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Decentralising mathematics: Mutual development of spontaneous and mathematical concepts via informal reasoning

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Abstract
This paper aims to shed light on an overlooked but essential aspect of informal reasoning and its radical implication to mathematics education research: Decentralising mathematics. We start to problematise that previous studies on informal reasoning implicitly overfocus on what students infer. Based on Walton's distinction between reasoning and argument, and Ernest's concept of intrapersonal dialogue, we propose two theoretical perspectives for understanding the roles of informal reasoning in argumentation: the semi-formal, and the negotiation perspectives. From the latter perspective, we can say that informal reasoning involves creating alternatives, eschewing the relatively unpromising ones, and choosing the most promising one. To illustrate the advantage of the negotiation perspective over the semi-formal perspective, we present two examples of students' statistical written reports from a previous study. These examples illustrate that spontaneous concepts influenced the students' creation of multiple alternatives, and choice of the most promising one, in informal reasoning. Therefore, to better understand the development of mathematical concepts, we need to recognise the role of spontaneous concepts through decentralising mathematics. Finally, we introduce inferentialism as an additional theoretical perspective for investigating both the mathematical development of spontaneous concepts, and the spontaneous development of mathematical concepts. The inferentialist idea of the game of giving and asking for reasons indicates how to empirically investigate the mutual development of spontaneous and mathematical concepts.

Keywords Decentralising mathematics · Inferentialism · Spontaneous concepts · Informal inferential reasoning in statistics · Negotiation

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1 Introduction

One of the primary concerns in statistics education research is informal reasoning, which is a process of making generalisations beyond the data at hand with informal statistical knowledge (Makar & Rubin, 2009; Zieffler et al., 2008). Previous studies have recommended that statistics education should start from experiences of informal reasoning within contexts, e.g., from the standpoint of a fish farmer who wants bigger fishes (Bakker & Derry, 2011) or in the experiment of determining which is better, short-wing or long-wing paper helicopters (Doerr et al., 2017; Schindler & Seidouvy, 2019).

One of the theoretical developments of informal reasoning started from a constructivist view of learning, which emphasises students' informal contextual knowledge (Zieffler et al., 2008). Students can engage in different thinking processes depending on their informal contextual knowledge and reach different conclusions (Pfannkuch, 2011). Kazak et al. (2023) show that some students tend to make inferences based on informal contextual knowledge rather than statistical knowledge with data when the data does not provide clear patterns. Constructivist researchers have consistently emphasised the variety of students' inferences. However, many empirical studies on inferential reasoning often do not explicitly mention the learning theories they are based on (Nilsson et al., 2018). This focus on diverse student inferences is not exclusive to Piagetian constructivists; it is also crucial for Vygotskian researchers, who seek a holistic understanding of the relationship between statistical and spontaneous concepts (Bakker & Derry, 2011).

Recently, there has been a line of mathematics education research on inferential reasoning from a social perspective (Bakker & Derry, 2011; Schindler & Seidouvy, 2019; Seidouvy et al., 2019). Informal contextual knowledge and social norms influence students' inferential reasoning (e.g., students may think the answer to a statistical question must be a single value rather than a confidence interval) (Schindler & Seidouvy, 2019). Previous studies gradually reveal what factors determine students' informal reasoning processes. However, we should also investigate *why students eschew certain claims* within mathematical activities. Here, by using the word 'eschew', we mean to avoid or abstain from something intentionally. Informal reasoning is a *choice* of the most plausible one from multiple potential alternatives in Morgan's (1996, 2006) sense: "Whenever an utterance is made, the speaker or writer makes choices (not necessarily consciously) between alternative structures and contents" (Morgan, 1996, p. 3). The result of informal reasoning can vary depending on what alternatives students create. We can conjecture that factors like informal contextual knowledge and social norms influence not only choosing a plausible conclusion but also eschewing an implausible one.

This paper aims to propose a new theoretical perspective for describing this overlooked aspect of informal reasoning and draws its implications from an emerging inferentialist perspective. The structure of this paper is as follows. First, referring to D. Walton's (1990, 2008) theory of informal logic, we propose two different theoretical perspectives: the semi-formal and the negotiation perspectives. Previous statistics education research from the former perspective may oversimplify the role of spontaneous concepts in informal reasoning. From the latter perspective, we suggest exploring the complexity of the roles of spontaneous and mathematical concepts in informal reasoning. Examining from a more general theoretical perspective, such as Walton's theory, effectively clarifies the issues in studying informal reasoning in mathematics education. Second, we illustrate how the semi-formal perspective is implicitly adopted in previous statistics education research on informal reasoning. Third, we argue the necessity of the negotiation perspective by showing illustrative examples of students' written statistical reports from Kazak et al. (2023) and show an important but radical implication from the negotiation perspective: Decentralising mathematics in mathematics education, i.e., acknowledging the critical role of spontaneous notions in conceptual development in mathematics learning. Fourth, we further introduce the inferentialist perspective, a philosophical viewpoint from the philosophy of language that has recently gained attention in mathematics education research (e.g., Bakker & Derry, 2011; Bakker & Hußmann, 2017; Nilsson, 2020; Uegatani & Otani, 2021, 2023). This perspective indicates what empirical investigations will achieve by decentralising mathematics. Finally, we draw further implications and contributions by discussing the connection of our proposed view with previous research and make conclusions from this paper.

