Moments of Dirichlet L-functions in Function Fields

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I certify that all material in this thesis which is not my own work has been identified and that any material that has previously been submitted and approved for the award of a degree by this or any other University has been acknowledged.

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Abstract

In this thesis, we compute several moments and mean values of Dirichlet L-functions in function fields, in both the odd and even characteristic setting.

Firstly, in Chapter 3, we use the techniques originally developed by Florea [Flo17a] to improve the asymptotic formula for the first moment of quadratic Dirichlet L-functions $L(s,\chi_D)$ at point $s=\frac{1}{2}$, where D runs over all monic, square-free polynomials of even degree in $\mathbb{A}=\mathbb{F}_q[T]$ and $q\equiv 1(\mathrm{mod}\ 4)$, which was first obtained by Jung [Jun13]. In particular, compared to the asymptotic formula obtained by Jung, we obtain a secondary main term and improve the error term.

In Chapter 4, we obtain an asymptotic formula for the first moment of $L(2,\chi_{\gamma D})$, where γ is a fixed generator of $\mathbb{F}_q^*$ and D runs over all monic, square-free polynomials of even degree in $\mathbb{A}$, where $q\equiv 1(\mathrm{mod}\ 4)$. As an application of this, we compute the average size of the algebraic group $K_2(\mathcal{O}_{\gamma D})$, where $\mathcal{O}_{\gamma D}$ denotes the integral closure of $\mathbb{A}$ in $k(\sqrt{\gamma D})$, where $k=\mathbb{F}_q(T)$.

In Chapter 5, we obtain a lower bound for the k^{th} moment of quadratic Dirichlet L-functions $L(s,\chi_P)$ at $s=\frac{1}{2}$, where k is an even natural number, P is a monic irreducible polynomial in $\mathbb{F}_q[T]$ and $q\equiv 1(\mathrm{mod}\ 4)$.

In Chapter 6, we formulate a conjecture for the integral moments of quadratic Dirichlet L-functions $L(s,\chi_u)$ at the central point $s=\frac{1}{2}$, where u runs over a specific family in $\mathbb{F}_q[T]$ and q is a power of 2. We also show that this conjecture agrees with the asymptotic formulas that have already been obtained. We also obtain the leading order asymptotic for the moments of $L\left(\frac{1}{2}+it,\chi_u\right)$ as we want to understand symmetry transitions of Dirichlet L-functions in the function field setting. In Chapter 7, we generalise the methods used in Chapter 6 to conjecture an asymptotic formula for the mean value of ratios of products of the Dirichlet L-functions $L(s,\chi_u)$. In Chapter 8, we present two applications of the Ratios conjecture in even characteristic. Namely, under the

condition of the Ratios conjecture, we derive a formula for the one-level density for the zeros of $L(s, \chi_u)$ and show that the proportion of $L(s, \chi_u)$ which do not vanish at $s = \frac{1}{2}$ is 100%.

Finally, in Chapter 9, we obtain an asymptotic formula for the second moments of $\left|L\left(\frac{1}{2}, \chi\right)\right|$ with one and two twists, when averaged over all primitive Dirichlet characters of modulus R, where R is a monic polynomial in $\mathbb{F}_q[T]$.

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Contents

Author's Declaration

I declare that the work done in this thesis is in accordance with the regulations of the University of Exeter. The work is original except where indicated.

The work done in this thesis was done under the supervision of Dr. Julio Andrade. In particular, the work done in Chapter 3 and Appendix A is joint with Julio Andrade and has been published in [AM21]. The work done in Chapter 4 is my own, however it has been combined with work done jointly with Julio Andrade and Rhianwen Davies and has been submitted for publication [ADM22]. Finally, the work done in Chapter 5 through to Chapter 9 was done by myself and it is my intention to submit these results for publication as sole authorship soon.

No part of this thesis has been submitted for any other academic degree and has not been presented to any other university for examination.

Notation

Most of the notation used in this thesis will be defined in Chapter 2. However, we will describe some of the notation that is used throughout this thesis here too.

O	$f(x) = O(g(x))$ if there exists a constant $c > 0$ such that $\lvert f(x)\rvert \leq cg(x)$ for all $x \geq x_0$.
$\ll$	$f(x) \ll g(x)$ if $f(x) = O(g(x))$.
o	$f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
$\sim$	$f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.
$\mathbb{F}_q$	A finite field with q elements.
$\mathbb{F}_q^*$	The multiplicative group of $\mathbb{F}_q$.
$\mathbb{A} = \mathbb{F}_q[T]$	The polynomial ring over $\mathbb{F}_q$.
$\mathbb{A}^+$	The set of monic polynomials in $\mathbb{A}$.
$k = \mathbb{F}_q(T)$	The rational function field over $\mathbb{F}_q$.
$\mathcal{H}_n$	The set of monic, square-free polynomials of degree n in $\mathbb{F}_q[T]$.
$\mathcal{P}_n$	The set of monic irreducible polynomials in $\mathbb{F}_q[T]$.
$\mathcal{I}_n$	The set, in even characteristic, defined in Section 2.7.1.
$\lvert f\rvert$	$\lvert f\rvert = q^{\deg(f)}$, the norm of $f \in \mathbb{F}_q[T]$.
γ	A fixed generator of $\mathbb{F}_q^*$.
P	A monic irreducible polynomial in $\mathbb{F}_q[T]$.
$\#G$	The size of the set G.
$\oint$	An integral over a closed contour.
$\int_{(c)}$	An integral along the line $\Re(s) = c$.
$\mu(f)$	The Möbius function for $\mathbb{A}$.
$\phi(f)$	The Euler-Totient function for $\mathbb{A}$.
$\omega(f)$	The number of distinct monic irreducible polynomial factors of f.
$d_k(f)$	The number of ways of writing the polynomial f as a product of k factors.

Chapter 1

Introduction

1.1 The Riemann zeta-function

One of the most important subjects in Analytic Number Theory is the study of the theory of the Riemann zeta-function, which was first introduced by Riemann [Rie59] and is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1.1}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$. In this region, the sum defining the Riemann zeta-function is convergent and in the region $\Re(s) \geq 1 + \delta$ the series is absolutely and uniformly convergent for every $\delta > 0$. Therefore the Riemann zeta-function is holomorphic for $\Re(s) > 1$. Furthermore Euler [Eul37] showed that, for $\Re(s) > 1$, the Riemann zeta-function can be represented as an Euler product, namely

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1.1.2}$$

where the product is over all primes p. Using this formula, we can see the importance of the Riemann zeta-function in the theory of prime numbers. For example Hadamard [Had96] and de la Vallée Poussin [dlVP96] independently proved that $\zeta(s) \neq 0$ on the line $\Re(s) = 1$ and used this to prove the Prime Number Theorem.

Theorem 1.1.1 (Prime Number Theorem)**.** *Let* $\pi(x) = \#\{p : p \text{ prime}, p \leq x\}$*. Then*

$$\pi(x) \sim \frac{x}{\log x} \tag{1.1.3}$$

as $x \to \infty$*.*

Further Riemann showed that the Riemann zeta-function has a meromorphic continuation to the whole complex plane with a simple pole at $s = 1$ with residue 1. He also showed that the Riemann zeta-function satisfies the following functional equation.

Theorem 1.1.2. *For all $s \in \mathbb{C}$ we have*

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s), \tag{1.1.4}$$

where $\Gamma(s)$ is the Gamma function defined as

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx.$$

From (1.1.2) and Hadamard and de la Vallée Poussin, we know that $\zeta(s) \neq 0$ in the region $\Re(s) \geq 1$. Thus $\zeta(1-s) \neq 0$ for $\Re(s) \leq 0$. Furthermore, it can be shown that $\Gamma(s)$ is non-zero for all s and is holomorphic in the region $\Re(s) > 0$. Therefore, from the functional equation (1.1.4), any zeros of $\zeta(s)$ which occur in the region $\Re(s) < 0$ arise from the zeros of $\sin\left(\frac{\pi s}{2}\right)$. Thus, in the region $\Re(s) < 0$, $\zeta(s) = 0$ when $s = -2n$ for all $n \in \mathbb{N}$, these are called the "trivial zeros" of the Riemann zeta-function. Therefore any "non-trivial" of the Riemann zeta-function lie in the critical strip $0 < \Re(s) < 1$. Riemann then stated the famous conjecture about the location of these "non-trivial" zeros.

Conjecture 1.1.3 (Riemann Hypothesis)**.** *All the non-trivial zeros of the Riemann-zeta function lie on the critical line $\Re(s) = \frac{1}{2}$.*

To this day, the Riemann hypothesis is still an open problem, and is one of the most important open problems in mathematics. Although it has not been proven explicitly, there is numerical evidence to support this conjecture. For example the hypothesis has been checked for the first 10,000,000,000 zeros (for a list of these zeros, see [LMF22]). Furthermore, we know, by Pratt, Robles, Zacharescu and Zeindler [PRZZ20] that more than five-twelfths of the non-trivial zeros of the Riemann zeta-function lie on the critical line $\Re(s) = \frac{1}{2}$.

Assuming the Riemann hypothesis, Gonek [Gon12] showed that the Riemann zeta-function is well approximated by short truncations of its Euler products in the region $\Re(s) > \frac{1}{2}$ and not too close to the critical line. Conversely, Gonek also showed that if the approximation of $\zeta(s)$ by Euler products is good in this region, then the Riemann zeta-function has at most a finite number of zeros in it.

1.2 Moments of the Riemann zeta-function

One interesting problem involving the Riemann zeta-function is to understand its growth rate on the critical line. Lindelöf [Lin08] conjectured the following result about this growth rate.

Conjecture 1.2.1 (Lindelöf Hypothesis). *For every $\epsilon > 0$ we have*

$$\zeta\left(\frac{1}{2}+it\right) \ll t^{\epsilon}. \tag{1.2.1}$$

Titchmarsh [Tit86] explained that the Lindelöf hypothesis is equivalent to showing that

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll T^{\epsilon} \tag{1.2.2}$$

for every integer k and any $\epsilon > 0$. Thus, a very important problem in Analytic Number Theory is to estimate these moments of the Riemann zeta-function. More specifically, we want to understand the asymptotic behaviour of

$$M_k(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt. \tag{1.2.3}$$

An asymptotic formula for the second moment of (1.2.3) was first obtained by Hardy and Littlewood [HL16] in which they proved the following result.

Theorem 1.2.2 (Hardy and Littlewood). *We have, as $T \to \infty$,*

$$M_1(T) \sim T \log T. \tag{1.2.4}$$

Ingham [Ing27] improved the asymptotic formula (1.2.4) by proving the following result.

Theorem 1.2.3 (Ingham). *We have*

$$M_1(T) = T\log\frac{T}{2\pi} + (2\gamma - 1)T + O\left(T^{\frac{1}{2}}\log T\right), \tag{1.2.5}$$

where γ is Euler's constant.

In the same paper, Ingham established an asymptotic formula for the fourth moment of (1.2.3). In particular he proved the following result.

Theorem 1.2.4 (Ingham). *We have, as $T \to \infty$,*

$$M_2(T) = \frac{1}{2\pi^2} T\log^4 T + O(T\log^3 T). \tag{1.2.6}$$

Subsequently, Heath-Brown [HB79] improved the asymptotic formula (1.2.6) by obtaining all the main terms for the asymptotic formula.

Theorem 1.2.5 (Heath-Brown). *There exist constants b_4, b_3, b_2, b_1 and b_0 such that for $T \geq 2$ and $\epsilon > 0$,*

$$M_2(T) = T\sum_{n=0}^{4} b_n \log^n T + O\left(T^{\frac{7}{8}+\epsilon}\right). \tag{1.2.7}$$

In particular, the constant $b_4 = \frac{1}{2\pi^2}$.

For the sixth moment, Conrey and Ghosh [CG98] conjectured that

$$M_3(T) \sim \frac{42}{9!} \prod_p \left(\left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right) T \log^9 T$$

and for the eighth moment, Conrey and Gonek [CG01] conjectured that

$$M_4(T) \sim \frac{24024}{16!} \prod_p \left(\left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right) \right).$$

In general, it is conjectured that

$$M_k(T) \sim \frac{a_k g_k}{(k^2)!} T (\log T)^{k^2} \tag{1.2.8}$$

where g_k is a positive integer and

$$a_k = \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^m} \right), \tag{1.2.9}$$

where $d_k(n)$ is the number of ways to write n as a product of k factors. Using Random Matrix Theory, Keating and Snaith [KS00b] conjectured a precise value for g_k, namely

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

Gonek, Hughes and Keating [GHK07] approximated $\zeta(s)$ as a hybrid Euler-Hadamard product. Namely, they showed that $\zeta(s) \approx P_X(s) Z_X(s)$, where $P_X(s)$ is a truncated Euler product and $Z_X(s)$ is a truncated Hadamard product. Using this, they conjectured that the $2k^{\text{th}}$ moment of $\zeta\left(\frac{1}{2} + it\right)$ is asymptotic to the $2k^{\text{th}}$ moment of $P_X\left(\frac{1}{2} + it\right)$ multiplied by the $2k^{\text{th}}$ moment of $Z_X\left(\frac{1}{2} + it\right)$, which is called the Splitting Conjecture. Furthermore, they showed that the $2k^{\text{th}}$ moment of $P_X\left(\frac{1}{2} + it\right)$ contributes the factor $a(k)$ and using Random Matrix Theory, they conjectured that the $2k^{\text{th}}$ moment of $Z_X\left(\frac{1}{2} + it\right)$ contributes the factor $g(k)$. Combining these results, they recovered the conjecture (1.2.8) and gave support for the Splitting Conjecture by showing that it holds when $k = 1$ and $k = 2$.

Conrey, Farmer, Keating, Rubinstein and Snaith [CFK+08] described a heuristic to conjecture all of the main terms for the $2k^{\text{th}}$ moments of the Riemann zeta-function and showed that the conjecture agrees with the results seen previously. In a series of papers, Conrey and Keating [CK15a, CK15b, CK15c, CK16, CK19] looked at the problem of obtaining asymptotic formulas for the $2k^{\text{th}}$ moment and shifted moments of the Riemann zeta-function on the critical line from a number-theoretic perspective. In particular, they gave new details about how the off-diagonal terms contribute to the main terms of the asymptotic formula.

One can also compute upper and lower bounds of (1.2.3) which can show that the conjecture (1.2.8) is of the correct order of magnitude. For example, Ramachandra [Ram80] obtained the following result for the lower bounds of (1.2.3).

Theorem 1.2.6 (Ramachandra)**.** *We have*

$$M_k(T) \gg T(\log T)^{k^2} \tag{1.2.10}$$

for natural numbers k.

Heath-Brown [HB81a] extended Ramachandra's result by showing that (1.2.10) holds for any rational number $k \geq 0$. This result was further improved by Radziwiłł and Soundararajan [RS13] and Heap and Soundararajan [HS22] who improved the result further by showing that for all large T and all real $k > 0$ we have

$$M_k(T) \geq C_k T(\log T)^{k^2} \tag{1.2.11}$$

for some constant C_k. For upper bounds Soundararajan [Sou09] showed that under the condition of the Riemann Hypothesis we have

$$M_k(T) \ll T(\log T)^{k^2+\epsilon}, \tag{1.2.12}$$

for every positive real k and every $\epsilon > 0$. Refining Soundararajan's method, Harper [Har13] removed the ϵ on the power of $\log T$ in (1.2.12) and thus, under the condition of the Riemann hypothesis, obtained upper bounds of the correct order of magnitude. In a recent paper, Heap, Radziwiłł and Soundararajan [HRS19] showed that, unconditionally we have

$$M_k(T) \ll T(\log T)^{k^2}$$

for $0 \leq k \leq 2$ and $T \geq 10$.

Another problem involving $\zeta(s)$ is what is called the mollified moments of the Riemann zeta-function. In other words, we want to understand the asymptotic behaviour of

$$I_y(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \left|M_y\left(\frac{1}{2}+it\right)\right|^2 dt, \tag{1.2.13}$$

where $M_y(s)$ is a mollifier of the form

$$M_y(s) = \sum_{n\leq y} \frac{a(n)}{n^s}$$

and for some specifically chosen coefficients $a(n)$. Computing mollified moments has applications to proving lower bounds on the proportion of the non-trivial zeros of $\zeta(s)$ that lie on the critical line. For example, Levinson [Lev74] showed that

$$\lim_{T\to\infty} \frac{I_{T^\theta}(T)}{T} = 1 + \frac{1}{\theta} \tag{1.2.14}$$

for $0 < \theta < \frac{1}{2}$ with $a(n) = \frac{\mu(n)\log(\frac{y}{n})}{\log y}$ and thus deduced that the proportion of the non-trivial zeros of $\zeta(s)$ that lie on the critical line is greater than $\frac{1}{3}$. Changing the coefficient to $a(n) = \mu(n)P\left(\frac{\log(\frac{y}{n})}{\log y}\right)$, where P is a polynomial satisfying $P(0) = 0$, Conrey [Con89] showed that (1.2.14) holds for $\theta < \frac{4}{7}$ and thus the proportion of the non-trivial zeros of $\zeta(s)$ is greater than $\frac{2}{5}$. Bettin and Gonek [BG17] showed that if $I_N(T) \ll_\epsilon T^{1+\epsilon}$ for $2 \leq N \leq T^\theta$ with θ arbitrarily large, then the Riemann hypothesis is true.

1.3 Dirichlet L-functions

In this section, we define a generalisation of the Riemann zeta-function, namely we state the definition of a Dirichlet L-function as well as state results about them. To do this, we first state the definition of a Dirichlet character.

Definition 1.3.1. Let q be a positive integer. Then a function $\chi : \mathbb{Z} \to \mathbb{C}$ is called a Dirichlet character modulo q if

i) $\chi(mn) = \chi(m)\chi(n), \quad \forall m, n \in \mathbb{Z}$,

ii) $\chi(n+q) = \chi(n), \quad \forall n \in \mathbb{Z}$,

iii) $\chi(1) = 1$,

iv) $\chi(n) = 0$, whenever $(n, q) > 1$.

We define the trivial character $\chi_0(n)$ by

$$\chi_0(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Definition 1.3.2. Let χ be a Dirichlet character modulo q. The Dirichlet L-function corresponding to χ is defined to be

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.3.1}$$

The Dirichlet L-function is absolutely convergent for $\Re(s) > 1$ and locally uniformly convergent, and thus $L(s,\chi)$ is holomorphic in this region. Furthermore in this region, these Dirichlet L-functions have an Euler product representation, namely

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}. \tag{1.3.2}$$

If χ is the trivial character modulo q, then $L(s,\chi_0)$ is regular for all $s \in \mathbb{C}$ except for a simple pole at $s = 1$ with residue $\frac{\phi(q)}{q}$. If χ is not the trivial character modulo q, then $L(s,\chi)$ can be defined for all $s \in \mathbb{C}$.

Definition 1.3.3. Let χ be a Dirichlet character modulo q and let $d|q$. The number d is called an induced modulus for χ if we have

$$\chi(a) = 1 \text{ whenever } (a,q) = 1 \text{ and } a \equiv 1 (\text{mod } d).$$

Definition 1.3.4. A Dirichlet character modulo q is said to be primitive modulo q if it has no induced modulus $d < q$.

Restricting χ to be a primitive Dirichlet character modulo q, we define the completed L-function as

$$\Lambda(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s,\chi), \tag{1.3.3}$$

where

$$\mathfrak{a} = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Then the completed L-function satisfies the functional equation

$$\Lambda(s,\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}} \Lambda(1-s,\bar{\chi}), \tag{1.3.4}$$

where $\tau(\chi)$ is the Gauss sum defined by

$$\tau(\chi) := \sum_{a (\text{mod } q)} \chi(a) e\left(\frac{a}{q}\right)$$

where $e(x) := \exp(2\pi i x)$. Furthermore, we know that (see [MV06, Chapter 10] for more details) $L(s,\chi) \neq 0$ for $\Re(s) > 1$ and for $\Re(s) < 0$, $L(s,\chi)$ has trivial zeros at $s = -2n$ if $\chi(-1) = 1$ and at $s = -2n-1$ if $\chi(-1) = -1$. For the non-trivial zeros, we have the following conjecture.

Conjecture 1.3.5 (Generalised Riemann Hypothesis)**.** *All the non-trivial zeros of Dirichlet L-functions lie on the critical line.*

1.4 Mean Value Theorems of Primitive Dirichlet L-functions

Similar to the Riemann zeta-function, we want to understand the asymptotic behaviour of moments of Dirichlet L-functions. One problem is to understand the $2k^{\text{th}}$ moment of $|L(s,\chi)|$ at the central point $s = \frac{1}{2}$, when summed over all primitive Dirichlet characters modulo q, which represents a q-analogue of the moments of the Riemann zeta-function on the critical line. More precisely, we want to understand asymptotic formulas for

$$\frac{1}{\phi^*(q)} \sum_{\chi (\bmod q)}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k}, \tag{1.4.1}$$

where $\phi^*(q)$ is the number of primitive Dirichlet characters modulo q and the sum is over all primitive Dirichlet characters modulo q. For the second moment of (1.4.1), Paley [Pal31] proved the following result.

Theorem 1.4.1 (Paley)**.** *We have, as $q \to \infty$,*

$$\frac{1}{\phi^*(q)} \sum_{\chi (mod\ q)}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2} \sim \frac{\phi(q)}{q} \log q.$$

For the fourth moment of (1.4.1) Heath-Brown [HB81b] proved the following result.

Theorem 1.4.2 (Heath-Brown)**.** *We have*

$$\frac{1}{\phi^*(q)} \sum_{\chi (mod\ q)}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{4} = \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 + O\left(\frac{2^{\omega(q)}}{\phi^*(q)} q (\log q)^3 \right), \tag{1.4.2}$$

where $\omega(q)$ is the number of prime divisors of q, where q is a positive integer.

The asymptotic formula (1.4.2) holds for almost all q. In particular, to ensure that the error term is smaller than the main term (as $q \to \infty$), q must be restricted in such a way that

$$\omega(q) < \frac{\log \log q}{\log 2}.$$

Soundararajan [Sou07] addressed this restriction by obtaining the following result.

Theorem 1.4.3 (Soundararajan)**.** *For all large q we have*

$$\frac{1}{\phi^*(q)} \sum_{\chi (mod\ q)}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{4}$$
$$= \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 \left(1 + O\left(\frac{\omega(q)}{\log q} \sqrt{\frac{q}{\phi(q)}} \right) \right) + O\left(\frac{q}{\phi^*(q)} (\log q)^{\frac{7}{2}} \right).$$

Further Young [You11a] obtained an asymptotic formula for all the main terms with a power savings for prime moduli. In particular he proved the following result.

Theorem 1.4.4 (Young). *For prime $q \neq 2$ we have*

$$\frac{1}{\phi^*(q)} \sum_{\chi (mod\ q)}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^4 = \sum_{n=0}^{4} c_n (\log q)^n + O\left(q^{-\frac{5}{512}+\epsilon}\right) \tag{1.4.3}$$

for some constants c_n.

Bloomer *et al* [BFK+17] improved the error term of (1.4.3) to $O\left(q^{-\frac{1}{20}+\epsilon}\right)$. Although no higher moments of (1.4.1) has been explicitly proven, Bui and Keating [BK07] conjectured an asymptotic formula for (1.4.1) for all real k.

Conjecture 1.4.5 (Bui and Keating). *For k fixed with $\Re(k) \geq 0$ we have*

$$\frac{1}{\phi^*(q)} \sum_{\chi (mod\ q)}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \sim a(k) \frac{G^2(k+1)}{G(2k+1)} \prod_{p|q} \left(\sum_{m \geq 0} \frac{d_k(p^m)^2}{p^m} \right)^{-1} (\log q)^{k^2}$$

where $a(k)$ is defined in (1.2.9) and G is the Barnes G-function.

Similar to the Riemann zeta-function, one can also obtain upper and lower bounds for the moments of primitive Dirichlet L-functions (1.4.1). For example Rudnick and Soundararajan [RS05] obtained the following result for lower bounds of (1.4.1).

Theorem 1.4.6 (Rudnick and Soundararajan). *Let k be a fixed natural number. Then for all large primes q we have*

$$\sum_{\chi (mod\ q)}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \gg_k q (\log q)^{k^2}. \tag{1.4.4}$$

This result was improved by Radziwiłł and Soundararajan [RS13] and Heap and Soundararajan [HS22] who showed that (1.4.4) holds for all real $k > 0$. For upper bounds Soundararajan [Sou09] showed that, under the condition of the Generalised Riemann hypothesis, we have

$$\frac{1}{\phi^*(q)} \sum_{\chi (\text{mod } q)}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \ll_{k,\epsilon} q (\log q)^{k^2+\epsilon}.$$

for all positive real k, $\epsilon > 0$ and q prime. Heath-Brown [HB10] showed that

$$\frac{1}{\phi^*(q)} \sum_{\chi (\text{mod } q)}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \ll_k (\log q)^{k^2}. \tag{1.4.5}$$

and thus obtaining upper bounds of the correct order of magnitude, under the condition of the Generalised Riemann Hypothesis. Unconditionally Heath-Brown showed that (1.4.5) holds for any k of the form $k = \frac{1}{n}$ with $n \in \mathbb{N}$. In a recent paper Gao

[Gao21a] showed that (1.4.5) holds, unconditionally, for all real $0 \le k \le 1$.

Another problem is to understand the asymptotic behaviour of twisted moments of Dirichlet L-functions. That is, for coprime integers h and $p > 0$, where p and h are both primes, we want to find an asymptotic formula for

$$M(p,h) = \sum_{\chi(\mathrm{mod}\ p)}^{*} \left| L\left(\frac{1}{2},\chi\right)\right|^2 \chi(h), \tag{1.4.6}$$

where the sum is over all primitive Dirichlet characters modulo p. Conrey [Con07] proved a reciprocity formula involving $M(p,h)$ and $M(h,-p)$ where p and h primes with $h < p^{\frac{2}{3}}$. Young [You11b] improved Conrey's reciprocity formula by proving the following result for $h < p$.

Theorem 1.4.7 (Young)**.** *Suppose $h < p$ are primes. Then*

$$M(p,h) - \frac{\sqrt{p}}{\sqrt{h}} M(h,-p) = \frac{p}{\sqrt{h}}\left(\log\left(\frac{p}{h}\right) + \gamma - \log(8\pi)\right) + \zeta\left(\frac{1}{2}\right)^2 \sqrt{p} + \mathcal{E}(p,h) \tag{1.4.7}$$

where $\mathcal{E}(p,h) \ll p^{-\frac{1}{2}+\epsilon} h + h^{-\frac{1}{2}+\epsilon} p^{\frac{1}{2}}$.

Bettin [Bet16] studied the error term $\mathcal{E}(p,h)$ and showed that $\mathcal{E}\left(\frac{p}{h}\right) := \mathcal{E}(p,h)$ extends to a continuous function $\mathcal{E}(x)$ of the non-negative real numbers, which is $O(x)$ as $x \to 0^+$. In particular $\mathcal{E}\left(\frac{p}{h}\right) \ll \frac{p}{h}$ for $h \ll p$.

Similarly, we also want to understand the asymptotic behaviour of the second moment of Dirichlet L-functions with two twists. Namely, one problem is to understand the asymptotic behaviour of

$$M_{\pm}(p;h,k) = \frac{p^{\frac{1}{2}}}{\phi(p)} \sum_{\substack{\chi(\mathrm{mod}\ p)\\ \chi(-1)=\pm 1}}^{*} \left| L\left(\frac{1}{2},\chi\right)\right|^2 \chi(h)\bar{\chi}(k). \tag{1.4.8}$$

This problem was first considered by Selberg [Sel46] who showed that

$$M_{\pm}(q;h,k) \sim \frac{1}{2}\left(\frac{q}{hk}\right)^{\frac{1}{2}} \left(\log\frac{q}{hk} + \gamma - \log(8\pi) \mp \frac{\pi}{2}\right),$$

where q is a prime and for some conditions on h and k. For different primes q, h and k with $q \ge 4hk$, Bettin [Bet16] proved a triple reciprocity formula involving $M(q;h,k)$, $M(k;h,q)$ and $M(h;k,q)$.

Theorem 1.4.8 (Bettin)**.** *Let h, k and q be different primes and let $q \ge 4hk$. Then*

$$\begin{aligned} M_{\pm}(q;h,k) &= \pm M(k;h,q) \pm M(h;k,q) \\ &\quad + \frac{1}{2}\left(\frac{q}{hk}\right)^{\frac{1}{2}}\left(\log\left(\frac{q}{hk}\right) + \gamma - \log(8\pi) \mp \frac{\pi}{2}\right) + O(\log q). \end{aligned}$$

For the twisted fourth moment Hough [Hou16] obtained an asymptotic formula for

$$\frac{1}{\phi^*(q)} \sum_{\chi (\bmod q)}^* \left| L\left(\frac{1}{2}, \chi\right)\right|^4 \chi(\ell_1)\bar{\chi}(\ell_2) \tag{1.4.9}$$

where ℓ_1 and ℓ_2 are coprime and square-free with $1 \leq \ell_1, \ell_2 \leq q^\theta$ and $\theta < \frac{1}{32}$. Zacharias [Zac19] obtained an asymptotic formula for (1.4.9) where ℓ_1 and ℓ_2 are coprime and cubefree with $(\ell_1\ell_2, q) = 1$. His result allows for the application to non-vanishing results.

1.5 Mean Value Theorems of Quadratic Dirichlet L-functions

We start this section by defining a quadratic Dirichlet L-function. To do this, we need the following definition.

Definition 1.5.1. The number $d \neq 1$ is called a fundamental discriminant if either $d \equiv 1 (\bmod 4)$ or $d = 4N$, where N is square-free and $N \equiv 2, 3 (\bmod 4)$.

Let χ_d be the Dirichlet character defined by the Kronecker's symbol $\chi_d(n) = \left(\frac{d}{n}\right)$ with d being restricted to fundamental discriminants. Then the character χ_d only takes the real values −1, 0 or 1. The quadratic Dirichlet L-function is then defined as the Dirichlet L-function corresponding to the Dirichlet character χ_d. In this setting, a problem is to understand the asymptotic behaviour of

$$\sum_{0<d\leq D} L\left(\frac{1}{2}, \chi_d\right)^k, \tag{1.5.1}$$

where the sum is over over fundamental discriminants d as $D \to \infty$. Jutila [Jut81] obtained the following result for the first moment of (1.5.1).

Theorem 1.5.2 (Jutila)**.** *We have*

$$\sum_{0<d\leq D} L\left(\frac{1}{2}, \chi_d\right) = \frac{P(1)}{4\zeta(2)} D \left(\log\left(\frac{D}{\pi}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right) + 4\gamma - 1 + 4\frac{P'}{P}(1)\right) + O\left(D^{\frac{3}{4}+\epsilon}\right) \tag{1.5.2}$$

where

$$P(s) = \prod_p \left(1 - \frac{1}{p^s(p+1)}\right).$$

Goldfeld and Hoffstein [GH85] improved the error term of (1.5.2) to $O\left(D^{\frac{19}{32}+\epsilon}\right)$ and Young [You09] showed that the error term is bounded by $D^{\frac{1}{2}+\epsilon}$ when considering the smoothed first moment. In the same paper, Jutila obtained an asymptotic formula for the second moment of (1.5.1), namely he proved the following result.

Theorem 1.5.3 (Jutila). *We have*

$$\sum_{0<d\le D} L\left(\frac{1}{2},\chi_d\right)^2 = \frac{c}{\zeta(2)} D\log^3 D + O\left(D(\log D)^{\frac{5}{2}+\epsilon}\right)$$

where

$$c = \frac{1}{48}\prod_p \left(1 - \frac{4p^2-3p+1}{p^4+p^3}\right).$$

Restricting d to be an odd, square-free integer, then χ_{8d} is a real primitive character with conductor $8d$ and with $\chi_{8d}(-1) = 1$, Soundararajan [Sou00] proved the following result.

Theorem 1.5.4 (Soundararajan). *There exists polynomials Q and R of degree 3 and 6 respectively such that*

$$\sum_{0<d\le D} L\left(\frac{1}{2},\chi_{8d}\right)^2 = DQ(\log D) + O\left(D^{\frac{5}{6}+\epsilon}\right) \tag{1.5.3}$$

and

$$\sum_{0<d\le D} L\left(\frac{1}{2},\chi_{8d}\right)^3 = DR(\log D) + O\left(D^{\frac{11}{12}+\epsilon}\right), \tag{1.5.4}$$

where the sums is over fundamental discriminants $8d$.

In the same paper, Soundararajan proved that for at least 87.5% of odd square-free integers $d \ge 0$, $L\left(\frac{1}{2},\chi_{8d}\right) \ne 0$. By a conjecture of Chowla [Cho65], it is believed that $L\left(\frac{1}{2},\chi\right) \ne 0$ for all quadratic characters χ.

Using multiple Dirichlet series, Diaconu, Goldfeld and Hoffstein [DGH03] improved the bound in (1.5.4), namely they proved that.

Theorem 1.5.5 (Diaconu, Goldfeld and Hoffstein). *For d summed over fundamental discriminants and any $\epsilon > 0$ we have*

$$\sum_{|d|\le D} L\left(\frac{1}{2},\chi_d\right)^3 = D\sum_{i=0}^{6} c_i(\log D)^i + O_\epsilon\left(D^{\theta+\epsilon}\right) \tag{1.5.5}$$

for some computable constants c_i and $\theta \sim 0.853$.

Furthermore, Diaconu and Whitehead [DW21] considered the smoothed third moment and proved the existence of a secondary main term of size $D^{\frac{3}{4}}$ and showed that the error term is bounded by $D^{\frac{2}{3}+\delta}$ for every $\delta > 0$.

For the fourth moment, Shen [She21] proved an asymptotic formula under the condition of the Riemann hypothesis and the Generalised Riemann hypothesis. Namely he proved the following result.

Theorem 1.5.6 (Shen)**.** *Assume the Generalised Riemann hypothesis for $L(s,\chi_d)$ for all fundamental discriminants d and the Riemann hypothesis for $\zeta(s)$. Then*

$$\sum_{\substack{0<d\le D\\ d \text{ square-free}\\ (d,2)=1}} L\left(\frac{1}{2},\chi_{8d}\right)^4 = aD(\log D)^{10} + O\left(D(\log D)^{\frac{39}{4}+\epsilon}\right).$$

for some constant a.

Although no higher moments have been explicitly proven, it is conjectured that

$$\sum_{0<d\le D} L\left(\frac{1}{2},\chi_d\right)^k \sim \frac{a_k g_k}{\left(\frac{1}{2}k(k+1)\right)!}(\log D)^{\frac{k(k+1)}{2}}$$

for some value g_k and

$$a_k = \prod_p \frac{\left(1-\frac{1}{p}\right)^{\frac{k(k+1)}{2}}}{1+\frac{1}{p}}\left(\frac{1}{2}\left(\left(1-\frac{1}{\sqrt{p}}\right)^{-k}+\left(1+\frac{1}{\sqrt{p}}\right)^{-k}\right)+\frac{1}{p}\right).$$

Using random matrix theory, Keating and Snaith [KS00a] conjectured that

$$g_k = \left(\frac{1}{2}k(k+1)\right)!\prod_{j=1}^{k}\left(\frac{j!}{(2j)!}\right).$$

Conrey, Farmer, Keating, Rubinstein and Snaith [CFK+05] developed a heuristic to conjecture all of the main terms of (1.5.1). Namely they conjectured the following.

Conjecture 1.5.7 (Conrey, Farmer, Keating, Rubinstein and Snaith)**.** *Let $X_d(s) = |d|^{\frac{1}{2}-s}X(s,a)$ where $a=0$ if $d>0$ and $a=1$ if $d<1$, and*

$$X(s,a) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1+a-s}{2}\right)}{\Gamma\left(\frac{s+a}{2}\right)}.$$

That is, $X_d(s)$ is the factor in the functional equation for the quadratic Dirichlet L-function

$$L(s,\chi_d) = \epsilon_d X_d(s) L(1-s,\chi_d).$$

Summing over fundamental discriminants d we have

$$\sum_d L\left(\frac{1}{2},\chi_d\right)^k = \sum_d Q_k(\log|d|)(1+o(1)), \tag{1.5.6}$$

where Q_k is the polynomial of degree $\frac{1}{2}k(k+1)$ given by the k-fold residue

$$Q_k(x) = \frac{(-1)^{\frac{k(k-1)}{2}}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\cdots\oint\frac{G(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2 e^{\frac{x}{2}\sum_{j=1}^k z_j}}{\prod_{j=1}^k z_j^{2k-1}}dz_1\ldots dz_k,$$

$$G(z_1,\dots,z_k) = A_k(z_1,\dots,z_k)\prod_{j=1}^{k} X\left(\frac{1}{2}+z_j, a\right)\prod_{1\le i\le j\le k}\zeta(1+z_i+z_j),$$

$\Delta(z_1,\dots,z_k)$ *is the Vandermonde determinant given by*

$$\Delta(z_1,\dots,z_k) = \prod_{1\le i<j\le k}(z_j - z_i)$$

and A_k *is the Euler product, absolutely convergent for* $|\Re(z_j)| < \frac{1}{2}$ *defined as*

$$A_k(z_1,\dots,z_k) = \prod_p \prod_{1\le i\le j\le k}\left(1-\frac{1}{p^{1+z_i+z_j}}\right)$$
$$\times\left(\frac{1}{2}\left(\prod_{j=1}^{k}\left(1-\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}+\prod_{j=1}^{k}\left(1+\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}\right)+\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^{-1}.$$

Keating and Odgers [KO08] conjectured the leading order for the smoothed $2k^{\text{th}}$ moment of $\left|L\left(\frac{1}{2}+it,\chi_d\right)\right|$ as $D\to\infty$, when summed over all fundamental discriminants d with $d<0$ and $d\equiv 0 (\text{mod } 8)$.

Similarly, one can also obtain upper and lower bounds for moments of quadratic Dirichlet L-functions (1.5.1). For example, Rudnick and Soundararajan [RS06] obtained the following result for the lower bounds of (1.5.1).

Theorem 1.5.8 (Rudnick and Soundararajan)**.** *For every even natural number* k *we have*

$$\sum_{|d|\le D} L\left(\frac{1}{2},\chi_d\right)^k \gg_k D(\log D)^{\frac{k(k+1)}{2}}. \tag{1.5.7}$$

This result has been improved by Radziwiłł and Soundararajan [RS13] and Heap and Soundararajan [HS22] who showed that (1.5.7) holds for all real $k>0$. For upper, bounds Soundararajan [Sou09] showed that, under the condition of the Generalised Riemann Hypothesis we have

$$\sum_{|d|\le D} L\left(\frac{1}{2},\chi_d\right)^k \ll_{k,\epsilon} D(\log D)^{\frac{k(k+1)}{2}+\epsilon} \tag{1.5.8}$$

for all positive real k and $\epsilon>0$. In a recent paper, Gao [Gao21b] proved, unconditionally, that for every $0\le k\le 2$ we have

$$\sum_{\substack{0<d<X\\ d \text{ square-free}\\ (d,2)=1}} \left|L\left(\frac{1}{2},\chi_{8d}\right)\right|^k \ll_k D(\log D)^{\frac{k(k+1)}{2}}.$$

Conrey, Farmer and Zirnbauer [CFZ08] presented a generalisation of the heuristic arguments given in [CFK+05] to conjecture asymptotic formulas for the ratios of products of Dirichlet L-functions.

Conjecture 1.5.9 (Conrey, Farmer and Zirnbauer)**.** *Let* $\mathfrak{D}^+ = \{L(s,\chi_d) : d > 0\}$ *be the symplectic family of L-functions associated with the quadratic character* χ_d *and suppose that the real parts of* α_k *and* γ_q *are positive. Then*

$$\sum_{0<d\le D} \frac{\prod_{k=1}^{K} L\left(\frac{1}{2}+\alpha_k,\chi_d\right)}{\prod_{q=1}^{Q} L\left(\frac{1}{2}+\gamma_q,\chi_d\right)}$$
$$= \sum_{0<d\le D}\sum_{\epsilon\in\{-1,1\}^K} \left(\frac{|d|}{\pi}\right)^{\frac{1}{2}\sum_{k=1}^{K}(\epsilon_k\alpha_k-\alpha_k)} \prod_{k=1}^{K} g_+\left(\frac{1}{2}+\frac{\alpha_k-\epsilon_k\alpha_k}{2}\right)$$
$$\times Y_D(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K;\gamma)A_D(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K;\gamma)(1+o(1))$$

where

$$g_+(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)},$$

$$A_D(\alpha;\gamma) = \prod_p \frac{\prod_{1\le j\le k\le K}\left(1-\frac{1}{p^{1+\alpha_j+\alpha_k}}\right)\prod_{1\le q<r\le Q}\left(1-\frac{1}{p^{1+\gamma_q+\gamma_r}}\right)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\left(1-\frac{1}{p^{1+\alpha_k+\gamma_q}}\right)}$$
$$\times\left(1+\left(1+\frac{1}{p}\right)^{-1}\sum_{0<\sum_k a_k+\sum_q c_q \text{ is even}} \frac{\prod_{q=1}^{Q}\mu(p^{c_q})}{p^{\sum_k\left(\frac{1}{2}+\alpha_k\right)+\sum_q c_q\left(\frac{1}{2}+\gamma_q\right)}}\right)$$

and

$$Y_D(\alpha;\gamma) = \frac{\prod_{1\le j\le k\le K}\zeta(1+\alpha_j+\alpha_k)\prod_{1\le q<r\le Q}\zeta(1+\gamma_q+\gamma_r)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\zeta(1+\alpha_k+\gamma_q)}.$$

As an application of the Ratios Conjecture 1.5.9, Conrey and Snaith [CS07] used the conjecture to compute the one-level density function for the zeros of quadratic Dirichlet L-functions complete with lower order terms. Namely, conditional on the Ratios Conjecture 1.5.9, they proved the following result.

Theorem 1.5.10 (Conrey and Snaith)**.** *Assuming the Ratios Conjecture* 1.5.9*, the one-level density for the zeros of the family of quadratic Dirichlet L-functions associated with the quadratic Dirichlet character* χ_d *is given by*

$$\sum_{d\le D}\sum_{\gamma_d} f(\gamma_d) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\sum_{d\le D}\left(\log\left(\frac{d}{\pi}\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-\frac{it}{2}\right)\right.$$
$$+2\left(\frac{\zeta'(1+2it)}{\zeta(1+2it)}+A_D'(it;it)\right.$$
$$\left.\left.-\left(\frac{d}{\pi}\right)^{-it}\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}\zeta(1-2it)A_D(-it;it)\right)\right)dt(1+o(1)),$$

where γ_d *is the ordinate of a generic zero of* $L(s,\chi_d)$ *on the half-line,*

$$A_D(-r;r) = \prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\left(1-\frac{1}{p}\right)^{-1},$$

$$A'_D(r;r) = \sum_p \frac{\log p}{(p+1)(p^{1+2r}-1)}$$

and f is a function such that $f(z)$ is holomorphic throughout the strip $|\Re(z)| < 2$, is real on the real line, even and $f(x) \ll (1+x^2)^{-1}$ as $x \to \infty$.

Conrey and Snaith also presented another application of the Ratios Conjecture 1.5.9, namely under the condition of the Ratio Conjecture 1.5.9, they obtained an asymptotic formula for the second mollified moment, where the mollifier is defined as

$$M(\chi_d, P) := \sum_{n \le y} \frac{\mu(n)\chi_d(n) P\left(\frac{\log(\frac{y}{n})}{\log y}\right)}{n^{\frac{1}{2}}},$$

where P is a polynomial satisfying $P(0) = 0$ and $P'(0) = 0$, $y = X^\theta$ for $\theta > 0$. In particular, if we let

$$\xi\left(\frac{1}{2}+\alpha, \chi_d\right) := \left(\frac{d}{\pi}\right)^{\frac{\alpha}{2}} \Gamma\left(\frac{1}{4}+\frac{\alpha}{2}\right) L\left(\frac{1}{2}+\alpha, \chi_d\right) = \xi\left(\frac{1}{2}-\alpha, \chi_d\right), \tag{1.5.9}$$

then, conditionally on the Ratios Conjecture 1.5.9 Conrey and Snaith proved the following result.

Theorem 1.5.11 (Conrey and Snaith)**.** *Assuming the Ratios Conjecture 1.5.9, we have for even polynomials Q_1 and Q_2 and polynomials P_1 and P_2 satisfying $P_1(0) = P_2(0) = P_1'(0) = P_2'(0) = 0$, and $y = X^\theta$ with any $\theta > 0$,*

$$Q_1\left(\frac{2}{\log D}\frac{d}{d\alpha}\right) Q_2\left(\frac{2}{\log D}\frac{d}{d\beta}\right) \sum_{0<d\le D} \xi\left(\frac{1}{2}+\alpha, \chi_d\right)\xi\left(\frac{1}{2}+\beta, \chi_d\right) M(\chi_d, P_1) M(\chi_d, P_2)\Bigg|_{\alpha=\beta=0}$$

$$= D^*\Bigg(\frac{1}{8\theta}\int_0^1\int_0^1 \left(\frac{1}{\theta}P_1''(r)\tilde{Q}_1(u) - 4\theta P_1(r)Q_1'(u)\right)\left(\frac{1}{\theta}P_2''(r)\tilde{Q}_2(u) - 4\theta P_2(r)Q_2'(u)\right) du dr$$

$$+ \frac{1}{4}\left(\frac{1}{\theta}P_1'(1)\tilde{Q}_1(1) + 2P_1(1)Q_1(1)\right)\left(\frac{1}{\theta}P_2'(1)\tilde{Q}_2(1) + 2P_2(1)Q_2(1)\right) + O\left(\frac{1}{\log D}\right)\Bigg),$$

where D^* denotes the number of fundamental discriminants less than or equal to D and

$$\tilde{Q}(u) = \int_0^u Q(t) dt.$$

1.6 Random Matrix Theory

Random Matrix Theory is the study of matrices whose elements are random variables, and in particular we study the properties of the eigenvalues and eigenvectors of these matrices. Random Matrix Theory has many applications, for example in Statistics and Nuclear Physics, however in this section we will briefly discuss its applications to

Number Theory and, in particular, its applications to the Riemann zeta-function and Dirichlet L-functions.

We first present some of the random matrices that have applications in Number Theory.

- **A Unitary Matrix** A is an $N \times N$ matrix such that $A\bar{A}^T = \bar{A}^T A = I_N$, where $\bar{A}^T$ denotes the complex transpose of A and I_N is the $N \times N$ identity matrix. The group of all Unitary matrices is called the Unitary group and is denoted by $U(N)$.
- **A Special Orthogonal Matrix** A is an $N \times N$ unitary matrix of dimension N such that $AA^T = A^T A = I_N$ with $\deg(A) = 1$. The group of all $N \times N$ special orthogonal matrices is called the Special orthogonal group and is called $SO(N)$.
- **A Unitary Symplectic matrix** A is an $2N \times 2N$ unitary matrix of dimension $2N$ such that $AJA^T = A$ where
$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$
The group of all $2N \times 2N$ unitary matrices is called the Unitary symplectic group and is denoted by $US_p(2N)$.

Each group has an attached Haar measure. The Haar measure for the Unitary group $U(N)$ is
$$dA := \frac{1}{N!(2\pi)^N} \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \dots d\theta_N,$$
and for the Special Orthogonal group $SO(2N)$ and the Unitary Symplectic group $US_p(2N)$
$$dA := \frac{1}{N!(4\pi)^N} \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{-i\theta_j}|^2 |e^{i\theta_k} - e^{i\theta_j}|^2 \prod_{k=1}^{N} |e^{i\theta_k} - e^{-i\theta_k}| d\theta_1 \dots d\theta_N,$$
where $e^{i\theta_n}$ are the eigenvalues of the associated matrices.

The connection between Analytic Number Theory and Random Matrix Theory was first observed by Montgomery and Dyson. Assuming the Riemann Hypothesis, Montgomery [Mon73] studied the pair correlation of the zeros of the Riemann zeta-function and put forward the following conjecture.

Conjecture 1.6.1 (Montgomery)**.** *Let γ be a generic ordinate of a zero of the Riemann zeta-function, then for a suitable test function f we have*
$$\lim_{T \to \infty} \frac{1}{N(T)} \sum_{0 < \gamma_m, \gamma_n \le T} f\left(\frac{\gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi} - \frac{\gamma_m}{2\pi} \log \frac{\gamma_m}{2\pi} \right) = \int_{-\infty}^{\infty} f(x) \left(\delta(x) + 1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2 \right) dx, \tag{1.6.1}$$
where $N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi}$ and δ is the Dirac's delta function.

In Random Matrix Theory, if $A \in U(N)$ then the eigenvalues of A are $e^{i\theta_n}$ for $1 \le n \le N$ and $0 \le \theta_n < 2\pi$. Similarly if $A \in SO(2N)$ or $A \in USp(2N)$, then the eigenvalues occur in conjugate pairs, thus the eigenvalues are $e^{\pm i\theta_n}$ where $1 \le n \le N$ and $0 \le \theta_n < 2\pi$. Dyson [Dys62] proved the following result for the pair correlation for the matrices $A \in U(N)$.

Theorem 1.6.2 (Dyson)**.** *For a nice test function f we have*

$$\lim_{N\to\infty} \int_{U(N)} \left(\frac{1}{N} \sum_{1 \le m,n \le N} f\left(\theta_n \frac{N}{2\pi} - \theta_m \frac{N}{2\pi}\right)\right) dA = \int_{-\infty}^{\infty} f(x) \left(\delta(x) + 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 \right) dx, \tag{1.6.2}$$

where δ is Dirac's delta function.

Comparing (1.6.1) and (1.6.2), we see that the pair correlation for $U(N)$ is the same as the conjecture for the pair correlation for the Riemann zeta-function.

Similarly, we can compute moments of the characteristic polynomials of the matrices defined above. For $A \in U(N)$, we define the characteristic polynomial of A by

$$\Lambda_A(s) = \det(I - \bar{A}^T s) = \prod_{n=1}^{N} (1 - s e^{-i\theta_n}) \tag{1.6.3}$$

and for either $A \in SO(2N)$ or $A \in USp(2N)$, the characteristic polynomial of A is defined as

$$\Lambda_A(s) = \det(I - \bar{A}^T s) = \prod_{n=1}^{N} (1 - s e^{i\theta_n})(1 - s e^{-i\theta_n}). \tag{1.6.4}$$

Keating and Snaith [KS00a, KS00b, KS03] proved the following results about the moments of these characteristic polynomials.

Theorem 1.6.3 (Keating and Snaith)**.** *We have*

$$\int_{U(N)} \left|\Lambda_A(e^{i\theta})\right|^{2k} \sim N^{k^2} \prod_{j=0}^{k-1} \left(\frac{j!}{(j+k)!} \right), \tag{1.6.5}$$

$$\int_{SO(2N)} \left|\Lambda_A(e^{0})\right|^{k} \sim N^{\frac{k(k-1)}{2}} 2^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \left(\frac{j!}{(2j)!} \right) \tag{1.6.6}$$

and

$$\int_{USp(2N)} \left|\Lambda_A(e^{0})\right|^{k} \sim N^{\frac{k(k+1)}{2}} 2^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \left(\frac{j!}{(2j)!} \right). \tag{1.6.7}$$

Under the correspondence $N = \log\left(\frac{T}{2\pi}\right)$ and $N = \frac{1}{2}\log D$, Keating and Snaith were able to formulate conjectures for the moments of the Riemann zeta-function and Dirichlet L-functions as stated in Section 1.2 and Section 1.5 respectively. In particular, they were able to conjecture a precise value for the term g_k which had not been obtained previously.

1.7 Overview of Thesis

In this thesis, the main focus has been the study of analytic number theory over function fields, specifically the study of several problems involving moments and mean values of Dirichlet L-functions in the function field context. In Chapter 2, we give all the necessary background in function fields that is needed for this thesis.

In Chapter 3 we prove a theorem about the first moment of quadratic Dirichlet L-functions at the critical point $s = \frac{1}{2}$ in function fields. In particular, we prove, compared to the result known previously, a secondary main term and improve the error term. In the following Chapter, Chapter 4, we prove a theorem about the average size of the algebraic group $K_2(\mathcal{O})$ in function fields and in Chapter 5, we use the methods of Rudnick and Soundararajan [RS06] to prove a theorem about the lower bounds of quadratic Dirichlet L-functions at $s = \frac{1}{2}$ in function field.

In the next Chapter, Chapter 6, we adapt the recipe for conjecturing moments of Dirichlet L-functions, Conjecture 1.5.7, to moments of Dirichlet L-functions in function fields in even characteristic. In Chapter 7, we adapt the recipe for conjecturing ratios of products of Dirichlet L-functions, Conjecture 1.5.9, to ratios of products of Dirichlet L-functions in function fields in even characteristic. Then in the following Chapter, Chapter 8, we present two applications of the Ratios conjecture in even characteristic which adapt to function fields in even characteristic the methods used to obtain Theorem 1.5.10 and Theorem 1.5.11.

Finally, in Chapter 9, we prove a result about the second moment of Dirichlet L-functions with one and two twists, when averaged over primitive Dirichlet characters of a certain modulus in function fields.

Chapter 2

Background on Function Fields

In this chapter, we will give some background on Number Theory in Function Fields as well as state some preliminary results. Most of these facts are stated in [Ros02].

2.1 Function Field Preliminaries

Let $\mathbb{F}_q$ denote a finite field with q elements and let $\mathbb{A} = \mathbb{F}_q[T]$ denote the polynomial ring over $\mathbb{F}_q$. This ring has many common properties with the ring of integers $\mathbb{Z}$, which can be found in [Ros02, Chapter 1]. Thus, many of the number theoretic questions that are asked about $\mathbb{Z}$ have analogues in $\mathbb{A}$.

An element $f(T) \in \mathbb{A}$ can be written as

$$f(T) = \alpha_n T^n + \ldots + \alpha_0,$$

where $\alpha_i \in \mathbb{F}_q$ for all $i = 0, \ldots, n$. If $\alpha_n \neq 0$, we say that f has degree n, which will be denoted as $\deg(f) = n$. We will denote by $\mathbb{A}_n$ and $\mathbb{A}_{\leq n}$ the set of all polynomials of degree n and degree at most n in $\mathbb{A}$ respectively. Furthermore, if $\alpha_n \neq 0$, we define the sign of f, $\operatorname{sgn}(f)$, to be equal to $\alpha_n \in \mathbb{F}_q^*$, where $\mathbb{F}_q^*$ denotes the set of all non-zero elements in $\mathbb{F}_q$. The following proposition states some important properties about $\deg(f)$ and $\operatorname{sgn}(f)$.

Proposition 2.1.1 ([Ros02, p.1]). *Let f and g be two non-zero polynomials in $\mathbb{A}$. Then we have*

i) $deg(fg) = deg(f) + deg(g)$.

ii) $sgn(fg) = sgn(f)sgn(g)$.

iii) $deg(f+g) \leq \max\{deg(f), deg(g)\}$, *with equality holding if* $deg(f) \neq deg(g)$.

Definition 2.1.2. A polynomial $f \in \mathbb{A}$ is called monic if $sgn(f) = 1$.

We will let $\mathbb{A}^+$ denote the set of all monic polynomials in $\mathbb{A}$. Furthermore, we will denote by $\mathbb{A}_n^+$ and $\mathbb{A}_{\leq n}^+$ to be the set of all monic polynomials of degree n and degree at most n in $\mathbb{A}$ respectively.

Definition 2.1.3. A polynomial $f \in \mathbb{A}$ is called reducible if we can write $f(T) = g(T)h(T)$, where $\deg(g) > 0$ and $\deg(h) > 0$. Otherwise f is called irreducible.

We will let the letter P denote a monic irreducible polynomial in $\mathbb{A}$. Similarly, we will let $\mathcal{P}$ and $\mathcal{P}_n$ denote the set of all monic irreducible polynomials and the set of all monic irreducible polynomials of degree n in $\mathbb{A}$ respectively. The next theorem is the function field analogue of the Prime Number Theorem.

Theorem 2.1.4 (Prime Polynomial Theorem, [Ros02, Theorem 2.2]). *Let $\#\mathcal{P}_n$ denote the number of monic irreducible polynomials of degree n in $\mathbb{A}$, then*

$$\#\mathcal{P}_n = \frac{q^n}{n} + O\left(\frac{q^{\frac{n}{2}}}{n}\right). \tag{2.1.1}$$

Remark 2.1.5. *Monic polynomials play the role of positive integers and monic irreducible polynomials play the role of prime numbers. We will also define $sgn(0) = 0$ and $deg(0) = -\infty$.*

Since the ring $\mathbb{A}$ has the unique factorisation property, we have that every $f \in \mathbb{A}$, $f \neq 0$ can be uniquely written in the form

$$f = \alpha P_1^{e_1} \dots P_r^{e_r}, \tag{2.1.2}$$

where $\alpha \in \mathbb{F}_q^*$ and each P_i is a monic irreducible polynomial, $P_i \neq P_j$ for $i \neq j$ and e_i is a non-negative integer for $i = 1, \dots, r$.

Definition 2.1.6. For a polynomial $f \in \mathbb{A}$, we define the norm of f to be

$$|f| := \begin{cases} q^{\deg(f)} & \text{if } f \neq 0, \\ 0 & \text{if } f = 0. \end{cases}$$

Definition 2.1.7. The zeta function of $\mathbb{A}$, which is denoted by $\zeta_{\mathbb{A}}(s)$, is defined by the infinite series

$$\zeta_{\mathbb{A}}(s) := \sum_{f \in \mathbb{A}^+} \frac{1}{|f|^s} = \prod_P \left(1 - \frac{1}{|P|^s}\right)^{-1} \qquad \Re(s) > 1. \tag{2.1.3}$$

There are q^n monic polynomials of degree n in $\mathbb{A}$, therefore

$$\zeta_{\mathbb{A}}(s) = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{1 - q^{1-s}}. \tag{2.1.4}$$

From this geometric series, we can see that $\zeta_{\mathbb{A}}(s)$ is never zero and thus in this setting the Riemann hypothesis is true.

We will make use of the change of variables $u = q^{-s}$, so that $\mathcal{Z}(u) = \zeta_{\mathbb{A}}(s)$ and so,

$$\mathcal{Z}(u) = \frac{1}{1 - qu}. \tag{2.1.5}$$

Definition 2.1.8. The Gamma function of $\mathbb{A}$ is defined to be

$$\Gamma_{\mathbb{A}}(s) := \frac{1}{1 - q^{-s}}.$$

Using (2.1.5), we can prove the following result.

Theorem 2.1.9 ([Ros02, p.12])**.** *The zeta function $\zeta_{\mathbb{A}}(s)$ can be continued to a meromorphic function to the whole of the complex plane with a simple pole at $s = 1$ with residue $\frac{1}{\log q}$. If we define $\xi_{\mathbb{A}}(s) = q^{-s}\Gamma_{\mathbb{A}}(s)\zeta_{\mathbb{A}}(s)$, then*

$$\xi_{\mathbb{A}}(s) = \xi_{\mathbb{A}}(1 - s). \tag{2.1.6}$$

2.2 Multiplicative Functions on $\mathbb{F}_q[T]$

In this section, we will define some multiplicative functions for $\mathbb{A}$ and state some preliminary results which will be used throughout this thesis.

Definition 2.2.1. The Möbius function for $\mathbb{A}$ is defined as

$$\mu(f) := \begin{cases} (-1)^r & \text{if } f = \alpha P_1 \dots P_r, \\ 0 & \text{otherwise,} \end{cases}$$

where the P_j are distinct monic polynomials.

Taking Euler products, we see that for all $s \in \mathbb{C}$ and all $R \in \mathbb{A}$, we have

$$\sum_{E|R} \frac{\mu(E)}{|E|^s} = \prod_{P|R} \left(1 - \frac{1}{|P|^s}\right) \tag{2.2.1}$$

and differentiating (2.2.1), we see that for all $s \in \mathbb{C}\backslash\{0\}$, we have

$$\sum_{E|R} \frac{\mu(E)\deg(E)}{|E|^s} = -\left(\prod_{P|R} 1 - \frac{1}{|P|^s}\right)\left(\sum_{P|R} \frac{\deg(P)}{|P|^s - 1}\right). \tag{2.2.2}$$

Definition 2.2.2. For all $R \in \mathbb{A}$, we define $\omega(R)$ to be the number of distinct prime factors of R.

Lemma 2.2.3 ([AY21, Lemma 4.5]). *Let $R \in \mathbb{A}^+$. We have that*

$$\sum_{P|R} \frac{deg(P)}{|P|-1} \ll \log \omega(R). \tag{2.2.3}$$

Lemma 2.2.4 ([Yia20, Lemma A.2.3]). *For $deg(R) > 1$, we have*

$$\omega(R) \ll \frac{\log_q |R|}{\log_q \log_q |R|}, \tag{2.2.4}$$

where the implied constant is independent of q.

Lemma 2.2.5. *We have*

$$2^{\omega(R)} = \sum_{E|R} |\mu(E)|. \tag{2.2.5}$$

Also, for any $\epsilon > 0$ we have

$$2^{\omega(R)} \ll_\epsilon |R|^\epsilon. \tag{2.2.6}$$

Proof. The first part of the lemma follows from the definition of $\mu(f)$ and $\omega(f)$. For the second part, notice that

$$2^{\omega(R)} = \sum_{E|R} |\mu(E)| \le \sum_{E|R} 1 = d(R),$$

where $d(f)$ denotes the divisor function for $\mathbb{A}$. The proof follows from using the fact that $d(f) \ll_\epsilon |f|^\epsilon$ for every $\epsilon > 0$. ■

Definition 2.2.6. The Euler-Totient function for $\mathbb{A}$ is defined as

$$\phi(f) := \sum_{\substack{g \in \mathbb{A}^+ \\ \deg(g) < \deg(f) \\ (f,g)=1}} 1. \tag{2.2.7}$$

Lemma 2.2.7 ([Yia20, Lemma A.2.4]). *For $deg(R) > q$ we have*

$$\phi(R) \gg \frac{|R|}{\log_q \log_q |R|}. \tag{2.2.8}$$

For $f \in \mathbb{A}^+$, let $d_k(f)$ represent the number of ways to write f as a product of k factors. Then we have the following results about $d_k(f)$.

Lemma 2.2.8. *We have*

$$\sum_{f \in \mathbb{A}^+_{\le z}} \frac{d_k(f)}{|f|} \sim c(k) z^k \tag{2.2.9}$$

for some positive constant $c(k)$.

Proof. The proof is similar to that given in [Ros02, Proposition 2.5]. ■

Lemma 2.2.9. *We have*

$$\sum_{f \in \mathbb{A}^+_{\le z}} \frac{d_k(f^2)}{|f|} \sim C(k) z^{\frac{k(k+1)}{2}} \tag{2.2.10}$$

for some positive constant $C(k)$.

Proof. The proof is similar to that given in [And16, p. 12]. ■

2.3 Dirichlet characters in Function Fields

In this section, we will discuss some properties about Dirichlet characters over function fields.

Definition 2.3.1. Let $Q \in \mathbb{A}^{+}$. Then a Dirichlet character modulo Q is defined to be a function $\chi : \mathbb{A} \to \mathbb{C}$ which satisfies the following properties:

i) $\chi(f + gQ) = \chi(f),\ \forall f, g \in \mathbb{A}$.

ii) $\chi(fg) = \chi(f)\chi(g),\ \forall f, g \in \mathbb{A}$.

iii) $\chi(f) \neq 0 \iff (f, Q) = 1$, where the symbol (f, Q) denotes the greatest common divisor of the functions f and Q.

A Dirichlet character modulo Q induces a homomorphism from $(\mathbb{A}/Q\mathbb{A})^{*} \to \mathbb{C}$ and conversely, given such a homomorphism, there is a uniquely corresponding Dirichlet character. The trivial character χ_0 is defined to be

$$\chi_0(f) := \begin{cases} 1 & \text{if } (f, Q) = 1, \\ 0 & \text{if } (f, Q) > 1. \end{cases}$$

The number of Dirichlet characters modulo Q is equal to $\phi(Q)$. For a Dirichlet character modulo Q, we have the following result.

Proposition 2.3.2 (Orthogonality Relations, [Ros02, Proposition 4.2])**.** *Let χ and ψ be two Dirichlet characters modulo Q and let f and g be two elements in $\mathbb{A}$ which are relatively prime to Q. Then*

i)

$$\sum_{f} \chi(f)\psi(f) = \begin{cases} \phi(Q) & \text{if } \chi = \psi, \\ 0 & \text{if } \chi \neq \psi. \end{cases}$$

ii)

$$\sum_{\chi} \chi(f)\bar{\chi}(g) = \begin{cases} \phi(Q) & \text{if } f \equiv g (\text{mod } Q), \\ 0 & \text{if } f \not\equiv g (\text{mod } Q). \end{cases}$$

The first sum is over any representatives of $(\mathbb{A}/Q\mathbb{A})$ and the second sum is over all Dirichlet characters modulo Q.

Definition 2.3.3. A Dirichlet character χ modulo Q is even if $\chi(cA) = \chi(A)$ for all $c \in \mathbb{F}_q^{*}$ and all $A \in \mathbb{F}_q[T]$. Otherwise χ is said to be odd.

We will let $\phi^{+}(Q)$ and $\phi^{-}(Q)$ denote the number of even and odd characters modulo Q respectively. From [KR14], we know that $\phi^{+}(Q) = \frac{\phi(Q)}{q-1}$ and $\phi^{-}(Q) = \frac{q-2}{q-1}\phi(Q)$. Furthermore, when the sum is restricted to odd and even Dirichlet characters, we have the following result.

Lemma 2.3.4 ([DĐLV21, Lemma 2.1]). *For polynomials $A, B \in \mathbb{F}_q[T]$ relatively prime to Q with $deg(Q) \geq 1$, we have*

$$\sum_{\substack{\chi(mod\ Q)\\ \chi\ even}} \chi(A)\bar{\chi}(B) = \begin{cases} \frac{\phi(Q)}{q-1} & \textit{if } A \equiv cB(mod\ Q), \quad c \in \mathbb{F}_q^*, \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$\sum_{\substack{\chi(mod\ Q)\\ \chi\ odd}} \chi(A)\bar{\chi}(B) = \begin{cases} \frac{q-2}{q-1}\phi(Q) & \textit{if } A \equiv B(mod\ Q) \\ -\frac{\phi(Q)}{q-1} & \textit{if } A \equiv cB(mod\ Q), \quad c \in \mathbb{F}_q^* \\ 0 & \textit{otherwise}. \end{cases}$$

Definition 2.3.5. Let $Q \in \mathbb{A}^{+}, S|Q$ and χ be a character of modulus Q. We say that S is an induced modulus of χ if there exists a character χ_1 of modulus S such that

$$\chi(A) = \begin{cases} \chi_1(A) & \text{if } (A, Q) = 1, \\ 0 & \text{otherwise}. \end{cases}$$

We say χ is primitive if there is no induced modulus of strictly smaller norm than Q. Otherwise χ is said to be non-primitive.

