}\log\log T\right)^{\frac{k}{2}}+O_k((\log\log T)^{\frac{k}{2}-1+\epsilon}).\end{aligned}$$

By (2.20) we conclude that $\Re(\mathcal{P}_{a_1,a_2}(\sigma_0+it))$ has an approximately normal distribution with mean 0 and variance $\frac{a_1^2+a_2^2}{2}\log\log T$.

Now it remains to connect $\mathcal{L}_{a_1,a_2}(\sigma_0+it)$ with $\mathcal{P}_{a_1,a_2}(\sigma_0+it)$. From the similar argument of Proposition 3.2.4 and Proposition 3.2.5, $|L(f_j,\sigma_0+it)| = (1+o(1))\exp(\Re(\mathcal{P}_{a_1,a_2}(\sigma_0+it)))$, except for a possible set of measure $o(T)$, for each j. Since, $\mathcal{L}_{a_1,a_2}(s) = |L(f_1,s)|^{a_1}|L(f_2,s)|^{a_2}$ and $\mathcal{P}_{a_1,a_2}(s) = a_1\mathcal{P}(f_1,s)+a_2\mathcal{P}(f_2,s)$, we have

$$\mathcal{L}_{a_1,a_2}(\sigma_0+it) = (1+o(1))\exp(\Re(\mathcal{P}_{a_1,a_2}(\sigma_0+it))).$$

It shows $\log|\mathcal{L}_{a_1,a_2}(\sigma_0+it)|$ has Gaussian distribution with mean 0 and variance $\frac{a_1^2+a_2^2}{2}\log\log T$, combining with (5.5) we conclude the proof. □

Finally, we are ready to prove Theorem 5.1.1.

Proof of Theorem 5.1.1: Let $\mathbf{X} := (X_j)_{j=1}^2 = \log|L(f_j,\frac{1}{2}+it)|_{j=1}^2$. By Theorem 3.1.1, we have X_1 and X_2 both are approximately normally distributed with mean 0 and variance $\frac{1}{2}\log\log T$. By applying Theorem 5.2.1 and Lemma 2.4.5, we have that the linear combination of X_1 and X_2 an approximate bi-variate normal distribution.

Putting $a_1 = a_2 = 1$ in Theorem 5.2.1, we get $\rho(X_1, X_2) \to 0$ as $T \to \infty$ and

$$\mathrm{Var}(X_1+X_2) = \frac{1+1}{2}\log\log T = \mathrm{Var}(X_1)+\mathrm{Var}(X_2).$$

Let Y_1 and Y_2 be normally distributed with mean 0 and variance $\frac{1}{2}\log\log T$ and X_1 and X_2 converges to Y_1 and Y_2 respectively, in distribution. Therefore, Y_1 and Y_2 are independent. Hence, by Remark 2.4.6, we conclude that X_1 and X_2 are asymptotically independent. □

5.2.2 Proof of Theorem 5.1.2

In this section, we prove Theorem 5.1.2. The proof follows similarly to the proof of Theorem 5.1.2. Since we are proving a more generalized version of it we need to face more complicated computations.

For $(a_j)_{j=1}^N \subset \mathbb{R}^N$ we consider,

$$\mathcal{P}_{a_1,a_2,\dots,a_N,0}(s) = \sum_{p\le X} \frac{a_1\lambda_{f_1}(p) + \cdots + a_N\lambda_{f_N}(p)}{p^s}.$$

Following the tradition to prove Theorem 5.1.2 we need the next lemma.

Lemma 5.2.3. *Let $(f_j)_{j=1}^N$ be a sequence of distinct primitive holomorphic cusp forms. Assume that k,ℓ are non-negative integers with $X^{k+\ell} \le T$. Then for any real numbers $(a_j)_{j=1}^N$ (with $a_j \neq 0$, for $j = 1,\dots,N$), we have for $k \neq \ell$*

$$\int_T^{2T} \mathcal{P}_{a_1,a_2,\dots,a_N,0}(\sigma_0+it)^k \overline{\mathcal{P}_{a_1,a_2,\dots,a_N,0}(\sigma_0+it)^\ell} \ll_{(|a_j|)_{j=1}^N} T.$$

