THE IMPACT OF THINKING FAST AND SLOW ON TEACHING AND LEARNING STRATEGIES IN MATHEMATICS

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INTRODUCTION

Mathematics at university level can be challenging: students are introduced to statements, proofs, and a variety of methodologies that are unfamiliar. The extent to which students engage with such topics greatly moulds their abilities and progress along their learning journey. Instructional strategies for teaching mathematics at university require a variety of approaches since students' abilities, thought process, motivations and personalities differ. Knowing more about how the mathematical mindset works, particularly at the stage of undergraduate students, allows the educator to come up with effective ways to communicate the theory as well as guide the student to develop a successful learning style. The goal of mathematics educators at university level is to teach students how to tackle knowledge with competence and experience in steering, and not (only) to deposit facts and taught procedures in their memory bank.

Daniel Kahneman's celebrated book *Thinking Fast and Slow*[1], describes *System 1* (fast thinking) and *System 2* (slow thinking) processes and discusses their roles in understanding socio-economic interaction and decision making. Kahneman makes the case that presence of these two cognitive systems influences everything we do, including how we make decisions, and how we learn. In the past decade, and especially since the COVID-19 pandemic, there has been a rise in innovative digital approaches to education to engage students in creativity and collaborative ways. Although this may be an enriching way to motivate and teach students mathematics, a gap remains around helping students to train their strategic and disciplined thinking skills. Training a student to become a strategic thinker (capable of solving a problem or proving a theorem) does not happen instinctively but happens knowingly and intentionally. The benefit of thinking in such a way has benefits, as it becomes continual, and one is doing it consciously, thus developing a long-term skill.

There is a clear belief in the education literature on the growth mindset that student abilities to grasp mathematical concepts can be developed achievement.[2] There are a variety of positive behavioural and academic outcomes associated with such a belief which can be developed by interventions in teaching, assessment, and learning strategies. However, what do those interventions look like and why might they work? Based on Systems 1 and 2, as outlined in Daniel Kahneman's book *Thinking, Fast and Slow*, this paper explores how, and to what extent, the mathematical mindset of undergraduate students may be shaped by a layering of slow and fast cognitive processes. When are the two systems used in learning new mathematical concepts, and when using what they already know in the process? And how does the use of these systems influence problem solving, students' thoughts and their ability to progress? This paper aims to use these questions to develop a rationale behind teaching strategies that reflect these insights and to discuss some limitations and difficulties that may arise while using such strategies.

AN OVERVIEW OF SYSTEM 1 AND SYSTEM 2

Kahneman's book made popular the idea that decision making in events occurring in our everyday lives can be seen as using two cognitive systems, which he termed System 1 and System 2. [3]

- *System 1* corresponds to a fast, automatic, and decisive thought process. This process does not actively require logical or rational thinking and relies on heuristics, instinctive and learned behaviour. Making decisions while in System 1 is easy and quick, but can be subject to quite severe biases, especially when untrained, as it typically makes a decision based on a small amount of available information. However, when well-trained, System 1 can produce a skilful but automatic response.
- *System 2* corresponds to slow and considered, rational cognitive thought processes. Such processes are effort-full, language-based and error-prone: they require search of memory for previously deposited facts and careful resolution of paradoxical information that often results in cognitive dissonance. For these reasons, System 2 is slow to come to conclusions and tends to be used in fairly limited circumstances in everyday life. However, it is a vital skill to develop for mathematical problem solving.

Note that these systems are more than simply *slow* and *fast*. They can broadly be thought of as independent agents, each with abilities, limitations, functions, and assumptions.[4] Kahneman and his collaborators' research has done much to clarify these limitations, especially in the area of socio-economic decision making. The systems correspond to the *dual-process model of cognition*[5] whereby System 1 is the automatic (unconscious) mode and System 2 is the deliberate (analytical) mode. In this model, decision making works as the two Systems independently, and often in parallel, try to reach a conclusion. Particular problems that might arise using System 1 thinking include anchoring, the availability trap, loss aversion, framing, and the sunk loss fallacy. One might say that one of the main aims of mathematical training is to develop and extend the ability for applying System 2 thinking in a wide range of problems, but we suggest it is more complex than this. In this paper we highlight particular effects associated with this approach can be used to understand problems and fallacies that may occur when attempting to teach or learn a new mathematical topic.