We let $\phi^*(Q)$ denote the number of primitive characters of modulus Q. We have the following result about $\phi^*(Q)$.

Lemma 2.3.6 ([Yia20, Lemma A.2.5.]). *For $deg(R) > q$, we have*

$$\phi^*(R) \gg \frac{\phi(R)}{\log_q \log_q |R|}. \tag{2.3.1}$$

If Q is a monic irreducible polynomial, then every character modulo Q is primitive except the trivial character. We will denote the sum over primitive characters of modulo Q by $\sum^*_{\chi(\text{mod } Q)}$. For the sum of primitive characters, we have the following results.

Lemma 2.3.7. *Let P be a monic, irreducible polynomial and let $A, B \in \mathbb{A}$. Then we have*

$$\frac{1}{\phi(P)} \sideset{}{^*}\sum_{\chi(mod\ P)} \chi(A)\bar{\chi}(B) = \begin{cases} 1 - \frac{1}{\phi(P)} & \textit{if } A \equiv B(mod\ P), \\ -\frac{1}{\phi(P)} & \textit{otherwise}. \end{cases}$$

Proof. Using Proposition 2.3.2 and the arguments stated above proves the Lemma. ∎

Lemma 2.3.8 ([AY21, Lemma 3.7]). *Let $R \in \mathbb{A}^+$ and $A, B \in \mathbb{A}$. Then*

$$\sum_{\chi (mod\ R)}^{*} \chi(A)\bar{\chi}(B) = \begin{cases} \sum_{\substack{EF=R \\ F|(A-B)}} \mu(E)\phi(F) & \text{if } (AB, R) = 1, \\ 0 & \text{otherwise} \end{cases}$$

As a Corollary we have the following result.

Corollary 2.3.9 ([AY21, Corollary 3.8]). *For all $R \in \mathbb{A}^+$ we have that*

$$\phi^*(R) = \sum_{EF=R} \mu(E)\phi(F). \tag{2.3.2}$$

2.4 Dirichlet L-functions in Function Fields

In this section, we will define the Dirichlet L-function corresponding to the Dirichlet character defined in Section 2.3.

Definition 2.4.1. Let χ be a Dirichlet character modulo Q. Then the Dirichlet L-series corresponding to χ is defined by

$$L(s,\chi) := \sum_{f \in \mathbb{A}^+} \frac{\chi(f)}{|f|^s} \tag{2.4.1}$$

which converges absolutely for $\Re(s) > 1$.

Since the Dirichlet characters are multiplicative, we have

$$L(s,\chi) = \prod_{P} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1} \tag{2.4.2}$$

for $\Re(s) > 1$. For the trivial character, we have

$$L(s,\chi_0) = \prod_{P|Q} \left(1 - \frac{1}{|P|^s}\right) \zeta_{\mathbb{A}}(s),$$

which shows that $L(s,\chi_0)$ can be analytically continued to the whole $\mathbb{C}$ and has a simple pole at $s = 1$. When χ is a non-trivial character, we have the following result.

Proposition 2.4.2 ([Ros02, Proposition 4.3]). *Let χ be a non-trivial Dirichlet character modulo Q. Then $L(s,\chi)$ is a polynomial in $u = q^{-s}$ of degree at most $deg(Q) - 1$.*

From Proposition 2.4.2, we have the following Corollary.

Corollary 2.4.3 ([Ros02, p.36]). *If χ is a non-trivial Dirichlet character modulo Q, then $L(s,\chi)$ can be analytically continued to an entire function to the whole complex plane $\mathbb{C}$.*

Using the change of variables $u = q^{-s}$, we have, for a non-trivial Dirichlet character modulo Q, that

$$\mathcal{L}(u,\chi) = \sum_{f \in \mathbb{A}^+_{\leq \deg(Q)-1}} \chi(f) u^{\deg(f)}. \tag{2.4.3}$$

2.5 Zeta functions associated with curves

For any algebraic curve C of genus $g \geq 1$ over $\mathbb{F}_q$, the zeta function to C was first introduced by Artin [Art24] and is defined as

$$Z_C(u) := \exp\left(\sum_{n=1}^{\infty} N_n(C)\frac{u^n}{n}\right) \qquad |u| < \frac{1}{q}, \tag{2.5.1}$$

where $N_n(C)$ is the number of $\mathbb{F}_{q^n}$ rational points on C. Weil [Wei48] showed that $Z_C(u)$ is a rational function of the form

$$Z_C(u) = \frac{P_C(u)}{(1-u)(1-qu)}, \tag{2.5.2}$$

where $P_C(u)$ is a polynomial of degree $2g$ with integer coefficients satisfying the functional equation

$$P_C(u) = (qu^2)^g P_C\left(\frac{1}{qu}\right). \tag{2.5.3}$$

Weil also proved the Riemann Hypothesis for function fields, namely that all of the zeros of $P_C(u)$ lie on the circle $|u| = q^{-\frac{1}{2}}$.

2.6 Quadratic Function Field in Odd characteristic

In this section, we assume that q is odd. Most of the facts stated in this section are stated in [AK12].

2.6.1 Characters and the Reciprocity Law

Let P be a monic, irreducible polynomial in $\mathbb{A}$. Then [Ros02, Proposition 1.10] tells us that if $f \in \mathbb{A}$ and $P \nmid f$, then the congruence $X^d \equiv f(\text{mod } P)$ is solvable if and only if

$$f^{\frac{|P|-1}{d}} \equiv 1(\text{mod } P),$$

where d is a divisor of $q-1$. Therefore if $P \nmid f$, then there is a unique element $\left(\frac{f}{P}\right)_d \in \mathbb{F}_q^*$ such that

$$f^{\frac{|P|-1}{d}} \equiv \left(\frac{f}{P}\right)_d (\text{mod } P),$$

otherwise we define $\left(\frac{f}{P}\right)_d = 0$. Thus, we can define the quadratic residue symbol $\left(\frac{f}{P}\right) \in \{\pm 1\}$ by

$$\left(\frac{f}{P}\right) \equiv f^{\frac{|P|-1}{2}} (\text{mod } P) \tag{2.6.1}$$

if f is coprime to P. If $P|f$, then $\left(\frac{f}{P}\right) = 0$. We can also define the Jacobi symbol for arbitrary monic Q. Let f be coprime to Q and $Q = P_1^{e_1} \dots P_r^{e_r}$, then the Jacobi symbol is defined as

$$\left(\frac{f}{Q}\right) = \prod_{i=1}^{r} \left(\frac{f}{P_i}\right)^{e_i}. \tag{2.6.2}$$

Theorem 2.6.1 (Quadratic Reciprocity, [Ros02, Theorem 3.3])**.** *Let $A, B \in \mathbb{A}$ be relatively prime and let $A \neq 0$ and $B \neq 0$. Then*

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right)(-1)^{\frac{q-1}{2} deg(A) deg(B)}. \tag{2.6.3}$$

When $q \equiv 1 (\text{mod } 4)$, Theorem 2.6.1 gives

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right). \tag{2.6.4}$$

Thus, in the rest of this section and in Chapters 3, 4 and 5 we will further restrict q to $q \equiv 1 (\text{mod } 4)$.

2.6.2 Quadratic Dirichlet L-functions

Definition 2.6.2. Let $D \in \mathbb{A}$ be square-free. We define the quadratic character χ_D using the quadratic residue symbol for $\mathbb{A}$ by

$$\chi_D(f) = \left(\frac{D}{f}\right). \tag{2.6.5}$$

Therefore if P is a monic irreducible polynomial in $\mathbb{A}$, we have

$$\chi_D(P) = \begin{cases} 0 & \text{if } P|D, \\ 1 & \text{if } P \nmid D \text{ and } D \text{ is a square modulo } P, \\ -1 & \text{if } P \nmid D \text{ and } D \text{ is a non-square modulo } P. \end{cases}$$

Definition 2.6.3. The L-function corresponding to the quadratic character χ_D is defined as

$$L(s, \chi_D) := \sum_{f \in \mathbb{A}^+} \frac{\chi_D(f)}{|f|^s}, \qquad \Re(s) > 1. \tag{2.6.6}$$

For the change of variables $u = q^{-s}$, we have

$$L(s, \chi_D) = \mathcal{L}(u, \chi_D) = \sum_{f \in \mathbb{A}^+} \chi_D(f) u^{\deg(f)} = \prod_P \left(1 - \chi_D(P) u^{\deg(P)}\right)^{-1}. \tag{2.6.7}$$

From Proposition 2.4.2, we know that $\mathcal{L}(u, \chi_D)$ is a polynomial in u of degree at most $\deg(D) - 1$. For P a monic irreducible polynomial, we can define the quadratic character χ_P and the L-function corresponding to χ_P in a similar way.

2.6.3 The Hyperelliptic Ensemble

Let

$$\mathcal{H}_n = \{D \in \mathbb{A} : D \text{ monic, square-free, } \deg(D) = n\}. \tag{2.6.8}$$

Then, by [Ros02, Proposition 2.3], for $g \geq 1$, we have

$$\#\mathcal{H}_{2g+1} = q^{2g}(q-1) = \frac{|D|}{\zeta_{\mathbb{A}}(2)} \text{ and } \#\mathcal{H}_{2g+2} = q^{2g+1}(q-1) = \frac{|D|}{\zeta_{\mathbb{A}}(2)}. \tag{2.6.9}$$

Let $k = \mathbb{F}_q(T)$ denote the rational function field over $\mathbb{F}_q$ and let ∞_k be the infinite prime associated with $\frac{1}{T}$. If $D \in \mathcal{H}_{2g+1}$, ∞_k ramifies in $k(\sqrt{D})$, i.e., $K_D := k(\sqrt{D})$ is an ramified quadratic extension of k. If $D \in \mathcal{H}_{2g+2}$, then ∞_k splits in $k(\sqrt{D})$, i.e., K_D is a real quadratic extension of k. Let γ be a fixed generator of $\mathbb{F}_q^*$, then for any $D \in \mathcal{H}_{2g+2}$, ∞_k is an inert in $k(\sqrt{\gamma D})$, i.e., $K_{\gamma D}$ is an inert imaginary quadratic extension of k.

From [Rud10] we know that for any $D \in \mathcal{H}_{2g+2}$, $\mathcal{L}(u, \chi_D)$ has a trivial zero at $u = 1$, and for a fixed generator γ of $\mathbb{F}_q^*$, $\mathcal{L}(u, \chi_{\gamma D})$ has a trivial zero at $u = -1$. Therefore if K_D is ramified, we can define the complete L-function $\mathcal{L}^*(u, \chi_D)$ as

$$\mathcal{L}^*(u, \chi_D) = \mathcal{L}(u, \chi_D). \tag{2.6.10}$$

Similarly, if K_D is real, we can define the complete L-function $\mathcal{L}^*(u, \chi_D)$ as

$$\mathcal{L}^*(u, \chi_D) = (1-u)^{-1}\mathcal{L}(u, \chi_D). \tag{2.6.11}$$

Finally, if $K_{\gamma D}$ is a inert imaginary, we can define the complete L-function $\mathcal{L}^*(u, \chi_{\gamma D})$ as

$$\mathcal{L}^*(u, \chi_{\gamma D}) = (1+u)^{-1}\mathcal{L}(u, \chi_{\gamma D}). \tag{2.6.12}$$

These complete L-functions are all polynomials in u of degree $2g$ which satisfies the functional equation

$$\mathcal{L}^*(u, \chi_{\tilde{D}}) = (qu^2)^g \mathcal{L}^*\left(\frac{1}{qu}, \chi_{\tilde{D}}\right), \tag{2.6.13}$$

where $\tilde{D} = D$ if $K_{\tilde{D}}$ is ramified or real and $\tilde{D} = \gamma D$ if $K_{\tilde{D}}$ is inert imaginary. In his thesis, Artin proved that these $\mathcal{L}^*(u, \chi_{\tilde{D}})$ are equal to $P_{C_D}(u)$, where $P_C(u)$ is defined in Section 2.5, D is a monic, square-free polynomial of degree $2g+1$ or $2g+2$ and the affine equation $y^2 = D(T)$ defines a projective and hyperelliptic curve C_D of genus g over $\mathbb{F}_q$.

For $D \in \mathcal{H}_{2g+1}$, let

$$\mathcal{X}_D(s) = |D|^{\frac{1}{2}-s} X(s)$$

where

$$X(s) = q^{-\frac{1}{2}+s}$$

Then we define the completed L-function $\Lambda(s, \chi_D)$ by

$$\Lambda(s, \chi_D) = \mathcal{X}_D(s)^{-\frac{1}{2}} L(s, \chi_D). \tag{2.6.14}$$

The completed L-function satisfies the following functional equation

$$\Lambda(s, \chi_D) = \Lambda(1-s, \chi_D). \tag{2.6.15}$$

2.7 Quadratic Function Field in Even characteristic

For this section, let q be a power of 2. Most of the facts stated in this section are stated in [ABJ16, BJ18].

2.7.1 Quadratic extensions of k

Any separable quadratic extension K of k is of the form $K = K_u := k(x_u)$, where x_u is a zero of $X^2 + X + u = 0$ for some $u \in k$. Two elements u and v are equivalent if $K_u = K_v$. Furthermore, they are also equivalent if and only if $u + v = \rho(w)$, where $w \in k$ and $\rho : k \to k$ is an additive homomorphism defined by $\rho(x) = x^2 + x$ (for more information see [Has35, HL10]). For $\xi \in \mathbb{F}_q \backslash \rho(\mathbb{F}_q)$, the following Theorem is due to Y. Li, but a proof is given in [BJ18].

Lemma 2.7.1 ([BJ18, Lemma 2.2])**.** *Any separable quadratic extension K of k is of the form $K = K_u$, where $u \in k$ can be uniquely normalised to satisfy the following conditions:*

$$u = \sum_{i=1}^{m} \sum_{j=1}^{e_i} \frac{Q_{i,j}}{P_i^{2j-1}} + \sum_{\ell=1}^{n} \alpha_\ell T^{2\ell-1} + \alpha, \tag{2.7.1}$$

where each $P_i \in \mathcal{P}$ are distinct, $Q_{i,j} \in \mathbb{A}$ with deg$(Q_{i,j}) <$ deg(P_i), $Q_{i,e_i} \neq 0$, $\alpha \in \{0, \xi\}, \alpha_\ell \in \mathbb{F}_q$ and $\alpha_n \neq 0$ for $n > 0$.

Let $u \in k$ be normalised as in (2.7.1). The infinite prime $\infty_k = \left(\frac{1}{T}\right)$ of k splits, is inert or is ramified in K_u according to if $n = 0$ and $\alpha = 0$, $n = 0$ and $\alpha = \xi$ and $n > 0$. Then the field K_u is called real, inert imaginary or ramified imaginary respectively. The discriminant D_u of K_u is given by

$$D_u = \begin{cases} \prod_{i=1}^{m} P_i^{2e_i} & \text{if } n = 0, \\ \prod_{i=1}^{m} P_i^{2e_i} \left(\frac{1}{T}\right)^{2n} & \text{if } n > 0. \end{cases}$$

By the Hurwitz genus formula ([Sti93, Theorem III 4.12]), the genus g_u of K_u is given by

$$g_u = \frac{1}{2}\deg(D_u) - 1. \tag{2.7.2}$$

For $M \in \mathbb{A}^+$, let $r(M) = \prod_{P|M} P$ and $t(M) = M \times r(M)$. For $P \in \mathcal{P}$, let $\mathfrak{v}_P$ be the normalised valuation at P, that is $\mathfrak{v}_P(M) = e$, where $P^e \| M$. Let $\mathcal{B}$ be the set of monic polynomials M such that $\mathfrak{v}_P(M) = 0$ or odd for any $P \in \mathcal{P}$. Thus for $M \in \mathcal{B}$, $t(M)$ is a square. Also, for $M \in \mathcal{B}$, let $\ell_P = \frac{1}{2}(\mathfrak{v}_P(M) + 1)$ and

$$\tilde{M} = \prod_{P|M} P^{\ell_P} = \sqrt{t(M)}. \tag{2.7.3}$$

Furthermore, let $\mathcal{C}$ be the set of rational functions $\frac{D}{M} \in k$ such that $D \in \mathbb{A}$, $M \in \mathcal{B}$ and $\deg(D) < \deg(M)$. Also, let $\mathcal{E}$ be the set of rational functions of the form

$$\frac{D}{M} = \sum_{P|M} \sum_{i=1}^{\ell_P} \frac{A_{P,i}}{P^{2i-1}},$$

where $\deg(A_{P,i}) \leq \deg(P)$ for any $P|M$ and for all $1 \leq i \leq \ell_P$. Note that for $\frac{D}{M} \in \mathcal{E}$, $\gcd(D, M) = 1$ if and only if $A_{P,\ell_P} \neq 0$ for any $P|M$. Thus let

$$\mathcal{F} = \left\{ \frac{D}{M} \in \mathcal{E} : A_{P,\ell_P} \neq 0 \text{ for any } P|M \right\} \tag{2.7.4}$$

and

$$\mathcal{F}' = \{u + \xi : u \in \mathcal{F}\}. \tag{2.7.5}$$

For any positive integer s, let $\mathcal{G}_s$ be the set of polynomials $F(T) \in \mathbb{A}$ of the form

$$F(T) = \alpha + \sum_{i=1}^{s} \alpha_i T^{2i-1}$$

where $\alpha \in \{0, \xi\}$, $\alpha_i \in \mathbb{F}_q$ and $\alpha_s \neq 0$ and let $\mathcal{G} = \cup_{s\geq 1} \mathcal{G}_s$. Then let

$$\mathcal{I} = \{u + F : u \in \hat{\mathcal{F}}, G \in \mathcal{G}\} \tag{2.7.6}$$

where $\hat{\mathcal{F}} = \mathcal{F} \cup \mathcal{F}_0$ and $\mathcal{F}_0 = \{0\}$. Then by the normalisation given in (2.7.1), we see that $u \mapsto K_u$ defines a one-to-one correspondence between $\mathcal{I}$, $\mathcal{F}$ and $\mathcal{F}'$ and the set of all ramified imaginary, real and inert imaginary quadratic extensions of k respectively.

Furthermore, for a positive integer n, let

$$\mathcal{B}_n = \{M \in \mathcal{B} : \deg(t(M)) = 2n\}, \quad \mathcal{C}_n = \left\{ \frac{D}{M} \in \mathcal{C} : M \in \mathcal{B}_n \right\}$$
$$\mathcal{E}_n = \mathcal{E} \cap \mathcal{C}_n, \quad \mathcal{F}_n = \mathcal{F} \cap \mathcal{E}_n \text{ and } \mathcal{F}'_n = \{u + \xi : u \in \mathcal{F}_n\}.$$

Also, for any integers $r \geq 0$ and $s \geq 1$, let

$$\mathcal{I}_{(r,s)} = \{u + F : u \in \mathcal{F}_r, F \in \mathcal{G}_s\}. \tag{2.7.7}$$

Then, for any integer $n \geq 1$, let $\mathcal{I}_n$ be the union of all $\mathcal{I}_{(r,s)}$, where (r,s) runs through all pairs of non-negative integers r and s with $s > 0$ and $r + s = n$. Then under the correspondence $u \mapsto K_u$, $\mathcal{I}_n$, $\mathcal{F}_n$ and $\mathcal{F}'_n$ corresponds to the set of ramified imaginary, real and inert imaginary separable quadratic extensions K_u of k with genus $n-1$ respectively.

Remark 2.7.2. *The map $\mathcal{B}_n \to \mathbb{A}_n^+$ defined by $M \to \tilde{M}$ and the map $\mathbb{A}_n^+ \to \mathcal{B}_n$ defined by $N \to N^* = N^2/r(N)$ are inverses of each other.*

We also have the following result about the sizes of the sets $\mathcal{B}_n$, $\mathcal{E}_n$, $\mathcal{F}_n$ and $\mathcal{I}_n$.

Lemma 2.7.3 ([BJ18], Lemma 2.3)**.** *For positive integers n, we have $\#\mathcal{B}_n = q^n$, $\#\mathcal{E}_n = q^{2n}$, $\#\mathcal{F}_n = \zeta_{\mathbb{A}}(2)^{-1}q^{2n}$ and $\#\mathcal{I}_n = 2\zeta_{\mathbb{A}}(2)^{-1}q^{2n-1}$.*

For each $M \in \mathcal{B}$, let $\mathcal{C}_M$ be the set of rational functions $u \in \mathcal{C}$ whose denominator divides M, $\mathcal{E}_M = \mathcal{E} \cap \mathcal{C}_M$ and $\mathcal{F}_M = \mathcal{F} \cap \mathcal{C}_M$. Furthermore we have that $\mathcal{C}_M$ and $\mathcal{E}_M$ are abelian groups under addition and $\#\mathcal{E}_M = |\tilde{M}|$ and $\#\mathcal{F}_M = \phi(\tilde{M})$.

We also let $\tilde{\mathcal{F}}$ be the set of rational functions $u \in \mathcal{F}$ whose denominator is a monic irreducible polynomial. In other words, we let

$$\tilde{\mathcal{F}} = \left\{u = \frac{A}{P} \in \mathcal{F} : P \in \mathcal{P}, 0 \neq A \in \mathbb{A} \text{ and } \deg(A) < \deg(P)\right\}.$$

Also, let

$$\tilde{\mathcal{F}}' = \{u + \xi : u \in \tilde{\mathcal{F}}\}$$

and

$$\tilde{\mathcal{I}} = \{u + F : u \in \tilde{\mathcal{F}}, F \in \mathcal{G}\}.$$

Then under the correspondence $u \mapsto K_u$, $\tilde{\mathcal{I}}$, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ corresponds to the set of ramified imaginary, real and inert imaginary separable quadratic extensions of k whose discriminant is a square of a prime polynomial respectively. Furthermore we let

$$\tilde{\mathcal{F}}_n = \{u \in \tilde{\mathcal{F}} : P \in \mathcal{P}_n\} \quad \text{and} \quad \tilde{\mathcal{F}}'_n = \{u + \xi : u \in \tilde{F}_n\}$$

Also, for $r, s \geq 1$, let

$$\tilde{\mathcal{I}}_{(r,s)} = \{u + F : u \in \tilde{\mathcal{F}}_r, F \in \mathcal{G}_s\}.$$

Then for integer $n \geq 1$, let $\tilde{\mathcal{I}}_n$ be the union of all $\tilde{\mathcal{I}}_{(r,s)}$ where (r,s) runs through all positive integers r and s with $r + s = n$. Then, under the correspondence $u \mapsto K_u$, $\tilde{\mathcal{I}}_{g+1}$, $\tilde{\mathcal{F}}_{g+1}$ and $\tilde{\mathcal{F}}'_{g+1}$ corresponds to the set of ramified imaginary, real and inert imaginary quadratic extensions of k whose discriminant is a square of a prime polynomial with genus g.

Remark 2.7.4. *Comparing to the odd characteristic case, the sets $\mathcal{I}_{g+1}$, $\mathcal{F}_{g+1}$ and $\mathcal{F}'_{g+1}$ correspond to the sets $\mathcal{H}_{2g+1}$, $\mathcal{H}_{2g+2}$ and $\gamma\mathcal{H}_{2g+2}$ respectively, where γ is a fixed generator of $\mathbb{F}_q^*$. Similarly, the sets $\tilde{\mathcal{I}}_{g+1}$, $\tilde{\mathcal{F}}_{g+1}$ and $\tilde{\mathcal{F}}'_{g+1}$ correspond to the sets $\mathcal{P}_{2g+1}$, $\mathcal{P}_{2g+2}$ and $\gamma\mathcal{P}_{2g+2}$ respectively.*

2.7.2 Hasse Symbol

Definition 2.7.5. Let $P \in \mathcal{P}$. For $u \in k$ whose denominator is not divisible by P, the Hasse symbol $[u, P)$ with values in $\mathbb{F}_2$ is defined by

$$[u, P) := \begin{cases} 0 & \text{if } X^2 + X \equiv u (\text{mod } P) \text{ is solvable in } \mathbb{A}, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.7.6. For $N \in \mathbb{A}$ prime to the denominator of u, write $N = sgn(N) \prod_{i=1}^{s} P_i^{e_i}$ where $P_i \in \mathcal{P}$ are distinct and $e_i \geq 1$. Then

$$[u, N) := \sum_{i=1}^{s} e_i [u, P_i). \tag{2.7.8}$$

Definition 2.7.7. For $u \in k$ and $0 \neq N \in \mathbb{A}$, we define the quadratic symbol $\left\{\frac{u}{N}\right\}$ by

$$\left\{\frac{u}{N}\right\} := \begin{cases} (-1)^{[u,N)} & \text{if } N \text{ is prime to the denominator of } u, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.7.8. *The symbol $[u, n)$ is additive and the quadratic symbol $\left\{\frac{u}{N}\right\}$ is multiplicative.*

2.7.3 Quadratic L-functions

Definition 2.7.9. For the field K_u, the character χ_u, the character χ_u on $\mathbb{A}^+$ is defined as $\chi_u(f) = \left\{\frac{u}{f}\right\}$. For $\Re(s) > 1$, the L-function associated with χ_u is defined by

$$L(s, \chi_u) := \sum_{f \in \mathbb{A}^+} \frac{\chi_u(f)}{|f|^s} = \prod_P \left(1 - \frac{\chi_u(P)}{|P|^s}\right)^{-1}. \tag{2.7.9}$$

Using the change of variables $z = q^{-s}$, we have

$$\mathcal{L}(z, \chi_u) = \sum_{f \in \mathbb{A}^+} \chi_u(f) z^{\deg(f)} = \prod_P \left(1 - \chi_u(P) z^{\deg(P)}\right)^{-1} \qquad \left(|z| < \frac{1}{q}\right). \tag{2.7.10}$$

Similarly, from [Rud10], we know that $\mathcal{L}(z, \chi_u)$ has a trivial zero at $z = 1$ if and only if K_u is real and $\mathcal{L}(z, \chi_u)$ has a zero at $z = -1$ if and only if K_u is inert imaginary. We thus define the complete L-function $\mathcal{L}^*(z,\chi_u)$ as

$$\mathcal{L}^*(z,\chi_u) = \begin{cases} \mathcal{L}(z,\chi_u) & \text{if } K_u \text{ is ramified,} \\ (1-z)^{-1}\mathcal{L}(z,\chi_u) & \text{if } K_u \text{ is real,} \\ (1+z)^{-1}\mathcal{L}(z,\chi_u) & \text{if } K_u \text{ is inert imaginary,} \end{cases}$$

which is a polynomial of degree $2g_u$, where g_u denotes the genus of K_u which is defined by (2.7.2), which satisfies the functional equation

$$\mathcal{L}^*(z,\chi_u) = (qz^2)^{g_u}\mathcal{L}^*\left(\frac{1}{qz},\chi_u\right).$$

For the hyperelliptic curve $C_u : X^2 + X + u = 0$, we have that $\mathcal{L}^*(z,\chi_u) = P_{C_u}(z)$, where $P_C(z)$ was defined in Section 2.5.

For $u \in \mathcal{I}_{g+1}$, let

$$\mathcal{X}_u(s) = (q^{2g+1})^{\frac{1}{2}-s}X(s)$$

where

$$X(s) = q^{-\frac{1}{2}+s}$$

Then we define the completed L-function $\Lambda(s,\chi_u)$ as

$$\Lambda(s,\chi_u) = \mathcal{X}_u(s)^{-\frac{1}{2}}L(s,\chi_u). \tag{2.7.11}$$

The completed L-function also satisfies the following functional equation

$$\Lambda(s,\chi_u) = \Lambda(1-s,\chi_u). \tag{2.7.12}$$

Chapter 3

The First Moment of $L\left(\frac{1}{2},\chi\right)$ for Real Quadratic Function Fields

The work done in this chapter is a joint work with my supervisor, Dr. Julio Andrade and has been published in Acta Arithmetica [AM21].

3.1 Introduction and Statement of Result

A problem in function fields is to understand the asymptotic behaviour of

$$\sum_{D\in\mathcal{H}_{2g+1}} L(s,\chi_D)^k, \tag{3.1.1}$$

for various values of s and k, as $|D|\to\infty$, where $q\equiv 1(\text{mod } 4)$, $L(s,\chi_D)$ is the quadratic Dirichlet L-function defined in Section 2.6.2 and $\mathcal{H}_{2g+1}$ is the hyperelliptic ensemble defined in Section 2.6.3. Since we are letting $|D|\to\infty$, there are two limits to consider, the first is to fix g and let $q\to\infty$ and the second is to fix q and let $g\to\infty$. Katz and Sarnak [KS99a, KS99b] used equidistribution results to relate the q limit of (3.1.1) to a random matrix integral, which was then computed by Keating and Snaith [KS00a]. We will thus concentrate on the other limit, namely when we fix q and let $g\to\infty$. In this setting Andrade and Keating [AK12] computed the first moment of (3.1.1), when $s=\frac{1}{2}$, which is seen to be the function field analogue of Jutila's result Theorem 1.5.2

Theorem 3.1.1 (Andrade and Keating)**.** *Let q be the fixed cardinality of the ground field $\mathbb{F}_q$ and assume that $q\equiv 1(mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2},\chi_D\right)=\frac{P(1)}{2\zeta_{\mathbb{A}}(2)}|D|\left[\log_q|D|+1+\frac{4}{\log q}\frac{P'}{P}(1)\right]+O\left(|D|^{\frac{3}{4}+\frac{1}{2}\log_q 2}\right), \tag{3.1.2}$$

where

$$P(s)=\prod_P\left(1-\frac{1}{|P|^s(|P|+1)}\right). \tag{3.1.3}$$

Motivated by Young's [You09] number field result, Florea [Flo17a] improved the asymptotic formula (3.1.2) by obtaining a secondary main term of size $gq^{\frac{2g+1}{3}}$ and bounded the error term by $q^{\frac{g}{2}(1+\epsilon)}$.

Theorem 3.1.2 (Florea)**.** *Let q be a prime with $q \equiv 1(mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2},\chi_D\right) = \frac{P(1)}{2\zeta_{\mathbb{A}}(2)} q^{2g+1}\left[(2g+1)+1+\frac{4}{\log q}\frac{P'}{P}(1)\right] + q^{\frac{2g+1}{3}}R(2g+1) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right), \tag{3.1.4}$$

where R is a polynomial of degree 1 that can be explicitly computed.

Florea [Flo17b, Flo17c] then computed the second, third and fourth moments of (3.1.1) at $s=\frac{1}{2}$. Namely, she proved the following results.

Theorem 3.1.3 (Florea)**.** *Let q be a prime with $q \equiv 1(mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2},\chi_D\right)^2 = q^{2g+1}P(2g+1) + O\left(q^{g(1+\epsilon)}\right), \tag{3.1.5}$$

where $P(x)$ is a polynomial of degree 3 that can explicitly be calculated.

Theorem 3.1.4 (Florea)**.** *Let q be a prime with $q \equiv 1(mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2},\chi_D\right)^3 = q^{2g+1}Q(2g+1) + O\left(q^{\frac{3g}{2}(1+\epsilon)}\right), \tag{3.1.6}$$

where $Q(x)$ is a polynomial of degree 6 that can explicitly be calculated.

Theorem 3.1.5 (Florea)**.** *Let q be a prime with $q \equiv 1(mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2},\chi_D\right)^4 = q^{2g+1}(a_{10}g^{10}+a_9g^9+a_8g^8) + O\left(q^{2g+1}g^{7+\frac{1}{2}+\epsilon}\right), \tag{3.1.7}$$

where the coefficients a_{10}, a_9 and a_8 are arithmetic factors that can be written down explicitly.

In [And12], Andrade obtained an asymptotic formula for the first moment of (3.1.1) at $s=1$. In particular, he proved the following result.

Theorem 3.1.6 (Andrade)**.** *Let $\mathbb{F}_q$ be a fixed finite field with $q \equiv 1(mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+1}} L(1,\chi_D) = |D|P(2) + O((2q)^g). \tag{3.1.8}$$

Andrade and Jung [AJ18], using the techniques presented by Florea [Flo17a], improved the asymptotic formula (3.1.8) by obtaining a secondary main term of size $q^{\frac{g}{3}}$ and bounded the error term by $q^{g\epsilon}$ for any $\epsilon > 0$.

Theorem 3.1.7 (Andrade and Jung). *Let $\mathbb{F}_q$ be a fixed finite field with q a prime number such that $q \equiv 1(mod\ 4)$ and $\epsilon > 0$. Then as $g \to \infty$, we have*

$$\sum_{D\in\mathcal{H}_{2g+1}} L(1,\chi_D) = P(2)q^{2g+1} + cq^{\frac{g}{3}} + O(q^{g\epsilon}), \tag{3.1.9}$$

where c is a constant that can be explicitly calculated.

Bae and Jung [BJ19] used the techniques presented by Florea to improve the asymptotic formula for the second derivative of $L(s,\chi_D)$ at $s=\frac{1}{2}$, when summed over $D\in\mathcal{H}_{2g+1}$, which was first calculated by Andrade and Rajagopal [AR16]. Compared to the asymptotic formula obtained by Andrade and Rajagopal, Bae and Jung obtained a secondary main term of size $g^3q^{\frac{2g+1}{3}}$ and bounded the error term by $q^{\frac{g}{2}(1+\epsilon)}$.

Another problem in function fields is to understand the asymptotic behaviour of

$$\sum_{D\in\mathcal{H}_{2g+2}} L(s,\chi_D)^k \tag{3.1.10}$$

as $|D|\to\infty$, where $\mathcal{H}_{2g+2}$ is the hyperelliptic ensemble defined in Section 2.6.3. In the setting of fixing q and letting $g\to\infty$, Jung [Jun13] obtained an asymptotic formula for the first moment of (3.1.10) at $s=\frac{1}{2}$.

Theorem 3.1.8 (Jung). *Assume that q is odd and greater than 3. Then we have*

$$\sum_{D\in\mathcal{H}_{2g+2}} L\left(\frac{1}{2},\chi_D\right) = \frac{P(1)}{2\zeta_{\mathbb{A}}(2)}|D|\left[\log_q|D| + \frac{4}{\log q}\frac{P'}{P}(1) + 2\zeta_{\mathbb{A}}\left(\frac{1}{2}\right)\right] + O\left(|D|^{\frac{3}{4}+\frac{1}{2}\log_q 2}\right). \tag{3.1.11}$$

In this chapter, we will use the techniques presented by Florea to improve the asymptotic formula (3.1.11), by obtaining a secondary main term of size $gq^{\frac{2g+2}{3}}$ and bound the error term by $q^{\frac{g}{2}(1+\epsilon)}$. In particular our goal for this chapter is to prove the following theorem.

Theorem 3.1.9. *Let q be a prime with $q\equiv 1(mod\ 4)$. Then*

$$\sum_{D\in\mathcal{H}_{2g+2}} L\left(\frac{1}{2},\chi_D\right) = \frac{P(1)}{2\zeta_{\mathbb{A}}(2)}q^{2g+2}\left[(2g+2) + \frac{4}{\log q}\frac{P'}{P}(1) + 2\zeta_{\mathbb{A}}\left(\frac{1}{2}\right)\right] + q^{\frac{2g+2}{3}}\mathcal{R}(2g+2) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right), \tag{3.1.12}$$

where $\mathcal{R}$ is a polynomial of degree 1 that can explicitly be calculated (see formula (3.6.25)).

3.2 Overview of Chapter

The techniques presented in this chapter follows the techniques presented in [Flo17a]. We will first use a form of the Poisson summation formula over $\mathbb{F}_q[T]$ to split the sum up to different formulas, which correspond to whether the degree of f is odd or even. These formulas are presented in Section 3.4. In Section 3.5, we will express sums over square polynomials f as contour integrals.

In Section 3.6, we will evaluate the non-square polynomials f using the Poisson summation formula, which will analyse the contribution of the square polynomials V. In the imaginary quadratic function field case, the contribution to the main terms from the square polynomials V come from when the degree of f is even. However in the real quadratic function field case, the contribution to the main terms from the square polynomials V come from when the degree of f is both odd and even. Thus, compared to the calculations done by Florea, there are extra terms to calculate and evaluate.

In Section 3.7, we will bound the contribution of the non-square polynomials V by $q^{\frac{g}{2}(1+\epsilon)}$. Finally, in Section 3.8, we will show how the results obtained in the previous sections combine to establish the desired asymptotic formula.

3.3 Preliminary Lemmas

In this section, we will state some results which will be used to prove Theorem 3.1.9.

For $D \in \mathcal{H}_{2g+2}$, the "approximate" functional equation was first proved in [Jun13], however it has been corrected to match that stated in [RW15].

Lemma 3.3.1 ("Approximate" Functional Equation, [Jun13, Lemma 3.1])**.** *Let χ_D be a quadratic character, where $D \in \mathcal{H}_{2g+2}$. Then*

$$L\left(\frac{1}{2},\chi_D\right) = \sum_{n=0}^{g}\sum_{f\in\mathbb{A}_n^+}\chi_D(f)q^{-\frac{n}{2}} - q^{-\frac{g+1}{2}}\sum_{n=0}^{g}\sum_{f\in\mathbb{A}_n^+}\chi_D(f) + \sum_{n=0}^{g-1}\sum_{f\in\mathbb{A}_n^+}\chi_D(f)q^{-\frac{n}{2}} - q^{-\frac{g}{2}}\sum_{n=0}^{g-1}\sum_{f\in\mathbb{A}_n^+}\chi_D(f). \tag{3.3.1}$$

Using Lemma 3.3.1, we have

$$\sum_{D\in\mathcal{H}_{2g+2}} L\left(\frac{1}{2},\chi_D\right) = \sum_{f\in\mathbb{A}_{\le g}^+}\frac{1}{\sqrt{|f|}}\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f) - q^{-\frac{g+1}{2}}\sum_{f\in\mathbb{A}_{\le g}^+}\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f) + \sum_{f\in\mathbb{A}_{\le g-1}^+}\frac{1}{\sqrt{|f|}}\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f) - q^{-\frac{g}{2}}\sum_{f\in\mathbb{A}_{\le g-1}^+}\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f). \tag{3.3.2}$$

The next results will be used in later sections.

Lemma 3.3.2. *Let $f \in \mathbb{A}^+$. Then*

$$\sum_{D\in\mathcal{H}_{2g+2}} \chi_D(f) = \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\leq g+1}}} \sum_{h\in\mathbb{A}^+_{2g+2-2deg(C)}} \chi_f(h) - q \sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\leq g}}} \sum_{h\in\mathbb{A}^+_{2g-2deg(C)}} \chi_f(h), \tag{3.3.3}$$

where $C|f^\infty$ means that any prime factor of C is among the prime factors of f.

Proof. The proof is similar to that given in [Flo17a, Lemma 2.2]. ∎

Next, we will state a version of the Poisson summation formula over function fields. To do this, we need to recall the exponential function and the generalised Gauss sum which was introduced by Hayes [Hay66, EH91] and Florea [Flo17a]. We know that each $a \in \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$ can be written uniquely as

$$a = \sum_{i=-\infty}^{\infty} a_i \left(\frac{1}{T}\right)^i, \tag{3.3.4}$$

with $a_i \in \mathbb{F}_q$ such that all but finitely many of the a_i's with $i < 0$ are non-zero. If $a \neq 0$ and a has the Laurent expansion (3.3.4), then one can define the valuation

$$v(a) = \text{ smallest } i \text{ such that } a_i \neq 0.$$

For $a \in \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$ the exponential sum (see [Hay66]) is defined as

$$e(a) = e^{\frac{2\pi i a_1}{q}},$$

where a_1 is the coefficient of $\frac{1}{T}$ in the Laurent expansion (3.3.4). By [Hay66, Theorem 3.3], we know that for $a, b \in \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$, we have $e(a+b) = e(a)e(b)$ and for $A, B, H \in \mathbb{F}_q[T]$, we have $e(A) = 1$ and $e\left(\frac{A}{H}\right) = e\left(\frac{B}{H}\right)$ if $A \equiv B(\text{mod } H)$. For a general character χ modulo f, the generalised Gauss sum is defined as

$$G(V,\chi) := \sum_{A \text{ mod } f} \chi(A) e\left(\frac{AV}{f}\right).$$

Then the following result holds.

Lemma 3.3.3 (Poisson Summation Formula, [Flo17a, Lemma 3.1])**.** *Let $f \in \mathbb{A}^+$ and let m be a positive integer.*

1. *If the degree of f is odd, we have*

$$\sum_{g\in\mathbb{A}^+_m} \chi_f(g) = \frac{q^{m+\frac{1}{2}}}{|f|} \sum_{V\in\mathbb{A}^+_{deg(f)-m-1}} G(V,\chi_f). \tag{3.3.5}$$

2. If the degree of f is even, we have

$$\sum_{g\in\mathbb{A}_m^+}\chi_f(g)=\frac{q^m}{|f|}\left(G(0,\chi_f)+(q-1)\sum_{V\in\mathbb{A}^+_{\leq deg(f)-m-2}}G(V,\chi_f)-\sum_{V\in\mathbb{A}^+_{deg(f)-m-1}}G(V,\chi_f)\right). \tag{3.3.6}$$

Remark 3.3.4. *$G(0,\chi_f)$ is non-zero if and only if f is a square, in which case, $G(0,\chi_f)=\phi(f)$, where $\phi(f)$ the Euler-Totient function defined in Section 2.2.*

The last result that we will state in this section is the function field analogue of Perron's formula.

Lemma 3.3.5 (The function field analogue of Perron's formula,[AJ18, Lemma 4.1])**.** *If the power series*

$$H(u)=\sum_{f\in\mathbb{A}^+}a(f)u^{deg(f)} \tag{3.3.7}$$

converges absolutely for $|u|\leq R<1$, then

$$\sum_{f\in\mathbb{A}_n^+}a(f)=\frac{1}{2\pi i}\oint_{|u|=R}\frac{H(u)}{u^{n+1}}du \tag{3.3.8}$$

and

$$\sum_{f\in\mathbb{A}_{\leq n}^+}a(f)=\frac{1}{2\pi i}\oint_{|u|=R}\frac{H(u)}{(1-u)u^{n+1}}du. \tag{3.3.9}$$

3.4 Setup of the Problem

Using the "approximate" functional equation (3.3.2) and Lemma 3.3.2, we write

$$\sum_{D\in\mathcal{H}_{2g+2}}L\left(\frac{1}{2},\chi_D\right)=\mathcal{S}_{g,1}-\mathcal{S}_{g,2}+\mathcal{S}_{g-1,1}-\mathcal{S}_{g-1,2} \tag{3.4.1}$$

where

$$\mathcal{S}_{g,1}=\sum_{f\in\mathbb{A}^+_{\leq g}}\frac{1}{\sqrt{|f|}}\sum_{\substack{C|f^\infty\\ C\in\mathbb{A}^+_{\leq g+1}}}\sum_{h\in\mathbb{A}^+_{2g+2-2\deg(C)}}\chi_f(h)-q\sum_{f\in\mathbb{A}^+_{\leq g}}\frac{1}{\sqrt{|f|}}\sum_{\substack{C|f^\infty\\ C\in\mathbb{A}^+_{\leq g}}}\sum_{h\in\mathbb{A}^+_{2g-2\deg(C)}}\chi_f(h),$$

$$\mathcal{S}_{g,2}=q^{-\frac{g+1}{2}}\sum_{f\in\mathbb{A}^+_{\leq g}}\sum_{\substack{C|f^\infty\\ C\in\mathbb{A}^+_{\leq g+1}}}\sum_{h\in\mathbb{A}^+_{2g+2-2\deg(C)}}\chi_f(h)-q^{-\frac{g-1}{2}}\sum_{f\in\mathbb{A}^+_{\leq g}}\sum_{\substack{C|f^\infty\\ C\in\mathbb{A}^+_{\leq g}}}\sum_{h\in\mathbb{A}^+_{2g-2\deg(C)}}\chi_f(h),$$

$$\mathcal{S}_{g-1,1}=\sum_{f\in\mathbb{A}^+_{\leq g-1}}\frac{1}{\sqrt{|f|}}\sum_{\substack{C|f^\infty\\ C\in\mathbb{A}^+_{\leq g+1}}}\sum_{h\in\mathbb{A}^+_{2g+2-2\deg(C)}}\chi_f(h)-q\sum_{f\in\mathbb{A}^+_{\leq g-1}}\frac{1}{\sqrt{|f|}}\sum_{\substack{C|f^\infty\\ C\in\mathbb{A}^+_{\leq g}}}\sum_{h\in\mathbb{A}^+_{2g-2\deg(C)}}\chi_f(h)$$

and

$$\mathcal{S}_{g-1,2} = q^{-\frac{g}{2}} \sum_{f \in \mathbb{A}^+_{\le g-1}} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\le g+1}}} \sum_{h \in \mathbb{A}^+_{2g+2-2\deg(C)}} \chi_f(h) - q^{-\frac{g}{2}+1} \sum_{f \in \mathbb{A}^+_{\le g-1}} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\le g}}} \sum_{h \in \mathbb{A}^+_{2g-2\deg(C)}} \chi_f(h).$$

From [Flo17a, Section 4], we have

$$\sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{g+1}}} 1 \ll q^{g\epsilon},$$

thus we see that the terms in $\mathcal{S}_{g,1}, \mathcal{S}_{g,2}, \mathcal{S}_{g-1,1}$ and $\mathcal{S}_{g-1,2}$ corresponding to $C \in \mathbb{A}^+_{g+1}$ are bounded by $O\left(q^{\frac{g}{2}(1+\epsilon)}\right)$. Therefore, for $k \in \{g, g-1\}$, we have

$$\mathcal{S}_{k,1} = \sum_{f \in \mathbb{A}^+_{\le k}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\le g}}} \left(\sum_{h \in \mathbb{A}^+_{2g+2-2\deg(C)}} \chi_f(h) - q \sum_{h \in \mathbb{A}^+_{2g-2\deg(C)}} \chi_f(h) \right) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right)$$

and

$$\mathcal{S}_{k,2} = q^{-\frac{k+1}{2}} \sum_{f \in \mathbb{A}^+_{\le k}} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\le g}}} \left(\sum_{h \in \mathbb{A}^+_{2g+2-2\deg(C)}} \chi_f(h) - q \sum_{h \in \mathbb{A}^+_{2g-2\deg(C)}} \chi_f(h) \right) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right).$$

For $\ell \in \{1, 2\}$, write

$$\mathcal{S}_{k,\ell} = \mathcal{S}^o_{k,\ell} + \mathcal{S}^e_{k,\ell} + O\left(q^{\frac{g}{2}(1+\epsilon)}\right), \tag{3.4.2}$$

where $\mathcal{S}^o_{k,\ell}$ and $\mathcal{S}^e_{k,\ell}$ denote the sum over $f \in \mathbb{A}^+_{\le k}$ of odd and even degree respectively. If the degree of f is odd, then using Lemma 3.3.3, we have

$$\mathcal{S}^o_{k,1} = q^{2g+\frac{5}{2}} \sum_{\substack{f \in \mathbb{A}^+_{\le k} \\ \deg(f) \text{ odd}}} \frac{1}{|f|} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\le g}}} \frac{1}{|C|^2} S^o(V; f, C)$$

and

$$\mathcal{S}^o_{k,2} = q^{\frac{4g-k}{2}+2} \sum_{\substack{f \in \mathbb{A}^+_{\le k} \\ \deg(f) \text{ odd}}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\le g}}} \frac{1}{|C|^2} S^o(V; f, C)$$

where

$$S^o(V; f, C) = \sum_{V \in \mathbb{A}^+_{\deg(f)-2g-3+2\deg(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}} - \frac{1}{q} \sum_{V \in \mathbb{A}^+_{\deg(f)-2g-1+2\deg(C)}} \frac{G(V, \chi_f)}{\sqrt{|f|}}. \tag{3.4.3}$$

If the degree of f is even, then using Lemma 3.3.3, we rewrite $\mathcal{S}^e_{k,\ell}$ as

$$\mathcal{S}^e_{k,\ell} = M_{k,\ell} + \mathcal{S}^e_{k,\ell,1} + \mathcal{S}^e_{k,\ell,2}. \tag{3.4.4}$$

Using Remark 3.3.4, we have that

$$M_{k,1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{L \in \mathbb{A}^+_{\leq [\frac{k}{2}]}} \frac{\phi(L^2)}{|L|^3} \sum_{\substack{C | L^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \frac{1}{|C|^2} \tag{3.4.5}$$

and

$$M_{k,2} = \frac{q^{\frac{4g-k+3}{2}}}{\zeta_{\mathbb{A}}(2)} \sum_{L \in \mathbb{A}^+_{\leq [\frac{k}{2}]}} \frac{\phi(L^2)}{|L|^2} \sum_{\substack{C | L^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \frac{1}{|C|^2}. \tag{3.4.6}$$

Similarly, for $j \in \{1,2\}$, we have

$$\mathcal{S}^e_{k,1,j} = q^{2g+2} \sum_{\substack{f \in \mathbb{A}^+_{\leq g} \\ \deg(f) \text{ even}}} \frac{1}{|f|} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \frac{1}{|C|^2} S^e_j(V; f, C)$$

and

$$\mathcal{S}^e_{k,2,j} = q^{\frac{4g-k+3}{2}} \sum_{\substack{f \in \mathbb{A}^+_{\leq k} \\ \deg(f) \text{ even}}} \frac{1}{\sqrt{|f|}} \sum_{\substack{C | f^\infty \\ C \in \mathbb{A}^+_{\leq g}}} \frac{1}{|C|^2} S^e_j(V; f, C),$$

where

$$S^e_1(V; f, C) = (q-1) \sum_{V \in \mathbb{A}^+_{\leq \deg(f)-2g-4+2\deg(C)}} \frac{G(V,\chi_f)}{\sqrt{|f|}} - \frac{q-1}{q} \sum_{V \in \mathbb{A}^+_{\leq \deg(f)-2g-2+2\deg(C)}} \frac{G(V,\chi_f)}{\sqrt{|f|}} \tag{3.4.7}$$

and

$$S^e_2(V; f, C) = \frac{1}{q} \sum_{V \in \mathbb{A}^+_{\deg(f)-2g-1+2\deg(f)}} \frac{G(V,\chi_f)}{\sqrt{|f|}} - \sum_{V \in \mathbb{A}^+_{\deg(f)-2g-3+2\deg(C)}} \frac{G(V,\chi_f)}{\sqrt{|f|}}. \tag{3.4.8}$$

For $i \in \{o, e\}$, define $\mathcal{S}^i_{k,\ell}(V = \square)$ to be the sum over V square and $\mathcal{S}^i_{k,\ell}(V \neq \square)$ to be the sum over non-square V. Since the degree of f is even in (3.4.8), then the degree of V is odd and so V cannot be a square. Furthermore, since the degree of f is odd in (3.4.3), then the degree of V is even, so there is a contribution to the main terms when the degree of f is odd, which does not occur when working in the imaginary function field case, i.e., in Florea's calculation [Flo17a].

3.5 Contribution from M term

Let

$$M = M_{g,1} - M_{g,2} + M_{g-1,1} - M_{g-1,2}. \tag{3.5.1}$$

Then, in this section, we evaluate the main term M. The main result in this section is the following result.

Proposition 3.5.1. *For any $\epsilon > 0$ we have*

$$M = M_1 + M_2 + M_3 + M_4 + O(q^{g\epsilon}), \tag{3.5.2}$$

where

$$M_1 = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{[\frac{g}{2}]}} du,$$

$$M_2 = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{[\frac{g-1}{2}]}} du,$$

$$M_3 = -\frac{q^{\frac{3g+3}{2}}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{[\frac{g}{2}]}} du$$

and

$$M_4 = -\frac{q^{\frac{3g}{2}+2}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{[\frac{g-1}{2}]}} du,$$

with $r < q^{-1}$ and

$$\mathcal{C}(u) = \prod_P \left(1 - \frac{u^{deg(P)}}{|P|+1}\right). \tag{3.5.3}$$

Remark 3.5.2. *$\mathcal{C}(u)$ is analytic in the region $|u| < 1$. We may further write*

$$\begin{aligned} \mathcal{C}(u) &= \mathcal{Z}\left(\frac{u}{q}\right)^{-1} \prod_P \left(1 + \frac{u^{deg(P)}}{(1+|P|)(|P| - u^{deg(P)})}\right) \\ &= (1-u) \prod_P \left(1 + \frac{u^{deg(P)}}{(1+|P|)(|P| - u^{deg(P)})}\right), \end{aligned} \tag{3.5.4}$$

which furnishes an analytic continuation of $\mathcal{C}(u)$ to the region $|u| < q$.

Proof of Proposition 3.5.1. From (3.4.5) and (3.4.6) and using the facts that (see [Flo17a, Section 5])

$$\sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g}}} \frac{1}{|C|^2} = \prod_{P|f} (1 - |P|^{-2})^{-1} + O(q^{-g(2-\epsilon)}) \quad \text{and} \quad \frac{\phi(L^2)}{|L|^2} = \prod_{P|L} (1 - |P|^{-1}),$$

we have

$$M_{k,1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{L\in\mathbb{A}^+_{\le[\frac{k}{2}]}} \frac{1}{|L|} \prod_{P|L} \frac{|P|}{|P|+1} + O(q^{g\epsilon})$$

and

$$M_{k,2} = \frac{q^{\frac{4g-k+3}{2}}}{\zeta_{\mathbb{A}}(2)} \sum_{L\in\mathbb{A}^+_{\le[\frac{k}{2}]}} \prod_{P|L} \frac{|P|}{|P|+1} + O(q^{g\epsilon}).$$

Using the function field version of Perron's formula, we have

$$M_{k,1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{A}(u)}{u(1-qu)(qu)^{[\frac{k}{2}]}} du + O(q^{g\epsilon}) \tag{3.5.5}$$

and

$$M_{k,2} = \frac{q^{\frac{4g-k+3}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=r}\frac{\mathcal{A}(u)}{u(1-u)u^{[\frac{k}{2}]}}du + O(q^{g\epsilon}), \tag{3.5.6}$$

where $r < q^{-1}$ and

$$\mathcal{A}(u) = \sum_{L\in\mathbb{A}^+} u^{\deg(L)}\prod_{P|L}\frac{|P|}{|P|+1}. \tag{3.5.7}$$

By multiplicativity, we may write (3.5.7) as

$$\mathcal{A}(u) = \prod_{P}\left(1+\frac{|P|}{|P|+1}\frac{u^{\deg(P)}}{1-u^{\deg(P)}}\right) = \mathcal{Z}(u)\mathcal{C}(u) = \frac{\mathcal{C}(u)}{(1-qu)}. \tag{3.5.8}$$

Inserting (3.5.8) into (3.5.5) and (3.5.6), we have that

$$M_{k,1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=r}\frac{C(u)}{u(1-qu)^2(qu)^{[\frac{k}{2}]}}du + O(q^{g\epsilon}) \tag{3.5.9}$$

and

$$M_{k,2} = \frac{q^{\frac{4g-k+3}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=r}\frac{C(u)}{u(1-u)(1-qu)u^{[\frac{k}{2}]}}du + O(q^{g\epsilon}). \tag{3.5.10}$$

Observing (3.5.9) and (3.5.10), we see that the terms $M_{g,1}, M_{g-1,1}, M_{g,2}$ and $M_{g-1,2}$ are precisely the terms $M_1, M_2, -M_3$ and $-M_4$ stated in Proposition 3.5.1 respectively. By (3.5.1) we deduce the Proposition. ■

3.6 Contribution from V Square

3.6.1 Main Result

Let

$$\mathcal{S}(V=\square) = \mathcal{S}^o(V=\square) + \mathcal{S}^e(V=\square) \tag{3.6.1}$$

where

$$\mathcal{S}^o(V=\square) = \mathcal{S}^o_{g,1}(V=\square) - \mathcal{S}^o_{g,2}(V=\square) + \mathcal{S}^o_{g-1,1}(V=\square) - \mathcal{S}^o_{g-1,2}(V=\square) \tag{3.6.2}$$

and

$$\mathcal{S}^e(V=\square) = \mathcal{S}^e_{g,1}(V=\square) - \mathcal{S}^e_{g,2}(V=\square) + \mathcal{S}^e_{g-1,1}(V=\square) - \mathcal{S}^e_{g-1,2}(V=\square). \tag{3.6.3}$$

In this section, we will evaluate the term $\mathcal{S}(V=\square)$. The main result in this section is the following Proposition.

Proposition 3.6.1. *Using the same notation as before, we have*

$$\begin{aligned}\mathcal{S}(V=\square) &= \mathcal{S}_1(V=\square) + \mathcal{S}_2(V=\square) + \mathcal{S}_3(V=\square) + \mathcal{S}_4(V=\square)\\ &\quad + q^{\frac{2g+2}{3}}\mathcal{R}(2g+2) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),\end{aligned} \tag{3.6.4}$$

where

$$\mathcal{S}_1(V=\square) = -\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{[\frac{g}{2}]}}du,$$

$$\mathcal{S}_2(V=\square) = -\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{[\frac{g-1}{2}]}}du,$$

$$\mathcal{S}_3(V=\square) = \frac{q^{\frac{3g+3}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{[\frac{g}{2}]}}du$$

and

$$\mathcal{S}_4(V=\square) = \frac{q^{\frac{3g}{2}+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{[\frac{g-1}{2}]}}du,$$

with $1<R<q$ *and*

$$\mathcal{C}(u) = \prod_P\left(1-\frac{u^{deg(P)}}{|P|+1}\right) = (1-u)\prod_P\left(1+\frac{u^{deg(P)}}{(|P|+1)(|P|-u^{deg(P)})}\right).$$

Furthermore, $\mathcal{R}$ *is a linear polynomial that can be explicitly calculated (see formula (3.6.25)).*

3.6.2 Notation and Preliminary Results

To prove Proposition 3.6.1, we first need the following notations and results, which are stated and proved in [Flo17a, Section 6]. For $|z|>q^{-2}$, let

$$\mathcal{B}(z,w) = \sum_{f\in\mathbb{A}^+} w^{\deg(f)}A_f(z)\prod_{P|f}\left(1-\frac{1}{|P|^2z^{\deg(P)}}\right)^{-1}$$

where

$$A_f(z) = \sum_{l\in\mathbb{A}^+} z^{\deg(l)}\frac{G(l^2,\chi_f)}{\sqrt{|f|}}.$$

Then we have the following results.

Lemma 3.6.2 ([Flo17a, Lemma 6.2])**.** *For* $|z|>q^{-2}$*, we have*

$$\mathcal{B}(z,w) = \mathcal{Z}(z)\mathcal{Z}(w)\mathcal{Z}(qw^2z)\prod_P\mathcal{B}_P(z,w) \tag{3.6.5}$$

where

$$\mathcal{B}_P(z,w) = 1+\frac{1}{z^{deg(P)}|P|^2-1}\Big(w^{deg(P)}-(zw^2)^{deg(P)}|P|^2-(z^2w)^{deg(P)}|P|^2$$
$$+(z^2w^3)^{deg(P)}|P|^2+(zw^2)^{deg(P)}|P|-(zw^3)^{deg(P)}|P|\Big)$$

Moreover $\prod_P\mathcal{B}_P(z,w)$ *converges absolutely for* $|w|<q|z|, |w|<q^{-\frac{1}{2}}$ *and* $|wz|<q^{-1}$*.*

Lemma 3.6.3 ([Flo17a, Lemma 6.3]). *We have*

$$\prod_P \mathcal{B}_P(z,w) = \mathcal{Z}\left(\frac{w}{q^2 z}\right)\mathcal{Z}(w^2)^{-1}\prod_P \mathcal{D}_P(z,w), \tag{3.6.6}$$

where

$$\mathcal{D}_P(z,w) = 1 + \frac{1}{(|P|^2 z^{deg(P)}-1)(1+w^{deg(P)})}\Bigg(-w^{2deg(P)} - \frac{w^{3deg(P)}}{|P|} + \frac{w^{deg(P)}}{|P|^2 z^{deg(P)}}$$
$$+ (zw^2)^{deg(P)}|P| + (zw^2)^{deg(P)} - (z^2w)^{deg(P)}|P|^2 + (zw^3)^{deg(P)} - (z^2w^2)^{deg(P)}|P|^2\Bigg).$$

Moreover $\prod_P \mathcal{D}_P(z,w)$ *converges absolutely for* $|w|^2 < q|z|, |w| < q^3|z|^2, |w| < 1$ *and* $|wz| < q^{-1}$.

3.6.3 Outline of the proof of Proposition 3.6.1

From the Poisson summation formula, the sum over square polynomials V will occur when the degree of f is even and when the degree of f is odd. In Section 3.6.4 and Section 3.6.5, we will obtain two integrals for each $\mathcal{S}^e_{k,\ell}(V=\square)$ and $\mathcal{S}^o_{k,\ell}(V=\square)$ respectively, which correspond to simple poles $w = q^{-1}$ and $w = qz$. In Section 3.6.6, we will manipulate the integrals corresponding to the pole at $w = q^{-1}$, similar to what was done in [Flo17a, Section 6] which will yield the main terms $\mathcal{S}_1(V=\square), \mathcal{S}_2(V=\square), \mathcal{S}_3(V=\square)$ and $\mathcal{S}_4(V=\square)$. In Section 3.6.7, we will evaluate the integrals corresponding to the pole at $w = qz$, which will yield the secondary main term.

3.6.4 Degree f even

In this subsection, we prove the following result.

