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Article

Fourier Series Related to p-Trigonometric Functions

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Abstract: In this paper, we introduce the concept of generalized Fourier series, generated by the p-trigonometric functions, namely cosp and sinp, recently introduced related to the generalized complex numbers systems. The aim of this study is to represent a periodic signal as a sum of p-sine and p-cosine functions. In order to achieve this, we first present the integrals of the product of the same or different family of p-trigonometric functions over the full period of these functions to understand the orthogonality properties. Next, we use these integrals to derive the coefficients of the generalized p-Fourier series along with a few examples. The generalized Fourier series can be used to expand an arbitrary forcing function in the solution of a non-homogeneous linear ordinary differential equation (ODE) with constant coefficients.

Keywords: p-complex number; generalized Fourier series; special functions; differential equations

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1. Introduction

Generalized complex numbers comprise two component numbers of the following form:

$$z = \mu + i\gamma \ (\mu, \gamma \in \mathbb{R}),$$

where [1,2]

$$i^2 = p + iq \ (p, q \in \mathbb{R}).$$

A one-parameter family of generalized complex numbers system is

$$\mathbb{C}_p := \{\mu + i\gamma : \mu; \gamma \in \mathbb{R}; i^2 = p; p \in \mathbb{R}\},$$

which was studied in [3]. When $p < 0$, $\mathbb{C}_p(p < 0)$ is referred to as an elliptical complex number system. For elliptical complex numbers $\xi_1 = \mu_1 + i\gamma_1$ and $\xi_2 = \mu_2 + i\gamma_2 \in \mathbb{C}_p$, addition and multiplication operators are defined by

$$\xi_1 + \xi_2 = (\mu_1 + i\gamma_1) + (\mu_2 + i\gamma_2) = (\mu_1 + \mu_2) + i(\gamma_1 + \gamma_2),$$

and

$$\xi_1\xi_2 = (\mu_1\mu_2 + p\gamma_1\gamma_2) + i(\mu_1\gamma_2 + \mu_2\gamma_1).$$

As it is well known, $\mathbb{C}_p$ is a field under these two operations [3]. On the other hand, the p-magnitude of $\xi = \mu + i\gamma \in \mathbb{C}_p$ is $\|\xi\|_p = \sqrt{\mu^2 - p\gamma^2}$. The unit circle in $\mathbb{C}_p$ is an Euclidean ellipse which is given by the equation $\mu^2 - p\gamma^2 = 1$. Specially, if $p = -1$, this ellipse matches the Euclidean unit circle.

Let $\xi = \mu + i\gamma \in \mathbb{C}_p$, in which the number ξ can be expressed with a position vector, as was observed in [3]. The arc of ellipse between this vector and the real axis determines an elliptic angle θ_p. This angle is called the p-argument of ξ. Generalized complex numbers and elliptical complex numbers in the literature can be found in [4–9] and the references therein. The authors in [3] have introduced, in $\mathbb{C}_p$, the p-trigonometric functions p-cosine, p-sine, and p-tangent as

$$\text{cosp}(\theta_p) = \cos\left(\sqrt{|p|}\theta_p\right), \tag{1}$$

$$\text{sinp}(\theta_p) = \frac{1}{\sqrt{|p|}} \sin\left(\sqrt{|p|}\theta_p\right), \tag{2}$$

$$\text{tanp}(\theta_p) = \frac{\text{sinp}(\theta_p)}{\text{cosp}(\theta_p)}. \tag{3}$$

Recently, in [10], we have introduced the following p-trigonometric functions:

$$\text{cotp}(\theta_p) = \frac{\text{cosp}(\theta_p)}{\text{sinp}(\theta_p)}, \tag{4}$$

$$\text{secp}(\theta_p) = \frac{1}{\text{cosp}(\theta_p)}, \tag{5}$$

$$\text{cosecp}(\theta_p) = \frac{1}{\text{sinp}(\theta_p)}, \tag{6}$$

along with their connection with the p-hyperbolic family of functions.

In [10], we studied some important identities related to bridging the family of p-trigonometric and p-hyperbolic functions, involving p-complex numbers. More details on the concept of p-complex numbers and their connections to p-trigonometric functions can be found in [2–4,11,12]. The extension of these properties to logarithmic functions with complex arguments can be found in [10]. The study of these special functions will also help in the development of unknown properties and identities involving other classes of p-trigonometric series [13].

A generalized Fourier series is a series expansion of a periodic function based on the special properties of a complete orthogonal system of functions [14–17]. The typical example of such a series is the classical Fourier series, which is based on the bi-orthogonality property of trigonometric functions. This can be extended with $\text{cosp}(nx)$ and $\text{sinp}(nx)$ functions, which form a complete bi-orthogonal system under integration over the range of their full period [18,19]. The generalized Fourier series plays a similar role as the classical Fourier series [13,20], with some additional tuning knobs. By expressing a function as a sum of p-sine and p-cosine functions, many complicated real world problems involving these functions become easier to analyze because p-trigonometric functions are not very well understood and applied in real-world data modeling. For example, p-Fourier series can be used to find solutions of some ordinary differential equations (ODEs). With periodic forcing, this application is possible because the derivatives of p-trigonometric functions fall into simpler patterns. The p-Fourier series cannot be used to approximate arbitrary non-periodic functions because most functions have infinitely many terms in their Fourier series, and the series do not always converge.

2. Integration of p-Trigonometric Functions

The integration of generalized p-trigonometric functions involves basic simplification techniques. These techniques use different p-trigonometric identities, which can be written in an alternative form that is more amenable to the list of integrations.

