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Dynamic Analysis of the Damped Trend Proportional OUT Policy

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Abstract

Using the Kalman filter we build a state-space model of the proportional order-up-to (POUT) policy with an autoregressive integrated moving average (ARIMA) demand process. The POUT policy is closely related to the order-to-up (OUT) policy with the addition of a proportional feedback controller in the inventory and work-in-progress feedback loops. Our modelling approach allows us to analyse the behaviour of the damped trend POUT policy when the damped trend forecasting method predicts ARIMA(1,1,2) demand. We derive and analyse the demand and inventory variances. We also find the covariance between the demand forecast and the inventory forecast in an attempt to obtain the order variance. Both the demand and the order variances are infinite under the non-stationary ARIMA(1,1,2) process. Thus, the traditional bullwhip measure (the ratio of the order variance divided by the demand variance) is indeterminate. However, we can study the difference between order and demand variance and the differnece between the OUT and POUT policies responses.

Keywords: Damped trend forecasting, Proportional OUT policy, Bullwhip, Inventory, Kalman filter, Control theory, Eigenvalue analysis

1. Introduction

The bullwhip effect, where the order variance is amplified as the orders proceed up the supply chain, has been observed in many industries for decades. There are many studies on how to reduce the cost of bullwhip, which often includes the cost of excessive capacity and inventory investments, emergency transport and machine setup, and the hiring and firing of employees. Extensive research has been devoted to quantifying the bullwhip under different stochastic demand patterns, forecasting methods, and replenishment policies. Demand forecasting and ordering policies have been found to be two of the most important causes of the bullwhip effect, Wang and Disney (2016). The order-up-to (OUT) policy and the proportional order-up-to (POUT) policy are two of the most common ordering algorithms in the literature.

The OUT policy is popular for regulating production and distribution in high volume settings as it minimises inventory holding and backlog costs while maintaining customer service levels. Some researchers quantified the bullwhip effect in supply chains with an autoregressive (AR) demand process. For example, Zhang (2004) study AR(1) demand, whereas Luong and Phien (2007) consider AR(2) demand. Alwan et al. (2003) studied the bullwhip effect resulting from the order-up-to (OUT) replenishment policy with optimal forecasts for first-order autoregressive and moving average, ARMA(1,1), demand. Rostami-Tabar and Disney (2023) investigate the impact of a first-order integer auto-regressive, INAR(1), demand process on the bullwhip generated in the OUT policy. Findings from these studies all indicate the existence of the bullwhip effect in the OUT policy under different demand patterns, even with optimal forecasts. Despite that, the bullwhip effect can be avoided for some demand processes by the OUT policy, Alwan et al. (2003), Gaalman et al. (2022).

The POUT policy adds a proportional feedback controller to the OUT policy to alter the trade-off between inventory and capacity costs. The POUT policy's effectiveness at reducing the bullwhip effect is widely recognised. A steady stream of research on the POUT policy assumes stationary demand. Hosoda and Disney (2006) quantified and compared the bullwhip between the POUT and OUT policies with optimal forecasts for AR(1) demand. The POUT policy with ARMA(p,q) demand was studied in Gaalman (2006). Gaalman and Disney (2009) identified the bullwhip effect in the POUT policy for ARMA(2,2) demand processes. Less attention has been given to non-stationary demand. Of the few that do, Graves (1999) quantified the bullwhip effect for a first order integrated moving average demand, IMA(0,1,1), and Boute et al. (2022) considered non-stationary demand in dual sourcing setting with SpeedFactories.

Nevertheless, bullwhip avoidance is possible when the OUT policy employs damped trend forecasts, Li et al. (2014). Bullwhip was eliminated when the forecasting parameters where selected from a special region in the parameter space under an independently and identically distributed (i.i.d.) demand, a feat never previously reported for the OUT policy. Li et al. (2023) further studied the system behaviour of the damped trend OUT (DT-OUT) policy finding that the dynamic behaviour of the DT-OUT policy was equivalent to the POUT policy.

Although the damped trend forecasting method can be applied to any demand process, it is optimal for the ARIMA(1,1,2) process (Gardner and McKenzie 1985). However, OUT and POUT bullwhip performance under ARIMA(1,1,2) demand remains unexplored. The purpose of this paper is to study the performance of using the damped trend forecasting method to predict ARIMA(1,1,2) demand process in both OUT and POUT policies. We build a state-space model with the Kalman filter and provide the expressions for the order variance and the inventory variance in eigenvalue form. We measure the bullwhip effect as the difference between order and demand variance to deal with the non-stationary demand. We compare the bullwhip in the two policies and show:

- The bullwhip effect can be larger in the POUT policy than in the OUT policy for certain types of ARIMA(1,1,2) demand, even when the proportional feedback controller lies in $0 \leq f < 1$.
- For other types of ARIMA(1,1,2) demand, the POUT policy ($0 \leq f < 1$) always generates less bullwhip than the OUT policy.
- In a final case, the proportional controller f needs to be carefully tuned in order to generate less bullwhip in the POUT policy than the OUT policy.

To address this optimal inventory control challenge, we present a generalizable approach for understanding the impact of the feedback controller, based on an analysis of the demand eigenvalues. The remainder of the paper is organised as follows. In §2, we present an approach to modelling the ARMA(p,q) processes. We adapt this approach to ARIMA(1,1,2) processes. Inventory policies are modelled in state-space forms in §3, and the variances for orders and inventory are derived in §4. §5 investigates the bullwhip produced by the OUT and POUT policies. §6 concludes.