2 Theoretical perspectives: The semi-formal and the negotiation perspectives for informal reasoning

D. Walton is a prominent theorist in the field of informal logic, and his contributions have frequently been applied to educational research (Rapanta, 2022). His key contribution can be found in Walton (1990), where he distinguishes between reasoning and argument. According to him, reasoning occurs in activities such as playing chess or understanding explanations, whereas argument arises within goal-oriented social activities, stating that "*[a]rgument* is a social and verbal means of trying to resolve, or at least to contend with, a conflict or difference that has arisen or exists between two (or more) parties. An argument necessarily involves a claim that is advanced by at least one of the parties" (Walton, 1990, p. 411, italics in the original).

In his framework, reasoning, argument, and dialogue form a hierarchical structure. He suggests that reasoning can occur within an argument, and that an argument can occur within a larger context of dialogue. Building on this hierarchical structure, he proposes the following eight types of argumentative dialogue: critical discussion, debate, inquiry, negotiation, planning committee, pedagogical, and quarrel.

Based on this classification, we would like to point out that previous studies on informal reasoning have implicitly treated informal reasoning as occurring within the argumentative dialogues of critical discussion and inquiry. A critical discussion is an argumentative dialogue where "there are two participants, each of whom has a thesis (conclusion) to prove" (Walton, 2008, p. 4). This type of dialogue is further explained as "my obligation should be to prove that thesis from premises that you accept or are committed to. Your obligation is to prove your thesis from premises that I accept or am committed to" (Walton, 2008, p. 4). Additionally, an inquiry is a dialogue where "premises can only be propositions that are known to be true, that have been established as reliable knowledge to the satisfaction of all parties to the inquiry" (Walton, 2008, p. 5). One example of this type of dialogue, as cited by Walton himself, is a report on an air crash disaster. As Walton (1990) also points out that the strong distinction between formal logic and informal logic is an illusion, the two types of argumentative dialogue, critical discussion and inquiry, are formulated under the influence of the framework used in formal logic, which involves correctly deriving conclusions from premises. For this reason, we understand that mathematics education research also implicitly characterises informal reasoning under the formal logic framework. Therefore, we will refer to this implicit perspective as the *semi-formal perspective*. However, in this paper, we propose an alternative possibility: that informal reasoning can be considered a form of argumentative dialogue, which we call *(self-)negotiation*.

Walton (1990) suggests that arguments can exist without being in dialogue. However, as Ernest (2016) pointed out, a competent mathematician must internalise both the roles of proposer and critic; we believe that in certain mathematical activities, even argumentation conducted by a single individual can exhibit aspects of dialogue. Thus, we propose extending Walton's type of argumentative dialogue(negotiation), which originally assumes negotiation between two or more parties, to include self-negotiation within an individual.

We introduce this *negotiation perspective* as a new theoretical perspective on informal reasoning. According to Walton (1990), "[i]n contrast to persuasion dialogue, negotiation is a form of interest-based bargaining where the goal is to 'get the best deal'" (p. 412). As we will illustrate in later sections, statistical arguments can play a role not only in solving given problems but also in identifying current issues and proposing better solutions, necessitating the reconciliation of conflicting interests. A similar tendency can be observed in arguments involving mathematical modelling. Therefore, we argue that when investigating informal reasoning in mathematics education research, it is essential to view it as a process that includes reconciling multiple conflicting interests.

Our concerns about the semi-formal perspective align with Dawkins and Karunakaran (2016). Specifically, using a framework that understands informal reasoning as akin to formal logic carries the following risks: "(1) researchers filter as noise some aspect of student behaviour that is in some way essential to understanding the emergent phenomenon or (2) researchers' framing and theory impose structures on student behaviour that are not native to it" (p. 73). Hence, we aim to illustrate how statistical arguments are constructed through informal reasoning and how this is appropriately seen from the negotiation perspective.

Figure 1 schematically contrasts the semi-formal perspective with the negotiation perspective. The semi-formal perspective envisions informal reasoning within statistical arguments as a process where one conclusion is derived from prior knowledge and data through reasoning. In contrast, the negotiation perspective envisions a process where multiple possible results are derived from prior knowledge and data, followed by an additional step of negotiating and selecting a better option. Based on Walton's (1990)

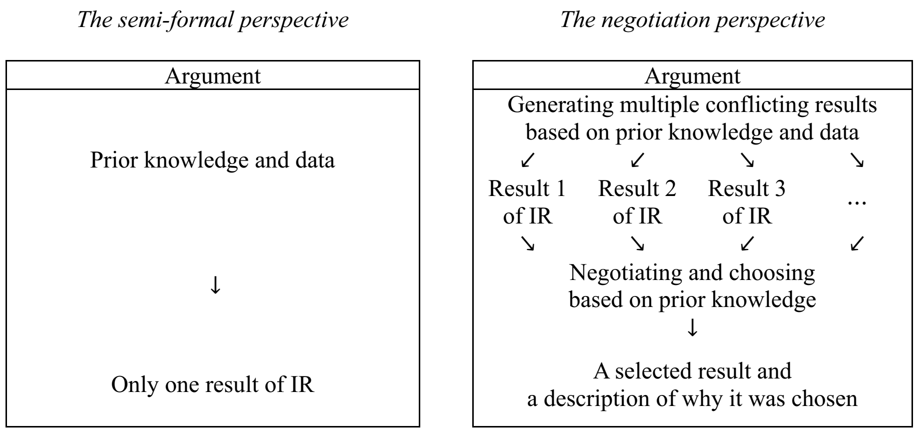

Fig. 1 The semi-formal and the negotiation perspectives for understanding the roles of informal reasoning in arguments (Note: IR in the table stands for informal reasoning.)

distinction between reasoning and argument, Fig. 1 illustrates that the negotiation perspective not only recognises the fact that informal reasoning constructs arguments but also shows that within an argument, there can be informal reasoning that produces results not adopted as conclusions.