Theorem 1. *Below is the list of few formulas for the integration of trigonometric functions:*

$$\int \text{sinp}(\psi)\, d\psi = \frac{1}{p}\,\text{cosp}(\psi) + C, \tag{7}$$

$$\int \text{cosp}(\psi)\, d\psi = \frac{1}{\sqrt{|p|}}\, \text{sinp}(\psi) + C, \tag{8}$$

$$\int \text{sinp}(\omega\psi)\, d\psi = \frac{1}{p\omega}\, \text{cosp}(\psi) + C, \quad \omega \neq 0, \tag{9}$$

$$\int \text{cosp}(\omega\psi)\, d\psi = \frac{1}{\omega\sqrt{|p|}}\, \text{sinp}(\psi) + C, \quad \omega \neq 0, \tag{10}$$

$$\int \text{tanp}(\psi)\, d\psi = \frac{1}{p}\ln(\text{secp}(\psi)) + C, \tag{11}$$

$$\int \text{cotp}(\psi)\, d\psi = \ln(\text{sinp}(\psi)) + C, \tag{12}$$

$$\int \text{secp}(\psi)\, d\psi = \ln|\text{tanp}(\psi) + \text{secp}(\psi)| + C, \tag{13}$$

$$\int \text{cosecp}(\psi)\, d\psi = \ln|\text{tanp}(\psi) - \text{cotp}(\psi)| + C, \tag{14}$$

$$\int \text{secp}(\psi)^2\, d\psi = \text{tanp}(\psi) + C, \tag{15}$$

$$\int \text{cosecp}(\psi)^2\, d\psi = -\text{cotp}(\psi) + C. \tag{16}$$

Here, C is the constant of integration.

Lemma 1. *(i) If $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is an even function, then*

$$\int_{-a}^{a} \psi(\theta)d\theta = 2\int_0^a \psi(\theta)d\theta \quad \forall\, a > 0. \tag{17}$$

(ii) If $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is an odd function, then

$$\int_{-a}^{a} \psi(\theta)d\theta = 0 \quad \forall\, a > 0. \tag{18}$$

(iii) If $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is a periodic function with period T, then

$$\int_0^T \psi(\theta)d\theta = \int_{-\frac{T}{2}}^{\frac{T}{2}} \psi(\theta)d\theta = \int_{\alpha}^{\beta} \psi(\theta)d\theta \qquad (where, \beta - \alpha = T). \tag{19}$$

Example 1. *(1) Set $\psi(\theta) = \text{sinp}(n\theta)\text{sinp}(k\theta)$. We observe that ψ is even and $\frac{2\pi}{\sqrt{|p|}}$ is periodic. According to statements (i) and (iii), we may write the following:*

$$\int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{sinp}(n\theta)\text{sinp}(k\theta)d\theta = \int_{-\frac{\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \underbrace{\text{sinp}(n\theta)\text{sinp}(k\theta)}_{\frac{2\pi}{\sqrt{|p|}} - periodic} d\theta = 2\int_0^{\frac{\pi}{\sqrt{|p|}}} \underbrace{\text{sinp}(n\theta)\text{sinp}(k\theta)}_{even function} d\theta.$$

(2) Set $\psi(\theta) = \text{cosp}(n\theta)\text{cosp}(k\theta)$. We observe that ψ is even and $\frac{2\pi}{\sqrt{|p|}}$ is periodic. According to statements (i) and (iii), we may write the following:

$$\int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{cosp}(n\theta)\text{cosp}(k\theta)d\theta = \int_{-\frac{\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \underbrace{\text{cosp}(n\theta)\text{cosp}(k\theta)}_{\frac{2\pi}{\sqrt{|p|}} - periodic} d\theta = 2\int_0^{\frac{\pi}{\sqrt{|p|}}} \underbrace{\text{cosp}(n\theta)\text{cosp}(k\theta)}_{even function} d\theta.$$

(3) Set $\psi(\theta) = \text{cosp}(n\theta)\text{sinp}(k\theta)$*. We observe that* ψ *is odd and* $\dfrac{2\pi}{\sqrt{|p|}}$ *is periodic. According to statements (ii), we may write the following:*

$$\int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{cosp}(n\theta)\text{sinp}(k\theta)d\theta = \int_{-\frac{\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \underbrace{\text{cosp}(n\theta)\text{sinp}(k\theta)}_{\frac{2\pi}{\sqrt{|p|}} - \textit{periodic}} d\theta = 0.$$

Based on the above relationships, it is enough to study the orthogonality of p-trigonometric functions on $\left[0, \dfrac{\pi}{\sqrt{|p|}}\right]$. Therefore, their orthogonality on $\left[0, \dfrac{2\pi}{\sqrt{|p|}}\right]$ or on $\left[-\dfrac{\pi}{\sqrt{|p|}}, \dfrac{\pi}{\sqrt{|p|}}\right]$ is an obvious consequence.

Now, we propose the following theorem by establishing some orthogonality properties of sinp and cosp functions, which will be useful later to derive p-Fourier series.

Theorem 2. *Integrals of the product of same p-trigonometric functions:*

The family $\{\text{cosp}(k\psi), \text{sinp}(k\psi);\ k = 1, 2, \cdots,\ \}$ *satisfies the following properties of orthogonality for* $p < 0$*:*

$$\int_0^{\frac{\pi}{\sqrt{|p|}}} \text{sinp}(n\psi)\ .\ \text{sinp}(k\psi)\ d\psi \ = 0 \quad for \quad k \neq n, \tag{20}$$

$$\int_0^{\frac{\pi}{\sqrt{|p|}}} \text{sinp}(n\psi)\ .\ \text{sinp}(n\psi) d\psi \ = \frac{\pi}{2\,|p|\,\sqrt{|p|}}, \tag{21}$$

$$\int_0^{\frac{\pi}{\sqrt{|p|}}} \text{cosp}(n\psi)\ .\ \text{cosp}(k\psi)\ d\psi = 0 \quad for \quad k \neq n, \tag{22}$$

$$\int_0^{\frac{\pi}{\sqrt{|p|}}} \text{cosp}(n\psi)\ .\ \text{cosp}(k\psi)\ d\psi = \frac{\pi}{2\sqrt{|p|}} \quad for \quad k = n. \tag{23}$$

Proof. We establish the identity (20) by means of (8) as

$$\begin{aligned}
& \int_0^{\frac{\pi}{\sqrt{|p|}}} \text{sinp}(n\psi)\ .\ \text{sinp}(k\psi)\ d\psi \\
= \ & \frac{1}{|p|} \int_0^{\frac{\pi}{\sqrt{|p|}}} \sin\left(\sqrt{|p|}\ n\psi\right)\ .\ \sin\left(\sqrt{|p|}\ k\psi\right)\ d\psi \\
= \ & \frac{1}{2\,|p|} \int_0^{\frac{\pi}{\sqrt{|p|}}} \left(\cos\left(\sqrt{|p|}\ (n-k)\ \psi\right) \ - \ \cos\left(\sqrt{|p|}\ (n+k)\ \psi\right)\ \right) d\psi \\
= \ & 0.
\end{aligned}$$