2. The demand and the forecast

Let us consider the general ARMA(p,q) case before we consider the ARIMA(1,1,2) case. (Box et al. 2008) define the ARMA process as $z_{t+1} - \phi_1 z_t - \cdots - \phi_p z_{t+1-m} = \eta_{t+1} - \theta_1 \eta_t - \cdots - \theta_q \eta_{t+1-m}$, where z_{t+1} is the demand at time $t+1$, η_{t+j} is an i.i.d. random process (white noise) and $m = \max(p,q)$. Several state space forms of the ARMA process exist; there is no unique form. We follow Gaalman (2006) and Gaalman and Disney (2009) and use a state y_t and a (left) canonical form of the system matrix $\mathbf{D}$,

$$\left.\begin{aligned} z_{t+1} &= \mathbf{M} y_{t+1} + \eta_{t+1} \\ y_{t+1} &= \mathbf{D} y_t + \mathbf{G} \eta_t \end{aligned}\right\}, \tag{1}$$

where

$$\mathbf{D} = \begin{pmatrix} \phi_1 & 1 & 0 & 0 \\ \phi_2 & 0 & \ddots & 0 \\ \vdots & 0 & \ddots & 1 \\ \phi_m & 0 & 0 & 0 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} \phi_1 - \theta_1 \\ \phi_2 - \theta_2 \\ \vdots \\ \phi_m - \theta_m \end{pmatrix}, \mathbf{M} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{2}$$

The system matrix $\mathbf{D}$ contains only auto-regressive coefficients; sometimes $\mathbf{D}$ is denoted as $\mathbf{D}^{\phi}$. The characteristic polynomial of $\mathbf{D}$ is

$$\det(\mathbf{D}-\lambda\mathbf{I}) = \lambda^m - \phi_1\lambda^{m-1} - \cdots - \phi_{m-1}\lambda - \phi_m = \prod_{j-1}^{m}\left(\lambda - \lambda_j^{\phi}\right). \tag{3}$$

$\lambda = \lambda_j^{\phi}$ indicates the poles of the ARMA process and λ_j^{ϕ} are related to the AR coefficients.

Remark 1. *The poles are distinct.*

The conditional expectation of the demand can be found from the one-period-ahead forecast using the Kalman filter approach (*a priori demand estimation*),

$$\left.\begin{aligned} \hat{z}_{t+1,t} &= \mathbf{M}\hat{y}_{t+1,t} \\ \hat{y}_{t+1,t} &= \mathbf{D}\hat{y}_{t,t-1} + \mathbf{K}_t(z_t - \hat{z}_{t,t-1}) \end{aligned}\right\} \tag{4}$$

with the gain $\mathbf{K}_t$.

Since the parameters of the state space expression are time-independent and infinite past demand measurements are present, the Kalman gain $\mathbf{K}_t$ converts to a constant $\mathbf{K}$, which can be determined by the stationary discrete time matrix Ricatti equation. However, as the system's structure has only one error η_t, the gain $\mathbf{K}$ can be derived directly by minimising the variance of the state space error $v_{t+1} = (y_{t+1} - \hat{y}_{t+1,t})$. Hyndman et al. (2008) call this a *single source of error* (SSOE) model. However, our derivation is more intuitive.

The state space error at time t satisfies

$$v_{t+1} = (y_{t+1} - \hat{y}_{t+1,t}) = \mathbf{D}(y_t - \hat{y}_{t,t-1}) - \mathbf{K}(z_t - \hat{z}_{t,t-1}) + \mathbf{G}\eta_t, \tag{5}$$

resulting in the variance expression

$$\mathbb{V}[v_{t+1}] = (\mathbf{D}-\mathbf{KM})\mathbb{V}[v_t](\mathbf{D}-\mathbf{KM})^T + (\mathbf{G}-\mathbf{K})(\mathbf{G}-\mathbf{K})^T\mathbb{V}[\eta]. \tag{6}$$

This expression holds for each t, as infinite past demand observations are considered. Iteration backwards reveals that $(\mathbf{D}-\mathbf{KM})$ is stable. At time t the minimum variance of v_t exists when $\mathbf{K} = \mathbf{G}$ and can even be zero, $\mathbf{K} = \mathbf{G} = 0$. Note, v_t is the innovation form of the Kalman filter. The state space one-period-ahead forecast $\hat{y}_{t+1,t}$ then becomes

$$\hat{y}_{t+1,t} = \mathbf{D}\hat{y}_{t,t-1} + \mathbf{K}(z_t - \hat{z}_{t,t-1}) = \mathbf{D}\hat{y}_{t,t-1} + \mathbf{GM}(y_t - \hat{y}_{t,t-1}) = \mathbf{D}\hat{y}_{t,t-1} + \mathbf{G}\eta_t. \tag{7}$$

The $(\mathbf{D}-\mathbf{KM})$ is

$$(\mathbf{D}-\mathbf{KM}) = (\mathbf{D}-\mathbf{GM}) = \begin{pmatrix} \theta_1 & 0 & \cdots & 0 \\ \theta_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 1 \\ \theta_m & 0 & \cdots & 0 \end{pmatrix} = \mathbf{D}^{\theta}. \tag{8}$$

This shows that $\mathbf{D}^{\theta}$ is always stable and invertible. The characteristic polynomial is

$$\lambda^m - \theta_1 \lambda^{m-1} - \cdots - \theta_m = \prod_{j-1}^{m} \left(\lambda - \lambda_j^{\theta}\right). \tag{9}$$

Remark 2. *The derivation is based on the assumption that the zeros are distinct.*