Note that Fig. 1 is somewhat simplified to represent the core aspects of informal reasoning from the negotiation perspective. For example, suppose that there is data on students' mathematics test scores. In this case, multiple conflicting results could be generated regarding the implications from the mean of this data, as well as multiple conflicting results regarding the implications from the median. After resolving each conflict, there could be a higher-level choice of whether to select only the implications from the mean, only the implications from the median, or the implications from both. Figure 1 represents only a single level of negotiation and choice to avoid complicating the figure.

The semi-formal perspective is a more abstract view that abstracts away the negotiation process present in the negotiation perspective. In this sense, these perspectives only indicate differences in the resolution of the phenomenon and can be chosen based on the research objectives. However, the semi-formal perspective carries the risk of giving the following two impressions: (1) informal contextual knowledge contributes only to the derivation of conclusions; (2) informal reasoning can be analysed based on its form. In this paper, from the negotiation perspective, we would like to emphasise the following two points to address the risk: (1) informal contextual knowledge contributes both to deriving the final conclusion and to eschewing conflicting possibilities; (2) informal reasoning is a negotiation-based process that depends on what conflicting options are proposed, and it can be analysed based on the content of these options rather than the form of reasoning. We believe that the negotiation perspective is useful in describing how the content of informal contextual knowledge influences students' informal reasoning. In the following sections, we will demonstrate how previous studies implicitly adopt the semi-formal perspective and how we can better describe actual students' statistical reports from the negotiation perspective.

3 The semi-formal perspective for informal reasoning in previous studies

We can point out two characteristics of the semi-formal perspective: (a) Mathematical and statistical knowledge should play a key role in formal and informal reasoning in a normative sense; (b) Only one result is drawn from data, prior knowledge, and norms within an argument. Let us raise two examples of previous studies implicitly based on this perspective.

As the first example, Zieffler et al. (2008) articulated that informal inferential reasoning (IIR), which is a kind of informal reasoning, involves what students make judgments, claims, or predictions.

> [W]e proposed a working definition of IIR that comprises three components: (1) making judgments, claims, or predictions about populations based on samples, but not using formal statistical procedures and methods (e.g., p-value, *t* tests); (2) drawing on, utilizing, and integrating prior knowledge to the extent that this knowledge is available; and (3) articulating evidence-based arguments for the judgments, claims, and predictions about populations based on samples. (pp. 52-53)

Zieffler et al. (2008) associate this working definition directly with what students make judgments, claims, or predictions based on samples. It also emphasised the need for formal statistical procedures and methods.

As the second example, Pfannkuch (2011) documented informal inferential reasoning (IIR) in classroom interaction between the teacher and the students. She argues:

> The data-context […] emerged spontaneously in the development of IIR concepts and took learners' attention away from the properties being revealed. Prior knowledge about how to use data-context in statistical enquiry emerged and took precedence in the evidence presented. This implies that instruction may need to deliberately suppress the data-context at salient moments. (pp. 42-43)

In this view, the data-context, which includes "contextual knowledge of the real-world situation, that is, the subject matter knowledge" (p. 28), has a more essential influence on the results of students' informal reasoning than mathematics and statistics. Kazak et al. (2023) also observe a similar tendency in students' written statistical reports.

These examples reveal that informal contextual knowledge plays a significant role in determining the result of IIR. They indicate that students' reasoning is often evidence-based but statistically informal due to informal contextual knowledge when they generalise a result inferred from samples to a population. However, this view implicitly depends on the *semi-formal perspective*. The perspective implies that informal contextual knowledge can impede students' appropriate generalisation from samples to populations. We could assume the following counterfactual explanation: If students formally applied mathematical and statistical knowledge to given situations, students could obtain appropriate results; but, because they tend to apply informal contextual knowledge to given situations, they obtain only idiosyncratic results. This view implies the following two claims: (a) Mathematical and statistical knowledge should play a key role in formal and informal reasoning in a normative sense, but informal contextual knowledge actually plays a significant role; (b) informal contextual knowledge as a counterfactor against mathematical and statistical knowledge functions as a provider of idiosyncratic inference rules to students and leads to only one result of reasoning within an argument.

Even research on informal reasoning from a social perspective often depends on the semi-formal perspective. For example, Schindler and Seidouvy (2019) view a social norm as a provider of idiosyncratic inference rules to students. Such a rule-based conceptualisation of informal inferential reasoning is analogous to the mechanistic application of mathematical and statistical knowledge to given situations.

Thus, explanations of students' informal reasoning are often not completely detached from their specific context but rather are abstract enough to be somewhat applicable to other situations. This way of explanation assumes the existence of implicit rules governing informal reasoning. This means that researchers are exploring how informal reasoning can be described in terms of its form. As Walton (1990) argued, the study of informal logic tends to become partially formal. Such an explanation obscures the influence that informal reasoning, which led to results not being adopted as a conclusion, had on the conclusion within an argument.

We appreciate the semi-formal perspective because it revealed that students generalise differently, not due to the lack of appropriate knowledge but due to the possession of different knowledge. However, the semi-formal perspective overlooks which claims students eschew, thereby imposing a particular structure on their behaviours.

4 The negotiation perspective for informal reasoning in students' statistical reports

In this section, we argue the necessity of the negotiation perspective by illustrating interesting examples of students from Kazak et al. (2023). In this study, the two students, S1 and S10 (17–18-year-old male students studying for the UK Business and Technology Education Council (BTEC) qualifications), were chosen from 42 students because they took contrasting approaches to the same statistical task. These students were given a statistical task of exploring airborne pollutants in Exeter (UK) and were asked to answer four questions as their final reports. The four questions were: (1) Have particulate matter (PM)10 levels reached a dangerous level? (2) Are PM10 levels rising over time? (3) What time of day are PM10 levels the highest? (4) Is there any correlation between PM10 levels at the two sites? These questions were posed as their written report for the summative assessment at the end of the 9-hour instruction for developing their statistical literacy. Here, we will show their written arguments to the first question by S1 and S10. After that, we will argue the limitation of the semi-formal perspective and the necessity of the negotiation perspective. In the following, based on the negotiation perspective, their arguments are seen as dialogues that negotiate the multiple results of informal reasoning.