Next, we establish Equation (21) by means of Equation (8) as

$$\begin{aligned}
& \int_0^{\frac{\pi}{\sqrt{|p|}}} \text{sinp}(n\psi)\ .\ \text{sinp}(n\psi) d\psi \\
= \ & \frac{1}{|p|} \int_0^{\frac{\pi}{\sqrt{|p|}}} \sin^2\left(\sqrt{|p|}\ \ n\psi\right) d\psi \\
= \ & \frac{1}{|p|} \int_0^{\frac{\pi}{\sqrt{|p|}}} \frac{1 - \cos(2\ \sqrt{|p|} n\psi)}{2} d\psi \\
= \ & \frac{\pi}{2\,|p|\,\sqrt{|p|}}.
\end{aligned}$$

We then establish Equation (22) by means of Equation (8), as follows:

$$\begin{aligned}
& \int_0^{\frac{\pi}{\sqrt{|p|}}} \operatorname{cosp}(n\psi)\, .\, \operatorname{cosp}(k\psi)\ \ d\psi \\
= & \int_0^{\frac{\pi}{\sqrt{|p|}}} \cos\Big(\sqrt{|p|}\ \ n\psi\Big)\cos\Big(\sqrt{|p|}\, k\psi\Big)\ d\psi \\
= & \ \frac{1}{2}\int_0^{\frac{\pi}{\sqrt{|p|}}} \Big(\cos\Big(\sqrt{|p|}(n+k)\ \psi\Big)\ +\ \cos\Big(\sqrt{|p|}\ (n-k)\ \psi\Big)\ \Big)d\psi \\
= & \ \ 0.
\end{aligned}$$

We then establish Equation (23) by means of Equation (8) as

$$\begin{aligned}
& \int_0^{\frac{\pi}{\sqrt{|p|}}} \operatorname{cosp}(n\psi)\, .\, \operatorname{cosp}(n\psi) d\psi \\
= & \int_0^{\frac{\pi}{\sqrt{|p|}}} \cos\Big(\sqrt{|p|}n\psi\Big)\, .\, \cos\Big(\sqrt{|p|}n\psi\Big)d\psi \\
= & \int_0^{\frac{\pi}{\sqrt{|p|}}} \cos^2\Big(\sqrt{|p|}n\psi\Big)d\psi \\
= & \ \frac{1}{2}\int_0^{\frac{\pi}{\sqrt{|p|}}} \Big(1+\cos\Big(2\sqrt{|p|}\ n\psi\Big)\Big)d\psi \\
= & \ \frac{\pi}{2\ \sqrt{|p|}}.
\end{aligned}$$

□

Corollary 1. *From the above Theorem 2, we immediately derive the following consequence:*

$$\int_0^{\frac{2\pi}{\sqrt{|p|}}} \operatorname{sinp}(n\psi)\, .\, \operatorname{sinp}(k\psi)\ d\psi\ \ = 0 \quad \text{for} \quad k \neq n, \tag{24}$$

$$\int_0^{\frac{2\pi}{\sqrt{|p|}}} \operatorname{sinp}(n\psi)\, .\, \operatorname{sinp}(n\psi) d\psi\ \ = \frac{\pi}{|p|\ \sqrt{|p|}}\ , \tag{25}$$

$$\int_0^{\frac{2\pi}{\sqrt{|p|}}} \operatorname{cosp}(n\psi)\, .\, \operatorname{cosp}(k\psi)\ d\psi = 0 \quad \text{for} \quad k \neq n, \tag{26}$$

$$\int_0^{\frac{2\pi}{\sqrt{|p|}}} \operatorname{cosp}(n\psi)\, .\, \operatorname{cosp}(k\psi)\ d\psi = \frac{\pi}{\sqrt{|p|}} \quad \text{for} \quad k = n. \tag{27}$$

Theorem 3. *Integrals of the product of different p-trigonometric functions:*
The family $\{\operatorname{cosp}(k\psi), \operatorname{sinp}(k\psi);\ k = 1, 2, \cdots,\ \}$ *satisfies the following identities:*

$$\int_0^{\frac{2\pi}{\sqrt{|p|}}} \operatorname{cosp}(n\psi)\, .\, \operatorname{sinp}(kx) d\psi = 0 \qquad \forall\ k\ , n. \tag{28}$$

Proof. We establish Equation (28) by means of Equation (7). Firstly, if $n \neq k$, in accordance with the definitions of cosp and sinp, in terms of standard sin and cos functions within the integration, we may write the following:

$$\begin{aligned} & \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{cosp}(n\psi).\text{sinp}(k\psi)\ d\psi \\ = & \frac{1}{\sqrt{|p|}} \int_0^{\frac{2\pi}{\sqrt{|p|}}} \cos\left(\sqrt{|p|}\ n\psi\right).\sin\left(\sqrt{|p|}\ k\psi\right) d\psi \\ = & \frac{1}{2\sqrt{|p|}} \int_0^{\frac{2\pi}{\sqrt{|p|}}} \sin\left(\sqrt{|p|}\ (n+k)\ \psi\right) + \sin\left(\sqrt{|p|}(n-k)\ \psi\right) d\psi \\ = & \frac{1}{2} \int_0^{\frac{2\pi}{\sqrt{|p|}}} \Big(\text{sinp}(\ (n+k)\ \psi) + \text{sinp}((n-k)\ \psi)\Big)\ d\psi \\ = & 0. \end{aligned}$$

Secondly, if $n = k$, we have

$$\begin{aligned} \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{cosp}(n\psi)\ .\ \text{sinp}(n\psi) &= \int_0^{\frac{2\pi}{\sqrt{|p|}}} \cos\left(\sqrt{|p|}\ n\psi\right).\frac{\sin\left(\sqrt{|p|}\ n\psi\right)}{\sqrt{|p|}}\ d\psi \\ &= \frac{-1}{\sqrt{|p|}\ n} \int_0^{\frac{2\pi}{\sqrt{|p|}}} \cos\left(\sqrt{|p|}\ n\psi\right).\left(\cos\left(\sqrt{|p|}\ n\psi\right)\right)'\ d\psi \\ &= 0. \end{aligned}$$

□

3. Generalized *p*-Fourier Series

In this section, we introduce the concept of generalized Fourier series. The coefficients of Fourier series are determined by integrals of the function multiplied by p-trigonometric functions, which are described in classical forms of the Fourier series [14–17].