2.1. The ARMA demand process in eigenvector form

The companion matrix $\mathbf{D} = \mathbf{D}^{\phi}$ has m eigenvalues and m “left hand eigenvectors”. This is written as $\mathbf{U}\mathbf{D}^{\phi} = \Lambda^{\phi}\mathbf{U}$, where Λ^{ϕ} is the (m by m) diagonal matrix of eigenvalues and $\mathbf{U}$ is the Vandermonde matrix that consists of a m rows of m eigenvectors:

$$\Lambda^{\phi} = \begin{pmatrix} (\lambda_1^{\phi}) & 0 & \cdots & 0 \\ 0 & (\lambda_2^{\phi}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\lambda_m^{\phi}) \end{pmatrix}, \mathbf{U} = \begin{pmatrix} (\lambda_1^{\phi})^{m-1} & (\lambda_1^{\phi})^{m-2} & \cdots & 1 \\ (\lambda_2^{\phi})^{m-1} & (\lambda_2^{\phi})^{m-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_m^{\phi})^{m-1} & (\lambda_m^{\phi})^{m-2} & \cdots & 1 \end{pmatrix}. \tag{10}$$

Note, the eigenvalues should be distinct (but they can be conjugate complex) otherwise, the inverse $\mathbf{U}^{-1}$ does not exist. However, if some eigenvalues are common, then extra independent eigenvectors are required. These can be found using the Jordan form (see Wikipedia (2023) or Kailath (1979) for example). An alternative to the Kailath’s approach is the z-transformation of an ARMA demand process and using partial fraction properties. The ARMA demand process is written as a transfer function of the ratio of output (zeros) to the input (poles), Gaalman et al. (2022).

To write $\mathbf{D}^{\phi}$ as a function of the eigenvectors, we get

$$\mathbf{D}^{\phi} = \mathbf{U}^{-1}\Lambda^{\phi}\mathbf{U}. \tag{11}$$

We now need to determine $\mathbf{U}^{-1}$. There are many approaches available; we follow Kailath

(1979) and Antsaklis and Michel (1997). Consider the matrix $\mathbf{V}$ of right eigenvectors ($\mathbf{V} \neq \mathbf{U}^{-1}$),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\phi_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\phi_{m-1} & \cdots & -\phi_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ (\lambda_1^\phi) & (\lambda_2^\phi) & \cdots & (\lambda_m^\phi) \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1^\phi)^{m-1} & (\lambda_2^\phi)^{m-1} & \cdots & (\lambda_2^\phi)^{m-1} \end{pmatrix}. \tag{12}$$

Then,

$$\mathbf{U}^{-1} = \mathbf{V} \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & s_l & 0 \\ 0 & \cdots & 0 & s_m \end{pmatrix}, \; s_l = \frac{1}{\prod_{\substack{j=1 \\ j\neq l}}^{m} \left(\lambda_l^\phi - \lambda_j^\phi\right)}. \tag{13}$$

Next the state space form of ARMA process (1) will be transformed to an eigenvector form $(\mathbf{U}y_{t+1}) = \Lambda^\phi(\mathbf{U}\mathbf{D}^\phi y_t) + \mathbf{U}\mathbf{G}\eta_t$ and further simplified to

$$\upsilon_{t+1} = \Lambda^\phi \upsilon_t + \mathbf{G}^\lambda \eta_t, \;\; \text{where } \; \mathbf{G}^\lambda = \mathbf{U}\mathbf{G} = \begin{pmatrix} \prod_{j=1}^{m}\left(\lambda_1^\phi - \lambda_j^\theta\right) \\ \prod_{j=1}^{m}\left(\lambda_2^\phi - \lambda_j^\theta\right) \\ \vdots \\ \prod_{j=1}^{m}\left(\lambda_m^\phi - \lambda_j^\theta\right) \end{pmatrix}. \tag{14}$$

Then, the demand is rewritten as $z_{t+1} = \mathbf{M}y_{t+1} + \eta_{t+1} = \mathbf{M}\mathbf{U}^{-1}(\mathbf{U}y_{t+1}) + \eta_{t+1}$. Let $\mathbf{M}^\lambda = \mathbf{M}\mathbf{U}^{-1}$, we get $z_{t+1} = \mathbf{M}^\lambda \upsilon_{t+1} + \eta_{t+1}$, $\mathbf{M}^\lambda = \mathbf{M}\mathbf{U}^{-1} = \begin{pmatrix} s_1 & s_2 & \cdots & s_m \end{pmatrix}$.

Finally, we can rewrite the ARMA(p,q) demand impulse response, p_{t+1}, in eigenvector form as

$$p_{t+1} = \sum_{j=1}^{m} r_j (\lambda_j^\phi)^t, \text{ where } r_i = \frac{\prod_{j=1}^{m}\left(\lambda_i^\phi - \lambda_j^\theta\right)}{\prod_{\substack{j=1 \\ j\neq i}}^{m}\left(\lambda_i^\phi - \lambda_j^\phi\right)}. \tag{15}$$

Remark 3. *This derivation assumes the AR eigenvalues are distinct, see Remark 1. It also shows the eigenvalues do not need to be stable (*$-1 < \lambda_j^\phi < 1$*).*

For the OUT policy, Gaalman and Disney (2009) introduced the inventory gain component,

$$E(0) = 1; \; E(l) = \sum_{j=0}^{l} p_j = 1 + \sum_{j=0}^{l-1} \mathbf{M}(\mathbf{D}^j)\mathbf{G}, \tag{16}$$

which we also use here. We can then rewrite the impulse response (15) as

$$p_0 = 1;\ p_t = \mathbf{M}\mathbf{D}^{t-1}\mathbf{G} = E(t) - E(t-1),\ E(t) = 1 + \mathbf{M}(\mathbf{I}-\mathbf{D})^{-1}(\mathbf{I}-\mathbf{D}^t)\mathbf{G}. \tag{17}$$

Note, $p_{j+1} = (\mathbf{M}\mathbf{D}^j\mathbf{G})$.