Lemma 3.6.4. *We have*

$$\mathcal{S}^e(V=\square) = \mathcal{A}^e_{g,1} - \mathcal{A}^e_{g,2} + \mathcal{A}^e_{g-1,1} - \mathcal{A}^e_{g-1,2} + \mathcal{B}^e_{g,1} - \mathcal{B}^e_{g,2} + \mathcal{B}^e_{g-1,1} - \mathcal{B}^e_{g-1,2} + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where $\mathcal{A}^e_{k,\ell}$ *and* $\mathcal{B}^e_{k,\ell}$ *are the integrals stated at the end of the subsection.*

Proof. From (3.4.7) and using the function field analogue of Perron's formula, we obtain

$$S^e_1(l^2;f,C) = \frac{1}{2\pi i}\oint_{|z|=q^{-1-\epsilon}} \frac{z^g(q-1)(qz-1)A_f(z)}{q(1-z)z^{\frac{\deg(f)}{2}+\deg(C)}}dz.$$

Using the fact that (see [Flo17a, Proof of Lemma 6.1])

$$\sum_{\substack{C|f^\infty \\ C\in\mathbb{A}^+_{\le g}}} \frac{1}{|C|^2 z^{\deg(C)}} = \prod_{P|f}(1-|P|^{-2}z^{-\deg(P)})^{-1} + O\left(q^{g(\epsilon-1)}\right)$$

we have, for $k \in \{g, g-1\}$

$$\mathcal{S}^e_{k,1}(V = \square) = \frac{q^{2g+2}}{2\pi i} \oint_{|z|=q^{-1-\epsilon}} \frac{z^g(q-1)(qz-1)}{q(1-z)} H^e_{k,1}(z) dz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$H^e_{k,1}(z) = \sum_{\substack{f \in \mathbb{A}^+_{\leq k} \\ \deg(f) \text{ even}}} \frac{A_f(z)}{|f| z^{\frac{\deg(f)}{2}}} \prod_{P|f} (1 - |P|^{-2} z^{-\deg(P)})^{-1}.$$

Similarly, we have

$$\mathcal{S}^e_{k,2}(V = \square) = \frac{q^{\frac{4g-k+3}{2}}}{2\pi i} \oint_{|z|=q^{-1-\epsilon}} \frac{z^g(q-1)(qz-1)}{q(1-z)} H^e_{k,2}(z) dz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$H^e_{k,2}(z) = \sum_{\substack{f \in \mathbb{A}^+_{\leq k} \\ \deg(f) \text{ even}}} \frac{A_f(z)}{\sqrt{|f|} z^{\frac{\deg(f)}{2}}} \prod_{P|f} (1 - |P|^{-2} z^{-\deg(P)})^{-1}.$$

Furthermore, we have

$$\begin{aligned} H^e_{k,\ell}(z) &= \sum_{\substack{f \in \mathbb{A}^+_{\leq k} \\ \deg(f) \text{ even}}} \frac{A_f(z)}{|f|^{\frac{3-\ell}{2}} z^{\frac{\deg(f)}{2}}} \prod_{P|f} (1 - |P|^{-2} z^{-\deg(P)})^{-1} \\ &= \sum_{\substack{n=0 \\ n=2m}}^{k} \sum_{f \in \mathbb{A}^+_n} \frac{A_f(z)}{|f|^{\frac{3-\ell}{2}} z^{\frac{\deg(f)}{2}}} \prod_{P|f} (1 - |P|^{-2} z^{-\deg(P)})^{-1} \\ &= \sum_{m=0}^{[\frac{k}{2}]} \frac{1}{q^{m(3-\ell)} z^m} \sum_{f \in \mathbb{A}^+_{2m}} A_f(z) \prod_{P|f} (1 - |P|^{-2} z^{-\deg(P)})^{-1}. \end{aligned}$$

Using the function field analogue of Perron's formula, we have

$$\begin{aligned} H^e_{k,\ell}(z) &= \frac{1}{2\pi i} \oint_{|w|=r_2} \frac{\mathcal{B}(z,w)}{w} \sum_{m=0}^{[\frac{k}{2}]} \frac{1}{q^{m(3-\ell)} z^m w^{2m}} dw \\ &= \frac{1}{2\pi i} \oint_{|w|=r_2} \frac{\mathcal{B}(z,w)}{w(1-q^{3-\ell}zw^2)(q^{3-\ell}zw^2)^{[\frac{k}{2}]}} dw - \frac{1}{2\pi i} \oint_{|w|=r_2} \frac{q^{3-\ell}zw\mathcal{B}(z,w)}{1-q^{3-\ell}zw^2} dw. \end{aligned} \tag{3.6.7}$$

For each $k \in \{g, g-1\}$ and each $\ell \in \{1, 2\}$, the second integral in (3.6.7) is zero since the integrands have no poles inside the circle $|w| = r_2 < q^{-1}$. Therefore

$$\mathcal{S}^e_{k,1}(V = \square) = \frac{q^{2g+2}}{(2\pi i)^2} \oint_{|z|=q^{-1-\epsilon}} \oint_{|w|=r_2} \frac{z^g(q-1)(qz-1)\mathcal{B}(z,w)}{qw(1-z)(1-q^2zw^2)(q^2zw^2)^{[\frac{k}{2}]}} dw dz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right)$$

and

$$\begin{aligned} \mathcal{S}^e_{k,2}(V = \square) = \frac{q^{\frac{4g-k+3}{2}}}{(2\pi i)^2} \oint_{|z|=q^{-1-\epsilon}} \oint_{|w|=r_2} \frac{z^g(q-1)(qz-1)\mathcal{B}(z,w)}{qw(1-z)(1-qzw^2)(qzw^2)^{[\frac{k}{2}]}} dw dz \\ + O\left(q^{\frac{g}{2}(1+\epsilon)}\right). \end{aligned}$$

Using (3.6.5) from Lemma 3.6.2, we obtain

$$\begin{aligned}&\mathcal{S}^e_{k,1}(V=\square)\\&=-\frac{q^{2g+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{z^g(q-1)\prod_P\mathcal{B}_P(z,w)}{qw(1-z)(1-qw)(1-q^2zw^2)^2(q^2zw^2)^{\left[\frac{k}{2}\right]}}dwdz+O\left(q^{\frac{g}{2}(1+\epsilon)}\right)\end{aligned}$$

and

$$\begin{aligned}&\mathcal{S}^e_{k,2}(V=\square)\\&=-\frac{q^{\frac{4g-k+3}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{z^g(q-1)\prod_P\mathcal{B}_P(z,w)}{qw(1-z)(1-qw)(1-qzw^2)(1-q^2zw^2)(qzw^2)^{\left[\frac{k}{2}\right]}}dwdz\\&+O\left(q^{\frac{g}{2}(1+\epsilon)}\right).\end{aligned}$$

Using (3.6.6), from Lemma 3.6.3 we obtain

$$\begin{aligned}&\mathcal{S}^e_{k,1}(V=\square)\\&=-\frac{q^{2g+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\frac{z^g(q-1)(1-qw^2)\prod_P\mathcal{D}_P(z,w)}{qw(1-z)(1-qw)\left(1-\frac{w}{qz}\right)(1-q^2zw^2)^2(q^2zw^2)^{\left[\frac{k}{2}\right]}}dwdz\\&+O\left(q^{\frac{g}{2}(1+\epsilon)}\right).\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}^e_{k,2}(V=\square)=&-\frac{q^{\frac{4g-k+3}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}\\&\times\frac{z^g(q-1)(1-qw^2)\prod_P\mathcal{D}_P(z,w)}{qw(1-z)(1-qw)\left(1-\frac{w}{qz}\right)(1-qzw^2)(1-q^2zw^2)(qzw^2)^{\left[\frac{k}{2}\right]}}dwdz\\&+O\left(q^{\frac{g}{2}(1+\epsilon)}\right).\end{aligned}$$

For each $S^e_{k,\ell}(V=\square)$, write

$$\mathcal{S}^e_{k,1}(V=\square)=-\frac{q^{2g+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}F^e_{k,1}(z,w)dwdz+O\left(q^{\frac{g}{2}(1+\epsilon)}\right)$$

and

$$\mathcal{S}^e_{k,2}(V=\square)=-\frac{q^{\frac{4g-k+3}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}F^e_{k,1}(z,w)dwdz+O\left(q^{\frac{g}{2}(1+\epsilon)}\right).$$

Shrinking the contour $|z|=q^{-1-\epsilon}$ to $|z|=q^{-\frac{3}{2}}$, we do not encounter any poles. Enlarging the contour $|w|=r_2<q^{-1}$ to $|w|=q^{-\frac{1}{4}-\epsilon}$, we encounter two simple poles, one at $w=q^{-1}$ and one at $w=qz$. Thus

$$\mathcal{S}^e_{k,\ell}(V=\square)=\mathcal{A}^e_{k,\ell}+\mathcal{B}^e_{k,\ell}+\mathcal{C}^e_{k,\ell}+O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$\mathcal{A}^e_{k,1}=\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\text{Res}\left(F^e_{k,1}(z,w);w=q^{-1}\right)dz,$$

$$\mathcal{B}^e_{k,1} = \frac{q^{2g+2}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} \operatorname{Res}\left(F^e_{k,1}(z,w); w = qz\right) dz$$

and

$$\mathcal{C}^e_{k,1} = -\frac{q^{2g+2}}{(2\pi i)^2} \oint_{|z|=q^{-\frac{3}{2}}} \oint_{|w|=q^{-\frac{1}{4}-\epsilon}} F^e_{k,1}(z,w) dw dz.$$

We can write down a similar expression for the terms $\mathcal{A}^e_{k,2}$, $\mathcal{B}^e_{k,2}$ and $\mathcal{C}^e_{k,2}$. We evaluate the residues at $w = q^{-1}$ and $w = qz$ in the following way. For example, we have

$$\begin{aligned}
\operatorname{Res}\left(F^e_{g,1}(z,w); w = q^{-1}\right) &= \lim_{w\to q^{-1}} \frac{z^g (w-q^{-1})(q-1)\prod_P \mathcal{B}_P(z,w)}{qw(1-z)(1-qw)(1-q^2zw^2)^2(q^2zw^2)^{[\frac{g}{2}]}} \\
&= -\lim_{w\to q^{-1}} \frac{z^g (1-qw)(q-1)\prod_P \mathcal{B}_P(z,w)}{q^2w(1-z)(1-qw)(1-q^2zw^2)(q^2zw^2)^{[\frac{g}{2}]}} \\
&- \frac{z^g(q-1)\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^3 z^{[\frac{g}{2}]}}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Res}(F^e_{g,1}(z,w); w = qz) &= \lim_{w\to qz} \frac{z^g (w-qz)(q-1)(1-qw^2)\prod_P \mathcal{D}_p(z,w)}{qw(1-z)(1-qw)\left(1-\frac{w}{qz}\right)(1-q^2zw^2)^2(q^2zw^2)^{[\frac{g}{2}]}} \\
&= -\lim_{w\to qz} \frac{qz^{g+1}\left(1-\frac{w}{qz}\right)(q-1)(1-qw^2)\prod_P \mathcal{D}_P(z,w)}{qw(1-z)(1-qw)\left(1-\frac{w}{qz}\right)(1-q^2zw^2)^2(q^2zw^2)^{[\frac{g}{2}]}} \\
&= -\frac{z^g(q-1)(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{q(1-z)(1-q^2z)(1-q^4z^3)^2(q^4z^3)^{[\frac{g}{2}]}}.
\end{aligned}$$

We can evaluate the residues of $F^e_{k,\ell}(z,w)$ at $w = q^{-1}$ and $w = qz$ in a similar way. Furthermore, we use Lemma 3.6.3 to show that $\mathcal{C}^e_{k,\ell} \ll q^{\frac{g}{2}(1+\epsilon)}$. Thus, for each $k \in \{g, g-1\}$ and $\ell \in \{1,2\}$, we have

$$\mathcal{S}^e_{k,\ell}(V = \square) = \mathcal{A}^e_{k,\ell} + \mathcal{B}^e_{k,\ell} + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$\mathcal{A}^e_{g,1} = -\frac{q^{2g+2}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} \frac{z^g(q-1)\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^3 z^{[\frac{g}{2}]}} dz,$$

$$\mathcal{B}^e_{g,1} = -\frac{q^{2g+2}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} \frac{z^g(q-1)(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{q(1-z)(1-q^2z)(1-q^4z^3)^2(q^4z^3)^{[\frac{g}{2}]}} dz,$$

$$\mathcal{A}^e_{g,2} = -\frac{q^{\frac{3g+3}{2}}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} \frac{z^g(q-1)\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^2(1-q^{-1}z)(q^{-1}z)^{[\frac{g}{2}]}} dz,$$

$$\mathcal{B}^e_{g,2} = -\frac{q^{\frac{3g+3}{2}}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} \frac{z^g(q-1)(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{q(1-z)(1-q^2z)(1-q^3z^3)(1-q^4z^3)(q^3z^3)^{[\frac{g}{2}]}} dz,$$

$$\mathcal{A}^e_{g-1,1} = -\frac{q^{2g+2}}{2\pi i} \oint_{|z|=q^{-\frac{3}{2}}} \frac{z^g(q-1)\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^3 z^{[\frac{g-1}{2}]}}$$

$$\mathcal{B}^{e}_{g-1,1} = -\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^{g}(q-1)(1-q^{3}z^{2})\prod_{P}\mathcal{D}_{P}(z,qz)}{q(1-z)(1-q^{2}z)(1-q^{4}z^{3})^{2}(q^{4}z^{3})^{\left[\frac{g-1}{2}\right]}}dz,$$

$$\mathcal{A}^{e}_{g-1,2} = -\frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^{g}(q-1)\prod_{P}\mathcal{B}_{P}(z,q^{-1})}{q(1-z)^{2}(1-q^{-1}z)(q^{-1}z)^{\left[\frac{g-1}{2}\right]}}dz$$

and

$$\mathcal{B}^{e}_{g-1,2} = -\frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^{g}(q-1)(1-q^{3}z^{2})\prod_{P}\mathcal{D}_{P}(z,qz)}{q(1-z)(1-q^{2}z)(1-q^{3}z^{3})(1-q^{4}z^{3})(q^{3}z^{3})^{\left[\frac{g-1}{2}\right]}}dz.$$

Finally, using (3.6.3) proves the Lemma. ∎

3.6.5 Degree f Odd

In this subsection, we will prove the following result.

Lemma 3.6.5. *We have*

$$\mathcal{S}^{o}(V=\square) = \mathcal{A}^{o}_{g,1} - \mathcal{A}^{o}_{g,2} + \mathcal{A}^{o}_{g-1,1} - \mathcal{A}^{o}_{g-1,2} + \mathcal{B}^{o}_{g,1} - \mathcal{B}^{o}_{g,2} + \mathcal{B}^{o}_{g-1,1} - \mathcal{B}^{o}_{g-1,2} + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where $\mathcal{A}^{o}_{k,\ell}$ and $\mathcal{B}^{o}_{k,\ell}$ are the terms stated at the end of the subsection.

Proof. From (3.4.3) and using the function field analogue of Perron's formula, we have

$$S^{o}(l^{2};f,C) = \frac{1}{2\pi i}\oint_{|z|=q^{-1-\epsilon}}\frac{A_{f}(z)z^{g-\frac{1}{2}}(qz-1)}{qz^{\frac{\deg(f)}{2}+\deg(C)}}dz.$$

Using the fact that (see [Flo17a, Proof of Lemma 6.1])

$$\sum_{\substack{C|f^{\infty}\\ C\in\mathbb{A}^{+}_{\leq g}}}\frac{1}{|C|^{2}z^{\deg(C)}} = \prod_{P|f}(1-|P|^{-2}z^{-\deg(P)})^{-1} + O(q^{g(\epsilon-1)}),$$

we have

$$\mathcal{S}^{o}_{k,1}(V=\square) = \frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-1-\epsilon}}\frac{z^{g-\frac{1}{2}}(qz-1)}{q}H^{o}_{k,1}(z)dz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$H^{o}_{k,1}(z) = \sum_{\substack{f\in\mathbb{A}^{+}_{\leq k}\\ \deg(f)\text{ odd}}}\frac{A_{f}(z)}{|f|z^{\frac{\deg(f)}{2}}}\prod_{P|f}(1-|P|^{-2}z^{-\deg(P)})^{-1}.$$

Similarly, we have

$$\mathcal{S}^{o}_{k,2}(V=\square) = \frac{q^{\frac{4g-k}{2}+2}}{2\pi i}\oint_{|z|=q^{-1-\epsilon}}\frac{z^{g-\frac{1}{2}}(qz-1)}{q}H^{o}_{k,2}(z)dz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$H^{o}_{k,2}(z) = \sum_{\substack{f\in\mathbb{A}^+_{\leq k} \\ \deg(f) \text{ odd}}} \frac{A_f(z)}{\sqrt{|f|}z^{\frac{\deg(f)}{2}}} \prod_{P|f}(1-|P|^{-2}z^{-\deg(P)})^{-1}.$$

Using similar methods to that seen in the previous subsection and the function field analogue of Perron's formula, we have

$$\begin{aligned} H^{o}_{g,\ell}(z) = &\frac{1}{2\pi i}\oint_{|w|=r_2} \frac{\mathcal{B}(z,w)}{q^{\frac{3-\ell}{2}}z^{\frac{1}{2}}w^2(1-q^{3-\ell}zw^2)(q^{3-\ell}zw^2)^{[\frac{g-1}{2}]}}dw \\ &-\frac{1}{2\pi i}\oint_{|w|=r_2} \frac{q^{\frac{3-\ell}{2}}z^{\frac{1}{2}}\mathcal{B}(z,w)}{1-q^{3-\ell}zw^2}dw \end{aligned} \tag{3.6.8}$$

and

$$H^{o}_{g-1,\ell}(z) = \frac{1}{2\pi i}\oint_{|w|=r_2} \frac{q^{\frac{3-\ell}{2}}z^{\frac{1}{2}}\mathcal{B}(z,w)}{(1-q^{3-\ell}zw^2)(q^{3-\ell}zw^2)^{[\frac{g}{2}]}}dw - \frac{1}{2\pi i}\oint_{|w|=r_2} \frac{q^{\frac{3-\ell}{2}}z^{\frac{1}{2}}\mathcal{B}(z,w)}{1-q^{3-\ell}zw^2}dw. \tag{3.6.9}$$

For each $\ell \in \{1,2\}$, the second integrals in (3.6.8) and (3.6.9) are zero since the integrands have no poles inside the circle $|w| = r_2 < q^{-1}$. Therefore

$$\begin{aligned} \mathcal{S}^{o}_{k,1}(V=\square) = &\frac{q^{2g+\frac{5}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} \\ &\times \frac{z^{k-(-1)^{g-k}}(qz-1)\mathcal{B}(z,w)}{q^{2(k-g+1)}w^{2(k-g+1)}(1-q^2zw^2)(q^2zw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right) \end{aligned}$$

and

$$\begin{aligned} &\mathcal{S}^{o}_{k,2}(V=\square) \\ &= \frac{q^{\frac{4g-k}{2}+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} \frac{z^{k-(-1)^{g-k}}(qz-1)\mathcal{B}(z,w)}{q^{k-g+\frac{3}{2}}w^{2(k-g+1)}(1-qzw^2)(qzw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right). \end{aligned}$$

Using (3.6.5) from Lemma 3.6.2, we obtain

$$\begin{aligned} \mathcal{S}^{o}_{k,1}(V=\square) = &-\frac{q^{2g+\frac{5}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} \\ &\times \frac{z^{k-(-1)^{g-k}}\prod_P \mathcal{B}_P(z,w)}{q^{2(k-g+1)}w^{2(k-g+1)}(1-qw)(1-q^2zw^2)^2(q^2zw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}^{o}_{k,2}(V=\square) = &-\frac{q^{\frac{4g-k}{2}+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} \\ &\times \frac{z^{k-(-1)^{g-k}}\prod_P \mathcal{B}_P(z,w)}{q^{k-g+\frac{3}{2}}w^{2(k-g+1)}(1-qw)(1-qzw^2)(1-q^2zw^2)(qzw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right). \end{aligned}$$

Using (3.6.6) from Lemma 3.6.3 we obtain

$$\mathcal{S}^o_{k,1}(V=\square) = -\frac{q^{2g+\frac{5}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}$$
$$\times \frac{z^{k-(-1)^{g-k}}(1-qw^2)\prod_P \mathcal{D}_P(z,w)}{q^{2(k-g+1)}w^{2(k-g+1)}(1-qw)\left(1-\frac{w}{qz}\right)(1-q^2zw^2)^2(q^2zw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz + O\left(q^{\frac{g}{2}(1+\epsilon)}\right)$$

and

$$\mathcal{S}^o_{k,2}(V=\square) = -\frac{q^{\frac{4g-k}{2}+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2}$$
$$\times \frac{z^{k-(-1)^{g-k}}(1-qw^2)\prod_P \mathcal{D}_P(z,w)}{q^{k-g+\frac{3}{2}}w^{2(k-g+1)}(1-qw)(1-\frac{w}{qz})(1-qzw^2)(1-q^2zw^2)(qzw^2)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}dwdz$$
$$+ O\left(q^{\frac{g}{2}(1+\epsilon)}\right).$$

For each $\mathcal{S}^o_{k,\ell}(V=\square)$ write

$$\mathcal{S}^o_{k,1}(V=\square) = -\frac{q^{2g+\frac{5}{2}}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} F^o_{k,1}(z,w)dwdz$$

and

$$\mathcal{S}^o_{k,2}(V=\square) = -\frac{q^{\frac{4g-k}{2}+2}}{(2\pi i)^2}\oint_{|z|=q^{-1-\epsilon}}\oint_{|w|=r_2} F^o_{k,2}(z,w)dwdz.$$

Shrinking the contour $|z|=q^{-1-\epsilon}$ to $|z|=q^{-\frac{3}{2}}$ we do not encounter any poles. Enlarging the contour $|w|=r_2<q^{-1}$ to $|w|=q^{-\frac{1}{4}-\epsilon}$, we encounter two simple poles, one at $w=q^{-1}$ and one at $w=qz$. Thus

$$\mathcal{S}^o_{k,\ell}(V=\square) = \mathcal{A}^o_{k,\ell}+\mathcal{B}^o_{k,\ell}+\mathcal{C}^o_{k,\ell}+O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$\mathcal{A}^o_{k,1} = \frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\operatorname{Res}\left(F^o_{k,1}(z,w);w=q^{-1}\right)dz,$$
$$\mathcal{B}^o_{k,1} = \frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\operatorname{Res}\left(F^o_{k,1}(z,w);w=qz\right)dz$$

and

$$\mathcal{C}^o_{k,1} = -\frac{q^{2g+2}}{(2\pi i)^2}\oint_{|z|=q^{-\frac{3}{2}}}\oint_{|w|=q^{-\frac{1}{4}-\epsilon}} F^o_{k,1}(z,w)dwdz.$$

We can write down similar expressions for the terms $\mathcal{A}^o_{k,2}$, $\mathcal{B}^o_{k,2}$ and $\mathcal{C}^o_{k,2}$. We evaluate the residues at $w=q^{-1}$ and $w=qz$ in the following way. For example, we have

$$\begin{aligned}\operatorname{Res}(F^o_{g,1}(z,w);w=q^{-1}) &= \lim_{w\to q^{-1}}\frac{z^{g-1}(w-q^{-1})\prod_P \mathcal{B}_P(z,w)}{q^2w^2(1-qw)(1-q^2w^2)^2(q^2zw^2)^{\left[\frac{g-1}{2}\right]}}\\ &= -\lim_{w\to q^{-1}}\frac{z^{g-1}(1-qw)\prod_P \mathcal{B}_P(z,w)}{q^3w^2(1-qw)(1-q^2zw^2)^2(q^2zw^2)^{\left[\frac{g-1}{2}\right]}}\\ &= -\frac{z^{g-1}\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^2z^{\left[\frac{g-1}{2}\right]}}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Res}(F^o_{g,1}(z,w); w=qz) &= \lim_{w\to qz} \frac{z^{g-1}(w-qz)(1-qw^2)\prod_P \mathcal{D}_P(z,w)}{q^2w^2(1-qw)\left(1-\frac{w}{qz}\right)(1-q^2zw^2)^2(q^2zw^2)^{\left[\frac{g-1}{2}\right]}}\\
&= -\lim_{w\to qz} \frac{z^{g}\left(1-\frac{w}{qz}\right)(1-qw^2)\prod_P \mathcal{D}_P(z,w)}{qw^2(1-qw)\left(1-\frac{w}{qz}\right)(1-q^2zw^2)^2(q^2zw^2)^{\left[\frac{g-1}{2}\right]}}\\
&= -\frac{z^{g-2}(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{q^3(1-q^2z)(1-q^4z^3)^2(q^4z^3)^{\left[\frac{g-1}{2}\right]}}.\end{aligned}$$

We can evaluate the residues of $F^o_{k,\ell}$ at $w=q^{-1}$ and $w=qz$ in a similar way. Furthermore, we can use Lemma 3.6.3 to show that $C^o_{k,\ell} \ll q^{\frac{g}{2}(1+\epsilon)}$. Thus, for each $k\in\{g,g-1\}$ and $\ell\in\{1,2\}$ we have

$$\mathcal{S}^o_{k,\ell} = \mathcal{A}^o_{k,\ell} + \mathcal{B}^o_{k,\ell} + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$\mathcal{A}^o_{g,1} = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g-1}\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^2 z^{\left[\frac{g-1}{2}\right]}}dz,$$

$$\mathcal{B}^o_{g,1} = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g-2}(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{q^3(1-q^2z)(1-q^4z^3)^2(q^4z^3)^{\left[\frac{g-1}{2}\right]}}dz,$$

$$\mathcal{A}^o_{g,2} = -\frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g-1}\prod_P \mathcal{B}_P(z,q^{-1})}{q^{\frac{1}{2}}(1-z)(1-q^{-1}z)(q^{-1}z)^{\left[\frac{g-1}{2}\right]}}dz,$$

$$\mathcal{B}^o_{g,2} = -\frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g-2}(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{q^{\frac{5}{2}}(1-q^2z)(1-q^3z^3)(1-q^4z^3)(q^3z^3)^{\left[\frac{g-1}{2}\right]}}dz,$$

$$\mathcal{A}^o_{g-1,1} = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g}\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)^2 z^{\left[\frac{g}{2}\right]}}dz,$$

$$\mathcal{B}^o_{g-1,1} = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{qz^{g+1}(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{(1-q^2z)(1-q^4z^3)^2(q^4z^3)^{\left[\frac{g}{2}\right]}}dz,$$

$$\mathcal{A}^o_{g-1,2} = -\frac{q^{\frac{3g+5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{z^{g}\prod_P \mathcal{B}_P(z,q^{-1})}{q^{\frac{3}{2}}(1-z)(1-q^{-1}z)(q^{-1}z)^{\left[\frac{g}{2}\right]}}dz,$$

and

$$\mathcal{B}^o_{g-1,2} = -\frac{q^{\frac{3g+5}{2}}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} \frac{q^{\frac{1}{2}}z^{g+1}(1-q^3z^2)\prod_P \mathcal{D}_P(z,qz)}{(1-q^2z)(1-q^3z^3)(1-q^4z^3)(q^3z^3)^{\left[\frac{g}{2}\right]}}dz.$$

Using (3.6.2) proves the Lemma. ∎

3.6.6 Contribution From $\mathcal{A}$ terms

In this subsection, we will focus on evaluating the $\mathcal{A}$ terms which will give the terms $\mathcal{S}_1(V=\square)$, $\mathcal{S}_2(V=\square)$, $\mathcal{S}_3(V=\square)$ and $\mathcal{S}_4(V=\square)$ in Proposition 3.6.1. Let

$$\mathcal{A}=\mathcal{A}_{g,1}^{e}-\mathcal{A}_{g,2}^{e}+\mathcal{A}_{g-1,1}^{e}-\mathcal{A}_{g-1,2}^{e}+\mathcal{A}_{g,1}^{o}-\mathcal{A}_{g,2}^{o}+\mathcal{A}_{g-1,1}^{o}-\mathcal{A}_{g-1,2}^{o}, \tag{3.6.10}$$

then the main result in this subsection is the following lemma.

Lemma 3.6.6. *Using the same notation as before, we have*

$$\mathcal{A}=\mathcal{S}_1(V=\square)+\mathcal{S}_2(V=\square)+\mathcal{S}_3(V=\square)+\mathcal{S}_4(V=\square), \tag{3.6.11}$$

where, in particular, the terms $\mathcal{S}_1(V=\square)$, $\mathcal{S}_2(V=\square)$, $\mathcal{S}_3(V=\square)$ and $\mathcal{S}_4(V=\square)$ are the terms stated in Proposition 3.6.1.

Proof. For each $k\in\{g,g-1\}$, write

$$\mathcal{A}_{k,1}^{e}=-\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^{g}\left(q-\frac{1}{z}+\frac{1}{z}-1\right)\prod_P\mathcal{B}_P(z,q^{-1})}{q(1-z)^3z^{\left[\frac{k}{2}\right]}}dz.$$

Let $\mathcal{A}_{k,1}^{e}=\mathcal{A}_{k,1,1}^{e}+\mathcal{A}_{k,1,2}^{e}$, where

$$\mathcal{A}_{k,1,1}^{e}=-\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^{g}\left(1-\frac{1}{qz}\right)\prod_P\mathcal{B}_P(z,q^{-1})}{(1-z)^3z^{\left[\frac{k}{2}\right]}}dz$$

and

$$\mathcal{A}_{k,1,2}^{e}=-\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}}\frac{z^{g-1}\prod_P\mathcal{B}_P(z,q^{-1})}{q(1-z)^2z^{\left[\frac{k}{2}\right]}}dz.$$

After the change of variables $z=(qu)^{-1}$, the contour of integration becomes a circle around the origin $|u|=\sqrt{q}$. Note that, from Lemma 3.6.2, $\prod_P\mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)$ is absolutely convergent for $q^{-1}<|u|<q$. Thus

$$\mathcal{A}_{g,1,1}^{e}=-\frac{q^{2g+2}}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{(1-u)\prod_P\mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)\left(1-\frac{1}{qu}\right)^{-1}}{u(1-qu)^2(qu)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}du.$$

Using the fact (see [Flo17a, Section 6]) that

$$(1-u)\prod_P\mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)\left(1-\frac{1}{qu}\right)^{-1}=\frac{\mathcal{C}(u)}{\zeta_{\mathbb{A}}(2)}, \tag{3.6.12}$$

we get

$$\mathcal{A}_{g,1,1}^{e}=-\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{\left[\frac{g-1}{2}\right]}}du \tag{3.6.13}$$

and

$$\mathcal{A}^{e}_{g-1,1,1} = -\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{[\frac{g}{2}]}}du. \tag{3.6.14}$$

We see that (3.6.13) and (3.6.14) are precisely the terms $\mathcal{S}_1(V=\square)$ and $\mathcal{S}_2(V=\square)$ in the statement of Lemma 3.6.6. Similarly, using the substitution $z=(qu)^{-1}$, we have

$$\mathcal{A}^{e}_{k,1,2} = -\frac{q^{2g+2}}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(1-qu)^2(qu)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}du,$$

and

$$\mathcal{A}^{o}_{k,1} = -\frac{q^{2g+\frac{5}{2}}}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(qu)^{g-k}(1-qu)^2(qu)^{[\frac{k}{2}]}}du.$$

Using (3.6.12), we have

$$\mathcal{A}^{e}_{k,1,2} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{qu(1-u)(1-qu)(qu)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}du$$

and

$$\mathcal{A}^{o}_{k,1} = \frac{q^{2g+\frac{5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{(qu)^{g-k+1}(1-u)(1-qu)(qu)^{[\frac{k}{2}]}}du.$$

Rewrite $\mathcal{A}^{o}_{g-1,1}$ as

$$\mathcal{A}^{o}_{g-1,1} = \frac{q^{2g+\frac{5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)(1-qu+qu)}{q^2u^2(1-u)(1-qu)(qu)^{[\frac{g-1}{2}]}}du.$$

Then, we let $\mathcal{A}^{o}_{g-1,1} = \mathcal{A}^{o}_{g-1,1,1} + \mathcal{A}^{o}_{g-1,1,2}$, where

$$\mathcal{A}^{o}_{g-1,1,1} = \frac{q^{2g+\frac{5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{qu(1-u)(1-qu)(qu)^{[\frac{g-1}{2}]}}du$$

and

$$\mathcal{A}^{o}_{g-1,1,2} = \frac{q^{2g+\frac{5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{(1-u)(qu)^{[\frac{g-1}{2}]+2}}du.$$

Combining $\mathcal{A}^{o}_{g,1}$ and $\mathcal{A}^{e}_{g-1,1,2}$, we have

$$\mathcal{A}^{o}_{g,1} + \mathcal{A}^{e}_{g-1,1,2} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)(1+q^{\frac{1}{2}})}{qu(1-u)(1-qu)(qu)^{[\frac{g}{2}]}}du.$$

Using the fact that (see [Jun13, Proof of Main Theorem])

$$1+q^{\frac{1}{2}} = q^{-\frac{g-1}{2}+[\frac{g}{2}]} + q^{-\frac{g}{2}+[\frac{g-1}{2}]+1}, \tag{3.6.15}$$

we have

$$\mathcal{A}^{o}_{g,1} + \mathcal{A}^{e}_{g-1,1,2} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)\left(q^{-\frac{g-1}{2}+[\frac{g}{2}]} + q^{-\frac{g}{2}+[\frac{g-1}{2}]+1}\right)}{qu(1-u)(1-qu)(qu)^{[\frac{g}{2}]}}du.$$

Let $\mathcal{A}^o_{g,1}+\mathcal{A}^e_{g-1,1,2}=\hat{\mathcal{A}}_1+\hat{\mathcal{A}}_2$, where

$$\hat{\mathcal{A}}_1=\frac{q^{\frac{3g+3}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{[\frac{g}{2}]}}du \tag{3.6.16}$$

and

$$\hat{\mathcal{A}}_2=\frac{q^{\frac{3g}{2}+3+[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{(1-u)(1-qu)(qu)^{[\frac{g}{2}]+1}}du.$$

Similarly, combining $\mathcal{A}^o_{g-1,1,1}+\mathcal{A}^e_{g,1,2}$, and using (3.6.15), we have

$$\mathcal{A}^o_{g-1,1,1}+\mathcal{A}^e_{g,1,2}=\tilde{\mathcal{A}}_1+\tilde{\mathcal{A}}_2,$$

where

$$\tilde{\mathcal{A}}_1=\frac{q^{\frac{3g}{2}+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{[\frac{g-1}{2}]}}du$$

and

$$\tilde{\mathcal{A}}_2=\frac{q^{\frac{3g+5}{2}+[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{\mathcal{C}(u)}{(1-u)(1-qu)(qu)^{[\frac{g-1}{2}]+1}}du.$$

We see that $\hat{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_1$ are precisely the terms $\mathcal{S}_3(V=\square)$ and $\mathcal{S}_4(V=\square)$ given in the statement of Lemma 3.6.6. From (3.5.4), we see that $\mathcal{C}(1)=0$, thus inside the circle $|u|=\sqrt{q}$, the integrand of $\mathcal{A}^o_{g-1,1,2}$ has a pole of order $\left[\frac{g-1}{2}\right]+2$ at $u=0$. Using the Residue Theorem and calculations of residues seen in the proof of Lemma 3.6.4 and Lemma 3.6.5 we have

$$\mathcal{A}^o_{g-1,1,2}=\frac{q^{2g+\frac{1}{2}-[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\sum_{n=0}^{[\frac{g-1}{2}]+1}\frac{\mathcal{C}^{(n)}(0)}{n!}.$$

Similarly, inside the circle $|u|=\sqrt{q}$, the integrands of $\hat{\mathcal{A}}_2$ and $\tilde{\mathcal{A}}_2$ have a simple pole at $u=q^{-1}$ and a pole at $u=0$ of order $\left[\frac{g}{2}\right]+1$ and $\left[\frac{g-1}{2}\right]+1$ respectively. Thus we have

$$\hat{\mathcal{A}}_2=\frac{q^{\frac{5g}{2}+1-2[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)}\sum_{n=0}^{[\frac{g}{2}]}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{[\frac{g}{2}]-n}q^k-\frac{q^{\frac{3g}{2}+3+[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{(q-1)}$$

and

$$\tilde{\mathcal{A}}_2=\frac{q^{\frac{5g+1}{2}-2[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\sum_{n=0}^{[\frac{g-1}{2}]}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{[\frac{g-1}{2}]-n}q^k-\frac{q^{\frac{3g+5}{2}+[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{(q-1)}.$$

For the remaining integrals, we write

$$\mathcal{A}^e_{k,2}=\mathcal{A}^e_{k,2,1}+\mathcal{A}^e_{k,2,2},$$

where

$$\mathcal{A}^e_{k,2,1}=-\frac{q^{\frac{4g-k+3}{2}}}{2\pi i}\oint_{|u|=\sqrt{q}}\frac{z^g\left(1-\frac{1}{qz}\right)\prod_P\mathcal{B}_P(z,q^{-1})}{(1-z)^2(1-q^{-1}z)(q^{-1}z)^{[\frac{k}{2}]}}dz$$

and

$$\mathcal{A}^e_{k,2,2} = -\frac{q^{\frac{4g-k+3}{2}}}{2\pi i}\oint_{|u|=\sqrt{q}} \frac{z^{g-1}\prod_P \mathcal{B}_P(z,q^{-1})}{q(1-z)(1-q^{-1}z)(q^{-1}z)^{[\frac{k}{2}]}}dz.$$

Using the substitution $z = (qu)^{-1}$, we have

$$\mathcal{A}^e_{k,2,1} = \frac{q^{\frac{6g-k+5}{2}}}{2\pi i}\oint_{|u|=\sqrt{q}} \frac{(1-u)\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(1-qu)^2(1-q^2u)(q^2u)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}du,$$

$$\mathcal{A}^e_{k,2,2} = -\frac{q^{\frac{6g-k+3}{2}}}{2\pi i}\oint_{|u|=\sqrt{q}} \frac{\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(1-qu)(1-q^2u)(q^2u)^{\left[\frac{k-(-1)^{g-k}}{2}\right]}}du$$

and

$$\mathcal{A}^o_{k,2} = -\frac{q^{\frac{6g-k+5}{2}}}{2\pi i}\oint_{|u|=\sqrt{q}} \frac{\prod_P \mathcal{B}_P\left(\frac{1}{qu},\frac{1}{q}\right)}{(q^2u)^{g-k}(1-qu)(1-q^2u)(q^2u)^{[\frac{k}{2}]}}du.$$

Using (3.6.12), we have

$$\mathcal{A}^e_{k,2,1} = -\frac{q^{\frac{6g-k+7}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}} \frac{\mathcal{C}(u)}{(1-qu)(1-q^2u)(q^2u)^{\left[\frac{k-(-1)^{g-k}}{2}\right]+1}}du,$$

$$\mathcal{A}^e_{k,2,2} = \frac{q^{\frac{6g-k+5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}} \frac{\mathcal{C}(u)}{(1-u)(1-q^2u)(q^2u)^{\left[\frac{k-(-1)^{g-k}}{2}\right]+1}}du$$

and

$$\mathcal{A}^o_{k,2} = \frac{q^{\frac{6g-k+7}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=\sqrt{q}} \frac{\mathcal{C}(u)}{(q^2u)^{g-k}(1-u)(1-q^2u)(q^2u)^{[\frac{k}{2}]+1}}du.$$

Inside the circle $|u| = \sqrt{q}$, the integrands have poles at $u = 0$, $u = q^{-1}$ and $u = q^{-2}$ of varying orders. Thus using the Residue Theorem we have

$$\mathcal{A}^e_{g,2,1} = -\frac{q^{\frac{5g+3}{2}-2[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\sum_{n=0}^{[\frac{g-1}{2}]}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=[\frac{g-1}{2}]-n}^{2([\frac{g-1}{2}]-n)} q^k - \frac{q^{\frac{5g+3}{2}-[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{q-1} + \frac{q^{\frac{5g+5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q-1},$$

$$\mathcal{A}^e_{g,2,2} = \frac{q^{\frac{5g+1}{2}-2[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\sum_{n=0}^{[\frac{g-1}{2}]}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{[\frac{g-1}{2}]-n} q^{2k} - \frac{q^{\frac{5g+5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q^2-1},$$

$$\mathcal{A}^e_{g-1,2,1} = -\frac{q^{\frac{5g}{2}+2-2[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\sum_{n=0}^{[\frac{g}{2}]}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=[\frac{g}{2}]-n}^{2([\frac{g}{2}]-n)} q^k - \frac{q^{\frac{5g}{2}+2-[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{q-1} + \frac{q^{\frac{5g}{2}+3}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q-1},$$

$$\mathcal{A}^e_{g-1,2,2} = \frac{q^{\frac{5g}{2}+1-2[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)}\sum_{n=0}^{[\frac{g}{2}]}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{[\frac{g}{2}]-n} q^{2k} - \frac{q^{\frac{5g}{2}+3}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q^2-1},$$

$$\mathcal{A}^o_{g,2} = \frac{q^{\frac{5g+3}{2}-2[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)}\sum_{n=0}^{[\frac{g}{2}]}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{[\frac{g}{2}]-n} q^{2k} - \frac{q^{\frac{5g+7}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q^2-1}$$

and

$$\mathcal{A}^o_{g-1,2} = \frac{q^{\frac{5g}{2}-2[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)} \sum_{n=0}^{[\frac{g-1}{2}]+1} \frac{\mathcal{C}^{(n)}(0)}{n!} \sum_{k=0}^{[\frac{g-1}{2}]+1-n} q^{2k} - \frac{q^{\frac{5g}{2}+4}}{\zeta_{\mathbb{A}}(2)} \frac{\mathcal{C}(q^{-2})}{q^2-1}.$$

Therefore combining everything seen in this proof with (3.6.10), we have

$$\begin{aligned}\mathcal{A} &= \mathcal{S}_1(V=\square) + \mathcal{S}_2(V=\square) + \mathcal{S}_3(V=\square) + \mathcal{S}_4(V=\square)\\ &\quad + \mathcal{A}^o_{g-1,1,2} + \hat{\mathcal{A}}_2 + \tilde{\mathcal{A}}_2 - \mathcal{A}^e_{g,2,1} - \mathcal{A}^e_{g,2,2} - \mathcal{A}^e_{g-1,2,1} - \mathcal{A}^e_{g-1,2,2} - \mathcal{A}^o_{g,2} - \mathcal{A}^o_{g-1,2}\end{aligned}$$

Thus, to complete the proof, we want to show that

$$\mathcal{A}^o_{g-1,1,2} + \hat{\mathcal{A}}_2 + \tilde{\mathcal{A}}_2 - \mathcal{A}^e_{g,2,1} - \mathcal{A}^e_{g,2,2} - \mathcal{A}^e_{g-1,2,1} - \mathcal{A}^e_{g-1,2,2} - \mathcal{A}^o_{g,2} - \mathcal{A}^o_{g-1,2} \tag{3.6.17}$$

equals zero. For the terms corresponding to the residue at $u = q^{-2}$, we have that (3.6.17) is equal to

$$-\frac{q^{\frac{5g+5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q-1} + \frac{q^{\frac{5g+5}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q^2-1} - \frac{q^{\frac{5g}{2}+3}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q-1} + \frac{q^{\frac{5g}{2}+3}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q^2-1} + \frac{q^{\frac{5g+7}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q^2-1} + \frac{q^{\frac{5g}{2}+4}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-2})}{q^2-1},$$

which clearly equals zero. For the terms corresponding to the residue at $u = q^{-1}$, we have that (3.6.17) is equal to

$$-\frac{q^{\frac{3g}{2}+3+[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{(q-1)} - \frac{q^{\frac{3g+5}{2}+[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{(q-1)} + \frac{q^{\frac{5g+3}{2}-[\frac{g-1}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{q-1} + \frac{q^{\frac{5g}{2}+2-[\frac{g}{2}]}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{q-1}. \tag{3.6.18}$$

Rearranging (3.6.18), we see that it is equal to

$$\frac{q^{\frac{3g}{2}+2}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{(q-1)}\left(q^{g-[\frac{g}{2}]} - q^{[\frac{g-1}{2}]+1}\right) + \frac{q^{\frac{3g+3}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{\mathcal{C}(q^{-1})}{(q-1)}\left(q^{g-[\frac{g-1}{2}]} - q^{[\frac{g}{2}]+1}\right). \tag{3.6.19}$$

Using the fact that (see [Jun14, Section 1])

$$q^{g-[\frac{g-1}{2}]} - q^{[\frac{g}{2}]+1} = 0 \quad \text{and} \quad q^{g-[\frac{g}{2}]} - q^{[\frac{g-1}{2}]+1} = 0 \tag{3.6.20}$$

we see that (3.6.19) is equal to zero. Finally, in Appendix A, we show that the terms corresponding to the residue at $u = 0$ is equals zero. Thus (3.6.17) equals zero which completes the proof of Lemma 3.6.6. ∎

3.6.7 Contribution from $\mathcal{B}$ terms

In this subsection, we will focus on evaluating the $\mathcal{B}$ terms, which will give the secondary main term of Proposition 3.6.1. Let

$$\mathcal{B} = \mathcal{B}^e_{g,1} - \mathcal{B}^e_{g,2} + \mathcal{B}^e_{g-1,1} - \mathcal{B}^e_{g-1,2} + \mathcal{B}^o_{g,1} - \mathcal{B}^o_{g,2} + \mathcal{B}^o_{g-1,1} - \mathcal{B}^o_{g-1,2} \tag{3.6.21}$$

Then, the main result in this subsection is the following lemma.

Lemma 3.6.7. *Using the same notion as before, we have*

$$\mathcal{B} = q^{\frac{2g+2}{3}}\mathcal{R}(2g+2) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right), \tag{3.6.22}$$

where $\mathcal{R}$ is a polynomial of degree 1 which can be explicitly be calculated.

Proof. From Section 3.6.4 and Section 3.6.5, we can write each $\mathcal{B}_{k,\ell}^j$ as

$$\mathcal{B}_{k,1}^j = -\frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} F_{k,1}^j(z)dz \quad \text{and} \quad \mathcal{B}_{k,2}^j = -\frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-\frac{3}{2}}} F_{k,2}^j(z)dz,$$

where each $F_{k,\ell}^j(z)$ correspond to the integrands of $\mathcal{B}_{k,\ell}^j$ that are stated in Section 3.6.4 and Section 3.6.5. Enlarging the contour $|z| = q^{-\frac{3}{2}}$ to $|z| = q^{-1-\epsilon}$ we encounter a double pole at $z = q^{-\frac{4}{3}}$ of $F_{k,1}^j(z)$ and a simple pole at $z = q^{-\frac{4}{3}}$ of $F_{k,2}^j(z)$. From Lemma 3.6.3, $\prod_P \mathcal{D}_P(z, qz)$ is absolutely convergent when $q^{-2} < |z| < q^{-1}$. Then

$$\mathcal{B}_{k,1}^j = q^{2g+2}\text{Res}\left(F_{k,1}^j(z); z = q^{-\frac{4}{3}}\right) - \frac{q^{2g+2}}{2\pi i}\oint_{|z|=q^{-1-\epsilon}} F_{k,1}^j(z)dz \tag{3.6.23}$$

and

$$\mathcal{B}_{k,2}^j = q^{\frac{3g}{2}+2}\text{Res}\left(F_{k,2}^j(z); z = q^{-\frac{4}{3}}\right) - \frac{q^{\frac{3g}{2}+2}}{2\pi i}\oint_{|z|=q^{-1-\epsilon}} F_{k,2}^j(z)dz, \tag{3.6.24}$$

where the second terms in (3.6.23) and (3.6.24) are bounded by $O\left(q^{\frac{g}{2}(1+\epsilon)}\right)$. Computing the residues, we see that

$$\mathcal{B}_{g,1}^e = q^{\frac{2g+2}{3}}Q_1(g) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right), \quad \mathcal{B}_{g-1,1}^e = q^{\frac{2g+2}{3}}Q_2(g) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$
$$\mathcal{B}_{g,1}^o = q^{\frac{2g+2}{3}}Q_3(g) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right), \quad \mathcal{B}_{g-1,1}^o = q^{\frac{2g+2}{3}}Q_4(g) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$Q_1(g) = \frac{(q-1)\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)}{9q^{\frac{4}{3}}\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\prod_P \mathcal{D}_P\left(q^{-\frac{4}{3}}, q^{-\frac{1}{3}}\right)$$
$$\times\left(\left(g - 3\left[\frac{g}{2}\right]\right) + \frac{1}{q^{\frac{4}{3}}}\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)\left(3 + 2q^{\frac{1}{3}} + q^{\frac{2}{3}} + 2q - q^{\frac{4}{3}}\right) + \frac{1}{q^{\frac{4}{3}}}\frac{\frac{d}{dz}\prod_P \mathcal{D}_P(z, qz)}{\prod_P \mathcal{D}_P(z, qz)}\Bigg|_{z=q^{-\frac{4}{3}}}\right),$$

$$Q_2(g) = \frac{(q-1)\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)}{9q^{\frac{4}{3}}\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\prod_P \mathcal{D}_P\left(q^{-\frac{4}{3}}, q^{-\frac{1}{3}}\right)$$
$$\times\left(\left(g - 3\left[\frac{g-1}{2}\right]\right) + \frac{1}{q^{\frac{4}{3}}}\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)\left(3 + 2q^{\frac{1}{3}} + q^{\frac{2}{3}} + 2q - q^{\frac{4}{3}}\right) + \frac{1}{q^{\frac{4}{3}}}\frac{\frac{d}{dz}\prod_P \mathcal{D}_P(z, qz)}{\prod_P \mathcal{D}_P(z, qz)}\Bigg|_{z=q^{-\frac{4}{3}}}\right),$$

$$Q_3(g) = \frac{\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)}{9q^{\frac{1}{6}}\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\prod_P \mathcal{D}_P\left(q^{-\frac{4}{3}}, q^{-\frac{1}{3}}\right)$$
$$\times\left(\left(g - 3\left[\frac{g-1}{2}\right]\right) + \frac{1}{q^{\frac{2}{3}}}\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\left(4 + 2q^{\frac{1}{3}} - 3q^{\frac{2}{3}}\right) + \frac{1}{q^{\frac{4}{3}}}\frac{\frac{d}{dz}\prod_P \mathcal{D}_P(z, qz)}{\prod_P \mathcal{D}_P(z, qz)}\Bigg|_{z=q^{-\frac{4}{3}}}\right)$$

and

$$Q_4(g)=\frac{\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)}{9q^{\frac{1}{6}}\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\prod_P\mathcal{D}_P\left(q^{-\frac{4}{3}},q^{-\frac{1}{3}}\right)$$
$$\times\left(\left(g-3\left[\frac{g}{2}\right]\right)+\frac{1}{q^{\frac{2}{3}}}\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\left(1+2q^{\frac{1}{3}}\right)+\frac{1}{q^{\frac{4}{3}}}\frac{\frac{d}{dz}\prod_P\mathcal{D}_P(z,qz)}{\prod_P\mathcal{D}_P(z,qz)}\bigg|_{z=q^{-\frac{4}{3}}}\right).$$

Let

$$Q(2g+2)=Q_1(g)+Q_2(g)+Q_3(g)+Q_4(g),$$

then

$$Q(x)=\frac{\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)}{9q^{\frac{4}{3}}\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\prod_P\mathcal{D}_P\left(q^{-\frac{4}{3}},q^{-\frac{1}{3}}\right)\left(\frac{x}{2}C_1+C_2-\frac{2C_1}{q^{\frac{4}{3}}}\frac{\frac{d}{dz}\prod_P\mathcal{D}_P(z,qz)}{\prod_P\mathcal{D}_P(z,qz)}\bigg|_{z=q^{-\frac{4}{3}}}\right),$$

where

$$C_1=1-q-q^{\frac{7}{6}}+q^{-\frac{1}{6}}$$

and

$$C_2=\frac{2}{q^{\frac{4}{3}}}\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)(q-1)\left(3+2q^{\frac{1}{3}}+q^{\frac{2}{3}}+2q-q^{\frac{4}{3}}\right)+\frac{1}{q^{\frac{2}{3}}}\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\left(q^{\frac{7}{6}}-q^{-\frac{1}{6}}\right)\left(5+4q^{\frac{1}{3}}-3q^{\frac{2}{3}}\right)-4C_1.$$

Similarly, computing the residue, we see that

$$\mathcal{B}^e_{g,2}=-q^{\frac{g}{6}+\left[\frac{g}{2}\right]}C^e_g+O\left(q^{\frac{g}{2}(1+\epsilon)}\right),\quad \mathcal{B}^e_{g-1,2}=-q^{\frac{g}{6}+\left[\frac{g-1}{2}\right]}C^e_{g-1}+O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$
$$\mathcal{B}^o_{g,2}=-q^{\frac{g}{6}+\left[\frac{g-1}{2}\right]}C^o_g+O\left(q^{\frac{g}{2}(1+\epsilon)}\right)),\quad \mathcal{B}^o_{g-1,2}=-q^{\frac{g}{6}+\left[\frac{g}{2}\right]}C^o_{g-1}+O\left(q^{\frac{g}{2}(1+\epsilon)}\right),$$

where

$$C^e_g=\frac{(q-1)\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)\zeta_{\mathbb{A}}(2)}{3q^{\frac{7}{6}}\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\prod_P\mathcal{D}_P\left(q^{-\frac{4}{3}},q^{-\frac{1}{3}}\right),$$

$$C^e_{g-1}=\frac{(q-1)\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)\zeta_{\mathbb{A}}(2)}{3q^{\frac{2}{3}}\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\prod_P\mathcal{D}_P\left(q^{-\frac{4}{3}},q^{-\frac{1}{3}}\right),$$

$$C^o_g=\frac{q^{\frac{1}{2}}\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\zeta_{\mathbb{A}}(2)}{3\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\prod_P\mathcal{D}_P\left(q^{-\frac{4}{3}},q^{-\frac{1}{3}}\right)$$

and

$$C^o_{g-1}=\frac{\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\zeta_{\mathbb{A}}(2)}{3\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\prod_P\mathcal{D}_P\left(q^{-\frac{4}{3}},q^{-\frac{1}{3}}\right).$$

Let

$$C_3=C^e_g+C^o_{g-1}\quad\text{and}\quad C_4=C^o_g+C^e_{g-1},$$

then

$$C_3=\frac{\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\left(1+q^{-\frac{1}{6}}-q^{-\frac{7}{6}}-q^{-\frac{4}{3}}\right)\prod_P\mathcal{D}_P\left(q^{-\frac{4}{3}},q^{-\frac{1}{3}}\right)$$

and

$$C_4 = \frac{\zeta_{\mathbb{A}}\left(\frac{5}{3}\right)\zeta_{\mathbb{A}}\left(\frac{7}{3}\right)\zeta_{\mathbb{A}}(2)}{\zeta_{\mathbb{A}}\left(\frac{4}{3}\right)}\left(q^{\frac{1}{3}} + q^{\frac{1}{2}} - q^{-\frac{2}{3}} - q^{-\frac{5}{6}}\right)\prod_P \mathcal{D}_P\left(q^{-\frac{4}{3}}, q^{-\frac{1}{3}}\right).$$

Moreover

$$\prod_P \mathcal{D}_P\left(q^{-\frac{4}{3}}, q^{-\frac{1}{3}}\right) = \prod_P\left(1 - \frac{|P|^{\frac{4}{3}} + |P|^{\frac{2}{3}} + |P|^{\frac{1}{3}} + 1}{(|P|^{\frac{4}{3}} + |P|)^2}\right)$$

and

$$\frac{1}{q^{\frac{4}{3}}}\frac{\frac{d}{dz}\prod_P \mathcal{D}_P(z,qz)}{\prod_P \mathcal{D}_P(z,qz)}\Bigg|_{z=q^{-\frac{4}{3}}} = -\sum_P \frac{\deg(P)(|P|-1)\left(|P|^{\frac{1}{3}}-1\right)}{\left(|P|^{\frac{1}{3}}-1\right)\left(|P|^{\frac{4}{3}}+|P|\right)^2}.$$

Thus, combining the above with (3.6.21), we have that

$$\mathcal{B} = q^{\frac{2g+2}{3}}Q(2g+2) + C_3 q^{\frac{g}{6}+\left[\frac{g}{2}\right]} + C_4 q^{\frac{g}{6}+\left[\frac{g-1}{2}\right]} + O\left(q^{\frac{g}{2}(1+\epsilon)}\right).$$

Letting

$$q^{\frac{2g+2}{3}}\mathcal{R}(2g+2) = q^{\frac{2g+2}{3}}Q(2g+2) + C_3 q^{\frac{g}{6}+\left[\frac{g}{2}\right]} + C_4 q^{\frac{g}{6}+\left[\frac{g-1}{2}\right]} \tag{3.6.25}$$

completes the proof of Lemma 3.6.7. ■

3.7 Error from non-square V

Let

$$\mathcal{S}(V \neq \square) = \mathcal{S}^o(V \neq \square) + \mathcal{S}^e(V \neq \square) \tag{3.7.1}$$

where

$$\mathcal{S}^o(V \neq \square) = \mathcal{S}^o_{g,1}(V \neq \square) - \mathcal{S}^o_{g,2}(V \neq \square) + \mathcal{S}^o_{g-1,1}(V \neq \square) - \mathcal{S}^o_{g-1,2}(V \neq \square) \tag{3.7.2}$$

and

$$\mathcal{S}^e(V \neq \square) = \mathcal{S}^e_{g,1}(V \neq \square) - \mathcal{S}^e_{g,2}(V \neq \square) + \mathcal{S}^e_{g-1,1}(V \neq \square) - \mathcal{S}^e_{g-1,2}(V \neq \square). \tag{3.7.3}$$

In this section, we will bound the term $\mathcal{S}(V \neq \square)$. The next proposition is the main result in this section.

Proposition 3.7.1. *Using the same notation described previously, we have, for any* $\epsilon > 0$,

$$\mathcal{S}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}. \tag{3.7.4}$$

To prove Proposition 3.7.1, we will need the following results from [Flo17a, Section 7]. We have

$$\sum_{\substack{C|f^\infty \\ C\in\mathbb{A}_m^+}} \frac{1}{|C|^2} = \frac{1}{2\pi i}\oint_{|u|=r_1} \frac{1}{q^{2m}u^{m+1}\prod_{P|f}(1-u^{\deg(P)})}du \tag{3.7.5}$$

with $r_1 < 1$. For a non-square $V \in \mathbb{A}^+$ and positive integer n, let

$$\delta_{V;n}(n) = \sum_{f \in \mathbb{A}_n^+} \frac{G(V,\chi_f)}{\sqrt{|f|}\prod_{P|f}(1-u^{\deg(P)})}.$$

If $|u| = q^{-\epsilon}$, then we have

$$|\delta_{V;n}(u)| \ll q^{\frac{n}{2}(1+\epsilon)}|V|^{\epsilon}. \tag{3.7.6}$$

3.7.1 Bounding $\mathcal{S}^e(V \neq \square)$

For each $k \in \{g, g-1\}$ and $\ell \in \{1,2\}$, we have

$$\mathcal{S}^e_{k,\ell}(V \neq \square) = \mathcal{S}^e_{k,\ell,1}(V \neq \square) + \mathcal{S}^e_{k,\ell,2}(V \neq \square).$$

For each $j \in \{1,2\}$, write

$$\mathcal{S}^e_{k,\ell,j}(V \neq \square) = \tilde{\mathcal{S}}^e_{k,\ell,j}(V \neq \square) - \hat{\mathcal{S}}^e_{k,\ell,j}(V \neq \square),$$

where $\tilde{\mathcal{S}}^e_{k,\ell,1}(V \neq \square)$ and $\hat{\mathcal{S}}^e_{k,\ell,1}(V \neq \square)$ denote the sums over non-square V with $\deg(V) \leq \deg(f) - 2g - 4 + 2\deg(C)$ and $\deg(V) \leq \deg(f) - 2g - 2 + 2\deg(C)$ respectively. Similarly, $\tilde{\mathcal{S}}^e_{k,\ell,2}(V \neq \square)$ and $\hat{\mathcal{S}}^e_{k,\ell,2}(V \neq \square)$ denotes the sums over non-square V with $\deg(V) = \deg(f) - 2g - 1 + 2\deg(C)$ and $\deg(V) = \deg(f) - 2g - 3 + 2\deg(C)$ respectively. Then by (3.7.5), we have

$$\tilde{\mathcal{S}}^e_{g,1,1}(V \neq \square) = \frac{(q-1)q^{2g+2}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g}{2}]} \frac{1}{q^{2n}} \sum_{m=g-n+2}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{\square \neq V \in \mathbb{A}^+_{\leq 2n-2g-4+2m}} \delta_{V;2n}(u)du,$$

$$\hat{\mathcal{S}}^e_{g,1,1}(V \neq \square) = \frac{(q-1)q^{2g+1}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g}{2}]} \frac{1}{q^{2n}} \sum_{m=g-n+1}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{\square \neq V \in \mathbb{A}^+_{\leq 2n-2g-2+2m}} \delta_{V;2n}(u)du,$$

$$\tilde{\mathcal{S}}^e_{g,1,2}(V \neq \square) = \frac{q^{2g+1}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g}{2}]} \frac{1}{q^{2n}} \sum_{m=g-n+1}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{\square \neq V \in \mathbb{A}^+_{2n-2g-1+2m}} \delta_{V;2n}(u)du$$

and

$$\hat{\mathcal{S}}^e_{g,1,2}(V \neq \square) = \frac{q^{2g+2}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g}{2}]} \frac{1}{q^{2n}} \sum_{m=g-n+2}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{\square \neq V \in \mathbb{A}^+_{2n-2g-3+2m}} \delta_{V;2n}(u)du,$$

with $r_1 < 1$. We can bound $\delta_{V;2n}(u)$ by (3.7.6) and the sum over V by using the fact that $\#\mathbb{A}_n^+ = q^n$. Thus we get that $\tilde{\mathcal{S}}^e_{g,1,1}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$, $\hat{\mathcal{S}}^e_{g,1,1}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$, $\tilde{\mathcal{S}}^e_{g,1,2}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$ and $\hat{\mathcal{S}}^e_{g,1,2}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$ and so $\mathcal{S}^e_{g,1}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$. Using similar calculations, we can bound $\mathcal{S}^e_{g,2}(V \neq \square)$, $\mathcal{S}^e_{g-1,1}(V \neq \square)$ and $\mathcal{S}^e_{g-1,2}(V \neq \square)$ by $q^{\frac{g}{2}(1+\epsilon)}$ and so by (3.7.3) we have

$$\mathcal{S}^e(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}. \tag{3.7.7}$$

3.7.2 Bounding $\mathcal{S}^o(V \neq \square)$

For $k \in \{g, g-1\}$ and $\ell \in \{1, 2\}$, let

$$\mathcal{S}^o_{k,\ell}(V \neq \square) = \tilde{\mathcal{S}}^o_{k,\ell}(V \neq \square) - \hat{\mathcal{S}}^o_{k,\ell}(V \neq \square),$$

where $\tilde{\mathcal{S}}^o_{k,\ell}(V \neq \square)$ and $\hat{\mathcal{S}}^o_{k,\ell}(V \neq \square)$ denotes the sums over non-square V with $\deg(V) = \deg(f) - 2g - 3 + 2\deg(C)$ and $\deg(V) = \deg(f) - 2g - 1 + 2\deg(C)$ respectively. Then using (3.7.5), we have

$$\tilde{\mathcal{S}}^o_{g,1}(V \neq \square) = \frac{q^{2g+\frac{5}{2}}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g-1}{2}]} \frac{1}{q^{2n+1}} \sum_{m=g-n+1}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{\square \neq V \in \mathbb{A}^+_{2n-2g-2+2m}} \delta_{V;2n+1}(u) du$$

and

$$\hat{\mathcal{S}}^o_{g,1}(V \neq \square) = \frac{q^{2g+\frac{3}{2}}}{2\pi i} \oint_{|u|=r_1} \sum_{n=0}^{[\frac{g-1}{2}]} \frac{1}{q^{2n+1}} \sum_{m=g-n}^{g} \frac{1}{q^{2m}u^{m+1}} \sum_{\square \neq V \in \mathbb{A}^+_{2n-2g+2m}} \delta_{V;2n+1}(u) du$$

with $r_1 < 1$. We can bound $\delta_{V;2n+1}(u)$ by (3.7.6) and the sum over V by using the fact that $\#\mathbb{A}^+_n = q^n$. Thus we get that $\tilde{\mathcal{S}}^o_{g,1}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$ and $\hat{\mathcal{S}}^o_{g,1}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$ and so $\mathcal{S}^o_{g,1}(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}$. Similar calculations can show that $\mathcal{S}^o_{g,2}(V \neq \square)$, $\mathcal{S}^o_{g-1,1}(V \neq \square)$ and $\mathcal{S}^o_{g-1,2}(V \neq \square)$ are bounded by $q^{\frac{g}{2}(1+\epsilon)}$. Therefore, by (3.7.2), we get

$$\mathcal{S}^o(V \neq \square) \ll q^{\frac{g}{2}(1+\epsilon)}. \tag{3.7.8}$$

Thus combining (3.7.7) and (3.7.8) with (3.7.1) proves Proposition 3.7.1.

3.8 Proof of Theorem 3.1.9

We combine results from the previous sections to prove Theorem 3.1.9.

Proof of Theorem 3.1.9. From (3.4.1), we have

$$\sum_{D \in \mathcal{H}_{2g+2}} L\left(\frac{1}{2}, \chi_D\right) = \mathcal{S}_{g,1} - \mathcal{S}_{g,2} + \mathcal{S}_{g-1.1} - \mathcal{S}_{g-1,2}. \tag{3.8.1}$$

From the arguments stated in Section 3.4, we can rewrite (3.8.1) as

$$\sum_{D \in \mathcal{H}_{2g+2}} L\left(\frac{1}{2}, \chi_D\right) = M + \mathcal{S}(V = \square) + \mathcal{S}(V \neq \square). \tag{3.8.2}$$

Using Proposition 3.5.1, Proposition 3.6.1 and Proposition 3.7.1, we have

$$\begin{aligned}\sum_{D \in \mathcal{H}_{2g+2}} L\left(\frac{1}{2}, \chi_D\right) = {} & M_1 + M_2 + M_3 + M_4 \\ & + \mathcal{S}_1(V = \square) + \mathcal{S}_2(V = \square) + \mathcal{S}_3(V = \square) + \mathcal{S}_4(V = \square) \\ & + q^{\frac{2g+2}{3}}\mathcal{R}(2g+2) + O\left(q^{\frac{g}{2}(1+\epsilon)}\right).\end{aligned} \tag{3.8.3}$$

Furthermore, from Proposition 3.5.1, and Proposition 3.6.1 we have that

$$\begin{aligned}\mathcal{S}_1(V=\square)+M_1=&-\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{\left[\frac{g}{2}\right]}}du\\&+\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=r}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{\left[\frac{g}{2}\right]}}du,\end{aligned}\tag{3.8.4}$$

$$\begin{aligned}\mathcal{S}_2(V=\square)+M_2=&-\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{\left[\frac{g-1}{2}\right]}}du\\&+\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=r}\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{\left[\frac{g-1}{2}\right]}}du,\end{aligned}\tag{3.8.5}$$

$$\begin{aligned}\mathcal{S}_3(V=\square)+M_3=&\frac{q^{\frac{3g+3}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{\left[\frac{g}{2}\right]}}du\\&-\frac{q^{\frac{3g+3}{2}}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=r}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{\left[\frac{g}{2}\right]}}du\end{aligned}\tag{3.8.6}$$

and

$$\begin{aligned}\mathcal{S}_4(V=\square)+M_4=&\frac{q^{\frac{3g}{2}+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=R}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{\left[\frac{g-1}{2}\right]}}du\\&-\frac{q^{\frac{3g}{2}+2}}{\zeta_{\mathbb{A}}(2)}\frac{1}{2\pi i}\oint_{|u|=r}\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{\left[\frac{g-1}{2}\right]}}du,\end{aligned}\tag{3.8.7}$$

where $r<\frac{1}{q}$ and $1<R<q$. By Remark 3.5.2, $\mathcal{C}(u)$ has an analytic continuation for $|u|<q$ and $\mathcal{C}(1)=0$. Therefore, between the circles $|u|=r$ and $|u|=R$, the integrands corresponding to the terms M_1, M_2, $\mathcal{S}_1(V=\square)$ and $\mathcal{S}_2(V=\square)$ have a double pole at $u=q^{-1}$. Similarly, the integrands corresponding to the terms M_3, M_4, $\mathcal{S}_3(V=\square)$ and $\mathcal{S}_4(V=\square)$ have a simple pole at $u=q^{-1}$. We can compute the residue as follows:

$$\begin{aligned}\operatorname{Res}\left(\frac{\mathcal{C}(u)}{u(1-qu)^2(qu)^{\left[\frac{k}{2}\right]}};u=q^{-1}\right)&=\lim_{u\to q^{-1}}\frac{d}{du}\frac{(u-q^{-1})^2\mathcal{C}(u)}{u(1-qu)^2(qu)^{\left[\frac{k}{2}\right]}}\\&=\lim_{u\to q^{-1}}\frac{d}{du}\frac{\mathcal{C}(u)}{q(qu)^{\left[\frac{k}{2}\right]+1}}\\&=\lim_{u\to q^{-1}}\frac{\mathcal{C}'(u)}{q(qu)^{\left[\frac{k}{2}\right]+1}}-\lim_{u\to q^{-1}}\frac{\mathcal{C}(u)\left(\left[\frac{k}{2}\right]+1\right)}{(qu)^{\left[\frac{k}{2}\right]+2}}\\&=\frac{\mathcal{C}'\left(q^{-1}\right)}{q}-\mathcal{C}\left(q^{-1}\right)\left(\left[\frac{k}{2}\right]+1\right).\end{aligned}$$

Similarly we have

$$\begin{aligned}\operatorname{Res}\left(\frac{\mathcal{C}(u)}{u(1-u)(1-qu)u^{\left[\frac{k}{2}\right]}};u=q^{-1}\right) &= \lim_{u\to q^{-1}}\frac{(u-q^{-1})\mathcal{C}(u)}{u(1-u)(1-qu)u^{\left[\frac{k}{2}\right]}}\\ &= -\lim_{u\to q^{-1}}\frac{\mathcal{C}(u)}{qu(1-u)u^{\left[\frac{k}{2}\right]}}\\ &= -\frac{\mathcal{C}\left(q^{-1}\right)q^{\left[\frac{k}{2}\right]+1}}{(q-1)}.\end{aligned}$$

Using the substitution $u=q^{-s}$, we have have that $\mathcal{C}(u)=P(s)$ and $\mathcal{C}'(u)=\frac{q^s}{\log q}P'(s)$. Thus

$$M_1+\mathcal{S}_1(V=\square)=\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\left(P(1)\left(\left[\frac{g}{2}\right]+1\right)+\frac{P'(1)}{\log q}\right),$$

$$M_2+\mathcal{S}_2(V=\square)=\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\left(P(1)\left(\left[\frac{g-1}{2}\right]+1\right)+\frac{P'(1)}{\log q}\right),$$

$$M_3+\mathcal{S}_3(V=\square)=-\frac{q^{\frac{3g+5}{2}+\left[\frac{g-1}{2}\right]}P(1)}{\zeta_{\mathbb{A}}(2)(q-1)}$$

and

$$M_4+\mathcal{S}_4(V=\square)=-\frac{q^{\frac{3g}{2}+3+\left[\frac{g}{2}\right]}P(1)}{\zeta_{\mathbb{A}}(2)(q-1)}.$$

Using the fact that

$$\left[\frac{g}{2}\right]+\left[\frac{g-1}{2}\right]=g-1$$

we have that

$$M_1+M_2+\mathcal{S}_1(V=\square)+\mathcal{S}_2(V=\square)=\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\left(P(1)(g+1)+\frac{2}{\log q}P'(1)\right). \tag{3.8.8}$$

Furthermore, using the fact that (see [Jun13, Proof of Main Theorem])

$$1+q^{\frac{1}{2}}=q^{-\frac{g-1}{2}+\left[\frac{g}{2}\right]}+q^{-\frac{g}{2}+\left[\frac{g-1}{2}\right]+1},$$

we have

$$\begin{aligned}M_3+M_4+\mathcal{S}_3(V=\square)+\mathcal{S}_4(V=\square) &= -\frac{P(1)}{\zeta_{\mathbb{A}}(2)(q-1)}\left(q^{\frac{3g+5}{2}+\left[\frac{g-1}{2}\right]}+q^{\frac{3g}{2}+3+\left[\frac{g}{2}\right]}\right)\\ &= -\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\frac{1+q^{\frac{1}{2}}}{q-1}P(1)\\ &= \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}P(1)\zeta_{\mathbb{A}}\left(\frac{1}{2}\right).\end{aligned} \tag{3.8.9}$$

Putting (3.8.8) and (3.8.9) into (3.8.3) completes the proof of Theorem 3.1.9. ■

Remark 3.8.1. *When revising the paper [AM21] and writing this chapter, we came across a recent paper by Jung [Jun20], where he computed, independently, the mean value of* $L\left(\frac{1}{2},\chi_D\right)$ *when summing over all monic, square-free polynomials of degree* $2g+2$ *as* $g \to \infty$ *using similar calculations to those used by Florea [Flo17a]. Compared to Jung's paper, we explicitly go into more detail about how to calculate the asymptotic formula, especially when analysing the contribution from the square polynomials* V*.*

Chapter 4

The Mean Value of $|K_2(\mathcal{O})|$ in the Inert Imaginary Quadratic Function Fields

4.1 The Algebraic K group $K_2(\mathcal{O})$

In this section, we will give some details about the algebraic K group $K_2(\mathcal{O})$, which is mainly stated in Rosen [Ros95, Section 2] and relate their size to the number $L(2,\chi)$.