For $k = \ell$ and $\epsilon > 0$,

$$\int_T^{2T} |\mathcal{P}_{a_1,a_2,\dots,a_N,0}(\sigma_0+it)|^{2k} dt = k!T\left(\left(\sum_{j=1}^N a_j^2\right)\log\log T\right)^k + O_k(T(\log\log T)^{k-1+\epsilon}).$$

Proof. Similarly like the proof of Lemma 5.2.2 we start the proof by setting

$$\psi(p) := \sum_{j=1}^N a_j\lambda_{f_j}(p)$$

and

$$\Psi_k(n) := \prod_{j=1}^r \psi(p_j)^{\alpha_j}$$

where $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ and $\alpha_1+\cdots+\alpha_r = k$. Then we define

$$\mathcal{P}_{a_1,a_2,\dots,a_N,0}(\sigma_0+it)^k = \sum_n \frac{a_k(n)\Psi_k(n)}{n^{\sigma_0+it}}$$

where $a_k(n)$ is defined in (3.4). Therefore,

$$\begin{aligned} &\int_T^{2T} \mathcal{P}_{a_1,a_2,\dots,a_N,0}(\sigma_0+it)^k \overline{\mathcal{P}_{a_1,a_2,\dots,a_N,0}(\sigma_0+it)^\ell} dt \qquad (5.6)\\ =&\int_T^{2T} \sum_n \frac{a_k(n)\Psi(n)}{n^s} \sum_m \frac{a_\ell(m)(\bar{\Psi}(m))}{m^{\bar{s}}} \\ =&T\sum_n \frac{a_k(n)a_\ell(n)\Psi_k(n)\overline{\Psi_\ell(n)}}{n^{2\sigma_0}} + O\left(\sum_{n\neq m} \frac{a_k(n)a_\ell(m)|\Psi_k(n)\overline{\Psi_\ell(m)}|}{(nm)^{\sigma_0-\frac{1}{2}}}\right). \end{aligned}$$

From the definition of $\Psi_k(n)$ for $n = \prod_{j=1}^r$ we have (with a similar argument given for the proof of Lemma 5.2.2)

$$|\Psi_k(n)| = \prod_{j=1}^r |\psi(p_j)|^{\alpha_j} \le \prod_{j=1}^r (|a_1\lambda_{f_1}(p_j)| + \cdots + |a_N\lambda_{f_N}(p_j)|)^{\alpha_j} \ll X^{k\epsilon}(|a_1|+\cdots+|a_N|)^k.$$

Thus the big-O term in (5.6) contributes

$$X^{(k+\ell)\epsilon}(|a_1|+\cdots+|a_N|)^{k+\ell}\sum_{\substack{n\neq m\\ m\leq X^k\\ n\leq X^\ell}} a_k(n)a_\ell(m) \ll_{(|a_j|)_{j=1}^N} X^{(k+\ell)(1+\epsilon)} \ll_{(|a_j|)_{j=1}^N} T.$$

As we know that by the definition $a_k(n)a_\ell(n)$ is 0 if $k\neq\ell$, we conclude the first assertion of the lemma.

Now we need to prove the next part of the lemma which is the case for $k=\ell$. For $n=\prod_{j=1}^r p_j^{\alpha_j}$ we write

$$\Psi_k(n)\overline{\Psi_k(n)}=\prod_{j=1}^r\left(\left(\sum_{j=1}^N a_j^2|\lambda_{f_j}(p_j)|^2\right)+\sum_{i=1}^N\sum_{i\neq i'} a_ia_{i'}(\lambda_{f_i}\bar\lambda_{f_{i'}}(p_j)+\lambda_{f_{i'}}\bar\lambda_{f_i}(p_j))\right)^{\alpha_j}. \tag{5.7}$$