2.2. The ARIMA(1,1,2) demand process and damped trend forecasts

The ARIMA(1,1,2) demand process, $z_{t+1} - z_t - \varphi(z_t - z_{t-1}) = \eta_{t+1} - \theta_1\eta_t - \theta_2\eta_{t-1}$, can be written as an ARMA process where $m = 2$:

$$z_{t+1} - (1+\varphi)z_t + \varphi z_{t-1} = \eta_{t+1} - \theta_1\eta_t - \theta_2\eta_{t-1}. \tag{18}$$

Note, to map the ARMA(2,2) parameters into ARIMA(1,1,2) parameters we use $\phi_1 = 1+\varphi$ and $\phi_2 = -\varphi$. There are two AR eigenvalues: $\lambda_1^\phi = \varphi$ and $\lambda_2^\phi = 1$, Li et al. (2023).

Our study uses the damped trend forecasting method to predict ARIMA(1,1,2) demand, as damped trend forecasts are optimal for the ARIMA(1,1,2) demand process, Gardner and McKenzie (1985). We follow our previous procedures to derive the eigenvector form of the demand process for later analysis. Departing from the state space model

$$y_{t+1} = \mathbf{D}^\phi y_t + \mathbf{G}\eta_t, \tag{19}$$

where

$$\mathbf{D}^\phi = \begin{pmatrix} 1+\varphi & 1 \\ -\varphi & 0 \end{pmatrix},\ \mathbf{G} = \begin{pmatrix} 1+\varphi-\theta_1 \\ -\varphi-\theta_2 \end{pmatrix} = \begin{pmatrix} 1+\varphi-(\lambda_1^\theta+\lambda_2^\theta) \\ -\varphi+\lambda_1^\theta\lambda_2^\theta \end{pmatrix}. \tag{20}$$

Let $\upsilon_t = \mathbf{U}y_t$ with the Vandermonde matrix

$$\mathbf{U} = \begin{pmatrix} \lambda_1^\phi & 1 \\ \lambda_2^\phi & 1 \end{pmatrix} = \begin{pmatrix} \varphi & 1 \\ 1 & 1 \end{pmatrix} \tag{21}$$

and its inverse

$$\mathbf{U}^{-1} = \frac{1}{\lambda_2^\phi - \lambda_1^\phi}\begin{pmatrix} -1 & 1 \\ \lambda_2^\phi & -\lambda_1^\phi \end{pmatrix} = \frac{1}{1-\varphi}\begin{pmatrix} -1 & 1 \\ 1 & -\varphi \end{pmatrix}. \tag{22}$$

Thus, the eigenvector form of ARIMA(1,1,2) is:

$$\upsilon_{t+1} = \Lambda^\phi\upsilon_t + \mathbf{G}^\lambda\eta_t, \tag{23}$$

$$z_{t+1} = \mathbf{M}^\lambda\upsilon_{t+1} + \eta_{t+1}, \tag{24}$$

with

$$\mathbf{G}^{\lambda} = \mathbf{UG} = \begin{pmatrix} (\lambda_1^{\phi} - \lambda_1^{\theta})(\lambda_1^{\phi} - \lambda_2^{\theta}) \\ (\lambda_2^{\phi} - \lambda_1^{\theta})(\lambda_2^{\phi} - \lambda_2^{\theta}) \end{pmatrix} = \begin{pmatrix} (\varphi - \lambda_1^{\theta})(\varphi - \lambda_2^{\theta}) \\ (1 - \lambda_1^{\theta})(1 - \lambda_2^{\theta}) \end{pmatrix}, \tag{25}$$

and

$$\mathbf{M}^{\lambda} = \mathbf{U}^{-1}\mathbf{M} = \begin{pmatrix} s_1 & s_2 \end{pmatrix}, \; s_1 = \frac{1}{\varphi - 1}, \; s_2 = \frac{1}{1 - \varphi}. \tag{26}$$

The impulse response for n-step ahead forecast is

$$\begin{aligned} p_t(n) &= \mathbf{MD}^{t+n}\mathbf{G} = \mathbf{M}^{\lambda}(\Lambda^{\phi})^{t+n}\mathbf{G}^{\lambda} \\ &= s_1(\varphi)^{t+n}(\varphi - \lambda_1^{\theta})(\varphi - \lambda_2^{\theta}) + s_2(1 - \lambda_1^{\theta})(1 - \lambda_2^{\theta}) = r_1(\varphi)^{t+n} + r_2, \end{aligned} \tag{27}$$

where

$$r_1 = \frac{(\varphi - \lambda_1^{\theta})(\varphi - \lambda_2^{\theta})}{(\varphi - 1)}, \; r_2 = \frac{(1 - \lambda_1^{\theta})(1 - \lambda_2^{\theta})}{(1 - \varphi)}. \tag{28}$$

3. Formulation of the inventory policy

While the order decision is made at the end of period t with a lead time k and a review period, that order is realised and influences the inventory at time $t+k+1$,

$$i_{t+k+1} = i_{t+k} + o_t - z_{t+k+1}. \tag{29}$$

Here o_t is the order quantity issued in period t and received at the beginning of period $t+k+1$. The inventory forecast is then

$$\hat{i}_{t+k+1,t+1} = \hat{i}_{t+k,t} + o_t - \hat{z}_{t+k+1,t} - E(k)\eta_{t+1}. \tag{30}$$

From (4) and (7), we can write the demand state forecast made at time $t+1$ for $k+1$ periods ahead:

$$\hat{y}_{t+k+2,t+1} = \mathbf{D}\hat{y}_{t+k+1,t} + \mathbf{D}^k\mathbf{G}\eta_{t+1}. \tag{31}$$

By this, the forecast state space system for inventory and demand, Gaalman and Disney (2009), can be written as

$$\begin{pmatrix} \hat{i}_{t+k+1,t+1} \\ \hat{y}_{t+k+2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & -\mathbf{M} \\ 0 & \mathbf{D} \end{pmatrix} \begin{pmatrix} \hat{i}_{t+k,t} \\ \hat{y}_{t+k+1,t} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} o_t + \begin{pmatrix} -E(k) \\ \mathbf{D}^k\mathbf{G} \end{pmatrix} \eta_{t+1} \tag{32}$$

The order in the POUT policy is

$$o_t = \begin{pmatrix} -f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i}_{t+k,t} \\ \hat{z}_{t+k+1,t} \end{pmatrix}. \tag{33}$$

f is the proportional feedback controller. $0 \leq f < 2$ is required for stability. When $f = 1$, the POUT policy (33) degenerates into the OUT policy.