4.1 S1's argument

As a response to the question (*Have PM10 levels reached a dangerous level?*), S1 concluded that the PM10 levels had not reached a dangerous level, stating as follows:

> From the data provided, I can safely say that overall the levels of PM10 have not reached a dangerous level. The government has provided a table that lists the different boundaries of air quality: [Tables].

S1 used the adversative conjunction "however" to contrast his conclusion with a potential opposite view, that the levels were dangerous.

> Using the colour scale and the data provided, I was able to pick out that in Alphington St. on 03/11/2017 20:00 the PM10 levels were up to 78.776 which puts it in the index 7 and is relatively high. *However*, it should not be anything to worry about considering it was only for one hour and it jumped back down to index 5 at a value of 58.855. (Emphasis added)

He supposed that one could judge that the observed relatively high PM10 level of 78.776 was dangerous, while he showed his opposite opinion that it was not dangerous because "it was only for one hour and it jumped back down". He continued and provided his conclusion as follows:

> I would not consider this to be a dangerous level, especially due to the fact it only lasted an hour. This only happened on one other occasion, on 29/11/2016 19:00 in RAMM [a museum in Exeter] which the level was recorded at 79.536 and puts it in the index 7 in the high band. Like the instance on Alphington St. in 2017, this only lasted an hour before it jumped back down to 57. Considering how rare this is to happen, it could either simply be an anomaly or an error in the recording of the data, which seems to be likely bearing in mind how much missing data there is.

4.2 S10's argument

S10 did not clearly say that the levels are dangerous. He first mentioned that there are two criteria for PM10 safe levels. By using an adversative conjunction "however", he compared the difference between the World Health Organization (WHO)'s and European Union (EU)'s criteria:

> PM10 safe levels are set by two organizations, the World Health Organization and the European Union. *However*, the differ in size, with the EU setting a 40 μg/m3 safe limit while the world health organization set their limit at 20 μg/m3. (Emphasis added)

Next, S10 used the adversative conjunction "however" again to consider applying the WHO criterion to the given issue. He understood the spiked levels as rare and the third quartile as higher than the criterion.

> These are both annual averages. The mean records are all below the EU limit, whereas one site – Alphington 2017 – is over the WHO safe guidelines by almost .5 μg/m3. PM10 has spiked up to three or even four times safe guidelines in rare occurrences, *however* the third quartile is often around the 20-25 μg/m3 mark showing the higher end of results are mostly over the WHO limit. (Emphasis added)

4.3 Comparison between S1's and S10's arguments

The two students, S1 and S10, drew different conclusions from the same data provided by their teacher. By comparing their arguments, we can identify two differences in their understanding of the given situation.

First, S1 understood the relatively high PM10 levels as potentially dangerous and the moderate values as safe, while S10 did not explicitly judge whether the levels would be dangerous. Generally speaking, the meanings of height and danger (moderate and safe) differ. These connections between height and danger (and between moderate and safe) are not inevitable and thus represent how S1 understood the potential danger in this context. In contrast, S10 recognised that he should not explicitly judge whether the levels would be dangerous. We can consider his attitude influenced by the fact that the two criteria provided different results. The following quotation is a part of the concluding section of S10's written report, albeit not a direct response to the question. By using an adversative conjunction "however" while he appreciated the value of the data, he argued the necessity of a sufficient amount of the data.

> The data Exeter City Council provided was invaluable in studying the levels of PM10 around the city. *However*, there are several ways that I believe the data could be further improved. Firstly, having more than two sensors across the city would be a massive step in the right direction. More sensors mean more datapoints, which will only improve the information that can be gained. Spreading these sensors out across the city would also give us a good idea of what areas are differently affected and how exactly PM10 acts as a whole across the city. This data could then be cross referenced with traffic, weather and other pollution data to get a much more advanced idea of the problem of pollution in Exeter. While it would be expensive, pollution is an evergrowing issue across the world and knowing more about it is the first step in combating the problem. (Emphasis added)

Second, S1 understood some observed data as potentially dangerous but simultaneously not as actually dangerous, while S10 negated that rare occurrences of high levels could indicate safety. S1 viewed some data as an anomaly or an error. He understood a rarely observed value far from its temporally neighbour values as an anomaly and the amount of missing data as the likelihood of an error occurrence. He connected the situation with a probabilistic idea of "rare". In contrast, S10 appealed to the third quartile rather than the spiked values. He connected the same situation with a statistical idea of "quartile".

Both S1 and S10 commonly recognised the same fact that the relatively high index was sometimes but only rarely recorded. Nevertheless, S1 distinguished between potential and actual dangerousness. S10 recognised that some people, like S1, could view the PM10 levels as safe due to the rare occurrence of the relatively high index. Notably, S10 also noticed one more possibility: The data was insufficient to conclude whether safe or dangerous. Using the adversative conjunction, "however" both of them compared multiple possibilities rather than obtain only one possibility through the mechanistic application of their informal contextual knowledge to the given data. They *eschewed* the relatively unpromising possibilities and adopted the most promising one. The same situation activated their different mathematical and everyday ideas and led to their different conclusions.