Definition 1. *Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be a $\frac{2\pi}{\sqrt{|p|}}$-periodic function; thus, the generated p-Fourier series is given by*

$$\omega_0 + \sum_{n=1}^{\infty} \Big(\omega_n\ \text{cosp}(n\psi) + \chi_n\ \text{sinp}(n\psi)\Big). \tag{29}$$

Theorem 4. *If T is a $\frac{2\pi}{\sqrt{|p|}}$-periodic and continuous function, then the Fourier coefficients are given by*

$$\omega_0 = \frac{\sqrt{|p|}}{2\pi} \int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi) d\psi, \tag{30}$$

$$\omega_n = \frac{\sqrt{|p|}}{\pi} \int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\ \text{cosp}(n\psi) d\psi \qquad ,\ n\ \geq 1, \tag{31}$$

$$\chi_n = \frac{|p|\ \sqrt{|p|}}{\pi} \int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\ \text{sinp}(n\psi) d\psi \qquad ,\ n\ \geq 1. \tag{32}$$

Proof. According to the equality

$$T(\psi) = \omega_0 + \sum_{n=1}^{\infty} \Big(\omega_n\ \text{cosp}(n\psi) + \chi_n\ \text{sinp}(n\psi)\Big)$$

and Equations (7) and (8), we have

$$\begin{aligned}\int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)d\psi &= \omega_0 \int_0^{\frac{2\pi}{\sqrt{|p|}}} d\psi + \sum_{n=1}^{\infty} \omega_n \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{cosp}(n\psi)d\psi + \chi_n \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{sinp}(n\psi)\ d\psi \\ &= \omega_0 \frac{2\pi}{\sqrt{|p|}}.\end{aligned}$$

Therefore, $\omega_0 = \dfrac{\sqrt{|p|}}{2\pi} \displaystyle\int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)d\psi$.

According to

$$T(\psi)\ .\ \text{cosp}(n\psi) = \omega_0\ \text{cosp}(nx) + \sum_{k=1}^{\infty} \omega_k\ \text{cosp}(k\psi)\ .\text{cosp}(n\psi) + \chi_k\ \text{sinp}(k\psi)\ .\ \text{cosp}(n\psi)$$

and Equations (8), (22), (23), and (28), we have

$$\begin{aligned}&\int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\ \text{cosp}(n\psi)d\psi \\ =\ &\int_0^{\frac{2\pi}{\sqrt{|p|}}} \omega_0\text{cosp}(n\psi)d\psi + \sum_{k=1}^{\infty} a_k \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{cosp}(k\psi).\text{cosp}(n\psi)d\psi + \chi_n \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{sinp}(k\psi).\text{cosp}(n\psi)d\psi \\ =\ &0 + \omega_n \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{cosp}(n\psi).\ \text{cosp}(n\psi)d\psi \\ =\ &\omega_n\ .\frac{\pi}{\sqrt{|p|}}.\end{aligned}$$

Hence,

$$\omega_n = \frac{\sqrt{|p|}}{\pi} \int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\ \text{cosp}(n\psi)d\psi.$$

According to equality

$$T(\psi)\ .\ \text{sinp}(n\psi) = \omega_0\ \text{sinp}(n\psi) + \sum_{k=1}^{\infty} \omega_k\ \text{cosp}(k\psi)\ .\text{sinp}(n\psi) + \chi_n\ \text{sinp}(k\psi)\ .\ \text{sinp}(n\psi)$$

and Equations (7), (20), (21), and (28), we have

$$\begin{aligned}&\int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\ \text{sinp}(n\psi)\ d\psi \\ =\ &\omega_0 \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{sinp}(n\psi)\ d\psi + \sum_{k=1}^{\infty} \omega_k \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{cosp}(k\psi).\text{sinp}(n\psi)d\psi \\ &+\chi_k \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{sinp}(k\psi).\text{sinp}(n\psi)d\psi \\ =\ &0\ + 0\ + \chi_n \int_0^{\frac{2\pi}{\sqrt{|p|}}} \text{sinp}(n\psi)\ .\ \text{sinp}(n\psi)\ d\psi \\ =\ &\chi_n \frac{\pi}{|p|\sqrt{|p|}}.\end{aligned}$$

Therefore,

$$\chi_n = \frac{|p|\ \sqrt{|p|}}{\pi} \int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\ \text{sinp}(n\psi)\ d\psi.$$

□

Remark 1. *When $p = -1$, Equations (30), (31), and (32) are reduced to*

$$\omega_0 = \frac{1}{2\pi}\int_0^{2\pi} T(\psi)d\psi, \tag{33}$$

$$\omega_n = \frac{1}{\pi}\int_0^{2\pi} T(\psi)\cos(n\psi)\,d\psi \qquad , n \geq 1, \tag{34}$$

$$\chi_n = \frac{1}{\pi}\int_0^{2\pi} T(\psi)\sin(n\psi)\,d\psi \qquad , n \geq 1. \tag{35}$$

Example 2. *Find the p-generalized Fourier series of the function*

$$T(\psi) = \begin{cases} 1 & \text{if} \quad 0 \leq \psi \leq \frac{\pi}{\sqrt{|p|}}, \\ 0 & \text{if} \quad \frac{\pi}{\sqrt{|p|}} \leq \psi \leq \frac{2\pi}{\sqrt{|p|}}. \end{cases}$$