As $\mathbf{M} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ for ARIMA(1,1,2) demand processes, substituting (33) into (32) yields the complete forecast state space recursion for the damped trend POUT policy:

$$\begin{pmatrix} \hat{i}_{t+k+1,t+1} \\ \hat{y}_{t+k+2,t+1} \end{pmatrix} = \begin{pmatrix} (1-f) & 0 \\ 0 & \mathbf{D} \end{pmatrix} \begin{pmatrix} \hat{i}_{t+k,t} \\ \hat{y}_{t+k+1,t} \end{pmatrix} + \begin{pmatrix} -E(k) \\ \mathbf{D}^k \mathbf{G} \end{pmatrix} \eta_{t+1}. \tag{34}$$

The eigenvalues of this system can be found from the determinant,

$$\det \begin{pmatrix} (1-f) - \lambda & 0 \\ 0 & \mathbf{D} - \mathbf{I}_m \lambda \end{pmatrix} = 0. \tag{35}$$

Solving $((1-f)-\lambda)(\mathbf{D}-\mathbf{I}_m\lambda) = 0$ gives us the eigenvalue $\lambda_1 = (1-f)$ and all m eigenvalues of $\mathbf{D}$: λ_j^{ϕ}. Since $0 < f < 1$, $0 < \lambda_1 = (1-f) < 1$ means the real λ_1 lies within the stability area.

4. Expressions for variances

Before deriving the order and inventory variance expressions we first study the variance of demand. The demand at time $t+k+1$, z_{t+k+1}, can be formulated as a function of the forecast $\hat{z}_{t+k+1,t}$ made at time t for z_{t+k+1},

$$z_{t+k+1} = \hat{z}_{t+k+1,t+k} + \eta_{t+k+1} = \hat{z}_{t+k+1,t} + \eta_{t+k+1} + \sum_{j=0}^{k-1} \mathbf{M}(\mathbf{D}^j)\mathbf{G}\eta_{t+k-j} \tag{36}$$

Remark 4. *As $\hat{z}_{t+k+1,t}$ is a function of $\{\eta_t, \eta_{t-1}, \eta_{t-2}, \ldots\}$ and the error terms in* (36) *are a function of $\{\eta_{t+k+1}, \eta_{t+k}, \ldots, \eta_{t+1}\}$ the components in* (36) *are uncorrelated.*

Due to Remark 4 the demand variance in period $t+k+1$

$$\mathbb{V}[z_{t+k+1}] = \mathbb{V}[\hat{z}_{t+k+1,t}] + \mathbb{V}[\eta] + \sum_{j=0}^{k-1} (\mathbf{M}\mathbf{D}^j\mathbf{G})^2 \mathbb{V}[\eta] = \mathbb{V}[\hat{z}_{t+k+1,t}] + \sum_{j=0}^{k} p_j^2 \mathbb{V}[\eta]. \tag{37}$$

4.1. Inventory variance

Eq. (34) shows that the inventory forecast $\hat{i}_{t+k+1,t+1} = (1-f)\hat{i}_{t+k,t} - E(k)\eta_{t+1}$ is stable. As $\mathbb{V}[\hat{i}_{t+k+1,t+1}] \equiv \mathbb{V}[\hat{i}_{t+k,t}]$, the variance of $\hat{i}_{t+k+1,t+1}$ is

$$\mathbb{V}[\hat{i}_{t+k+1,t+1}] = \left(\frac{1}{f(2-f)}\right)(E(k))^2\mathbb{V}[\eta]. \tag{38}$$

Inventory can be written as

$$i_{t+k+1} = \hat{i}_{t+k+1,t} + (i_{t+k+1} - \hat{i}_{t+k+1,t}), \tag{39}$$

where $(i_{t+k+1} - \hat{i}_{t+k+1,t}) = -\sum_{j=1}^{k+1}(d_{t+j} - \hat{d}_{t+j,t}) = -\sum_{j=1}^{k+1} E(k+1-j)\eta_{t+j}$. The component, $(i_{t+k+1} - \hat{i}_{t+k+1,t})$, is the inventory forecast error and is uncorrelated with the inventory forecast, Gaalman and Disney (2009). Then we obtain the variance for the inventory forecast error

$$\mathbb{V}[i_{t+k+1} - \hat{i}_{t+k+1,t}] = \sum_{l=0}^{k}(E(l))^2\mathbb{V}[\eta]. \tag{40}$$

In the OUT case, $\hat{i}_{t+k+1,t} = 0$, however in the POUT case $\hat{i}_{t+k+1,t} \neq 0$. Consider (34) and (40), the inventory forecast variance in the POUT policy can be obtained

$$\mathbb{V}[\hat{i}_{t+k+1,t}] = (1-f)^2\mathbb{V}[\hat{i}_{t+k,t}] = \left(\frac{(1-f)^2}{f(2-f)}\right)(E(k))^2\mathbb{V}[\eta]. \tag{41}$$

The POUT policy's inventory variance then becomes

$$\mathbb{V}[i_{t+k+1}] = \left(\frac{1}{f(2-f)}\right)(E(k))^2\mathbb{V}[\eta] + \sum_{l=0}^{k-1}(E(l))^2\mathbb{V}[\eta]. \tag{42}$$

Let $f=1$, $\mathbb{V}[i_{t+k+1}] = \sum_{l=0}^{k}(E(l))^2\mathbb{V}[\eta]$, which concurs to the OUT case.