We set $\lambda_f(n)=\sum_{i,i'=1}^N\sum_{i\neq i'}\lambda_{f_i}\bar\lambda_{f_{i'}}$. By (5.7) and the fact that $\sum_j\alpha_j=k$ we have

$$\left|\sum_{n \text{ non square free}}\frac{a_k(n)a_k(n)\Psi_k(n)\overline{\Psi_k(n)}}{n^{2\sigma_0}}\right|\leq T^\epsilon\left(\sum_{j=1}^N|a_j|\right)^{2k}\sum_{n\text{ non square free}}\frac{a_k(n)^2\lambda_f(n)^2}{n^{2\sigma_0}}.$$

As argued in the proof of Lemma 5.2.2 the non square free n terms in (5.7) give $O_k((\log\log T)^{k-1+\epsilon})$. Similarly like (5.2) for square free n we write

$$\sum\left(\sum_{\substack{i_1+\cdots+i_N=\beta\\ i_1,\ldots,i_N\geq 1}}\binom{\beta}{i_1,\ldots,i_N}\prod_{t=1}^N {a_t}^{2i_t}|\lambda_{f_t}(p_{i_1}\cdots p_{i_t})|^2\right)$$
$$\cdot(a_1a_2)^{\Omega(m_1)+\Omega(m_1')}\cdots(a_{N-1}a_N)^{\Omega(m_{(N-1)N/2})+\Omega(m'_{(N-1)N/2})}$$
$$\cdot\lambda_{f_1}\bar\lambda_{f_2}(m_1)\lambda_{f_2}\bar\lambda_{f_1}(m_1')\cdots\lambda_{f_N}\bar\lambda_{f_{N-1}}(m'_{(N-1)N/2}))$$

where the sum runs over $\beta+\sum_{j=1}^{(N-1)N/2}(\Omega(m_j)+\Omega(m_j'))=k$ such that $0\leq\beta\leq k$ and $n=n'\prod_{j=1}^{(N-1)N/2}m_jm_j'$, where n has the prime composition of $p_1,\ldots,p_k$.

For the terms involving ${a_j}^{2\beta}$ (for $j=1,2,\ldots,N$) the above expression can be written as

$$\sum\left(\sum_{j=1}^{N} {a_j}^{2\beta}|\lambda_{f_j}(n')|^2\right)(a_1a_2)^{\Omega(m_1)+\Omega(m_1')}\cdots(a_{N-1}a_N)^{\Omega(m_{(N-1)N/2})+\Omega(m'_{(N-1)N/2})}$$
$$\cdot\lambda_{f_1}\bar{\lambda}_{f_2}(m_1)\lambda_{f_2}\bar{\lambda}_{f_1}(m_1')\cdots\lambda_{f_N}\bar{\lambda}_{f_{N-1}}(m'_{(N-1)N/2)})$$
$$+\sum\left(\sum_{\substack{i_1+\cdots+i_N=\beta-1\\ i_1,\ldots,i_N\geq 1}}\binom{\beta-1}{i_1,\ldots,i_N}\prod_{t=1}^{N}{a_t}^{2i_t}|\lambda_{f_t}(p_{i_1}\cdots p_{i_t})|^2\right)$$
$$\cdot(a_1a_2)^{\Omega(m_1)+\Omega(m_1')}\cdots(a_{N-1}a_N)^{\Omega(m_{(N-1)N/2})+\Omega(m'_{(N-1)N/2})}$$
$$\cdot\lambda_{f_1}\bar{\lambda}_{f_2}(m_1)\lambda_{f_2}\bar{\lambda}_{f_1}(m_1')\cdots\lambda_{f_N}\bar{\lambda}_{f_{N-1}}(m'_{(N-1)N/2)}). \tag{5.8}$$

For each j, we can express j-th term of the first sum of (5.8) as

$$\sum_{0\leq\beta\leq k} {a_j}^{2\beta}(a_1a_2)^{\Omega(m_1)+\Omega(m_1')}\cdots(a_{N-1}a_N)^{\Omega(m_{(N-1)N/2})+\Omega(m'_{(N-1)N/2})}$$
$$\cdot\sum_n\frac{|\lambda_{f_j}(n')|^2\lambda_{f_1}\bar{\lambda}_{f_2}(m_1)\lambda_{f_2}\bar{\lambda}_{f_1}(m_1')\cdots\lambda_{f_N}\bar{\lambda}_{f_{N-1}}(m'_{(N-1)N/2)})}{n^{2\sigma_0}} \tag{5.9}$$