Solution: T generated a p-generalized Fourier series, which is given by

$$\omega_0 + \sum_{k=1}^{\infty} \omega_k \operatorname{cosp}(k\psi) + \chi_k \operatorname{sinp}(k\psi),$$

where

$$\begin{aligned} \omega_0 &= \frac{\sqrt{|p|}}{2\pi}\int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)d\psi \\ &= \frac{\sqrt{|p|}}{2\pi}\left(\int_0^{\frac{\pi}{\sqrt{|p|}}} T(\psi)d\psi + \int_{\frac{\pi}{\sqrt{|p|}}}^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)d\psi\right) \\ &= \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} \omega_n &= \frac{\sqrt{|p|}}{\pi}\int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\,.\operatorname{cosp}(n\psi)\,d\psi \qquad , n \geq 1 \\ &= \frac{\sqrt{|p|}}{\pi}\left(\int_0^{\frac{\pi}{\sqrt{|p|}}} T(\psi)\,.\operatorname{cosp}(n\psi)\,d\psi + \int_{\frac{\pi}{\sqrt{|p|}}}^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\,.\operatorname{cosp}(n\psi)\right) d\psi \\ &= 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \chi_n &= \frac{|p|\sqrt{|p|}}{\pi}\int_0^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\,.\operatorname{sinp}(n\psi)\,d\psi \;, \qquad n \geq 1 \\ &= \frac{|p|\sqrt{|p|}}{\pi}\int_0^{\frac{\pi}{\sqrt{|p|}}} T(\psi)\,.\operatorname{sinp}(n\psi)\,d\psi + \int_{\frac{\pi}{\sqrt{|p|}}}^{\frac{2\pi}{\sqrt{|p|}}} T(\psi)\,.\operatorname{sinp}(n\psi)\,d\psi \\ \chi_n &= \frac{-\sqrt{|p|}}{n\pi}\Big((-1)^n - 1\Big). \end{aligned}$$

The series is given by

$$\frac{1}{2} + \frac{2\sqrt{|p|}}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\operatorname{sinp}\Big((2n+1)\psi\Big).$$

Theorem 5. *Let T be a $\frac{2\pi}{\sqrt{|p|}}$-periodic and continuous function and*

$$T(\psi) = \omega_0 + \sum_{k=1}^{\infty} \Big(\omega_k \operatorname{cosp}(k\psi) + \chi_k \operatorname{sinp}(k\psi) \Big).$$

Then, the generalized Parseval's identity holds, as follows:

$$\frac{\sqrt{|p|}}{2\pi} \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)^2 \, d\psi = \omega_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} \Big(\omega_k^2 + \frac{1}{|p|} \chi_k^2 \Big) . \tag{36}$$

Proof. Assume that

$$T(\psi) \approx \omega_0 + \sum_{k=1}^{N} \Big(\omega_k \operatorname{cosp}(k\psi) + \chi_k \operatorname{sinp}(k\psi) \Big),$$

and let

$$F(\psi) = A_0 + \sum_{k=1}^{N} \omega_k \operatorname{cosp}(k\psi) + b_k \operatorname{sinp}(k\psi).$$

Set

$$E = \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \Big(T(\psi) - F(\psi) \Big)^2 d\psi \, .$$

We have

$$E = \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \Big(T(\psi) \Big)^2 d\psi \, - 2 \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} F(\psi).T(\psi) \, d\psi \, + \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \Big(F(\psi) \Big)^2 d\psi \, .$$

We observe that

$$\begin{aligned}
&\Bigg(\Bigg(A_0 + \sum_{k=1}^{N} \Big(A_k \operatorname{cosp}(k\psi) + B_k \operatorname{sinp}(k\psi) \Big) \Bigg) \\
&. \Bigg(A_0 + \sum_{k=1}^{N} \Big(A_k \operatorname{cosp}(k\psi) + B_k \operatorname{sinp}(k\psi) \Big) \Bigg) \Bigg) \\
= \;& A_0^2 + 2A_0 \sum_{k=1}^{N} \Big(A_k \operatorname{cosp}(k\psi) + B_k \operatorname{sinp}(k\psi) \Big) \\
&+ \sum_{j=1}^{N} A_j \Bigg(\sum_{k=1}^{N} \big(A_k \operatorname{cosp}(j\psi)\operatorname{cosp}(k\psi) + B_k \operatorname{cosp}(j\psi)\operatorname{sinp}(k\psi) \big) \Bigg) \\
&+ \sum_{j=1}^{N} B_j \Bigg(\sum_{k=1}^{N} \big(A_k \operatorname{sinp}(j\psi)\operatorname{cosp}(k\psi) + B_k \operatorname{sinp}(j\psi)\operatorname{sinp}(k\psi) \big) \Bigg).
\end{aligned}$$

By taking into account Equations (20), (21), (22), (23), and (28), we get

$$\begin{aligned}
\int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \Big(F(\psi) \Big)^2 d\psi &= \frac{2\pi}{\sqrt{|p|}} .A_0^2 + \sum_{k=1}^{N} \Bigg(\frac{\pi}{\sqrt{|p|}} (A_k)^2 + \frac{\pi}{|p| \sqrt{|p|}} (B_k)^2 \Bigg) \\
&= \frac{2\pi}{\sqrt{|p|}} \Bigg(A_0^2 + \frac{1}{2} \sum_{k=1}^{N} \big(A_k^2 + \frac{1}{|p|} B_k^2 \big) \Bigg).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& F(\psi)T(\psi) \\
= & \left(\left(A_0+\sum_{k=1}^{N}\left(A_k \operatorname{cosp}(k\psi)+B_k \operatorname{sinp}(k\psi)\right)\right).\left(\omega_0+\sum_{k=1}^{N}\left(\omega_k \operatorname{cosp}(k\psi)+\xi_k \operatorname{sinp}(k\psi)\right)\right)\right) \\
= & A_0\omega_0 + A_0\sum_{k=1}^{N}\left(\omega_k \operatorname{cosp}(k\psi)+\chi_k \operatorname{sinp}(k\psi)\right)+\omega_0\sum_{k=1}^{N}\left(A_k \operatorname{cosp}(k\psi)+B_k \operatorname{sinp}(k\psi)\right) \\
& +\sum_{j=1}^{N}A_j\left(\sum_{k=1}^{N}\left(\omega_k \operatorname{cosp}(j\psi)\operatorname{cosp}(k\psi)+\chi_k \operatorname{cosp}(j\psi)\operatorname{sinp}(k\psi)\right)\right) \\
& +\sum_{j=1}^{N}B_j\left(\sum_{k=1}^{N}\left(\omega_k \operatorname{sinp}(j\psi)\operatorname{cosp}(k\psi)+\chi_k \operatorname{sinp}(j\psi)\operatorname{sinp}(k\psi)\right)\right).
\end{aligned}
$$