MAPPING SYSTEMS 1 AND 2 ONTO LEARNING MATHEMATICS IN HE

When approaching a mathematical problem, such as solving a differential equation or proving a theorem, there is usually a desire to reach the outcome. The use of System 1 is rapid and instinctive but will tend to use heuristics to come to a conclusion;[6] if these heuristics are not well adapted to the situation, this may result in failure to provide a valid solution. On the other hand, the effortful and slow nature of System 2 can lead to demotivation and abandonment of the problem. This is particularly the case if the student is not familiar with bringing an extended use of System 2 to a successful conclusion.

Implications for Problem Solving

When attempting an unseen problem, a student may have difficulties in decoding a question, comprehending what is being asked, sorting relevant and essential information from the extraneous parts, recalling, or deriving a method to solve the problem, working through each step, and knowing what calculations are relevant. How do we problem solve? There have been many models in literature on the stages of problem solving. One model involves the "four steps":[7] understanding and exploring the problem; devising a plan; carrying out the plan; reflecting on the solution. Having steps to follow allows students to construct their own knowledge within the steps and gives them agency to collaborate actively on the problem.[8]

Frequently, some problems require students to move back and forth and across these steps. *Loss aversion* within System 1 can prevent students from making the right decisions, particularly when solving complex problems or when reaching a difficult stage within the problem-solving steps. The *sunk loss fallacy* is a similar problem - if a student has invested time in effort into a particular method that may not be appropriate, they may persist instead of switching to another more appropriate method. Both of these may cause students to disengage with a new mathematical approach and seek a way around solving the problem using methods they are used to.

Students can also switch back and forth between intuitive (System 1), deliberate, analytic (System 2) approaches during problem solving depending on their state of mind, progress within the problem and input from the educator. Encouraging students devote time and effort to System 2 can lead to "A-ha" moments[9] where knowledge gets successfully embedded into the student's skilful System 1, after which becomes accessible without the need for conscious processing.[10]

The effectiveness of a teaching approach is enhanced when its design and aims are aligned with the six levels of Bloom's Taxonomy[11]. Bloom's Taxonomy is a well-known tool used by many educators to classify learning objectives and is built on the cognitive domain that assumes that learning should start from the basics and progress towards higher level concepts using six level as shown in Figure 1. We suggest that motivational and integrational skills require extensive use of System 1, while intermediate levels require extensive use of System 2.

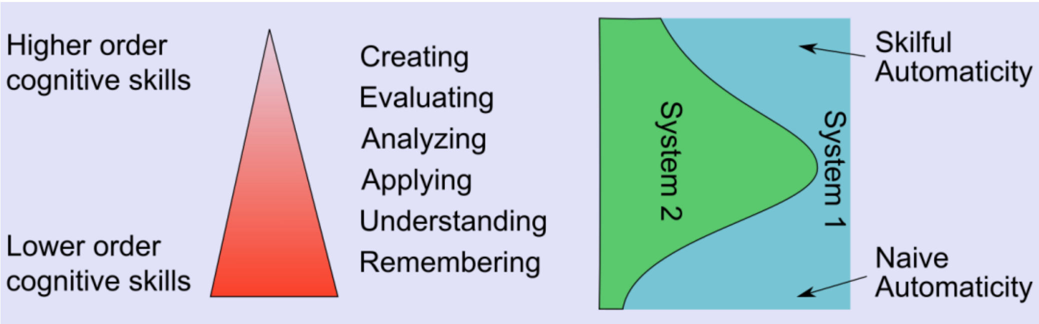

Figure 1: The role of System 1 (fast) and System 2 (slow) thinking within Bloom's Taxonomy. From lower to higher level thinking skills (i) remembering: memorization and recollection of facts without needing to understand; (ii) understanding: having deeper knowledge of the topic; (iii) applying: using what we know in solving and implementation; (iv) analyzing: this involves examining and breaking down information into smaller components, then determining how the parts relate to each other; (v)

evaluating: this involves analyzing while critiquing and comparing; and (VI) creating; generalizing and extending the end result. We suggest that System 2 needs to become most active at the intermediate levels of the taxonomy.

We suggest that a common problem for students arises from the temptation to use System 1 excessively at intermediate levels of the taxonomy. It is essential to help students activate their System 2 thinking, via slow and careful exposition of topics such as mathematical logic: this gives students confidence that a cognitive excursion into System 2 is likely to be worthwhile by giving comprehensible building blocks and a likely outcome when faced with an unseen concept or problem.