Let $F = \mathbb{F}_q$ and let K/F be a function field in one variable with a finite constant field F. Furthermore, we will let the primes in K be denoted by $\mathfrak{v}$ and the valuation ring at $\mathfrak{v}$ by $\mathcal{O}_\mathfrak{v}$. Also, we let $\mathcal{P}_\mathfrak{v}$ denote the maximal ideal of $\mathcal{O}_\mathfrak{v}$ and $\bar{F}_\mathfrak{v}$ denote the residue class field at $\mathfrak{v}$. The tame symbol $(*,*)_\mathfrak{v}$ is a mapping from $K^* \times K^*$ to $\bar{F}_\mathfrak{v}^*$ defined by

$$(a,b)_\mathfrak{v} = (-1)^{\mathfrak{v}(a)\mathfrak{v}(b)} a^{\mathfrak{v}(b)}/b^{\mathfrak{v}(a)} \text{ modulo } \mathcal{P}_\mathfrak{v}. \tag{4.1.1}$$

This symbol is bimultiplicative and has the property that $(a, 1-a)_\mathfrak{v} = 1$ for all $a \in K^*$ with $a \neq 0, 1$.

The group $K_2(K)$ can be defined as $K^* \otimes K^*$ modulo the subgroup generated by the elements $a \otimes (1-a)$ for all $a \in K^*$ with $a \neq 0,1$. There is a map $\lambda_\mathfrak{v} : K_2(K) \to \bar{F}_\mathfrak{v}^*$ which is induced by $\lambda_\mathfrak{v}(a \otimes b) = (a,b)_\mathfrak{v}$. If we let $\lambda : K_2(K) \to \oplus_\mathfrak{v} \bar{F}_\mathfrak{v}^*$ be the sum of the tame symbol maps and $\mu : \oplus_\mathfrak{v} \bar{F}_\mathfrak{v}^* \to F^*$ be the map given by $\mu(\ldots, a_\mathfrak{v}, \ldots,) = \prod_\mathfrak{v} a_\mathfrak{v}^{m_\mathfrak{v}/m}$ where $m_\mathfrak{v} = N\mathcal{P}_\mathfrak{v} - 1 = \#\bar{F}_\mathfrak{v}^*$ and $m = q - 1 = \#F$, then Moore (see [Tat71]) proved that the sequence

$$(0) \to \mathrm{Ker}(\lambda) \to K_2(K) \xrightarrow{\lambda} \bigoplus_\mathfrak{v} \bar{F}_\mathfrak{v}^* \to F^* \to (0) \tag{4.1.2}$$

is exact. Tate [Tat71] proved the Birch-Tate conjecture concerning the size of $\mathrm{Ker}(\lambda)$. More precisely, he proved that

$$\#\mathrm{Ker}(\lambda) = (q-1)(q^2-1)\zeta_K(-1)$$

where

$$\zeta_K(s) = \prod_{\mathfrak{v}}(1 - N\mathcal{P}_{\mathfrak{v}}^{-s})^{-1}$$

and the product is over all primes $\mathfrak{v}$ of the function field K. Furthermore let $S = (\mathcal{P}_1, \dots, \mathcal{P}_t)$ be a finite set of primes of K and let $\mathcal{O}_S$ denote the set of S-integers of K, i.e. the elements of K whose poles lie in S. Then using a theorem of Quillen [Qui72] we have that

$$(0) \to K_2(\mathcal{O}_S) \to K_2(K) \xrightarrow{\lambda'} \bigoplus_{\mathfrak{v}\notin S} \bar{F}^* \to (0)$$

where the map λ' is the truncation of the map λ. If we define the S-zeta function of K to be

$$\zeta_S(s) = \prod_{v\notin S}(1 - N\mathcal{P}_{\mathfrak{v}}^{-s})^{-1}, \tag{4.1.3}$$

then Rosen [Ros95, Proposition 1] says that

$$K_2(\mathcal{O}_S) = (-1)^t(q^2-1)\zeta_S(-1). \tag{4.1.4}$$

Let $m \in \mathbb{A}$ be square-free then we define $K_m = k(\sqrt{m})$ and $\mathcal{O}_m$ be the integral closure of $\mathbb{A}$. We also define the zeta function of the ring to be

$$\zeta_{\mathcal{O}_m}(s) = \sum_{\mathfrak{a}} N\mathfrak{a}^{-s},$$

where $\mathfrak{a}$ runs through the non-zero ideals of $\mathcal{O}_m = \mathbb{A}[\sqrt{m}]$ and $N\mathfrak{a}$ is the number of elements in $\mathcal{O}_m/\mathfrak{a}$. Then, from [Ros95, Proposition 17.7] we have

$$\zeta_{\mathcal{O}_m}(s) = \zeta_{\mathbb{A}}(s)L(s,\chi_m), \tag{4.1.5}$$

where $L(s,\chi_m)$ is the L-function defined in Section 2.6.2. If we let $S = S_m(\infty)$ be the primes in K above ∞, then $\mathcal{O}_S$ is $\mathcal{O}_m$ and thus combining (4.1.4) and (4.1.5) we have

$$\#K_2(\mathcal{O}_m) = (-1)^{t-1}L(-1,\chi_m), \tag{4.1.6}$$

where t is the number of primes in K_m above ∞. Finally, proving a relationship between $L(-1,\chi_m)$ and $L(2,\chi_m)$, Rosen proved a relation between the size of the group $K_2(\mathcal{O}_m)$ and the number $L(2,\chi_m)$.

Proposition 4.1.1 ([Ros95, Proposition 2])**.** *Let $K_m = k(\sqrt{m})$, where m is a square-free polynomial of degree M in $\mathbb{A}$. We have the following:*

a) If M is odd, then

$$\#K_2(\mathcal{O}_m) = q^{\frac{3M}{2}} q^{-\frac{3}{2}} L(2, \chi_m). \tag{4.1.7}$$

b) If M is even and the leading coefficient of m is a square, then

$$\#K_2(\mathcal{O}_m) = q^{\frac{3M}{2}} q^{-2} \frac{\zeta_{\mathbb{A}}(3)}{\zeta_{\mathbb{A}}(2)} L(2, \chi_m). \tag{4.1.8}$$

c) If M is even and the leading coefficient of m is not a square, then

$$\#K_2(\mathcal{O}_m) = q^{\frac{3M}{2}} q^{-2} \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(5)}{\zeta_{\mathbb{A}}^2(3)} L(2, \chi_m). \tag{4.1.9}$$

4.2 The Mean Value of $|K_2(\mathcal{O})|$ in Function Fields

Using Proposition 4.1.1 and [HR92, Theorem 0.8], Rosen proved the following result.

Theorem 4.2.1 (Rosen)**.** *Let $\epsilon > 0$ be given.*

a) Suppose M is odd, then

$$(q-1)^{-1}(q^M - q^{M-1})^{-1} \sum_{\substack{m \in \mathbb{A}_M \\ m \text{ square-free}}} \#K_2(\mathcal{O}_m) = q^{\frac{3M}{2}} q^{-\frac{3}{2}} \zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(4)c(2) + O_\epsilon\left(q^{M(1+\epsilon)}\right). \tag{4.2.1}$$

b) Suppose M is even and the leading coefficient of m is a square, then

$$2(q-1)^{-1}(q^M - q^{M-1})^{-1} \sum_{\substack{m \in \mathbb{A}_M \\ m \text{ square-free}}} \#K_2(\mathcal{O}_m) = q^{\frac{3M}{2}} q^{-2} \zeta_{\mathbb{A}}(3)\zeta_{\mathbb{A}}(4)c(2) + O_\epsilon\left(q^{M(1+\epsilon)}\right). \tag{4.2.2}$$

c) Suppose M is even and the leading coefficient of M is not a square, then

$$\begin{aligned} 2(q-1)^{-1}(q^M - q^{M-1})^{-1} &\sum_{\substack{m \in \mathbb{A}_M \\ m \text{ square-free}}} \#K_2(\mathcal{O}_m) \\ &= q^{\frac{3M}{2}} q^{-2} \frac{\zeta_{\mathbb{A}}^2(2)\zeta_{\mathbb{A}}(4)\zeta_{\mathbb{A}}(5)}{\zeta_{\mathbb{A}}^2(3)} c(2) + O_\epsilon\left(q^{M(1+\epsilon)}\right). \end{aligned} \tag{4.2.3}$$

The constant $c(2)$ is given by

$$c(2) = \prod_P \left(1 - |P|^{-2} - |P|^{-5} + |P|^{-6}\right).$$

Restricting the sum to monic, square-free polynomials of a certain degree, Andrade [And15] established an asymptotic formula for the size of the group $K_2(\mathcal{O}_D)$ for $D \in \mathcal{H}_{2g+1}$, where $\mathcal{H}_{2g+1}$ is the hyperelliptic ensemble defined in Section 2.6.3. In particular he proved the following result.

Theorem 4.2.2 (Andrade)**.** *Let $\epsilon > 0$ be given. Then*

$$\frac{1}{\#\mathcal{H}_{2g+1}} \sum_{D \in \mathcal{H}_{2g+1}} \#K_2(\mathcal{O}_D) = q^{\frac{3}{2}(2g+1)} q^{-\frac{3}{2}} \zeta_{\mathbb{A}}(4) P(4) + O_\epsilon \left(q^{(2g+1)(1+\epsilon)} \right), \tag{4.2.4}$$

where

$$P(s) = \prod_P \left(1 - \frac{1}{|P|^s(|P|+1)} \right). \tag{4.2.5}$$

In this chapter, we will use the methods of Andrade to calculate the average size of the group $K_2(\mathcal{O}_{\gamma D})$, where D is a monic, square-free polynomial of degree $2g+2$ and γ is a fixed generator of $\mathbb{F}_q^*$. The main result in this Chapter is the following:

Theorem 4.2.3. *Let $\epsilon > 0$ be given and let γ be a fixed generator of $\mathbb{F}_q^*$. Then*

$$\frac{1}{\#\mathcal{H}_{2g+2}} \sum_{D \in \mathcal{H}_{2g+2}} \#K_2(\mathcal{O}_{\gamma D}) = q^{\frac{3}{2}(2g+2)} q^{-2} \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(4)\zeta_{\mathbb{A}}(5)}{\zeta_{\mathbb{A}}^2(3)} P(4) + O_\epsilon \left(q^{(2g+2)(1+\epsilon)} \right), \tag{4.2.6}$$

where $P(s)$ is given in (4.2.5).

Remark 4.2.4. *The asymptotic for $\sum_{D \in \mathcal{H}_{2g+2}} \#K_2(\mathcal{O}_D)$ was obtained jointly with Andrade and Davies in a paper, [ADM22], that is currently submitted for publication. As the calculations are similar, I will only include the calculations to prove an asymptotic for $\sum_{D \in \mathcal{H}_{2g+2}} \#K_2(\mathcal{O}_{\gamma D})$ here.*

4.3 Preliminaries

In this section, we will state and prove results that will be needed to prove Theorem 4.2.3. We will start by proving the following result.

Lemma 4.3.1. *Let $\chi_{\gamma D}$ be a quadratic character, where γ is a fixed generator of $\mathbb{F}_q^*$ and $D \in \mathcal{H}_{2g+2}$. Then*

$$\mathcal{L}(q^{-2}, \chi_{\gamma D}) = \sum_{f \in \mathbb{A}^+_{\leq 2g}} (-1)^{deg(f)} \frac{\chi_D(f)}{|f|^2} + q^{-4g-2} \sum_{f \in \mathbb{A}^+_{\leq 2g}} \chi_D(f). \tag{4.3.1}$$

Proof. From Section 2.4 and Section 2.6.2, we know that the L-function $\mathcal{L}(u, \chi_{\tilde{D}})$ is written as

$$\mathcal{L}(u, \chi_{\tilde{D}}) = \sum_{n=0}^{2g+1} A_{\tilde{D}}(n) u^n \tag{4.3.2}$$

where $\tilde{D} = \gamma D$ and

$$A_D(n) = \sum_{f \in \mathbb{A}^+_n} \chi_D(f). \tag{4.3.3}$$

From Section 2.6.3, we know that the complete L-function $\mathcal{L}^*(u,\chi_{\tilde{D}})$ can be written as

$$\mathcal{L}^*(u,\chi_{\tilde{D}}) = \sum_{n=0}^{2g} A^*_{\tilde{D}}(n)u^n. \tag{4.3.4}$$

From (2.6.12) we know that

$$A^*_{\tilde{D}}(n) = \sum_{i=0}^{n}(-1)^{n-i}A_{\tilde{D}}(i).$$

Furthermore, since $\left(\frac{\gamma}{f}\right) = (-1)^{\deg(f)}$ then $\left(\frac{\tilde{D}}{f}\right) = (-1)^{\deg(f)}\left(\frac{D}{f}\right)$ and so $A_{\tilde{D}}(n) = (-1)^n A_D(n)$. Therefore

$$A^*_{\tilde{D}}(n) = \sum_{i=0}^{n}(-1)^n A_D(i). \tag{4.3.5}$$

Thus combining (4.3.4) and (4.3.5) we get

$$\mathcal{L}^*(q^{-2},\chi_{\tilde{D}}) = \sum_{n=0}^{2g}\sum_{i=0}^{n}(-1)^n A_D(i)q^{-2n}. \tag{4.3.6}$$

Interchanging the sums we get

$$\mathcal{L}^*(q^{-2},\chi_{\tilde{D}}) = \sum_{n=0}^{2g}\sum_{i=n}^{2g}(-1)^i A_D(n)q^{-2i}.$$

Thus

$$\mathcal{L}^*(q^{-2},\chi_{\tilde{D}}) = \sum_{n=0}^{2g} A_D(n)\left(\frac{(-1)^n q^{-2n} + q^{-4g-2}}{1+q^{-2}}\right). \tag{4.3.7}$$

Using (2.6.12) in (4.3.7) proves the Lemma. ■

Proposition 4.3.2. *Let $\ell \in \mathbb{A}$ be a monic polynomial. Then for all $\epsilon > 0$ we have*

$$\sum_{\substack{D\in\mathcal{H}_{2g+2}\\(D,\ell)=1}} 1 = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\prod_{P\mid\ell}\frac{|P|}{|P|+1} + O\left(q^g|\ell|^\epsilon\right). \tag{4.3.8}$$

Proof. The proof is similar to that given in [AK12, Proposition 5.2]. ■

Lemma 4.3.3 ([AK12, Lemma 5.7])**.** *We have*

$$\sum_{\ell\in\mathbb{A}_m^+}\prod_{P\mid\ell}\frac{|P|}{|P|+1} = q^m\sum_{d\in\mathbb{A}_{\le m}^+}\frac{\mu(d)}{|d|}\prod_{P\mid d}\frac{1}{|P|+1}. \tag{4.3.9}$$

Lemma 4.3.4. *For $s=1$ or $s=4$ we have*

$$\sum_{d\in\mathbb{A}_{\le g}^+}\frac{\mu(d)}{|d|^s}\prod_{P\mid d}\frac{1}{|P|+1} = P(s) + O(q^{-sg}), \tag{4.3.10}$$

where $P(s)$ is given by (4.2.5).

Proof. When $s=1$, the proof is given in [Jun14, Lemma 3.3]. Using similar methods, we have that when $s=4$ we can use the definition of the Möbius function and the Euler product formula to obtain

$$\sum_{d\in\mathbb{A}^+}\frac{\mu(d)}{|d|^4}\prod_{P|d}\frac{1}{|P|+1}=\prod_P\left(1-\frac{1}{|P|^4(|P|+1)}\right)=P(4).$$

We also have

$$\begin{aligned}\sum_{\substack{d\in\mathbb{A}^+\\ \deg(d)>g}}\frac{\mu(d)}{|d|^4}\prod_{P|d}\frac{1}{|P|+1}&\le\sum_{\substack{d\in\mathbb{A}^+\\ \deg(d)>g}}\frac{\mu^2(d)}{|d|^4}\prod_{P|d}\frac{1}{|P|}\\&\le\sum_{\substack{d\in\mathbb{A}^+\\ \deg(d)>g}}\frac{1}{|d|^5}=\sum_{n=g+1}^{\infty}q^{-4n}\ll q^{-4g}.\end{aligned}$$

Thus

$$\sum_{d\in\mathbb{A}^+_{\le g}}\frac{\mu(d)}{|d|^4}\prod_{P|d}\frac{1}{|P|+1}=\sum_{d\in\mathbb{A}^+}\frac{\mu(d)}{|d|^4}\prod_{P|d}\frac{1}{|P|+1}-\sum_{\substack{d\in\mathbb{A}^+\\ \deg(d)>g}}\frac{\mu(d)}{|d|^4}\prod_{P|d}\frac{1}{|P|+1}=P(4)+O\left(q^{-4g}\right).$$

■

Lemma 4.3.5. *If $f\in\mathbb{A}$ is not a perfect square then*

$$\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(f)\ll q^g|f|^{\frac{1}{4}}\tag{4.3.11}$$

Proof. The proof is similar to that given in [And15, Lemma 4.3]. ■

4.4 Proof of Theorem 4.2.3

In this section, we will prove Theorem 4.2.3. Firstly, we split each term of (4.3.1) in two, the first over all polynomials of degree at most $2g$ which are a square and the second over all polynomials of degree at most $2g$ which are not a square. Thus we get

$$\begin{aligned}\sum_{D\in\mathcal{H}_{2g+2}}L(2,\chi_{\gamma D})&=\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{f\in\mathbb{A}^+_{\le 2g}\\ f=\ell^2=\square}}\frac{\chi_D(f)}{|f|^2}+\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{f\in\mathbb{A}^+_{\le 2g}\\ f\ne\square}}(-1)^{\deg(f)}\frac{\chi_D(f)}{|f|^2}\\&+q^{-4g-2}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{f\in\mathbb{A}^+_{\le 2g}\\ f=\ell^2=\square}}\chi_D(f)+q^{-4g-2}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{f\in\mathbb{A}^+_{\le 2g}\\ f\ne\square}}\chi_D(f).\end{aligned}\tag{4.4.1}$$

We will evaluate each term of (4.4.1) separately and then use Proposition 4.1.1 to prove Theorem 4.2.3.

Proposition 4.4.1. *For all $\epsilon>0$, we have*

$$\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{f\in\mathbb{A}^+_{\le 2g}\\ f=\square}}\frac{\chi_D(f)}{|f|^2}=\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\zeta_{\mathbb{A}}(4)P(4)+O\left(q^{g(1+\epsilon)}\right).\tag{4.4.2}$$

Proof. From the definition of quadratic Dirichlet characters stated in Section 2.6.2, we know that $\chi_D(\ell^2) = 1$ if $(D,\ell) = 1$ and 0 otherwise. Thus

$$\sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{f\in\mathbb{A}^+_{\leq 2g} \\ f=\square}} \frac{\chi_D(f)}{|f|^2} = \sum_{\ell\in\mathbb{A}^+_{\leq g}} \frac{1}{|\ell|^4} \sum_{\substack{D\in\mathcal{H}_{2g+2} \\ (D,\ell)=1}} 1.$$

Using Proposition 4.3.2, we have

$$\sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{f\in\mathbb{A}^+_{\leq 2g} \\ f=\square}} \frac{\chi_D(f)}{|f|^2} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{m=0}^{g} q^{-4m} \sum_{\ell\in\mathbb{A}^+_m} \prod_{P|\ell} \frac{|P|}{|P|+1} + O\left(q^g \sum_{m=0}^{2g} q^{m\epsilon-m}\right).$$

Invoking Lemma 4.3.3 we have

$$\sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{f\in\mathbb{A}^+_{\leq 2g} \\ f=\square}} \frac{\chi_D(f)}{|f|^2} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{m=0}^{g} q^{-3m} \sum_{d\in\mathbb{A}^+_{\leq m}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} + O\left(q^{g(1+\epsilon)}\right). \tag{4.4.3}$$

Rearranging (4.4.3) we get

$$\begin{aligned}\sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{f\in\mathbb{A}^+_{\leq 2g} \\ f=\square}} \frac{\chi_D(f)}{|f|^2} &= \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{d\in\mathbb{A}^+_{\leq g}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} \sum_{\deg(d)\leq m\leq g} q^{-3m} \\ &= \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\zeta_{\mathbb{A}}(4) \sum_{d\in\mathbb{A}^+_{\leq g}} \frac{\mu(d)}{|d|^4} \prod_{P|d} \frac{1}{|P|+1} \\ &\quad - \frac{q^{2-g}}{q^3-1} \frac{1}{\zeta_{\mathbb{A}}(2)} \sum_{d\in\mathbb{A}^+_{\leq g}} \frac{\mu(d)}{|d|} \prod_{P|d} \frac{1}{|P|+1} + O\left(q^{g(1+\epsilon)}\right).\end{aligned}$$

Using Lemma 4.3.4 proves the Proposition. ∎

Lemma 4.4.2. *We have*

$$q^{-4g-2} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{f\in\mathbb{A}^+_{\leq 2g} \\ f=\square}} \chi_D(f) \ll q^{-g}.$$

Proof. Trivially bounding the quadratic Dirichlet character, we have

$$q^{-4g-2} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{f\in\mathbb{A}^+_{\leq 2g} \\ f=\square}} \chi_D(f) \ll q^{-4g} \sum_{\substack{f\in\mathbb{A}^+_{\leq 2g} \\ f=\square}} \sum_{D\in\mathcal{H}_{2g+2}} 1 \ll q^{-2g} \sum_{L\in\mathbb{A}^+_{\leq g}} 1 \ll q^{-g}.$$

∎

Lemma 4.4.3. *We have*

$$\sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{f\in\mathbb{A}^+_{\leq 2g} \\ f\neq\square}} \chi_D(f) \ll q^g.$$

Proof. Using Lemma 4.3.5, we have

$$\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{f\in\mathbb{A}^+_{\leq 2g}\\ f\neq\square}}\frac{\chi_D(f)}{|f|^2}\ll q^g\sum_{f\in\mathbb{A}^+_{\leq 2g}}|f|^{-\frac{7}{4}}\ll q^g\sum_{m=0}^{2g}q^{-\frac{3m}{4}}\ll q^g.$$

■

Lemma 4.4.4. *We have*

$$q^{-4g-2}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{f\in\mathbb{A}^+_{\leq 2g}\\ f\neq\square}}\chi_D(f)\ll q^{-\frac{g}{2}}.$$

Proof. From Lemma 4.3.5 we have

$$q^{-4g-2}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{f\in\mathbb{A}^+_{\leq 2g}\\ f\neq\square}}\chi_D(f)\ll q^{-3g}\sum_{f\in\mathbb{A}^+_{\leq 2g}}|f|^{\frac{1}{4}}\ll q^{-3g}\sum_{n=0}^{2g}q^{\frac{5m}{4}}\ll q^{-\frac{g}{2}}.$$

■

Thus using Proposition 4.4.1, Lemma 4.4.2, Lemma 4.4.3 and Lemma 4.4.4 in (4.4.1), we have, for γ a fixed generator of $\mathbb{F}_q^*$, that

$$\sum_{D\in\mathcal{H}_{2g+2}}L(2,\chi_{\gamma D})=\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\zeta_{\mathbb{A}}(4)P(4)+O\left(q^{g(1+\epsilon)}\right). \tag{4.4.4}$$

Using (2.6.9) and Proposition 4.1.1 part c) proves Theorem 4.2.3.

■

Chapter 5

Rudnick and Soundararajan's Theorem over Prime Polynomials for the Rational Function Field

5.1 Lower bounds of Dirichlet L-functions in Function Fields

As previously discussed, a fundamental problem in Analytic Number Theory is to understand the asymptotic behaviour of moments of Dirichlet L-functions in function fields. Andrade and Keating [AK14] conjectured that

$$\frac{1}{\#\mathcal{H}_{2g+1}} \sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right)^k \sim P_k(\log_q |D|) \tag{5.1.1}$$

where P_k is an explicit polynomial of degree $\frac{1}{2}k(k+1)$, where $\mathcal{H}_{2g+1}$ is the hyperelliptic ensemble and $L(s, \chi_D)$ is the Dirichlet L-function defined in Section 2.6.3 and Section 2.6.2 respectively. The first four moments have been explicitly been computed by Andrade and Keating [AK12] and Florea [Flo17a, Flo17b, Flo17c] and have also verified the conjecture for these cases.

Furthermore, Andrade [And16] established lower bounds for the moments of Dirichlet L-functions in function fields, which is seen to be the function field analogue of Rudnick and Soundararajan's [RS06] result Theorem 1.5.8. Namely Andrade proved the following result.

Theorem 5.1.1. *For every even natural number k and $n = 2g+1$ or $n = 2g+2$, we have*

$$\frac{1}{\#\mathcal{H}_n} \sum_{D \in \mathcal{H}_n} L\left(\frac{1}{2}, \chi_D\right)^k \gg_k (\log_q |D|)^{\frac{k(k+1)}{2}}. \tag{5.1.2}$$

For the family of Dirichlet L-functions associated with the Dirichlet character χ_P, where P is a monic, irreducible polynomial in $\mathbb{F}_q[T]$, Andrade, Jung and Shamesaldeen [AJS21] conjectured

$$\frac{1}{\#\mathcal{P}_{2g+1}} \sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right)^k \sim Q_k(\log_q |P|), \tag{5.1.3}$$

where Q_k is an explicit polynomial of degree $\frac{1}{2}k(k+1)$. The first and second moments have explicitly been calculated by Andrade and Keating [AK13] and Bui and Florea [BF20].

In this chapter, we will use the methods of Rudnick and Soundararajan [RS06] and Andrade [And16] to establish lower bounds for the moments of Dirichlet L-functions associated with the Dirichlet character χ_P.

Theorem 5.1.2. *For every even natural number k and $n = 2g+1$ and $n = 2g+2$, we have*

$$\frac{1}{\#\mathcal{P}_n} \sum_{P \in \mathcal{P}_n} L\left(\frac{1}{2}, \chi_P\right)^k \gg_k (\log_q |P|)^{\frac{k(k+1)}{2}}. \tag{5.1.4}$$

Remark 5.1.3. *Recently, Gao and Zhao [GZ22b] used the work of Radziwiłł and Soundararajan [RS13] and Heap and Soundararajan [HS22] to show that Theorem* 5.1.1 *and Theorem* 5.1.2 *holds for all real $k > 0$.*

5.2 Preliminary Lemmas

In this section, we will state some preliminary Lemmas which will be used to prove Theorem 5.1.2.

Lemma 5.2.1 ("Approximate Functional Equation")**.** *For $P \in \mathcal{P}_{2g+1}$, we have*

$$L\left(\frac{1}{2}, \chi_P\right) = \sum_{f \in \mathbb{A}^+_{\leq g}} \frac{\chi_P(f)}{\sqrt{|f|}} + \sum_{f \in \mathbb{A}^+_{\leq g-1}} \frac{\chi_P(f)}{\sqrt{|f|}}. \tag{5.2.1}$$

Proof. The proof is similar to that given in [AK12, Lemma 3.3]. ∎

Proposition 5.2.2 ([Rud10, (2.5)])**.** *For $f \in \mathbb{A}^+$ with $deg(f) > 0$ and f not a perfect square, we have*

$$\left| \sum_{P \in \mathcal{P}_n} \chi_P(f) \right| \ll \frac{deg(f)}{n} q^{\frac{n}{2}}. \tag{5.2.2}$$

5.3 Proof of Theorem 5.1.2

In this section, we prove Theorem 5.1.2. We will prove the result for $n = 2g+1$, but similar methods can be used to prove the result for $n = 2g+2$.

5.3.1 Set Up of the Proof

Let k be an even natural number and let $x = \frac{2(2g)}{5k}$.

Remark 5.3.1. *This is the maximum and simplest choice of x so that the error term in (5.3.14) is bounded by $|P|^{1-\epsilon}$ for some $\epsilon > 0$.*

For $P \in \mathcal{P}_{2g+1}$, we define

$$A(P) = \sum_{f \in \mathbb{A}^+_{\le x}} \frac{\chi_P(f)}{\sqrt{|f|}} \tag{5.3.1}$$

and let

$$S_1 := \sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right) A(P)^{k-1} \tag{5.3.2}$$

and

$$S_2 := \sum_{P \in \mathcal{P}_{2g+1}} A(P)^k. \tag{5.3.3}$$

Using the triangle inequality and Hölders inequality we get

$$\begin{aligned}\left| \sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right) A(P)^{k-1} \right| &\le \sum_{P \in \mathcal{P}_{2g+1}} \left| L\left(\frac{1}{2}, \chi_P\right)\right| |A(P)|^{k-1} \\ &\le \left(\sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right)^k \right)^{\frac{1}{k}} \left(\sum_{P \in \mathcal{P}_{2g+1}} A(P)^k \right)^{\frac{1}{k}}.\end{aligned}$$

Thus

$$\sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right)^k \ge \frac{\left(\sum_{P \in \mathcal{P}_{2g+1}} L\left(\frac{1}{2}, \chi_P\right) A(P)^{k-1}\right)^k}{\left(\sum_{P \in \mathcal{P}_{2g+1}} A(P)^k\right)^{k-1}} = \frac{S_1^k}{S_2^{k-1}}. \tag{5.3.4}$$

Therefore to prove Theorem 5.1.2, we need to obtain estimates for S_1 and S_2.

5.3.2 Estimating S_2

We have

$$S_2 = \sum_{P \in \mathcal{P}_{2g+1}} A(P)^k = \sum_{n_1, \ldots, n_k \in \mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|n_1 \ldots n_k|}} \sum_{P \in \mathcal{P}_{2g+1}} \chi_P(n_1 \ldots n_k).$$

We split the sum up in two, where the first is over all polynomials $n_1, \ldots, n_k$ of degree at most x for which the product $n_1 \ldots n_k$ is a square and the second sum is over all polynomials $n_1, \ldots, n_k$ of degree at most x for which the product $n_1 \ldots n_k$ is not a square. Furthermore, from the definition of the quadratic Dirichlet character given in Section 2.6.2, we know that $\chi_P(\ell^2) = 1$ if $(P, \ell) = 1$ and 0 otherwise. Also, if $\deg(P) > \deg(\ell)$, then all polynomials ℓ are coprime to P. Since $\deg(n_1\dots n_k) \le kx = \frac{2(2g)}{5} < 2g+1 = \deg(P)$, then combining the above we get

$$S_2 = \sum_{\substack{n_1,\dots,n_k\in\mathbb{A}^+_{\le x}\\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1\dots n_k|}} \sum_{P\in\mathcal{P}_{2g+1}} 1 + \sum_{\substack{n_1,\dots,n_k\in\mathbb{A}^+_{\le x}\\ n_1\dots n_k\ne\square}} \frac{1}{\sqrt{|n_1\dots n_k|}} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(n_1\dots n_k).$$

Using the Prime Polynomial Theorem, Theorem 2.1.4, and Proposition 5.2.2, we have

$$\begin{aligned} S_2 = \frac{|P|}{\log_q|P|} &\sum_{\substack{n_1,\dots,n_k\in\mathbb{A}^+_{\le x}\\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1\dots n_k|}} + \sum_{\substack{n_1,\dots,n_k\in\mathbb{A}^+_{\le x}\\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1\dots n_k|}} O\left(\frac{|P|^{\frac{1}{2}}}{\log_q|P|}\right) \\ &+ \sum_{\substack{n_1,\dots,n_k\in\mathbb{A}^+_{\le x}\\ n_1\dots n_k\ne\square}} \frac{1}{\sqrt{|n_1\dots n_k|}} O\left(\frac{|P|^{\frac{1}{2}}}{\log_q|P|}\deg(n_1\dots n_k)\right). \end{aligned} \tag{5.3.5}$$

Since $x = \frac{2(2g)}{5k}$, then for the second term in (5.3.5) we have

$$\begin{aligned} \frac{|P|^{\frac{1}{2}}}{\log_q|P|} \sum_{\substack{n_1,\dots,n_k\in\mathbb{A}^+_{\le x}\\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1\dots n_k|}} &\le \frac{|P|^{\frac{1}{2}}}{\log_q|P|} \sum_{n_1\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|n_1|}} \cdots \sum_{n_k\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|n_k|}} \\ &\ll \frac{|P|^{\frac{1}{2}}}{\log_q|P|} q^{\frac{kx}{2}} = \frac{|P|^{\frac{7}{10}}}{\log_q|P|}. \end{aligned} \tag{5.3.6}$$

Similarly, for the last term in (5.3.5) we have

$$\begin{aligned} \frac{|P|^{\frac{1}{2}}}{\log_q|P|} \sum_{\substack{n_j\in\mathbb{A}^+_{\le x}\\ j=1,\dots,k\\ n_1\dots n_k\ne\square}} \frac{\deg(n_1\dots n_k)}{\sqrt{|n_1\dots n_k|}} &\le \frac{|P|^{\frac{1}{2}}}{\log_q|P|} kx \sum_{n_1\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|n_1|}} \cdots \sum_{n_k\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|n_k|}} \\ &\ll \frac{|P|^{\frac{1}{2}}}{\log_q|P|} kx q^{\frac{kx}{2}} \ll |P|^{\frac{7}{10}}. \end{aligned} \tag{5.3.7}$$

Therefore combining (5.3.5), (5.3.6) and (5.3.7), we have

$$S_2 = \frac{|P|}{\log_q|P|} \sum_{\substack{n_1,\dots,n_k\in\mathbb{A}^+_{\le x}\\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1\dots n_k|}} + O\left(|P|^{\frac{7}{10}}\right). \tag{5.3.8}$$

Writing $n_1\dots n_k = m^2$, then from [And16, (4.18)], we have that

$$\sum_{m\in\mathbb{A}^+_{\le\frac{x}{2}}} \frac{d_k(m^2)}{|m|} \le \sum_{\substack{n_1,\dots,n_k\in\mathbb{A}^+_{\le x}\\ n_1\dots n_k=\square}} \frac{1}{\sqrt{|n_1\dots n_k|}} \le \sum_{m\in\mathbb{A}^+_{\le\frac{kx}{2}}} \frac{d_k(m^2)}{|m|}, \tag{5.3.9}$$

where $d_k(f)$ is defined in Section 2.2. From Lemma 2.2.9, we have

$$\sum_{m\in\mathbb{A}^+_{\le\frac{x}{2}}} \frac{d_k(m^2)}{|m|} \sim C(k)\left(\frac{2g}{5k}\right)^{\frac{k(k+1)}{2}} \tag{5.3.10}$$

and

$$\sum_{m\in\mathbb{A}^+_{\le\frac{kx}{2}}} \frac{d_k(m^2)}{|m|} \sim C(k)\left(\frac{2g}{5}\right)^{\frac{k(k+1)}{2}}. \tag{5.3.11}$$

Thus, using (5.3.8), (5.3.10) and (5.3.11), we have that

$$S_2 \ll |P|(\log_q|P|)^{\frac{k(k+1)}{2}-1}. \tag{5.3.12}$$

5.3.3 Estimating S_1

Using Lemma 5.2.1, we have

$$\begin{aligned} S_1 &= \sum_{P\in\mathcal{P}_{2g+1}} L\left(\frac{1}{2},\chi_P\right)A(P)^{k-1} \\ &= \sum_{\substack{f\in\mathbb{A}^+_{\le g} \\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(fn_1\dots n_{k-1}) \\ &+ \sum_{\substack{f\in\mathbb{A}^+_{\le g-1} \\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(fn_1\dots n_{k-1}). \end{aligned} \tag{5.3.13}$$

The two sums in the last equality of (5.3.13) are the same apart from the size of the sum. Thus, we will only estimate the first sum, as the second follows from replacing g with $g-1$.

We split up the sum in two, where the first sum is over all polynomials f of degree at most g and all polynomials $n_1,\dots,n_{k-1}$ of degree at most x for which the product $fn_1\dots n_{k-1}$ is a square and the second sum is over all polynomials f of degree at most g and all polynomials $n_1,\dots,n_{k-1}$ of degree at most x for which the product $fn_1\dots n_{k-1}$ is not a square. Furthermore using the arguments given in Section 5.3.2 and the above we have

$$\begin{aligned} &\sum_{\substack{f\in\mathbb{A}^+_{\le g} \\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}}} \frac{1}{\sqrt{|fn_1\dots n_k|}} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(fn_1\dots n_{k-1}) \\ &= \sum_{\substack{f\in\mathbb{A}^+_{\le g} \\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x} \\ fn_1\dots n_{k-1}=\square}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} \sum_{P\in\mathcal{P}_{2g+1}} 1 + \sum_{\substack{f\in\mathbb{A}^+_{\le g} \\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x} \\ fn_1\dots n_k\neq\square}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(fn_1\dots n_{k-1}). \end{aligned}$$

Using the Prime Polynomial Theorem and Proposition 5.2.2 we have

$$\sum_{\substack{f\in\mathbb{A}^+_{\le g}\\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(fn_1\dots n_{k-1})$$
$$= \frac{|P|}{\log_q|P|} \sum_{\substack{f\in\mathbb{A}^+_{\le g}\\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}\\ fn_1\dots n_{k-1}=\square}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} + \sum_{\substack{f\in\mathbb{A}^+_{\le g}\\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}\\ fn_1\dots n_{k-1}=\square}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} O\left(\frac{|P|^{\frac{1}{2}}}{\log_q|P|}\right)$$
$$+ \sum_{\substack{f\in\mathbb{A}^+_{\le g}\\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}\\ fn_1\dots n_{k=1}\neq\square}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} O\left(\frac{|P|^{\frac{1}{2}}}{\log_q|P|}\deg(fn_1\dots n_{k-1})\right). \tag{5.3.14}$$

Since $x = \frac{2(2g)}{5k}$, then for the second term in (5.3.14), we have

$$\frac{|P|^{\frac{1}{2}}}{\log_q|P|} \sum_{\substack{f\in\mathbb{A}^+_{\le g}\\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}\\ fn_1\dots n_{k-1}=\square}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} \le \frac{|P|^{\frac{1}{2}}}{\log_q|P|} \sum_{f\in\mathbb{A}^+_{\le g}} \frac{1}{\sqrt{|f|}} \sum_{n_1\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|n_1|}} \cdots \sum_{n_{k-1}\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|n_{k-1}|}}$$
$$\ll \frac{|P|^{\frac{1}{2}}}{\log_q|P|} q^{\frac{g}{2}} q^{(k-1)\frac{x}{2}} \le \frac{|P|^{\frac{1}{2}}}{\log_q|P|} q^{\frac{g}{2}} q^{\frac{2g}{5}} = \frac{|P|^{\frac{19}{20}}}{\log_q|P|} \tag{5.3.15}$$

Similarly, for the final term in (5.3.14), we have

$$\frac{|P|^{\frac{1}{2}}}{\log_q|P|} \sum_{\substack{f\in\mathbb{A}^+_{\le g}\\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}\\ fn_1\dots n_{k-1}\neq\square}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} \deg(fn_1\dots n_{k-1})$$
$$\le \frac{|P|^{\frac{1}{2}}}{\log_q|P|}(g+(k-1)x) \sum_{f\in\mathbb{A}^+_{\le g}} \frac{1}{\sqrt{|f|}} \sum_{n_1\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|n_1|}} \cdots \sum_{n_{k-1}\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|n_{k-1}|}}$$
$$\ll \frac{|P|^{\frac{1}{2}}}{\log_q|P|}(g+(k-1)x) q^{\frac{g}{2}} q^{\frac{(k-1)x}{2}} \ll \frac{|P|^{\frac{1}{2}}}{\log_q|P|} g q^{\frac{g}{2}} q^{\frac{2g}{5}} \ll |P|^{\frac{19}{20}}. \tag{5.3.16}$$

Thus, combining (5.3.14), (5.3.15) and (5.3.16) we get

$$\sum_{\substack{f\in\mathbb{A}^+_{\le g}\\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} \sum_{P\in\mathcal{P}_{2g+1}} \chi_P(fn_1\dots n_{k-1})$$
$$= \frac{|P|}{\log_q|P|} \sum_{\substack{f\in\mathbb{A}^+_{\le g}\\ n_1,\dots,n_{k-1}\in\mathbb{A}^+_{\le x}\\ j=1,\dots,k-1\\ fn_1\dots n_{k=1}=\square}} \frac{1}{\sqrt{|fn_1\dots n_{k-1}|}} + O\left(|P|^{\frac{19}{20}}\right). \tag{5.3.17}$$

For the main term of (5.3.17), we write $n_1 \dots n_{k-1} = rh^2$ where $r, h \in \mathbb{A}^+$ with r square-free. Thus, if $fn_1 \dots n_{k-1}$ is a square, then $f = rl^2$ for some $l \in \mathbb{A}^+$. Then

$$\sum_{\substack{f \in \mathbb{A}^+_{\leq g} \\ n_1, \dots, n_{k-1} \in \mathbb{A}^+_{\leq x} \\ fn_1 \dots n_{k-1} = \square}} \frac{1}{\sqrt{|fn_1 \dots n_{k-1}|}} = \sum_{\substack{n_1, \dots, n_{k-1} \in \mathbb{A}^+_{\leq x} \\ n_1 \dots n_{k-1} = rh^2}} \frac{1}{|rh|} \sum_{l \in \mathbb{A}^+_{\leq \frac{g - \deg(r)}{2}}} \frac{1}{|l|}.$$

Now, we have that

$$\sum_{l \in \mathbb{A}^+_{\leq \frac{g - \deg(r)}{2}}} \frac{1}{|l|} \sim \frac{1}{2} (\log_q |P|).$$

Thus using (5.3.9) we have

$$\begin{aligned} \frac{|P|}{\log_q |P|} \sum_{\substack{n_1, \dots, n_{k-1} \in \mathbb{A}^+_{\leq x} \\ n_1 \dots n_{k-1} = rh^2}} \frac{1}{|rh|} \sum_{l \in \mathbb{A}^+_{\leq \frac{g - \deg(r)}{2}}} \frac{1}{|l|} &\gg |P| \sum_{\substack{r, h \in \mathbb{A}^+ \\ \deg(rh^2) \leq x}} \frac{d_{k-1}(rh^2)}{|rh|} \\ &\gg |P| (\log_q |P|)^{\frac{k(k+1)}{2} - 1}, \end{aligned} \tag{5.3.18}$$

where the final bound in (5.3.18) follows from Lemma 2.2.8 and Lemma 2.2.9. The same argument applies for the second sum in (5.3.13) replacing g with $g - 1$. Therefore we have

$$S_1 \gg |P| (\log_q |P|)^{\frac{k(k+1)}{2} - 1}. \tag{5.3.19}$$

Combining (5.3.4), (5.3.12) and (5.3.19) and using the Prime Polynomial Theorem completes the proof of Theorem 5.1.2.

■

Chapter 6

Integral Moments of L-functions in Even Characteristic

6.1 Moments of Dirichlet L-functions in Function Fields

As mentioned in Chapter 3, a problem in function fields is to understand the asymptotic behaviour of

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2},\chi_D\right)^k \tag{6.1.1}$$

as $|D|\to\infty$, for positive integer values k, where $q\equiv 1(\mathrm{mod}\ 4)$, $L(s,\chi_D)$ is the quadratic Dirichlet L-function and $\mathcal{H}_{2g+1}$ is the hyperelliptic ensemble which are defined in Section 2.6.2 and Section 2.6.3 respectively. In the case of fixing q and letting $g\to\infty$, the first four moments of (6.1.1) have been calculated by Andrade and Keating [AK12] and Florea [Flo17a, Flo17b, Flo17c]. Furthermore Andrade and Keating [AK14] adapted the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith [CFK+05] to conjecture the integral moments of quadratic L-functions in function fields. Their conjecture reads.

Conjecture 6.1.1 (Andrade and Keating)**.** *Suppose that q odd is the fixed cardinality of the finite field $\mathbb{F}_q$ and let $\mathcal{X}_D(s)=|D|^{\frac{1}{2}-s}X(s)$ and*

$$X(s)=q^{-\frac{1}{2}+s}.$$

That is $\mathcal{X}_D(s)$ is the factor in the functional equation

$$L(s,\chi_D)=\mathcal{X}_D(s)L(1-s,\chi_D).$$

Summing over fundamental discriminants $D\in\mathcal{H}_{2g+1}$, we have

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2},\chi_D\right)^k=\sum_{D\in\mathcal{H}_{2g+1}} Q_k(\log_q|D|)(1+o(1)), \tag{6.1.2}$$

where Q_k is the polynomial of degree $\frac{1}{2}k(k+1)$ given by the k-fold residue

$$Q_k(x) = \frac{(-1)^{\frac{k(k-1)}{2}} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{G(z_1, \ldots, z_k)\Delta(z_1^2, \ldots, z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} q^{\frac{x}{2}\sum_{j=1}^k z_j} dz_1 \ldots dz_k,$$

where $\Delta(z_1, \ldots, z_k)$ is the Vandermonde determinant given by

$$\Delta(z_1, \ldots, z_k) = \prod_{1 \le i < j \le k} (z_j - z_i), \tag{6.1.3}$$

$$G(z_1, \ldots, z_k) = A\left(\frac{1}{2}; z_1, \ldots, z_k\right) \prod_{j=1}^k X\left(\frac{1}{2} + z_j\right)^{-\frac{1}{2}} \prod_{1 \le i \le j \le k} \zeta_{\mathbb{A}}(1 + z_i + z_j)$$

and $A\left(\frac{1}{2}; z_1, \ldots, z_k\right)$ is the Euler product, absolutely convergent for $|\Re(z_j)| < \frac{1}{2}$, defined by

$$A\left(\frac{1}{2}; z_1, \ldots, z_k\right) = \prod_P \prod_{1 \le i \le j \le k} \left(1 - \frac{1}{|P|^{1+z_i+z_j}}\right) \times \left(\frac{1}{2}\left(\prod_{j=1}^k \left(1 - \frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1} + \prod_{j=1}^k \left(1 + \frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1}\right) + \frac{1}{|P|}\right)\left(1 + \frac{1}{|P|}\right)^{-1}.$$

Florea [Flo17a, Flo17b, Flo17c] showed that the asymptotic formulas that she obtained, see Theorem 3.1.2, to Theorem 3.1.5, agree with Conjecture 6.1.1. Furthermore, for the third moment, Diaconu [Dia19] proved the existence of a secondary main term of size $|D|^{\frac{3}{4}}$ in the asymptotic formula and for higher moments, Diaconu and Twiss [DT20] conjectured that there exists additional terms which occur in the asymptotic formula. Rubinstein and Wu [RW15] provided numerical evidence for Conjecture 6.1.1, namely they numerically computed the moments for $k \le 10$ and $d \le 18$, where $d = 2g + 1$ for various values of q.

Understanding negative moments of Dirichlet L-functions in function fields is a harder problem due to the zeros of the L-functions on the critical line. Considering shifted negative moments, Bui, Florea and Keating [BFK21a] showed that for $\beta \ll g^{-\frac{1}{2k}+\epsilon}$ and k a positive integer we have

$$\frac{1}{\#\mathcal{H}_{2g+1}} \sum_{D \in \mathcal{H}_{2g+1}} \frac{1}{\left|L\left(\frac{1}{2} + \beta, \chi_D\right)\right|^k} \ll \left(\frac{1}{\beta}\right)^{\frac{k(k-1)}{2}} (\log g)^{\frac{k(k+1)}{2}}. \tag{6.1.4}$$

Florea [Flo21] proved an upper bound for

$$\frac{1}{\#\mathcal{H}_{2g+1}} \sum_{D \in \mathcal{H}_{2g+1}} \frac{1}{\left|L\left(\frac{1}{2} + \beta + it, \chi_D\right)\right|^k}$$

for $k > 0$ and $\beta > 0$ such that $\log\left(\frac{1}{\beta}\right) \ll \log g$. Additionally, Florea proved that for k a positive integer, $\epsilon > 0$ and $\mathfrak{R}(\beta) \gg g^{-1}(\log g)^{1-\frac{1}{2k}+\epsilon}$ we have

$$\frac{1}{\#\mathcal{H}_{2g+1}} \sum_{D\in\mathcal{H}_{2g+1}} \frac{1}{L\left(\frac{1}{2}+\beta,\chi_D\right)^k} = \zeta_{\mathbb{A}}(1+2\beta)^{\binom{k}{2}} A(\beta) + O\left(q^{-\frac{2g\mathfrak{R}(\beta)(2-\epsilon)}{k}} g^{\frac{k^2}{2}+\frac{k}{2}+\epsilon}\right),$$

where $A(\beta)$ is a specific constant.

Andrade, Jung and Shamesaldeen [AJS21] conjectured the integral moments of quadratic Dirichlet L-functions over monic irreducible polynomials in $\mathbb{F}_q[T]$ and showed that their conjecture agrees with the asymptotic formulas obtained by Andrade and Keating [AK13] and Bui and Florea [BF20]. Their conjecture reads.

Conjecture 6.1.2 (Andrade, Jung and Shamesaldeen)**.** *Suppose that $q \equiv 1 (mod\ 4)$ is the fixed cardinality of the finite field $\mathbb{F}_q$ and let $\mathcal{X}_P(s) = |P|^{\frac{1}{2}-s} X(s)$ where*

$$X(s) = q^{-\frac{1}{2}+s}.$$

That is $\mathcal{X}_P(s)$ is the factor in the functional equation

$$L(s,\chi_P) = \mathcal{X}_P(s) L(1-s,\chi_P).$$

Summing over primes $P \in \mathcal{P}_{2g+1}$, we have

$$\sum_{P\in\mathcal{P}_{2g+1}} L\left(\frac{1}{2},\chi_P\right)^k = \sum_{P\in\mathcal{P}_{2g+1}} Q_k(\log_q |P|)(1+o(1)), \tag{6.1.5}$$

where Q_k is the polynomial of degree $\frac{1}{2}k(k+1)$ given by the k-fold residue

$$Q_k(x) = \frac{(-1)^{\frac{k(k-1)}{2}} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{G(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} q^{\frac{x}{2}\sum_{i=1}^k z_i} dz_1 \ldots dz_k,$$

where $\Delta(z_1,\ldots,z_k)$ is the Vandermonde determinant defined in (6.1.3),

$$G(z_1,\ldots,z_k) = A_k\left(\frac{1}{2};z_1,\ldots,z_k\right) \prod_{i=1}^k X\left(\frac{1}{2}+z_i\right)^{-\frac{1}{2}} \prod_{1\le i\le j\le k} \zeta_{\mathbb{A}}(1+z_i+z_j)$$

and A_k is the Euler product, absolutely convergent for $|\mathfrak{R}(z_i)| < \frac{1}{2}$ defined by

$$A_k\left(\frac{1}{2};z_1,\ldots,z_k\right) = \prod_P \prod_{1\le i\le j\le k}\left(1-\frac{1}{|P|^{1+z_i+z_j}}\right) \times \frac{1}{2}\left(\prod_{i=1}^k\left(1-\frac{1}{|P|^{\frac{1}{2}+z_i}}\right)^{-1} + \prod_{i=1}^k\left(1+\frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1}\right).$$

When q is a power of 2, a problem in function fields is to understand the asymptotic behaviour of

$$\sum_{u \in \mathcal{I}_{g+1}} L(s, \chi_u)^k, \qquad \sum_{u \in \mathcal{F}_{g+1}} L(s, \chi_u)^k \quad \text{and} \quad \sum_{u \in \mathcal{F}'_{g+1}} L(s, \chi_u)^k \tag{6.1.6}$$

when q is fixed and $g \to \infty$ for various values of s and k, where $L(s, \chi_u)$ is the quadratic Dirichlet L-function defined in Section 2.7.3 and $\mathcal{I}_{g+1}$, $\mathcal{F}_{g+1}$ and $\mathcal{F}'_{g+1}$ are the sets defined in Section 2.7.1. In this setting, Bae and Jung [BJ18] computed an asymptotic formula for the first moment of (6.1.6) for almost all $s \in \mathbb{C}$ with $\Re(s) \geq \frac{1}{2}$. For the interests of this chapter we will only state their result for when $s = \frac{1}{2}$ and when $s = \frac{1}{2} + it$ where $t \neq 0$, where the sum is over all $u \in \mathcal{I}_{g+1}$.

Theorem 6.1.3 (Bae and Jung)**.** *Suppose that q is a power of 2. Then we have*

$$\sum_{u \in \mathcal{I}_{g+1}} L\left(\frac{1}{2}, \chi_u\right) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)} P(1) \left(g + 1 + \frac{2}{\log q} \frac{P'}{P}(1)\right) + O\left(g 2^{\frac{g}{2}} q^{\frac{3g}{2}}\right) \tag{6.1.7}$$

and for $t \neq 0$ we have

$$\sum_{u \in \mathcal{I}_{g+1}} L\left(\frac{1}{2} + it, \chi_u\right) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \zeta_{\mathbb{A}}(1+2it)(P(1+2it) - q^{-2it(g+1)} P(1-2it)) + O\left(g 2^{\frac{g}{2}} q^{g(\frac{3}{2}-it)}\right). \tag{6.1.8}$$

where

$$P(s) = \prod_{P} \left(1 - \frac{1}{|P|^s(|P|+1)}\right).$$

A further problem is to understand the asymptotic behaviour of

$$\sum_{u \in \tilde{\mathcal{I}}_{g+1}} L(s, \chi_u)^k, \qquad \sum_{u \in \tilde{\mathcal{F}}_{g+1}} L(s, \chi_u)^k \quad \text{and} \quad \sum_{u \in \tilde{\mathcal{F}}'_{g+1}} L(s, \chi_u)^k \tag{6.1.9}$$

where $\tilde{\mathcal{I}}_{g+1}$, $\tilde{\mathcal{F}}_{g+1}$ and $\tilde{\mathcal{F}}'_{g+1}$ are the sets defined in Section 2.7.1. In this setting Andrade, Bae and Jung [ABJ16] computed an asymptotic formula for the first moment of (6.1.9) for all $s \in \mathbb{C}$ with $\Re(s) \geq \frac{1}{2}$ and an asymptotic formula for the second moment of (6.1.9) when $s = \frac{1}{2}$.

6.2 Statement of Result

In this chapter, we develop to even characteristic the heuristic developed in [CFK+05, AK14, AJS21]. The main result is the following Conjecture.

Conjecture 6.2.1. *Suppose that q is a power of 2 which is the fixed cardinality of the finite field $\mathbb{F}_q$ and let*

$$\mathcal{X}_u(s) = (q^{2g+1})^{\frac{1}{2}-s} X(s)$$

where

$$X(s) = q^{-\frac{1}{2}+s}.$$

That is $\mathcal{X}_u(s)$ *is the factor of the functional equation*

$$L(s,\chi_u) = \mathcal{X}_u(s)L(1-s,\chi_u).$$

Summing over fundamental discriminants $u \in \mathcal{I}_{g+1}$, *we have*

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right)^k = \sum_{u\in\mathcal{I}_{g+1}} Q_k(2g+1)(1+o(1)), \tag{6.2.1}$$

where $Q_k(x)$ *is the polynomial of degree* $\frac{1}{2}k(k+1)$ *given by the k-fold residue*

$$Q_k(x) = \frac{(-1)^{\frac{k(k-1)}{2}}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\cdots\oint\frac{G(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}}q^{\frac{x}{2}\sum_{j=1}^k z_j}dz_1\ldots dz_k,$$

where $\Delta(z_1,\ldots,z_k)$ *is the Vandermonde determinant defined in (6.1.3),*

$$G(z_1,\ldots,z_k) = A\left(\frac{1}{2};z_1,\ldots,z_k\right)\prod_{j=1}^k X\left(\frac{1}{2}+z_j\right)^{-\frac{1}{2}}\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+z_i+z_j)$$

and $A\left(\frac{1}{2};z_1,\ldots,z_k\right)$ *is the Euler product, absolutely convergent for* $|\Re(z_j)|<\frac{1}{2}$ *defined by*

$$\begin{aligned}A\left(\frac{1}{2};z_1,\ldots,z_k\right) = \prod_P\prod_{1\le i\le j\le k}&\left(1-\frac{1}{|P|^{1+z_i+z_j}}\right)\\ &\times\left(\frac{1}{2}\left(\prod_{j=1}^k\left(1-\frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1}+\prod_{j=1}^k\left(1+\frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1}\right)+\frac{1}{|P|}\right)\left(1+\frac{1}{|P|}\right)^{-1}.\end{aligned}$$

Remark 6.2.2. *To obtain Conjecture* 6.2.1, *we will use the methods seen in [AK14, AJS21]. However, the main difference is when averaging over the family* $\mathcal{I}_{g+1}$. *For this we need to use the calculations seen in [BJ18], which is done in Lemma* 6.4.2. *We will also show that our conjecture agrees with (6.1.7) and (6.1.8), the latter of which has not been done in either [AK14] or [AJS21], since asymptotic formulas for*

$$\sum_{D\in\mathcal{H}_{2g+1}} L\left(\frac{1}{2}+it,\chi_D\right)^k \quad \textit{or} \quad \sum_{P\in\mathcal{P}_{2g+1}} L\left(\frac{1}{2}+it,\chi_P\right)^k$$

have not been explicitly obtained. Finally we will use our conjecture to obtain explicit conjectural formulae for higher moments.

6.3 Preliminary Lemmas

In this section, we will state results which will be used in this chapter.

Lemma 6.3.1 ("Approximate" Functional Equation, [BJ18, Lemma 3.1]). *Let $s \in \mathbb{C}$ with $\Re(s) \geq \frac{1}{2}$, then for $u \in \mathcal{I}$, we have*

$$L(s, \chi_u) = \sum_{f \in \mathbb{A}^+_{\leq g}} \frac{\chi_u(f)}{|f|^s} + \mathcal{X}_u(s) \sum_{f \in \mathbb{A}^+_{\leq g-1}} \frac{\chi_u(f)}{|f|^{1-s}}, \tag{6.3.1}$$

where $\mathcal{X}_u(s) = q^{g(1-2s)}$.

Lemma 6.3.2 ([BJ18, Lemma 3.3]). *Let $L \in \mathbb{A}^+$. Given any $\epsilon > 0$, we have*

$$\sum_{\substack{f \in \mathbb{A}^+_n \\ (f,L)=1}} \phi(f) = \frac{q^{2n}}{\zeta_{\mathbb{A}}(2)} \prod_{P|L} (1 + |P|^{-1})^{-1} + O(q^{(1+\epsilon)n}). \tag{6.3.2}$$

Proposition 6.3.3 ([BJ18, Proposition 3.20]). *For any $f \in \mathbb{A}^+_n$ with $n \leq g$ which is not a perfect square, we have*

$$\sum_{u \in \mathcal{I}_{g+1}} \chi_u(f) \ll g 2^{\frac{n}{2}} q^g. \tag{6.3.3}$$

6.4 Heuristic Derivation of the Conjecture

In this section, we present the details for conjecturing moments of L-functions associated to quadratic Dirichlet L-function $L(s, \chi_u)$ with $u \in \mathcal{I}_{g+1}$ as $g \to \infty$, where $\mathbb{F}_q$ is a fixed finite field with q a power of 2. As in [AK14, AJS21] we will adjust the recipe first presented in [CFK+05] to the even characteristic setting.

6.4.1 Analogies between Classical L-functions and L-functions over Function Fields

Let $u \in \mathcal{I}_{g+1}$. For a fixed positive integer k, we want to obtain an asymptotic expression for

$$\sum_{u \in \mathcal{I}_{g+1}} L\left(\frac{1}{2}, \chi_u\right)^k \tag{6.4.1}$$

as $g \to \infty$. To achieve this, we consider a more general expression obtained by introducing small shifts $\alpha_1, \ldots, \alpha_k$. Thus we seek to achieve an asymptotic expression for

$$\sum_{u \in \mathcal{I}_{g+1}} L\left(\frac{1}{2} + \alpha_1, \chi_u\right) \ldots L\left(\frac{1}{2} + \alpha_k, \chi_u\right). \tag{6.4.2}$$

The introduction of these shifts reveals hidden structures and the calculations are simplified. In the end we will let each $\alpha_1, \ldots, \alpha_k$ tend to zero in (6.4.2) to obtain an asymptotic expression for (6.4.1).

The first step to obtain the conjecture for integral moments is to use the "Approximate" functional equation, Lemma 6.3.1. Here we note that $\mathcal{X}_u(s)$ can be written as

$$\mathcal{X}_u(s) = (q^{2g+1})^{\frac{1}{2}-s} X(s), \tag{6.4.3}$$

where $X(s) = q^{-\frac{1}{2}+s}$. Throughout this chapter, we will use the following results about $\mathcal{X}_u(s)$.

Lemma 6.4.1. *We have that*

$$\mathcal{X}_u(s)^{\frac{1}{2}} = \mathcal{X}_u(1-s)^{-\frac{1}{2}} \tag{6.4.4}$$

and

$$\mathcal{X}_u(s)\mathcal{X}_u(1-s) = 1. \tag{6.4.5}$$

Proof. The proof follows directly from the definition of $\mathcal{X}_u(s)$. ∎

Recall, from Section 2.7.3 that for $u \in \mathcal{I}_{g+1}$, we defined the completed L-function $\Lambda(s, \chi_u)$ as

$$\Lambda(s, \chi_u) = \mathcal{X}_u(s)^{-\frac{1}{2}} L(s, \chi_u). \tag{6.4.6}$$

We will apply the recipe to the completed L-function since it simplifies the calculations and it satisfies the functional equation

$$\Lambda(s, \chi_u) = \Lambda(1-s, \chi_u). \tag{6.4.7}$$

Thus, our goal is to obtain an asymptotic formula for the k-shifted moment

$$L_u(s) = \sum_{u \in \mathcal{I}_{g+1}} Z(s; \alpha_1, \ldots, \alpha_k) \tag{6.4.8}$$

where

$$Z(s; \alpha_1, \ldots, \alpha_k) = \prod_{j=1}^{k} \Lambda(s + \alpha_j, \chi_u). \tag{6.4.9}$$

Using Lemma 6.3.1, Lemma 6.4.1 and (6.4.6), we have

$$\Lambda(s, \chi_u) = \mathcal{X}_u(s)^{-\frac{1}{2}} \sum_{f \in \mathbb{A}^+_{\leq g}} \frac{\chi_u(f)}{|f|^s} + \mathcal{X}_u(1-s)^{-\frac{1}{2}} \sum_{f \in \mathbb{A}^+_{\leq g-1}} \frac{\chi_u(f)}{|f|^{1-s}}. \tag{6.4.10}$$

6.4.2 Applying the recipe for L-functions in Even characteristic

In this subsection, we will present the recipe which follows from [CFK+05, AK14, AJS21] with the necessary modifications for the family $L(s,\chi_u)$.

1. **Write the product of k-shifted L-functions:**
$$Z\left(\frac{1}{2};\alpha_1,\dots,\alpha_k\right)=\Lambda\left(\frac{1}{2}+\alpha_1,\chi_u\right)\dots\Lambda\left(\frac{1}{2}+\alpha_k,\chi_u\right).$$

2. **Replace each L-function by its "approximate" functional equation (6.4.10).** Hence we obtain
$$\begin{aligned}
&Z\left(\frac{1}{2};\alpha_1,\dots,\alpha_k\right)\\
&=\prod_{j=1}^{k}\left(\mathcal{X}_u\left(\frac{1}{2}+\alpha_j\right)^{-\frac{1}{2}}\sum_{\substack{n_j \text{ monic}\\ \deg(n_j)\le g}}\frac{\chi_u(n_j)}{|n_j|^{\frac{1}{2}+\alpha_j}}+\mathcal{X}_u\left(\frac{1}{2}-\alpha_j\right)^{-\frac{1}{2}}\sum_{\substack{n_j \text{ monic}\\ \deg(n_j)\le g-1}}\frac{\chi_u(n_j)}{|n_j|^{\frac{1}{2}-\alpha_j}}\right)\\
&=\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}\sum_{\substack{n_j \text{ monic}\\ \deg(n_j)\le f(\epsilon_j)}}\frac{\chi_u(n_j)}{|n_j|^{\frac{1}{2}+\epsilon_j\alpha_j}},
\end{aligned}\tag{6.4.11}$$
where $f(1)=g$ and $f(-1)=g-1$.