The function $(f(2-f))^{-1}$ is positive and is convex in f with a minimum at $f=1$ and asymptotes to infinity at $f=0$ and $f=2$. Thus, the OUT policy's inventory variance is the minimal case of the POUT policy. The derivative, $d\left(\mathbb{V}[i_{t+k+1}]\right)/df$ also confirms this.

4.2. Covariance of demand forecast and inventory forecast

The state space expression (34) shows that both $\hat{i}_{t+k+1,t+1}$ and $\hat{y}_{t+k+2,t+1}$ have the same error component η_{t+1}. Then we have

$$\text{cov}[\hat{i}_{t+k+1,t+1}, \hat{y}_{t+k+2,t+1}] = ((1-f)\mathbf{D})\,\text{cov}[\hat{i}_{t+k,t}, \hat{y}_{t+k+1,t}]. \tag{43}$$

The inventory is a scalar, thus, an alternative expression is

$$\mathrm{cov}[\hat{y}_{t+k+2,t+1}, \hat{i}_{t+k+1,t+1}] = ((1-f)\mathbf{D})\,\mathrm{cov}[\hat{y}_{t+k+1,t}, \hat{i}_{t+k,t}] - (\mathbf{D}^k\mathbf{G})E(k)\mathbb{V}[\eta]. \tag{44}$$

In the OUT case ($f = 1$), $\mathrm{cov}[\hat{y}_{t+k+2,t+1}, \hat{i}_{t+k+1,t+1}] = -(\mathbf{D}^k\mathbf{G})E(k)\mathbb{V}[\eta]$. Given infinite past observations, $\mathrm{cov}[\hat{y}_{t+k+2,t+1}, \hat{i}_{t+k+1,t+1}] \equiv \mathrm{cov}[\hat{y}_{t+k+1,t}, \hat{i}_{t+k,t}]$ holds. Then we obtain

$$\mathrm{cov}[\hat{y}_{t+k+1,t}, \hat{i}_{t+k,t}] = -(\mathbf{I}_m - (1-f)\mathbf{D})^{-1}(\mathbf{D}^k\mathbf{G})E(k)\mathbb{V}[\eta]. \tag{45}$$

Thus, the covariance between the demand forecast and the inventory forecast is

$$\mathrm{cov}[\hat{z}_{t+k+1,t}, \hat{i}_{t+k,t}] = \mathbf{M}\mathrm{cov}[\hat{y}_{t+k+1,t}, \hat{i}_{t+k,t}] = -W(f,k)E(k)\mathbb{V}[\eta], \tag{46}$$

where

$$W(f,k) = \mathbf{M}((\mathbf{I}_m - (1-f)\mathbf{D})^{-1}\mathbf{D}^k)\mathbf{G}. \tag{47}$$

Before we work out $W(f,k)$, we consider the order variance as $W(f,k)$ is also used there.

4.3. Order variance

From (33), the order variance expression can be written as

$$\mathbb{V}[o_t] = \mathbb{V}[\hat{z}_{t+k+1,t}] - 2f\mathrm{cov}[\hat{z}_{t+k+1,t}, \hat{i}_{t+k,t}] + f^2\mathbb{V}[\hat{i}_{t+k,t}]. \tag{48}$$

We have shown that $\hat{z}_{t+k+1,t}$ and $\hat{i}_{t+k,t}$ are correlated. Substitution of the inventory forecast variance (38) and the covariance (46) results in

$$\mathbb{V}[o_t] = \mathbb{V}[\hat{z}_{t+k+1,t}] + 2fW(f,k)E(k)\mathbb{V}[\eta] + \left(\frac{f}{2-f}\right)(E(k))^2\mathbb{V}[\eta]. \tag{49}$$

Next we work out the $W(f,k)$. The $(\mathbf{I}_m - (1-f)\mathbf{D})^{-1}$ in (47) is the Woodbury matrix identity, Woodbury (1950), $(A-B)^{-1} = \sum_{j=0}^{\infty}(A^{-1}B)^jA^{-1}$. Let $A = \mathbf{I}_m$ and $B = (1-f)\mathbf{D}$, then

$$(\mathbf{I}_m - (1-f)\mathbf{D})^{-1} = \sum_{j=0}^{\infty}((1-f)\mathbf{D})^j. \tag{50}$$

Substituting (11), we have

$$(\mathbf{I}_m-(1-f)\mathbf{D})^{-1}=\mathbf{U}^{-1}\sum_{j=0}^{\infty}\begin{pmatrix}((1-f)\lambda_1^{\phi})^j & 0 & \cdots & 0\\ 0 & ((1-f)\lambda_2^{\phi})^j & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & ((1-f)\lambda_m^{\phi})^j\end{pmatrix}\mathbf{U}. \tag{51}$$

As $-1<(1-f)\lambda_l^{\phi}<1$, the infinite summation is $(1-(1-f)\lambda_l^{\phi})^{-1}$. Thus,

$$W(f,k)=\mathbf{M}((\mathbf{I}_m-(1-f)\mathbf{D})^{-1}\mathbf{D}^k)\mathbf{G}=\sum_{l=1}^{m}\frac{r_l(\lambda_l^{\phi})^k}{1-(1-f)\lambda_l^{\phi}} \tag{52}$$

Substituting (28) into (52), the $W(f,k)$ in the POUT policy when the Damped Trend produces optimal forecasts of the ARIMA(1,1,2) demand process becomes

$$W(f,k)=\left(\frac{(\varphi-\lambda_1^{\theta})(\varphi-\lambda_2^{\theta})}{(\varphi-1)}\right)\left(\frac{(\varphi)^k}{1-(1-f)\varphi}\right)+\left(\frac{(1-\lambda_1^{\theta})(1-\lambda_2^{\theta})}{(1-\varphi)(1-(1-f))}\right). \tag{53}$$