where the second sum is over $n=n'\prod_{j=1}^{(N-1)N/2} m_jm_j'$ and n', m_j, m_j' are pairwise co-prime for all $j=1,\ldots,N$. Also,

$$\beta+\sum_{j=1}^{(N-1)N/2}(\Omega(m_j)+\Omega(m_j'))=k.$$

Since each $\lambda_{f_j}(n')$ is a real multiplicative function and by the change of variables $\Omega(m_j)\mapsto\gamma_{2j-1}$, $\Omega(m_j')\mapsto\gamma_{2j}$ we can write the inner sum of (5.9) as

$$k!\sum_{0\leq\gamma_1,\ldots,\gamma_{(N-1)N}\leq k-\beta}\frac{k!}{\beta!\gamma_1!\cdots\gamma_{(N-1)N}!(k-\gamma_1!-\cdots-\gamma_{(N-1)N})!}$$
$$\cdot\sum_{\substack{p_1,\ldots,p_k\leq X\\ p_j\text{'s are distinct}\\ (p_j,q)=1}}\frac{|\lambda_{f_j}(p_1\cdots p_\beta)|^2\cdots\lambda_{f_N}\bar{\lambda}_{f_{N-1}}(p_{\beta+\gamma_1+\cdots+\gamma_{(N-1)}+1}\cdots p_{\beta+\gamma_1+\cdots+\gamma_{N(N-1)}})}{(p_1\cdots p_k)^{2\sigma_0}}$$
$$=k!\sum_{0\leq\gamma_1,\ldots,\gamma_{(N-1)N}\leq k-\beta}\frac{k!}{\beta!\gamma_1!\cdots\gamma_{(N-1)N}!(k-\gamma_1!-\cdots-\gamma_{(N-1)N})!}$$
$$\cdot\left(\sum_{p\leq X}\frac{|\lambda_{f_j}(p)|^2}{p^{2\sigma_0}}\right)^{\beta}\left(\sum_{p\leq X}\frac{\lambda_{f_1}\bar{\lambda}_{f_2}(p)}{p^{2\sigma_0}}\right)^{\gamma_1}\cdots\left(\sum_{p\leq X}\frac{\lambda_{f_N}\bar{\lambda}_{f_{N-1}}(p)}{p^{2\sigma_0}}\right)^{\gamma_{(N-1)N}}. \tag{5.10}$$

For each $j = 1, \ldots, N$ we get a term like (5.10). Similarly, we can compute the contribution of the second sum of (5.8). Adding up for all $j = 1, \ldots, N$ and the similar argument of the proof of Lemma 5.2.2, the first sum of in (5.6) is mainly contributed by $k = \beta$ which is

$$k!\left(\left(\sum_{j=1}^{N} a_j^2\right)\log\log T\right)^k - k!\left(\sum_{\substack{i_1+\cdots+i_N=k-1\\ i_1,\ldots,i_1\geq 1}} \binom{k-1}{i_1,\ldots,i_N}\prod_{t=1}^{N} {a_t}^{i_t}(\log\log T)^{k-1}\right)$$
$$+ k!\left(\sum_{\substack{i_1+\cdots+i_N=k-1\\ i_1,\ldots,i_1\geq 1}} \binom{k-1}{i_1,\ldots,i_N}\prod_{t=1}^{N} {a_t}^{i_t}(\log\log T)^{k-1}\right) + O_k((\log\log T)^{k-1+\epsilon})$$

where $\binom{k}{i_1,\ldots,i_N} = \frac{k!}{i_1!\cdots i_N!}$. The terms involving $(\log\log T)^{k-1}$ will be subsumed by the error term. We conclude that the above sum equals

$$k!\left(\left(\sum_{j=1}^{N} a_j^2\right)\log\log T\right)^k + O_k((\log\log T)^{k-1+\epsilon}),$$

completing the proof of the lemma. $\square$

In Lemma 5.2.3, we have computed the moments of the auxiliary series $\mathcal{P}_{a_1,\ldots,a_N,0}(s)$, with this calculation in hand in the next theorem we study the joint distribution of the automorphic L-functions.