   By multiplying out, we get a sum over all monic polynomials $n_1,\dots,n_k$, then we can write (6.4.11) as
$$Z\left(\frac{1}{2};\alpha_1,\dots,\alpha_k\right)=\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}\sum_{\substack{n_1,\dots,n_k\\ n_j \text{ monic}}}\frac{\chi_u(n_1\dots n_k)}{\prod_{j=1}^{k}|n_j|^{\frac{1}{2}+\epsilon_j\alpha_j}}.\tag{6.4.12}$$

3. **Average the sign of the functional equation.** Note that in this case, the ϵ_f-signs of the functional equations are all equal to 1 and therefore do not produce any effect on the final result.

4. **Replace each summand by its expected value when averaged over $\mathcal{I}_{g+1}$.** For this we have the following result.

   Lemma 6.4.2. *Let*
$$a_m=\prod_{P|m}\left(1+\frac{1}{|P|}\right)^{-1},$$
   then
$$\lim_{g\to\infty}\frac{1}{\#\mathcal{I}_{g+1}}\sum_{u\in\mathcal{I}_{g+1}}\chi_u(m)=\begin{cases}a_m & \textit{if } m \textit{ is a square of a polynomial,}\\ 0 & \textit{otherwise.}\end{cases}$$

Remark 6.4.3. *The same factor a_m when averaging over fundamental discriminants $D \in \mathcal{H}_{2g+1}$, which is why Conjecture* 6.2.1 *is similar to [AK14, Conjecture 5].*

Proof of Lemma 6.4.2. We start by considering the case when m is a square. For $m = \square = \ell^2$, and by the definition of $\mathcal{I}$ given in Section 2.7.1, we know that $\mathcal{I}_{g+1}$ is the disjoint union of the $\mathcal{I}_{(r,g+1-r)}$'s for $0 \le r \le g$. Thus we have

$$\frac{1}{\#\mathcal{I}_{g+1}} \sum_{u \in \mathcal{I}_{g+1}} \chi_u(m = \ell^2) = \frac{1}{\#\mathcal{I}_{g+1}} \sum_{r=0}^{g} \sum_{u \in \mathcal{I}_{(r,g+1-r)}} \chi_u(m = \ell^2).$$

Note that $\mathcal{I}_{(0,g+1)} = \mathcal{G}_{g+1}$. For $1 \le r \le g$, we let

$$\mathcal{I}_M = \{v + F : v \in \mathcal{F}_M, F \in \mathcal{G}_{g+1-r}\},$$

where $\mathcal{F}_M$ and $\mathcal{G}_n$ are the sets defined in Section 2.7.1. Then $\mathcal{I}_{(r,g+1-r)}$ is the disjoint union of $\mathcal{I}_M$'s, where M runs over $\mathcal{B}_r$ and $\mathcal{B}_r$ is the set defined in Section 2.7.1. Hence we have

$$\frac{1}{\#\mathcal{I}_{g+1}} \sum_{r=0}^{g} \sum_{u \in \mathcal{I}_{(r,g+1-r)}} \chi_u(\ell^2) = \frac{1}{\#\mathcal{I}_{g+1}} \sum_{F \in \mathcal{G}_{g+1}} \chi_F(\ell^2) + \frac{1}{\#\mathcal{I}_{g+1}} \sum_{r=1}^{g} \sum_{M \in \mathcal{B}_r} \sum_{u \in \mathcal{I}_M} \chi_u(\ell^2).$$

For $u \in \mathcal{I}_M$ with $M \in \mathcal{B}_r$, we have, from Definition 2.7.7, that

$$\chi_u(\ell^2) = \begin{cases} 1 & \text{if } (M, \ell) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have

$$\frac{1}{\#\mathcal{I}_{g+1}} \sum_{r=0}^{g} \sum_{u \in \mathcal{I}_{(r,g+1-r)}} \chi_u(\ell^2) = \frac{1}{\#\mathcal{I}_{g+1}} \sum_{F \in \mathcal{G}_{g+1}} 1 + \frac{1}{\#\mathcal{I}_{g+1}} \sum_{r=1}^{g} \sum_{\substack{M \in \mathcal{B}_r \\ (M,\ell)=1}} \sum_{u \in \mathcal{I}_M} 1. \tag{6.4.13}$$

Using Lemma 2.7.3 and the fact that (see [BJ18, Proof of Proposition 4.1]) $\#\mathcal{G}_n = 2\zeta_{\mathbb{A}}(2)^{-1} q^n$, we have

$$\frac{1}{\#\mathcal{I}_{g+1}} \sum_{F \in \mathcal{G}_{g+1}} 1 = \frac{2\zeta_{\mathbb{A}}(2)^{-1} q^{g+1}}{2\zeta_{\mathbb{A}}(2)^{-1} q^{2g+1}} = q^{-g} \to 0 \text{ as } g \to \infty.$$

Using the above and the arguments stated in Section 2.7.1, we know that, for $M \in \mathcal{B}_r$, $\#\mathcal{I}_M = \#\mathcal{F}_M \#\mathcal{G}_{g+1-r} = 2\zeta_{\mathbb{A}}(2)^{-1} q^{g+1-r} \phi(\tilde{M})$, where $\tilde{M}$ is defined in Section 2.7.1. Furthermore, from Remark 2.7.2, we know that the map $\mathcal{B}_n \to \mathbb{A}_n^+$ defined by $M \mapsto \tilde{M}$ is a bijection and $(M, f) = 1$ if and only if $(\tilde{M}, f) = 1$. Thus, using the above arguments and Lemma 2.7.3 we have

$$\frac{1}{\#\mathcal{I}_{g+1}} \sum_{r=1}^{g} \sum_{\substack{M \in \mathcal{B}_r \\ (M,\ell)=1}} \sum_{u \in \mathcal{I}_M} 1 = q^{-g} \sum_{r=1}^{g} q^{-r} \sum_{\substack{\tilde{M} \in \mathbb{A}_r^+ \\ (\tilde{M},\ell)=1}} \phi(\tilde{M}).$$

Invoking Lemma 6.3.2, we have

$$q^{-g}\sum_{r=1}^{g} q^{-r} \sum_{\substack{\tilde{M}\in\mathbb{A}_r^+\\ (\tilde{M},\ell)=1}} \phi(\tilde{M}) = \frac{q^{-g}}{\zeta_{\mathbb{A}}(2)} \prod_{P|\ell}\left(1+\frac{1}{|P|}\right)^{-1} \sum_{r=1}^{g} q^{r} + O\left(q^{-g}\sum_{r=1}^{g} q^{\epsilon r}\right)$$

$$= \frac{q^{-g}}{\zeta_{\mathbb{A}}(2)} \prod_{P|\ell}\left(1+\frac{1}{|P|}\right)^{-1} \frac{q}{q-1}(q^{g}-1) + O\left(q^{-g(1-\epsilon)}\right).$$

As $g\to\infty$, the main term becomes a_m and the second and error terms tend to zero. Therefore if m is a square we have

$$\lim_{g\to\infty}\frac{1}{\#\mathcal{I}_{g+1}}\sum_{u\in\mathcal{I}_{g+1}}\chi_u(m) = a_m. \tag{6.4.14}$$

For m not a perfect square, we have, by Proposition 6.3.3

$$\frac{1}{\#\mathcal{I}_{g+1}}\sum_{u\in\mathcal{I}_{g+1}}\chi_u(m) \ll g2^{\frac{g}{2}}q^{-g} \to 0 \text{ as } g\to\infty. \tag{6.4.15}$$

Combining (6.4.14) and (6.4.15) completes the proof of Lemma 6.4.2. ∎

Using Lemma 6.4.2, we have that

$$\lim_{g\to\infty}\frac{1}{\#\mathcal{I}_{g+1}}\sum_{u\in\mathcal{I}_{g+1}}\sum_{\substack{n_1,\dots,n_k\\ n_j \text{ monic}}}\frac{\chi_u(n_1\dots n_k)}{\prod_{j=1}^{k}|n_j|^{\frac{1}{2}+\epsilon_j\alpha_j}} = \sum_{\substack{n_1,\dots,n_k\\ n_j\text{ monic}\\ n_1\dots n_k=m^2}}\frac{a_{m^2}}{\prod_{j=1}^{k}|n_j|^{\frac{1}{2}+\epsilon_j\alpha_j}}$$

$$= \sum_{m\text{ monic}}\sum_{\substack{n_1,\dots,n_k\\ n_j\text{ monic}\\ n_1\dots n_k=m^2}}\frac{a_{m^2}}{\prod_{j=1}^{k}|n_j|^{\frac{1}{2}+\epsilon_j\alpha_j}}.$$

5. **Extend each of $n_1,\dots,n_k$ sum for all monic polynomials and denote the sum $M(s;\alpha_1,\dots,\alpha_k)$.**
   If we let

$$R_k\left(\frac{1}{2};\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k\right) = \sum_{m\text{ monic}}\sum_{\substack{n_1,\dots,n_k\\ n_j\text{ monic}\\ n_1\dots n_k=m^2}}\frac{a_{m^2}}{\prod_{j=1}^{k}|n_j|^{\frac{1}{2}+\epsilon_j\alpha_j}}, \tag{6.4.16}$$

   then the extended sum produced by the recipe is

$$M\left(\frac{1}{2};\alpha_1,\dots,\alpha_k\right) = \sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}} R_k\left(\frac{1}{2};\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k\right). \tag{6.4.17}$$

6. **The conjecture is**

$$\sum_{u\in\mathcal{I}_{g+1}} Z\left(\frac{1}{2};\alpha_1,\dots,\alpha_k\right) = \sum_{u\in\mathcal{I}_{g+1}} M\left(\frac{1}{2};\alpha_1,\dots,\alpha_k\right)(1+o(1)). \tag{6.4.18}$$

6.4.3 Putting the Conjecture into a more useful form

In this subsection, we will put Conjecture (6.4.18) into a more useful form since the conjecture is problematic in this form because of the individual terms have poles that cancel when summed. More specifically, we will write R_k as an Euler product and then factor out the appropriate zeta factors, which helps us identify the poles.

First note that a_m is multiplicative since

$$a_{mn} = a_m a_n \qquad \text{whenever } (m,n) = 1,$$

where

$$a_m = \prod_{P|m}\left(1 + \frac{1}{|P|}\right)^{-1}.$$

Define

$$\psi(x) := \sum_{\substack{n_1,\dots,n_k \\ n_j \text{ monic} \\ n_1\dots n_k = x}} \frac{1}{|n_1|^{s+\alpha_1}\dots|n_k|^{s+\alpha_k}}, \tag{6.4.19}$$

so that $\psi(m^2)$ is multiplicative on m, Therefore

$$\begin{aligned}\sum_{m \text{ monic}} \sum_{\substack{n_1,\dots,n_k \\ n_j \text{ monic} \\ n_1\dots n_k = m^2}} \frac{a_{m^2}}{|n_1|^{s+\alpha_1}\dots|n_k|^{s+\alpha_k}} &= \sum_{m \text{ monic}} a_{m^2} \sum_{\substack{n_1,\dots,n_k \\ n_j \text{ monic} \\ n_1\dots n_k = m^2}} \frac{1}{|n_1|^{s+\alpha_1}\dots|n_k|^{s+\alpha_k}} \\ &= \sum_{m \text{ monic}} a_{m^2}\psi(m^2) \end{aligned} \tag{6.4.20}$$

Taking the Euler product of (6.4.20), we have

$$\sum_{m \text{ monic}} \sum_{\substack{n_1,\dots,n_k \\ n_j \text{ monic} \\ n_1\dots n_k = m^2}} \frac{a_{m^2}}{|n_1|^{s+\alpha_1}\dots|n_k|^{s+\alpha_k}} = \prod_P \left(1 + \sum_{j=1}^{\infty} a_{P^{2j}}\psi(P^{2j})\right), \tag{6.4.21}$$

where

$$\psi(P^{2j}) = \sum_{\substack{n_1,\dots,n_k \\ n_j \text{ monic} \\ n_1\dots n_k = P^{2j}}} \frac{1}{|n_1|^{s+\alpha_1}\dots|n_k|^{s+\alpha_k}}. \tag{6.4.22}$$

Since we have $n_1\dots n_k = P^{2j}$, then for each $i = 1,\dots,k$, write $n_i = P^{e_i}$ for some $e_i \geq 0$ and $e_1 + \dots + e_k = 2j$. Thus (6.4.22) becomes

$$\psi(P^{2j}) = \sum_{\substack{e_1,\dots,e_k \geq 0 \\ e_1+\dots+e_k = 2j}} \prod_{i=1}^{k} \frac{1}{|P|^{e_i(s+\alpha_i)}}. \tag{6.4.23}$$

Therefore, combining (6.4.21) and (6.4.23) in (6.4.16) we have

$$R_k(s;\alpha_1,\ldots,\alpha_k) = \prod_P \left(1 + \sum_{j=1}^{\infty} a_{P^{2j}} \psi(P^{2j})\right)$$
$$= \prod_P \left(1 + \sum_{j=1}^{\infty} a_{P^{2j}} \sum_{\substack{e_1,\ldots,e_k \geq 0 \\ e_1+\ldots+e_k=2j}} \prod_{i=1}^{k} \frac{1}{|P|^{e_i(s+\alpha_i)}}\right). \tag{6.4.24}$$

Furthermore, we know that

$$a_{P^{2j}} = (1+|P|^{-1})^{-1},$$

and thus (6.4.24) becomes

$$R_k(s;\alpha_1,\ldots,\alpha_k) = \prod_P \left(1 + (1+|P|^{-1})^{-1} \sum_{j=1}^{\infty} \sum_{\substack{e_1,\ldots,e_k \geq 0 \\ e_1+\ldots+e_k=2j}} \prod_{i=1}^{k} \frac{1}{|P|^{e_i(s+\alpha_i)}}\right)$$
$$= \prod_P R_{k,P}. \tag{6.4.25}$$

Also, we know that

$$(1+|P|^{-1})^{-1} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{|P|^{\ell}},$$

thus

$$R_{k,P} = 1 + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{\substack{e_1,\ldots,e_k \geq 0 \\ e_1+\ldots+e_k=2j}} \prod_{i=1}^{k} \frac{(-1)^{\ell}}{|P|^{e_i(s+\alpha_i)+\ell}}. \tag{6.4.26}$$

When $\alpha_i = 0$ and $s = \frac{1}{2}$, only terms with $e_1 + \ldots + e_k = 2$ give rise to poles. Isolating the term with $\ell = 0$ and $j = 1$, we have

$$R_{k,P} = 1 + \sum_{e_1+\ldots+e_k=2} \prod_{i=1}^{k} \frac{1}{|P|^{e_i(s+\alpha_i)}} + (\text{lower order terms})$$
$$= 1 + \sum_{1 \leq i \leq j \leq k} \frac{1}{|P|^{2s+\alpha_i+\alpha_j}} + (\text{lower order terms}).$$

Thus, for $\mathfrak{R}(\alpha_i)$ sufficiently small, we have (by [CFK+05, p.87]),

$$R_{k,P} = 1 + \sum_{1 \leq i \leq j \leq k} \frac{1}{|P|^{2s+\alpha_i+\alpha_j}} + O\left(|P|^{-1-2s+\epsilon}\right) + O\left(|P|^{-3s+\epsilon}\right).$$

Expressing $R_{k,P}$ as an Euler product, we have

$$R_{k,P} = \prod_{1 \leq i \leq j \leq k} \left(1 + \frac{1}{|P|^{2s+\alpha_i+\alpha_j}}\right)\left(1 + O\left(|P|^{-1-2s+\epsilon}\right) + O\left(|P|^{-3s+\epsilon}\right)\right). \tag{6.4.27}$$

Furthermore since

$$\prod_P \left(1 + \frac{1}{|P|^{2s}}\right) = \frac{\zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(4s)} \tag{6.4.28}$$

has a simple pole at $s = \frac{1}{2}$ and

$$\prod_P \left(1 + O\left(|P|^{-1-2s+\epsilon}\right) + O\left(|P|^{-3s+\epsilon}\right)\right)$$

is analytic in $\Re(s) > \frac{1}{3}$, we see that $\prod_P R_{k,P}$ has a pole at $s = \frac{1}{2}$ of order $\frac{1}{2}k(k+1)$ if $\alpha_1 = \ldots = \alpha_k = 0$.

Since we have identified the leading order poles, we can now factor out the appropriate zeta factors and thus put Conjecture (6.4.18) into a more desirable form in order to obtain Conjecture 6.2.1. Using (6.4.25) and (6.4.27) we have that

$$R_k(s;\alpha_1,\ldots,\alpha_k) = \prod_P \left(\prod_{1\le i\le j\le k} \left(1 + \frac{1}{|P|^{2s+\alpha_i+\alpha_j}}\right) \left(1 + O\left(|P|^{-1-2s+\epsilon}\right) + O\left(|P|^{-3s+\epsilon}\right)\right)\right).$$

Using (6.4.28), we have

$$R_k(s;\alpha_1,\ldots,\alpha_k) = \prod_{1\le i\le j\le k} \frac{\zeta_{\mathbb{A}}(2s+\alpha_i+\alpha_j)}{\zeta_{\mathbb{A}}(2(2s+\alpha_i+\alpha_j))} \prod_P \left(1 + O\left(|P|^{-1-2s+\epsilon}\right) + O\left(|P|^{-3s+\epsilon}\right)\right)$$

Using (2.1.3), we have

$$\begin{aligned} R_k(s;\alpha_1,\ldots,\alpha_k) = &\prod_{1\le i\le j\le k} \zeta_{\mathbb{A}}(2s+\alpha_i+\alpha_j) \prod_P \left(1 - \frac{1}{|P|^{2s+\alpha_i+\alpha_j}}\right)\left(1 + \frac{1}{|P|^{2s+\alpha_i+\alpha_j}}\right) \\ &\times \left(1 + O\left(|P|^{-1-2s+\epsilon}\right) + O\left(|P|^{-3s+\epsilon}\right)\right). \end{aligned}$$

Finally, using (6.4.27), we have

$$\begin{aligned} R_k(s;\alpha_1,\ldots,\alpha_k) &= \prod_{1\le i\le j\le k} \zeta_{\mathbb{A}}(2s+\alpha_i+\alpha_j) \prod_P R_{k,P}(s;\alpha_1,\ldots,\alpha_k)\left(1 - \frac{1}{|P|^{2s+\alpha_i+\alpha_j}}\right) \\ &= \prod_{1\le i\le j\le k} \zeta_{\mathbb{A}}(2s+\alpha_i+\alpha_j) A(s;\alpha_1,\ldots,\alpha_k), \end{aligned}$$

where

$$A(s,\alpha_1,\ldots,\alpha_k) = \prod_P R_{k,P}(s;\alpha_1,\ldots,\alpha_k)\left(1 - \frac{1}{|P|^{2s+\alpha_i+\alpha_j}}\right). \tag{6.4.29}$$

Here $A(s,\alpha_1,\ldots,\alpha_k)$ defines an absolutely convergent Dirichlet series for $\Re(s) > \frac{1}{2}+\delta$ for some $\delta > 0$ and for all α_j's in some sufficiently small neighbourhood of 0. Furthermore, we can write $A(s;\alpha_1,\ldots,\alpha_k)$ by using the following Lemma.

Lemma 6.4.4. *Using the notation described previously, we have*

$$\begin{aligned} A\left(\frac{1}{2};z_1,\ldots,z_k\right) = &\prod_P \prod_{1\le i\le j\le k} \left(1 - \frac{1}{|P|^{1+z_i+z_j}}\right) \\ &\times \left(\frac{1}{2}\left(\prod_{j=1}^k \left(1 - \frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1} + \prod_{j=1}^k \left(1 + \frac{1}{|P|^{\frac{1}{2}+z_j}}\right)^{-1}\right) + \frac{1}{|P|}\right)\left(1 + \frac{1}{|P|}\right)^{-1}. \end{aligned} \tag{6.4.30}$$

Proof. From (6.4.29), we have that

$$A\left(\frac{1}{2};z_1,\ldots,z_k\right)=\prod_P\left(R_{k,P}\left(\frac{1}{2};z_1,\ldots,z_k\right)\prod_{1\le i\le j\le k}\left(1-\frac{1}{|P|^{1+z_i+z_j}}\right)\right), \tag{6.4.31}$$

where

$$R_{k,P}\left(\frac{1}{2};z_1,\ldots,z_k\right)=1+(1+|P|^{-1})^{-1}\sum_{j=1}^{\infty}\sum_{\substack{e_1,\ldots,e_k\ge 0\\ e_1+\ldots+e_k=2j}}\prod_{i=1}^{k}\frac{1}{|P|^{e_i(\frac{1}{2}+z_i)}}.$$

Furthermore, we have

$$R_{k,P}\left(\frac{1}{2};z_1,\ldots,z_k\right)=\left(1+|P|^{-1}\right)^{-1}\left(\left(1+|P|^{-1}\right)+\sum_{j=1}^{\infty}\sum_{\substack{e_1,\ldots,e_k\ge 0\\ e_1+\ldots+e_k=2j}}\prod_{i=1}^{k}\frac{1}{|P|^{e_i(\frac{1}{2}+z_i)}}\right). \tag{6.4.32}$$

Thus

$$\begin{aligned}1+\sum_{j=1}^{\infty}\sum_{\substack{e_1,\ldots,e_k\ge 0\\ e_1+\ldots+e_k=2j}}\prod_{i=1}^{k}\frac{1}{|P|^{e_i(\frac{1}{2}+z_i)}}&=\frac{1}{2}\sum_{j=0}^{\infty}\sum_{\substack{e_1,\ldots,e_k\ge 0\\ e_1+\ldots+e_k=2j}}2\prod_{i=1}^{k}\frac{1}{|P|^{e_i(\frac{1}{2}+z_i)}}\\&=\frac{1}{2}\left(\prod_{i=1}^{k}\sum_{e_i\ge 0}\left(\frac{1}{|P|^{\frac{1}{2}+z_i}}\right)^{e_i}+\prod_{i=1}^{k}\sum_{e_i\ge 0}(-1)^{e_1+\ldots+e_k}\left(\frac{1}{|P|^{\frac{1}{2}+z_i}}\right)^{e_i}\right)\\&=\frac{1}{2}\left(\prod_{i=1}^{k}\left(1-\frac{1}{|P|^{\frac{1}{2}+z_i}}\right)^{-1}+\prod_{i=1}^{k}\left(1+\frac{1}{|P|^{\frac{1}{2}+z_i}}\right)^{-1}\right).\end{aligned} \tag{6.4.33}$$

Putting (6.4.33) and (6.4.32) into (6.4.31) completes the proof of Lemma 6.4.4. ∎

From (6.4.17) and (6.4.29) we have

$$\begin{aligned}M\left(\frac{1}{2};\alpha_1,\ldots,\alpha_k\right)&=\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}R_k\left(\frac{1}{2};\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k\right)\\&=\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}\\&\times\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)A\left(\frac{1}{2};\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k\right).\end{aligned}$$

Therefore, from (6.4.18) the conjectured asymptotic takes the form

$$\begin{aligned}&\sum_{u\in\mathcal{I}_{g+1}}Z\left(\frac{1}{2};\alpha_1,\ldots,\alpha_k\right)\\&=\sum_{u\in\mathcal{I}_{g+1}}\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}A\left(\frac{1}{2};\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k\right)\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)(1+o(1)).\end{aligned}$$

Using the definition of $\mathcal{X}_u(s)$, we have that

$$\mathcal{X}_u\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}=\left(q^{2g+1}\right)^{\frac{\epsilon_j\alpha_j}{2}}X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}.$$

Thus we arrive at the following conjecture:

$$\sum_{u\in\mathcal{I}_{g+1}} \Lambda\left(\frac{1}{2}+\alpha_1,\chi_u\right)\ldots\Lambda\left(\frac{1}{2}+\alpha_k,\chi_u\right)$$
$$=\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k} X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}\sum_{u\in\mathcal{I}_{g+1}}(q^{2g+1})^{\frac{1}{2}\sum_{j=1}^{k}\epsilon_j\alpha_j} R_k\left(\frac{1}{2};\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k\right)(1+o(1)). \tag{6.4.34}$$

6.4.4 The Integral Representation of the Conjecture

In this subsection, we will write Conjecture (6.4.34) as contour integrals. To do this, we will need the following lemma.

Lemma 6.4.5 ([CFK+05, Lemma 2.5.2])**.** *Suppose F is a symmetric function in k variables, regular near $(0,\ldots,0)$ and $f(s)$ has a simple pole of residue 1 at $s=0$ and is otherwise analytic in a neighbourhood of $s=0$, and let*

$$K(a_1,\ldots,a_k)=F(a_1,\ldots,a_k)\prod_{1\le i\le j\le k} f(a_i+a_j)$$

or

$$K(a_1,\ldots,a_k)=F(a_1,\ldots,a_k)\prod_{1\le i<j\le k} f(a_i+a_j).$$

If $\alpha_i+\alpha_j$ is are contained in the region of analyticity of $f(s)$ then

$$\sum_{\epsilon_j=\pm1} K(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k)=\frac{(-1)^{\frac{k(k-1)}{2}}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\ldots\oint K(z_1,\ldots,z_k)$$
$$\times\frac{\Delta(z_1^2,\ldots,z_k^2)^2\prod_{j=1}^{k} z_j}{\prod_{i=1}^{k}\prod_{j=1}^{k}(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\ldots dz_k$$

and

$$\sum_{\epsilon_j=\pm1}\left(\prod_{j=1}^{k}\epsilon_j\right) K(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k)=\frac{(-1)^{\frac{k(k-1)}{2}}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\ldots\oint K(z_1,\ldots,z_k)$$
$$\times\frac{\Delta(z_1^2,\ldots,z_k^2)^2\prod_{j=1}^{k} \alpha_j}{\prod_{i=1}^{k}\prod_{j=1}^{k}(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\ldots dz_k,$$

where the path of integration encloses the $\pm\alpha_j$'s.

First recall that

$$\sum_{u\in\mathcal{I}_{g+1}}\prod_{j=1}^{k}\Lambda\left(\frac{1}{2}+\alpha_j,\chi_u\right)=\sum_{u\in\mathcal{I}_{g+1}}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\alpha_j\right)^{-\frac{1}{2}}L\left(\frac{1}{2}+\alpha_j,\chi_u\right) \tag{6.4.35}$$

and

$$\sum_{u\in\mathcal{I}_{g+1}} Z\left(\frac{1}{2};\alpha_1,\dots,\alpha_k\right)$$
$$=\sum_{u\in\mathcal{I}_{g+1}}\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}} A\left(\frac{1}{2};\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k\right)\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)(1+o(1)). \tag{6.4.36}$$

Since $\mathcal{X}_u\left(\frac{1}{2}+\alpha_j\right)^{-\frac{1}{2}}$ does not depend on u, we can factor it out and, using (6.4.35) and (6.4.36), we have

$$\sum_{u\in\mathcal{I}_{g+1}}\prod_{j=1}^{k} L\left(\frac{1}{2}+\alpha_j,\chi_u\right)$$
$$=\sum_{u\in\mathcal{I}_{g+1}}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\alpha_j\right)^{\frac{1}{2}}\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\mathcal{X}_u\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}} A\left(\frac{1}{2};\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k\right)$$
$$\times\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)(1+o(1)).$$

Using the definition of $\mathcal{X}_u(s)$ we have

$$\sum_{u\in\mathcal{I}_{g+1}}\prod_{j=1}^{k} L\left(\frac{1}{2}+\alpha_j,\chi_u\right)$$
$$=\sum_{u\in\mathcal{I}_{g+1}}\prod_{j=1}^{k}\left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{j=1}^{k}\alpha_j} X\left(\frac{1}{2}+\alpha_j\right)^{\frac{1}{2}}\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{j=1}^{k}\epsilon_j\alpha_j} X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}$$
$$\times A\left(\frac{1}{2};\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k\right)\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)(1+o(1)).$$

Multiplying and dividing by $(\log q)^{\frac{k(k+1)}{2}}$, we have

$$\sum_{u\in\mathcal{I}_{g+1}}\prod_{j=1}^{k} L\left(\frac{1}{2}+\alpha_j,\chi_u\right)$$
$$=\sum_{u\in\mathcal{I}_{g+1}}\frac{\prod_{j=1}^{k}\left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{j=1}^{k}\alpha_j} X\left(\frac{1}{2}+\alpha_j\right)^{\frac{1}{2}}}{(\log q)^{\frac{k(k+1)}{2}}}\sum_{\epsilon_j=\pm1}\prod_{j=1}^{k}\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{j=1}^{k}\epsilon_j\alpha_j} X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}$$
$$\times A\left(\frac{1}{2};\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k\right)\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)(\log q)(1+o(1)). \tag{6.4.37}$$

If we call

$$F(\alpha_1,\dots,\alpha_k)=\prod_{j=1}^{k}\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{j=1}^{k}\alpha_j} X\left(\frac{1}{2}+\alpha_j\right)^{-\frac{1}{2}} A\left(\frac{1}{2};\alpha_1,\dots,\alpha_k\right) \tag{6.4.38}$$

and

$$f(s)=\zeta_{\mathbb{A}}(1+s)(\log q) \tag{6.4.39}$$

so that

$$f(\alpha_i + \alpha_j) = \zeta_{\mathbb{A}}(1 + \alpha_i + \alpha_j)(\log q), \tag{6.4.40}$$

then $f(s)$ has a simple pole at $s = 0$ with residue 1. Denoting

$$K(\alpha_1, \ldots, \alpha_k) = F(\alpha_1, \ldots, \alpha_k) \prod_{1 \le i \le j \le k} f(\alpha_i + \alpha_j), \tag{6.4.41}$$

then, using (6.4.37), we have that

$$\begin{aligned}
&\sum_{u \in \mathcal{I}_{g+1}} \prod_{j=1}^{k} L\left(\frac{1}{2} + \alpha_j, \chi_u\right) \\
&= \left(\sum_{u \in \mathcal{I}_{g+1}} \frac{\prod_{j=1}^{k} \left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{j=1}^{k} \alpha_j} X\left(\frac{1}{2} + \alpha_j\right)^{\frac{1}{2}}}{(\log q)^{\frac{k(k+1)}{2}}} \sum_{\epsilon_j = \pm 1} K(\epsilon_1 \alpha_1, \ldots, \epsilon_k \alpha_k) \right) (1 + o(1)).
\end{aligned} \tag{6.4.42}$$

Thus applying Lemma 6.4.5, we have

$$\begin{aligned}
&\sum_{u \in \mathcal{I}_{g+1}} \prod_{j=1}^{k} L\left(\frac{1}{2} + \alpha_j, \chi_u\right) \\
&= \sum_{u \in \mathcal{I}_{g+1}} \frac{\prod_{j=1}^{k} \left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{j=1}^{k} \alpha_j} X\left(\frac{1}{2} + \alpha_j\right)^{\frac{1}{2}}}{(\log q)^{\frac{k(k+1)}{2}}} \frac{(-1)^{\frac{k(k-1)}{2}} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint K(z_1, \ldots, z_k) \\
&\times \frac{\Delta(z_1^2, \ldots, z_k^2)^2 \prod_{j=1}^{k} z_j}{\prod_{i=1}^{k} \prod_{j=1}^{k} (z_i - \alpha_j)(z_i + \alpha_j)} dz_1 \ldots dz_k + o\left(q^{2g+1}\right).
\end{aligned}$$

Using (6.4.38), (6.4.40) and (6.4.41) we have

$$\begin{aligned}
&\sum_{u \in \mathcal{I}_{g+1}} \prod_{j=1}^{k} L\left(\frac{1}{2} + \alpha_j, \chi_u\right) \\
&= \sum_{u \in \mathcal{I}_{g+1}} \left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{j=1}^{k} \alpha_j} X\left(\frac{1}{2} + \alpha_j\right)^{\frac{1}{2}} \\
&\times \frac{(-1)^{\frac{k(k-1)}{2}} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \prod_{j=1}^{k} \left(q^{2g+1}\right)^{\frac{1}{2}\sum_{j=1}^{k} \alpha_j} X\left(\frac{1}{2} + \alpha_j\right)^{-\frac{1}{2}} A\left(\frac{1}{2}; \alpha_1, \ldots, \alpha_k\right) \\
&\times \prod_{1 \le i \le j \le k} \zeta_{\mathbb{A}}(1 + z_i + z_j) \frac{\Delta(z_1^2, \ldots, z_k^2)^2 \prod_{j=1}^{k} z_j}{\prod_{i=1}^{k} \prod_{j=1}^{k} (z_i - \alpha_j)(z_i + \alpha_j)} dz_1 \ldots dz_k + o\left(q^{2g+1}\right).
\end{aligned}$$

If we let

$$G(z_1, \ldots, z_k) = \prod_{j=1}^{k} X\left(\frac{1}{2} + z_j\right)^{-\frac{1}{2}} A\left(\frac{1}{2}; z_1, \ldots, z_k\right) \prod_{1 \le i \le j \le k} \zeta_{\mathbb{A}}(1 + z_i + z_j),$$

then we have

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2}+\alpha_1,\chi_u\right)\dots L\left(\frac{1}{2}+\alpha_k,\chi_u\right)$$
$$=\sum_{u\in\mathcal{I}_{g+1}}\prod_{j=1}^{k}\left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{j=1}^{k}\alpha_j} X\left(\frac{1}{2}+\alpha_j\right)^{\frac{1}{2}}\frac{(-1)^{\frac{k(k-1)}{2}}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\dots\oint G(z_1,\dots,z_k)$$
$$\times\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{j=1}^{k}z_j}\frac{\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^{k}z_j}{\prod_{i=1}^{k}\prod_{j=1}^{k}(z_i-\alpha_j)(z_i+\alpha_j)}dz_1\dots dz_k+o\left(q^{2g+1}\right). \tag{6.4.43}$$

Finally, let

$$Q_k(x)$$
$$=\frac{(-1)^{\frac{k(k-1)}{2}}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\dots\oint\frac{G(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^{k}z_j}{\prod_{i=1}^{k}\prod_{j=1}^{k}(z_i-\alpha_j)(z_i+\alpha_j)}q^{\frac{x}{2}\sum_{j=1}^{k}z_j}dz_1\dots dz_k, \tag{6.4.44}$$

then setting $\alpha_j=0$ we obtain the formulae stated in Conjecture 6.2.1.

6.5 Some Conjectural Formulae for Moments of $L\left(\frac{1}{2},\chi_u\right)$

In this section, we use Conjecture 6.2.1 to obtain explicit conjectural formulae for the first three moments of quadratic Dirichlet L-functions in even characteristic. In particular we will show that, for the first moment, Conjecture 6.2.1 agrees with (6.1.7).

6.5.1 First Moment

We will use Conjecture 6.2.1 to determine the asymptotic formula for the first moment of our family of Dirichlet L-functions and compare it with (6.1.7). For the first moment Conjecture 6.2.1 predicts that

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right)=\sum_{u\in\mathcal{I}_{g+1}} Q_1(2g+1)(1+o(1)),$$

where $Q_1(x)$ is a polynomial of degree 1. From Conjecture 6.2.1, we have

$$Q_1(x)=\frac{1}{\pi i}\oint\frac{G(z_1)\Delta(z_1^2)^2q^{\frac{x}{2}z_1}}{z_1}dz_1, \tag{6.5.1}$$

where

$$G(z_1)=A\left(\frac{1}{2};z_1\right)X\left(\frac{1}{2}+z_1\right)^{-\frac{1}{2}}\zeta_{\mathbb{A}}(1+2z_1).$$

From the definition of the Vandermonde determinant and the definition of $X(s)$, we have that

$$\Delta(z_1^2)^2=1\quad\text{and}\quad X\left(\frac{1}{2}+z_j\right)^{-\frac{1}{2}}=q^{-\frac{z_1}{2}}.$$

Therefore (6.5.1) becomes

$$Q_1(x) = \frac{1}{\pi i} \oint \frac{A\left(\frac{1}{2}; z_1\right) \zeta_{\mathbb{A}}(1+2z_1) q^{\frac{x}{2}z_1} q^{-\frac{z_1}{2}}}{z_1} dz_1. \tag{6.5.2}$$

From Lemma 6.4.4 we have

$$A\left(\frac{1}{2}; z_1\right) = \prod_P \left(1 - \frac{1}{|P|^{1+2z_1}}\right)\left(\frac{1}{2}\left(\left(1 - \frac{1}{|P|^{\frac{1}{2}+z_1}}\right)^{-1} + \left(1 + \frac{1}{|P|^{\frac{1}{2}+z_1}}\right)^{-1}\right) + \frac{1}{|P|}\right)\left(1 + \frac{1}{|P|}\right)^{-1}.$$

We want to compute the integral (6.5.2), where the contour is a small circle around the origin. For this we need to locate the poles of the integrand. Let

$$f(z_1) = \frac{A\left(\frac{1}{2}; z_1\right) \zeta_{\mathbb{A}}(1+2z_1) q^{\frac{x}{2}z_1} q^{-\frac{z_1}{2}}}{z_1}, \tag{6.5.3}$$

then $f(z_1)$ has a double pole at $z_1 = 0$. To compute the residue, we expand $f(z_1)$ as a Laurent series and pick up the coefficient of z_1^{-1}. Expanding the numerator of (6.5.3) around $z_1 = 0$, we have

$$A\left(\frac{1}{2}; z_1\right) = A\left(\frac{1}{2}; 0\right) + A'\left(\frac{1}{2}; 0\right) z_1 + \frac{1}{2} A''\left(\frac{1}{2}; 0\right) z_1^2 + \dots,$$

$$\zeta_{\mathbb{A}}(1+2z_1) = \frac{1}{2\log q}\frac{1}{z_1} + \frac{1}{2} + \frac{1}{6}(\log q) z_1 - \frac{1}{90}(\log q)^3 z_1^3 + \dots,$$

$$q^{\frac{x}{2}z_1} = 1 + \frac{1}{2}(\log q) x z_1 + \frac{1}{8}(\log q)^2 x^2 z_1^2 + \dots$$

and

$$q^{-\frac{z_1}{2}} = 1 - \frac{1}{2}(\log q) z_1 + \frac{1}{8}(\log q)^2 z_1^2 + \dots .$$

Thus we have

$$\begin{aligned} f(z_1) = \frac{1}{z_1} &\left(A\left(\frac{1}{2}; 0\right) + A'\left(\frac{1}{2}; 0\right) z_1 + \frac{1}{2} A''\left(\frac{1}{2}; 0\right) z_1^2 + \dots\right) \\ &\times \left(1 - \frac{1}{2}(\log q) z_1 + \frac{1}{8}(\log q) z_1^2 + \dots\right) \\ &\times \left(\frac{1}{2\log q}\frac{1}{z_1} + \frac{1}{2} + \frac{1}{6}(\log q) z_1 - \frac{1}{90}(\log q)^3 z_1^3 + \dots\right) \\ &\times \left(1 + \frac{1}{2}(\log q) x z_1 + \frac{1}{8}(\log q) x^2 z_1^2 + \dots\right) \end{aligned}$$

Multiplying the above expression and collecting the terms corresponding to z_1^{-1}, we see that

$$\operatorname{Res}(f(z_1); z_1 = 0) = \frac{1}{4} A\left(\frac{1}{2}; 0\right) + \frac{1}{4} A\left(\frac{1}{2}; 0\right) x + \frac{1}{2\log q} A'\left(\frac{1}{2}; 0\right).$$

We know that

$$A\left(\frac{1}{2}; 0\right) = P(1) \quad \text{and} \quad A'\left(\frac{1}{2}; 0\right) = 2P'(1),$$

where

$$P(s) = \prod_P \left(1 - \frac{1}{|P|^s(|P|+1)}\right).$$

Thus

$$\frac{1}{\pi i}\oint \frac{A\left(\frac{1}{2};z_1\right)\zeta_{\mathbb{A}}(1+2z_1)q^{\frac{x}{2}z_1}q^{-\frac{z_1}{2}}}{z_1}dz_1 = \frac{1}{2}P(1)\left(x+1+\frac{4}{\log q}\frac{P'}{P}(1)\right).$$

We therefore have

$$\begin{aligned}\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right) &= \sum_{u\in\mathcal{I}_{g+1}} Q_1(2g+1)(1+o(1))\\ &= \sum_{u\in\mathcal{I}_{g+1}} P(1)\left(g+1+\frac{2}{\log q}\frac{P'}{P}(1)\right)(1+o(1))\\ &= P(1)\left(g+1+\frac{2}{\log q}\frac{P'}{P}(1)\right)\sum_{u\in\mathcal{I}_{g+1}}1 + o\left(q^{2g+1}\right).\end{aligned}$$

Using Lemma 2.7.3, we conclude that, for the first moment, Conjecture 6.2.1 predicts

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}P(1)\left(g+1+\frac{2}{\log q}\frac{P'}{P}(1)\right) + o\left(q^{2g+1}\right). \quad (6.5.4)$$

Comparing (6.1.7) and (6.5.4), we see that the main and the principal lower order terms are the same. Hence Theorem 6.1.3 proves Conjecture 6.2.1 with an error of $O\left(g2^{\frac{g}{2}}q^{\frac{3g}{2}}\right)$ when $k=1$.

6.5.2 Second Moment

For the second moment, Conjecture 6.2.1 predicts that

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right)^2 = \sum_{u\in\mathcal{I}_{g+1}} Q_2(2g+1)(1+o(1)),$$

where

$$Q_2(x) = -\frac{2}{(2\pi i)^2}\oint\oint \frac{G(z_1,z_2)\Delta(z_1^2,z_2^2)^2 q^{\frac{x}{2}(z_1+z_2)}}{z_1^3 z_2^3}dz_1dz_2.$$

From Conjecture 6.2.1, we have

$$\begin{aligned}G(z_1,z_2) &= A\left(\frac{1}{2};z_1,z_2\right)\prod_{j=1}^{2} X\left(\frac{1}{2}+z_j\right)^{-\frac{1}{2}} \prod_{1\le i\le j\le 2}\zeta_{\mathbb{A}}(1+z_i+z_j)\\ &= A\left(\frac{1}{2};z_1,z_2\right)X\left(\frac{1}{2}+z_1\right)^{-\frac{1}{2}}X\left(\frac{1}{2}+z_2\right)^{-\frac{1}{2}}\zeta_{\mathbb{A}}(1+2z_1)\zeta_{\mathbb{A}}(1+z_1+z_2)\zeta_{\mathbb{A}}(1+2z_2).\end{aligned}$$

From the definition of the Vandermonde determinant and the definition of $X(s)$, we have

$$\Delta(z_1^2,z_2^2)^2 = (z_2^2-z_1^2)^2 \quad \text{and} \quad X\left(\frac{1}{2}+z_1\right)^{-\frac{1}{2}}X\left(\frac{1}{2}+z_2\right)^{-\frac{1}{2}} = q^{-\frac{1}{2}(z_1+z_2)}.$$

Thus

$$Q_2(x) = -\frac{2}{(2\pi i)^2}\oint\oint \frac{A\left(\frac{1}{2};z_1,z_2\right)\zeta_{\mathbb{A}}(1+2z_1)\zeta_{\mathbb{A}}(1+z_1+z_2)\zeta_{\mathbb{A}}(1+2z_2)}{z_1^3z_2^3} \times (z_2^2-z_1^2)^2 q^{\frac{x}{2}(z_1+z_2)}q^{-\frac{1}{2}(z_1+z_2)}dz_1dz_2.$$

Using MATHEMATICA, we have that

$$\begin{aligned} Q_2(x) = \frac{1}{24\log^3(q)}\Bigg[&(x^3+6x^2+11x+6)A\left(\frac{1}{2};0,0\right)\log^3(q) \\ &+(3x^2+12x+11)\log^2(q)\left(A_1\left(\frac{1}{2};0,0\right)+A_2\left(\frac{1}{2};0,0\right)\right) \\ &+12(2+x)A_{12}\left(\frac{1}{2};0,0\right)\log(q) \\ &-2\left(A_{111}\left(\frac{1}{2};0,0\right)-3A_{112}\left(\frac{1}{2};0,0\right)-3A_{122}\left(\frac{1}{2};0,0\right)+A_{222}\left(\frac{1}{2};0,0\right)\right)\Bigg], \end{aligned}$$

where A_j denotes the partial derivative, evaluated at zero of the function $A\left(\frac{1}{2};z_1,\ldots,z_k\right)$ with respect to the j^{th} variable. Hence the leading order asymptotic for the second moment for this family of L-functions can be written conjecturally as

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right)^2 \sim \frac{2}{3}\frac{q^{2g+1}}{\zeta_{\mathbb{A}}(2)}g^3A\left(\frac{1}{2};0,0\right)$$

when $g\to\infty$, where

$$A\left(\frac{1}{2};0,0\right) = \prod_P\left(1-\frac{4|P|^2-3|P|+1}{|P|^3(|P|+1)}\right).$$

6.5.3 Third Moment

For the third moment, Conjecture 6.2.1 predicts that

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right)^3 = \sum_{u\in\mathcal{I}_{g+1}} Q_3(2g+1)(1+o(1)),$$

where

$$Q_3(x) = -\frac{4}{3}\frac{1}{(2\pi i)^3}\oint\oint\oint \frac{G(z_1,z_2,z_3)\Delta(z_1^2,z_2^2,z_3^2)^2q^{\frac{x}{2}(z_1+z_2+z_3)}}{z_1^5z_2^5z_3^5}dz_1dz_2dz_3.$$

From Conjecture 6.2.1 we have

$$\begin{aligned} G(z_1,z_2,z_3) &= A\left(\frac{1}{2};z_1,z_2,z_3\right)\prod_{j=1}^3 X\left(\frac{1}{2}+z_j\right)^{-\frac{1}{2}}\prod_{1\le i\le j\le 3}\zeta_{\mathbb{A}}(1+z_i+z_j) \\ &= A\left(\frac{1}{2};z_1,z_2,z_3\right)X\left(\frac{1}{2}+z_1\right)^{-\frac{1}{2}}X\left(\frac{1}{2}+z_2\right)^{-\frac{1}{2}}X\left(\frac{1}{2}+z_3\right)^{-\frac{1}{2}} \\ &\times\zeta_{\mathbb{A}}(1+2z_1)\zeta_{\mathbb{A}}(1+z_1+z_2)\zeta_{\mathbb{A}}(1+z_1+z_3) \\ &\times\zeta_{\mathbb{A}}(1+2z_2)\zeta_{\mathbb{A}}(1+z_2+z_3)\zeta_{\mathbb{A}}(1+2z_3). \end{aligned}$$

From the definition of the Vandermonde determinant and the definition of $X(s)$, we have

$$\Delta(z_1^2,z_2^2,z_3^2)^2=(z_2^2-z_1^2)^2(z_3^2-z_1^2)^2(z_3^2-z_2^2)^2$$

and

$$X\left(\frac{1}{2}+z_1\right)^{-\frac{1}{2}}X\left(\frac{1}{2}+z_2\right)^{-\frac{1}{2}}X\left(\frac{1}{2}+z_3\right)^{-\frac{1}{2}}=q^{-\frac{1}{2}(z_1+z_2+z_3)}.$$

Thus

$$Q_3(x)=-\frac{4}{3}\frac{1}{(2\pi i)^3}\oint\oint\oint f(z_1,z_2,z_3)dz_1dz_2dz_3,$$

where

$$\begin{aligned}f(z_1,z_2,z_3)&=\frac{A\left(\frac{1}{2};z_1,z_2,z_3\right)(z_2^2-z_1^2)^2(z_3^2-z_1^2)^2(z_3^2-z_2^2)^2q^{\frac{x}{2}(z_1+z_2+z_3)}q^{-\frac{1}{2}(z_1+z_2+z_3)}}{z_1^5z_2^5z_3^5}\\&\times\zeta_{\mathbb{A}}(1+2z_1)\zeta_{\mathbb{A}}(1+z_1+z_2)\zeta_{\mathbb{A}}(1+z_1+z_3)\\&\times\zeta_{\mathbb{A}}(1+2z_2)\zeta_{\mathbb{A}}(1+z_2+z_3)\zeta_{\mathbb{A}}(1+2z_3).\end{aligned}$$

Using MATHEMATICA, we have

$$\begin{aligned}&Q_3(x)\\&=\frac{1}{8640\log^6(q)}\Bigg[3(3+x)^2(x^4+12x^3+49x^2+78x+40)A\left(\frac{1}{2};0,0,0\right)\log^6(q)\\&+4(3x^5+45x^4+260x^3+720x^2+949x+471)\log^5(q)\\&\times\left(A_1\left(\frac{1}{2};0,0,0\right)+A_2\left(\frac{1}{2};0,0,0\right)+A_3\left(\frac{1}{2};0,0,0\right)\right)+4(949+1440x+780x^2+180x^3+15x^4)\\&\times\log^4(q)\left(A_{23}\left(\frac{1}{2};0,0,0\right)+A_{13}\left(\frac{1}{2};0,0,0\right)+A_{12}\left(\frac{1}{2};0,0,0\right)\right)-10(24+26x+9x^2+x^3)\\&\times\log^3(q)\Bigg(2A_{333}\left(\frac{1}{2};0,0,0\right)-3A_{233}\left(\frac{1}{2};0,0,0\right)-3A_{223}\left(\frac{1}{2};0,0,0\right)+2A_{222}\left(\frac{1}{2};0,0,0\right)\\&-3A_{133}\left(\frac{1}{2};0,0,0\right)-36A_{123}\left(\frac{1}{2};0,0,0\right)-3A_{122}\left(\frac{1}{2};0,0,0\right)-3A_{113}\left(\frac{1}{2};0,0,0\right)\\&-3A_{112}\left(\frac{1}{2};0,0,0\right)+2A_{111}\left(\frac{1}{2};0,0,0\right)\Bigg)-20(26+18x+3x^2)\log^2(q)\Bigg(A_{2333}\left(\frac{1}{2};0,0,0\right)\\&+A_{2223}\left(\frac{1}{2};0,0,0\right)+A_{1333}\left(\frac{1}{2};0,0,0\right)-6A_{1233}\left(\frac{1}{2};0,0,0\right)-6A_{1223}\left(\frac{1}{2};0,0,0\right)\\&+A_{1222}\left(\frac{1}{2};0,0,0\right)-6A_{1123}\left(\frac{1}{2};0,0,0\right)+A_{1113}\left(\frac{1}{2};0,0,0\right)+A_{1112}\left(\frac{1}{2};0,0,0\right)\Bigg)\\&+6(3+x)\log(q)\Bigg(2A_{33333}\left(\frac{1}{2};0,0,0\right)-5A_{23333}\left(\frac{1}{2};0,0,0\right)-10A_{22333}\left(\frac{1}{2};0,0,0\right)\\&-10A_{22233}\left(\frac{1}{2};0,0,0\right)-5A_{22223}\left(\frac{1}{2};0,0,0\right)+2A_{22222}\left(\frac{1}{2};0,0,0\right)-5A_{13333}\left(\frac{1}{2};0,0,0\right)\\&+60A_{12233}\left(\frac{1}{2};0,0,0\right)-5A_{12222}\left(\frac{1}{2};0,0,0\right)-10A_{11333}\left(\frac{1}{2};0,0,0\right)+60A_{11233}\left(\frac{1}{2};0,0,0\right)\end{aligned}$$

$$+60A_{11223}\left(\frac{1}{2};0,0,0\right)-10A_{11222}\left(\frac{1}{2};0,0,0\right)-10A_{11133}\left(\frac{1}{2};0,0,0\right)-10A_{11122}\left(\frac{1}{2};0,0,0\right)$$
$$-5A_{11113}\left(\frac{1}{2};0,0,0\right)-5A_{11112}\left(\frac{1}{2};0,0,0\right)+2A_{11111}\left(\frac{1}{2};0,0,0\right)\Bigg)+4\Bigg(3A_{233333}\left(\frac{1}{2};0,0,0\right)$$
$$-20A_{222333}\left(\frac{1}{2};0,0,0\right)+3A_{222223}\left(\frac{1}{2};0,0,0\right)+3A_{133333}\left(\frac{1}{2};0,0,0\right)-30A_{123333}\left(\frac{1}{2};0,0,0\right)$$
$$+30A_{122333}\left(\frac{1}{2};0,0,0\right)+30A_{122233}\left(\frac{1}{2};0,0,0\right)-30A_{122223}\left(\frac{1}{2};0,0,0\right)$$
$$+3A_{122222}\left(\frac{1}{2};0,0,0\right)+30A_{112333}\left(\frac{1}{2};0,0,0\right)+30A_{112223}\left(\frac{1}{2};0,0,0\right)-20A_{111333}\left(\frac{1}{2};0,0,0\right)$$
$$+30A_{111233}\left(\frac{1}{2};0,0,0\right)+30A_{111223}\left(\frac{1}{2};0,0,0\right)-20A_{111222}\left(\frac{1}{2};0,0,0\right)$$
$$-30A_{111123}\left(\frac{1}{2};0,0,0\right)+3A_{111113}\left(\frac{1}{2};0,0,0\right)+3A_{111112}\left(\frac{1}{2};0,0,0\right)\Bigg)\Bigg],$$

where A_j denotes the partial derivative, evaluated at zero of the function $A\left(\frac{1}{2};z_1,\dots,z_k\right)$ with respect to the j^{th} variable. Hence the leading order asymptotic for the third moment for this family of L-functions can be written as

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right)^3 \sim \frac{2}{45}\frac{q^{2g+1}}{\zeta_{\mathbb{A}}(2)}g^6 A\left(\frac{1}{2};0,0,0\right),$$

when $g\to\infty$, where

$$A\left(\frac{1}{2};0,0,0\right)=\prod_P\left(1-\frac{12|P|^5-23|P|^4+23|P|^3-15|P|^2+6|P|-1}{|P|^6(|P|+1)}\right).$$

6.6 Leading order Asymptotic for the Moments of $L\left(\frac{1}{2},\chi_u\right)$

In this section, we will show how to obtain an explicit conjecture for the leading order asymptotic of the moments for a general integer k. We will also use the conjecture to calculate the leading order of the asymptotic for the fourth and fifth moments.

6.6.1 Leading order for general k

To obtain a formula for the leading order asymptotic, we need the following lemma.

Lemma 6.6.1 ([AK14, Lemma 5]). *Suppose F is a symmetric function of k variables, regular near $(0,\dots,0)$ and $f(s)$ has a simple pole of residue 1 at $=0$ and is otherwise analytic in a neighbourhood of $s=0$. Let*

$$K\left(q^{2g+1};w_1,\dots,w_k\right)=\sum_{\epsilon_j=\pm1} e^{\frac{1}{2}\log\left(q^{2g+1}\right)\sum_{j=1}^k \epsilon_j w_j} F(\epsilon_1 w_1,\dots,\epsilon_k w_k)\prod_{1\le i\le j\le k} f(\epsilon_i w_i+\epsilon_j w_j)$$

and define $I\left(q^{2g+1},k,w=0\right)$ *to be the value of* K *when* $w_1,\ldots,w_k=0$. *We have that*

$$I\left(q^{2g+1},k,w=0\right)\sim\left(\frac{1}{2}\log\left(q^{2g+1}\right)\right)^{\frac{k(k+1)}{2}}F(0,\ldots,0)2^{\frac{k(k+1)}{2}}\left(\prod_{j=1}^{k}\frac{j!}{(2j)!}\right).$$

Recall from (6.4.42), we have that

$$\begin{aligned}&\sum_{u\in\mathcal{I}_{g+1}}L\left(\frac{1}{2}+\alpha_1,\chi_u\right)\ldots L\left(\frac{1}{2}+\alpha_k,\chi_u\right)\\&=\left(\sum_{u\in\mathcal{I}_{g+1}}\frac{\prod_{j=1}^{k}\left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{j=1}^{k}\alpha_j}X\left(\frac{1}{2}+\alpha_j\right)^{\frac{1}{2}}}{(\log q)^{\frac{k(k+1)}{2}}}\sum_{\epsilon_j=\pm1}K(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k)\right)(1+o(1)),\end{aligned}$$

where

$$\begin{aligned}K(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k)=&\prod_{j=1}^{k}\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{j=1}^{k}\epsilon_j\alpha_j}X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}\\&\times A\left(\frac{1}{2};\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k\right)\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)(\log q).\end{aligned}$$

Applying Lemma 6.6.1 with

$$f(s)=\zeta_{\mathbb{A}}(1+s)\log q,$$

$$F(\alpha_1,\ldots,\alpha_k)=\prod_{j=1}^{k}X\left(\frac{1}{2}+\alpha_j\right)^{-\frac{1}{2}}A\left(\frac{1}{2};\alpha_1,\ldots,\alpha_k\right)$$

and

$$K\left(q^{2g+1};\alpha_1,\ldots,\alpha_k\right)=\sum_{\epsilon_j=\pm1}\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{j=1}^{k}\epsilon_j\alpha_j}F(\epsilon_1\alpha_1,\ldots,\epsilon_k\alpha_k)\prod_{1\le i\le j\le k}f(\epsilon_i\alpha_i+\epsilon_j\alpha_j),$$

and letting $\alpha_1,\ldots,\alpha_k\to0$ we obtain

$$\sum_{u\in\mathcal{I}_{g+1}}L\left(\frac{1}{2},\chi_u\right)^k\sim\sum_{u\in\mathcal{I}_{g+1}}\frac{1}{(\log q)^{\frac{k(k+1)}{2}}}\left(\frac{1}{2}\log\left(q^{2g+1}\right)\right)^{\frac{k(k+1)}{2}}A\left(\frac{1}{2};0,\ldots,0\right)2^{\frac{k(k+1)}{2}}\prod_{j=1}^{k}\frac{j!}{(2j)!}.$$

Using Lemma 2.7.3 we obtain the following result.

Proposition 6.6.2. *Conditional on Conjecture* 6.2.1, *we have that, as* $g\to\infty$, *the following holds*

$$\sum_{u\in\mathcal{I}_{g+1}}L\left(\frac{1}{2},\chi_u\right)^k\sim2^{\frac{k(k+1)}{2}+1}\frac{q^{2g+1}}{\zeta_{\mathbb{A}}(2)}g^{\frac{k(k+1)}{2}}A\left(\frac{1}{2};0,\ldots,0\right)\prod_{j=1}^{k}\frac{j!}{(2j)!}.$$

6.6.2 Fourth Moment

Proposition 6.6.2 implies that the leading order for the fourth moment is given by

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right)^4 \sim \frac{2}{4725}\frac{q^{2g+1}}{\zeta_{\mathbb{A}}(2)}g^{10}A\left(\frac{1}{2};0,0,0,0\right),$$

where

$$A\left(\frac{1}{2};0,0,0,0\right) = \prod_P\left(1-\frac{h_4(|P|)}{|P|^{10}(|P|+1)}\right)$$

and

$$h_4(x) = 30x^9 - 109x^8 + 210x^7 - 274x^6 + 272x^5 - 210x^4 + 119x^3 - 45x^2 + 10x - 1.$$

6.6.3 Fifth Moment

Proposition 6.6.2 implies that the leading order for the fifth moment is given by

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right)^5 \sim \frac{2}{4465125}\frac{q^{2g+1}}{\zeta_{\mathbb{A}}(2)}g^{15}A\left(\frac{1}{2};0,0,0,0,0\right),$$

where

$$A\left(\frac{1}{2};0,0,0,0,0\right) = \prod_P\left(1-\frac{h_5(|P|)}{|P|^{15}(|P|+1)}\right)$$

and

$$\begin{aligned}h_5(x) = {}& 65x^{14} - 385x^{13} + 1220x^{12} - 2613x^{11} + 4263x^{10} - 5725x^9 + 6540x^8\\ &- 6275x^7 + 4875x^6 - 2965x^4 + 1360x^4 - 455x^3 + 105x^2 - 15x + 1.\end{aligned}$$

6.7 Conjectural Asymptotic Formulae for the moments of $L\left(\frac{1}{2}+it,\chi_u\right)$

In this section, we will use Conjecture (6.4.43) to write down an asymptotic formula for

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2}+it,\chi_u\right)^k, \tag{6.7.1}$$

where $t \neq 0$ is real and fixed. Using techniques similar to that done in Section 6.5.1, we will show that, for the first moment, (6.4.43) agrees with (6.1.8) when $\alpha = it$. We will then use the methods of [KO08, Lemma 3] to show how to obtain an explicit conjecture for the leading order asymptotic of (6.7.1) for a general integer k.

6.7.1 Conjectured Asymptotic Formulae

Letting $\alpha_1 = \ldots = \alpha_k = it$ in (6.4.43), we have that

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2}+it,\chi_u\right)^k = \sum_{u\in\mathcal{I}_{g+1}} q^{-gikt} Q_k(2g+1) + o\left(q^{2g+1}\right), \tag{6.7.2}$$

where

$$Q_k(x) = \frac{(-1)^{\frac{k(k-1)}{2}}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\ldots\oint \frac{G(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)\prod_{j=1}^k z_j}{\prod_{j=1}^k (z_j-it)^k(z_j+it)^k} q^{\frac{x}{2}\sum_{j=1}^k z_j} dz_1\ldots dz_k, \tag{6.7.3}$$

$\Delta(z_1,\ldots,z_k)$ is the Vandermonde determinant defined in (6.1.3),

$$G(z_1,\ldots,z_k) = \prod_{j=1}^k X\left(\frac{1}{2}+z_j\right)^{-\frac{1}{2}} A\left(\frac{1}{2};z_1,\ldots,z_k\right)\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+z_i+z_j). \tag{6.7.4}$$

and the path of integration encloses the $\pm it$'s.

6.7.2 First Moment

For the first moment, (6.7.2) predicts that

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2}+it,\chi_u\right) = \sum_{u\in\mathcal{I}_{g+1}} q^{-git} Q_1(2g+1) + o\left(q^{2g+1}\right),$$

where

$$Q_1(x) = \frac{1}{\pi i}\oint \frac{G(z_1)\Delta(z_1^2)^2 z_1 q^{\frac{x}{2}z_1}}{(z_1-it)(z_1+it)} dz_1.$$

From (6.7.4) we have that

$$G(z_1) = A\left(\frac{1}{2};z_1\right) X\left(\frac{1}{2}+z_1\right)^{-\frac{1}{2}}\zeta_{\mathbb{A}}(1+2z_1).$$

From the definition of the Vandermonde determinant and the definition of $X(s)$, we have that

$$\Delta(z_1^2)^2 = 1 \quad \text{and} \quad X\left(\frac{1}{2}+z_j\right)^{-\frac{1}{2}} = q^{-\frac{z_1}{2}}.$$

Thus

$$Q_1(x) = \frac{1}{\pi i}\oint \frac{A\left(\frac{1}{2};z_1\right)\zeta_{\mathbb{A}}(1+2z_1) z_1 q^{\frac{x}{2}z_1} q^{-\frac{z_1}{2}}}{(z_1-it)(z_1+it)} dz_1. \tag{6.7.5}$$

From Lemma 6.4.4 we have

$$A\left(\frac{1}{2};z_1\right) = \prod_P \left(1-\frac{1}{|P|^{1+2z_1}}\right)\left(\frac{1}{2}\left(\left(1-\frac{1}{|P|^{\frac{1}{2}+z_1}}\right)^{-1} + \left(1+\frac{1}{|P|^{\frac{1}{2}+z_1}}\right)^{-1}\right)+\frac{1}{|P|}\right)\left(1+\frac{1}{|P|}\right)^{-1}.$$

We want to compute the integral (6.7.5), where the contour is a small circle around the origin that encloses the $\pm it$'s. Let

$$g(z_1) = \frac{A\left(\frac{1}{2}; z_1\right)\zeta_{\mathbb{A}}(1+2z_1)z_1 q^{\frac{x}{2}z_1} q^{-\frac{z_1}{2}}}{(z_1 it)(z_1+it)},$$

then $g(z_1)$ has a simple pole at $z_1 = it$ and $z_1 = -it$ (there is no pole at $z_1 = 0$ since the z_1 term in the numerator cancels the simple pole that comes from the zeta function). We have that

$$\begin{aligned}\mathrm{Res}(g(z_1); z_1 = 0) &= \lim_{z\to it} \frac{(z-it)A\left(\frac{1}{2}; z_1\right)\zeta_{\mathbb{A}}(1+2z_1)z_1 q^{\frac{x}{2}z_1} q^{-\frac{z_1}{2}}}{(z_1+it)(z_i-it)} \\ &= \frac{A\left(\frac{1}{2}; it\right)\zeta_{\mathbb{A}}(1+2it)q^{\frac{it}{2}(x-1)}}{2}.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}\mathrm{Res}(g(z_1); z_1 = -it) &= \lim_{z\to -it} \frac{(z+it)A\left(\frac{1}{2}; z_1\right)\zeta_{\mathbb{A}}(1+2z_1)z_1 q^{\frac{x}{2}z_1} q^{-\frac{z_1}{2}}}{(z_1+it)(z_1-it)} \\ &= \frac{A\left(\frac{1}{2}; -it\right)\zeta_{\mathbb{A}}(1-2it)q^{-\frac{it}{2}(x-1)}}{2}.\end{aligned}$$

Furthermore, we know that

$$A\left(\frac{1}{2}; it\right) = P(1+2it) \quad \text{and} \quad A\left(\frac{1}{2}; -it\right) = P(1-2it),$$

and from the definition of $\zeta_{\mathbb{A}}(s)$ we see that

$$\zeta_{\mathbb{A}}(1-2it) = -q^{-2it}\zeta_{\mathbb{A}}(1+2it).$$

Using the residue theorem we have

$$\begin{aligned}&\frac{1}{\pi i}\oint \frac{A\left(\frac{1}{2}; z_1\right)\zeta_{\mathbb{A}}(1+2z_1)z_1 q^{\frac{x}{2}z_1} q^{-\frac{z_1}{2}}}{(z_1-it)(z_1+it)} dz_1 \\ &= \zeta_{\mathbb{A}}(1+2it)\left(q^{\frac{it}{2}(x-1)}P(1+2it) - q^{-\frac{it}{2}(x-1)-2it}P(1-2it)\right).\end{aligned}$$

We therefore have

$$\begin{aligned}\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2}+it, \chi_u\right) &= \sum_{u\in\mathcal{I}_{g+1}} q^{-git}Q_1(2g+1) + o\left(q^{2g+1}\right) \\ &= q^{-git}\zeta_{\mathbb{A}}(1+2it)\left(q^{git}P(1+2it) - q^{-it(g+2)}P(1-2it)\right)\sum_{u\in\mathcal{I}_{g+1}} 1 \\ &\quad + o\left(q^{2g+1}\right).\end{aligned}$$

Using Lemma 2.7.3, we conclude that, for the first moment, (6.7.2) predicts that

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2}+it, \chi_u\right) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\zeta_{\mathbb{A}}(1+2it)\left(P(1+2it) - q^{-2it(g+1)}P(1-2it)\right) + o\left(q^{2g+1}\right). \tag{6.7.6}$$

Comparing (6.7.6) and (6.1.8), we see that the main and the principal lower order terms are the same. Hence Theorem 6.1.3 proves (6.7.2), for $k=1$, with an error term $O\left(g2^{\frac{g}{2}}q^{g\left(\frac{3}{2}-it\right)}\right)$.