4.4 A limitation of the semi-formal perspective and the necessity of the negotiation perspective

The contrast between the two students indicates the limitation of the semi-formal perspective. Reasoning with informal contextual knowledge is not simply deriving a conclusion from data. It is comparing multiple alternatives and choosing the most plausible one between them. Hence, the conclusion of informal reasoning is always relative to what alternatives students can imagine. In fact, unlike S10, S1 did not consider the possibility that the data was insufficient for judging safety. A critical difference exists between choosing one from two alternatives (safe and dangerous) and three (safe, dangerous, and undeterminable). The semi-formal perspective cannot describe this relativity because of its formal nature.

The negotiation perspective, which we developed based on Walton's theory (1990, 2008), overcomes this limitation of the semi-formal perspective due to the following two characteristics: (a) Informal contextual knowledge plays a crucial role in creating multiple alternatives and negotiating what alternative is the most promising; and (b) Mathematical and statistical knowledge works for choosing and negotiating the most promising alternative within an argument (see also Fig. 1). When we, as researchers, explore students' informal reasoning, this characterisation is important because it emphasises the creation of alternatives. We agree that previous studies investigate negotiation among students (or between students and teachers). However, we argue that even when a student makes a statistical argument without the teacher's and the classmates' support, the student needs to create alternatives and negotiate which is a better conclusion. In fact, the aforementioned example of S1 and 10 clearly demonstrates how each engaged in an internal negotiation of multiple possibilities, eschewing the relatively unpromising ones. *Self-negotiation is an essential process of informal reasoning*.

The theoretical key to this perspective is, of course, negotiation. Vygotskian theories in mathematics education have revealed the role of negotiation of meaning in learning mathematics. For example, Ernest (1998), who generalised Lakatos's logic of mathematical discovery (Lakatos, 1976), describes proofs and refutations as negotiations both in the history of mathematics and in mathematics classrooms. Roth (2016) argues that mathematical reasoning arises from negotiating norms between people. Radford (2016) describes mathematical knowledge as a common work of joint labour between teachers and students. These Vygotskian theories share the common idea that the meanings of words are *negotiated* and, hence, *adjusted* through social communication. The same holds in educational research on students' application of mathematics to real-world situations, such as using statistics and mathematical modelling.

Let us elaborate on this point. Vygotsky (1987) referred to ways of thinking that arise from everyday practices rather than explicit instruction as "spontaneous concepts" or "everyday concepts", distinguishing them from "scientific concepts" (see Chapter 6, and the Editor's note on p. 168). He then argues, "*scientific concepts develop differently than everyday concepts*" (p. 172, italics in the original) and "[t]he scientific concept blazes the trail for the everyday concept" (p. 169). For example, S1 attempts to negotiate the meaning of "safe" by associating the spontaneous concept of "safe" with the probabilistic concept of "rare" rather than associating the spontaneous concept of "danger" with the mathematical concept of "relatively high". Similarly, S10 attempts to negotiate the meaning of "danger" by associating the spontaneous concept of "danger" with the concept of "third quartile" rather than associating the spontaneous concept of "safe" with the probabilistic concept of "rare". Here, they adjusted the meanings of the concepts "safe" and "danger" based on the mathematical concepts. Of course, S1 and S10 did not create the meanings of "safe" and "danger" for the first time in this context; rather, they synthesised the meanings of these concepts from their previous understanding and the mathematical concepts they chose to use in this context, thereby updating the meanings of "safe" and "danger" to what they consider valid in this context. At the same time, these students are also updating the meanings of the mathematical concepts "relatively high", "rare", and "third quartile" by considering how they should be used in the context of air pollution. As Vygotsky (1987) stated, "the development of the corresponding concept is not completed but only beginning at the moment a new word is learned" (p. 241), the meanings of words are adjusted through negotiation whenever they are used in new contexts.

In the case of S1 and S10, the questions posed by the teacher triggered their internal negotiation between multiple possibilities. The eschewal of less promising options and the choice of the most promising one emerged socially. Any statistical inference by students, whether formal or informal, is their product of negotiation between multiple alternatives in social domains.

Negotiation should be essential by the very nature of argument with informal reasoning, even though existing literature may not emphasise it. If formal statistics methodologies do not validate an argument, it is a result of choice by the argumentator in Morgan's (1996, 2006) sense. It unavoidably involves screening and choosing alternative arguments. As Ernest (2016) points out in his discussion on the unit of analysis in mathematics education, one cannot become a competent mathematician without internalising both the roles of proposer and critic. When analysing a statistical argument, we should consider the self-negotiation expressed within the report as the unit of analysis.

Note that the negotiation perspective is different from the existing attempt to apply the Toulmin model to statistical reasoning (Gómez-Blancarte & Tobías-Lara, 2018) because Rebuttals in the Toulmin model refer to uncertainty and limitation of reasoning rather than to alternative arguments. Therefore, we should pay attention to what conclusion the students finally chose and what they considered in arguments but did not choose. As described in Fig. 1, the negotiation perspective can show a different role of students' knowledge in informal reasoning than the semi-formal perspective.

If we switch our perspective for informal reasoning from the semi-formal to the negotiation perspectives, the roles of knowledge in informal reasoning change drastically: Informal contextual knowledge plays a critical role in creating potential alternatives rather than in producing only one result; mathematical and statistical knowledge plays a vital role in eschewing unpromising alternatives and choosing the most promising one rather than in deriving only one rational claim from given data. Therefore, this shift has an essential but radical implication for mathematics education research on conceptual development: Decentralising mathematics.

In this paper, we refer to the attitude of studying the development of a mathematical concept without over-focusing on it, and instead paying attention to various surrounding concepts, including spontaneous concepts, as "decentralising mathematics". However, decentralising mathematics only indicates an attitude toward research and does not directly specify what kind of empirical research would fulfil this attitude. In the next section, we will discuss how the philosophy of inferentialism can contribute to the realisation of decentralising mathematics.