The temptation for students seeing a problem for the first time is to jump from an initial understanding straight to problem solving in the hope of gaining a reward. The problem posed by this temptation is that it springs over the necessary use of System 2 shown in the intermediate level of the taxonomy in Figure 1. This can be aggrevated by a knowledgable (but impatient) instructor who fails to highlight necessary System 2 involvement and who may have forgotten the importance of this involvement when learning this for the first time. Only through the involvement of System 2 in the solution to the problem leads to an approach that then becomes a reliable part of their System 1,[12] and after which the student requires reduced need for the effortful System 2 intervention with that particular approach.[13]

Implications for Reasoning

System 2 allows the mind to carefully identify benefits and drawbacks of using a particular approach. One way to influence the use of analogical reasoning and critical thinking is to set the problem to be solved in a way that compares two separate approaches with one leading to the wrong answer or conclusion and the other leading to the correct one. This may be particularly useful if the incorrect method highlights a common (but problematic) System 1 heuristic. When learning a new algorithm, it is useful to split the algorithm into distinct parts and to spend time with the parts in a sequential yet thorough manner. This allows the students to exercise their decompositional reasoning.[14] When doing this, it is important to slow down with the explanation when it comes to a more abstract, or unseen concept.

Occasionally, students fall into the *availability trap* whereby they apply an incorrect approach simply because they know how to apply it and believe it is somewhat relevant to the question. This can often take place even before the question is well understood. A useful exercise is to ask students to reflect and write down why the method they used is not applicable, rather than discarding it. This reflection helps them think more clearly about the goal of a problem and what *other* approaches are available to them and so stimulating System 2 deliberations. One way to deepen understanding of a theorem is to ask students to explain which hypotheses are broken in cases where the conclusion of a theorem does not hold.

How can we use *framing* to make an approach more appealing and convincing? Before introducing a new formal definition, start off by saying it using words only, and then explain the meaning behind that statement, again using words and images if possible. After the students have a good understanding of that definition, then they are ready to see the formal definition in terms of the mathematical notation and to launch into System 2 processes. It can be occasionally difficult to use various approaches to framing a mathematics statement, particularly in abstract maths, however speaking about its use or derivation (using words and images) to accompany the statement makes it easier to understand and removes the monotony of formal definition writing.

Implications for Student Engagement

One major challenge for the mathematics educator is to maintain student engagement across a range of backgrounds and expectations within a class. System 2 thinking is effortful and laborious, and as it does not lend itself to (lecture) theatrics or PowerPoint presentations, there is a temptation for the educator to leave it to students' independent study. This is especially the case in a cohort of students with mixed experiences in engaging System 2 to solve novel problems[15]. An experienced and successful student becomes used to the need to engage System 2 and to delay judgement on its outcome, in the knowledge that the effort is likely to be worthwhile. However, it can easily become a source of frustration and disillusionment, if attempts to use System 2 thinking to solve a problem are not successful even for "easier" material. This may be compounded by more experienced tutor or peers who may have learned advanced and accurate heuristics and who are able to solve a problem using predominantly System 1 thinking. In such a case it is important to devise strategies that ensure students value, and have access to, detailed (possibly pedestrian) System 2 arguments. If a learner "gives up" investing time in System 2 they may prioritise System 1 heuristics to get them through the assessment. In the worst case they may lose engagement with the learning altogether.

Implications for Assessment

Mathematics is a lot about ideas and processes, but this internalizes when students attempt examples and try to engage more deeply. A variety of assessments can make a difference, as does positive feedback, listening to different approaches to the questions, and having more open-ended questions, set yourself a question as a student and do it that way.

The concept of *loss aversion* certainly plays a key role in the students' approach to revision and assessments. Most students start with an aspiration to achieve high marks in their exams and course as a whole. As soon as the pace of the course speeds up and they settle in into university life, students start to have to prioritise their activities and set their own personal objectives. For some students, this is getting a first-class mark, for others it is simply passing each module. By focusing too much on certain objectives, students may miss the fundamental point that learning and training the mind for future (harder) topics takes more than just doing what is required to get a first-class mark or a pass. It requires slowing down with learning new concepts and training the high-level cognitive processes. Studies show that when students set their goal on learning rather than on achieving high marks, their problem solving skills and reasoning skills are enhanced considerably.[16]

Spending time on System 2 may feel risky for a student worried about gaining marks in exams. However, this has a long-term impact on their learning strategies. To that end, we argue that assessment should be designed in a way that approaches using System 1 (e.g. presenting comparable questions) is not enough. Assessment problems should be designed in a way that makes the revision of the topics deep and thorough, and that ensures System 2 is activated during the learning process. The paradox here is that assessing the outcomes using time limited examinations strongly incentivizes use of System 1 heuristics that are only likely to be reliable if they have been developed with the benefit of System 2! We suggest it is useful to consider which processes are engaged during assessment of an intended learning outcomes, and whether this is optimal. Overall, assessments should help students gain confidence that they can successfully engage System 2 thinking as and when needed. In addition to demonstrating and understanding of the basics of the topic, more challenging questions that require a slower thinking approach and a combination of concepts and theories are good ways to do this.