6.7.3 Leading order for general k

In this subsection, we use the methods of Keating and Odgers [KO08] to obtain the leading order asymptotic for the moments of $L\left(\frac{1}{2}+it,\chi_u\right)$ for a general integer k. To do this will first need to prove the following lemma.

Lemma 6.7.1. *Suppose F is a symmetric function of k variables regular near $(0,\dots,0)$ and $f(s)$ has a simple pole of residue 1 at $s=0$ and is otherwise analytic in a neighbourhood of $s=0$. Let*

$$K\left(q^{2g+1};w_1,\dots,w_k\right)=\sum_{\epsilon_j=\pm1}e^{\frac{1}{2}\log\left(q^{2g+1}\right)\sum_{j=1}^{k}\epsilon_jw_j}F(\epsilon_1w_1,\dots,\epsilon_kw_k)\prod_{1\le i\le j\le k}f(\epsilon_iw_i+\epsilon_jw_j)$$

and define $I\left(q^{2g+1};k,i\beta\right)$ to be the value of K when $w_1=\ldots=w_k=i\beta$ for a fixed real $\beta\neq0$. Then we have

$$I(q^{2g+1};k,i\beta)\sim\left(q^{2g+1}\right)^{\frac{ki\beta}{2}}F(i\beta,\dots,i\beta)f(2i\beta)^{\frac{k(k+1)}{2}}. \tag{6.7.7}$$

Proof. Let

$$G\left(q^{2g+1};w_1,\dots,w_k\right)=e^{\frac{1}{2}\log\left(q^{2g+1}\right)\sum_{j=1}^{k}w_j}F(w_1,\dots,w_k)\prod_{1\le i\le j\le k}f(w_i+w_j), \tag{6.7.8}$$

then by Lemma 6.4.5 we have

$$\begin{aligned}\sum_{\epsilon_j=\pm1}G\left(q^{2g+1};\epsilon_1w_1,\dots,\epsilon_kw_k\right)=&\frac{(-1)^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\frac{2^k}{k!}\oint\cdots\oint G\left(q^{2g+1};z_1,\dots,z_k\right)\\&\times\frac{\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^{k}z_j}{\prod_{i=1}^{k}\prod_{j=1}^{k}(z_i-w_j)(z_i+w_j)}dz_1\dots dz_k.\end{aligned}$$

Thus

$$\begin{aligned}I\left(q^{2g+1};k;i\beta\right)=&\frac{(-1)^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\frac{2^k}{k!}\oint\cdots\oint G\left(q^{2g+1};z_1,\dots,z_k\right)\\&\times\frac{\Delta(z_1^2,\dots,z_k^2)^2\prod_{j=1}^{k}z_j}{\prod_{j=1}^{k}(z_j-i\beta)^k(z_j+i\beta)^k}dz_1\dots dz_k.\end{aligned} \tag{6.7.9}$$

For each $\beta\neq0$ each contour in the integral $I\left(q^{2g+1},k,i\beta\right)$ can be continuously deformed to two smaller circular contours centered at the poles $\pm i\beta$ connected by two straight lines (whose contribution cancel). Therefore we can consider the multiple contour integral as a sum of 2^k integrals in which each z_j runs over one of the smaller circular contours.

With $\epsilon_j = \pm 1$, let $\Gamma_{\epsilon_j i\beta}$ be a circle with center $\epsilon_j i\beta$ and radius less than $|\beta|$. Thus let $J\left(q^{2g+1}; k, i\beta; \Gamma_{\epsilon_1 i\beta}, \ldots, \Gamma_{\epsilon_k i\beta}\right)$ be the value of the multiple contour (6.7.9) but with the z_j contour changed to $\Gamma_{\epsilon_j i\beta}$. Hence

$$J\left(q^{2g+1}; k, i\beta; \Gamma_{\epsilon_1 i\beta}, \ldots, \Gamma_{\epsilon_k i\beta}\right) = \frac{(-1)^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \frac{2^k}{k!} \oint_{\Gamma_{\epsilon_1 i\beta}} \cdots \oint_{\Gamma_{\epsilon_k i\beta}} G\left(q^{2g+1}; z_1, \ldots, z_k\right) \times \frac{\Delta(z_1^2, \ldots, z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{j=1}^k (z_j - i\beta)^k (z_j + i\beta)^k} dz_1 \ldots dz_k$$

and

$$I\left(q^{2g+1}; k, i\beta\right) = \sum_{\epsilon_j = \pm 1} J\left(q^{2g+1}; k, i\beta; \Gamma_{\epsilon_1 i\beta}, \ldots, \Gamma_{\epsilon_k i\beta}\right).$$

Using the change of variables $z_j = \frac{2v_j}{\log(q^{2g+1})} + \epsilon_j i\beta$, we have

$$dz_j = \frac{2dv_j}{\log\left(q^{2g+1}\right)},$$

$$G\left(q^{2g+1}; z_1, \ldots, z_k\right) = G\left(q^{2g+1}; \frac{2v_1}{\log\left(q^{2g+1}\right)} + \epsilon_1 i\beta, \ldots, \frac{2v_k}{\log\left(q^{2g+1}\right)} + \epsilon_k i\beta\right),$$

$$\Delta(z_1^2, \ldots, z_k^2)^2 = \Delta\left(\left(\frac{2v_1}{\log\left(q^{2g+1}\right)} + \epsilon_1 i\beta\right)^2, \ldots, \left(\frac{2v_k}{\log\left(q^{2g+1}\right)} + \epsilon_k i\beta\right)^2\right)^2$$

and

$$(z_j - i\beta)^k (z_j + i\beta)^k = \left(\frac{2v_j}{\log\left(q^{2g+1}\right)}\right)^k \left(\frac{2v_j}{\log\left(q^{2g+1}\right)} + 2\epsilon_j i\beta\right)^k$$

Thus

$$\begin{aligned} &J\left(q^{2g+1}; k, i\beta; \Gamma_{\epsilon_1 i\beta}, \ldots, \Gamma_{\epsilon_k i\beta}\right) \\ &= \frac{(-1)^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \frac{2^k}{k!} \oint_{\Gamma_0} \cdots \oint_{\Gamma_0} G\left(q^{2g+1}; \frac{2v_1}{\log\left(q^{2g+1}\right)} + \epsilon_1 i\beta, \ldots, \frac{2v_k}{\log\left(q^{2g+1}\right)} + \epsilon_k i\beta\right) \\ &\times \frac{\Delta\left(\left(\frac{2v_1}{\log(q^{2g+1})} + \epsilon_1 i\beta\right)^2, \ldots, \left(\frac{2v_k}{\log(q^{2g+1})} + \epsilon_k i\beta\right)^2\right)^2 \prod_{j=1}^k \left(\frac{2v_j}{\log(q^{2g+1})} + \epsilon_j i\beta\right)}{\prod_{j=1}^k \left(\frac{2v_j}{\log(q^{2g+1})}\right)^k \left(\frac{2v_j}{\log(q^{2g+1})} + 2\epsilon_j i\beta\right)^k} \\ &\times \frac{2dv_1}{\log\left(q^{2g+1}\right)} \cdots \frac{2dv_k}{\log\left(q^{2g+1}\right)}, \end{aligned}$$

where Γ_0 is a circle centered at the origin with radius less than $|\beta|$. Furthermore from

(6.7.8) we have

$$
\begin{aligned}
&G\left(q^{2g+1};\frac{2v_1}{\log\left(q^{2g+1}\right)}+\epsilon_1 i\beta,\dots,\frac{2v_k}{\log\left(q^{2g+1}\right)}+\epsilon_k i\beta\right)\\
&=e^{\frac{1}{2}\log\left(q^{2g+1}\right)\sum_{j=1}^{k}\left(\frac{2v_j}{\log\left(q^{2g+1}\right)}+\epsilon_j i\beta\right)}F\left(\frac{2v_1}{\log\left(q^{2g+1}\right)}+\epsilon_1 i\beta,\dots,\frac{2v_k}{\log\left(q^{2g+1}\right)}+\epsilon_k i\beta\right)\\
&\times\prod_{1\le i\le j\le k}f\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}+(\epsilon_i+\epsilon_j)i\beta\right).
\end{aligned}
$$

Also, from the definition of the Vandermonde determinant, we have

$$
\begin{aligned}
&\Delta\left(\left(\frac{2v_1}{\log\left(q^{2g+1}\right)}+\epsilon_1 i\beta\right)^2,\dots,\left(\frac{2v_k}{\log\left(q^{2g+1}\right)}+\epsilon_k i\beta\right)^2\right)^2\\
&=\prod_{1\le i<j\le k}\left(\left(\frac{2v_j}{\log\left(q^{2g+1}\right)}+\epsilon_j i\beta\right)^2-\left(\frac{2v_i}{\log\left(q^{2g+1}\right)}+\epsilon_i i\beta\right)^2\right)^2\\
&=\prod_{1\le i<j\le k}\left(\frac{2v_j+2v_i}{\log\left(q^{2g+1}\right)}+(\epsilon_j+\epsilon_i)i\beta\right)^2\left(\frac{2v_j-2v_i}{\log\left(q^{2g+1}\right)}+(\epsilon_j-\epsilon_i)i\beta\right)^2.
\end{aligned}
$$

We also have

$$
\frac{2dv_1}{\log\left(q^{2g+1}\right)}\cdots\frac{2dv_k}{\log\left(q^{2g+1}\right)}=\left(\frac{2}{\log\left(q^{2g+1}\right)}\right)^k dv_1\dots dv_k
$$

and

$$
\prod_{j=1}^{k}\left(\frac{2v_j}{\log\left(q^{2g+1}\right)}\right)^k=\left(\frac{2}{\log\left(q^{2g+1}\right)}\right)^{k^2}\prod_{j=1}^{k}v_j^k.
$$

Combining the above we have

$$
\begin{aligned}
&J\left(q^{2g+1};k,i\beta;\Gamma_{\epsilon_1 i\beta},\dots,\Gamma_{\epsilon_k i\beta}\right)\\
&=\left(\frac{1}{2}\log\left(q^{2g+1}\right)\right)^{k(k-1)}e^{\frac{1}{2}\log\left(q^{2g+1}\right)\sum_{j=1}^{k}\epsilon_j i\beta}\frac{(-1)^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\frac{2^k}{k!}\oint_{\Gamma_0}\cdots\oint_{\Gamma_0}e^{\sum_{j=1}^{k}v_j}\\
&\times F\left(\frac{2v_1}{\log\left(q^{2g+1}\right)}+\epsilon_1 i\beta,\dots,\frac{2v_k}{\log\left(q^{2g+1}\right)}+\epsilon_k i\beta\right)\\
&\times\prod_{1\le i<j\le k}f\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}+(\epsilon_i+\epsilon_j)i\beta\right)\\
&\times\prod_{1\le i<j\le k}\left(\frac{2v_j+2v_i}{\log\left(q^{2g+1}\right)}+(\epsilon_j+\epsilon_i)i\beta\right)^2\left(\frac{2v_j-2v_i}{\log\left(q^{2g+1}\right)}+(\epsilon_j-\epsilon_i)i\beta\right)^2\\
&\times\prod_{j=1}^{k}f\left(\frac{2(2v_j)}{\log\left(q^{2g+1}\right)}+2\epsilon_j i\beta\right)\frac{\prod_{j=1}^{k}\left(\frac{2v_j}{\log\left(q^{2g+1}\right)}+\epsilon_j i\beta\right)}{\prod_{j=1}^{k}v_j^k\left(\frac{2v_j}{\log\left(q^{2g+1}\right)}+2\epsilon_j i\beta\right)^k}dv_1\dots dv_k.
\end{aligned}
$$

Also we have

$$
\prod_{1\le i<j\le k}f\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}+(\epsilon_i+\epsilon_j)i\beta\right)=\prod_{\substack{1\le i<j\le k\\ \epsilon_j=\epsilon_i}}f\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}+2\epsilon_j i\beta\right)\prod_{\substack{1\le i<j\le k\\ \epsilon_j=-\epsilon_i}}f\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}\right)
$$

and

$$\prod_{\substack{1\le i<j\le k\\ \epsilon_j=-\epsilon_i}} f\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}\right) = \prod_{\substack{1\le i<j\le k\\ \epsilon_j=-\epsilon_i}} f\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}\right)\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}\right)\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}\right)^{-1}.$$

Thus with β fixed, real and away from 0, we have, as $g\to\infty$ that

$$\begin{aligned}
&J\left(q^{2g+1};k,i\beta;\Gamma_{\epsilon_1 i\beta},\dots,\Gamma_{\epsilon_k i\beta}\right)\\
&\sim \left(\frac{1}{2}\log\left(q^{2g+1}\right)\right)^{k(k-1)} e^{\frac{1}{2}\log\left(q^{2g+1}\right)\sum_{j=1}^k \epsilon_j i\beta} F(\epsilon_1 i\beta,\dots,\epsilon_k i\beta)\\
&\times \prod_{\substack{1\le i<j\le k\\ \epsilon_j=\epsilon_i}} f(2\epsilon_j i\beta)\prod_{j=1}^k f(2\epsilon_j i\beta)(2i\beta)^{k(k-1)}\frac{\prod_{j=1}^k(\epsilon_j i\beta)}{\prod_{j=1}^k(2\epsilon_j i\beta)^k}\frac{(-1)^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\frac{2^k}{k!}\oint_{\Gamma_0}\dots\oint_{\Gamma_0} e^{\sum_{j=1}^k v_j}\\
&\times \prod_{\substack{1\le i<j\le k\\ \epsilon_j=\epsilon_i}}\left(\frac{2v_j-2v_i}{\log\left(q^{2g+1}\right)}\right)^2 \prod_{\substack{1\le i<j\le k\\ \epsilon_j=-\epsilon_i}}\left(\frac{2v_j+2v_i}{\log\left(q^{2g+1}\right)}\right)\frac{1}{\prod_{j=1}^k v_j^k}dv_1\dots dv_k,
\end{aligned}$$

where we have also used the fact that $\epsilon_j^2=1$ for all j. Simplifying we get

$$\begin{aligned}
&J\left(q^{2g+1};k,i\beta;\Gamma_{\epsilon_1 i\beta},\dots,\Gamma_{\epsilon_k i\beta}\right)\\
&\sim \left(\frac{1}{2}\log\left(q^{2g+1}\right)\right)^{k(k-1)} e^{\frac{1}{2}\log\left(q^{2g+1}\right)\sum_{j=1}^k \epsilon_j i\beta} F(\epsilon_1 i\beta,\dots,\epsilon_k i\beta)\prod_{j=1}^k \epsilon_j^{1-k}\\
&\times \prod_{\substack{1\le i<j\le k\\ \epsilon_j=\epsilon_i}} f(2\epsilon_j i\beta)\prod_{j=1}^k f(2\epsilon_j i\beta)\frac{(-1)^{\frac{k(k-1)}{2}}}{k!}\frac{1}{(2\pi i)^k}\oint_{\Gamma_0}\dots\oint_{\Gamma_0} e^{\sum_{j=1}^k v_j}\\
&\times \prod_{\substack{1\le i<j\le k\\ \epsilon_j=\epsilon_i}}\left(\frac{2v_i+2v_j}{\log\left(q^{2g+1}\right)}\right)^2 \prod_{\substack{1\le i<j\le k\\ \epsilon_j=-\epsilon_i}}\left(\frac{2v_j+2v_i}{\log\left(q^{2g+1}\right)}\right)\frac{1}{\prod_{j=1}^k v_j^k}dv_1\dots dv_k.
\end{aligned}$$

The leading order term arises when $\epsilon_j=1$ for all j. Thus

$$\begin{aligned}
I\left(q^{2g+1};k,i\beta\right) &\sim e^{\frac{1}{2}\log\left(q^{2g+1}\right)\sum_{j=1}^k i\beta} F(i\beta,\dots,i\beta) f(2i\beta)^{\frac{k(k+1)}{2}}\\
&\times \frac{(-1)^{\frac{k(k-1)}{2}}}{k!}\frac{1}{(2\pi i)^k}\oint_{\Gamma_0}\dots\oint_{\Gamma_0} e^{\sum_{j=1}^k v_j}\frac{\prod_{1\le i<j\le k}(v_j-v_i)^2}{\prod_{j=1}^k v_j^k}dv_1\dots dv_k.
\end{aligned}$$

The proof follows by using the fact that

$$\frac{(-1)^{\frac{k(k-1)}{2}}}{k!}\frac{1}{(2\pi i)^k}\oint\dots\oint e^{\sum_{j=1}^k v_j}\frac{\Delta(z_1,\dots,z_k)^2}{\prod_{j=1}^k v_j^k}dv_1\dots dv_k = 1.$$

■

Recall from (6.4.42), we have that

$$\begin{aligned}
&\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2}+\alpha_1,\chi_u\right)\dots L\left(\frac{1}{2}+\alpha_k,\chi_u\right)\\
&=\left(\sum_{u\in\mathcal{I}_{g+1}}\frac{\prod_{j=1}^k\left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{j=1}^k\alpha_j}X\left(\frac{1}{2}+\alpha_j\right)^{\frac{1}{2}}}{(\log q)^{\frac{k(k+1)}{2}}}\sum_{\epsilon_j=\pm1}K(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k)\right)(1+o(1)),
\end{aligned}$$

where

$$K(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k)=\prod_{j=1}^{k}\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{j=1}^{k}\epsilon_j\alpha_j}X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{-\frac{1}{2}}$$
$$\times A\left(\frac{1}{2};\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k\right)\prod_{1\le i\le j\le k}\zeta_{\mathbb{A}}(1+\epsilon_i\alpha_i+\epsilon_j\alpha_j)(\log q).$$

Applying Lemma 6.7.1 with

$$f(s)=\zeta_{\mathbb{A}}(1+s)\log q,$$
$$F(\alpha_1,\dots,\alpha_k)=\prod_{j=1}^{k}X\left(\frac{1}{2}+\alpha_j\right)^{-\frac{1}{2}}A\left(\frac{1}{2};\alpha_1,\dots,\alpha_k\right)$$

and

$$K\left(q^{2g+1};\alpha_1,\dots,\alpha_k\right)=\sum_{\epsilon_j=\pm1}\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{j=1}^{k}\epsilon_j\alpha_j}F(\epsilon_1\alpha_1,\dots,\epsilon_k\alpha_k)\prod_{1\le i\le j\le k}f(\epsilon_i\alpha_i+\epsilon_j\alpha_j),$$

and letting $\alpha_1=\ldots=\alpha_k=it$, where $t\neq0$ is real and fixed, we have that

$$\sum_{u\in\mathcal{I}_{g+1}}L\left(\frac{1}{2}+it,\chi_u\right)^k$$
$$\sim\sum_{u\in\mathcal{I}_{g+1}}\frac{\left(q^{2g+1}\right)^{\frac{-kit}{2}}X\left(\frac{1}{2}+it\right)^{\frac{k}{2}}}{(\log q)^{\frac{k(k+1)}{2}}}\left(q^{2g+1}\right)^{\frac{kit}{2}}X\left(\frac{1}{2}+it\right)^{-\frac{k}{2}}A\left(\frac{1}{2};it,\dots,it\right)$$
$$\times\zeta_{\mathbb{A}}(1+2it)^{\frac{k(k+1)}{2}}(\log q)^{\frac{k(k+1)}{2}}.$$

Simplifying and using Lemma 2.7.3 we obtain the following result.

Proposition 6.7.2. *Conditional on Conjecture* 6.2.1 *and for a fixed real* $t\neq0$ *we have that, as* $g\to\infty$ *the following holds*

$$\sum_{u\in\mathcal{I}_{g+1}}L\left(\frac{1}{2}+it,\chi_u\right)^k\sim\frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}A\left(\frac{1}{2};it,\dots,it\right)\zeta_{\mathbb{A}}(1+2it)^{\frac{k(k+1)}{2}}.\tag{6.7.10}$$

Chapter 7

Autocorrelation of Ratios of L-functions in Even characteristic

7.1 Autocorrelation of Ratios of L-functions over the Rational Function Field

The generalisation of (6.1.1) is to conjecture an asymptotic formula for the mean-value of ratios of products of L-functions. More precisely, the generalised problem of (6.1.1) is to understand the asymptotic behaviour of

$$\sum_{D\in\mathcal{H}_{2g+1}} \frac{\prod_{k=1}^{K} L\left(\frac{1}{2}+\alpha_k,\chi_D\right)}{\prod_{q=1}^{Q} L\left(\frac{1}{2}+\gamma_q,\chi_D\right)}, \tag{7.1.1}$$

when $q \equiv 1 (\mathrm{mod}\ 4)$ is fixed and $g\to\infty$, where $L(s,\chi_D)$ is the quadratic Dirichlet L-function and $\mathcal{H}_{2g+1}$ is the hyperelliptic ensemble defined in Section 2.6.2 and Section 2.6.3 respectively. In this setting, Andrade and Keating [AK14] adapted the recipe of Conrey, Farmer and Zirnbauer [CFZ08] to conjecture ratios of products of quadratic Dirichlet L-functions in function fields, which is seen to be the generalisation of Conjecture 6.1.1. Their conjecture reads.

Conjecture 7.1.1 (Andrade and Keating)**.** *Suppose that the real parts of α_k and γ_q are positive and that q odd is the fixed cardinality of the finite field $\mathbb{F}_q$. Let $\mathfrak{D} = \{L(s,\chi_D) : D\in\mathcal{H}_{2g+1}\}$ be the family of L-functions associated with the quadratic character χ_D. Furthermore, let $\mathcal{X}_D(s) = |D|^{\frac{1}{2}-s}X(s)$ where*

$$X(s) = q^{-\frac{1}{2}+s}.$$

That is $\mathcal{X}_D(s)$ is the factor in the functional equation

$$L(s,\chi_D) = \mathcal{X}_D(s)L(1-s,\chi_D).$$

Then we have

$$\sum_{D\in\mathcal{H}_{2g+1}} \frac{\prod_{k=1}^{K} L\left(\frac{1}{2}+\alpha_k,\chi_D\right)}{\prod_{q=1}^{Q} L\left(\frac{1}{2}+\gamma_q,\chi_D\right)}$$
$$= \sum_{D\in\mathcal{H}_{2g+1}} \sum_{\epsilon\in\{-1,1\}^K} |D|^{\frac{1}{2}\sum_{k=1}^{K}(\epsilon_k\alpha_k-\alpha_k)} \prod_{k=1}^{K} X\left(\frac{1}{2}+\frac{\alpha_k-\epsilon_k\alpha_k}{2}\right)$$
$$\times Y_{\mathfrak{D}}(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K;\gamma)A_{\mathfrak{D}}(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K;\gamma)+o(|D|),$$

where

$$A_{\mathfrak{D}}(\alpha;\gamma)=\prod_P \frac{\prod_{1\le j\le k\le K}\left(1-\frac{1}{|P|^{1+\alpha_j+\alpha_k}}\right)\prod_{1\le q<r\le Q}\left(1-\frac{1}{|P|^{1+\gamma_q+\gamma_r}}\right)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\left(1-\frac{1}{|P|^{1+\alpha_k+\gamma_q}}\right)}$$
$$\times\left(1+\left(1+\frac{1}{|P|}\right)^{-1}\sum_{0<\sum_k a_k+\sum_q c_q \text{ is even}} \frac{\prod_{q=1}^{Q}\mu(P^{c_q})}{|P|^{\sum_k a_k\left(\frac{1}{2}+\alpha_k\right)+\sum_q c_q\left(\frac{1}{2}+\gamma_q\right)}}\right)$$

and

$$Y_{\mathfrak{D}}(\alpha;\gamma)=\frac{\prod_{1\le j\le k\le K}\zeta_{\mathbb{A}}(1+\alpha_j+\alpha_k)\prod_{1\le q<r\le Q}\zeta_{\mathbb{A}}(1+\gamma_q+\gamma_r)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\zeta_{\mathbb{A}}(1+\alpha_k+\gamma_q)}.$$

If we let

$$H_{D,|D|,\alpha,\gamma}(w)=|D|^{\frac{1}{2}\sum_{k=1}^{K}w_k}\prod_{k=1}^{K}X\left(\frac{1}{2}+\frac{\alpha_k-w_k}{2}\right)\times Y_{\mathfrak{D}}(w_1,\dots,w_K;\gamma)A_{\mathfrak{D}}(w_1,\dots,w_K;\gamma),$$

then the conjecture may be formulated as

$$\sum_{D\in\mathcal{H}_{2g+1}} \frac{\prod_{k=1}^{K} L\left(\frac{1}{2}+\alpha_k,\chi_D\right)}{\prod_{q=1}^{Q} L\left(\frac{1}{2}+\gamma_q,\chi_D\right)}$$
$$=\sum_{D\in\mathcal{H}_{2g+1}} |D|^{-\frac{1}{2}\sum_{k=1}^{K}\alpha_k} \sum_{\epsilon\in\{-1,1\}^K} H_{D,|D|,\alpha,\gamma}(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K;\gamma)+o(|D|).$$

In a recent paper, Bui, Florea and Keating [BFK21a] used the upper bounds on negative moments of Dirichlet L-functions (6.1.4) to prove special cases of the Ratios Conjecture 7.1.1. More specifically they proved the following result.

Theorem 7.1.2 (Bui, Florea and Keating)**.** *Let* $0<\Re(\beta_j)<\frac{1}{2}$ *for* $1\le j\le k$. *Let* $\alpha=\max\{|\Re(\alpha_1)|,\dots,|\Re(\alpha_k)|\}$ *and* $\beta=\min\{\Re(\beta_1),\dots,\Re(\beta_k)\}$. *Then Conjecture* 7.1.1 *holds for* $1\le k\le 3$ *with the error term* E_k *where*

$$E_1\ll_\epsilon \begin{cases} q^{-g\beta(3+2\alpha)+\epsilon g\beta} & \text{if } 0\le\Re(\alpha_1)<\frac{1}{2} \text{ and } \beta\gg g^{-\frac{1}{2}+\epsilon},\\ q^{-g\beta(3-4\alpha)+\epsilon g\beta} & \text{if } -\frac{1}{2}<\Re(\alpha_1)<0 \text{ and } \beta\gg g^{-\frac{1}{2}+\epsilon},\end{cases}$$

$$E_2\ll_\epsilon q^{-g\beta\min\left\{\frac{1-4\alpha}{1+\beta},\frac{1-2\alpha}{2+\beta}\right\}+\epsilon g\beta} \qquad \text{if } \alpha<\frac{1}{4} \text{ and } \beta\gg g^{-\frac{1}{4}+\epsilon},$$

and

$$E_3\ll_\epsilon q^{-g\beta\min\left\{\frac{\frac{1}{4}-4\alpha}{\beta},\frac{\frac{1}{2}-4\alpha}{3+\beta}\right\}+\epsilon g\beta} \qquad \text{if } \alpha<\frac{1}{16} \text{ and } \beta\gg g^{-\frac{1}{6}+\epsilon}.$$

Andrade, Jung and Shamesaldeen [AJS21] also stated a conjecture for the ratios of products of Dirichlet L-functions with the quadratic character χ_P, where P is a monic irreducible polynomial in $\mathbb{F}_q[T]$, which is seen to be the generalisation of Conjecture 6.1.2. Their conjecture reads.

Conjecture 7.1.3 (Andrade, Jung and Shamesaldeen)**.** *Suppose that the real parts of α_k and γ_q are positive and that q odd is the fixed cardinality of the finite field $\mathbb{F}_q$. Let $\mathfrak{P} = \{L(s,\chi_P) : P \in \mathcal{P}_{2g+1}\}$ be the family of L-functions associated with the quadratic character χ_P. Furthermore, let $\mathcal{X}_P(s) = |P|^{\frac{1}{2}-s}X(s)$ where*

$$X(s) = q^{-\frac{1}{2}+s}.$$

That is $\mathcal{X}_P(s)$ is the factor in the functional equation

$$L(s,\chi_P) = \mathcal{X}_P(s)L(1-s,\chi_P).$$

Then we have

$$\begin{aligned}
&\sum_{P\in\mathcal{P}_{2g+1}} \frac{\prod_{k=1}^{K} L\left(\frac{1}{2}+\alpha_k,\chi_P\right)}{\prod_{q=1}^{Q} L\left(\frac{1}{2}+\gamma_q,\chi_P\right)} \\
&= \sum_{P\in\mathcal{P}_{2g+1}} \sum_{\epsilon\in\{-1,1\}^K} |P|^{-\frac{1}{2}\sum_{k=1}^{K}(\epsilon_k\alpha_k-\alpha_k)} \prod_{k=1}^{K} X\left(\frac{1}{2}+\frac{\alpha_k-\epsilon_k\alpha_k}{2}\right) \\
&\times Y_{\mathfrak{P}}(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K;\gamma)A_{\mathfrak{P}}(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K;\gamma) + o(|P|),
\end{aligned}$$

where

$$\begin{aligned}
A_{\mathfrak{P}}(\alpha;\gamma) = \prod_P &\frac{\prod_{1\le j\le k\le K}\left(1-\frac{1}{|P|^{1+\alpha_j+\alpha_k}}\right)\prod_{1\le m<r\le Q}\left(1-\frac{1}{|P|^{1+\gamma_q+\gamma_r}}\right)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\left(1-\frac{1}{|P|^{1+\alpha_k+\gamma_q}}\right)} \\
&\times\left(1+\sum_{0<\sum_k a_k+\sum_q c_q \text{ is even}} \frac{\prod_{q=1}^{Q}\mu(P^{c_q})}{|P|^{\frac{1}{2}\sum_k a_k\left(\frac{1}{2}+\alpha_k\right)+\sum_q c_q\left(\frac{1}{2}+\gamma_q\right)}}\right)
\end{aligned}$$

and

$$Y_{\mathfrak{P}}(\alpha;\gamma) = \frac{\prod_{1\le j\le k\le K}\zeta_{\mathbb{A}}(1+\alpha_j+\alpha_k)\prod_{1\le q<r\le Q}\zeta_{\mathbb{A}}(1+\gamma_q+\gamma_r)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\zeta_{\mathbb{A}}(1+\alpha_k+\gamma_q)}.$$

In this chapter, we will develop to even characteristic the heuristic developed in [CFZ08, AK14, AJS21] which will lead to a conjecture for the ratios of products of Dirichlet L-functions $L(s,\chi_u)$ with $u\in\mathcal{I}_{g+1}$, where $L(s,\chi_u)$ is the Dirichlet L-function defined in Section 2.7.3 and $\mathcal{I}_{g+1}$ is the set defined in Section 2.7.1. The main result in this chapter is the following.

Conjecture 7.1.4. *Suppose that the real parts of α_k and γ_q are positive and that q is a power of 2 which is the fixed cardinality of the finite field $\mathbb{F}_q$. Let $\mathcal{U} = \{L(s,\chi_u) : u\in\mathcal{I}_{g+1}\}$*

be the family of L-functions associated with the quadratic character χ_u. Furthermore, let $\mathcal{X}_u(s) = (q^{2g+1})^{\frac{1}{2}-s} X(s)$ where

$$X(s) = q^{-\frac{1}{2}+s}.$$

That is $\mathcal{X}_u(s)$ is the factor in the functional equation

$$L(s,\chi_u) = \mathcal{X}_u(s) L(1-s,\chi_u).$$

Then we have

$$\begin{aligned}
&\sum_{u\in\mathcal{I}_{g+1}} \frac{\prod_{k=1}^{K} L\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^{Q} L\left(\frac{1}{2}+\gamma_q,\chi_u\right)} \\
&= \sum_{u\in\mathcal{I}_{g+1}} \sum_{\epsilon\in\{-1,1\}^K} \left(q^{2g+1}\right)^{\frac{1}{2}\sum_{k=1}^{K}(\epsilon_k\alpha_k-\alpha_k)} \prod_{k=1}^{K} X\left(\frac{1}{2}+\frac{\alpha_k-\epsilon_k\alpha_k}{2}\right) \\
&\times A_{\mathcal{U}}(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K;\gamma) Y_{\mathcal{U}}(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K;\gamma) + o\left(q^{2g+1}\right),
\end{aligned}$$

where

$$\begin{aligned}
A_{\mathcal{U}}(\alpha;\gamma) &= \prod_{P} \frac{\prod_{1\le j\le k\le K}\left(1-\frac{1}{|P|^{1+\alpha_j+\alpha_k}}\right)\prod_{1\le q<r\le Q}\left(1-\frac{1}{|P|^{1+\gamma_q+\gamma_r}}\right)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\left(1-\frac{1}{|P|^{1+\alpha_k+\gamma_q}}\right)} \\
&\times\left(1+\left(1+\frac{1}{|P|}\right)^{-1} \sum_{0<\sum_k a_k+\sum_q c_q \text{ is even}} \frac{\prod_{q=1}^{Q}\mu(P^{c_q})}{|P|^{\sum_k a_k\left(\frac{1}{2}+\alpha_k\right)+\sum_q c_q\left(\frac{1}{2}+\gamma_q\right)}}\right)
\end{aligned}$$

and

$$Y_{\mathcal{U}}(\alpha;\gamma) = \frac{\prod_{1\le j\le k\le K}\zeta_{\mathbb{A}}(1+\alpha_j+\alpha_k)\prod_{1\le q<r\le Q}\zeta_{\mathbb{A}}(1+\gamma_q+\gamma_r)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\zeta_{\mathbb{A}}(1+\alpha_k+\gamma_q)}.$$

If we let

$$H_{\mathcal{I},\alpha,\gamma}(w) = \left(q^{2g+1}\right)^{\frac{1}{2}\sum_{k=1}^{K} w_k} \prod_{k=1}^{K} X\left(\frac{1}{2}+\frac{\alpha_k-w_k}{2}\right) A_{\mathcal{U}}(w_1,\dots,w_K;\gamma) Y_{\mathcal{U}}(w_1,\dots,w_K;\gamma),$$

then the Conjecture may be formulated as

$$\begin{aligned}
&\sum_{u\in\mathcal{I}_{g+1}} \frac{\prod_{k=1}^{K} L\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^{Q} L\left(\frac{1}{2}+\gamma_q,\chi_u\right)} \\
&= \sum_{u\in\mathcal{I}_{g+1}} \left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{k=1}^{K}\alpha_k} \sum_{\epsilon\in\{-1,1\}^K} H_{\mathcal{I},\alpha,\gamma}(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K) + o\left(q^{2g+1}\right).
\end{aligned}$$

7.2 Applying the Ratios Conjecture for L-functions in Even characteristic

In this section, we will obtain a conjectural asymptotic formula for

$$\sum_{u\in\mathcal{I}_{g+1}} \frac{\prod_{k=1}^{K} L\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^{Q} L\left(\frac{1}{2}+\gamma_q,\chi_u\right)}, \tag{7.2.1}$$

where the set $\mathcal{I}_{g+1}$ is defined in Section 2.7.1 and the family $\mathcal{U} = \{L(s,\chi_u) : u \in \mathcal{I}_{g+1}\}$ is a symplectic family. By the "approximate" functional equation, Lemma 6.3.1, the L-functions in the numerator can be written as

$$L(s,\chi_u) = \sum_{f \in \mathbb{A}^+_{\leq g}} \frac{\chi_u(f)}{|f|^s} + \mathcal{X}_u(s) \sum_{f \in \mathbb{A}^+_{\leq g-1}} \frac{\chi_u(f)}{|f|^{1-s}}, \tag{7.2.2}$$

where $\mathcal{X}_u(s) = q^{g(1-2s)}$ and, by (2.4.2), those L-functions in the denominator can be written as

$$\frac{1}{L(s,\chi_u)} = \prod_P \left(1 - \frac{\chi_u(P)}{|P|^s}\right) = \sum_{f \in \mathbb{A}^+} \frac{\mu(f)\chi_u(f)}{|f|^s}, \tag{7.2.3}$$

where $\mu(f)$ is the Möbius function defined in Section 2.2. In the numerator, we replace $L(s,\chi_u)$ with the completed L-function $\Lambda(s,\chi_u)$, which is defined in Section 2.7.3. Thus we will apply the recipe to

$$\sum_{u \in \mathcal{I}_{g+1}} \frac{\prod_{k=1}^K \Lambda\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^Q L\left(\frac{1}{2}+\gamma_q,\chi_u\right)}. \tag{7.2.4}$$

We will recover Conjecture 7.1.4 by using the fact that

$$\Lambda(s,\chi_u) = \mathcal{X}_u(s)^{-\frac{1}{2}} L(s,\chi_u). \tag{7.2.5}$$

Using (7.2.3), we have

$$\begin{aligned}
&\sum_{u \in \mathcal{I}_{g+1}} \frac{\prod_{k=1}^K \Lambda\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^Q L\left(\frac{1}{2}+\gamma_q,\chi_u\right)} \\
&= \sum_{u \in \mathcal{I}_{g+1}} \prod_{k=1}^K \Lambda\left(\frac{1}{2}+\alpha_k,\chi_u\right) \sum_{\substack{h_1,\dots,h_Q \\ h_i \text{ monic}}} \frac{\mu(h_1)\dots\mu(h_Q)\chi_u(h_1)\dots\chi_u(h_Q)}{\prod_{q=1}^Q |h_q|^{\frac{1}{2}+\gamma_q}}.
\end{aligned}$$

From (6.4.11), we further know that

$$\prod_{k=1}^K \Lambda\left(\frac{1}{2}+\alpha_k,\chi_u\right) = \sum_{\epsilon \in \{-1,1\}^K} \prod_{k=1}^K \mathcal{X}_u\left(\frac{1}{2}+\epsilon_k\alpha_k\right)^{-\frac{1}{2}} \sum_{\substack{m_1,\dots,m_K \\ m_j \text{ monic}}} \frac{\chi_u(m_1\dots m_K)}{\prod_{k=1}^K |m_k|^{\frac{1}{2}+\epsilon_k\alpha_k}},$$

thus

$$\begin{aligned}
&\sum_{u \in \mathcal{I}_{g+1}} \frac{\prod_{k=1}^K \Lambda\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^Q L\left(\frac{1}{2}+\gamma_q,\chi_u\right)} \\
&= \sum_{u \in \mathcal{I}_{g+1}} \sum_{\epsilon \in \{-1,1\}^K} \prod_{k=1}^K \mathcal{X}_u\left(\frac{1}{2}+\epsilon_k\alpha_k\right)^{-\frac{1}{2}} \sum_{\substack{m_1,\dots,m_K \\ h_1,\dots,h_Q \\ m_j,h_i \text{ monic}}} \frac{\prod_{q=1}^Q \mu(h_q)\chi_u\left(\prod_{k=1}^K m_k \prod_{q=1}^Q h_q\right)}{\prod_{k=1}^K |m_k|^{\frac{1}{2}+\epsilon_k\alpha_k} \prod_{q=1}^Q |m_q|^{\frac{1}{2}+\gamma_q}}.
\end{aligned}$$

Following the recipe, we average the summand over fundamental discriminants $u \in \mathcal{I}_{g+1}$. Thus, using Lemma 6.4.2, we have that

$$\lim_{g\to\infty} \frac{1}{\#\mathcal{I}_{g+1}} \sum_{u\in\mathcal{I}_{g+1}} \sum_{\epsilon\in\{-1,1\}^K} \prod_{k=1}^{K} \mathcal{X}_u\left(\frac{1}{2}+\epsilon_k\alpha_k\right)^{-\frac{1}{2}} \sum_{\substack{m_1,\dots,m_K\\ h_1,\dots,h_Q\\ m_j,h_i \text{ monic}}} \frac{\prod_{q=1}^{Q}\mu(h_q)\chi_u\left(\prod_{k=1}^{K} m_k \prod_{q=1}^{Q} h_q\right)}{\prod_{k=1}^{K}|m_k|^{\frac{1}{2}+\epsilon_k\alpha_k}\prod_{q=1}^{Q}|h_q|^{\frac{1}{2}+\gamma_q}}$$
$$= \sum_{\epsilon\in\{-1,1\}^K} \prod_{k=1}^{K} \mathcal{X}_u\left(\frac{1}{2}+\epsilon_k\alpha_k\right)^{-\frac{1}{2}} \sum_{\substack{m_1,\dots,m_K\\ h_1,\dots,h_Q\\ m_j,h_i \text{ monic}}} \frac{\prod_{q=1}^{Q}\mu(h_q)\delta\left(\prod_{k=1}^{K} m_k \prod_{q=1}^{Q} h_q\right)}{\prod_{k=1}^{K}|m_k|^{\frac{1}{2}+\epsilon_k\alpha_k}\prod_{q=1}^{Q}|h_q|^{\frac{1}{2}+\gamma_q}},$$

where

$$\delta(n) = \begin{cases} \prod_{P|n}\left(1+\frac{1}{|P|}\right)^{-1} & \text{if } n \text{ is a square,}\\ 0 & \text{otherwise.}\end{cases}$$

Let

$$G_{\mathcal{U}}(\alpha;\gamma) = \sum_{\substack{m_1,\dots,m_K\\ h_1,\dots,h_Q\\ m_j,h_i \text{ monic}}} \frac{\prod_{q=1}^{Q}\mu(h_q)\delta\left(\prod_{k=1}^{K} m_k \prod_{q=1}^{Q} h_q\right)}{\prod_{k=1}^{K}|m_k|^{\frac{1}{2}+\epsilon_k\alpha_k}\prod_{q=1}^{Q}|h_q|^{\frac{1}{2}+\gamma_q}},$$

then we can express $G_{\mathcal{U}}(\alpha;\gamma)$ as a convergent Euler product, provided that $\Re(\alpha_k) > 0$ and $\Re(\gamma_q) > 0$. Thus

$$G_{\mathcal{U}}(\alpha;\gamma) = \prod_P\left(1+\left(1+\frac{1}{|P|}\right)^{-1} \sum_{0<\sum_k a_k+\sum_q c_q \text{ is even}} \frac{\prod_{q=1}^{Q}\mu(P^{c_q})}{|P|^{\sum_k a_k(\frac{1}{2}+\alpha_k)+\sum_q c_q(\frac{1}{2}+\gamma_q)}}\right).$$

We can use the Euler product expression to write $G_{\mathcal{U}}$ in terms of the zeta function which will enable us to locate the zeros and poles. We have

$$G_{\mathcal{U}}(\alpha;\gamma) = \prod_P\Bigg(1+\left(1+\frac{1}{|P|}\right)^{-1}$$
$$\times\left[\sum_{1\le j\le k\le K}\frac{1}{|P|^{(\frac{1}{2}+\alpha_j)+(\frac{1}{2}+\alpha_k)}} + \sum_{1\le q<r\le Q}\frac{\mu(P)^2}{|P|^{(\frac{1}{2}+\gamma_q)+(\frac{1}{2}+\gamma_r)}} + \sum_{k=1}^{K}\sum_{q=1}^{Q}\frac{\mu(P)}{|P|^{(\frac{1}{2}+\alpha_k)+(\frac{1}{2}+\gamma_q)}} + \dots\right]\Bigg)$$

where $\dots$ indicates that the terms converge. Since

$$\zeta_{\mathbb{A}}(s) = \prod_P\left(1-\frac{1}{|P|^s}\right)^{-1} = \prod_P\sum_{j=0}^{\infty}\left(\frac{1}{|P|^s}\right)^j,$$

then the terms with $\sum_{k=1}^{K} a_k + \sum_{q=1}^{Q} c_q = 2$ contribute to the poles and zeros. The poles come from the terms with $a_j = a_k = 1$ for $1 \le j \le k \le K$ and $c_q = c_r = 1$ for $1 \le q < r \le Q$. The terms with $a_k = c_q = 1$ for $1 \le k \le K$ and $1 \le q \le Q$ contribute to the zeros. Thus, the contribution of all these zeros and poles is

$$Y_{\mathcal{U}}(\alpha;\gamma) = \frac{\prod_{1\le j\le k\le K}\zeta_{\mathbb{A}}(1+\alpha_j+\alpha_k)\prod_{1\le q<r\le Q}\zeta_{\mathbb{A}}(1+\gamma_q+\gamma_r)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\zeta_{\mathbb{A}}(1+\alpha_k+\gamma_q)}.$$

When factoring out $Y_{\mathcal{U}}$ from $G_{\mathcal{U}}$, we are left with an Euler product $A_{\mathcal{U}}$, where

$$A_{\mathcal{U}}(\alpha;\gamma) = \prod_P \frac{\prod_{1\le j\le k\le K}\left(1-\frac{1}{|P|^{1+\alpha_k+\gamma_q}}\right)\prod_{1\le q<r\le Q}\left(1-\frac{1}{|P|^{1+\gamma_q+\gamma_r}}\right)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\left(1-\frac{1}{|P|^{1+\alpha_k+\gamma_q}}\right)}$$
$$\times\left(1+\left(1+\frac{1}{|P|}\right)^{-1}\sum_{0<\sum_k a_k+\sum_q \text{ is even}}\frac{\prod_{q=1}^{Q}\mu(P^{c_q})}{|P|^{\sum_k a_k\left(\frac{1}{2}+\alpha_k\right)+\sum_q c_q\left(\frac{1}{2}+\gamma_q\right)}}\right). \tag{7.2.6}$$

Furthermore, $A_{\mathcal{U}}$ is absolutely convergent for all the variables in the small discs around zero. Combining all this, we get

$$\sum_{u\in\mathcal{I}_{g+1}}\frac{\prod_{k=1}^{K}\Lambda\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^{Q}L\left(\frac{1}{2}+\gamma_q,\chi_u\right)} = \sum_{u\in\mathcal{I}_{g+1}}\sum_{\epsilon\in\{-1,1\}^K}\prod_{k=1}^{K}\mathcal{X}_u\left(\frac{1}{2}+\epsilon_k\alpha_k\right)^{-\frac{1}{2}}$$
$$\times A_{\mathcal{U}}(\epsilon_1\alpha_1,\ldots,\epsilon_K\alpha_K;\gamma)Y_{\mathcal{U}}(\epsilon_1\alpha_1,\ldots,\epsilon_K\alpha_K;\gamma)+o\left(q^{2g+1}\right).$$

Using (7.2.5) and the definition of $\mathcal{X}_u(s)$, we have

$$\sum_{u\in\mathcal{I}_{g+1}}\frac{\prod_{k=1}^{K}L\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^{Q}L\left(\frac{1}{2}+\gamma_q,\chi_u\right)}$$
$$=\sum_{u\in\mathcal{I}_{g+1}}\sum_{\epsilon\in\{-1,1\}^K}\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{k=1}^{K}(\epsilon_k\alpha_k-\alpha_k)}\prod_{k=1}^{K}X\left(\frac{1}{2}+\alpha_k\right)^{\frac{1}{2}}X\left(\frac{1}{2}+\epsilon_k\alpha_k\right)^{-\frac{1}{2}}$$
$$\times A_{\mathcal{U}}(\epsilon_1\alpha_1,\ldots,\epsilon_K\alpha_K;\gamma)Y_{\mathcal{U}}(\epsilon_1\alpha_1,\ldots,\epsilon_K\alpha_K;\gamma)+o\left(q^{2g+1}\right). \tag{7.2.7}$$

To obtain the formulae stated in Conjecture 7.1.4, we require the following Lemma.

Lemma 7.2.1. *We have*

$$X\left(\frac{1}{2}+\alpha_k\right)^{\frac{1}{2}}X\left(\frac{1}{2}+\epsilon_k\alpha_k\right)^{-\frac{1}{2}} = X\left(\frac{1}{2}+\frac{\alpha_k-\epsilon_k\alpha_k}{2}\right).$$

Proof. The proof follows directly from the definition of $X(s)$. ∎

Therefore, using Lemma 7.2.1, we have that (7.2.7) becomes

$$\sum_{u\in\mathcal{I}_{g+1}}\frac{\prod_{k=1}^{K}L\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^{Q}L\left(\frac{1}{2}+\gamma_q,\chi_u\right)}$$
$$=\sum_{u\in\mathcal{I}_{g+1}}\sum_{\epsilon\in\{-1,1\}^K}\left(q^{2g+1}\right)^{\frac{1}{2}\sum_{k=1}^{K}(\epsilon_k\alpha_k-\alpha_k)}\prod_{k=1}^{K}X\left(\frac{1}{2}+\frac{\alpha_k-\epsilon_k\alpha_k}{2}\right)$$
$$\times A_{\mathcal{U}}(\epsilon_1\alpha_1,\ldots,\epsilon_K\alpha_K;\gamma)Y_{\mathcal{U}}(\epsilon_1\alpha_1,\ldots,\epsilon_K\alpha_K;\gamma)+o\left(q^{2g+1}\right).$$

If we let

$$H_{\mathcal{I},\alpha,\gamma}(w) = \left(q^{2g+1}\right)^{\frac{1}{2}\sum_{k=1}^{K}w_k}\prod_{k=1}^{K}X\left(\frac{1}{2}+\frac{\alpha_k-w_k}{2}\right)A_{\mathcal{U}}(w_1,\ldots,w_K;\gamma)Y_{\mathcal{U}}(w_1,\ldots,w_K;\gamma),$$

then

$$\sum_{u\in\mathcal{I}_{g+1}} \frac{\prod_{k=1}^{K} L\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^{Q} L\left(\frac{1}{2}+\gamma_q,\chi_u\right)}$$
$$= \sum_{u\in\mathcal{I}_{g+1}} \left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{k=1}^{K}\alpha_k} \sum_{\epsilon\in\{-1,1\}^K} H_{\mathcal{I},\alpha,\gamma}(\epsilon_1\alpha_1,\ldots,\epsilon_K\alpha_K;\gamma) + o\left(q^{2g+1}\right),$$

which is precisely the formulae given in Conjecture 7.1.4.

7.3 Refinements of the Conjecture

In this section, we derive a closed form expression for $A_{\mathcal{U}}(\alpha;\gamma)$. The main results in the section are the following.

Lemma 7.3.1. *We have*

$$1+\left(1+\frac{1}{|P|}\right)^{-1} \sum_{0<\sum_k a_k+\sum_q c_q \textit{ is even}} \frac{\prod_{q=1}^{Q}\mu(P^{c_q})}{|P|^{\sum_k a_k\left(\frac{1}{2}+\alpha_k\right)+\sum_q c_q\left(\frac{1}{2}+\gamma_q\right)}}$$
$$= \frac{1}{1+\frac{1}{|P|}}\left(\frac{1}{2}\frac{\prod_{q=1}^{Q}\left(1-\frac{1}{|P|^{\frac{1}{2}+\gamma_q}}\right)}{\prod_{k=1}^{K}\left(1-\frac{1}{|P|^{\frac{1}{2}+\alpha_k}}\right)}+\frac{1}{2}\frac{\prod_{q=1}^{Q}\left(1+\frac{1}{|P|^{\frac{1}{2}+\gamma_q}}\right)}{\prod_{k=1}^{K}\left(1+\frac{1}{|P|^{\frac{1}{2}+\alpha_k}}\right)}+\frac{1}{|P|}\right).$$

Using Lemma 7.3.1 and (7.2.6), we immediately obtain Corollary 7.3.2, which states the closed form expression of $A_{\mathcal{U}}(\alpha;\gamma)$, which is the same expression given in Conjecture 7.1.4.

Corollary 7.3.2. *We have*

$$A_{\mathcal{U}}(\alpha;\gamma) = \prod_P \frac{\prod_{1\le j\le k\le K}\left(1-\frac{1}{|P|^{1+\alpha_j+\alpha_k}}\right)\prod_{1\le q<r\le Q}\left(1-\frac{1}{|P|^{1+\gamma_q+\gamma_r}}\right)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\left(1-\frac{1}{|P|^{1+\alpha_k+\gamma_q}}\right)}$$
$$\times\frac{1}{1+\frac{1}{|P|}}\left(\frac{1}{2}\frac{\prod_{q=1}^{Q}\left(1-\frac{1}{|P|^{\frac{1}{2}+\gamma_q}}\right)}{\prod_{k=1}^{K}\left(1-\frac{1}{|P|^{\frac{1}{2}+\alpha_k}}\right)}+\frac{1}{2}\frac{\prod_{q=1}^{Q}\left(1+\frac{1}{|P|^{\frac{1}{2}+\gamma_q}}\right)}{\prod_{k=1}^{K}\left(1+\frac{1}{|P|^{\frac{1}{2}+\alpha_k}}\right)}+\frac{1}{|P|}\right).$$

Proof of Lemma 7.3.1. Suppose that

$$f(x) = 1+\sum_{n=1}^{\infty} u_n x^n,$$

then

$$\sum_{0<n \text{ is even}} u_n x^n = \frac{1}{2}\left(f(x)+f(-x)-2\right)$$

and so

$$1+\left(1+\frac{1}{|P|}\right)^{-1}\sum_{0<n\text{ is even}}u_nx^n=1+\left(1+\frac{1}{|P|}\right)^{-1}\left(\frac{1}{2}\left(f(x)+f(-x)-2\right)\right)$$
$$=\frac{1}{1+\frac{1}{|P|}}\left(\frac{f(x)+f(-x)}{2}+\frac{1}{|P|}\right). \tag{7.3.1}$$

Thus if we let

$$f\left(\frac{1}{|P|}\right)=\sum_{a_k,c_q}\frac{\prod_{q=1}^{Q}\mu(P^{c_q})}{|P|^{\sum_k a_k(\frac{1}{2}+\alpha_k)+\sum_q c_q(\frac{1}{2}+\gamma_q)}},$$

then

$$\begin{aligned}f\left(\frac{1}{|P|}\right)&=\sum_{a_k}\frac{1}{|P|^{\sum_k a_k(\frac{1}{2}+\alpha_k)}}\sum_{c_q}\frac{\prod_{q=1}^{Q}\mu(P^{c_q})}{|P|^{\sum_q c_q(\frac{1}{2}+\gamma_q)}}\\&=\sum_{a_k}\prod_{k=1}^{K}\frac{1}{|P|^{a_k(\frac{1}{2}+\alpha_k)}}\sum_{c_q}\prod_{q=1}^{Q}\frac{\mu(P^{c_q})}{|P|^{c_q(\frac{1}{2}+\gamma_q)}}\\&=\frac{\prod_{q=1}^{Q}\left(1-\frac{1}{|P|^{\frac{1}{2}+\gamma_q}}\right)}{\prod_{k=1}^{K}\left(1-\frac{1}{|P|^{\frac{1}{2}+\alpha_k}}\right)}.\end{aligned} \tag{7.3.2}$$

Combining (7.3.1) and (7.3.2) proves the lemma. ■

7.4 The final form of the Conjecture

In this section, we present a form of the Ratios Conjecture 7.1.4 using contour integrals. To do this, we will need the following Lemma.

Lemma 7.4.1 ([CFZ08, Lemma 6.8])**.** *Suppose that $F(z)=F(z_1,\dots,z_K)$ is a function of K variables, which is symmetric and regular near $(0,\dots,0)$. Suppose further that $f(s)$ has a simple pole of residue 1 at $s=0$ but is otherwise analytic in $|s|\le 1$. Let either*

$$H(z_1,\dots,z_K)=F(z_1,\dots,z_K)\prod_{1\le j\le k\le K}f(z_j+z_k)$$

or

$$H(z_1,\dots,z_K)=F(z_1,\dots,z_K)\prod_{1\le j<k\le K}f(z_j+z_k).$$

If $|\alpha_k|<1$, then

$$\sum_{\epsilon\in\{-1,1\}^K}H(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K)$$
$$=\frac{(-1)^{\frac{K(K-1)}{2}}2^K}{K!}\frac{1}{(2\pi i)^K}\int_{|z_i|=1}\frac{H(z_1,\dots,z_K)\Delta(z_1^2,\dots,z_K^2)^2\prod_{k=1}^{K}z_k}{\prod_{j=1}^{K}\prod_{k=1}^{K}(z_k-\alpha_j)(z_k+\alpha_j)}dz_1\dots dz_K,$$

and

$$\sum_{\epsilon\in\{-1,1\}^K} sgn(\epsilon)H(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K)$$
$$= \frac{(-1)^{\frac{K(K-1)}{2}}2^K}{K!}\frac{1}{(2\pi i)^K}\int_{|z_i|=1}\frac{H(z_1,\dots,z_K)\Delta(z_1^2,\dots,z_K^2)^2\prod_{k=1}^K\alpha_k}{\prod_{j=1}^K\prod_{k=1}^K(z_k-\alpha_j)(z_k+\alpha_j)}dz_1\dots dz_K.$$

We are now in a position to present the final form of the Ratios Conjecture for Dirichlet L-functions in even characteristic using the integrals introduced in Lemma 7.4.1. If we let

$$F(z_1,\dots,z_K;\gamma) = \left(q^{2g+1}\right)^{\frac{1}{2}\sum_{k=1}^K z_k}\prod_{k=1}^K X\left(\frac{1}{2}+\frac{\alpha_k-z_k}{2}\right)A_{\mathcal{U}}(z_1,\dots,z_K;\gamma)$$

and

$$\prod_{1\le j\le k\le K} f(z_j+z_k) = Y_{\mathcal{U}}(z_1,\dots,z_K;\gamma),$$

then using the same notation as given in Conjecture 7.1.4, we let

$$H_{\mathcal{I},\alpha,\gamma}(z_1,\dots,z_K)$$
$$= \left(q^{2g+1}\right)^{\frac{1}{2}\sum_{k=1}^K z_k}\prod_{k=1}^K X\left(\frac{1}{2}+\frac{\alpha_k-z_k}{2}\right)A_{\mathcal{U}}(z_1,\dots,z_K;\gamma)Y_{\mathcal{U}}(z_1,\dots,z_K;\gamma),$$

where

$$\sum_{u\in\mathcal{I}_{g+1}}\frac{\prod_{k=1}^K L\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^Q L\left(\frac{1}{2}+\gamma_q,\chi_u\right)}$$
$$= \sum_{u\in\mathcal{I}_{g+1}}\left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{k=1}^K\alpha_k}\sum_{\epsilon\in\{-1,1\}^K}H_{\mathcal{I},\alpha,\gamma}(\epsilon_1\alpha_1,\dots,\epsilon_K\alpha_K)+o\left(q^{2g+1}\right).$$

Thus, using Lemma 7.4.1, Conjecture 7.1.4 can be written as follows.

Conjecture 7.4.2. *Suppose that the real parts of α_k and γ_q are positive. Then*

$$\sum_{u\in\mathcal{I}_{g+1}}\frac{\prod_{k=1}^K L\left(\frac{1}{2}+\alpha_k,\chi_u\right)}{\prod_{q=1}^Q L\left(\frac{1}{2}+\gamma_q,\chi_u\right)} = \sum_{u\in\mathcal{I}_{g+1}}\left(q^{2g+1}\right)^{-\frac{1}{2}\sum_{k=1}^K\alpha_k}$$
$$\frac{(-1)^{\frac{K(K-1)}{2}}2^K}{K!}\frac{1}{(2\pi i)^K}\int_{|z_i|=1}\frac{H_{\mathcal{I},\alpha,\gamma}(z_1,\dots,z_K)\Delta(z_1^2,\dots,z_K^2)^2\prod_{k=1}^K z_k}{\prod_{j=1}^K\prod_{k=1}^K(z_k-\alpha_j)(z_k+\alpha_j)}dz_1\dots dz_K+o\left(q^{2g+1}\right).$$

Chapter 8

Applications of the Ratios Conjecture in Even characteristic

8.1 Applications of the Ratios Conjecture in Function Fields

8.1.1 The One-Level Density

As an application of the Ratios Conjecture 7.1.1 Andrade and Keating [AK14] used the methods of Conrey and Snaith [CS07] to derive a formula for the one-level density for the zeros of quadratic Dirichlet L-functions $L(s,\chi_D)$ with $D \in \mathcal{H}_{2g+1}$, where $\mathcal{H}_{2g+1}$ is the hyperelliptic ensemble and $L(s,\chi_D)$ is the Dirichlet L-function which are defined in Section 2.6.2 and Section 2.6.3 respectively. In particular, they obtained the following result.

Theorem 8.1.1 (Andrade and Keating). *Assuming the Ratios Conjecture* 7.1.1*, the one-level density for the zeros of the family of quadratic Dirichlet L-functions associated with hyperelliptic curves given by the affine equation $C_D : y^2 = D(T)$, where $D \in \mathcal{H}_{2g+1}$, is given by*

$$\sum_{D\in\mathcal{H}_{2g+1}} \sum_{\gamma_D} f(\gamma_D)$$
$$= \frac{1}{2\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} f(t) \sum_{D\in\mathcal{H}_{2g+1}} \left(\log|D| - \frac{X'}{X}\left(\frac{1}{2}-it\right) + 2\left(\frac{\zeta_{\mathbb{A}}'(1+2it)}{\zeta_{\mathbb{A}}(1+2it)}\right.\right.$$
$$\left.\left.+ A_{\mathfrak{D}}'(it;it) - (\log q)|D|^{-it} X\left(\frac{1}{2}+it\right)\zeta_{\mathbb{A}}(1-2it)A_{\mathfrak{D}}(-it;it)\right)\right)dt + o(|D|),$$

where γ_D is the ordinate of a generic zero of $L(s,\chi_D)$, f is an even and periodic nice *test function,* $X(s) = q^{-\frac{1}{2}+s}$,

$$A_{\mathfrak{D}}(-r;r) = \prod_P \left(1 - \frac{1}{|P|}\right)^{-1} \left(1 - \frac{1}{(|P|+1)|P|^{1-2r}} - \frac{1}{|P|+1}\right)$$

and

$$A'_{\mathfrak{D}}(r;r) = \sum_P \frac{\log|P|}{(|P|^{1+2r}-1)(|P|+1)}.$$

Bui and Florea [BF18] studied the one-level density of the zeros of quadratic Dirichlet L-functions $L(s,\chi_D)$, when averaged over the hyperelliptic ensemble $\mathcal{H}_{2g+1}$. Specifically, when the Fourier transform of the test function is restricted in some interval, they computed some lower order terms which is not predicted by the Ratios Conjecture 7.1.1. In a recent paper Bui, Florea and Keating [BFK21b] used the Ratios Conjecture 7.1.1 to write down formulas for the one and two level densities of the zeros of quadratic Dirichlet L-functions in function fields. More precisely, they used the Ratios Conjecture 7.1.1 to predict the Type-0 and Type-I terms for the one-level density and the Type-0, Type-I and Type-II for the two-level density. For a certain range, they also rigorously computed the Type-0 and Type-I for each of the one and two level densities and showed they agree with the predicted conjecture.

Andrade, Jung and Shamesaldeen [AJS21] used the Ratios Conjecture 7.1.3 to derive a formula for the one-level density of the zeros of quadratic Dirichlet L-functions associated with monic, irreducible polynomials in $\mathbb{F}_q[T]$. In particular they obtained the following result. ,

Theorem 8.1.2 (Andrade, Jung and Shamesaldeen)**.** *Assuming the Ratios Conjecture* 7.1.3 *we have that*

$$\begin{aligned}&\sum_{P\in\mathcal{P}_{2g+1}} \sum_{\gamma_P} f(\gamma_P)\\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} f(t) \sum_{P\in\mathcal{P}_{2g+1}} \left(\log|P| - \frac{X'}{X}\left(\frac{1}{2}-it\right)\right.\\ &\left.+2\left(\frac{\zeta'_{\mathbb{A}}(1+2it)}{\zeta_{\mathbb{A}}(1+2it)} - (\log q)|P|^{-it}X\left(\frac{1}{2}+it\right)\zeta_{\mathbb{A}}(1-2it)\right)\right)dt + o(|P|),\end{aligned}$$

where γ_P is the ordinate of a generic zero of $L(s,\chi_P)$, f is an even and periodic suitable test function and $X(s) = q^{-\frac{1}{2}+s}$.

8.1.2 Non-Vanishing of $L\left(\frac{1}{2},\chi\right)$

In the function field setting, we want to obtain results about non-vanishing of Dirichlet L-functions $L(s,\chi)$ at the central point $s=\frac{1}{2}$. In this setting Li [Li18] showed that

$L(s,\chi)$ vanish infinitely often at $s=\frac{1}{2}$, and thus showing that the function field analogue of Chowla's conjecture is false. However, the conjectures of Katz and Sarnak [KS99a] predict that $L\left(\frac{1}{2},\chi_D\right)\neq 0$ for 100% of discriminants D. Using their one-level density results, Bui and Florea [BF18] proved, unconditionally, that the proportion of $L(s,\chi_D)$ which do not vanish at $s=\frac{1}{2}$ is greater than 94%.

In a recent paper, Andrade and Best [AB22] used the Ratios Conjecture 7.1.1 and mollified moments to show that, conditional on the Ratios Conjecture 7.1.1, the proportion of $L(s,\chi_D)$ which do not vanish at $s=\frac{1}{2}$ is 100%. In particular they obtained the following result.

Theorem 8.1.3 (Andrade and Best)**.** *Conditional on the Ratios Conjecture* 7.1.1*, we have*

$$\frac{1}{\#\mathcal{H}_{2g+1}}\sum_{\substack{D\in\mathcal{H}_{2g+1}\\ L\left(\frac{1}{2},\chi_D\right)\neq 0}}1\geq 1+o(1)$$

as $g\to\infty$.

8.2 Statement of Main Results

In this chapter, we will present two applications of the Ratios Conjecture 7.1.4. Firstly, we will use the conjecture to obtain a formula for the one-level density for the zeros of quadratic Dirichlet L-function $L(s,\chi_u)$ with $u\in\mathcal{I}_{g+1}$, where $\mathcal{I}_{g+1}$ and $L(s,\chi_u)$ are defined in Section 2.7.1 and Section 2.7.3 respectively. In particular, assuming the Ratios Conjecture 7.1.4 we obtain the following result.