5 Implementing decentralising mathematics: An inferentialist approach

Researchers often believe that the specialty of mathematics education research stems from the nature of mathematics (Heid, 2010; Sierpinska & Kilpatrick, 1998). However, as the *negotiation perspective* suggests, informal contextual knowledge is more important for producing potential alternatives than mathematical and statistical knowledge. Although mathematical and statistical knowledge is still crucial for negotiating and choosing the most promising alternative if mathematics teachers want to foster their students' ability to use statistics, the negotiation perspective implies that they should support their students to establish informal contextual knowledge besides mathematical and statistical knowledge. Students can improve their statistical reports through their ability to connect informal contextual knowledge with mathematical and statistical knowledge rather than through their ability to apply mathematical and statistical knowledge simply to contexts. Hence, mathematics education research should also focus on developing informal contextual knowledge with everyday notions. The trigger for activating mathematical knowledge is not mathematical knowledge itself but *everyday notions*.

Inferentialism, one of the contemporary philosophies originally proposed by Brandom (1994, 2000), is well-suited for the new research purpose of investigating the empirical relationship between everyday notions and mathematical and statistical notions from the negotiation perspective. It is philosophical semantics (i.e., a theory of meaning) based on pragmatics (i.e., the actual use of words). Inferentialism is introduced in educational research by a Vygotskian researcher, Derry (2008, 2013a, 2013b, 2016). Through reassessing Vygotskian thoughts from the inferentialist standpoint, inferentialists deny the higher priority of scientific concepts than spontaneous ones (Derry, 2008, 2013b) and identify the interwoven development of both types of concepts in statistics learning (Bakker & Derry, 2011). The inferentialist ideas are now widely applied not only to statistics education but also to mathematics education (Bakker & Hußmann, 2017; Nilsson, 2018; Noorloos et al., 2017; Ryan, 2019; Ryan & Chronaki, 2020) and to science education (Causton, 2019).

The crucial characteristic of inferentialism is its complete reverse order of explaining conceptual understanding. From the inferentialist angle, how to use a concept determines what it means.

> An account of the conceptual might explain the *use* of concepts in terms of a prior understanding of conceptual *content*. Or it might pursue a complementary explanatory strategy, beginning with a story about the practice or activity of applying concepts, and elaborating on that basis an understanding of conceptual content. The first can be called a *platonist* strategy, and the second a *pragmatist* (in this usage, a species of functionalist) strategy. […] [Inferentialism] is a kind of conceptual pragmatism (broadly, a form of functionalism) in this sense. It offers an account of knowing (or believing, or saying) *that* such and such is the case in terms of knowing *how* (being able) to *do* something. (Brandom, 2000, p. 4, italics in the original)

In this sense, inferentialists do not explain students' ways of using mathematical terms based on their understanding. Instead, inferentialists explain their understanding based on their ways of using mathematical terms by reversing the order of explanations.

This change of views implies an inferentialist distinction between notions and concepts. Uegatani and Otani (2023) argue that notions (words) first appear in conceptual development, and concepts (the inferential connection between notions) appear second. For example, when the notion of *two* activates the notion of *prime numbers* in the form "two is a prime number", it determines the conceptual understanding of two and prime numbers in this context.

Let us illustrate this drastic change of views by using the two students, S1 and S10, shown in the previous sections again. For example, consider when they were asked whether the air pollution levels were dangerous. While S1 did not view the rare occurrence of relatively high PM10 levels as actually dangerous, S10 argued that the third quartile of the levels was more important than the rare occurrence of the relatively high levels because the third quartile was higher than the WHO limit. The concept of danger activated the different statistical concepts: rare occurrence for the two students and third quartile only for S10. This claim does not mean that the different understandings of the concept of danger activate different statistical concepts. Instead, this activation determines their conceptual understanding of danger. Using a notion determines its conceptual understanding at that moment.

This characteristic view on conceptual understanding implies how conceptualisation occurs. Inferentialists argue that to express is to conceptualise.

> [W]e might think of the process of expression in the more complex and interesting cases as a matter not of transforming what is inner into what is outer but of making *explicit* what is *implicit*. This can be understood in a pragmatist sense of turning something we can initially only *do* into something we can *say*: codifying some sort of knowing *how* in the form of a knowing *that*. (Brandom, 2000, p. 8, italics in the original)
>
> The process of explicitation is to be the process of applying concepts: conceptualising some subject matter. (Brandom, 2000, p. 8)

Although this view may be radical for some readers of this paper, it is consistent with Radford's (2009) semiotic idea that expressing is a genuine form of thinking. It is also compatible with Roth's (2016) sociogenetic view. Conceptual understanding develops in social domains rather than in psychological ones. Therefore, we should understand that new conceptualisation occurs when students use words (Uegatani & Otani, 2021, 2023;

for an empirical example, see Uegatani et al., 2023). From the negotiation perspective, this idea of conceptualisation implies that the creation of options for negotiation is inherently involved in the process of conceptualisation itself.

Inferentialists further describe communication as a game of assessing others' assertions to check if they can become premises for one's assertions. Inferentialists call this game *a game of giving and asking for reasons*.

> The context within which concern with what is thought and talked *about* arises is the assessment of how the judgements of one individual can serve as reasons for another. The representational content of claims and the beliefs they express reflect the social dimension of the game of giving and asking for reasons. (Brandom, 2000, p. 159, italics in the original)

In the game, players judge whether assertions are true or false and what the assertions are true *about*. The players' locutions "make the words 'of' and 'about' express the intentional directedness of thought and talk" (Brandom, 2000, p. 169). In inferentialism, the game of giving and asking for reasons, which is essentially a social practice, is intentionally directed by the words "of" and "about", and this function of the words "of" and "about" is called *aboutness*. Hence, by analysing what topics students derive from what topics, inferentialists explore how they conceptually understand the topic in a given communication.