Now the order variance in the POUT policy becomes

$$\mathbb{V}[o_t]=\mathbb{V}[\hat{z}_{t+k+1,t}]+2\sum_{l=1}^{2}\left(\frac{f}{1-(1-f)\lambda_l^{\phi}}\right)r_l(\lambda_l^{\phi})^kE(k)\mathbb{V}[\eta]+\left(\frac{f}{2-f}\right)(E(k))^2\mathbb{V}[\eta] \tag{54}$$

5. Comparison of Bullwhip in DT-POUT and DT-OUT

The impulse response for orders in the damped trend OUT (DT-OUT) case, p_t^O, is identical to the impulse response for $(k+1)$-period ahead damped trend forecast, $p_t(k+1)$,

$$p_t^O=\begin{cases}\sum_{i=0}^{k+1}p_i=E(k+1), & \text{if } t=0,\\ p_t(k)=r_1(\varphi)^{t+k}+r_2, & \text{if } t>0.\end{cases} \tag{55}$$

As the demand variance for ARIMA(1,1,2) is infinite, we measure bullwhip as the difference between the order variance and the demand variance, $CB(k)=\mathbb{V}[o_t]-\mathbb{V}[z_{t+k+1}]$. The bullwhip in the DT-OUT policy:

$$CB(k)^O=2(r_2+r_1(\varphi)^k)E(k)\mathbb{V}[\eta]+(E(k))^2\mathbb{V}[\eta]-\sum_{j=0}^{k}p_j^2\mathbb{V}[\eta]. \tag{56}$$

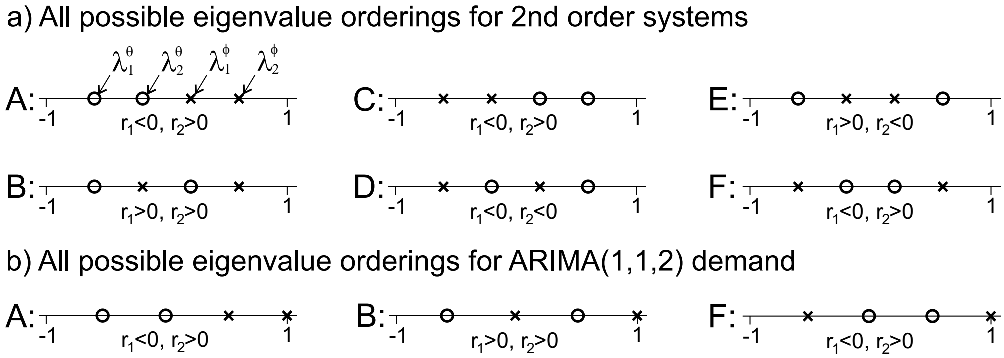

Figure 1: The possible eigenvalue orderings. Panel a: All possible eigenvalue ordering for second order systems. Panel b: Possible eigenvalue orderings for ARIMA(1,1,2).

For the DT-POUT policy, the bullwhip difference can be written as

$$CB(k)^P = 2fW(f,k)E(k)\mathbb{V}[\eta] + \left(\frac{f}{2-f}\right)(E(k))^2\mathbb{V}[\eta] - \sum_{j=0}^{k} p_j^2 \mathbb{V}[\eta] \tag{57}$$

$$= 2\sum_{l=1}^{2}\left(\frac{f}{1-(1-f)\lambda_l^{\phi}}\right) r_l(\lambda_l^{\phi})^k E(k)\mathbb{V}[\eta] + \left(\frac{f}{2-f}\right)(E(k))^2\mathbb{V}[\eta] - \sum_{j=0}^{k} p_j^2 \mathbb{V}[\eta]. \tag{58}$$

To compare the bullwhip between DT-POUT and DT-OUT policies, we can measure $CB(k)^O - CB(k)^P = \mathbb{V}[o_t]^O - \mathbb{V}[o_t]^P$. Using (56) and (57),

$$CB(k)^O - CB(k)^P = 2(1-f)E(k)\left(\frac{E(k)}{2-f} + \frac{r_1(1-\varphi)(\varphi)^k}{1-(1-f)\varphi}\right)\mathbb{V}[\eta]. \tag{59}$$

Note, when $\varphi < 0$ and odd-even lead time effect can be directly observed from (56), (57), and (59).

Li et al. (2023) show that ARIMA(1,1,2) eigenvalues (the poles and zeros) have three orderings, A, B, and F, out of 6 possible cases, see Figure 1, Gaalman et al. (2022). Note, the zeros of Type A and B, λ_1^{θ}, λ_2^{θ}, can be negative or positive. With knowledge of (28), we deduce

Lemma 5.1. *For Type A ARIMA(1,1,2) demand, $r_2 > 0$ and $r_1 < 0$; for Type B demand, $r_2 > 0$ and $r_1 > 0$; and for Type F, $r_2 > 0$ and $r_1 < 0$.*

Proof. The proof of Lemma 5.1 can be found in Gaalman et al. (2022). □

Lemma 5.2. *The demand impulse response p_t is always positive for Type A and Type B ARIMA(1,1,2) processes.*

Proof. Observe from Figure (Type A), we know that the distance between λ_2^{ϕ} and λ_1^{θ} is longer than the distance between λ_1^{ϕ} and λ_1^{θ}, and $(\lambda_2^{\phi} - \lambda_1^{\theta}) > (\lambda_1^{\phi} - \lambda_1^{\theta}) > 0$. Similarly, we have $(\lambda_2^{\phi} - \lambda_2^{\theta}) >$