Theorem 8.2.1. *Assuming the Ratio Conjecture* 7.1.4*, the one-level density for the zeros of quadratic Dirichlet L-functions associated with the quadratic character* χ_u *with* $u\in\mathcal{I}_{g+1}$ *is given by*

$$\begin{aligned}&\sum_{u\in\mathcal{I}_{g+1}}\sum_{\gamma_u}f(\gamma_u)\\&=\frac{1}{2\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}}f(t)\sum_{u\in\mathcal{I}_{g+1}}\left(\log\left(q^{2g+1}\right)-\frac{X'}{X}\left(\frac{1}{2}-it\right)+2\left(\frac{\zeta_{\mathbb{A}}'(1+2it)}{\zeta_{\mathbb{A}}(1+2it)}\right.\right.\\&\left.\left.+A_{\mathcal{U}}'(it;it)-(\log q)\left(q^{2g+1}\right)^{-it}X\left(\frac{1}{2}+it\right)\zeta_{\mathbb{A}}(1-2it)A_{\mathcal{U}}(-it;it)\right)\right)dt+o\left(q^{2g+1}\right),\end{aligned}$$

where γ_u *is the ordinate of a generic zero of* $L(s,\chi_u)$*,* f *is an even and periodic nice test function,* $X(s)=q^{-\frac{1}{2}+s}$*,*

$$A_{\mathcal{U}}(-r;r)=\prod_P\left(1-\frac{1}{|P|}\right)^{-1}\left(1-\frac{1}{(|P|+1)|P|^{1-2r}}-\frac{1}{|P|+1}\right)$$

and

$$A'_{\mathcal{U}}(r;r) = \sum_{P} \frac{\log|P|}{(|P|^{1+2r}-1)(|P|+1)}.$$

Also, we will use the Ratios Conjecture 7.1.4 and mollified moments to show that the proportion of $L(s,\chi_u)$ which do not vanish at $s=\frac{1}{2}$ is 100%. In particular, conditional on the Ratios Conjecture 7.1.4 we obtain the following result.

Theorem 8.2.2. *Conditional in the Ratios Conjecture* 7.1.4, *we have*

$$\frac{1}{\#\mathcal{I}_{g+1}} \sum_{\substack{u\in\mathcal{I}_{g+1}\\ L(\frac{1}{2},\chi_u)\neq 0}} 1 \geq 1+o(1) \tag{8.2.1}$$

as $g\to\infty$.

8.3 An Application of the Ratios Conjecture in Even Characteristic: The One-Level Density

In this section, we present an application of the Ratios Conjecture for Dirichlet L-functions in even characteristic, namely we derive a formula for the one-level density of the zeros of quadratic Dirichlet L-functions $L(s,\chi_u)$.

8.3.1 Applying the Ratios Recipe

In this subsection, we establish an asymptotic formula for

$$R_{\mathcal{U}}(\alpha;\gamma) = \sum_{u\in\mathcal{I}_{g+1}} \frac{L\left(\frac{1}{2}+\alpha,\chi_u\right)}{L\left(\frac{1}{2}+\gamma,\chi_u\right)}, \tag{8.3.1}$$

following the recipe described in Chapter 7. Following this recipe, we use Lemma 6.3.1 to express the Dirichlet L-function in the numerator as

$$L(s,\chi_u) = \sum_{m\in\mathbb{A}^+_{\leq g}} \frac{\chi_u(m)}{|m|^{\frac{1}{2}+\alpha}} + \mathcal{X}\left(\frac{1}{2}+\alpha\right) \sum_{m\in\mathbb{A}^+_{\leq g-1}} \frac{\chi_u(m)}{|m|^{\frac{1}{2}-\alpha}}, \tag{8.3.2}$$

and we write the Dirichlet L-function in the denominator as

$$\frac{1}{L(s,\chi_u)} = \sum_{h\in\mathbb{A}^+} \frac{\mu(h)\chi_u(h)}{|h|^s}. \tag{8.3.3}$$

From Lemma 6.4.2, we know that when averaging over the family $\#\mathcal{I}_{g+1}$, we retain only square terms, since

$$\sum_{u\in\mathcal{I}_{g+1}} \chi_u(n) = \begin{cases} a(n)\#\mathcal{I}_{g+1} + \text{small} & \text{if } n \text{ is a square,} \\ \text{small} & \text{if } n \text{ is not a square,} \end{cases} \tag{8.3.4}$$

where

$$a(n) = \prod_{P|n} \frac{|P|}{|P|+1}, \tag{8.3.5}$$

and, from Lemma 2.7.3, $\#\mathcal{I}_{g+1} = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}$. We compute the square terms and complete the sum by extending the range of the summation to include all monic polynomials. We then identify and factor out the appropriate zeta factors which are multiplied by an absolutely convergent Euler product. From the definition of $R_{\mathcal{U}}(\alpha,\gamma)$ and $\mathcal{X}(s)$, we have

$$\begin{aligned} R_{\mathcal{U}}(\alpha,\gamma) &= \sum_{u\in\mathcal{I}_{g+1}} \sum_{m\in\mathbb{A}^+_{\leq g}} \sum_{h\in\mathbb{A}^+} \frac{\chi_u(m)\mu(h)\chi_u(h)}{|m|^{\frac{1}{2}+\alpha}|h|^{\frac{1}{2}+\gamma}} \\ &+ \left(q^{2g+1}\right)^{-\alpha} X\left(\frac{1}{2}+\alpha\right) \sum_{u\in\mathcal{I}_{g+1}} \sum_{m\in\mathbb{A}^+_{\leq g-1}} \sum_{h\in\mathbb{A}^+} \frac{\chi_u(m)\mu(h)\chi_u(h)}{|m|^{\frac{1}{2}-\alpha}|h|^{\frac{1}{2}+\gamma}}. \end{aligned} \tag{8.3.6}$$

Considering the first sum in (8.3.6) and the terms $mh = \square$, we have, from (8.3.4),

$$\sum_{h,m\in\mathbb{A}^+} \sum_{u\in\mathcal{I}_{g+1}} \frac{\mu(h)\chi_u(mh)}{|m|^{\frac{1}{2}+\alpha}|h|^{\frac{1}{2}+\gamma}} = \#\mathcal{I}_{g+1} \sum_{\substack{m,h\in\mathbb{A}^+\\ mh=\square}} \frac{\mu(h)a(mh)}{|m|^{\frac{1}{2}+\alpha}|h|^{\frac{1}{2}+\gamma}}. \tag{8.3.7}$$

We want to express the sum on the right-hand side of (8.3.7) as an Euler product so that we can identify and factor out the appropriate zeta factors. To do this, we have

$$\begin{aligned} \sum_{\substack{m,h\in\mathbb{A}^+\\ mh=\square}} \frac{\mu(h)a(mh)}{|m|^{\frac{1}{2}+\alpha}|h|^{\frac{1}{2}+\gamma}} &= \sum_{j\in\mathbb{A}^+} \sum_{\substack{m,h\in\mathbb{A}^+\\ mh=\square=j^2}} \frac{\mu(h)a(mh)}{|m|^{\frac{1}{2}+\alpha}|h|^{\frac{1}{2}+\gamma}} \\ &= \sum_{j\in\mathbb{A}^+} a(j^2) \sum_{\substack{m,h\in\mathbb{A}^+\\ mh=\square=j^2}} \frac{\mu(h)a(mh)}{|m|^{\frac{1}{2}+\alpha}|h|^{\frac{1}{2}+\gamma}}. \end{aligned}$$

Let

$$\psi(j^2) = \sum_{\substack{m,h\in\mathbb{A}^+\\ mh=j^2}} \frac{\mu(h)}{|m|^{\frac{1}{2}+\alpha}|h|^{\frac{1}{2}+\gamma}},$$

then

$$\sum_{j\in\mathbb{A}^+} a(j^2)\psi(j^2) = \prod_P \left(1 + \sum_{\nu=1}^{\infty} a(P^{2\nu})\psi(P^{2\nu})\right).$$

Let $mh = P^{2\nu}$, and let $m = P^{e_1}$ and $h = P^{e_2}$. Then $mh = P^{2\nu}$ and $e_1 + e_2 = 2\nu$. Hence

$$\psi(p^{2\nu}) = \sum_{\substack{e_1,e_2\geq 0\\ e_1+e_2=2\nu}} \frac{\mu(P^{e_2})}{|P|^{e_1(\frac{1}{2}+\alpha)}|P|^{e_2(\frac{1}{2}+\gamma)}}.$$

Therefore we have

$$\begin{aligned} \sum_{j\in\mathbb{A}^+} a(j^2)\psi(j^2) &= \prod_P \left(1 + \sum_{\nu=1}^{\infty} a(P^{2\nu}) \sum_{\substack{e_1,e_2\geq 0\\ e_1+e_2=2\nu}} \frac{\mu(P^{e_1})}{|P|^{e_1(\frac{1}{2}+\alpha)}|P|^{e_2(\frac{1}{2}+\gamma)}}\right) \\ &= \prod_P \left(\sum_{\substack{e_1,e_2\geq 0\\ e_1+e_2 \text{ even}}} \frac{\mu(P^{e_2})a(P^{e_1+e_2})}{|P|^{e_1(\frac{1}{2}+\alpha)}|P|^{e_2(\frac{1}{2}+\gamma)}}\right). \end{aligned}$$

From the definition of the Möbius function, we know that $\mu(P^{e_2})$ equals zero except when e_2 equals 0 or 1. When e_2 equals 0, we have

$$\begin{aligned}\sum_{e_1 \text{ even}} \frac{a(P^{e_1})}{|P|^{e_1(\frac{1}{2}+\alpha)}} &= \sum_{e_1=0}^{\infty} \frac{a(P^{2e_1})}{|P|^{e_1(1+2\alpha)}} \\ &= 1 + \sum_{e_1=1}^{\infty} \frac{a(P^{2e_1})}{|P|^{e_1(1+2\alpha)}} \\ &= 1 + \frac{|P|}{|P|+1} \sum_{e_1=1}^{\infty} \left(\frac{1}{|P|^{1+2\alpha}}\right)^{e_1} \\ &= 1 + \frac{|P|}{|P|+1} \frac{1}{|P|^{1+2\alpha}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)}.\end{aligned}$$

Similarly, if $e_2 = 1$ we have

$$\begin{aligned}\sum_{e_1 \text{ odd}} \frac{a(P^{e_1+1})\mu(P)}{|P|^{e_1(\frac{1}{2}+\alpha)}|P|^{\frac{1}{2}+\gamma}} &= -\frac{1}{|P|^{\frac{1}{2}+\gamma}} \sum_{e_1=0}^{\infty} \frac{a(P^{2e_2+2})}{|P|^{(2e_1+1)(\frac{1}{2}+\alpha)}} \\ &= -\frac{|P|}{|P|+1} \frac{1}{|P|^{1+\alpha+\gamma}} \sum_{e_1=0}^{\infty} \left(\frac{1}{|P|^{1+2\alpha}}\right)^{e_1} \\ &= -\frac{|P|}{|P|+1} \frac{1}{|P|^{1+\alpha+\gamma}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)}.\end{aligned}$$

Combining this, we have

$$\sum_{j \in \mathbb{A}^+} a(j^2)\psi(j^2) = \prod_P \left(1 + \frac{|P|}{|P|+1} \frac{1}{|P|^{1+2\alpha}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)} - \frac{|P|}{|P|+1} \frac{1}{|P|^{1+\alpha+\gamma}} \frac{1}{\left(1 - \frac{1}{|P|^{1+2\alpha}}\right)}\right).$$

Factoring out the appropriate terms, we have that

$$\begin{aligned}&\sum_{\substack{m,h \in \mathbb{A}^+ \\ mh = \square}} \frac{\mu(h)a(mh)}{|m|^{\frac{1}{2}+\alpha}|h|^{\frac{1}{2}+\gamma}} \\ &= \frac{\zeta_{\mathbb{A}}(1+2\alpha)}{\zeta_{\mathbb{A}}(1+\alpha+\gamma)} \prod_P \left(1 - \frac{1}{|P|^{1+\alpha+\gamma}}\right)^{-1} \left(1 - \frac{1}{|P|^{1+2\alpha}(|P|+1)} - \frac{1}{|P|^{\alpha+\gamma}(|P|+1)}\right),\end{aligned}$$

where the product over monic irreducible polynomials P is absolutely convergent when $\Re(\alpha), \Re(\gamma) > \frac{1}{4}$. We can use similar methods for the other term in (8.3.4) which leads to the following conjecture.

Conjecture 8.3.1. *With* $-\frac{1}{4} < \Re(\alpha) < \frac{1}{4}$, $\frac{1}{\log(q^{2g+1})} \ll \Re(\gamma) < \frac{1}{4}$ *and* $\Im(\alpha), \Im(\gamma) \ll (q^{2g+1})^{1-\epsilon}$ *for every* $\epsilon > 0$, *we have*

$$\begin{aligned}R_{\mathcal{U}}(\alpha;\gamma) &= \sum_{u \in \mathcal{I}_{g+1}} \frac{L\left(\frac{1}{2}+\alpha, \chi_u\right)}{L\left(\frac{1}{2}+\gamma, \chi_u\right)} \\ &= \sum_{u \in \mathcal{I}_{g+1}} \left(A_{\mathcal{U}}(\alpha;\gamma) \frac{\zeta_{\mathbb{A}}(1+2\alpha)}{\zeta_{\mathbb{A}}(1+\alpha+\gamma)} + \left(q^{2g+1}\right)^{-\alpha} X\left(\frac{1}{2}+\alpha\right) A_{\mathcal{U}}(-\alpha;\gamma) \frac{\zeta_{\mathbb{A}}(1-2\alpha)}{\zeta_{\mathbb{A}}(1-\alpha+\gamma)}\right) \\ &\quad + o\left(q^{2g+1}\right), \qquad (8.3.8)\end{aligned}$$

where

$$A_{\mathcal{U}}(\alpha;\gamma) = \prod_P \left(1 - \frac{1}{|P|^{1+\alpha+\gamma}}\right)^{-1} \left(1 - \frac{1}{|P|^{1+2\alpha}(|P|+1)} - \frac{1}{|P|^{\alpha+\gamma}(|P|+1)}\right). \tag{8.3.9}$$

8.3.2 An Asymptotic formula for the Logarithmic Derivative of $L(s,\chi_u)$

To obtain the one-level density from the Ratios Conjecture, we need an asymptotic formula for

$$\sum_{u\in\mathcal{I}_{g+1}} \frac{L'\left(\frac{1}{2}+r,\chi_u\right)}{L\left(\frac{1}{2}+r,\chi_u\right)}. \tag{8.3.10}$$

First we note that

$$\sum_{u\in\mathcal{I}_{g+1}} \frac{L'\left(\frac{1}{2}+r,\chi_u\right)}{L\left(\frac{1}{2}+r,\chi_u\right)} = \frac{\partial}{\partial\alpha} R_{\mathcal{U}}(\alpha;\gamma)\Big|_{\alpha=\gamma=r}. \tag{8.3.11}$$

Next, we have that

$$\frac{\partial}{\partial\alpha}\frac{\zeta_{\mathbb{A}}(1+2\alpha)}{\zeta_{\mathbb{A}}(1+\alpha+\gamma)} A_{\mathcal{U}}(\alpha;\gamma)\Big|_{\alpha=\gamma=r} = \frac{\zeta_{\mathbb{A}}'(1+2r)}{\zeta_{\mathbb{A}}(1+2r)} A_{\mathcal{U}}(r;r) + A_{\mathcal{U}}'(r;r)$$

and

$$\begin{aligned}&\frac{\partial}{\partial\alpha}\left(\left(q^{2g+1}\right)^{-\alpha} X\left(\frac{1}{2}+\alpha\right)\frac{\zeta_{\mathbb{A}}(1-2\alpha)}{\zeta_{\mathbb{A}}(1-\alpha+\gamma)} A_{\mathcal{U}}(-\alpha;\gamma)\right)\Big|_{\alpha=\gamma=r}\\ &= -(\log q)\left(q^{2g+1}\right)^{-r} X\left(\frac{1}{2}+r\right)\zeta_{\mathbb{A}}(1-2r) A_{\mathcal{U}}(-r;r),\end{aligned}$$

where

$$A_{\mathcal{U}}(r;r) = 1,$$

$$A_{\mathcal{U}}(-r;r) = \prod_P \left(1 - \frac{1}{|P|}\right)^{-1}\left(1 - \frac{1}{(|P|+1)|P|^{1-2r}} - \frac{1}{|P|+1}\right),$$

and

$$A_{\mathcal{U}}'(r;r) = \sum_P \frac{\log|P|}{(|P|^{1+2r}-1)(|P|+1)}.$$

Thus, using the calculations stated above, we have that the Ratios Conjecture 7.1.4 implies the following.

Theorem 8.3.2. *Assuming Conjecture* 8.3.1, $\frac{1}{\log(q^{2g+1})} \ll \Re(r) < \frac{1}{4}$ *and* $\Im(r) \ll_\epsilon \left(q^{2g+1}\right)^{1-\epsilon}$, *we have*

$$\begin{aligned}&\sum_{u\in\mathcal{I}_{g+1}} \frac{L'\left(\frac{1}{2}+r,\chi_u\right)}{L\left(\frac{1}{2}+r,\chi_u\right)}\\ &= \sum_{u\in\mathcal{I}_{g+1}}\left(\frac{\zeta_{\mathbb{A}}'(1+2r)}{\zeta_{\mathbb{A}}(1+2r)} + A_{\mathcal{U}}'(r;r) - (\log q)\left(q^{2g+1}\right)^{-r} X\left(\frac{1}{2}+r\right)\zeta_{\mathbb{A}}(1-2r)A_{\mathcal{U}}(-r;r)\right) + o\left(q^{2g+1}\right).\end{aligned}$$

8.3.3 The One-Level Density

In this subsection, we derive Theorem 8.2.1, which states a formula for the one-level density for the zeros of Dirichlet L-functions $L(s,\chi_u)$, complete with lower order terms.

Let γ_u denote the ordinate of a generic zero of $L(s,\chi_u)$ on the half line. As $L(s,\chi_u)$ is a function in q^{-s} and so is periodic with period $\frac{2\pi i}{\log q}$, thus we can confine our analysis of the zeros to $-\frac{\pi i}{\log q} \le \Im(s) \le \frac{\pi i}{\log q}$. We consider the one-level density

$$S_1(f) = \sum_{u\in\mathcal{I}_{g+1}} \sum_{\gamma_u} f(\gamma_u), \tag{8.3.12}$$

where f is a $\frac{2\pi}{\log q}$-periodic even test function and holomorphic. By Cauchy's Theorem, we have that

$$S_1(f) = \sum_{u\in\mathcal{I}_{g+1}} \frac{1}{2\pi i}\left(\int_{(c)} - \int_{(1-c)}\right) \frac{L'(s,\chi_u)}{L(s,\chi_u)} f\left(-i\left(s-\frac{1}{2}\right)\right) ds, \tag{8.3.13}$$

where (c) denotes a vertical line from $c-\frac{\pi i}{\log q}$ to $c+\frac{\pi i}{\log q}$ and $\frac{1}{2}+\frac{1}{\log(q^{2g+1})} < c < \frac{3}{4}$. The integral along the (c)-line is equal to

$$\frac{1}{2\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} f\left(-i\left(c+it-\frac{1}{2}\right)\right) \sum_{u\in\mathcal{I}_{g+1}} \frac{L'(c+it,\chi_u)}{L(c+it,\chi_u)} dt. \tag{8.3.14}$$

Moving the path of integration to $c=\frac{1}{2}$ as the integral is regular at $t=0$ and using Theorem 8.3.2, we get that the integral along the (c)-line is equal to

$$\frac{1}{2\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} f(t) \sum_{u\in\mathcal{I}_{g+1}} \left(\frac{\zeta_{\mathbb{A}}'(1+2it)}{\zeta_{\mathbb{A}}(1+2it)} + A_{\mathcal{U}}'(it;it) - (\log q)\left(q^{2g+1}\right)^{-it} X\left(\frac{1}{2}+it\right)\zeta_{\mathbb{A}}(1-2it)A_{\mathcal{U}}(-it;it)\right)dt + o\left(q^{2g+1}\right). \tag{8.3.15}$$

For the integral along the $(1-c)$-line, we use the change $s\to 1-s$ and we use the functional equation

$$\frac{L'(1-s,\chi_u)}{L(1-s,\chi_u)} = \frac{\mathcal{X}_u'(s)}{\mathcal{X}_u(s)} - \frac{L'(s,\chi_u)}{L(s,\chi_u)}$$

where

$$\frac{\mathcal{X}_u'(s)}{\mathcal{X}_u(s)} = -\log\left(q^{2g+1}\right) + \frac{X'(s)}{X(s)}.$$

Thus, the integral along the $(1-c)$-line is equal to

$$\frac{1}{2\pi}\int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} f(t) \sum_{u\in\mathcal{I}_{g+1}} \Bigg(\log\left(q^{2g+1}\right) - \frac{X'}{X}\left(\frac{1}{2}-it\right) + \left(\frac{\zeta_{\mathbb{A}}'(1+2it)}{\zeta_{\mathbb{A}}(1+2it)} + A_{\mathcal{U}}'(it;it) - (\log q)\left(q^{2g+1}\right)^{-it} X\left(\frac{1}{2}+it\right)\zeta_{\mathbb{A}}(1-2it)A_{\mathcal{U}}(-it;it)\right)\Bigg)dt + o\left(q^{2g+1}\right). \tag{8.3.16}$$

Combining (8.3.12), (8.3.13), (8.3.15) and (8.3.16), we obtain Theorem 8.2.1.

8.3.4 The Scaled One-Level Density

Defining

$$f(t) = h\left(\frac{t(2g\log q)}{2\pi}\right)$$

and scaling the variable t from Theorem 8.2.1 as

$$\tau = \frac{t(2g\log q)}{2\pi},$$

we have that

$$\begin{aligned}
&\sum_{u\in\mathcal{I}_{g+1}}\sum_{\gamma_u} f\left(\gamma_u\frac{2g\log q}{2\pi}\right)\\
&= \frac{1}{2g\log q}\int_{-g}^{g} h(\tau)\sum_{u\in\mathcal{I}_{g+1}}\left(\log\left(q^{2g+1}\right) - \frac{X'}{X}\left(\frac{1}{2}-\frac{2\pi i\tau}{2g\log q}\right) + 2\left(\frac{\zeta_{\mathbb{A}}'\left(1+\frac{4\pi i\tau}{2g\log q}\right)}{\zeta_{\mathbb{A}}\left(1+\frac{4\pi i\tau}{2g\log q}\right)}\right.\right.\\
&+ A_{\mathcal{U}}'\left(\frac{2\pi i\tau}{2g\log q};\frac{2\pi i\tau}{2g\log q}\right) - (\log q)e^{-\frac{2\pi i\tau}{2g\log q}\log\left(q^{2g+1}\right)}\\
&\left.\left.\times X\left(\frac{1}{2}+\frac{2\pi i\tau}{2g\log q}\right)\zeta_{\mathbb{A}}\left(1-\frac{4\pi i\tau}{2g\log q}\right)A_{\mathcal{U}}\left(-\frac{2\pi i\tau}{2g\log q};\frac{2\pi i\tau}{2g\log q}\right)\right)\right)d\tau + o\left(q^{2g+1}\right). \qquad (8.3.17)
\end{aligned}$$

Writing

$$\zeta_{\mathbb{A}}(1+s) = \frac{1}{\log q}\frac{1}{s} + \frac{1}{2} + \frac{1}{12}(\log q)s + O(s^2)$$

we have

$$\frac{\zeta_{\mathbb{A}}'(1+s)}{\zeta_{\mathbb{A}}(1+s)} = -\frac{1}{s} + \frac{1}{2}\log q - \frac{1}{12}(\log q)^2 s + O(s^3).$$

Therefore as $g\to\infty$, only the $\log\left(q^{2g+1}\right)$ term, the $\frac{\zeta_{\mathbb{A}}'}{\zeta_{\mathbb{A}}}$ and the final term in the integral (8.3.17) contribute. Thus we have that

$$\begin{aligned}
&\sum_{u\in\mathcal{I}_{g+1}}\sum_{\gamma_u} f\left(\gamma_u\frac{2g\log q}{2\pi}\right)\\
&\sim \frac{1}{2g\log q}\int_{-\infty}^{\infty} h(\tau)\left((\#\mathcal{I}_{g+1})\log\left(q^{2g+1}\right) - (\#\mathcal{I}_{g+1})\frac{2g\log q}{2\pi i\tau} + (\#\mathcal{I}_{g+1})\frac{2g\log q}{2\pi i\tau}e^{-2\pi i\tau}\right)d\tau.
\end{aligned}$$

Since h is an even test function, the middle term drops out and the last term can be duplicated with a change of sign. Thus we get

$$\begin{aligned}
&\lim_{g\to\infty}\frac{1}{\#\mathcal{I}_{g+1}}\sum_{u\in\mathcal{I}_{g+1}}\sum_{\gamma_u} f\left(\gamma_u\frac{2g\log q}{2\pi}\right)\\
&= \int_{-\infty}^{\infty} h(\tau)\left(1+\frac{e^{-2\pi i\tau}}{4\pi i\tau} - \frac{e^{2\pi i\tau}}{4\pi i\tau}\right)d\tau\\
&= \int_{-\infty}^{\infty} h(\tau)\left(1+\frac{1}{4\pi i\tau}\left(-\left(\cos(2\pi\tau)+i\sin(2\pi\tau)\right)+\left(\cos(2\pi\tau)-i\sin(2\pi\tau)\right)\right)\right)d\tau\\
&= \int_{-\infty}^{\infty} h(\tau)\left(1-\frac{\sin(2\pi\tau)}{2\pi\tau}\right)d\tau.
\end{aligned}$$

Therefore for q a fixed power of 2 and as $g\to\infty$, the one-level density for the scaled zeros have the same form as the one-level density of the eigenvalues of matrices from $USp(2g)$ with respect with the Haar measure, which was obtained in [KS99b].

8.4 An Application of the Ratios Conjecture in Even Characteristic: Non-Vanishing of $L\left(\frac{1}{2},\chi_u\right)$

In this section, we present another application of the Ratios Conjecture 7.2.1 for Dirichlet L-functions in even characteristic, namely, conditionally on the Ratios Conjecture 7.1.4 we will prove that the proportion of quadratic L-functions $L\left(\frac{1}{2},\chi_u\right)$ which does not equal zero is 100%. To do this, we need to introduce a mollifier, similar to that done in [AB22], and prove results for the mollified first and second moments.

8.4.1 The Mollifier

Since the Riemann hypothesis for zeta functions associated with curves over finite fields has been proved (see Section 2.5 and Section 2.7.3 for more details), we know that

$$\frac{1}{L(s,\chi_u)}=\sum_{f\in\mathbb{A}^+}\frac{\mu(f)\chi_u(f)}{|f|^s} \tag{8.4.1}$$

is absolutely convergent for $\Re(s)>\frac{1}{2}$. Truncating this sum and multiplying by a smoothing function leads to the mollifier

$$M(\chi_u,P)=\sum_{\substack{f\in\mathbb{A}^+\\ |f|\le y}}\frac{\mu(f)\chi_u(f)}{\sqrt{|f|}}P\left(\frac{\log\left(\frac{y}{|f|}\right)}{\log y}\right), \tag{8.4.2}$$

where P is a polynomial satisfying $P(0)=0$ and $y=(q^{2g})^\theta$ for $\theta>0$.

Remark 8.4.1. *If we let $n=q^{deg(f)}$, then we see that the mollifier (8.4.2) is exactly the same as the mollifier used in the sumber field setting [CS07]. The mollifier (8.4.2) is also the same mollifier used in [AB22].*

We will write this sum as an integral using the following result.

Lemma 8.4.2 ([RM08, Exercise 4.1.6])**.** *For $c>0$ and every integer $n\ge1$ we have*

$$\frac{1}{2\pi i}\int_{(c)}\frac{x^z}{z^{n+1}}dz=\begin{cases}\frac{1}{n!}(\log x)^n & \text{if } x\ge1,\\ 0 & \text{if } x<1.\end{cases}$$

Let P be a polynomial satisfying $P(0) = 0$, then we can write it as $P(x) = \sum_{n\geq 1} p_n x^n$. Therefore we have

$$M(\chi_u, P) = \sum_{\substack{f\in\mathbb{A}^+ \\ |f|\leq y}} \frac{\mu(f)\chi_u(f)}{\sqrt{|f|}} \sum_{n\geq 1} \frac{p_n}{\log^n y} \log^n \left(\frac{y}{|f|}\right).$$

Thus, using Lemma 8.4.2 and the definition of the mollifier, we can rewrite (8.4.2) as

$$M(\chi_u, P) = \sum_{n\geq 1} \frac{p_n n!}{\log^n y} \frac{1}{2\pi i} \int_{(c)} \frac{y^z}{z^{n+1}} \frac{1}{L\left(\frac{1}{2}+z, \chi_u\right)} dz. \tag{8.4.3}$$

Remark 8.4.3. *To obtain the non-vanishing result Theorem* 8.2.2*, we first need to obtain, conditional on the Ratios Conjecture* 7.1.4*, asymptotic formulas for the first and second mollified moments which will be done in Section 8.4.2 and Section 8.4.3 respectively. Using these results, the Cauchy-Schwartz inequality and letting the length of the mollifier grow arbitrary large (i.e. $\theta \to \infty$) we obtain the result.*

8.4.2 The Mollified First Moment

In this subsection, we prove, conditional on the Ratios Conjecture 7.1.4, an asymptotic formula for the mollified first moment, which is defined as

$$\mathcal{M}(\alpha; P) := \sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2}+\alpha, \chi_u\right) M(\chi_u, P). \tag{8.4.4}$$

where $\alpha \ll \frac{1}{g}$. Thus, in this subsection, we prove the following result.

Theorem 8.4.4. *For Q an even polynomial, P a polynomial satisfying $P(0) = 0$ and for any $\theta > 0$ we have*

$$\begin{aligned} &Q\left(\frac{1}{g\log q}\frac{d}{d\alpha}\right) \sum_{u\in\mathcal{I}_{g+1}} \Lambda\left(\frac{1}{2}+\alpha, \chi_u\right) M(\chi_u, P)\bigg|_{\alpha=0} \\ &= \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \left(P(1)Q(1) + \frac{1}{2\theta}P'(1)\int_0^1 Q(t)dt + O(g^{-1})\right), \end{aligned} \tag{8.4.5}$$

where $\Lambda(s, \chi_u)$ is the completed L-function defined in Section 2.7.3.

Remark 8.4.5. *In Theorem* 8.4.4 *and Theorem* 8.4.7 *we only consider Q even since, by the functional equation (2.7.12), $\Lambda^{(k)}\left(\frac{1}{2}, \chi_u\right) = 0$ if k is odd.*

To prove Theorem 8.4.4 we first need to prove the following result.

Lemma 8.4.6. *For P a polynomial satisfying $P(0) = 0$, we have*

$$\mathcal{M}(\alpha, P) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \left(\frac{1+q^{-2g\alpha}}{2} P(1) + \frac{1-q^{-2g\alpha}}{2\alpha\log y} P'(1) + O\left(g^{-1}\right)\right) \tag{8.4.6}$$

uniformly for $\alpha \ll \frac{1}{g}$.

Proof. Let P be a polynomial satisfying $P(0)=0$, then we write $P(x)=\sum_{n\geq 1}p_n x^n$. Therefore, using (8.4.3) in (8.4.4) we have

$$\mathcal{M}(\alpha;P)=\sum_{n\geq 1}\frac{p_n n!}{\log^n y}\frac{1}{2\pi i}\int_{(c)}\frac{y^z}{z^{n+1}}\sum_{u\in\mathcal{I}_{g+1}}\frac{L\left(\frac{1}{2}+\alpha,\chi_u\right)}{L\left(\frac{1}{2}+z,\chi_u\right)}dz. \tag{8.4.7}$$

From Conjecture 8.3.1, we know that

$$\begin{aligned}&\sum_{u\in\mathcal{I}_{g+1}}\frac{L\left(\frac{1}{2}+\alpha,\chi_u\right)}{L\left(\frac{1}{2}+z,\chi_u\right)}\\&=\sum_{u\in\mathcal{I}_{g+1}}\left(A_{\mathcal{U}}(\alpha;z)\frac{\zeta_{\mathbb{A}}(1+2\alpha)}{\zeta_{\mathbb{A}}(1+\alpha+z)}+q^{-2g\alpha}A_{\mathcal{U}}(-\alpha;z)\frac{\zeta_{\mathbb{A}}(1-2\alpha)}{\zeta_{\mathbb{A}}(1-\alpha+z)}\right)+o\left(q^{2g+1}\right),\end{aligned} \tag{8.4.8}$$

where

$$A_{\mathcal{U}}(\alpha,z)=\prod_{p}\left(1-\frac{1}{|P|^{1+\alpha+z}}\right)^{-1}\left(1-\frac{1}{|P|^{1+2\alpha}(|P|+1)}-\frac{1}{|P|^{\alpha+z}(|P|+1)}\right).$$

Thus if we let

$$I_\alpha(y)=\zeta_{\mathbb{A}}(1+2\alpha)\sum_{n\geq 1}\frac{p_n n!}{\log^n y}J_\alpha(y) \tag{8.4.9}$$

where

$$J_\alpha(y)=\frac{1}{2\pi i}\int_{(c)}\frac{y^z}{z^{n+1}}\frac{A_{\mathcal{U}}(\alpha;z)}{\zeta_{\mathbb{A}}(1+\alpha+z)}dz, \tag{8.4.10}$$

then using Lemma 2.7.3 we have

$$\mathcal{M}(\alpha;P)=\frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(I_\alpha(y)+q^{-2g\alpha}I_{-\alpha}(y)+o(1)\right). \tag{8.4.11}$$

Since $\zeta_{\mathbb{A}}(s)\neq 0$ for all s, then $\frac{1}{\zeta_{\mathbb{A}}(s)}$ has no poles anywhere. Then moving the contour from $\Re(z)=c$ to $\Re(z)=-\delta$ where $\delta>0$ is sufficiently small so that the Euler product is absolutely convergent, we get that $J_\alpha(y)$ is given by the residue at $z=0$ plus the integral along the line $\Re(z)=-\delta$. We can write the residue at $z=0$ as a contour integral with the contour a circle of radius $\asymp\frac{1}{g}$ and for the integral along the line $\Re(z)=-\delta$, we have

$$\left|\frac{1}{2\pi i}\int_{(-\delta)}\frac{y^z}{z^{n+1}}\frac{A_{\mathcal{U}}(\alpha,z)}{\zeta_{\mathbb{A}}(1+\alpha+z)}dz\right|\ll y^{-\delta}\int_{-\infty}^{\infty}\frac{1}{|t|^{n+1}}dt \tag{8.4.12}$$

and since the integral on the right-hand side of (8.4.12) exists, then the contribution of the integral along the line $\Re(z)=-\delta$ is bounded above by $y^{-\delta}$. Thus combining all this we get

$$J_\alpha(y)=\frac{1}{2\pi i}\oint\frac{y^z}{z^{n+1}}\frac{A_{\mathcal{U}}(\alpha;z)}{\zeta_{\mathbb{A}}(1+\alpha+z)}dz+O\left(y^{-\delta}\right). \tag{8.4.13}$$

On the circular contour $|z|\asymp\frac{1}{g}$ and with $\alpha\asymp\frac{1}{g}$, we have the Taylor expansion

$$\frac{A_{\mathcal{U}}(\alpha;z)}{\zeta_{\mathbb{A}}(1+\alpha+z)}=(\alpha+z)A_{\mathcal{U}}(0;0)\log q+O\left(g^{-2}\right). \tag{8.4.14}$$

Furthermore, since $A_{\mathcal{U}}(0;0) = 1$, then we have

$$J_\alpha(y) = \frac{1}{2\pi i}\oint \frac{y^z}{z^{n+1}}\left(\log q(\alpha+z) + O(g^{-2})\right)dz.$$

Furthermore, on this circular contour and with $y = (q^{2g})^\theta$, we have, by the Estimation Lemma, [ST18, Lemma 6.41],

$$\left|\frac{g^{-2}}{2\pi i}\oint \frac{y^z}{z^{n+1}}dz\right| \ll g^{n-2}.$$

Thus

$$J_\alpha(y) = \frac{\log q}{2\pi i}\oint \frac{y^z}{z^{n+1}}(\alpha+z)dz + O\left(g^{n-2}\right). \tag{8.4.15}$$

Using the residue theorem, we have

$$\frac{1}{2\pi i}\oint \frac{y^z}{z^{n+1}}dz = \frac{\log^n y}{n!}$$

and so

$$\sum_{n\geq 1}\frac{p_n n!}{\log^n y}\frac{1}{2\pi i}\oint \frac{y^z}{z^{n+1}}dz = \sum_{n\geq 1}p_n = P(1) \tag{8.4.16}$$

and

$$\sum_{n\geq 1}\frac{p_n n!}{\log^n y}\frac{1}{2\pi i}\oint \frac{y^z}{z^{n}}dz = \sum_{n\geq 1}\frac{np_n}{\log y} = \frac{P'(1)}{\log y}. \tag{8.4.17}$$

Thus combining (8.4.9), (8.4.15), (8.4.16) and (8.4.17) together, we have

$$I_\alpha(y) = \log q\zeta_{\mathbb{A}}(1+2\alpha)\left(\alpha P(1) + \frac{1}{\log y}P'(1) + O\left(g^{-2}\right)\right). \tag{8.4.18}$$

For $\alpha \asymp \frac{1}{g}$, we have the Laurent expansion

$$\zeta_{\mathbb{A}}(1+2\alpha) = \frac{1}{2\alpha\log q} + O(1) \ll g$$

and so we have that

$$I_\alpha(y) = \frac{1}{2}P(1) + \frac{1}{2\alpha\log y}P'(1) + O\left(g^{-1}\right) \tag{8.4.19}$$

uniformly on any fixed annulus $\alpha \asymp \frac{1}{g}$. Using (8.4.19) we can rewrite (8.4.11) as

$$\mathcal{M}(\alpha;P) = \frac{1+q^{-2g\alpha}}{2}P(1) + \frac{1-q^{-2g\alpha}}{2\alpha\log y}P'(1) + O\left(g^{-1}\right). \tag{8.4.20}$$

Since $\mathcal{M}(\alpha;P)$ and the main term on the right hand side of (8.4.20) are holomorphic for $\alpha \ll \frac{1}{g}$, then the error term is also holomorphic in this region. Thus, by the maximum modulus principle, (8.4.20) holds uniformly for $\alpha \ll \frac{1}{g}$. ∎

Proof of Theorem 8.4.4. Define

$$\mathcal{N}(\alpha;P) := \sum_{u\in\mathcal{I}_{g+1}} \Lambda\left(\frac{1}{2}+\alpha,\chi_u\right) M(\chi_u,P). \tag{8.4.21}$$

From the definition of the completed L-function defined in Section 2.7.3, we know that $\Lambda\left(\frac{1}{2}+\alpha,\chi_u\right) = q^{g\alpha} L\left(\frac{1}{2}+\alpha,\chi_u\right)$. Thus by Lemma 8.4.6 we have

$$\mathcal{N}(\alpha;P) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(\frac{q^{g\alpha}+q^{-g\alpha}}{2}P(1) + \frac{q^{g\alpha}-q^{-g\alpha}}{2\alpha\log y}P'(1) + O\left(g^{-1}\right)\right) \tag{8.4.22}$$

uniformly for $\alpha \ll \frac{1}{g}$. Let $\alpha = \frac{a}{g\log q}$ then we have $q^{g\alpha} = q^{\frac{a}{\log q}} = e^{\frac{a}{\log q}\log q} = e^a$. Similarly $q^{-g\alpha} = e^{-a}$. Then by the definition of sinh and cosh and with $y = (q^{2g})^\theta$ we have

$$\mathcal{N}\left(\frac{a}{g\log q};P\right) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(P(1)\cosh a + \frac{\sinh a}{2a\theta}P'(1) + O\left(g^{-1}\right)\right). \tag{8.4.23}$$

Let Q be an even polynomial, then

$$Q\left(\frac{d}{da}\right)\cosh a\bigg|_{a=0} = Q(1) \tag{8.4.24}$$

and

$$Q\left(\frac{d}{da}\right)\frac{\sinh a}{a}\bigg|_{a=0} = Q\left(\frac{d}{da}\right)\int_0^1 \cosh(at)dt = \int_0^1 Q(t)dt. \tag{8.4.25}$$

Combining (8.4.23), (8.4.24) and (8.4.25), we have

$$Q\left(\frac{d}{da}\right)\mathcal{N}\left(\frac{a}{g\log q};P\right)\bigg|_{a=0} = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(P(1)Q(1) + \frac{1}{2\theta}P'(1)\int_0^1 Q(t)dt + O\left(g^{-1}\right)\right).$$

Using the change of variables $\alpha = \frac{a}{g\log q}$ so that $\frac{d}{da} = \frac{d\alpha}{da}\frac{d}{d\alpha} = \frac{1}{g\log q}\frac{d}{d\alpha}$ and (8.4.21) completes the proof of Theorem 8.4.4. ■

8.4.3 The Mollified Second Moment

In this subsection, we prove, conditional on the Ratios Conjecture 7.1.4, an asymptotic formula for the mollified second moment, which is defined as

$$\mathcal{M}(\alpha,\beta;P_1,P_2) := \sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2}+\alpha,\chi_u\right)L\left(\frac{1}{2}+\beta,\chi_u\right)M(\chi_u,P_1)M(\chi_u,P_2), \tag{8.4.26}$$

where $\alpha,\beta \ll \frac{1}{g}$. Thus, in this subsection, we prove the following result.

Theorem 8.4.7. *For even polynomials Q_1 and Q_2 and polynomials P_1 and P_2 satisfying $P_1(0) = P_1'(0) = P_2(0) = P_2'(0) = 0$ and for every $\theta > 0$ we have*

$$Q_1\left(\frac{1}{g\log q}\frac{d}{d\alpha}\right)Q_2\left(\frac{1}{g\log q}\frac{d}{d\beta}\right)\sum_{u\in\mathcal{I}_{g+1}}\Lambda\left(\frac{1}{2}+\alpha,\chi_u\right)\Lambda\left(\frac{1}{2}+\beta,\chi_u\right)M(\chi_u,P_1)M(\chi_u,P_2)\Bigg|_{\alpha=\beta=0}$$
$$= \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\Bigg(\frac{1}{8\theta}\int_0^1\int_0^1\left(\frac{1}{\theta}P_1''(r)\tilde{Q}_1(u)-4\theta P_1(r)Q_1'(u)\right)\left(\frac{1}{\theta}P_2''(r)\tilde{Q}_2(u)-4\theta P_2(r)Q_2'(u)\right)dudr$$
$$+\frac{1}{4}\left(\frac{1}{\theta}P_1'(1)\tilde{Q}_1(1)+2P_1(1)Q_1(1)\right)\left(\frac{1}{\theta}P_2'(1)\tilde{Q}_2(1)+2P_2(1)Q_2(1)\right)+O\left(g^{-1}\right)\Bigg), \tag{8.4.27}$$

where

$$\tilde{Q}(u) = \int_0^u Q(t)dt$$

and $\Lambda(s,\chi_u)$ is the completed L-function defined in Section 2.7.3.

To prove Theorem 8.4.7, we first need to prove the following result.

Lemma 8.4.8. *For polynomials P_1 and P_2 satisfying $P_1(0) = P_1'(0) = P_2(0) = P_2'(0) = 0$ we have*

$$\mathcal{M}(\alpha,\beta;P_1,P_2) \tag{8.4.28}$$
$$= \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\Bigg(\frac{\alpha\beta\log y}{4}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\alpha}-q^{-2g\beta}}{\alpha-\beta}\right)\int_0^1 P_1(r)P_2(r)dr$$
$$+\frac{1}{4}\left(1+q^{-2g\alpha}\right)\left(1+q^{-2g\beta}\right)\int_0^1(P_1(r)P_2'(r)+P_1'(r)P_2(r))dr$$
$$+\frac{1}{4\log y}\left(\frac{(1+q^{-2g\alpha})(1-q^{-2g\beta})}{\beta}+\frac{(1-q^{-2g\beta})(1+q^{-2g\beta})}{\alpha}\right)\int_0^1 P_1'(r)P_2'(r)dr$$
$$+\frac{1}{4\log y}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\beta}-q^{-2g\alpha}}{\alpha-\beta}\right)\int_0^1(P_1(r)P_2''(r)+P_1''(r)P_2(r))dr$$
$$+\frac{1}{4\alpha\beta\log^2 y}\left(1-q^{-2g\alpha}\right)\left(1-q^{-2g\beta}\right)\int_0^1(P_1'(r)P_2''(r)+P_1''(r)P_2'(r))dr$$
$$+\frac{1}{4\alpha\beta\log^3 y}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\alpha}-q^{-2g\beta}}{\alpha-\beta}\right)\int_0^1 P_1''(r)P_2''(r)dr+O\left(g^{-1}\right)\Bigg) \tag{8.4.29}$$

uniformly for $\alpha,\beta \ll \frac{1}{g}$.

Proof. Let P_1 and P_2 be polynomials satisfying $P_1(0) = P_1'(0) = P_2(0) = P_2'(0) = 0$. Then we can write the polynomials as $P_1(x) = \sum_{m\geq 2}p_{1,m}x^m$ and $P_2(x) = \sum_{n\geq 2}p_{2,n}x^n$.

Therefore, using (8.4.3) twice in (8.4.26) we have

$$\mathcal{M}(\alpha,\beta;P_1,P_2)=\sum_{m,n\geq 2}\frac{p_{1,m}p_{2,n}m!n!}{\log^{m+n}y}\times\frac{1}{(2\pi i)^2}\int_{(c)}\int_{(c)}\frac{y^{w+z}}{w^{m+1}z^{n+1}}\sum_{u\in\mathcal{I}_{g+1}}\frac{L\left(\frac{1}{2}+\alpha,\chi_u\right)L\left(\frac{1}{2}+\beta,\chi_u\right)}{L\left(\frac{1}{2}+w,\chi_u\right)L\left(\frac{1}{2}+z,\chi_u\right)}dwdz \tag{8.4.30}$$

for any $c>0$. From the Ratios Conjecture 7.1.4 we have that

$$\begin{aligned}&\sum_{u\in\mathcal{I}_{g+1}}\frac{L\left(\frac{1}{2}+\alpha,\chi_u\right)L\left(\frac{1}{2}+\beta,\chi_u\right)}{L\left(\frac{1}{2}+w,\chi_u\right)L\left(\frac{1}{2}+z,\chi_u\right)}\\&=\sum_{u\in\mathcal{I}_{g+1}}\Bigg(A_{\mathcal{U}}(\alpha,\beta;w,z)Y_{\mathcal{U}}(\alpha,\beta;w,z)+q^{-2g\alpha}A_{\mathcal{U}}(-\alpha,\beta;w,z)Y_{\mathcal{U}}(-\alpha,\beta;w,z)\\&+q^{-2g\beta}A_{\mathcal{U}}(\alpha,-\beta;w,z)Y_{\mathcal{U}}(\alpha,-\beta;w,z)+q^{-2g(\alpha+\beta)}A_{\mathcal{U}}(-\alpha,-\beta;w,z)Y_{\mathcal{U}}(-\alpha,-\beta;w,z)\Bigg)\\&+o\left(q^{2g+1}\right),\end{aligned}$$

where

$$\begin{aligned}&A_{\mathcal{U}}(\alpha,\beta;w,z)\\&=\prod_P\frac{\left(1-\frac{1}{|P|^{1+\alpha+\beta}}\right)\left(1-\frac{1}{|P|^{w+z}}\right)}{\left(1-\frac{1}{|P|^{1+\alpha+w}}\right)\left(1-\frac{1}{|P|^{1+\alpha+z}}\right)\left(1-\frac{1}{|P|^{1+\beta+w}}\right)\left(1-\frac{1}{|P|^{1+\beta+z}}\right)}\\&\times\Bigg(1+\frac{1}{|P|^{\alpha+\beta}(|P|+1)}+\frac{1}{|P|^{w+z}(|P|+1)}+\frac{1}{|P|^{1+\alpha+\beta+w+z}(|P|+1)}+\frac{1}{|P|^{2(1+\alpha+\beta)}(|P|+1)}\\&-\frac{1}{|P|^{\alpha+w}(|P|+1)}-\frac{1}{|P|^{\alpha+z}(|P|+1)}-\frac{1}{|P|^{\beta+w}(|P|+1)}\\&-\frac{1}{|P|^{\beta+z}(|P|+1)}-\frac{1}{|P|^{1+2\alpha}(|P|+1)}-\frac{1}{|P|^{1+2\beta}(|P|+1)}\Bigg)\end{aligned}\tag{8.4.31}$$

and

$$Y_{\mathcal{U}}(\alpha,\beta;w,z)=\frac{\zeta_{\mathbb{A}}(1+2\alpha)\zeta_{\mathbb{A}}(1+\alpha+\beta)\zeta_{\mathbb{A}}(1+2\beta)\zeta_{\mathbb{A}}(1+w+z)}{\zeta_{\mathbb{A}}(1+\alpha+w)\zeta_{\mathbb{A}}(1+\alpha+z)\zeta_{\mathbb{A}}(1+\beta+w)\zeta_{\mathbb{A}}(1+\beta+z)}.$$

Thus if we let

$$I_{\alpha,\beta}(y)=\zeta_{\mathbb{A}}(1+2\alpha)\zeta_{\mathbb{A}}(1+\alpha+\beta)\zeta_{\mathbb{A}}(1+2\beta)\sum_{m,n\geq 2}\frac{p_{1,m}p_{2,n}m!n!}{\log^{m+n}y}J_{\alpha,\beta}(y) \tag{8.4.32}$$

where

$$J_{\alpha,\beta}(y)=\frac{1}{(2\pi i)^2}\int_{(c)}\int_{(c)}\frac{y^{w+z}}{w^{m+1}z^{n+1}}\times\frac{\zeta_{\mathbb{A}}(1+w+z)A_{\mathcal{U}}(\alpha,\beta;w,z)}{\zeta_{\mathbb{A}}(1+\alpha+w)\zeta_{\mathbb{A}}(1+\alpha+z)\zeta_{\mathbb{A}}(1+\beta+w)\zeta_{\mathbb{A}}(1+\beta+z)}dwdz, \tag{8.4.33}$$

then using Lemma 2.7.3 we have

$$\mathcal{M}(\alpha,\beta;P_1,P_2) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(I_{\alpha,\beta}(y) + q^{-2g\alpha}I_{-\alpha,\beta}(y) + q^{-2g\beta}I_{\alpha,-\beta}(y) + q^{-2g(\alpha+\beta)}I_{-\alpha,-\beta}(y) + o(1)\right). \quad (8.4.34)$$

For $\Re(w+z) > 0$ we write

$$\frac{y^{w+z}}{w+z} = \int_0^y u^{w+z}\frac{du}{u},$$

thus

$$J_{\alpha,\beta}(y) = \frac{1}{(2\pi i)^2}\int_1^y \int_{(c)}\int_{(c)} \frac{u^{w+z}}{w^{m+1}z^{n+1}} \times \frac{(w+z)\zeta_{\mathbb{A}}(1+w+z)A_{\mathcal{U}}(\alpha,\beta;w,z)}{\zeta_{\mathbb{A}}(1+\alpha+w)\zeta_{\mathbb{A}}(1+\alpha+z)\zeta_{\mathbb{A}}(1+\beta+w)\zeta_{\mathbb{A}}(1+\beta+z)}dwdz\frac{du}{u}. \quad (8.4.35)$$

The integration in u is over $1 \le u \le y$ as for $u < 1$ the contours can be moved to the right so that the integrands in w and z equal zero. Since $\zeta_{\mathbb{A}}(s) \neq 0$ for all s , then $\frac{1}{\zeta_{\mathbb{A}}(s)}$ has no poles anywhere. Furthermore, $\zeta_{\mathbb{A}}(s)$ has a simple pole at $s = 1$, thus $\zeta_{\mathbb{A}}(1+w+z)$ has a pole at $w = -z$, therefore $(w+z)\zeta_{\mathbb{A}}(1+w+z)$ is analytic at $w = -z$. Hence the poles of the integrand of (8.4.35) occur when $w = z = 0$. Therefore moving the contours from $\Re(w) = \Re(z) = c$ to $\Re(w) = \Re(z) = -\delta$ where $\delta > 0$ is sufficiently small so that the Euler product (8.4.31) is absolutely convergent, we have that $J_{\alpha,\beta}(y)$ is given by the residue at $w = z = 0$ plus the integrals along the $\Re(w) = \Re(z) = -\delta$. We express the residue at $w = z = 0$ as contour integrals where the contour is a circle of radius $\asymp \frac{1}{g}$ and using similar calculations as those done in the proof of Lemma 8.4.6, we see that the integrals along the line $\Re(w) = -\delta$ and $\Re(z) = -\delta$ is bounded above by $u^{-\delta}$. Letting $2\delta = \epsilon$, we have that

$$\left|\int_1^y \int_{(-\delta)}\int_{(-\delta)} \frac{u^{w+z}}{w^{m+1}z^{n+1}} \times \frac{(w+z)\zeta_{\mathbb{A}}(1+w+z)A_{\mathcal{U}}(\alpha,\beta;w,z)}{\zeta_{\mathbb{A}}(1+\alpha+w)\zeta_{\mathbb{A}}(1+\alpha+z)\zeta_{\mathbb{A}}(1+\beta+w)\zeta_{\mathbb{A}}(1+\beta+z)}dwdz\frac{du}{u}\right| \ll \int_1^y u^{-\epsilon}\frac{du}{u} \ll 1.$$

On the circular contours $|w| \asymp \frac{1}{g}$ and $|z| \asymp \frac{1}{g}$ and with $\alpha \asymp \frac{1}{g}$ and $\beta \asymp \frac{1}{g}$, we have the Taylor expansion

$$\frac{(w+z)\zeta_{\mathbb{A}}(1+w+z)A_{\mathcal{U}}(\alpha,\beta;w,z)}{\zeta_{\mathbb{A}}(1+\alpha+w)\zeta_{\mathbb{A}}(1+\alpha+z)\zeta_{\mathbb{A}}(1+\beta+w)\zeta_{\mathbb{A}}(1+\beta+z)} = (\alpha+w)(\alpha+z)(\beta+w)(\beta+z)A_{\mathcal{U}}(0,0;0,0)\log^3 q + O\left(g^{-5}\right).$$

Using the fact that $A_{\mathcal{U}}(0,0;0,0)=1$, we have

$$J_{\alpha,\beta}(y) = \frac{1}{(2\pi i)^2}\int_1^y \oint\oint \frac{u^{w+z}}{w^{m+1}z^{n+1}}\left((\alpha+w)(\alpha+z)(\beta+w)(\beta+z)\log^3 q + O(g^{-5})\right)dwdz\frac{du}{u}.$$

On these circular contours and with $y=(q^{2g})^{\theta}$, we have, by the Estimation Lemma,

$$\left|\frac{g^{-5}}{(2\pi i)^2}\int_1^y\oint\oint\frac{u^{w+z}}{w^{m+1}z^{n+1}}dwdz\frac{du}{u}\right| \ll g^{m+n-5}\int_1^y u^{\frac{1}{g}}\frac{du}{u} \ll g^{m+n-4}.$$

Thus

$$J_{\alpha,\beta}(y) = \frac{\log^3 q}{(2\pi i)^2}\int_1^y\left(\oint\frac{u^w}{w^{m+1}}(\alpha+w)(\beta+w)dw\right)\left(\oint\frac{u^z}{z^{n+1}}(\alpha+z)(\beta+z)dz\right)\frac{du}{u} + O\left(g^{m+n-4}\right).$$

Expanding and computing the residue where we use the fact that for $k\in\{0,1,2\}$ and $i\in\{1,2\}$ we have

$$\sum_{n\geq 2}\frac{p_{i,n}n!}{\log^n y}\frac{1}{2\pi i}\oint\frac{u^z}{z^{n+1-k}}dz = \frac{1}{\log^k y}P_i^{(k)}\left(\frac{\log u}{\log y}\right),$$

we get that

$$\begin{aligned} I_{\alpha,\beta}(y) &= \zeta_{\mathbb{A}}(1+2\alpha)\zeta_{\mathbb{A}}(1+\alpha+\beta)\zeta_{\mathbb{A}}(1+2\beta)\log^3 q \\ &\times\left(\int_1^y\left(\alpha\beta P_1\left(\frac{\log u}{\log y}\right)+\frac{\alpha+\beta}{\log y}P_1'\left(\frac{\log u}{\log y}\right)+\frac{1}{\log^2 y}P_1''\left(\frac{\log u}{\log y}\right)\right)\right. \\ &\times\left.\left(\alpha\beta P_2\left(\frac{\log u}{\log y}\right)+\frac{\alpha+\beta}{\log y}P_2'\left(\frac{\log u}{\log y}\right)+\frac{1}{\log^2 y}P_2''\left(\frac{\log u}{\log y}\right)\right)\frac{du}{u}+O\left(g^{-4}\right)\right). \end{aligned}$$

Using the change $u=y^r$ we have

$$\begin{aligned} I_{\alpha,\beta}(y) &= \zeta_{\mathbb{A}}(1+2\alpha)\zeta_{\mathbb{A}}(1+\alpha+\beta)\zeta_{\mathbb{A}}(1+2\beta)\log^3 q\log y \\ &\times\left(\int_0^1\left(\alpha\beta P_1(r)+\frac{\alpha+\beta}{\log y}P_1'(r)+\frac{1}{\log^2 y}P_1''(r)\right)\right. \\ &\times\left.\left(\alpha\beta P_2(r)+\frac{\alpha+\beta}{\log y}P_2'(r)+\frac{1}{\log^2 y}P_2''(r)\right)dr+O\left(g^{-4}\right)\right). \end{aligned}$$

For $\alpha,\beta\asymp\frac{1}{g}$ and $|\alpha+\beta|\gg\frac{1}{g}$, we have the Laurent expansion

$$\zeta_{\mathbb{A}}(1+2\alpha)\zeta_{\mathbb{A}}(1+\alpha+\beta)\zeta_{\mathbb{A}}(1+2\beta) = \frac{1}{4\alpha\beta(\alpha+\beta)\log^3 q}+O\left(g^2\right)\ll g^3.$$

Thus

$$I_{\alpha,\beta}(y)=\frac{\log y}{4\alpha\beta(\alpha+\beta)}\int_0^1\left(\alpha\beta P_1(r)+\frac{\alpha+\beta}{\log y}P_1'(r)+\frac{1}{\log^2 y}P_1''(r)\right)$$
$$\times\left(\alpha\beta P_2(r)+\frac{\alpha+\beta}{\log y}P_2'(r)+\frac{1}{\log^2 y}P_2''(r)\right)dr+O\left(g^{-1}\right)$$

uniformly on any fixed annuli such that $\alpha,\beta\asymp\frac{1}{g}$ and $|\alpha+\beta|\gg\frac{1}{g}$. Multiplying out we have

$$\begin{aligned}I_{\alpha,\beta}(y)=&\frac{\alpha\beta\log y}{4(\alpha+\beta)}\int_0^1 P_1(r)P_2(r)dr\\&+\frac{1}{4}\int_0^1(P_1(r)P_2'(r)+P_1'(r)P_2(r))dr\\&+\frac{\alpha+\beta}{4\alpha\beta\log y}\int_0^1 P_1'(r)P_2'(r)dr\\&+\frac{1}{4(\alpha+\beta)\log y}\int_0^1(P_1(r)P_2''(r)+P_1''(r)P_2(r))dr\\&+\frac{1}{4\alpha\beta\log^2 y}\int_0^1(P_1'(r)P_2''(r)+P_1''(r)P_2'(r))dr\\&+\frac{1}{4\alpha\beta(\alpha+\beta)\log^3 y}\int_0^1 P_1''(r)P_2''(r)dr+O\left(g^{-1}\right).\end{aligned}\tag{8.4.36}$$

Using (8.4.36), we can rewrite (8.4.34) as

$$\begin{aligned}&\mathcal{M}(\alpha,\beta;P_1,P_2)\\&=\frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(\frac{\alpha\beta\log y}{4}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\alpha}-q^{-2g\beta}}{\alpha-\beta}\right)\int_0^1 P_1(r)P_2(r)dr\right.\\&+\frac{1}{4}(1+q^{-2g\alpha})(1+q^{-2g\beta})\int_0^1(P_1(r)P_2'(r)+P_1'(r)P_2(r))dr\\&+\frac{1}{4\log y}\left(\frac{(1+q^{-2g\alpha})(1-q^{-2g\beta})}{\beta}+\frac{(1-q^{-2g\alpha})(1+q^{-2g\beta})}{\alpha}\right)\int_0^1 P_1'(r)P_2'(r)dr\\&+\frac{1}{4\log y}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\beta}-q^{-2g\alpha}}{\alpha-\beta}\right)\int_0^1(P_1(r)P_2''(r)+P_1''(r)P_2(r))dr\\&+\frac{1}{4\alpha\beta\log^2 y}(1-q^{-2g\alpha})(1-q^{-2g\beta})\int_0^1(P_1'(r)P_2''(r)+P_1''(r)P_2'(r))dr\\&\left.+\frac{1}{4\alpha\beta(\alpha+\beta)\log^3 y}\left(\frac{1-q^{-2g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{-2g\alpha}-q^{-2g\beta}}{\alpha-\beta}\right)\int_0^1 P_1''(r)P_2''(r)dr+O\left(g^{-1}\right)\right).\end{aligned}\tag{8.4.37}$$

Since $\mathcal{M}(\alpha,\beta;P_1,P_2)$ and the main term on the right hand side of (8.4.37) are holomorphic for $\alpha,\beta\ll\frac{1}{g}$, then the error term is holomorphic in this region too. Thus by the maximum modulus principle (8.4.37) holds uniformly for $\alpha,\beta\ll\frac{1}{g}$. ∎

Proof of Theorem 8.4.7. Define

$$\mathcal{N}(\alpha,\beta;P_1,P_2) := \sum_{u\in\mathcal{I}_{g+1}} \Lambda\left(\frac{1}{2}+\alpha,\chi_u\right)\Lambda\left(\frac{1}{2}+\beta,\chi_u\right)M(\chi_u,P_1)M(\chi_u,P_2),$$

From the definition of the completed L-function, we know that $\Lambda\left(\frac{1}{2}+\alpha,\chi_u\right)\Lambda\left(\frac{1}{2}+\beta,\chi_u\right) = q^{g(\alpha+\beta)}L\left(\frac{1}{2}+\alpha,\chi_u\right)L\left(\frac{1}{2}+\beta,\chi_u\right)$, then by Lemma 8.4.8 we have

$$\begin{aligned}
&\mathcal{N}(\alpha,\beta;P_1,P_2)\\
&= \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\Bigg(\frac{\alpha\beta\log y}{4}\left(\frac{q^{g(\alpha+\beta)}-q^{-g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{g(\beta-\alpha)}-q^{g(\alpha-\beta)}}{\alpha-\beta}\right)\int_0^1 P_1(r)P_2(r)dr\\
&+\frac{1}{4}\left(q^{g\alpha}+q^{-g\alpha}\right)\left(q^{g\beta}+q^{-g\beta}\right)\int_0^1 (P_1(r)P_2'(r)+P_1'(r)P_2(r))dr\\
&+\frac{1}{4\log y}\left(\frac{\left(q^{g\alpha}+q^{-g\alpha}\right)\left(q^{g\beta}-q^{-g\beta}\right)}{\beta}+\frac{\left(q^{g\beta}+q^{-g\beta}\right)\left(q^{g\alpha}-q^{-g\alpha}\right)}{\alpha}\right)\int_0^1 P_1'(r)P_2'(r)dr\\
&+\frac{1}{4\log y}\left(\frac{q^{g(\alpha+\beta)}-q^{-g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{g(\alpha-\beta)}-q^{g(\beta-\alpha)}}{\alpha-\beta}\right)\int_0^1 (P_1(r)P_2''(r)+P_1''(r)P_2(r))dr\\
&+\frac{1}{4\alpha\beta\log^2 y}\left(q^{g\alpha}-q^{-g\alpha}\right)\left(q^{g\beta}-q^{-g\beta}\right)\int_0^1 (P_1'(r)P_2''(r)+P_1''(r)P_2'(r))dr\\
&+\frac{1}{4\alpha\beta\log^3 y}\left(\frac{q^{g(\alpha+\beta)}-q^{-g(\alpha+\beta)}}{\alpha+\beta}+\frac{q^{g(\beta-\alpha)}-q^{g(\alpha-\beta)}}{\alpha-\beta}\right)\int_0^1 P_1''(r)P_2''(r)dr+O\left(g^{-1}\right)\Bigg)
\end{aligned}$$

uniformly for $\alpha,\beta \ll \frac{1}{g}$. Let $\alpha = \frac{a}{g\log q}$ and $\beta = \frac{b}{\log q}$, then by the definition of sinh and cosh and with $y = (q^{2g})^\theta$ we have

$$\begin{aligned}
\mathcal{N}\left(\frac{a}{g\log q},\frac{b}{g\log q};P_1,P_2\right) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\Bigg(&2\theta\int_0^1 a\sinh(au)b\sinh(bu)du\int_0^1 P_1(r)P_2(r)dr\\
&+\cosh a\cosh b\int_0^1 (P_1(r)P_2'(r)+P_1'(r)P_2(r))dr\\
&+\frac{1}{2\theta}\left(\frac{\sinh a\cosh b}{a}+\frac{\sinh b\cosh a}{b}\right)\int_0^1 P_1'(r)P_2'(r)dr\\
&+\int_0^1\cosh(au)\cosh(bu)du\int_0^1 (P_1(r)P_2''(r)+P_1''(r)P_2(r))dr\\
&+\frac{1}{4\theta^2}\frac{\sinh a}{a}\frac{\sinh b}{b}\int_0^1 (P_1'(r)P_2''(r)+P_1''(r)P_2'(r))dr\\
&+\frac{1}{8\theta^3}\int_0^1\frac{\sinh(au)}{a}\frac{\sinh(bu)}{b}du\int_0^1 P_1''(r)P_2''(r)dr+O\left(g^{-1}\right)\Bigg).
\end{aligned}$$

Let Q be an even polynomial, then we have

$$Q\left(\frac{d}{da}\right)a\sinh(au)\bigg|_{a=0} = Q\left(\frac{d}{da}\right)\frac{d}{du}\cosh(au)\bigg|_{a=0} = \frac{d}{du}Q(u) = Q'(u),$$

$$Q\left(\frac{d}{da}\right)\cosh a\bigg|_{a=0} = Q(1)$$

and

$$Q\left(\frac{d}{da}\right)\frac{\sinh(au)}{a}\bigg|_{a=0} = Q\left(\frac{d}{da}\right)\int_0^u \cosh(at)dt\bigg|_{a=0} = \int_0^u Q(t)dt =: \tilde{Q}(u).$$

Thus, let Q_1 and Q_2 be even polynomials and define

$$\mathcal{N}(Q_1,Q_2;P_1,P_2) = Q_1\left(\frac{d}{da}\right)Q_2\left(\frac{d}{db}\right)\mathcal{N}\left(\frac{a}{g\log q},\frac{b}{g\log q};P_1,P_2\right)\bigg|_{a=b=0},$$

then we have

$$\begin{aligned}\mathcal{N}(Q_1,Q_2;P_1,P_2) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\Bigg(&2\theta\int_0^1 Q_1'(u)Q_2'(u)du\int_0^1 P_1(r)P_2(r)dr \\ &+ Q_1(1)Q_2(1)\int_0^1 (P_1(r)P_2'(r)+P_1'(r)P_2(r))dr \\ &+\frac{1}{2\theta}\left(Q_1(1)\tilde{Q}_2(1)+\tilde{Q}_1(1)Q_2(1)\right)\int_0^1 P_1'(r)P_2'(r)dr \\ &+\frac{1}{2\theta}\int_0^1 Q_1(u)Q_2(u)du\int_0^1(P_1(r)P_2''(r)+P_1''(r)P_2(r))dr \\ &+\frac{1}{4\theta^2}\tilde{Q}_1(1)\tilde{Q}_2(1)\int_0^1(P_1'(r)P_2''(r)+P_1''(r)P_2'(r))dr \\ &+\frac{1}{8\theta^3}\int_0^1\tilde{Q}_1(u)\tilde{Q}_2(u)du\int_0^1 P_1''(r)P_2''(r)dr + O\left(g^{-1}\right)\Bigg). \qquad (8.4.38)\end{aligned}$$

To write (8.4.38) in the form seen in the statement of Theorem 8.4.7 we need to recall the following identities, which all follow from integration by parts:

$$\int_0^1 (P_1(r)P_2'(r)+P_1'(r)P_2(r))dr = P_1(1)P_2(1),$$

$$\int_0^1 (P_1'(r)P_2''(r)+P_1''(r)P_2'(r))dr = P_1'(1)P_2'(1),$$

$$\int_0^1 P_1''(r)P_2(r)dr = P_1'(1)P_2(1) - \int_0^1 P_1'(r)P_2'(r)dr$$

and

$$\int_0^1 \tilde{Q}_1(u)Q_2'(u)du = \tilde{Q}_1(1)Q_2(1) - \int_0^1 Q_1(u)Q_2(u)du.$$

Using the above identities, we also have

$$\begin{aligned}
&\int_0^1 Q_1(u)Q_2(u)du \int_0^1 (P_1(r)P_2''(r) + P_1''(r)P_2(r))dr \\
&= \int_0^1\int_0^1 Q_1(u)Q_2(u)P_1(r)P_2''(r)dudr + \int_0^1\int_0^1 Q_1(u)Q_2(u)P_1''(r)P_2(r)dudr \\
&= \int_0^1 P_1(r)P_2''(r)dr\left(Q_1(1)\tilde{Q}_2(1) - \int_0^1 Q_1'(u)\tilde{Q}_2(u)du\right) \\
&+ \int_0^1 P_1''(r)P_2(r)dr\left(\tilde{Q}_1(1)Q_2(1) - \int_0^1 \tilde{Q}_1(u)Q_2'(u)du\right) \\
&= Q_1(1)\tilde{Q}_2(1)\int_0^1 P_1(r)P_2''(r)dr + \tilde{Q}_1(1)Q_2(1)\int_0^1 P_1''(r)P_2(r)dr \\
&- \int_0^1\int_0^1 P_1(r)P_2''(r)Q_1'(u)\tilde{Q}_2(u)dudr - \int_0^1\int_0^1 P_1''(r)P_2(r)\tilde{Q}_1(u)Q_2'(u)dudr \\
&= Q_1(1)\tilde{Q}_2(1)\left(P_1(1)P_2'(1) - \int_0^1 P_1'(r)P_2'(r)dr\right) \\
&+ \tilde{Q}_1(1)Q_2(1)\left(P_1'(1)P_2(1) - \int_0^1 P_1'(r)P_2'(r)dr\right) \\
&- \int_0^1\int_0^1 P_1(r)Q_1'(u)P_2''(r)\tilde{Q}_2(u)dudr - \int_0^1\int_0^1 P_1''(r)\tilde{Q}_1(u)P_2(r)Q_2(u)dudr.
\end{aligned}$$

Thus, combining the above results together, we have

$$\begin{aligned}
&\mathcal{N}(Q_1,Q_2;P_1,P_2) \\
&= \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\Bigg(2\theta\int_0^1\int_0^1 P_1(r)Q_1'(u)P_2(r)Q_2'(u)dudr + Q_1(1)Q_2(1)P_1(1)P_2(1) \\
&+ \frac{1}{2\theta}\left(P_1(1)Q_1(1)P_2'(1)\tilde{Q}_2(1) + P_1'(1)\tilde{Q}_1(1)P_2(1)Q_2(1)\right) \\
&- \frac{1}{2\theta}\left(\int_0^1\int_0^1\left(P_1(r)Q_1'(u)P_2''(r)\tilde{Q}_2(u) + P_1''(r)\tilde{Q}_1(u)P_2(r)Q_2'(u)\right)dudr\right) \\
&+ \frac{1}{4\theta^2}\tilde{Q}_1(1)\tilde{Q}_2(1)P_1'(1)P_2'(1) + \frac{1}{8\theta^3}\int_0^1\int_0^1 P_1''(r)\tilde{Q}_1(u)P_2''(r)\tilde{Q}_2(u)dudr + O\left(g^{-1}\right)\Bigg).
\end{aligned} \tag{8.4.39}$$

Factorising (8.4.39) completes the proof of Theorem 8.4.7. ■

8.4.4 Proof of Theorem 8.2.2

In this subsection, we combine Theorem 8.4.4 and Theorem 8.4.7 to prove Theorem 8.2.2.