Theorem 5.2.4. *Let $(f_j)_{j=1}^N$ be a sequence of distinct primitive holomorphic cusp forms. Let V be a fixed positive real number. As $T \to \infty$, for any non-zero real numbers $(a_j)_{j=1}^N$, for all $v \in [-V, V]$*

$$\frac{1}{T} meas\left\{t \in [T, 2T] : \log|\mathcal{L}_{a_1,\ldots,a_N}(\frac{1}{2}+it)| \geq v\sqrt{\frac{a_1^2+\cdots+a_N^2}{2}\log\log T}\right\}$$
$$\sim \frac{1}{\sqrt{2\pi}}\int_v^\infty e^{-\frac{x^2}{2}}dx,$$

where

$$\mathcal{L}_{a_1,\ldots,a_N}(s) = \mathcal{L}(f_1,\ldots,f_N,a_1,\ldots,a_N,s) := |L(f_1,s)|^{a_1}\cdots|L(f_N,s)|^{a_N}.$$

Proof. The proof of this theorem is the same as the proof of Theorem 5.2.1, instead of Lemma 5.2.2 we have to consider Lemma 5.2.3, that's why we can omit it without repeating the same steps. $\square$

Now we are ready to prove Theorem 5.1.2.

Proof of Theorem 5.1.2: Let J denotes a totally ordered set of distinct primitive holomorphic cusp forms. For any finite ordered subset $\{f_1, \ldots, f_N\}$ of J we consider $\mathbf{X} := (\log|L(f_j, \frac{1}{2}+it)|)_{j=1}^{N}$. From Theorem 5.2.4 and Lemma 2.4.3 it follows that $\mathbf{X}$ is an approximate N-variate normal distribution. Thus, we see that any finite linear combination of elements in $(\log|L(f, \frac{1}{2}+it)|)_{f\in J}$ is a multivariate normal distribution. Hence, $(\log|L(f, \frac{1}{2}+it)|)_{f\in J}$ forms a Gaussian process.

Let $Y_1, Y_2, \ldots Y_N$ be normally distributed with mean 0 and variance $\frac{1}{2}\log\log T$ and $X_1, X_2, \ldots, X_N$ converges to $Y_1, Y_2, \ldots Y_N$ respectively, in distribution. Therefore, $Y_1, Y_2, \ldots Y_N$ are independent. Hence, by Remark 2.4.6, we conclude that $X_1, X_2, \ldots, X_N$ are asymptotically independent.

Moreover, the components in $\mathbf{X}$ are mutually asymptotically independent since they are pair-wise independent by Theorem 5.1.1. □

5.3 Extension to a generalized notion

In this section, we generalize the notion of the independence of the families of L-functions. This chapter focuses on the independence of the automorphic L-functions. In the previous section, we have seen that distinct primitive holomorphic cusp forms associated with the L-functions form the Gaussian process and they are jointly independent. The notion we are going to study in this chapter is slightly different from the previous one. By the end of this section, we combine both of them and remark on their statistical behaviour.

Recall the concept introduced by Selberg, the purpose of studying the behaviour of logarithm of the Riemann zeta function in the critical strip. Since L-function is a generalized concept of ζ-function and the main focus of this thesis is to study the behaviour of the logarithm of the L-function but we first look at the independence of the Riemann zeta function.

We know that if we take the L-functions associated with two (or more) primitive cusp forms (or Maass forms) is statistically independent at $\frac{1}{2}+it$ for $t \in \mathbb{R}$. Then the question arises that how do we study the independence of the Riemann zeta function because the Fourier coefficient of the Riemann zeta function in the Dirichlet series is 1. Well, we study the independence of $\zeta(s)$ on the critical line when t varies which means we study the behaviour of $\zeta(\frac{1}{2}+it_1)$ and $\zeta(\frac{1}{2}+it_2)$ for $t_1 \neq t_2$ and t_1, t_2 are real numbers. Selberg's central limit theorem provides a lot of information in this context.