For example, the two students, S1 and S10, developed their understanding of danger by writing statistical reports. Although they were familiar with the concept of danger, they developed their understanding of it in the context because they made a new connection with the statistical concepts of rare occurrence and the third quartile. In this sense, students' written reports *about* the dangerousness of airborne pollutants as responses to whether the PM10 levels are dangerous are games of giving and asking for reasons. Their word choices made these communications statistical. *The spontaneous concept of danger develops mathematically*, and at the same time, *the mathematical concepts of rare occurrence and third quartile develop spontaneously* through experiences of writing these reports. Thus, the inferentialist view allows researchers working with the negotiation perspective to take a serious look at *the mathematical development of spontaneous concepts* and *the spontaneous development of mathematical concepts*.

We can explore the mutual development of spontaneous and mathematical concepts through the inferentialist idea of the game of giving and asking for reasons. For example, S1 used the probabilistic concept of "rare" to justify that air pollution was not in a dangerous state. Similarly, S10 showed the rationale for focusing on the statistical concept of "the third quartile", highlighting that annual averages do not indicate danger. As these examples show, spontaneous notions can be used as reasons for focusing on a mathematical notion and vice versa. While spontaneous and mathematical concepts may not intersect as knowledge systems, they do intersect and mutually develop within the game of giving and asking for reasons. Therefore, from the inferentialist perspective, examining how spontaneous and mathematical notions intersect can lead to a better understanding of the actual developmental process of mathematical concepts.

Previous research on informal reasoning problematically assumed that informal contextual knowledge might play only an auxiliary role in informal reasoning. Additionally, prior inferentialist research has not comparatively illustrated students who responded differently to the same tasks, merely suggesting the importance of spontaneous concepts in mathematics learning. However, as a unique contribution of this paper to mathematics education research, we argue for the potential benefit of focusing on the mathematical development of spontaneous concepts for studying the development of mathematical concepts through the inferentialist lens. We can explore how spontaneous notions generate options using mathematical and statistical notions in the negotiation process to understand the mutual development of these concepts. We can capture the development of mathematical concepts more precisely through decentralising mathematics in mathematics education research.

Note that the specific ways in which spontaneous and mathematical concepts constitute claims and reasons remain an open empirical question from the inferentialist perspective. For instance, the illustrative examples of S1 and S10 suggest that the concept of "danger" could elicit the concepts of "rare" or "third quartile", but it could also elicit other mathematical concepts. What possibilities exist is open to empirical investigation. Furthermore, researchers must consider how teachers can support students' informal contextual knowledge in mathematics classrooms since it differs from mathematical knowledge.

6 Further implications and contributions from the combination of the negotiation perspective and the inferentialist perspective

In this section, we move beyond informal reasoning to explore broader research perspectives, discussing how our view aligns with or differs from the theoretical insights in previous studies: (1) Even though we decentralise mathematics in mathematics education research, studying the interwoven and inseparable relationship between mathematical and spontaneous concepts is a still important research topic in this field; (2) We deny the existence of abstract rationality across a variety of situations; and (3) Our proposed perspective helps investigate how people develop mathematical and spontaneous concepts simultaneously in real societies.

First, mathematics education researchers should increasingly focus on both mathematical and spontaneous concepts (Bakker & Derry, 2011; Derry, 2008) because, as illustrated in this paper, the development of these concepts is interrelated. In addition, we argue that mathematical and spontaneous concepts are interwoven and inseparable in a more radical sense than existing inferentialist studies have argued. S1 and S10 used adversative conjunctions to consider opposite answers to their own. They explicitly contrasted multiple ways of modelling the current data mathematically. Their conclusion depended on their different ways of conceptualisation of danger. As this example indicates, researchers cannot judge which aspect is foreground in the problem, mathematical or spontaneous. In other words, we can describe these students' activities either as statistical problem-solving in an everyday context or as everyday problem-solving in a statistical context. It is excessively theory-laden to centralise statistical concepts for explaining the target phenomena. Researchers can understand the inseparability between texts and contexts (Bakker & Derry, 2011; Roth, 1996) because spontaneous notions can trigger mathematical notions, or vice versa.

We can make a more radical claim from this perspective than the existing inferentialist studies in mathematics education. For example, although Schindler and Seidouvy (2019) reveal that a normative way of answering statistical questions influences IIR, such a norm is not specific to mathematics. We might not have to call it a socio-mathematical norm (Yackel & Cobb, 1996). This view is a consequence of decentralising mathematics. There is a different possibility that a non-mathematical social norm dramatically influences what is a good way of answering. This means that students can eschew certain ways of answering and opt for the best one, based on the context.

For example, S10 suggested that the government should equip more sensors to improve data on PM10 levels. He avoided directly answering the question of whether the levels are dangerous. However, some people may feel that the rare occurrence of relatively high levels is sufficiently dangerous. Does this mean reporters follow different norms depending on the purposes of writing statistical reports? Even though they are not socio-mathematical, if they are related to reflective knowledge on how mathematics can be used (Skovsmose, 1990), various types of norms in the use of mathematics are still essential topics in mathematics education research.

As the field of mathematics education research expands (Wagner et al., 2023), there is a growing need to address not only issues like air pollution but also other political, economic, and ecological concerns within the scope of mathematics education research. These include problems related to infectious disease testing, such as COVID-19 (Uegatani et al., 2021), and the increasingly noted issue of climate change (Barwell & Hauge, 2021). Investigating the empirical interplay between informal contextual knowledge and mathematical knowledge will likely be one of the keys to effectively tackling these challenges.