Proof of Theorem 8.2.2. Using Theorem 8.4.4 with $P(x) = x^2$ and $Q(x) = 1$, and using the definition of the completed L-function we have that

$$\sum_{u\in\mathcal{I}_{g+1}} L\left(\frac{1}{2},\chi_u\right)M(\chi_u,P) = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(1+\frac{1}{\theta}+O\left(\frac{1}{g}\right)\right). \tag{8.4.40}$$

Similarly, using Theorem 8.4.7 with $P(x) = P_1(x) = P_2(x) = x^2$ and $Q(x) = Q_1(x) = Q_2(x) = 1$ and using the definition of the completed L-function we have that

$$\sum_{u \in \mathcal{I}_{g+1}} L\left(\frac{1}{2}, \chi_u\right)^2 M(\chi_u, P)^2 = \frac{2q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \left(\frac{1}{6\theta^3} + \left(1 + \frac{1}{\theta}\right)^2 + O\left(\frac{1}{g}\right)\right). \tag{8.4.41}$$

An application of the Cauchy-Schwartz inequality gives us that

$$\sum_{\substack{u \in \mathcal{I}_{g+1} \\ L\left(\frac{1}{2}, \chi_u\right) \neq 0}} 1 \geq \frac{\sum_{u \in \mathcal{I}_{g+1}} \left(L\left(\frac{1}{2}, \chi_u\right) M\left(\chi_u, P\right)\right)^2}{\sum_{u \in \mathcal{I}_{g+1}} \left|L\left(\frac{1}{2}, \chi_u\right) M\left(\chi_u, P\right)\right|^2}. \tag{8.4.42}$$

Thus using (8.4.40) and (8.4.41) in (8.4.42) we have, as $g \to \infty$ that

$$\frac{1}{\# \mathcal{I}_{g+1}} \sum_{\substack{u \in \mathcal{I}_{g+1} \\ L\left(\frac{1}{2}, \chi_u\right) \neq 0}} 1 \geq \frac{\left(1 + \frac{1}{\theta}\right)^2}{\frac{1}{6\theta^3} + \left(1 + \frac{1}{\theta}\right)^2} + o(1).$$

Letting the length of the mollifier grow arbitrary large (i.e. letting $\theta \to \infty$) proves the result. ∎

Chapter 9

The Twisted Second Moment of Dirichlet L-functions in $\mathbb{F}_q[T]$

9.1 Twisted Moments of Dirichlet L-functions in Function Fields

In function fields it is an interesting problem to understand the asymptotic behaviour of

$$\sum_{\chi(\text{mod } Q)}^{*} \left| L\left(\frac{1}{2},\chi\right)\right|^{2k}, \tag{9.1.1}$$

where Q is a polynomial in $\mathbb{F}_q[T]$ with q being a power of an odd prime, χ is a primitive Dirichlet character modulo Q, $L(s,\chi)$ is a Dirichlet L-function associated to the Dirichlet character χ, which are defined in Section 2.3 and Section 2.4 respectively, and the sum being over all primitive Dirichlet characters modulo Q.

For Q a monic irreducible polynomial in $\mathbb{F}_q[T]$, Tamam [Tam14] proved an asymptotic formula for the second and fourth moments of (9.1.1), where the sum is over all primitive Dirichlet characters modulo Q.

Theorem 9.1.1 (Tamam)**.** *Let Q be a monic irreducible polynomial in $\mathbb{F}_q[T]$, then we have*

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi(mod\ Q)\\ \chi\neq\chi_0}} \left| L\left(\frac{1}{2},\chi\right)\right|^{2} = deg(Q) - 1 - \frac{1}{(q^{\frac{1}{2}}-1)^2}\left(1 - \frac{2}{|Q|^{\frac{1}{2}}+1}\right)$$

and

$$\frac{1}{\phi(Q)} \sum_{\substack{\chi(mod\ Q)\\ \chi\neq\chi_0}} \left| L\left(\frac{1}{2},\chi\right)\right|^{4} = \frac{q-1}{12q}\left(deg(Q)\right)^4 + O\left(\left(deg(Q)\right)^3\right).$$

When summing over all primitive Dirichlet characters of modulus R, where R is a monic polynomial in $\mathbb{F}_q[T]$, Andrade and Yiasemides [AY21] proved an asymptotic formula for the second and fourth moments of (9.1.1).

Theorem 9.1.2 (Andrade and Yiasemides). *Let R be a monic polynomial in $\mathbb{F}_q[T]$, then we have*

$$\frac{1}{\phi^*(R)}\sum^*_{\chi(mod\ R)}\left|L\left(\frac{1}{2},\chi\right)\right|^2 = \frac{\phi(R)}{|R|}deg(R) + O\left(\log\omega(R)\right) \tag{9.1.2}$$

and

$$\sum^*_{\chi(mod\ R)}\left|L\left(\frac{1}{2},\chi\right)\right|^4 = \frac{1-q^{-1}}{12}\phi^*(R)\prod_{P|R}\left(\frac{(1-|P|^{-1})^3}{1+|P|^{-1}}\right)(deg(R))^4\left(1+O\left(\sqrt{\frac{\omega(R)}{deg(R)}}\right)\right) \tag{9.1.3}$$

where $\omega(R)$ is the number of prime divisors of R and $\phi^*(R)$ in the number of primitive Dirichlet characters of modulus R.

Yiasemides [Yia21] conjectured higher moments of (9.1.1) where the sum is over all primitive Dirichlet characters of modulus R, where R is a monic polynomial in $\mathbb{F}_q[T]$ and showed that the conjecture agrees with (9.1.2) and (9.1.3).

Conjecture 9.1.3 (Yiasemides). *For all non-negative integers k it is conjectured that*

$$\frac{1}{\phi^*(R)}\sum^*_{\chi(mod\ R)}\left|L\left(\frac{1}{2},\chi\right)\right|^{2k} \sim C_k(deg(R))^{k^2},$$

for some explicit constant C_k as $deg(R)\to\infty$.

Furthermore, Andrade and Yiasemides [AY21] proved an asymptotic formula for the second moment of (9.1.1) when the sum is over all primitive Dirichlet characters modulo R, where R is a square-full polynomial in $\mathbb{F}_q[T]$.

Theorem 9.1.4 (Andrade and Yiasemides). *Let R be a square-full polynomial, that is if $P|R$ then $P^2|R$ and let χ be a Dirichlet character modulo R. Then*

$$\sum^*_{\chi(mod\ R)}\left|L\left(\frac{1}{2},\chi\right)\right|^2 = \frac{\phi(R)^3}{|R|}deg(R) + \left(\frac{\phi(R)}{|R|^2} - \frac{\phi(R)^2}{|R|^2}\right)\sum_{P|R}\frac{deg(R)}{|P|-1}$$
$$+\frac{1}{(q^{\frac{1}{2}}-1)^2}\left(-\frac{\phi(R)^3}{|R|^2} + 2\frac{\phi(R)}{|R|^{\frac{1}{2}}}\prod_{P|R}\left(1-\frac{1}{|P|^{\frac{1}{2}}}\right)\right).$$

Another problem in function fields is to understand the asymptotic behaviour of twisted moments of Dirichlet L-functions, when averaged over primitive Dirichlet characters of modulus Q. If we let P and H be monic irreducible polynomials in $\mathbb{F}_q[T]$, then a problem is to establish an asymptotic formula for

$$S(P;H) = \sum_{\substack{\chi (\text{mod } P) \\ \chi \neq \chi_0}} \left| L\left(\frac{1}{2}, \chi\right)\right|^2 \chi(H). \tag{9.1.4}$$

Motivated by the methods of Young [You11b] in the number field setting, Djanković [Dja18] established a reciprocity formula involving $S(P;H)$ and $S(H;-P)$. In particular he proved the following result.

Theorem 9.1.5 (Djanković)**.** *For any two monic irreducible polynomials $P, H \in \mathbb{F}_q[T]$ with $H \neq P$ and $deg(H) \leq deg(P)$, we have the following reciprocity formula between the twisted second moments:*

$$\frac{|P|^{\frac{1}{2}}}{\phi(P)} S(P;H) - \frac{|H|^{\frac{1}{2}}}{\phi(H)} S(H;-P) = \frac{|P|^{\frac{1}{2}}}{|H|^{\frac{1}{2}}} \left(deg(P) - deg(H) - \zeta_{\mathbb{A}}\left(\frac{1}{2}\right)^2 \right) + \zeta_{\mathbb{A}}\left(\frac{1}{2}\right)^2 \left(1 - 2\frac{|P|^{\frac{1}{2}}}{\phi(P)} \left(1 - |P|^{-\frac{1}{2}}\right) + 2\frac{|H|^{\frac{1}{2}}}{\phi(H)} \left(1 - |H|^{-\frac{1}{2}}\right) \right).$$

Similarly, we also want to understand the second moment of Dirichlet L-functions with two twists when averaged over primitive Dirichlet characters modulo Q in function fields. If we let H, K and Q be monic irreducible polynomials in $\mathbb{F}_q[T]$ and restrict the sum further to be over all even or odd Dirichlet characters modulo Q, where the definition of an odd and even Dirichlet characters is stated in Section 2.3, then a problem is to establish an asymptotic formula for

$$S^{\pm}(Q;H,K) = \frac{|Q|^{\frac{1}{2}}}{\phi^{\pm}(Q)} \sideset{}{^{\pm}}\sum_{\substack{\chi (\text{mod } Q) \\ \chi \neq \chi_0}} \left| L\left(\frac{1}{2}, \chi\right)\right|^2 \chi(H)\bar{\chi}(K), \tag{9.1.5}$$

where $\phi^{\pm}(Q)$ denotes the number of even or odd Dirichlet characters modulo Q. Motivated by the methods of Bettin [Bet16], Djanković, Đokić and Lelas [DĐL21] established a triple reciprocity formula involving $S^{-}(Q;H,K)$, $S^{-}(H;K,-Q)$ and $S^{-}(K;H,-Q)$ and involving $S^{+}(Q;H,K)$, $S^{+}(H;K,Q)$ and $S^{+}(K;H,Q)$. In particular, they proved the following results.

Theorem 9.1.6 (Djanković, Đokić and Lelas)**.** *Let H, K and Q be distinct monic irreducible polynomials in $\mathbb{F}_q[T]$ such that $deg(H) + deg(K) \leq deg(Q)$. Then we have the following triple reciprocity formulas:*

$$S^{-}(Q;H,K) = S^{-}(H;K,-Q) + S^{-}(K;H,-Q) + \frac{|Q|^{\frac{1}{2}}}{|HK|^{\frac{1}{2}}} \left(deg(Q) - deg(H) - deg(K) \right)$$

and

$$\begin{aligned} S^+(Q;H,K) &= S^+(H;K,Q) + S^+(K;H,Q) \\ &\quad + \frac{|Q|^{\frac{1}{2}}}{|HK|^{\frac{1}{2}}}\left(deg(Q) - deg(H) - deg(K) - \zeta_{\mathbb{A}}\left(\frac{1}{2}\right)^2(q-1)\right) \\ &\quad - 2\zeta_{\mathbb{A}}\left(\frac{1}{2}\right)^2\left(\frac{|Q|^{\frac{1}{2}}-1}{\phi^+(Q)} - \frac{|H|^{\frac{1}{2}}-1}{\phi^+(H)} - \frac{|K|^{\frac{1}{2}}-1}{\phi^+(K)}\right). \end{aligned}$$

9.2 Statement of Main Results

In this chapter, we will obtain asymptotic formulas for

$$\frac{1}{\phi^*(R)}\sum^*_{\chi(\text{mod } R)}\left|L\left(\frac{1}{2},\chi\right)\right|^2\chi(H) \quad \text{and} \quad \frac{1}{\phi^*(R)}\sum^*_{\chi(\text{mod } R)}\left|L\left(\frac{1}{2},\chi\right)\right|^2\chi(H)\bar{\chi}(K) \tag{9.2.1}$$

where H, K and R are monic polynomials in $\mathbb{F}_q[T]$, $\phi^*(R)$ denotes the number of primitive Dirichlet characters modulo R and the sum is over all primitive Dirichlet characters modulo R. In particular we prove the following results.

Theorem 9.2.1. *Let H and R be monic polynomials in $\mathbb{F}_q[T]$ with $deg(H) < deg(R)$. Then*

$$\frac{1}{\phi^*(R)}\sum^*_{\chi(\text{mod } R)}\left|L\left(\frac{1}{2},\chi\right)\right|^2\chi(H) = |H|^{\frac{1}{2}}\frac{\phi(R)}{|R|}deg(HR) + O\left(|H|^{\frac{1}{2}}\log\omega(R)\right), \tag{9.2.2}$$

where $\omega(R)$ is the function defined in Section 2.2, $\phi^*(R)$ denotes the number of primitive Dirichlet characters modulo R and the sum is over all primitive Dirichlet characters modulo R.

Theorem 9.2.2. *Let H, K and R be monic polynomials in $\mathbb{F}_q[T]$ with $deg(H) + deg(K) < \deg(R)$. Then*

$$\frac{1}{\phi^*(R)}\sum^*_{\chi(\text{mod } R)}\left|L\left(\frac{1}{2},\chi\right)\right|^2\chi(H)\bar{\chi}(K) = |HK|^{\frac{1}{2}}\frac{\phi(R)}{|R|}deg(HKR) + O\left(|HK|^{\frac{1}{2}}\log\omega(R)\right), \tag{9.2.3}$$

where $\omega(R)$ is the function defined in Section 2.2, $\phi^*(R)$ denotes the number of primitive Dirichlet characters modulo R and the sum is over all primitive Dirichlet characters modulo R.

Remark 9.2.3. *In Theorem 9.2.1 and Theorem 9.2.2, we take the sum over all primitive Dirichlet characters of modulus R, where R is a monic polynomial in $\mathbb{F}_q[T]$. In particular, we do not restrict these sums to odd or even Dirichlet characters that was considered in Theorem 9.1.6.*

From Theorem 9.2.1 and Theorem 9.2.2 we immediately have the following corollaries.

Corollary 9.2.4. *Under the same assumptions as Theorem* 9.2.1*, we have*

$$\frac{1}{\phi^*(R)}\sum_{\chi(mod\ R)}^{*}\left|L\left(\frac{1}{2},\chi\right)\right|^2\chi(H) \sim |H|^{\frac{1}{2}}\frac{\phi(R)}{|R|}deg(HR) \tag{9.2.4}$$

as deg$(R)\to\infty$.

Corollary 9.2.5. *Under the same assumptions Theorem* 9.2.2*, we have*

$$\frac{1}{\phi^*(R)}\sum_{\chi(mod\ R)}^{*}\left|L\left(\frac{1}{2},\chi\right)\right|^2\chi(H)\bar{\chi}(K) \sim |HK|^{\frac{1}{2}}\frac{\phi(R)}{|R|}deg(HKR) \tag{9.2.5}$$

as deg$(R)\to\infty$.

Remark 9.2.6. *Using Lemma* 2.2.4 *and Lemma* 2.2.7 *we can see that* $\frac{\log\omega(R)}{\frac{\phi(R)}{|R|}deg(R)}$ *tends to zero as deg*(R) *tends to infinity and thus verifying the asymptotic formulas (9.2.2), (9.2.3), (9.2.4) and (9.2.5).*

9.3 Preliminary Lemmas

In this section, we state and prove results which will be needed to prove Theorem 9.2.1 and Theorem 9.2.2. We start by stating the approximate function equation for $\left|L\left(\frac{1}{2},\chi\right)\right|^2$.

Lemma 9.3.1 ([GZ22a, Lemma 2.5])**.** *Let χ be a primitive Dirichlet character of modulus R. Then we have*

$$\left|L\left(\frac{1}{2},\chi\right)\right|^2 = 2\sum_{\substack{A,B\in\mathbb{A}^+\\ deg(AB)<deg(R)}}\frac{\chi(A)\bar{\chi}(B)}{|AB|^{\frac{1}{2}}} + O\left(|R|^{-\frac{1}{2}+\epsilon}\right). \tag{9.3.1}$$

The next lemma will be used to obtain the main term of Theorem 9.2.1 and Theorem 9.2.2.

Lemma 9.3.2. *Let H and R be fixed monic polynomials in $\mathbb{F}_q[T]$ with deg*$(H) <$ *deg*(R) *and let x be a positive integer. If $x \geq$ deg(R) − deg(H), then*

$$\sum_{\substack{A\in\mathbb{A}^+_{\leq x}\\ (AH,R)=1}}\frac{1}{|A|} = |H|\frac{\phi(R)}{|R|}(x+deg(H)) + O\left(|H|\log\omega(R)\right). \tag{9.3.2}$$

Whereas if $x <$ deg(R) − deg(H), then

$$\sum_{\substack{A\in\mathbb{A}^+_{\leq x}\\ (AH,R)=1}}\frac{1}{|A|} = |H|\frac{\phi(R)}{|R|}(x+deg(H))+O\left(|H|\log\omega(R)\right)+O\left(\frac{2^{\omega(R)}(x+deg(H))}{q^x}\right). \tag{9.3.3}$$

Proof. We have

$$\sum_{\substack{A\in\mathbb{A}^+_{\leq x}\\(AH,R)=1}}\frac{1}{|A|}=\sum_{A\in\mathbb{A}^+_{\leq x}}\frac{1}{|A|}\sum_{E|(AH,R)}\mu(E)=\sum_{A\in\mathbb{A}^+_{\leq x}}\frac{1}{|A|}\sum_{\substack{E|AH\\E|R}}\mu(E)=\sum_{E|R}\mu(E)\sum_{\substack{A\in\mathbb{A}^+_{\leq x}\\E|AH}}\frac{1}{|A|}. \quad (9.3.4)$$

Since $E|AH$ then $EL=AH$ for some $L\in\mathbb{A}^+$ with $\deg(L)=\deg(A)+\deg(H)-\deg(E)\leq x+\deg(H)-\deg(E)$. Furthermore, since $EL=AH$, then $|EL|=|AH|$ and so $\frac{1}{|A|}=\frac{|H|}{|E||L|}$. Also, since there does not exist any polynomials $L\in\mathbb{A}^+$ with $\deg(L)<0$, then, for the sum over all $L\in\mathbb{A}^+$ with $\deg(L)\leq x+\deg(H)-\deg(E)$, we can restrict the sum over $E|R$ further to the sum over $E|R$ with $\deg(E)\leq x+\deg(H)$. Thus combining (9.3.4) and the above arguments we have

$$\sum_{\substack{A\in\mathbb{A}^+_{\leq x}\\(AH,R)=1}}\frac{1}{|A|}=|H|\sum_{\substack{E|R\\\deg(E)\leq x+\deg(H)}}\frac{\mu(E)}{|E|}\sum_{\substack{L\in\mathbb{A}^+\\\deg(L)\leq x+\deg(H)-\deg(E)}}\frac{1}{|L|}.$$

We know that, for a non-negative integer y,

$$\sum_{L\in\mathbb{A}^+_{\leq y}}\frac{1}{|L|}=\sum_{k=0}^{y}q^{-k}\sum_{L\in\mathbb{A}^+_k}1=\sum_{k=0}^{y}1=y+1,$$

and so

$$\begin{aligned}\sum_{\substack{A\in\mathbb{A}^+_{\leq x}\\(AH,R)=1}}\frac{1}{|A|}&=|H|\sum_{\substack{E|R\\\deg(E)\leq x+\deg(H)}}\frac{\mu(E)}{|E|}(x+\deg(H)-\deg(E)+1)\\&=|H|\sum_{E|R}\frac{\mu(E)}{|E|}(x+\deg(H)-\deg(E)+1)\\&\quad-|H|\sum_{\substack{E|R\\\deg(E)>x+\deg(H)}}\frac{\mu(E)}{|E|}(x+\deg(H)-\deg(E)+1). \quad (9.3.5)\end{aligned}$$

Using (2.2.1), (2.2.2) and Lemma 2.2.3 we have

$$\sum_{E|R}\frac{\mu(E)}{|E|}(x+\deg(H)-\deg(E)+1)=\frac{\phi(R)}{|R|}(x+\deg(H))+O(\log\omega(R)). \quad (9.3.6)$$

If $x+\deg(H)\geq\deg(R)$, then there is no $E|R$ with $\deg(E)>\deg(R)$ and so the final term on the right-hand side of (9.3.5) is empty. Thus for $x+\deg(H)\geq\deg(R)$

$$\sum_{\substack{E|R\\\deg(E)>x+\deg(H)}}\frac{\mu(E)}{|E|}(x+\deg(H)-\deg(E)+1)=0. \quad (9.3.7)$$

Whereas for $x+\deg(H)<\deg(R)$, we have that if $\deg(E)>x+\deg(H)$, then $x+\deg(H)-\deg(E)+1\leq\deg(E)-\deg(E)\leq\deg(E)$ and so

$$\sum_{\substack{E|R\\\deg(E)>x+\deg(H)}}\frac{\mu(E)}{|E|}(x+\deg(H)-\deg(E)+1)\ll\sum_{\substack{E|R\\\deg(E)>x+\deg(H)}}\frac{|\mu(E)|}{|E|}\deg(E). \quad (9.3.8)$$

As $\deg(E) \leq |E|$ for $\deg(E) > x + \deg(H)$, then

$$\sum_{\substack{E|R \\ \deg(E)>x+\deg(H)}} \frac{|\mu(E)|\deg(E)}{|E|} \ll \frac{x+\deg(H)}{q^{x+\deg(H)}} \sum_{\substack{E|R \\ \deg(E)>x+\deg(H)}} |\mu(E)|. \tag{9.3.9}$$

Furthermore, since $|\mu(E)| \geq 0$, then

$$\sum_{\substack{E|R \\ \deg(E)>x+\deg(H)}} |\mu(E)| \leq \sum_{E|R} |\mu(E)| = 2^{\omega(R)}, \tag{9.3.10}$$

where the final equality follows from Lemma 2.2.5. Combining (9.3.8), (9.3.9) and (9.3.10) we have

$$\sum_{\substack{E|R \\ \deg(E)>x+\deg(H)}} \frac{\mu(E)}{|E|}(x+\deg(H)-\deg(E)+1) \ll \frac{2^{\omega(R)}(x+\deg(H))}{q^{x+\deg(H)}} \tag{9.3.11}$$

which completes the proof. ■

Finally, the following lemmas will be used to create a suitable bound for the error term of Theorem 9.2.1 and Theorem 9.2.2.

Lemma 9.3.3. *Let F, H and R be fixed monic polynomials in $\mathbb{F}_q[T]$ where $F|R$ and let $z < deg(R)$. Then*

$$\sum_{\substack{A,B\in\mathbb{A}^+ \\ deg(AB)=z \\ AH\equiv B(mod\ F) \\ AH\neq B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{q^{\frac{z}{2}}(z+1)|H|}{|F|}. \tag{9.3.12}$$

Proof. We consider three cases, $\deg(AH) > \deg(B)$, $\deg(AH) < \deg(B)$ and $\deg(AH) = \deg(B)$ where $AH \neq B$.

If we first consider the case $\deg(AH) > \deg(B)$ and suppose that $\deg(A) = i$, then since $AH \equiv B(\text{mod } F)$ and $AH \neq B$ we have that $AH = LF + B$ for some $L \in \mathbb{A}$ where $\deg(AH) = \deg(LF+B)$. Furthermore since A, H, F and B are monic, then L is monic. Also, since $\deg(AH) > \deg(B)$, then $\deg(LF) > \deg(B)$ and so, by Proposition 2.1.1 $\deg(LF + B) = \deg(LF)$. Thus, using the above and Proposition 2.1.1 again, we have that $\deg(L) = \deg(A) + \deg(H) - \deg(F) = i + \deg(H) - \deg(F)$. Furthermore, since $\deg(AB) = z$ and $\deg(A) = i$, then $\deg(B) = z - \deg(A) = z - i$ where $0 \leq i \leq z$ and

$|AB|^{-\frac{1}{2}} = q^{-\frac{z}{2}}$. Combining all the above we have that

$$\sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)=z \\ \deg(AH)>\deg(B) \\ AH\equiv B(\text{mod } F) \\ AH\neq B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \leq q^{-\frac{z}{2}} \sum_{i=0}^{z} \sum_{\substack{L\in\mathbb{A}^+ \\ \deg=i+\deg(H)-\deg(F)}} \sum_{\substack{B\in\mathbb{A}^+ \\ \deg(B)=z-i}} 1$$

$$= q^{\frac{z}{2}} \sum_{i=0}^{z} q^{-i} \sum_{\substack{L\in\mathbb{A}^+ \\ \deg(L)=i+\deg(H)-\deg(F)}} 1 = \frac{q^{\frac{z}{2}}|H|}{|F|} \sum_{i=0}^{z} 1 = \frac{q^{\frac{z}{2}}(z+1)|H|}{|F|}. \tag{9.3.13}$$

Similarly, if we consider the case $\deg(AH) < \deg(B)$ and suppose that $\deg(B) = i$, then since $AH \equiv B(\text{mod } F)$ and $AH \neq B$, then we have that $B = LF + AH$ for some $L \in \mathbb{A}$ where $\deg(B) = \deg(LF + AH)$. Furthermore since A, H, F and B are monic, then L is monic. Also, since $\deg(B) > \deg(AH)$, then $\deg(LF) > \deg(AH)$ and so by Proposition 2.1.1 $\deg(LF+AH) = \deg(LF)$. Thus using the above and Proposition 2.1.1 we have that $\deg(L) = \deg(B) - \deg(F)$. Furthermore since $\deg(AB) = z$ and $\deg(B) = i$, then $|AB|^{-\frac{1}{2}} = q^{-\frac{z}{2}}$ and $\deg(A) = z - \deg(B) = z - i$ where $0 \leq i \leq z$. Thus combining the above we have

$$\sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)=z \\ \deg(B)>\deg(AH) \\ AH\equiv B(\text{mod } F) \\ AH\neq B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \leq q^{-\frac{z}{2}} \sum_{i=0}^{z} \sum_{\substack{L\in\mathbb{A}^+ \\ \deg(L)=i-\deg(F)}} \sum_{\substack{A\in\mathbb{A}^+ \\ \deg(A)=z-i}} 1$$

$$= q^{\frac{z}{2}} \sum_{i=0}^{z} q^{-i} \sum_{\substack{L\in\mathbb{A}^+ \\ \deg(L)=i-\deg(F)}} 1 = \frac{q^{\frac{z}{2}}}{|F|} \sum_{i=0}^{z} 1 = \frac{q^{\frac{z}{2}}(z+1)}{|F|}. \tag{9.3.14}$$

Finally, if we consider the case where $\deg(AH) = \deg(B) = i$, then $2i = \deg(ABH) = z + \deg(H)$ and so $\deg(B) = i = \frac{z+\deg(H)}{2}$. Furthermore since $AH \equiv B(\text{mod F})$ and $AH \neq B$, then $AH = LF + B$ where $L \in \mathbb{A}$ with $\deg(AH) = \deg(LF + B)$. Since $\deg(AH) = \deg(B)$ where A, H and B are monic, then by Proposition 2.1.1 and the above arguments we have that $\deg(LF) < \deg(B) = \frac{z+\deg(H)}{2}$. Thus combining the above and using the argument stated previously we have

$$\sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)=z \\ \deg(AH)=\deg(B) \\ AH\equiv B(\text{mod } F) \\ AH\neq B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \leq q^{-\frac{z}{2}} \sum_{\substack{B\in\mathbb{A}^+ \\ \deg(B)=\frac{z+\deg(H)}{2}}} \sum_{\substack{L\in\mathbb{A} \\ \deg(L)<\frac{z+\deg(H)}{2}-\deg(F)}} 1$$

$$\ll \frac{|H|^{\frac{1}{2}}}{|F|} \sum_{\substack{B\in\mathbb{A}^+ \\ \deg(B)=\frac{z+\deg(H)}{2}}} 1 = \frac{q^{\frac{z}{2}}|H|}{|F|}. \tag{9.3.15}$$

Combining all the cases proves the result. ■

Remark 9.3.4. *The inequality that occurs in (9.3.13), (9.3.14) and (9.3.15) comes from the removal of the condition of* $(ABH, R) = 1$. *Although, we could consider this condition, it maybe harder to evaluate and we obtain a desirable bound without it. We similarly do this in (9.3.17), (9.3.18) and (9.3.19) with the condition* $(ABHK, R) = 1$.

Lemma 9.3.5. *Let* F, H, K *and* R *be fixed monic polynomials in* $\mathbb{F}_q[T]$ *where* $F|R$ *and let* $z < deg(R)$. *Then*

$$\sum_{\substack{A,B\in\mathbb{A}^+ \\ deg(AB)=z \\ AH\equiv BK(mod\ F) \\ AH\neq BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{q^{\frac{z}{2}}(z+1)|HK|}{|F|}. \tag{9.3.16}$$

Proof. The proof is similar to the proof of Lemma 9.3.3 and [Yia21, Lemma 6.4], but will be presented here too. We consider three cases, $\deg(AH) > \deg(BK)$, $\deg(AH) < \deg(BK)$ and $\deg(AH) = \deg(BK)$ where $AH \neq BK$.

If we consider the first case $\deg(AH) > \deg(BK)$ and suppose that $\deg(A) = i$, then since $AH \equiv BK(\text{mod } F)$ and $AH \neq BK$ we have that $AH = LF + BK$ for some $L \in \mathbb{A}$ where $\deg(AH) = \deg(LF + BK)$. Furthermore, since A, H, K, F and B are all monic, then L is monic. Also, since $\deg(AH) > \deg(BK)$, then $\deg(LF) > \deg(BK)$ and so by Proposition 2.1.1 $\deg(LF + BK) = \deg(LF)$. Invoking Proposition 2.1.1 again and the above we see that $\deg(L) = \deg(AH) - \deg(F) = i + \deg(H) - \deg(F)$. Furthermore since $\deg(AB) = z$ and $\deg(A) = i$, then $\deg(B) = z - \deg(A) = z - i$ where $0 \leq i \leq z$ and $|AB|^{-\frac{1}{2}} = q^{-\frac{z}{2}}$. Combining the above we have

$$\begin{aligned}\sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)=z \\ \deg(AH)>\deg(BK) \\ AH\equiv BK(\text{mod } F) \\ AH\neq BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} &\leq q^{-\frac{z}{2}} \sum_{i=0}^{z} \sum_{\substack{L\in\mathbb{A}^+ \\ \deg(L)=i+\deg(H)-\deg(F)}} \sum_{\substack{B\in\mathbb{A}^+ \\ \deg(B)=z-i}} 1 \\ &= q^{\frac{z}{2}} \sum_{i=0}^{z} q^{-i} \sum_{\substack{L\in\mathbb{A}^+ \\ \deg(L)=i+\deg(H)-\deg(F)}} 1 = \frac{q^{\frac{z}{2}}|H|}{|F|} \sum_{i=0}^{z} 1 = \frac{q^{\frac{z}{2}}(z+1)|H|}{|F|}.\end{aligned} \tag{9.3.17}$$

Similarly, if we consider the case where $\deg(BK) > \deg(AH)$ and suppose that $\deg(B) = i$, then since $AH \equiv BK(\text{mod } F)$ and $AH \neq BK$, then $BK = LF + AH$ for some $L \in \mathbb{A}$ with $\deg(BK) = \deg(LF + AH)$. Furthermore, since A, H, K, F and B are all monic, then L is monic. Also, since $\deg(BK) > \deg(AH)$, then $\deg(LF) > \deg(AH)$ and so by Proposition 2.1.1 $\deg(LF+AH)=\deg(LF)$. Invoking Proposition 2.1.1 again and the above we see that $\deg(L)=\deg(B)+\deg(K)-\deg(F)=i+\deg(K)-\deg(F)$. Furthermore since $\deg(AB)=z$ and $\deg(B)=i$, then $\deg(A)=z-\deg(B)=z-i$ and $|AB|^{-\frac{1}{2}}=q^{-\frac{z}{2}}$. Combining the above and using the arguments stated previously we have

$$\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)=z\\ \deg(BK)>\deg(AH)\\ AH\equiv BK(\text{mod } F)\\ AH\neq BK\\ (ABHK,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}\leq q^{-\frac{z}{2}}\sum_{i=0}^{z}\sum_{\substack{L\in\mathbb{A}^+\\ \deg(L)=i+\deg(K)-\deg(F)}}\sum_{\substack{A\in\mathbb{A}^+\\ \deg(A)=z-i}}1$$

$$=q^{\frac{z}{2}}\sum_{i=0}^{z}q^{-i}\sum_{\substack{L\in\mathbb{A}^+\\ \deg(L)=i+\deg(K)-\deg(F)}}1=\frac{q^{\frac{z}{2}}|K|}{|F|}\sum_{i=0}^{z}1=\frac{q^{\frac{z}{2}}(z+1)|K|}{|F|}. \tag{9.3.18}$$

Finally, if we consider the case where $\deg(AH)=\deg(BK)=i$. Then $2i=\deg(ABHK)=z+\deg(HK)$ and so $\deg(B)=i-\deg(K)=\frac{z+\deg(H)-\deg(K)}{2}$. Furthermore since $AH\equiv BK(\text{mod } F)$ and $AH\neq BK$ then $AH=LF+BK$ where $\deg(AH)=\deg(LF+BK)$. Since $\deg(AH)=\deg(BK)$ where A, B, H and K are monic, then by Proposition 2.1.1 and the arguments stated above we have $\deg(LF)<\deg(BK)$ and so $\deg(L)<i-\deg(F)=\frac{z+\deg(HK)}{2}-\deg(F)$. Furthermore since $\deg(AB)=z$ then $|AB|^{-\frac{1}{2}}=q^{-\frac{z}{2}}$. Thus combining the above and using the arguments stated previously we have

$$\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)=z\\ \deg(AH)=\deg(BK)\\ AH\equiv BK(\text{mod } F)\\ AH\neq BK\\ (ABHK,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}\leq q^{-\frac{z}{2}}\sum_{\substack{B\in\mathbb{A}^+\\ \deg(B)=\frac{z+\deg(H)-\deg(K)}{2}}}\sum_{\substack{L\in\mathbb{A}\\ \deg(L)<\frac{z+\deg(HK)}{2}-\deg(F)}}1$$

$$\ll\frac{|HK|^{\frac{1}{2}}}{|F|}\sum_{\substack{B\in\mathbb{A}^+\\ \deg(B)=\frac{z+\deg(H)-\deg(K)}{2}}}1=\frac{q^{\frac{z}{2}}|H|}{|F|}. \tag{9.3.19}$$

Combining all the above cases proves the result. ∎

Lemma 9.3.6. *For all $R\in\mathbb{A}^+$ and $\epsilon>0$ we have*

$$\frac{2^{\omega(R)}|R|^{\frac{1}{2}}deg(R)}{\phi^*(R)}\ll_\epsilon|R|^{\epsilon-\frac{1}{2}}. \tag{9.3.20}$$

Proof. For $\deg(R)\leq q$ we know, by [Yia20, (A.2.3)] that $\frac{\phi^*(R)}{|R|}\gg 1$. Thus for $\deg(R)\leq q$ we have

$$\frac{2^{\omega(R)}|R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)}\ll\frac{2^{\omega(R)}\deg(R)}{|R|^{\frac{1}{2}}}\ll\frac{2^{\omega(R)}}{|R|^{\frac{1}{2}-\epsilon}}.$$

From Lemma 2.2.5 we know that $2^{\omega(R)}\ll|R|^\epsilon$, thus (9.3.20) holds for $\deg(R)\leq q$.

For $\deg(R) > q$ we know by Lemma 2.2.7 and Lemma 2.3.6 that

$$\phi^*(R) \gg \frac{\phi(R)}{\log_q \log_q |R|} \gg \frac{|R|}{(\log_q \log_q |R|)^2}.$$

Thus if $\deg(R) > q$, then

$$\frac{2^{\omega(R)}|R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)} \ll \frac{2^{\omega(R)}\deg(R)(\log_q \log_q |R|)^2}{|R|^{\frac{1}{2}}} \ll_\epsilon \frac{2^{\omega(R)}}{|R|^{\frac{1}{2}-\epsilon}}$$

Finally, from Lemma 2.2.5, we know that $2^{\omega(R)} \ll |R|^\epsilon$, then (9.3.20) holds for $\deg(R) > q$ and thus completes the proof. ∎

9.4 Proof of Theorem 9.2.1

In this section, we use results stated previously to prove Theorem 9.2.1.

Proof of Theorem 9.2.1. Using the approximate function equation Lemma 9.3.1 we have

$$\begin{aligned}&\frac{1}{\phi^*(R)} \sideset{}{^*}\sum_{\chi (\text{mod } R)} \left| L\left(\frac{1}{2}, \chi\right)\right|^2 \chi(H) \\ &= \frac{2}{\phi^*(R)} \sideset{}{^*}\sum_{\chi (\text{mod } R)} \sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB) < \deg(R)}} \frac{\chi(A)\bar{\chi}(B)\chi(H)}{|AB|^{\frac{1}{2}}} + O\left(\frac{|R|^{-\frac{1}{2}+\epsilon}}{\phi^*(R)} \sideset{}{^*}\sum_{\chi (\text{mod } R)} \chi(H)\right). \end{aligned} \tag{9.4.1}$$

Using the definition of Dirichlet characters and $\phi^*(R)$ we have

$$\frac{|R|^{-\frac{1}{2}+\epsilon}}{\phi^*(R)} \sideset{}{^*}\sum_{\chi (\text{mod } R)} \chi(H) \ll \frac{|R|^{-\frac{1}{2}+\epsilon}}{\phi^*(R)} \sideset{}{^*}\sum_{\chi (\text{mod } R)} 1 = |R|^{-\frac{1}{2}+\epsilon}.$$

Using the orthogonality relation Lemma 2.3.8, we have

$$\begin{aligned}&\frac{2}{\phi^*(R)} \sideset{}{^*}\sum_{\chi (\text{mod } R)} \sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB) < \deg(R)}} \frac{\chi(A)\bar{\chi}(B)\chi(H)}{|AB|^{\frac{1}{2}}} \\ &= \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB) < \deg(R) \\ AH \equiv B (\text{mod } F) \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}}. \end{aligned} \tag{9.4.2}$$

For the second sum on the right-hand side of (9.4.2), we will consider the contribution of the diagonal, $AH = B$, and the off-diagonal, $AH \neq B$, terms separately. Thus we write

$$\begin{aligned}
&\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH\equiv B(\text{mod } F) \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \\
&= \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH\equiv B(\text{mod } F) \\ AH=B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \\
&+ \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH\equiv B(\text{mod } F) \\ AH\neq B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}}.
\end{aligned}$$

Considering the contribution of the diagonal, $AH = B$, terms we have that $|AB|^{-\frac{1}{2}} = |H|^{-\frac{1}{2}}|A|^{-1}$ and $\deg(AB) = \deg(ABH) - \deg(H) = 2\deg(A) + \deg(H)$. Thus the double sum over $A, B \in \mathbb{A}^+$ with $\deg(AB) < \deg(R)$, $AH = B$ and $(ABH, R) = 1$ becomes a single sum over $A \in \mathbb{A}^+$ with $\deg(A) < \frac{1}{2}(\deg(R) - \deg(H))$ and $(AH, R) = 1$. Therefore using the arguments stated above and Corollary 2.3.9 we have

$$\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH=B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} = \frac{2}{|H|^{\frac{1}{2}}} \sum_{\substack{A\in\mathbb{A}^+ \\ \deg(A)<\frac{\deg(R)-\deg(H)}{2} \\ (AH,R)=1}} \frac{1}{|A|}. \tag{9.4.3}$$

Using Lemma 9.3.2 with $x = \frac{\deg(R)-\deg(H)}{2} - 1$ we have

$$\frac{2}{|H|^{\frac{1}{2}}} \sum_{\substack{A\in\mathbb{A}^+ \\ \deg(A)<\frac{\deg(R)-\deg(H)}{2} \\ (AH,R)=1}} \frac{1}{|A|} = |H|^{\frac{1}{2}} \frac{\phi(R)}{|R|}(\deg(H) + \deg(R)) + O\left(|H|^{\frac{1}{2}} \log \omega(R)\right). \tag{9.4.4}$$

For the contribution of the off-diagonal terms we have

$$\sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH\equiv B(\text{mod } F) \\ AH\neq B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} = \sum_{z=0}^{\deg(R)-1} \sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)=z \\ AH\equiv B(\text{mod } F) \\ AH\neq B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}}. \tag{9.4.5}$$

Using Lemma 9.3.3 we have

$$\sum_{\substack{A,B\in\mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH\equiv B(\text{mod } F) \\ AH\neq B \\ (ABH,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \sum_{z=0}^{\deg(R)-1} \frac{|H|q^{\frac{z}{2}}(z+1)}{|F|} \ll \frac{|H||R|^{\frac{1}{2}}\deg(R)}{|F|}. \tag{9.4.6}$$

Thus using (9.4.6) we have

$$\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\text{mod } F)\\ AH\neq B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}\ll\frac{|H||R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)}\sum_{EF=R}|\mu(E)|\frac{\phi(F)}{|F|}. \tag{9.4.7}$$

Combining (9.4.7), Lemma 2.2.5 and the fact that $\frac{\phi(R)}{|R|}\leq 1$ we have

$$\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\text{mod } F)\\ AH\neq B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}\ll\frac{2^{\omega(R)}|H||R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)}. \tag{9.4.8}$$

Furthermore, combining (9.4.8) and Lemma 9.3.6, we have

$$\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\text{mod } F)\\ AH\neq B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}\ll\frac{2^{\omega(R)}|H||R|^{\frac{1}{2}}\deg(R)}{\phi^*(R)}\ll|H||R|^{\epsilon-\frac{1}{2}}. \tag{9.4.9}$$

Since $\deg(H)<\deg(R)$, then there is some $\epsilon>0$ such that $\deg(H)\leq(1-2\epsilon)\deg(R)$. Thus $|H|^{\frac{1}{2}}|R|^{\epsilon-\frac{1}{2}}=q^{\frac{1}{2}\deg(H)+\left(\epsilon-\frac{1}{2}\right)\deg(R)}\leq q^{\frac{1}{2}(1-2\epsilon)\deg(R)+\left(\epsilon-\frac{1}{2}\right)\deg(R)}=1$. Therefore combining the above with (9.4.9), we get

$$\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv B(\text{mod } F)\\ AH\neq B\\ (ABH,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}\ll|H|^{\frac{1}{2}}. \tag{9.4.10}$$

Combining the above completes the proof of Theorem 9.2.1. ∎

9.5 Proof of Theorem 9.2.2

In this section we use similar methods to that seen in the proof of Theorem 9.2.1 to prove Theorem 9.2.2.

Proof of Theorem 9.2.2. Using the approximate functional equation, Lemma 9.3.1, we have

$$\begin{aligned}&\frac{1}{\phi^*(R)}\sum_{\chi(\mathrm{mod}\ R)}^{*}\left|L\left(\frac{1}{2},\chi\right)\right|^2\chi(H)\bar{\chi}(K)\\&=\frac{2}{\phi^*(R)}\sum_{\chi(\mathrm{mod}\ R)}^{*}\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)}}\frac{\chi(A)\bar{\chi}(B)\chi(H)\bar{\chi}(K)}{|AB|^{\frac{1}{2}}}+O\left(\frac{|R|^{-\frac{1}{2}+\epsilon}}{\phi^*(R)}\sum_{\chi(\mathrm{mod}\ R)}^{*}\chi(H)\bar{\chi}(K)\right).\end{aligned}\tag{9.5.1}$$

Using the definition of Dirichlet characters and $\phi^*(R)$ we have

$$\frac{|R|^{-\frac{1}{2}+\epsilon}}{\phi^*(R)}\sum_{\chi(\mathrm{mod}\ R)}^{*}\chi(H)\bar{\chi}(K)\ll\frac{|R|^{-\frac{1}{2}+\epsilon}}{\phi^*(R)}\sum_{\chi(\mathrm{mod}\ R)}^{*}1=|R|^{-\frac{1}{2}+\epsilon}.$$

Using the orthogonality relation Lemma 2.3.8, we have

$$\begin{aligned}&\frac{2}{\phi^*(R)}\sum_{\chi(\mathrm{mod}\ R)}^{*}\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)}}\frac{\chi(A)\bar{\chi}(B)\chi(H)\bar{\chi}(K)}{|AB|^{\frac{1}{2}}}\\&=\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv BK(\mathrm{mod}\ F)\\ (ABHK,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}.\end{aligned}\tag{9.5.2}$$

For the second sum on the right-hand side of (9.5.2) we will consider the contribution of the diagonal, $AH=BK$, and off-diagonal, $AH\neq BK$, terms separately. Thus we write

$$\begin{aligned}&\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv BK(\mathrm{mod}\ F)\\ (ABHK,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}\\&=\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv BK(\mathrm{mod}\ F)\\ AH=BK\\ (ABHK,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}\\&+\frac{2}{\phi^*(R)}\sum_{EF=R}\mu(E)\phi(F)\sum_{\substack{A,B\in\mathbb{A}^+\\ \deg(AB)<\deg(R)\\ AH\equiv BK(\mathrm{mod}\ F)\\ AH\neq BK\\ (ABHK,R)=1}}\frac{1}{|AB|^{\frac{1}{2}}}.\end{aligned}\tag{9.5.3}$$

Considering the contribution of the diagonal, $AH=BK$, terms we have that $|AB|^{-\frac{1}{2}}=|H|^{-\frac{1}{2}}|K|^{\frac{1}{2}}|A|^{-1}$ and $\deg(AB)=\deg(ABHK)-\deg(HK)=2\deg(A)+\deg(H)-\deg(K)$. Thus the double sum over $A,B\in\mathbb{A}^+$ with $\deg(AB)<\deg(R)$, $AH=BK$ and $(ABHK,R)=$

1 becomes a single sum over $A \in \mathbb{A}^+$ with $\deg(A) < \frac{1}{2}(\deg(R) + \deg(K) - \deg(H))$ and $(AH, R) = 1$. Therefore using the arguments stated above and Corollary 2.3.9 we have

$$\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH=BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} = \frac{2|K|^{\frac{1}{2}}}{|H|^{\frac{1}{2}}} \sum_{\substack{A \in \mathbb{A}^+ \\ \deg(A)<\frac{\deg(R)+\deg(K)-\deg(H)}{2} \\ (AH,R)=1}} \frac{1}{|A|}. \tag{9.5.4}$$

Using Lemma 9.3.2 with $x = \frac{1}{2}(\deg(R) + \deg(K) - \deg(H)) - 1$ we have

$$\begin{aligned} &\frac{2|K|^{\frac{1}{2}}}{|H|^{\frac{1}{2}}} \sum_{\substack{A \in \mathbb{A}^+ \\ \deg(A)<\frac{\deg(R)+\deg(K)-\deg(H)}{2} \\ (AH,R)=1}} \frac{1}{|A|} \\ &= |HK|^{\frac{1}{2}} \frac{\phi(R)}{|R|}(\deg(R) + \deg(H) + \deg(K)) + O\left(|HK|^{\frac{1}{2}} \log \omega(R)\right). \end{aligned}$$

For the contribution of the off-diagonal terms we have

$$\sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH \equiv BK (\text{mod } F) \\ AH \neq BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} = \sum_{z=0}^{\deg(R)-1} \sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB)=z \\ AH \equiv BK (\text{mod } F) \\ AH \neq BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}}. \tag{9.5.5}$$

Using Lemma 9.3.5 we have

$$\sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH \equiv BK (\text{mod } F) \\ AH \neq BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \sum_{z=0}^{\deg(R)-1} \frac{|HK| q^{\frac{z}{2}} (z+1)}{|F|} \ll \frac{|HK||R|^{\frac{1}{2}} \deg(R)}{|F|}. \tag{9.5.6}$$

Thus using (9.5.6) we have

$$\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH \equiv BK (\text{mod } F) \\ AH \neq BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{|HK||R|^{\frac{1}{2}} \deg(R)}{\phi^*(R)} \sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|}. \tag{9.5.7}$$

Combining (9.5.7), Lemma 2.2.5 and the fact that $\frac{\phi(R)}{|R|} \leq 1$ we have

$$\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH \equiv BK (\text{mod } F) \\ AH \neq BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{2^{\omega(R)} |HK||R|^{\frac{1}{2}} \deg(R)}{\phi^*(R)}. \tag{9.5.8}$$

Furthermore, combining (9.5.8) and Lemma 9.3.6 we have

$$\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH \equiv BK (\text{mod } F) \\ AH \neq BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll \frac{2^{\omega(R)} |HK| |R|^{\frac{1}{2}} \deg(R)}{\phi^*(R)} \ll |HK| |R|^{\epsilon - \frac{1}{2}}. \tag{9.5.9}$$

Since $\deg(H)+\deg(K)<\deg(R)$, then there is some $\epsilon>0$ such that $\deg(H)+\deg(K) \leq (1-2\epsilon)\deg(R)$. Thus $|HK|^{\frac{1}{2}} |R|^{\epsilon-\frac{1}{2}} = q^{\deg(H)+\deg(K)+(\epsilon-\frac{1}{2})\deg(R)} \leq q^{(1-2\epsilon)\deg(R)+(\epsilon-\frac{1}{2})\deg(R)} = 1$. Thus, combining the above and (9.5.9) we have

$$\frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{\substack{A,B \in \mathbb{A}^+ \\ \deg(AB)<\deg(R) \\ AH \equiv BK (\text{mod } F) \\ AH \neq BK \\ (ABHK,R)=1}} \frac{1}{|AB|^{\frac{1}{2}}} \ll |HK|^{\frac{1}{2}}. \tag{9.5.10}$$

Combining everything completes the proof of Theorem 9.2.2.

∎

Appendix A

Completing the proof of Lemma 3.6.6

A.1 Introduction

In this appendix we prove that

$$\mathcal{A}^{o}_{g-1,1,2} + \hat{\mathcal{A}}_2 + \tilde{\mathcal{A}}_2 - \mathcal{A}^{e}_{g,2,1} - \mathcal{A}^{e}_{g,2,2} - \mathcal{A}^{e}_{g-1,2,1} - \mathcal{A}^{e}_{g-1,2,2} - \mathcal{A}^{o}_{g,2} - \mathcal{A}^{o}_{g-1,2} \tag{A.1.1}$$

equals zero. For the terms corresponding to the residues at $u = q^{-1}$ and $u = q^{-2}$, it was already shown, in Chapter 3, that (A.1.1) equals zero, thus it remains to show that, for the terms corresponding to the residue at $u = 0$, (A.1.1) equals zero. To do this, we will use induction on g and consider two cases: g even and g odd. This appendix also appears in [AM21].

A.2 g even

Let $g = 2m$ for $m \in \mathbb{Z}$, we will show, by induction on m, that (A.1.1) equals zero for all $m \geq 1$. For the base case, $m = 1$, (A.1.1) is equalling to

$$\begin{aligned}\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(& q^{\frac{9}{2}}\big(\mathcal{C}(0) + \mathcal{C}'(0)\big) + q^4\big(\mathcal{C}(0)(1+q) + \mathcal{C}'(0)\big) + q^{\frac{11}{2}}\mathcal{C}(0) + q^{\frac{13}{2}}\mathcal{C}(0) - q^{\frac{11}{2}}\mathcal{C}(0) \\ & + q^5\big(\mathcal{C}(0)(q+q^2) + \mathcal{C}'(0)\big) - q^4\big(\mathcal{C}(0)(1+q^2) + \mathcal{C}'(0)\big) - q^{\frac{9}{2}}\big(\mathcal{C}(0)(1+q^2) + \mathcal{C}'(0)\big) \\ & - q^5\big(\mathcal{C}(0)(1+q^2) + \mathcal{C}'(0)\big)\Bigg).\end{aligned} \tag{A.2.1}$$

Rearranging (A.2.1), we see that it is equal to

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(\mathcal{C}(0)\left(\left(q^4+q^{\frac{9}{2}}+q^5+q^{\frac{11}{2}}+q^6+q^{\frac{13}{2}}+q^7\right)-\left(q^4+q^{\frac{9}{2}}+q^5+q^{\frac{11}{2}}+q^6-q^{\frac{13}{2}}+q^7\right)\right)$$
$$+\mathcal{C}'(0)\left(\left(q^4+q^{\frac{9}{2}}+q^5\right)-\left(q^4+q^{\frac{9}{2}}+q^5\right)\right)\Bigg),$$

which clearly equals zero. Assume that (A.1.1)= 0 for $m=t$. Then

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{3}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+1}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k+q^{3t+\frac{5}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-1-n}q^k$$
$$+q^{3t+\frac{7}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-1-n}^{2(t-1-n)}q^k-q^{3t+\frac{5}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-1-n}q^{2k}+q^{3t+2}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k$$
$$-q^{3t+1}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+\frac{3}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+2}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}\Bigg)=0. \tag{A.2.2}$$

For $m=t+1$, (A.1.1) equals

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{9}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^k+q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k$$
$$+q^{3t+\frac{13}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k-q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}+q^{3t+5}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t+1-n}^{2(t+1-n)}q^k$$
$$-q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}-q^{3t+\frac{9}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}-q^{3t+5}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}\Bigg). \tag{A.2.3}$$

Rearranging (A.2.3), we have that (A.1.1)=(A.2.4)+(A.2.5), where

$$\frac{q^3}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{3}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+1}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^k+q^{3t+\frac{5}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-1-n}q^k$$
$$+q^{3t+\frac{7}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-1-n}^{2(t-1-n)}q^k-q^{3t+\frac{5}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-1-n}q^{2k}+q^{3t+2}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^k$$
$$-q^{3t+1}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+\frac{3}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+2}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}\Bigg) \tag{A.2.4}$$

and

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{9}{2}}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}+q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t+1-n}+q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-n}$$
$$-q^{3t+\frac{13}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-1-n}+q^{3t+\frac{13}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)-1}(1+q)+q^{3t+\frac{13}{2}}\frac{\mathcal{C}^{(t)}(0)}{t!}$$
$$-q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}-q^{3t+5}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-n}+q^{3t+5}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)+1}(1+q)$$
$$+q^{3t+5}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}-q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}$$
$$-q^{3t+\frac{9}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+5}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}\Bigg) \tag{A.2.5}$$

Using the inductive hypothesis, we have that (A.2.4) equals zero, therefore it remains to show that (A.2.5) equals zero. Rearranging (A.2.5) we see that it is equal to

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(-q^{3t+\frac{13}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}+q^{3t+4}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}+q^{3t+\frac{11}{2}}\frac{\mathcal{C}^{(t)}(0)}{t!}$$
$$+q^{3t+\frac{11}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}+q^{3t+\frac{13}{2}}\sum_{n=0}^{t-1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}+q^{3t+\frac{13}{2}}\frac{\mathcal{C}^{(t)}(0)}{t!}$$
$$-q^{3t+\frac{11}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}+q^{3t+6}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}+q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}$$
$$-q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}-q^{3t+4}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}\Bigg). \tag{A.2.6}$$

Rearranging (A.2.6), we see that it is equal to

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{11}{2}}(1+q^{\frac{1}{2}}+q+q^{\frac{3}{2}})\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}-q^{3t+\frac{11}{2}}(1+q^{\frac{1}{2}}+q+q^{\frac{3}{2}})\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t-n)}\Bigg),$$

which equals zero. Thus (A.1.1)= 0 for $m=t+1$, and so, by induction, (A.1.1)= 0 for all $g\geq 1$ even.

A.3 g odd

Now let $g=2m+1$. We will show, by induction on m, (A.1.1) equals zero for all $m\geq 0$. For the base case, $m=0$, (A.1.1) is equal to

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{\frac{5}{2}}\big(\mathcal{C}(0)+\mathcal{C}'(0)\big)+q^{\frac{7}{2}}\mathcal{C}(0)+q^{3}\mathcal{C}(0)+q^{4}\mathcal{C}(0)-q^{3}\mathcal{C}(0)+q^{\frac{9}{2}}\mathcal{C}(0)-q^{\frac{7}{2}}\mathcal{C}(0)$$
$$-q^{4}\mathcal{C}(0)-q^{\frac{5}{2}}\big(\mathcal{C}(0)(1+q^{2})+\mathcal{C}'(0)\big)\Bigg). \tag{A.3.1}$$

Rearranging (A.3.1) we see that it is equal to

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\left(\mathcal{C}(0)\left(\left(q^{\frac{5}{2}}+q^{3}+q^{\frac{7}{2}}+q^{4}+q^{\frac{9}{2}}\right)-\left(q^{\frac{5}{2}}+q^{3}+q^{\frac{7}{2}}+q^{4}+q^{\frac{9}{2}}\right)\right)+\mathcal{C}'(0)\left(q^{\frac{5}{2}}-q^{\frac{5}{2}}\right)\right)$$

which clearly equals zero. Assume that (A.1.1)= 0 for $m=t$. Then

$$\begin{aligned}\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(&q^{3t+\frac{5}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+\frac{7}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{k}+q^{3t+3}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{k}\\&+q^{3t+4}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^{k}-q^{3t+3}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}+q^{3t+\frac{9}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^{k}\\&-q^{3t+\frac{7}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+4}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+\frac{5}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}\Bigg)=0.\end{aligned} \tag{A.3.2}$$

For $m=t+1$, (A.1.1) equals

$$\begin{aligned}\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(&q^{3t+\frac{11}{2}}\sum_{n=0}^{t+2}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{k}+q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{k}\\&+q^{3t+7}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t+1-n}^{2(t+1-n)}q^{k}-q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}+q^{3t+\frac{15}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t+1-n}^{2(t+1-n)}q^{k}\\&-q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}-q^{3t+7}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+1-n}q^{2k}-q^{3t+\frac{11}{2}}\sum_{n=0}^{t+2}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t+2-n}q^{2k}\Bigg).\end{aligned} \tag{A.3.3}$$

Rearranging (A.3.3), we see that (A.1.1)=(A.3.4)+(A.3.5), where

$$\begin{aligned}\frac{q^{3}}{\zeta_{\mathbb{A}}(2)}\Bigg(&q^{3t+\frac{5}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}+q^{3t+\frac{7}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{k}+q^{3t+3}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{k}\\&+q^{3t+4}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^{k}-q^{3t+3}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}+q^{3t+\frac{9}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=t-n}^{2(t-n)}q^{k}\\&-q^{3t+\frac{7}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+4}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}-q^{3t+\frac{5}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}\sum_{k=0}^{t-n}q^{2k}\Bigg)\end{aligned} \tag{A.3.4}$$

and

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+\frac{11}{2}}\frac{\mathcal{C}^{(t+2)}(0)}{(t+2)!}+q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t+1-n}+q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t+1-n}$$
$$-q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-n}+q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)-1}(1+q)+q^{3t+7}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}$$
$$-q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+\frac{15}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{t-n}$$
$$+q^{3t+\frac{15}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)-1}(1+q)+q^{3t+\frac{15}{2}}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}-q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}$$
$$-q^{3t+7}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+\frac{11}{2}}\sum_{n=0}^{t+2}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+2-n)}\Bigg). \tag{A.3.5}$$

By the inductive hypothesis (A.3.4) equals zero, therefore it remains to show that (A.3.5) equals zero. Rearranging (A.3.5), we see that it equals

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(-q^{3t+\frac{15}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}+q^{3t+\frac{13}{2}}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}+q^{3t+6}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}$$
$$+q^{3t+6}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}+q^{3t+7}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}+q^{3t+7}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}$$
$$-q^{3t+6}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}+q^{3t+\frac{13}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}+q^{3t+\frac{15}{2}}\sum_{n=0}^{t}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}$$
$$+q^{3t+\frac{15}{2}}\frac{\mathcal{C}^{(t+1)}(0)}{(t+1)!}-q^{3t+\frac{13}{2}}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+7}\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}\Bigg). \tag{A.3.6}$$

Rearranging (A.3.6) we see that it equals

$$\frac{1}{\zeta_{\mathbb{A}}(2)}\Bigg(q^{3t+6}(1+q^{\frac{1}{2}}+q+q^{\frac{3}{2}})\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}-q^{3t+6}(1+q^{\frac{1}{2}}+q+q^{\frac{3}{2}})\sum_{n=0}^{t+1}\frac{\mathcal{C}^{(n)}(0)}{n!}q^{2(t+1-n)}\Bigg),$$

which equals zero. Thus (A.1.1)= 0 for $m=t+1$, and so, by induction, (A.1.1) for all $g\geq 1$ odd. This completes the proof of Lemma 3.6.6.

■

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