CONCLUDING REMARKS

Learning mathematics requires the constructive use of both System 1, the fast, automatic (but heuristic and prone to bias) approach, as well as System 2, the slow, rational, language-based (but effortful and error-prone) approach. When teaching a new topic, a slow approach is a way to signal the need for System 2 thinking, however it is also important to recognize that as understanding of a topic matures, a System 2 technique may be transferred into skillful System 1 heuristics.

Going through the steps of problem solving in the slow manner of System 2 heavily relies on students identifying relevant details in problems, having the skill to identify what is irrelevant and what the meaning of each part is. With practice and time, repeated System 2 use on similar problems can create new, additional, skills for their System 1 thinking, thus giving them more areas of experiencing satisfaction that comes from success in solving problems.

It is important to make students aware of the variety of learning techniques, making sure they have some level of freedom in what and how they study. The assessments shouldn't be too rigid, too much or too little, also neither too hard nor too easy. And encourage students to experiment and have an open mind. Students who have cultivated a growth mindset are less likely to be demotivated by failing an assessment or by not quickly grasping the theory. On the other hand, they are more likely to put in more effort and try out various approaches to bridge the gaps and identify ways to succeed the next time. Ideally, students will use their losses and failures to learn how to use System 1 and 2 appropriately to gain and succeed in future.

NOTES

[1] Daniel Kahneman, *Thinking Fast and Slow*, Penguin, (1st edition 2012).

[2] Jo Boaler, "Ability and Mathematics", Forum, (2013) 55(1) pp. 143–152.

[3] We stick primarily to Kahneman's terminology in this paper, though note that System 1 corresponds to automaticity and System 2 to cognitive rational responses. See for example a discussion of the effects of automaticity on the pedagogical process, Feldon, D.F., Cognitive Load and Classroom Teaching: The Double-Edged Sword of Automaticity, (Educational Psychologist 2007), 42(3), pp. 123–137.

[4] Daniel Kahneman, *Thinking Fast and Slow*, Penguin, (1st edition 2012).

[5] David Feldon, "Cognitive Load and Classroom Teaching: The Double-Edged Sword of Automaticity", Educational Psychologist (2007), 42(3), pp. 123–137.

[6] Daniel Kahneman, *Thinking Fast and Slow*, Penguin, (1st edition 2012).

[7] Jason van Steenburgh, et al. "Insight In The Oxford Handbook of Thinking and Reasoning", ed. K Holyoak, R Morrison, New York: Oxford Univ. Press (2012), pp. 475–91.

[8] George Polya, *How to solve it, a new aspect of mathematical methods*, Stanford University Press (Doubleday Anchor Books, 1957).

[9] John Kounis and Mark Beeman, "The Aha! Moment: The Cognitive Neuroscience of Insight", Current Directions in Psychological Science, Sage Journals, (2009), 18(4) pp. 210-216.

[10] Brian Bottge, et al. "Impact of enhanced anchored instruction in inclusive math classrooms", Exceptional Children (2015) 81(2), pp. 158-175.

[11] Benjamin Samuel Bloom, *Taxonomy of educational objectives: The classification of educational goals: Handbook I, cognitive domain*, New York: Toronto, Longmans, Green, (1956).

[12] Daniel Kahneman, *Thinking Fast and Slow*, Penguin, (1st edition 2012).

[13] Richard Clark and David Feldon, “Five Common but Questionable Principles of Multimedia Learning”, Cambridge Handbooks in Psychology, Cambridge University Press, (2012) pp. 97-116.

[14] Jason van Steenburgh, et al. “Insight In The Oxford Handbook of Thinking and Reasoning”, ed. K Holyoak, R Morrison, New York: Oxford Univ. Press (2012), pp. 475–91.

[15] Linda Darling-Hammond and Jon Snyder, Authentic Assessment of Teaching in Context, Teaching and Teacher Education, (2000), 16(5-6) pp. 523-545.

[16] John Sweller, “Cognitive technology: Some procedures for facilitating learning and problem solving in mathematics and science”, Journal of Educational Psychology (1989) 81(4), 457–466.

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