By taking into account Equations (20), (21), (22), (23), and (28), we get

$$
\begin{aligned}
\int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)\,.\,F(\psi)d\psi &= \frac{2\pi}{\sqrt{|p|}}\,A_0\,\omega_0+\sum_{k=1}^{N}\left(\frac{\pi}{\sqrt{|p|}}\,A_k\omega_k\,+\frac{\pi}{|p|\,\sqrt{|p|}}\,B_k\,\chi_k\right) \\
&= \frac{\pi}{\sqrt{|p|}}\left(2A_0\omega_0+\sum_{k=1}^{N}\left(A_k\omega_k+\frac{1}{|p|}B_k\chi_k\right)\right).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
E = \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)^2 d\psi - \frac{2\pi}{\sqrt{|p|}}\left(2A_0\omega_0+\sum_{k=1}^{N}\left(A_k\omega_k+\frac{1}{|p|}\,B_k\chi_k\right)+\frac{2\pi}{\sqrt{|p|}}A_0^2\right. \\
\left.+\frac{\pi}{\sqrt{|p|}}\sum_{k=1}^{N}(A_k^2+\frac{1}{|p|}B_k^2)\right).
\end{aligned}
$$

By taking $A_k = \omega_k$ and $B_k = \chi_k$, we get

$$
E^* = \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)^2 - \frac{\pi}{\sqrt{|p|}}\left(2\,\omega_0^2\;+\sum_{k=1}^{N}\omega_k^2+\frac{1}{|p|}\,\chi_k^2\right),
$$

$$
E - E^* = \frac{\pi}{\sqrt{|p|}}\left(2(A_0-\omega_0)^2+\sum_{k=1}^{N}\left(A_k-\omega_k\right)^2+\frac{1}{|p|}\left(B_k-\chi_k\right)^2\right),
$$

$E - E^* \geq 0$ or $E \geq E^*$ and $E = E^*$ if and only if $A_k = \omega_k$ and $B_k = \chi_k$.

Since E^* is positive, we get

$$
2\,\omega_0^2\;+\sum_{k=1}^{N}\left(\omega_k^2+\frac{1}{|p|}\,\chi_k^2\right)\;\leq\;\frac{\sqrt{|p|}}{\pi}\;\int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)^2 d\psi.
$$

If $N \to \infty$, we get Bessel's inequalities, as follows:

$$
2\,\omega_0^2+\sum_{k=1}^{\infty}\left(\omega_k^2+\frac{1}{|p|}\,\chi_k^2\right)\;\leq\;\frac{\sqrt{|p|}}{\pi}\;\int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)^2 d\psi\,.
$$

If

$$
2\,\omega_0^2+\sum_{k=1}^{\infty}\left(\omega_k^2+\frac{1}{|p|}\,\chi_k^2\right)\;=\;\frac{\sqrt{|p|}}{\pi}\;\int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)^2\,d\psi,
$$

we get the generalized Parseval's identity. □

Example 3. *Consider T to be a $\frac{2\pi}{\sqrt{|p|}}$-periodic and continuous function, such that*

$$T(\psi) = \psi + \frac{\pi}{\sqrt{|p|}}, \; \frac{-\pi}{\sqrt{|p|}} \leq \psi \leq \frac{\pi}{\sqrt{|p|}}.$$

We have

$$E^* = \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)^2 d\psi - \frac{\pi}{\sqrt{|p|}} \left(2\,\omega_0^2 + \sum_{k=1}^{N} \left(\omega_k^2 + \frac{1}{|p|}\,\chi_k^2 \right) \right).$$

Solution:

$$\begin{aligned} \omega_0 &= \frac{\sqrt{|p|}}{2\pi} \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi) d\psi \; = \frac{\sqrt{|p|}}{2\pi} \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \left(\psi + \frac{\pi}{\sqrt{|p|}} \right) d\psi \\ &= \frac{\pi}{\sqrt{|p|}}. \end{aligned}$$

$$\begin{aligned} \omega_n &= \frac{\sqrt{|p|}}{\pi} \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)\,.\,\mathrm{cosp}(n\psi) d\psi \\ &= \frac{\sqrt{|p|}}{\pi} \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \left(\psi + \frac{\pi}{\sqrt{|p|}} \right) .\,\mathrm{cosp}(n\psi) d\psi \\ &= \frac{2}{\sqrt{|p|}} \sin(n\pi). \end{aligned}$$

Hence, $\omega_n = 0, \;\; n \;\; \geq 1$.

On the other hand,

$$\begin{aligned} \chi_n &= \frac{|p|\,\sqrt{|p|}}{\pi} \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)\,\mathrm{sinp}(n\psi) d\psi \\ &= \frac{|p|\,\sqrt{|p|}}{\pi} \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \left(\psi + \frac{\pi}{\sqrt{|p|}} \right) .\,\mathrm{sinp}(n\psi)\, d\psi \\ &= \frac{|p|\sqrt{|p|}}{\pi} \left(\frac{2\pi}{\sqrt{|p|}} (-1)^n \right) \\ &= \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)^2\, d\psi &= \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} \left(\psi + \frac{\pi}{\sqrt{|p|}} \right)^2 d\psi \\ &= \frac{8\pi^3}{3\,|p|\,\sqrt{|p|}}. \end{aligned}$$

From the above calculation, we get

$$E^* = \frac{8\,\pi^3}{3|p|\sqrt{|p|}} - \frac{\pi}{\sqrt{|p|}} \left(2\frac{\pi^2}{|p|} + \frac{4}{|p|} \sum_{k=1}^{N} \frac{1}{k^2} \right).$$

4. *p*-Fourier Series Solutions of Ordinary Differential Equations

Consider the second-order non-homogeneous differential equation:

$$\mathcal{Z}'' + \alpha\mathcal{Z}' + \beta\mathcal{Z} = T(x), \qquad \frac{-\pi}{\sqrt{|p|}} \leq \psi \leq \frac{\pi}{\sqrt{|p|}}, \tag{37}$$

where T satisfies the following conditions:

(1) T is $\frac{2\pi}{\sqrt{|p|}}$-periodic,

(2) T and T' are piecewise continuous differentiable of $\left[\frac{-\pi}{\sqrt{|p|}}, \frac{\pi}{\sqrt{|p|}}\right]$,

(3) α and β are constant with $\beta \neq 0$.