In other terms, Selberg central limit theorem can be written as

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{2T} \mathbb{I}\left\{\frac{\log\zeta(\frac{1}{2}+it)}{\sqrt{\frac{1}{2}\log\log T}} \in \Gamma\right\} dt = \frac{1}{2\pi}\int_\Gamma e^{-\frac{x^2+y^2}{2}}dxdy,$$

where Γ is a regular Borel measurable subset of $\mathbb{C}$ and $\mathbb{I}$ is the indicator function, and regular means that the boundary of Γ has zero Lebesgue measure.

In 2007, C. P. Hughes et al. [HNY07] convert the "static" concept of Selberg's central limit theorem into a more "dynamic" probabilistic result. This concept can be generalized for the families of L-functions belonging to the Selberg class.

If one let

$$\mathfrak{L}_\mu(N,u) := \frac{\log \zeta(\frac{1}{2} + iue^{N^\mu})}{\sqrt{\log N}}$$

then Selberg's central limit theorem[3] implies that

$$\lim_{N\to\infty} \int_1^2 \mathbb{I}\{\mathfrak{L}_\mu(N,u) \in \Gamma\} du = \mathbb{P}\{\mathfrak{G}_\mu \in \Gamma\}$$

where $\mathbb{I}$ is the indicator function and $\mathfrak{G}_\mu = \mathfrak{G}_\mu^{(1)} + i\mathfrak{G}_\mu^{(2)}$ is complex valued Gaussian random variable with mean 0 and variance $\mu/2$ which means $\mathfrak{G}_\mu^{(1)}$ and $\mathfrak{G}_\mu^{(2)}$ are independent, centered and $\mathbb{E}[(\mathfrak{G}_\mu^{(1)})^2] = \mathbb{E}[(\mathfrak{G}_\mu^{(2)})^2] = \frac{\mu}{2}$. The next theorem studies the asymptotic behaviour of $\mathfrak{L}_\mu(N,u)$ for different μ.

Theorem 5.3.1 (Hughes-Nikeghbali-Yor). *For $\mu_1 > \mu_2 > \cdots > \mu_k > 0$, for every $(\Gamma_i, i \le k)$ regular*

$$\lim_{N\to\infty} \int_1^2 \mathbb{I}\left\{\mathfrak{L}_{\mu_1}(N,u) \in \Gamma_1, \ldots, \mathfrak{L}_{\mu_k}(N,u) \in \Gamma_k\right\} du = \prod_{j=1}^k \mathbb{P}\{\mathfrak{G}_{\mu_j} \in \Gamma_j\}.$$

Now we generalize Theorem 5.3.1 for the families of L-functions belonging to the Selberg class.

Let f be a primitive holomorphic cusp form. Then

$$\mathfrak{L}_\kappa(f;N,u) := \frac{\log L(f, \frac{1}{2} + iue^{N^\kappa})}{\sqrt{\log N}}$$

Then Theorem 3.1.1 implies that

$$\lim_{N\to\infty} \int_1^2 \mathbb{I}\left\{\mathfrak{L}_\kappa(f;N,u) \in \Gamma\right\} du = \mathbb{P}\{\mathfrak{G}_\kappa \in \Gamma\}$$

where $\mathfrak{G}_\kappa = \mathfrak{G}_\kappa^{(1)} + i\mathfrak{G}_\kappa^{(2)}$ is complex valued Gaussian random variable with mean 0 and variance $\kappa/2$ which means $\mathfrak{G}_\kappa^{(1)}$ and $\mathfrak{G}_\kappa^{(2)}$ are independent, centered and $\mathbb{E}[(\mathfrak{G}_\kappa^{(2)})^2] = \mathfrak{G}_\kappa = \frac{\kappa}{2}$. So Theorem 5.3.1 becomes

Theorem 5.3.2. *For $\kappa_1 > \kappa_2 > \cdots > \kappa_k > 0$, for every $(\Gamma_i, i \le k)$ regular*

$$\lim_{N\to\infty} \int_1^2 \mathbb{I}\left\{\mathfrak{L}_{\kappa_1}(f;N,u) \in \Gamma_1, \ldots, \mathfrak{L}_{\kappa_k}(f;N,u) \in \Gamma_k\right\} du = \prod_{j=1}^k \mathbb{P}\{\mathfrak{G}_{\kappa_j} \in \Gamma_j\}.$$

[3] If we take $N = \log T$ and u has uniform distribution then $\mathfrak{L}_\mu(N,u)$ gives $\left\{\frac{\log \zeta(\frac{1}{2}+it)}{\sqrt{\log\log T}}\right\}$.