Second, following Derry (2008), we question the existence of abstract rationality. As illustrated in this paper, the mutual development of mathematical concepts with the spontaneous concepts used in that context is essential for their effective application. Abstract statistical knowledge itself does not offer rational decisions. The negotiation perspective suggests that depending on contexts, spontaneous concepts should always play a key role in making rational decisions.

As our inferentialist framework assumes, students have many default assumptions for their situations. They make sense of their situations not only based on their perceptions but also on their many default assumptions. The variation of their interpretations stems from such default assumptions. From this angle, the processes of negotiating meanings include making implicit default assumptions explicit and contrasting between their own and the others' assumptions (Ishibashi & Uegatani, 2022). Hence, we can pose the following new research question from the inferentialist perspective: How do the same students connect mathematical and spontaneous concepts differently to answer the same questions depending on the situation? Even though students try to answer the same question, a subtle difference between situations changes the aboutness of the discourse in the game of giving and asking for reasons. In this sense, experiences of negotiating conclusions with peers can change their default assumptions, and we expect them to develop their negotiating skills for some controversial topics even when they write statistical reports by themselves. In particular, we may be interested in how students alter their approach to eschewing certain options through such experiences. The existing literature argues that exchanging opinions and making their implicit default assumptions conscious is necessary for developing higher-order rationality (Fujita et al., 2019; Kazak et al., 2015). Because our inferentialist framework is consistent with a methodological framework of design research (Uegatani & Otani, 2021), this necessary condition could be tested through future design research (for more information on design research, see Bakker, 2018). For example, in a classroom setting, design research can be used to explore how and in what situations teachers can pose questions to make students' implicit default assumptions explicit.

Third, from our proposed perspective, researchers can examine how mathematics education can address "crises" (Skovsmose, 2019, 2021). Different paths of developing spontaneous concepts with different mathematical concepts lead to students' different conclusions. However, the negotiation perspective further emphasises spontaneous concepts more than mathematical concepts in producing potential alternatives. Existing literature argued the importance of reflective knowledge on the role of mathematics in society (Barbosa, 2006; Skovsmose, 1990) and called attention to not only what mathematical concepts to use but also how and when to use them (Lavie et al., 2019; Sfard & Lavie, 2005). By decentralising mathematics, we can investigate how people simultaneously develop mathematical and spontaneous concepts in societies. In addition, we should also investigate how people eschew the use of mathematical knowledge in societies. These investigations will provide fundamental data on reflective knowledge of mathematics.

7 Conclusion

In this paper, based on Walton's (1990, 2008) distinction between argument and reasoning and Ernest's (2016) concept of intrapersonal dialogue, we proposed the semi-formal and the negotiation perspectives and problematised the implicit adoption of the semi-formal perspective to informal reasoning in statistics education, which overfocuses on generating a single claim by applying a mechanistic application of statistical concepts. Instead, we enhanced the negotiation perspective approach, which characterises informal reasoning as creating multiple conflicting alternatives, eschewing less promising ones, and choosing the most promising one. Our comparison between the two students' statistical written reports illustrated that spontaneous notions could influence students' creation, eschewal, and choice of alternatives. In discussing further implications from the negotiation perspective, we introduced the inferentialist perspective, which proposes to reverse the order of explaining conceptual understanding: Inference determines understanding rather than understanding determines inference. In this view, spontaneous notions like "danger" activate mathematical notions like "rare" and "quartile", and making such inferential connections is conceptual development in discourses. Albeit ironically, mathematics education researchers should consider the mathematical development of spontaneous concepts and the spontaneous development of mathematical concepts to understand the development of mathematical concepts more deeply. Thus, we have titled this paper "decentralising mathematics". There are two aspects of conceptual development: Spontaneous concepts develop mathematically, and mathematical concepts develop spontaneously. Referring to the inferentialist idea of the game of giving and asking for reasons, we proposed an empirical investigation that embodies decentralising mathematics.

There are two future tasks in this line of research. First, we limited our discussion to the role of spontaneous concepts in statistical negotiation. However, further generalizing our implications could be crucial even in pure mathematics and mathematical modelling. For example, if we regard graphical inference and computer assistance as a part of pure mathematical practice (see Hanna & Larvor, 2020), non-mathematical notions also influence conceptual development in pure mathematics. From our proposed perspective, we must collect and analyse empirical data in pure mathematical practice. Second, we should investigate how students enter mathematical practice. The centralisation of mathematics in mathematics education research is a theoretical bias. From our perspective, the distinction between spontaneous and mathematical practices becomes more blurred than researchers have assumed. However, it does not mean that mathematical practice disappears. Mathematical practice actually exists, though we cannot provide a clear cut between spontaneous and mathematical practices. Akkerman and Bakker (2011) contrasted the concept of transfer with the concepts of boundary crossing and boundary objects, stating:

> Although transfer is mostly about one-time and one-sided transitions, primarily affecting an individual who moves from a context of learning to one of application (e.g., from school to work), concepts of boundary crossing and boundary objects are

used to refer to ongoing, two-sided actions and interactions between contexts. (p. 136)

By adopting this perspective of “ongoing, two-sided actions and interactions between contexts”, we can understand that an important mission of mathematics education research is to elucidate phenomena that arise not merely within mathematical contexts but from the interactions between mathematical and spontaneous contexts. Therefore, we need to find some notions that activate another practice and bridge between two practices, and to reveal their roles in mathematics learning.

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Declarations

Ethics approval This study has been approved by the Research Ethics Committee of the University of Exeter (Approval number: STF/17/18/07).

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