We are interested in finding a solution of the differential equation, which is $\frac{2\pi}{\sqrt{|p|}}$-periodic, which we shall denote by $\mathcal{Z}_p$ and which is of the following form:

$$\mathcal{Z}_p(\psi) = A_0 + \sum_{n=1}^{\infty}\big(A_n \operatorname{cosp}(n\psi) + B_n \operatorname{sinp}(n\psi)\big).$$

Proof.

$$\mathcal{Z}_p(\psi) = A_0 + \sum_{n=1}^{\infty}\big(A_n \operatorname{cosp}(n\psi) + B_n \operatorname{sinp}(n\psi)\big).$$

We have

$$\mathcal{Z}_p'(\psi) = \sum_{n=1}^{\infty}\big(A_n\, pn \operatorname{sinp}(n\psi) + B_n\, n \operatorname{cosp}(n\psi)\big),$$

and

$$\mathcal{Z}_p''(x) = \sum_{n=1}^{\infty}\big(A_n\, pn^2 \operatorname{cosp}(n\psi) + B_n\, pn^2 \operatorname{sinp}(n\psi)\big).$$

A straight forward calculation gives

$$\begin{aligned}\mathcal{Z}_p''(\psi) + \alpha\mathcal{Z}_p'(\psi) + \beta\mathcal{Z}_p(x) = \beta A_0 + \sum_{n=1}^{\infty}\Big(A_n\, pn^2 + \alpha B_n\, n + \beta A_n\Big)&\operatorname{cosp}(n\psi)\\ +(B_n\, pn^2 + \alpha A_n\, pn + \beta B_n) \operatorname{sinp}&(n\psi).\end{aligned}$$

On the other hand, we may write the following:

$$T(\psi) = \omega_0 + \sum_{n=1}^{\infty}\big(\omega_n \operatorname{cosp}(n\psi) + \chi_n \operatorname{sinp}(n\omega)\big).$$

From the differential equation

$$\mathcal{Z}_p''(\psi) + \alpha\mathcal{Z}_p'(\psi) + \beta\mathcal{Z}_p(\psi) = T(\psi),$$

we get $\beta\, A_0 = \omega_0$ and so

$$\rightarrow A_0 = \frac{\omega_0}{\beta}. \tag{38}$$

$$(pn^2 + \beta)A_n + \alpha n\, B_n = \omega_n \qquad (i)$$

and

$$\alpha\, pn\, A_n + (pn^2 + \beta)B_n = \chi_n \qquad (ii).$$

Multiply Equation (i) by $(pn^2 + \beta)$ and Equation (ii) by $n\alpha$ and substitute; thus, we get

$$\big((pn^2 + \beta)^2 - (\alpha^2\, pn)\big)A_n = (pn + \beta)\omega_n - \alpha n\, \chi_n$$

$$\Longrightarrow\ A_n = \frac{(pn+\beta)\omega_n - \alpha n\,\chi_n}{(pn^2+\beta)^2 - \alpha^2\,pn}. \tag{39}$$

Multiply Equation (i) by αp and Equation (ii) by $(pn^2+\beta)$ and substitute; thus, we get

$$B_n = \frac{(pn^2+\beta)\chi_n - \alpha p\,\omega_n}{(pn^2+\beta)^2 - \alpha^2\,pn}. \tag{40}$$

□

Example 4. *Solve the differential equation*

$$\mathcal{Z}'' + 3\mathcal{Z} = T(\psi) \tag{41}$$

where T is a $\frac{2\pi}{\sqrt{|p|}}$*-periodic and continuous function given by*

$$T(\psi) = \begin{cases} 1 & \text{if} \quad \frac{-\pi}{\sqrt{|p|}} \le \psi \le 0 \\ 2 & \text{if} \quad 0 < \psi < \frac{\pi}{\sqrt{|p|}}. \end{cases}$$

Solution: We find the p-Fourier series of T as

$$T(\psi) = \omega_0 + \sum_{n=1}^{\infty}\Big(\omega_n\,\mathrm{cosp}(n\psi) + \chi_n\,\mathrm{sinp}(n\psi)\Big).$$

We have

$$\begin{aligned}
\omega_0 &= \frac{\sqrt{|p|}}{2\pi}\int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} T(\psi)d\psi \\
&= \frac{\sqrt{|p|}}{2\pi}\left[\int_{\frac{-\pi}{\sqrt{|p|}}}^{0} T(\psi)d\psi + \int_0^{\frac{\pi}{\sqrt{|p|}}} T(\psi)d\psi\right] \\
&= \frac{\sqrt{|p|}}{2\pi}\left(\frac{\pi}{\sqrt{|p|}} + \frac{2\pi}{\sqrt{|p|}}\right) \\
&= \frac{3}{2},
\end{aligned}$$

$$\begin{aligned}
\omega_n &= \frac{\sqrt{|p|}}{\pi}\int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} f(\psi)\mathrm{cosp}(nx)d\psi \\
&= \frac{\sqrt{|p|}}{\pi}\left[\int_{\frac{-\pi}{\sqrt{|p|}}}^{0} \mathrm{cosp}(n\psi)d\psi + 2\int_0^{\frac{\pi}{\sqrt{|p|}}} \mathrm{cosp}(n\psi)d\psi\right] \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned} \chi_n &= \frac{|p|\sqrt{|p|}}{\pi} \int_{\frac{-\pi}{\sqrt{|p|}}}^{\frac{\pi}{\sqrt{|p|}}} f(\psi)\mathrm{sinp}(n\psi)d\psi \\ &= \frac{|p|\sqrt{|p|}}{\pi} \left[\int_{\frac{-\pi}{\sqrt{|p|}}}^{0} \mathrm{sinp}(n\psi)dx + 2\int_0^{\frac{\pi}{\sqrt{|p|}}} \mathrm{sinp}(n\psi)d\psi \right] \\ &= \frac{\sqrt{|p|}}{\pi n}\Big(1-(-1)^n\Big). \end{aligned}$$

We get

$$\chi_{2n} = 0 \quad \text{and} \quad \chi_{2n-1} = \frac{2\sqrt{|p|}}{\pi(2n-1)}.$$

Consider that

$$\mathcal{Z}(\psi) = A_0 + \sum_{n=1}^{\infty} \Big(A_n \,\mathrm{cosp}(n\psi) + B_n \,\mathrm{sinp}(n\psi) \Big).$$

According to Equations (38), (39), and (40), we obtain

$$\begin{cases} A_0 = \frac{\omega_0}{\beta}, \\ A_n = \dfrac{(pn+\beta)\omega_n - \alpha n\, \chi_n}{(pn^2+\beta)^2 - \alpha^2\, pn}, \\ B_n = \dfrac{(pn^2+\beta)\chi_n - \alpha p\, \omega_n}{(pn^2+\beta)^2 - \alpha^2\, pn}. \end{cases}$$