As remarked in [HNY07] the result of this theorem can be obtained with the help of the proof given in [HNY07].

Now let us think of the concept introduced in the previous section of this chapter in Theorem 5.1.2. Let $(f_j)_{j=1}^{k}$ be sequence of distinct holomorphic cusp forms. Consider the matrix

$$\begin{bmatrix} \mathfrak{L}_{\kappa_{11}}(f_1;N,u) & \cdots & \mathfrak{L}_{\kappa_{1k}}(f_1;N,u) \\ \vdots & \ddots & \vdots \\ \mathfrak{L}_{\kappa_{k1}}(f_k;N,u) & \cdots & \mathfrak{L}_{\kappa_{kk}}(f_k;N,u) \end{bmatrix} \tag{5.11}$$

with ij-th entry $\mathfrak{L}_{\kappa_{ij}}(f_i;N,u)$. From the definition of $\mathfrak{L}_{\kappa}(f_1;N,u)$ the matrix given in (5.11) is a random matrix. If we look at the row vectors of (5.11) it led us to Theorem 5.3.2 and if we look at the column vector then it is similar to Theorem 5.1.2. Observe that the ij-th entry is not equal to ji-th entry for $i \neq j$. So this is not a symmetric matrix.

The interesting thing about this matrix is if we choose two-elements from the same row (or from the same column) they are pair-wise independent. But if we choose two elements of the matrix randomly such that they do not lie in the same row (or in the same column) then we can study their joint distribution. Also, we can study the eigenvalues of this matrix and their connection with the pair correlation conjecture.

6. Conclusion

In chapter 3, we have estimated the mean square L-function using Hughes and Young's method. Also, we have taken the help of P. Kühn et al result. It is possible to obtain a better error term that we have obtained. But the main focus of this thesis concerns the proof of Selberg's central limit theorem for L-functions. Any improvement on the error term for the mean square estimate of L-function will not effect our result. The most important fact we have relied on is the Ramanujan-Peterson conjecture and the upper bound of the shifted convolution sum.

We have proved the main result for t and q-aspect. For the proof in t-aspect, we fix the cusp form f attached to the L-function and integrate over the interval $[T, 2T]$. In the q-aspect we fix the range of $t \in [-1, 1]$ and average over the Dirichlet characters. One can prove Selberg's central limit theorem on weight aspect. For such calculations Lau's [Lau05] work might be helpful.

As we have mentioned earlier that due to the lack of information on the shifted convolution problem we can not prove the result for higher degree L-functions. A recent work by G. Hu and G. Lü [HL20] might be applicable to obtain a mean value estimate for the higher degree L-functions. If one obtains such an estimate then it not only prove Selberg's central limit theorem for all L-functions belonging to *Selberg class* but also it will help to calculate the upper and lower bound of the moments of the L-functions.

Note that we have proved Selberg's central limit theorem for the real part of the L-functions not for the imaginary part. As mentioned in [RS17] the method we have used in this thesis can not be used to prove the result for the imaginary part. We have used the mollification technique here, for the imaginary part, the L-function and the mollifier is not inverse to each other. In fact, they differ with a substantial integer multiple of 2π.

Additionally, we have proved the independence of the automorphic L-functions associated with the sequence of primitive holomorphic cusp forms. Such a result can be proved for q-aspect as well by fixing the range of t. But hybrid aspect is not possible for the same reason we have mentioned earlier. Further, we have described a generalized notion of independence with the help of C. P. Hughes et al's work. They have explored the independence of the Riemann zeta function on the critical line for the different values of t. Such an idea can also be extended for the families of L-function as well. But when we combine these two ideas we get the random matrix (5.11). The study of the joint distribution of any two random elements (not chosen from the same row or column) of this matrix will be interesting.

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