$(\lambda_1^\phi - \lambda_2^\theta) > 0$. Rewriting (15) as

$$p_{t+1} = \frac{(\lambda_2^\phi - \lambda_1^\theta)(\lambda_2^\phi - \lambda_2^\theta) - (\lambda_1^\phi - \lambda_1^\theta)(\lambda_1^\phi - \lambda_2^\theta)(\lambda_1^\phi)^t}{(\lambda_2^\phi - \lambda_1^\phi)}, \tag{60}$$

and $p_0 = 1$, $p_t > 0$ is proved. For Type B, $(\lambda_2^\phi - \lambda_1^\theta)(\lambda_2^\phi - \lambda_2^\theta) > 0$ and $(\lambda_1^\phi - \lambda_1^\theta)(\lambda_1^\phi - \lambda_2^\theta) < 0$. Thus, $p_t > 0$ for Type B. □

Remark 5. *For Type A, Lemma 5.2 holds for positive and negative $-1 < \varphi < 1$; for Type B, $\varphi \geq 0$ is required. A positive demand impluse response means the OUT policy creates a bullwhip effect that always increases in the leadtime, Gaalman et al. (2022).*

Lemma 5.2 also indicates $E(k) > 0$ for Type A and Type B ARIMA(1,1,2) demand. This can be proved directly by $E(k) = \sum_{j=0}^{k} p_j$ and $p_j > 0$. Therefore, we have

Proposition 5.3. *For Type A and Type B ARIMA(1,1,2) demand, the DT-OUT's order impulse response is always positive.*

Proof. When $t = 0$, the order impulse response $p_t^O = E(k+1)$; $p_t^O = (p_t(k+1) > 0)$ which comes from Lemma 5.2. Thus, $\forall t$, $p_t^O > 0$ for Type A and Type B ARIMA(1,1,2) demand. □

For Type A ARIMA(1,1,2) demand, the proportional controller value needs to be carefully tuned based on the eigenvalues of the demand process, in order to reduce bullwhip. As $r_1 < 0$, $E(k) > 0$, it is possible that (59) is negative (that is, it is possible that $CB(k)^P > CB(k)^O$). This means the DT-POUT policy creates more bullwhip than the DT-OUT policy. The DT-POUT policy creates less bullwhip than the DT-OUT policy only when f satisfies

$$\frac{E(k)}{2-f} + \frac{r_1(1-\varphi)(\varphi)^k}{1-(1-f)\varphi} > 0. \tag{61}$$

As $r_2 > |r_1|$, we can derive the sufficient condition for $CB(k)^P < CB(k)^O$ with any $0 < f < 1$:

$$k_{\min} > \frac{\varphi - 2}{\varphi - 1} - \frac{W\left(\varphi^{\frac{1}{1-\varphi}+2} \log(\varphi)(1-\varphi)^{-1}\right)}{\log(\varphi)}. \tag{62}$$

Here $W(\cdot)$ is the Lambert W Function, Disney and Warburton (2012). For Type B ARIMA(1,1,2) demand, $0 < \varphi < 1$ and $0 < f < 1$, we know $1-f > 0$, $2-f > 0$, $1-(1-f)\varphi > 0$, $1-\varphi > 0$, $\varphi^k > 0$, and $r_1 > 0$; we also have shown $E(k) > 0$. Thus, we have:

Proposition 5.4. *$CB(k)^P < CB(k)^O$ for Type B ARIMA(1,1,2) demand when $0 < \varphi < 1$ and $0 < f < 1$.*

There is always less bullwhip in DT-POUT compared with DT-OUT when facing Type B ARIMA(1,1,2) demand when the proportional controller $0 \leq f < 1$.

Lemma 5.5. *For Type F ARIMA(1,1,2) demand with $\varphi > 0$, the demand impulse response is always positive if $1+\varphi > \theta_1$.*

Proof. When $\varphi > 0$, the F_1 case in Gaalman et al. (2022) is present; Gaalman et al. (2022) provides the proof. □

Remark 6. *When $\varphi < 0$, the conditions for a positive demand impluse response in case F is somwhat complex, we refer readers to Gaalman et al. (2022). However, when the demand impluse resposne is always positive the bullwhip effect in the OUT policy always increases in the lead time. The eigenvalue forms of p_t, $E(k)$ and the order impulse response suggest the same condition, $1+\varphi > \theta_1$, determines their sign.*

For Type F ARIMA(1,1,2) demand, when $1+\varphi > \theta_1$, the following relation determines when $CB(k)^P < CB(k)^O$,

$$\frac{E(k)}{2-f} + \frac{r_1(1-\varphi)(\varphi)^k}{1-(1-f)\varphi} > 0. \tag{63}$$

When $1+\varphi < \theta_1$, following relation determines when $CB(k)^P < CB(k)^O$,

$$\frac{E(k)}{2-f} + \frac{r_1(1-\varphi)(\varphi)^k}{1-(1-f)\varphi} < 0. \tag{64}$$

The DT-POUT bullwhip behaviour contradicts much existing bullwhip theory. The literature highlights the contribution of the proportional controller on bullwhip reduction and recommends switching from OUT policy to POUT policies. Our result shows that the POUT policy's order variance is sometimes larger than the OUT policy when demand is non-stationary.

6. Concluding remarks

We have studied the bullwhip in both the OUT and POUT policies with optimal forecasts for ARIMA(1,1,2) demand. We quantify the bullwhip effect for all possible ARIMA(1,1,2) demand processes under the OUT and POUT policies. Previous studies have demonstrated that the bullwhip can be reduced and even eliminated by using the proportional controller. However, our analysis shows that conventional values for the proportional controller f, can sometimes (Type A and Type F) create a larger bullwhip than the DT-OUT policy. For Type B, all vales of $0 < f < 1$ can reduce the bullwhip effect in the DT-POUT policy. Based on the eigenvalues of the demand, we provide the conditions where the DT-POUT policy is able to outperform the DT-OUT policy.

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