By taking into account that $\alpha = 0, \beta = 3, \omega_0 = \dfrac{3}{2}, \omega_n = 0\ n \geq 1,\ \chi_{2n} = 0$, and $\chi_{2n-1} = \dfrac{2\sqrt{|p|}}{\pi(2n-1)},\ n \geq 1$, we get

$$\begin{cases} A_0 = \frac{1}{2}, \\ A_n = 0 \quad n \geq 1, \\ B_{2n} = 0 \quad n \geq 1, \\ B_{2n-1} = \dfrac{1}{(pn^2+3)} \dfrac{2\sqrt{|p|}}{\pi(2n-1)}. \end{cases}$$

Thus, we can write the solution as

$$\mathcal{Z}(\psi) = \frac{1}{2} + \sum_{n\geq 1} \left(\frac{1}{(pn^2+3)} \frac{2\sqrt{|p|}}{\pi(2n-1)} \right) \mathrm{sinp}\big((2n-1)\psi\big).$$

5. Conclusions

We provide the definition of the generalized p-Fourier series and then study some of its properties, involving the sinp and cosp functions. We also present examples of solutions to some non-homogeneous differential equations using the proposed p-Fourier series. It is well known that sinp and cosp functions are capable of modeling damped oscillations and complex periodic shapes, similar to the Bessel and other special functions. This study was motivated by wavelet analysis, where basis functions represent local periodic fluctuations, as opposed to the infinitely long basis functions like the classical sin and cos functions.

This type of analysis is particularly useful for representing non-stationary functions with a complex geometric shape.

We started this investigation based on the foundational works on p-trigonometric functions, which are a significant contribution in the study of special trigonometric functions for modeling real-world phenomena. They are generalized versions of classical trigonometric functions for elliptic complex numbers, and only one special case with $p = -1$ coincides with standard trigonometric functions. The geometric nature of these special functions is different, exhibiting interesting kinds of oscillations, as revealed in [21]. Although the results presented here may seem to be similar to the classical Fourier series analysis, the p-Fourier series are able to model signals with different kinds of flexible basis functions beyond classical trigonometric functions.

In the future, we will validate this work using numerical simulations of stationary and non-stationary even and odd functions. However, the use of special functions, like Bessel, Legendre, etc., within Fourier series is an established method in obtaining analytical solutions of ODEs for boundary value problems (BVPs) and partial differential equations (PDEs) on certain geometric systems, which can be extended using the new classes of sinp and cosp special functions in the future.

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References

1. Yaglom, I.M. *A Simple Non-Euclidean Geometry and Its Physical Basis: An Elementary Account of Galilean Geometry and the Galilean Principle of Relativity*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2012.
2. Yaglom, I.M. *Complex Numbers in Geometry*; Academic Press: Cambridge, MA, USA, 2014.
3. Harkin, A.A.; Harkin, J.B. Geometry of generalized complex numbers. *Math. Mag.* **2004**, *77*, 118–129. [CrossRef]
4. Özen, K.E.; Tosun, M. p-Trigonometric approach to elliptic biquaternions. *Adv. Appl. Clifford Algebr.* **2018**, *28*, 62. [CrossRef]
5. Özen, K.E.; Tosun, M. Elliptic matrix representations of elliptic biquaternions and their applications. *Int. Electron. J. Geom.* **2018**, *11*, 96–103. [CrossRef]
6. Erişir, T.; Güngör, M.A. The Holditch-type theorem for the polar moment of inertia of the orbit curve in the generalized complex plane. *Adv. Appl. Clifford Algebr.* **2016**, *26*, 1179–1193. [CrossRef]
7. Erişir, T.; Güngör, M.A. Holditch-Type Theorem for Non-Linear Points in Generalized Complex Plane Cp. *Univers. J. Math. Appl.* **2018**, *1*, 239–243. [CrossRef]
8. Gürses, N.; Yüce, S. One-parameter planar motions in generalized complex number plane CJ. *Adv. Appl. Clifford Algebr.* **2015**, *25*, 889–903. [CrossRef]
9. Eren, K.; Ersoy, S. Burmester theory in Cayley–Klein planes with affine base. *J. Geom.* **2018**, *109*, 45. [CrossRef]
10. Alibrahim, A.H.; Das, S. Bridging the p-Special Functions between the Generalized Hyperbolic and Trigonometric Families. *Mathematics* **2024**, *12*, 1242. [CrossRef]
11. Özen, K.E. On the trigonometric and p-trigonometric functions of elliptical complex variables. *Commun. Adv. Math. Sci.* **2020**, *3*, 143–154. [CrossRef]
12. Zygmund, A. *Trigonometric Series*; Cambridge University Press: Cambridge, UK, 2002; Volume 1.
13. Trigub, R.M. Summability of trigonometric Fourier series at-points and a generalization of the Abel–Poisson method. *Izv. Math.* **2015**, *79*, 838. [CrossRef]
14. Bary, N.K. *A Treatise on Trigonometric Series: Volume 1*; Elsevier: Amsterdam, The Netherlands, 2014; Volume 1.
15. Folland, G.B. *Fourier Analysis and Its Applications*; American Mathematical Society: Providence, RI, USA, 2009; Volume 4.

16. Rønning, F. The role of Fourier series in mathematics and in signal theory. *Int. J. Res. Undergrad. Math. Educ.* **2021**, *7*, 189–210. [CrossRef]
17. Grafakos, L. *Classical and Modern Fourier Analysis*; Springer: Berlin/Heidelberg, Germany, 2008; Volume 2.
18. Hardy, G.H.; Rogosinski, W. *Fourier Series*; Courier Corporation: North Chelmsford, MA, USA, 1999; Volume 1.
19. Bôcher, M. Introduction to the theory of Fourier's series. *Ann. Math.* **1906**, *7*, 81–152. [CrossRef]
20. Trigub, R.M.; Belinsky, E.S. *Fourier Analysis and Approximation of Functions*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2012.
21. Alibrahim, A.H.; Das, S. The forgotten p-versine and p-coversine family of functions revisited. *PLoS ONE* **2024**, *19*, e0308529. [CrossRef] [PubMed]